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MASTERARBEIT

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Graded free resolutions and applications to combinatorics

Verfasser

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Introduction

Integer partitions are well-studied objects in the field of combinatorics and play a significant role in representation theory. A *partition* of some positive integer n is a weakly increasing sequence of positive integer, where the sum of all elements is equal to n. The number of partitions of n is denoted by $p(n)$.
A celebrated result is the Rogers-Ramanujan identity, stating the following remarkable fact: Denote by $a(n)$ the number of partitions of n, where all elements of the partition are 1 or 4 modulo 5, and denote by $b(n)$ the number of partitions of n, where any two elements of the partition differ by at least 2. Then $a(n) = b(n)$. This was proved by Rogers and Ramanujan, using power series manipulations. Up to this date, there is no combinatorial proof that shows the implicit connection between those two objects (see for example [6]).
The goal of this thesis is to devise new concepts to prove partition identities like the Rogers Ramanujan identity. The starting point is given by the observation that there is a one-to-one correspondence between partitions of n and monomials of degree n of $k[x_1, \ldots, x_n]$ graded via $\deg(x_i) = i$. This allows us to consider partitions in an algebraic setting. In order to work with partition identities, we will use factor rings of the polynomial ring. By breaking the factor ring into several pieces, we will be able to encode the number of partitions into smaller factor rings. Then, with some tools of commutative algebra (mostly free resolutions of the factor ring), we are able to give formulas for the partition numbers, which can result in quick proofs for partition identities.
For the Rogers-Ramanujan identity, the use of free resolutions is not enough to prove the partition identity. However, we can at least show that the numbers $a(n)$ fulfill the following recursion formula:

$$a(n) = p(n) - p(n-2) - p(n-3) + p(n-9) + p(n-11) - \ldots,$$

where the numbers in parentheses are exactly the numbers of the form $\frac{i(5i\pm1)}{2}$. We will also show a new proof of a famous and well-known formula by Euler, stating that

$$0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \ldots,$$

where the numbers in parentheses are exactly the numbers of the form $\frac{i(3i\pm1)}{2}$.

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Conventions

We will often denote rings by the letter R, while we denote fields by the letter k. If not otherwise stated, all rings are assumed to be commutative with unity, so all ideals are two-sided. We denote by $\mathbb{N}$ the set of positive integers, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote by $[n]$ the set $\{1, 2, \ldots, n\}$. If $f(t)$ is a power series in the variable t, we denote by $\langle t^k \rangle f(t)$ the coefficient of t^k in the power series expansion.

1 A brief summary of Gröbner bases and free modules

An advanced knowledge of Gröbner basis theory is helpful, but not needed (see for example [8],[10] or [13]).

1.1 Gröbner basis theory on $k\left[x_1, \ldots, x_n\right]^t$

This section is meant to provide a quick overview. For a detailed analysis of the topic, as well as all omitted proofs of the first section, the reader is adviced to consult [9]. For this section, we suppose that $R = k\left[x_1, \ldots, x_n\right]$.

Definition 1.1. Let $m = cx_1^{\alpha_1} \ldots x_n^{\alpha_n}$ be a monomial in R. A *monomial vector* (or *monomial*) in R^t is a vector me_i, where e_i is the i-th standard basis vector of R^t. We say that me_i *contains* the i-th standard basis vector of R^t.
The monomial $x^\alpha e_i$ is said to be a *multiple* of $x^\beta e_j$ if $i = j$ and x^α is divisible by x^β. If this is the case, we say that $x^\alpha e_i$ is *divisible* by $x^\beta e_j$.

Definition 1.2. If $x^\alpha e_i$ and $x^\beta e_j$ are monomial vectors in R^t, we define the *least common multiple* lcm of $x^\alpha e_i$ and $x^\beta e_j$ to be

$$\operatorname{lcm}\left(x^\alpha e_i, x^\beta e_j\right) = \delta_{i,j} \cdot \operatorname{lcm}\left(x^\alpha, x^\beta\right) e_i,$$

where $\delta_{i,j}$ is the Kronecker delta.

Definition 1.3. A *monomial order* on R^t is a total order $<$ on the set of monomials of R^t with coefficient 1 satisfying

- $<$ is a well-order.
- If $x^\alpha e_i$ and $x^\beta e_j$ are monomials in R^t with $x^\alpha e_i < x^\beta e_j$, then $x^{\gamma+\alpha} e_i < x^{\gamma+\beta} e_j$ holds for all monomials $x^\gamma \in R$.

Every vector $f \in R^t$ can be written as a sum of monomials, $f = \sum_{i=1}^s c_i m_i$, where m_i are monomials and $c_i \neq 0$. Given a monomial order $<$, we can then order the monomials in this expansion with respect to $<$.

Definition 1.4. The largest monomial in the above description that appears with respect to $<$ is called *leading monomial* $\operatorname{LM}_<(f)$ of f. If m_j is the leading monomial, we call $c_j m_j$ the *leading term* $\operatorname{LT}_<(f)$. If the monomial order is unambiguous, we will write $\operatorname{LT}(f)$.

Theorem 1.1. *Let $<$ be a monomial order on R^t. Let $f_1, \ldots, f_s \in R^t$ be an ordered tuple of vectors. Then every $f \in R^t$ can be written as a sum*

$$f = c_1 f_1 + \cdots + c_s f_s + r,$$

where $c_1, \ldots, c_s \in R$, $LT(c_i f_i) \leq LT(f)$ *and* $r = 0$ *or* r *is a linear combination of monomials of which none is divisible by any of* $LM(f_1), \ldots, LM(f_s)$. *The vector* r *is called the remainder upon division.*

Gröbner bases

Definition 1.5. Let M be a submodule of R^t and $<$ be a monomial order on R^t. We define $\mathrm{LT}(M)$ as the set of leading terms of all elements of M with respect to $<$ and $\langle \mathrm{LT}(M) \rangle$ as the submodule the leading terms generate. A finite subset $\mathcal{G} = \{g_1, \ldots, g_s\}$ of M is called *Gröbner basis* for M if $\langle \mathrm{LT}(M) \rangle = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_s) \rangle$.

The remainder r in Theorem 1.1 depends on the order of the f_i. One can prove that if $f_1, \ldots, f_n$ is a Gröbner basis, r is indeed independent of the order of the f_i.

Definition 1.6. Let $<$ be a monomial order on R^t, $f, f_1, \ldots, f_s \in R^t$ as above, and suppose that $f_1, \ldots, f_s$ form a Gröbner basis for M. We write $f \equiv r(f_1, \ldots, f_s)$ if r is the (unique) remainder upon division by $f_1, \ldots, f_s$.

Gröbner bases are of importance because they solve the module membership problem.

Theorem 1.2. *Let* $\mathcal{G}$ *be a Gröbner basis of* $M \subseteq R^t$ *with respect to a monomial order* $<$ *on* R^t *and* $f \in R^t$. *Then* $\mathcal{G}$ *generates* M *as a module and* $f \in M$ *if and only if* $f \equiv 0(\mathcal{G})$.

Existence of Gröbner bases, Buchberger algorithm

Definition 1.7. Let $<$ be a given monomial order on R^t, and let $f, g \in R^t$. Denote by

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(f)} f - \frac{\mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(g)} g \in R^t$$

the *S-vector* of f and g. Note that $S(f,g) = 0$ if the leading terms of f and g contain different standard basis vectors.

Theorem 1.3. *(Buchberger criterion) A set* $\mathcal{G} = \{g_1, \ldots, g_s\} \subset R^t$ *is a Gröbner basis for the module it generates if and only if* $S(g_i, g_j) \equiv 0(\mathcal{G})$ *for all* $1 \leq i < j \leq s$.

Theorem 1.4. *(Buchberger algorithm) Let* $\{g_1, \ldots, g_s\}$ *be a subset of* R^t *and let* M *be the module generated by* $g_1 \ldots, g_s$.. *Compute all remainders of the S-vectors* $S(g_i, g_j)$ *upon division by the set* $\{g_1, \ldots, g_s\}$. *Add all nonzero remainders (in the sense of Theorem 1.1) to the set* $\{g_1, \ldots, g_s\}$ *and continue with this process. This algorithm eventually stops, and the set constructed in the last step is a Gröbner basis for* M.

Reduced and monic Gröbner bases

Definition 1.8. A Gröbner basis $\{g_1, \ldots, g_s\}$ of M (with respect to some monomial order $<$) is called *reduced*, if for all $g_i \neq g_j$, no monomial in g_i is divisible by some $\mathrm{LT}(g_j)$. It is called *monic* if it is reduced and every leading coefficient $\mathrm{LC}(g_i)$ is 1.

Theorem 1.5. *Let* M *be a submodule of* R^t. *There exists a reduced Gröbner basis of* M *and an unique monic Gröbner basis of* M.

Remark 1.1. Every Gröbner basis can be transformed into a reduced Gröbner basis by using the division algorithm.

1.2 Free modules

The reader is assumed to know the notion of free modules. We will briefly introduce to the topic and give a summary of all properties of free modules that we will be used later on. More on free modules can be found in most books on algebra, see for example [7].

Definition 1.9. An R-module M is said to be a *free module over* R if it has an R-basis.

Lemma 1.1. *Let M be a free R-module and $n \in \mathbb{N}$. Then*

1. *R^n is a free R-module.*
2. *If a basis of M consists of m elements, we have $M \cong R^m$.*
3. *We have $R^n \cong R^m$ if and only if $m = n$.*
4. *If M has a finite basis, every basis has the same cardinality.*

Proof. Choose the standard basis $e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$ with $1 \leq i \leq n$, where 1 is in the i-th row and 0 elsewhere. This is clearly a basis, which proves the first claim. For the second claim, fix a basis. Define a homomorphism $\phi : M \to R^m$, sending the i-th basis element of M to the i-standard basis vector e_i of R. This is clearly an isomorphism.
For the third claim, let $\mathfrak{m}$ be a maximal ideal of R, then

$$\left(R/\mathfrak{m}\right)^m \cong R^m/\mathfrak{m}R^m \cong R/\mathfrak{m} \otimes_R R^m \cong R/\mathfrak{m} \otimes_R R^n \cong R^n/\mathfrak{m}R^n \cong \left(R/\mathfrak{m}\right)^n,$$

where $\otimes_R$ is the tensor product over R. Recall that, if k is a field, k^n is isomorphic to k^m if and only if $m = n$. Since $R/\mathfrak{m}$ is a field, the claim follows. The last property follows immediately. □

Definition 1.10. Let M be a free R-module. The number of elements of a basis of M is called the *rank* of M.

Remark 1.2. Free modules are easy to handle, because every element has a unique representation of a (finite) sum of R-multiples of basis elements. If M is not free, the situation is in general a lot harder. One can give a generating system for the module (in the worst case, the module itself), but the generating system need not be linearly independent. In this case, the *relations* between the generators are of big importance.

2 Graded Structures

2.1 Graded rings and modules

We closely follow [9] and [16] in this chapter.

Definition 2.1. Let $R = (R, +, \cdot)$ be a ring. R is called $\mathbb{Z}$-*graded* (or *graded*) if abelian subgroups $\{R_i = (R_i, +), i \in \mathbb{Z}\}$ of $(R, +)$ are given that satisfy $R = \bigoplus_{j \in \mathbb{Z}} R_j$ and are compatible with the ring multiplication, i.e. $R_m \cdot R_n \subseteq R_{m+n}$ holds for all $m, n \in \mathbb{Z}$. In this case, we call an element r of R_i *homogeneous* of degree i and write $\deg(r) = i$. The zero element $0 \in R$ is assumed to be homogeneous of any degree.

Example 2.1.

1. Let S be a ring. The polynomial ring $R = S[x_1, \ldots, x_n]$ is graded via $\deg(x_i) = d_i > 0$, where

$$R_i = \langle \text{monomials of degree } i \text{ in } k[x_1, \ldots, x_n] \rangle .$$

   If $S = k$ is a field, the graded pieces S_i are k-vector spaces. Also note that if $i < 0$, $R_i = 0$. Graded rings that fulfil this condition are called $\mathbb{N}_0$-*graded*.

2. Every ring R is trivially graded via $R = R_0$ and $R_i = 0$ if $i \neq 0$.

3. Let k be a field. The ring $k[x] / (x^2)$ is graded of the form $k \oplus k\overline{x}$ with $\overline{x}^2 = 0$.

4. If $R = \bigoplus_{i \in \mathbb{Z}} R_i$ and $S = \bigoplus_{i \in \mathbb{Z}} S_i$ are graded rings, so is $R \times S$. The homogeneous parts are given by the $R_j \times S_j$. Therefore, if R is a graded ring, so is $R^t = R \times \cdots \times R$.

5. Let R be a graded ring and S be a multiplicatively closed subset of homogeneous elements of R. Then R_S, the localization of R over S, is a graded ring, where the graded parts are given by

$$(R_S)_i = \left\{ \frac{r}{s} \in R_S, r, s \text{ homogeneous in } R, \deg(r) - \deg(s) = i \right\} .$$

Proposition 2.1. *Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a graded ring. Then R_0 is a subring of R and all homogeneous parts R_i are R_0-modules.*

Proof. Consider $1 \in R$ as a sum of homogeneous elements, i.e., $1 = \sum_{i \in \mathbb{Z}} f_i$, where $f_i \in R_i$ and only finitely many are nonzero. Then $f_j = 1 \cdot f_j = \sum_{i \in \mathbb{Z}} f_i f_j$ implies $f_j = f_j \cdot f_0$. Now

$$f_0 = 1 \cdot f_0 = \sum_{i \in \mathbb{Z}} f_i f_0 = \sum_{i \in \mathbb{Z}} f_i = 1,$$

hence $1 \in R_0$. Since $R_0 \cdot R_0 \subseteq R_0$, R_0 is a subring of R. Furthermore, $R_0 \cdot R_i \subseteq R_i$, so R_i is a R_0-module. □

Definition 2.2. Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a graded ring and M be an R-module. The module M is called a *graded R-module* if additive subgroups $\{M_j, j \in \mathbb{Z}\}$ of M exist that satisfy $M = \bigoplus_{j\in\mathbb{Z}} M_j$ and are compatible with the module multiplication, i.e., $R_i M_j \subseteq M_{i+j}$ holds for all $i, j \in \mathbb{Z}$. We call the elements of the graded part M_j *homogeneous of degree j*.

Remark 2.1. If $M = \bigoplus_{j\in\mathbb{Z}} M_j$ is a graded R-module (with $R = \bigoplus_{i\in\mathbb{Z}} R_i$), then M_j is an R_0-module for all $j \in \mathbb{Z}$.

Example 2.2.

1. Every graded ring R is a graded R-module over itself.
2. If M_1, M_2 are two graded R-modules, so is $M_1 \times M_2$.
3. Let $M = \bigoplus_{j\in\mathbb{Z}} M_j$ be a graded R-module and $n \in \mathbb{Z}$ an integer. Define the graded module $M(n)$ as follows: $M(n)$ is isomorphic to M as an R-module, but its graded pieces are *shifted*, defined as $M(n)_i = M_{n+i}$. Then $M(n)$ is a graded R-module.

Definition 2.3. Let $M = \bigoplus_{j\in\mathbb{Z}} M_j$ be a graded R-module and N be a submodule of M. The submodule N is called a *graded submodule* if

$$N = \bigoplus_{j\in\mathbb{Z}} N \cap M_j.$$

Remark 2.2. If N is a graded R-submodule of M, then N is itself a graded R-module, because

$$R_i (N \cap M_j) = R_i N \cap R_i M_j \subseteq N \cap M_{i+j}.$$

The question arises *which* submodules of graded modules are graded themselves.

Proposition 2.2. *Let $M = \bigoplus_{j\in\mathbb{Z}} M_j$ be a graded R-module and let N be a submodule of M. The following are equivalent:*

1. *N is a graded submodule of M.*
2. *An element $f \in M$ is in N if and only if all homogeneous components of f are in N.*
3. *N has a system of homogeneous generators.*

Proof. (1) $\Rightarrow$ (2): Suppose that $f \in M$, $f = f_1 + \cdots + f_k$, $f_i \in M_{d_i}$ is the unique way to write f as sum of homogeneous elements. If $f \in N$, then $f = \sum_{j\in\mathbb{Z}} g_j$ with $g_j \in N \cap M_j$. By uniqueness of the decomposition in homogeneous elements, $f_i = g_i$ if $1 \leq i \leq k$, and $g_i = 0$ otherwise. Therefore, $f_i \in N \cap M_{d_i}$, so $f_i \in N$.
(2) $\Rightarrow$ (1): If $f \in N$, $f = f_1 + \cdots + f_k$ is the (unique) homogeneous decomposition, we have $f_i \in M_{d_i} \cap N$, from which everything follows.
(2) $\Rightarrow$ (3): The set of homogeneous elements (with respect to the graded module M) in N fulfils the conditions.
(3) $\Rightarrow$ (2): Let $f \in N$ and suppose that $f = f_1 + \cdots + f_k$ is the unique decomposition in homogeneous parts, where $f_j \in M_{d_j}$. Suppose that S is a system of homogeneous generators

of N, then there exist $r_1, \ldots, r_n \in R, s_1, \ldots, s_n \in S$, $s_j \in M_{e_j}$, satisfying $f = \sum_{i=1}^{n} r_i s_i$. Since R is graded, there exists a unique decomposition into homogeneous elements $r_i = \sum_{j \in \mathbb{Z}} r_{i,j}$ for $1 \leq i \leq n$. Therefore

$$f = \sum_{i=1}^{n} \sum_{j \in \mathbb{Z}} \underbrace{r_{i,j} s_i}_{\in M_{e_i+j}} .$$

By uniqueness, every f_j is therefore a sum of the form $\sum r_{k,l} s_l$. Since $s_1, \ldots, s_n$ are elements of N, then f_j is for all $1 \leq j \leq k$. □

Corollary 2.1. *Let R be a graded ring and I be an ideal. The ideal I is graded (as submodule of R) if and only if I is generated by homogeneous elements.*

Proof. Put $M = R$ in Proposition 2.2. □

Example 2.3. If R is a noetherian ring, $R[x_1 \ldots, x_n]$ is noetherian by the Hilbert basis theorem. This ring is also graded via the degree function, i.e., a function $\deg : \{x_1, \ldots, x_n\} \to \mathbb{N}$, and the i-th graded piece is generated by all monomials of degree i. Graded ideals of these rings are ideals, that are generated by homogeneous polynomials. Since $R[x_1, \ldots, x_n]$ is noetherian, only a finite number of generators is needed. We therefore have a complete classification of graded ideals of the polynomial ring in finitely many variables over a noetherian ring.

Proposition 2.3. *Let $M = \bigoplus_{j \in \mathbb{Z}} M_j$ be a graded R-module and $N = \bigoplus_{j \in \mathbb{Z}} N_j$ be a graded submodule. Then M/N is a graded R-module, where*

$$\left(M/N\right)_i = M_i/N_i = M_i/(N \cap M_i) = (M_i + N)/N.$$

Proof. If A_i, $i \in \mathbb{Z}$ are R-modules and B_i are submodules of A_i for each i, it is well known that

$$\bigoplus A_i \Big/ \bigoplus B_i \cong \bigoplus A_i/B_i$$

(see for example [7] for a proof). This proves the condition on the direct sum.
To prove the compatibility with the module multiplication, note that

$$R_i \cdot \left(M/N\right)_j = (R_i M_j + N)/N \subseteq (M_{i+j} + N)/N = \left(M/N\right)_{i+j}$$

holds for all $i, j \in \mathbb{Z}$. □

Definition 2.4. Let $M = \bigoplus_{j \in \mathbb{Z}} M_j$ and $N = \bigoplus_{j \in \mathbb{Z}} N_j$ be graded R-modules and $f : M \to N$ be an R-module homomorphism. The map f is *graded of degree i*, if $f(M_j) \subseteq N_{i+j}$ holds for all $j \in \mathbb{Z}$. Denote by $\mathrm{Hom}_i(M, N)$ the set of all module homomorphisms $M \to N$ of degree i.
A homomorphism $g : M \to N$ is *graded* if an $i \in \mathbb{Z}$ exists such that $g \in \mathrm{Hom}_i(M, N)$.

It is convenient to work with homomorphisms of degree 0. Indeed, we can transform any given graded homomorphism to a homomorphism of degree 0:

Proposition 2.4. *Let M, N be two graded R-modules. Suppose that $f \in Hom_i(M, N)$, then $f \in Hom_0(M(-i), N)$, where $M(-i)$ is the graded R-module M shifted by i degrees.*

Proof. Clearly, $M(-i)$ is a graded module itself (see Example 2.2). Furthermore,

$$f(M(-i)_l) = f(M_{l-i}) \subseteq N_l,$$

so that $f \in \mathrm{Hom}_0(M(-i), N)$. □

We will shortly discuss three important examples of graded homomorphisms, which appear in graded free resolutions that we will encounter in Section 3. Let R be a graded ring and consider R^t as a graded R-module.

First, if A is an $s \times t$-matrix $\in R^{s \times t}$, where all entries are homogeneous of degree d, A defines a graded homomorphism $\phi : R^t \to R^s$ of degree d via the matrix multiplication, i.e. $f \mapsto A \cdot f$. We can shift R^t to get a graded homomorphism $\phi : R(-d)^t \to R^s$ of degree 0.
Second, if B is a $s \times t$-matrix with entries in R, where the entries of the j-th column are all homogeneous of the same degree d_j for $1 \leq j \leq t$, B induces a graded homomorphism

$$\phi : R(-d_1) \oplus \cdots \oplus R(-d_t) \to R^s,$$

that is again given by multiplication with B. The homomorphism is of degree 0.
Third, given a $s \times t$-matrix C where all entries are homogeneous in R and the entry c_{ij} is homogeneous of degree $d_j - d'_i$ for $1 \leq i \leq s$ and $1 \leq j \leq t$, C induces a graded homomorphism

$$\phi : R(-d_1) \oplus \cdots \oplus R(-d_t) \to R\left(-d'_1\right) \oplus \cdots \oplus R\left(-d'_s\right)$$

of degree 0. Clearly these matrices are useful to describe graded homomorphisms between free modules.

Definition 2.5. Let R be a graded ring. An $s \times t$-matrix C with entries in R is called a *graded matrix* if there exist $d_1, \ldots, d_t, d'_1, \ldots, d'_s \in \mathbb{Z}$ so that all entries c_{ij} of C are homogeneous of degree $d_j - d'_i$ for all $1 \leq i \leq s, 1 \leq j \leq t$.

By the above observation, graded matrices describe the graded homomorphisms between free finitely generated R-modules.

Structure theorem for finitely generated graded modules

Our goal is to show that every finitely generated graded R-module is isomorphic with degree 0 to a quotient module M/M', where M is a finite sum of shifted copies of R and M' is a graded submodule of M.

Proposition 2.5. *Let $M = \bigoplus_{j \in \mathbb{Z}} M_j, N = \bigoplus_{j \in \mathbb{Z}} N_j$ be graded R-modules and suppose that $f \in Hom_i(M, N)$ for some $i \in \mathbb{Z}$. If $m = m_1 + \cdots + m_k$ is the homogeneous decomposition of m in M, then $f(m) = f(m_1) + \cdots + f(m_k)$ is the homogeneous decomposition of $f(m)$ in N.*

Proof. This is the logical conclusion of f being a graded homomorphism. □

Proposition 2.6. *Let $f : M \to N$ be a graded homomorphism of R-modules. Then $Ker(f)$ is a graded submodule of M.*

Proof. Suppose that $m \in \operatorname{Ker}(f)$ has the unique decomposition $m = m_1 + \cdots + m_k$ into homogeneous parts in M. Then $0 = f(m) = f(m_1) + \cdots + f(m_k)$. The images $f(m_i)$ are homogeneous in N for $1 \leq i \leq k$, and therefore $f(m_i) = 0$ for $1 \leq i \leq k$, implying $m_i \in \operatorname{Ker}(f)$. By Proposition 2.2, $\operatorname{Ker}(f)$ is graded. □

Remark 2.3. One can also show that the image of a graded homomorphism is a graded submodule.

Theorem 2.1. *Let $N = \bigoplus_{j \in \mathbb{Z}} N_j$ be a finitely generated graded R-module. Then there exists a graded isomorphism $f : N \to M/M'$ of degree 0, where M is a finite direct sum of shifted copies of R and M' is a graded submodule of M.*

Proof. Choose a finite system $\{n_1, \ldots, n_k\}$ of homogeneous generators of N (this is possible by Proposition 2.2), and suppose that $n_i \in N_{d_i}$ for $1 \leq i \leq k$. Set

$$M := R(-d_1) \oplus \cdots \oplus R(-d_k).$$

As a finite direct sum of graded R-modules, M is a graded R-module. If $1 \in R$ is the unity element, then the element $1 \in R(-d_j)$ has degree d_j, call it e_j for $1 \leq j \leq k$.
The map

$$f' : M \to N, f(e_i) = n_i$$

is in $\operatorname{Hom}_0(M, N)$. Set $M' := \operatorname{Ker}(f')$, which is a graded submodule of M by Proposition 2.6. The isomorphy follows from the homomorphism theorem for modules. □

3 Syzygies and free resolutions

In this chapter, we will develop the tools we need to state and prove Hilbert's syzygy theorem (see for example [10],[9]). For a computational aspect of free resolutions, see [12] or [14]. We will closely follow [9].

3.1 Syzygies

Since modules do not possess a basis in general, computations are not as easy as in vector spaces. Indeed, by losing the linear independency, we also lose the uniqueness of the representation of the elements of m as sum of generating elements. In the case that we have a generating set $\{m_i, i \in I\}$ and two representations $f = \sum_{i \in I} r_i m_i$ and $g = \sum_{i \in I} s_i m_i$, we have $f = g$ if and only if $\sum_{i \in I} (r_i - s_i)\, m_i = 0$, which does not imply $r_i = s_i$ for $i \in I$ anymore. Therefore it is natural to give the following definition.

Definition 3.1. Let M be an R-module and $(m_1, \ldots, m_t) \in M^t$. A vector $(r_1, \ldots, r_t) \in R^t$ is called a *syzygy* (or *relation*) of $(m_1, \ldots, m_t)$ if

$$r_1 m_1 + \cdots + r_t m_t = 0.$$

Denote by $\operatorname{Syz}(m_1, \ldots, m_t)$ the set of all syzygies of $(m_1, \ldots, m_t)$, that is,

$$\operatorname{Syz}(m_1, \ldots, m_t) = \left\{ r \in R^t : r_1 m_1 + \cdots + r_t m_t = 0 \right\} \subseteq R^t.$$

Proposition 3.1. *Let M be an R-module and $(m_1, \ldots, m_t) \in M^t$. Then $Syz(m_1, \ldots, m_t)$ is a submodule of R^t.*

Proof. Trivial. □

Remark 3.1. Note that $\operatorname{Syz}(m_1, \ldots, m_t)$ is the kernel of the map $\phi : R^t \to M$, $e_i \to m_i$.

Let M be a finitely generated R-module over an noetherian ring R. We will prove in Proposition 4.2 that M is noetherian itself. Therefore, if R is noetherian, $\operatorname{Syz}(f_1, \ldots, f_t) \subseteq R^t$ is finitely generated.

Definition 3.2. Let R be a noetherian ring and $M = \langle m_1, \ldots, m_t \rangle_R$ be a finitely generated R-module. A *presentation matrix* for M is a matrix A with t rows and a finite number of columns so that the set of columns generate the syzygy-module $\operatorname{Syz}(m_1, \ldots, m_t)$.

Remark 3.2. A presentation matrix $A \in M_{t \times s}(R)$ of M induces an exact sequence

$$R^s \xrightarrow{\psi} R^t \xrightarrow{\phi} M \to 0,$$

where ψ maps the i-th standard basis vector of R^s to the i-th column of A and ϕ maps the j-th standard basis vector e_j of R^t to m_j. We call this exact sequence a *presentation*.

Note that a presentation matrix is not unique and depends on the choice of generators. If M is free, $A = (0, \ldots, 0) \in R^t$ is a presentation matrix for M.

Proposition 3.2. *Let A be a $t \times s$-matrix with entries in R, and M be an R-module. If A is a presentation matrix for M (with respect to some generating set), we have $M \cong R^t/AR^s$.*

Proof. First, note that AR^s is the submodule of R^t generated by the columns of A. Suppose that $\{e_1, \ldots, e_t\}$ is the standard basis of R^t. Then R^t/AR^s is generated by the set $\{e_1 + AR^s, \ldots, e_t + AR^s\}$ and $(r_1, \ldots, r_t)^T$ is in $\mathrm{Syz}\,(e_1 + AR^s, \ldots, e_t + AR^s)$ if and only if r^T is in the span of the columns of A. Hence, A is a presentation matrix for R^t/AR^s. One can easily see that two modules that have the same presentation matrix are isomorphic (if $m_1, \ldots, m_k$ and $n_1, \ldots, n_k$ are the generators of the modules, consider the natural map ϕ, that sends m_i to n_i. This is an isomorphism). Therefore, $M \cong R^t/AR^s$. □

The question is *how to compute* a presentation of a module. We will solve this problem if the underlying ring R is the polynomial ring in finitely many variables.

3.1.1 Computing a presentation

For this section, we suppose that R is a polynomial ring $k[x_1, \ldots, x_n]$ over a field k. Given elements $m_1, \ldots, m_t$ of some R-module M, our goal is to find a presentation for $\mathrm{Syz}\,(m_1, \ldots, m_t) \subseteq R^t$.

Syzygies of monomial submodules

Definition 3.3. A submodule N of R^t is called *monomial*, if it is generated by a set of monomial vectors $m_{ij}e_i$.

Clearly, finding the syzygy module for the generators of a monomial submodule is a lot easier than for any submodule.

Proposition 3.3. *(Generalization of Dickson's Lemma) Let N be a monomial submodule of R^t for some $t \in \mathbb{N}$. Then N is generated by a finite number of monomials.*

Proof. Consider the submodules $N_i = N \cap Re_i$ of N for $1 \leq i \leq t$. The N_i are of the form $I_i e_i$ for some ideal of R. Since N is monomial, so are the I_i. Dickson's Lemma tells us that finitely many monomials suffice to generate each I_i. If $I_i = \langle m_{i1}, \ldots, m_{ij} \rangle$, then N is generated by the set of all products $m_{ij}e_i$. □

We will now compute the syzygy module for a monomial submodule.

Proposition 3.4. *Let M be a monomial submodule of R^s, that is generated by the monomial vectors $m_1, \ldots, m_t$. Define for $1 \leq i < j \leq t$,*

$$\sigma_{ij} := \frac{lcm\,(m_i, m_j)}{m_i} \cdot e_i - \frac{lcm\,(m_i, m_j)}{m_j} \cdot e_j.$$

Then $Syz(m_1, \ldots, m_t) = \langle \sigma_{ij}, 1 \leq i < j \leq t \rangle_R$.

Proof. First, note that the σ_{ij} are indeed syzygies by trivial computation. Next, if $a \in R^t$ is an element of the syzygy module, we have $\sum_{i=1}^{t} a_i m_i = 0$ in R^t. Write this sum as $\sum_{i=1}^{t} f_i e_i = 0$, where e_i is the i-th standard basis vector. This implies $f_i = 0$ for $1 \leq i \leq t$, so that, without loss of generality, we can restrict ourselves to the case of monomial vectors containing only the standard basis vector e_1.
Without loss of generality, suppose that $m_1 = x^{\alpha_1} e_1, \ldots, m_r = x^{\alpha_r} e_1$ and let b be a syzygy of $m_1, \cdots_r$. By definition, $\sum_{i=1}^{r} b_i x^{\alpha_i} e_1 = 0$, which implies $\sum_{i=1}^{r} b_i x^{\alpha_i} = 0$. By comparison of coefficients, this is only possible if there exist $c_1, \ldots, c_r \in R$ and an α such that $b_i = c_i x^{\alpha - \alpha_i}$ and $\sum_{i=1}^{r} c_i = 0$. Therefore, b is of the form

$$b = \left(c_1 x^{\alpha - \alpha_1}, \ldots, c_r x^{\alpha - \alpha_r} \right).$$

We have to prove that $b \in \langle \sigma_{ij}, 1 \leq i < j \leq t \rangle_R$. To prove this, write b as

$$b = \sum_{m=1}^{r-1} \sigma_m,$$

where

$$\sigma_m = \left(0, \ldots, 0, \sum_{j=1}^{m} c_j x^{\alpha - \alpha_m}, -\sum_{j=1}^{m} c_j x^{\alpha - \alpha_{m+1}}, 0, \ldots, 0 \right).$$

Clearly, $\sigma_m \in \langle \sigma_{ij}, 1 \leq i < j \leq t \rangle_R$. Hence $b \in \langle \sigma_{ij}, 1 \leq i < j \leq t \rangle_R$. □

Syzygies of arbitrary submodules

We still assume $R = k[x_1, \ldots, x_n]$. Given an R-module M and some elements $m_1, \ldots, m_k$ of M, our goal is now to find a generating set for the syzygy module of $m_1, \ldots, m_k$. The idea is simple - we want to *lift* syzygies of the leading terms to syzygies of the generators.
If $\mathcal{G} = \{g_1, \ldots, g_s\}$ is a Gröbner basis for the module it generates, the division algorithm gives an equation of the form

$$S(g_i, g_j) = \sum_{k=1}^{s} a_{ijk} g_k,$$

where $\mathrm{LT}(a_{ijk} g_k) \leq \mathrm{LT}(S(g_i, g_j))$.
Furthermore, Buchberger algorithm is a way of *detecting syzygies*. One can easily see that

$$\frac{\mathrm{lcm}(\mathrm{LT}(g_i), \mathrm{LT}(g_j))}{\mathrm{LT}(g_i)} e_i - \frac{\mathrm{lcm}(\mathrm{LT}(g_i), \mathrm{LT}(g_j))}{\mathrm{LT}(g_j)} e_j$$

is a syzygy of the leading terms of the g_k. We want to construct a syzygy of $\{g_1, \ldots, g_s\}$ given a syzygy on the leading terms with this knowledge.
Define $a_{ij} <= (a_{ij1}, \ldots, a_{ijs})^T \in R^s$ and $m_{ij} = \mathrm{lcm}(\mathrm{LT}(g_i), \mathrm{LT}(g_j))$. We can lift the syzygy from above by considering

$$s_{ij} := \begin{cases} 0, & \text{if } m_{ij} = 0, \\ \frac{m_{ij}}{\mathrm{LT}(g_i)} e_i - \frac{m_{ij}}{\mathrm{LT}(g_j)} e_j - a_{ij}, & \text{otherwise.} \end{cases}$$

Proposition 3.5. *The vectors s_{ij} defined above are indeed syzygies of $\{g_1, \ldots, g_s\}$.*

Proof. Let $1 \leq i < j \leq s$. If $m_{ij} = 0$, the claim holds trivially, so assume that $m_{ij} \neq 0$. Then

$$\begin{aligned}(g_1, \ldots, g_s) \cdot s_{ij} =& g_i \frac{m_{ij}}{\mathrm{LT}(g_i)} - g_j \frac{m_{ij}}{\mathrm{LT}(g_j)} - g_1 a_{ij1} - \cdots - g_s a_{ijs} \\ =& S(g_i, g_j) - g_1 a_{ij1} - \cdots - g_s a_{ijs} = 0.\end{aligned}$$

□

Theorem 3.1. *Let $\mathcal{G} = \{g_1, \ldots, g_s\} \subseteq R^t$ be a Gröbner basis of the module it generates, with respect to a monomial order $>$ on R^t.*
Define an order $>_{\mathcal{G}}$ on R^s by

- *If $LT_>(x^\alpha g_i) > LT_>\left(x^\beta g_j\right)$, $x^\alpha e_i >_{\mathcal{G}} x^\beta e_j$, or*
- *If $LT_>(x^\alpha g_i) = LT_>\left(x^\beta g_j\right)$ and $i < j$, $x^\alpha e_i >_{\mathcal{G}} x^\beta e_j$.*

Then $>_{\mathcal{G}}$ is a monomial order on R^s and the s_{ij} form a Gröbner basis of $Syz(g_1, \ldots, g_s)$ with respect to $>_{\mathcal{G}}$.

Proof. First, $<_{\mathcal{G}}$ is easily seen to be a monomial order, since $<$ is. Furthermore, as $s_{ij} = -s_{ji}$, we may assume $i < j$. Define $m_{ij} = \mathrm{lcm}(\mathrm{LT}(g_i), \mathrm{LT}(g_j))$. Now, since

$$\mathrm{LT}\left(\frac{m_{ij}}{\mathrm{LT}(g_i)} g_i\right) > \mathrm{LT}\left(\frac{m_{ij}}{\mathrm{LT}(g_j)} g_j\right)$$

and, because of the cancellation property of the leading terms of the S-vector,

$$\mathrm{LT}\left(\frac{m_{ij}}{\mathrm{LT}(g_i)} g_i\right) > \mathrm{LT}(S(g_i, g_j)) \geq \mathrm{LT}(a_{ijk} g_k)$$

for all $1 \leq i, j, k \leq s$, we have

$$\mathrm{LT}_{>_{\mathcal{G}}}(s_{ij}) = \frac{m_{ij}}{\mathrm{LT}(g_i)} e_i.$$

Let $f = \sum_{i=1}^{s} f_i e_i \in R^s$ be a syzygy, let $\mathrm{LT}_{<_{\mathcal{G}}}(f_i e_i) = m_i e_i$ and let j be the unique number satisfying $\mathrm{LT}_{<_{\mathcal{G}}}(f) = m_j e_j$. We have to prove that $\mathrm{LT}_{<_{\mathcal{G}}}(f)$ is divisible by some $\mathrm{LT}_{<_{\mathcal{G}}}(s_{ij})$, making the s_{ij} a Gröbner basis of the syzygy module. Define

$$S := \{l : m_j \mathrm{LT}_<(g_j) = m_l \mathrm{LT}_<(g_l)\}.$$

By the definition of $<_{\mathcal{G}}$, we have $S \subseteq \{j, j+1, \ldots, s\}$. Now, consider the element

$$s = \sum_{l \in S} m_l e_l,$$

which is a syzygy on the set of the leading terms of the g_l for $l \in S$, with respect to $<$, since f is a syzygy on $g_1, \ldots, g_s$. By Proposition 3.4, s is in the submodule generated by the σ_{ij} with $i < j$ and $i, j \in S$, so $\mathrm{LT}_{<_{\mathcal{G}}}(s)$ is divisible by some $\mathrm{LT}_{<_{\mathcal{G}}}(s_{ij}) = \frac{m_{ij}}{\mathrm{LT}(g_i)} e_i$.
By construction, $\mathrm{LT}_{<_{\mathcal{G}}}(f) = \mathrm{LT}_{<_{\mathcal{G}}}(s)$ and the s_{ij} form a Gröbner basis with respect to $<_{\mathcal{G}}$. □

Therefore, a Gröbner basis of some module M contains enough information to find a presentation for M.
For Hilbert's syzygy theorem, we need an induction argument which will be contained in the following lemma.

Lemma 3.1. *Let $\mathcal{G}$ be a Gröbner basis for the module it generates with respect to some monomial order $<$ on R^t.*
Rearrange the elements of $\mathcal{G}$ to a tuple $\mathcal{G}' = (g_1, \ldots, g_s)$, so that whenever $i < j$ and the leading terms $LT_< (g_i)$ and $LT_< (g_j)$ contain the same e_k, we have $\frac{LM_<(g_i)}{e_k} >_{lex} \frac{LM_<(g_j)}{e_k}$, where lex is the lexicographic order on R satisfying $x_1 > \cdots > x_n$.
If the variables $x_1, \ldots, x_m$ do not appear in the leading terms of $\mathcal{G}'$, then $x_1, \ldots, x_{m+1}$ do not appear in the leading terms of $s_{ij} \in Syz(g_1, \ldots, g_s)$ with respect to $<_{\mathcal{G}}$.

Proof. It suffices to consider $g_i, g_j \in \mathcal{G}'$ where $\mathrm{LT}_< (g_i), \mathrm{LT}_< (g_j)$ contain the same standard basis vector and $i < j$. Then $\frac{\mathrm{LM}(g_i)}{e_k} = x_{m+1}^{\alpha_1} y_i$ and $\frac{\mathrm{LM}(g_j)}{e_k} = x_{m+1}^{\alpha_2} y_j$ with $\alpha_1 \geq \alpha_2$ and y_i, y_j are monomials in $x_{m+2}, \ldots, x_n$. But then

$$\mathrm{LT}_{>_{\mathcal{G}}} (s_{ij}) = \frac{\mathrm{lcm} (\mathrm{LT}_< (g_i), \mathrm{LT}_< (g_j))}{\mathrm{LT}_< (g_i)} e_i$$

does not contain $x_1, \ldots, x_{m+1}$. □

3.2 Free resolutions and the Hilbert Syzygy Theorem

We already saw that, given a system of generators of some module M, we gain information about M if we are also given the syzygy module of the generators. We know that, in certain cases, the syzygy module is finitely generated. Choose a finite generating system of the syzygy module. In order to understand the syzygy module, we also need the syzygies on the generators of the syzygy module - these are the so called *second syzygies*. By iterating this process, we will get a new module in each step, the i-th syzygy module. The question one has to ask is easy: Does this process always stop, for any choice of generators? In general, the answer is no and we will see shortly a counterexample. However, Hilbert proved that the above iteration stops if M is a finitely generated $k[x_1, \ldots, x_n]$-module.

Remark 3.3. Suppose that M is an R-module, and $M = \langle m_1, \ldots, m_t \rangle_R$. Consider the homomorphism $f_1 : R^t \to M$, given by $f_1 (e_i) = m_i$ for $1 \leq i \leq t$ and linear extension. This is clearly a surjective homomorphism, making the sequence

$$R^t \xrightarrow{f_1} M \to 0$$

exact. The kernel of f_1 is the set of all $r \in R^t$ satisfying $\sum_{i=1}^{t} r_i m_i = 0$, so $\mathrm{Ker} (f_1) = \mathrm{Syz} (m_1, \ldots, m_t)$.
Now, suppose that we want to extend this exact sequence to the left. Construct a homomorphism with the target R^t and image $\mathrm{Ker} (f_1)$. If $\mathrm{Syz} (m_1, \ldots, m_t) = \mathrm{Ker} (f_1)$ is finitely generated by s elements $\{n_1, \ldots, n_s\}$, we can choose the homomorphism $f_2 : R^s \to R^t$ that sends $e_i \in R^s$ to n_i for $1 \leq i \leq s$. The image $f_2 (R^s)$ is by definition the first syzygy module, such that we extended our exact sequence to

$$R^s \to R^t \to M \to 0.$$

One can iterate this process to gain a (potentially infinitely long) exact sequence to the module M with respect to a chosen system of generators - as long as the kernels of the homomorphisms are finitely generated.
Note that if M is a finitely generated R-module and R is noetherian, the kernels of these homomorphisms are finitely generated, and there exists an exact sequence as above. The above computation of kernels is a *free resolution* of M.

Definition 3.4. Let M be an R-module. A *free resolution* of M is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

of free finitely generated R-modules.
If for some l, we have $F_{l+i} = 0$ for all $i \geq 1$ and $F_l \neq 0$, we call the free resolution *finite* of length l. We will often write

$$\cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0.$$

In general, computing a free resolution of M is a non-trivial problem - there need not even exist a finite free resolution of M.

Example 3.1. Let $R = \mathbb{Z}/6\mathbb{Z}$. We claim that there is no finite free resolution of the module $M = 2\mathbb{Z}/6\mathbb{Z}$. Since M has one generator, we get an exact sequence

$$R \to M \to 0,$$

where $f_1 : R \to M$ is given by the multiplication with 2 and $\operatorname{Ker}(f_1) = 3\mathbb{Z}/6\mathbb{Z}$. We can extend the above exact sequence to the sequence

$$R \to R \to M \to 0,$$

where $f_2 : R \to R$ is given by the multiplication with 3, such that $\operatorname{Ker}(f_2) = 2\mathbb{Z}/6\mathbb{Z}$. This process of alternating kernels will not terminate. Hence there is no finite free resolution of M.

One now might ask for conditions an R-module M has to fulfil in order to possess a finite free resolution. A celebrated theorem of Hilbert gives at least a sufficient condition for M to possess a finite free resolution.

Theorem 3.2. *(Hilbert's Syzygy theorem) Let M be a finitely generated $k[x_1, \ldots, x_n]$-module. Then M has a finite free resolution of length $\leq n$.*

Proof. Let $\{f_1, \ldots, f_{t_0}\}$ be a generating set of M. As already seen, there is a sequence

$$F_1 \xrightarrow{\phi_1} R^{t_0} \xrightarrow{\phi_0} M \to 0.$$

Choose a reduced Gröbner basis $\mathcal{G}_0$ for $\operatorname{Syz}(f_0, \ldots, f_{t_0}) \subseteq R^{t_0}$, with respect to a monomial order on R^{t_0} and rearrange the elements as in Lemma 3.1 to a t_0-tuple $\mathcal{G}_0'$. Now, with Theorem 3.1, choose a reduced Gröbner basis $\mathcal{G}_1$ for $\operatorname{Syz}\left(\mathcal{G}_0'\right) \subset R^{|\mathcal{G}_0|}$. None of the leading terms of $\mathcal{G}_1$ will contain x_1 by Lemma 3.1. We can now continue the sequence from above to a sequence

$$R^{|\mathcal{G}_1|} \xrightarrow{\phi_2} R^{|\mathcal{G}_0|} \xrightarrow{\phi_1} R^{t_0} \xrightarrow{\phi_0} M \to 0.$$

Continue with this process until the leading terms of the Gröbner basis $\mathcal{G}_l$ of Syz $\left(\mathcal{G}'_{l-1}\right)$ do not contain any of the variables $x_1, \ldots, x_n$. By Lemma 3.1, this has to happen for some $l \leq n$. Then the Gröbner basis consists of products of nonzero constants times standard basis vectors of F_l, showing that Syz $\left(\mathcal{G}'_{l-1}\right)$ is free as a module, since the generators only possess trivial relations. This implies that we can extend the resolution with a zero to the left, i.e.

$$0 \to F_l \to F_{l-1} \to \cdots \to R^{|\mathcal{G}_1|} \xrightarrow{\phi_2} R^{|\mathcal{G}_0|} \xrightarrow{\phi_1} R^{t_0} \xrightarrow{\phi_0} M \to 0.$$

Since $l \leq n < \infty$, this is a finite free resolution of M. □

3.3 Graded free resolutions

If the module that is to be resolved is graded, it is natural to ask for resolutions that respect the grading.

Definition 3.5. Let R be a graded ring and M be a graded R-module. A *graded free resolution* of M is an exact sequence

$$\cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0,$$

with F_i free finitely generated graded R-modules, where the ϕ_i are graded of degree 0, such that F_i is a shifted free module $R(-d_1) \oplus \cdots \oplus R(-d_m)$ with $m = \text{rank}(F_i)$.
If for some l, $F_{l+i} = 0$ for all $i \geq 1$ and $F_l \neq 0$, we call the free resolution finite of length l.

Similar to the above question, one can naturally ask in which cases finite graded free resolutions exist. Indeed, we can adapt the Hilbert Syzygy theorem to prove that every finitely generated graded $k[x_1, \ldots, x_n]$-module (with $\deg(x_i) \in \mathbb{N}$) has a finite graded free resolution. For the remainder of this section, suppose that R is a polynomial ring in finitely many variables.

Lemma 3.2. *Let M be a submodule of R^t for some $t \in \mathbb{N}$. The following are equivalent:*

1. *M is a graded submodule of R^t.*
2. *A reduced Gröbner basis of M consists of vectors of homogeneous polynomials, with the addition that all components of each vector are of the same degree.*

Proof. The second implication follows directly from Proposition 2.2 (since Gröbner bases are generating systems).
For the other direction, since M is graded, there exist homogeneous generators $f_1, \ldots, f_m$ of M. Apply the Buchberger algorithm to get a Gröbner basis $\{g_1, \ldots, g_n\}$. Note that, if f and g are homogeneous, so is $S(f, g)$. By applying the division algorithm we see that $g_1, \ldots, g_n$ have to be homogeneous. Reduce this Gröbner basis - this is done by cancelling the leading terms (with respect to a given monomial order), which is equivalent to taking linear combinations of homogeneous vectors of the same degree. Hence, a reduced Gröbner basis consists of vectors of homogeneous elements of the same degree. □

Lemma 3.3. *Suppose we are given $d_1, \ldots, d_t$ integers, $m_1, \ldots, m_s \in R^t$, where the elements $m_1, \ldots, m_s$ are vectors $m_i = (m_{i1}, \ldots, m_{it})^T$ $(1 \leq i \leq s)$ of homogeneous polynomials that satisfy $deg(m_{ij}) - d_j = deg(m_{ik}) - d_k$ for $1 \leq i \leq s$ and $1 \leq j < k \leq t$. Then $m_1, \ldots, m_s$ are homogeneous elements of $F := R(d_1) \oplus \cdots \oplus R(d_t)$ and $\langle m_1, \ldots, m_s \rangle$ is a graded submodule of F.*

Proof. It suffices to prove that m_1 is homogeneous in F. To prove this, suppose that $m_1 = (m_{11}, \ldots, m_{1t}) \in R^t$ where $m_{1j} \in R_{c_j}$ for $1 \leq j \leq t$ and

$$m_{1j} \in R(d_j)_{c_j - d_j} = R(d_j)_{\deg(m_{1j}) - d_j}.$$

Hence,

$$m_1 \in (R(d_1) \oplus \cdots \oplus R(d_m))_{\deg(m_{11}) - d_1},$$

so that the m_i are homogeneous. By Proposition 2.2, $\langle m_1, \ldots, m_s \rangle$ is a graded submodule of F. □

Theorem 3.3. *(Hilbert syzygy theorem for graded modules) Let M be a finitely generated graded $k[x_1, \ldots, x_n]$-module. Then there exists a finite graded free resolution of M of length $\leq n$.*

Proof. The proof is a slight modification of the proof of the Hilbert syzygy theorem. Since M is a graded submodule of some R^t ($t \in \mathbb{N}$), there is by Lemma 3.2 a reduced Gröbner basis $\{g_1, \ldots, g_t\}$ of M where each element g_i is a vector of homogeneous polynomials, all of which have the same degree d_i.
Now choose a reduced Gröbner basis $\{m_1, \ldots, m_s\} \subseteq R^t$ as in the proof of the syzygy theorem. Since $m_1, \ldots, m_s \in \operatorname{Syz}(g_1, \ldots, g_t)$, we have

$$m_{i1} g_1 + \ldots m_{it} g_t = 0,$$

where m_{uv} is the v-th entry of m_u for $1 \leq u \leq s$, $1 \leq v \leq t$. Therefore, $\deg(m_{ij}) - d_j = \deg(m_{ik}) - d_k$ for $1 \leq i \leq s$ and $1 \leq j < k \leq t$. This shows that

$$\operatorname{Syz}(g_1, \ldots, g_t) = \langle m_1, \ldots, m_s \rangle$$

is a graded submodule of $R(d_1) \oplus \cdots \oplus R(d_t)$. Now apply the proof of Theorem 3.2 to get a finite graded free resolution of length at most n. □

Remark 3.4. In the algorithms of Theorem 3.2 and Theorem 3.3, one need not choose a Gröbner basis that is reduced in each step.

3.4 Minimal free resolutions and Betti numbers

By Hilbert's syzygy theorem, every finitely generated $k[x_1, \ldots, x_n]$-module M has a finite free resolution of length n or less. As in the proof of Theorem 3.2, choosing the Gröbner basis for the syzygy modules will have an impact on the resolution.

Definition 3.6. Let M be a finitely generated $k[x_1, \ldots, x_n]$-module, $\deg(x_i) \in \mathbb{N}$, and let

$$0 \to F_l \xrightarrow{\phi_l} \cdots \to F_0 \xrightarrow{\phi_0} M \to 0$$

be a graded free resolution of M. This resolution is *minimal*, if

$$\phi_i(F_i) \subseteq (x_1, \ldots, x_n) F_{i-1}$$

holds for $1 \leq i \leq l$, that is, all nonzero entries of the graded matrices A_i corresponding to ϕ_i for $1 \leq i \leq l$ have positive degree.

Theorem 3.4. *Let M be a finitely generated $k[x_1, \ldots, x_n]$-module as above and*

$$0 \to F_l \xrightarrow{\phi_l} \cdots \to F_0 \xrightarrow{\phi_0} M \to 0$$

be a graded free resolution of M. The following are equivalent.

1. *The above resolution is minimal.*
2. *For $1 \leq i \leq l$, ϕ_i maps a standard basis of F_i to a minimal generating system of $Im(\phi_i)$.*
3. *In every step of the algorithm in the proof of Theorem 3.2, choose a minimal system of generators of the kernel.*

Proof. For some $1 \leq i \leq l$, let $e_1, \ldots, e_m$ be the standard basis of F_i. Assume that the elements $\phi_i(e_1), \ldots, \phi_i(e_m)$ do not form a minimal generating system of $\mathrm{Im}(\phi_i)$, so that one of these can be expressed as a linear combination of the others. Without loss of generality, assume that it is $\phi_i(e_1)$. Then there are $r_2, \ldots, r_m \in k[x_1, \ldots, x_n]$ with

$$\phi_i(e_1) = \sum_{2 \leq j \leq m} r_j \phi_i(e_j),$$

implying $(1, -r_2, \ldots, -r_m)^T \in \mathrm{Ker}(\phi_i) = \mathrm{Im}(\phi_{i+1})$. The columns of the graded matrix A_{i+1} generate $\mathrm{Im}(\phi_{i+1})$. Since all nonzero entries of A_{i+1} have positive degree, the above vector is not a linear combination of the columns of A_{i+1}, which is a contradiction, thus the first implication holds. The equivalence of the second and third claim follows from the exactness. For the third implication, assume that the resolution is not minimal. Therefore, there exists an i, $1 \leq i \leq l$, satisfying $\phi_i(F_i) \not\subseteq (x_1, \ldots, x_n) F_{i-1}$. By exactness, there is an element $f \in \mathrm{Ker}(\phi_{i-1}) \setminus (x_1, \ldots, x_n) F_{i-1}$. Let $f_1, \ldots, f_s$ be a minimal generating system of $\mathrm{Ker}(\phi_{i-1})$. After renumbering the elements $f_1, \ldots, f_s$, if necessary, there is a relation $f = f_1 - \sum_{j=2}^{s} r_j f_j$ with $r_j \in k[x_1, \ldots, x_n]$ for $2 \leq j \leq n$. Since $f \in \mathrm{Ker}(\phi_{i-1})$,

$$\phi_{i-1}(f_1) = \sum_{j=2}^{s} r_j \phi_{i-1}(f_j).$$

This is a contradiction to the fact that we have chosen a minimal generating system for $\mathrm{Ker}(\phi_{i-2})$. □

The notion of minimal free resolutions is therefore intuitive. By always picking a minimal generating set of the kernels, we therefore guarantee a minimal free resolution. The natural question is to which degree a minimal free resolution is unique.

Definition 3.7. Let M be a finitely generated graded $k[x_1, \ldots, x_n]$-module as above and let

$$0 \to F_{l_1} \xrightarrow{\phi_{l_1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

and

$$0 \to G_{l_2} \xrightarrow{\psi_{l_2}} \cdots \xrightarrow{\psi_1} G_0 \xrightarrow{\psi_0} M \to 0$$

be two graded free resolutions of M. These resolutions are called *isomorphic* if there are graded isomorphisms $\alpha_i : F_i \to G_i$ of degree 0 for $i \geq 0$ such that $\psi_0 \circ \alpha_0 = \phi_0$ and $\alpha_{i-1} \circ \phi_i = \psi_i \circ \alpha_i$ holds for all $i \geq 0$.

Theorem 3.5. *Let M be a finitely generated graded $k[x_1, \ldots, x_n]$-module as above and let*

$$0 \to F_{l_1} \overset{\phi_{l_1}}{\to} \cdots \overset{\phi_1}{\to} F_0 \overset{\phi_0}{\to} M \to 0$$

and

$$0 \to G_{l_2} \overset{\psi_{l_2}}{\to} \cdots \overset{\psi_1}{\to} G_0 \overset{\psi_0}{\to} M \to 0$$

be two minimal graded free resolutions of M. Then the resolutions are isomorphic.

Proof. We start by constructing $\alpha_0 : F_0 \to G_0$. Let $e_1, \ldots, e_m$ be a standard basis of F_0. For $1 \leq i \leq m$, there is a $g_i \in G_0$ such that $\phi_0(e_i) = \psi_0(g_i)$. Set $\alpha_0(e_i) = g_i$. This is clearly a graded homomorphism of degree 0 which satisfies $\psi_0 \circ \alpha_0 = \phi_0$. Construct $\beta_0 : G_0 \to F_0$ in the same manner, which satisfies $\phi_0 \circ \beta_0 = \psi_0$. We have to prove that α_0 is an isomorphism.
We will first prove that $\beta_0 \circ \alpha_0 : F_0 \to F_0$ is an isomorphism. Note that

$$\phi_0 \circ (\mathrm{id}_{F_0} - \beta_0 \circ \alpha_0) = 0.$$

Since $\mathrm{Im}(\phi_1) \subseteq (x_1, \ldots, x_n) F_0$ and $(\mathrm{id}_{F_0} - \beta_0 \circ \alpha_0) \subseteq \mathrm{Im}(\phi_1)$ by the above consideration and exactness, $f - \beta_0 \circ \alpha_0(f)$ is an element of $(x_1, \ldots, x_n) F_0$ for all $f \in F_0$.
In order to prove that $\beta_0 \circ \alpha_0$ is an isomorphism, it suffices to prove that $\beta_0 \circ \alpha_0 \mid_{(F_0)_t} : (F_0)_t \to (F_0)_t$ is an isomorphism for each $t \geq 0$, where $(F_0)_t$ is the t-th graded part of F_0. Since the $(F_0)_t$ are finite dimensional vector spaces, it suffices to prove that these maps are surjective.
We prove that these maps are surjective by induction. If $t = 0$, $\beta_0 \circ \alpha_0 = \mathrm{id}_{F_0}$ follows from the minimality of the resolution. Furthermore, if $f \in M_t$, we have $f - \beta_0 \circ \alpha_0(f) \in (x_1, \ldots, x_n) \bigoplus_{i=1}^{t-1} M_i$, and since $\beta_0 \circ \alpha_0 \mid_{(F_0)_s} : (F_0)_s \to (F_0)_s$ are isomorphisms for $s \leq t$, the claim holds.
Therefore, $\beta_0 \circ \alpha_0$ is an isomorphism, so that α_0 is injective. By a similar argument, $\alpha_0 \circ \beta_0 : G_0 \to G_0$ is an isomorphism, implying that α_0 is surjective, so α_0 is an isomorphism.
We will now prove that $\alpha_0 \mid_{\mathrm{Ker}(\phi_0)} : \mathrm{Ker}(\phi_0) \to \mathrm{Ker}(\psi_0)$ is an isomorphism, induced by α_0. A simple computation shows that the map is well-defined. It is injective, since α_0 is. If $g \in \mathrm{Ker}(\psi_0)$, choose $f \in \mathrm{Ker}(\phi_0)$ with $\alpha_0(f) = g$. Since $\beta_0(\phi_0(f)) = \psi_0(\alpha_0(f)) = \psi_0(g) = 0$, $\phi_0(f) = 0$, hence $f \in \mathrm{Ker}(\phi_0)$, which guarantees surjectivity.
Note that

$$0 \to F_{l_1} \overset{\phi_{l_1}}{\to} \cdots \overset{\phi_1}{\to} \mathrm{Ker}(\phi_0) \to 0$$

and

$$0 \to G_{l_2} \overset{\psi_{l_2}}{\to} \cdots \overset{\psi_1}{\to} \mathrm{Ker}(\psi_0) \to 0$$

are minimal resolutions of $\mathrm{Ker}(\phi_0)$ and $\mathrm{Ker}(\psi_0)$ respectively. Considering the isomorphism $\alpha_0 \mid_{\mathrm{Ker}(\phi_0)} : \mathrm{Ker}(\phi_0) \to \mathrm{Ker}(\psi_0)$, the above argumentation constructs a graded homomorphism $\alpha_1 : F_1 \to G_1$ with $\alpha_0 \mid_{\mathrm{Ker}(\phi_0)} \circ \phi_1 = \psi_1 \circ \alpha_1$. Again, α_1 induces an isomorphism $\mathrm{Ker}(\phi_1) \to \mathrm{Ker}(\psi_1)$. Continuing inductively in this manner proves the statement. □

Betti numbers

If M is any finitely generated $k[x_1, \ldots, x_n]$-module, M has a unique minimal free resolution, up to isomorphy. Therefore, the ranks of the free modules in the minimal resolution contain a lot of information on M.

Definition 3.8. Let M be a finitely generated $k[x_1, \ldots, x_n]$-module and suppose that

$$0 \to R^{b_l} \to \cdots \to R^{b_0} \to M$$

is the minimal free resolution of M, up to isomorphy. The numbers $b_i = b_i(M)$ for $i \in \mathbb{N}_0$ are called *Betti numbers* of M.

Remark 3.5. By Theorem 3.2, $b_i = 0$ if $i > n$.

In the graded case, the free modules are shifted and contain more information than in the non-graded case.

Definition 3.9. Let M be a finitely generated graded $k[x_1, \ldots, x_n]$-module, $\deg(x_i) \in \mathbb{N}$. Suppose that

$$0 \to F_l \to \cdots \to F_0 \to M \to 0$$

is the minimal graded free resolution of M, where

$$F_i \cong \bigoplus_{j \in \mathbb{Z}} R(-j)^{b_{i,j}}.$$

The $b_{i,j}$ are called *graded Betti numbers* of M.

The information encoded in the Betti numbers will help us in later chapters.

4 Tools of commutative algebra

4.1 Aspects of homological algebra

We will closely follow [15] and [16], as well as a short excerpt of [10].

Complexes and homologies of modules

Definition 4.1. A *complex* $\mathcal{F}$ over a ring R is a sequence of R-modules

$$\mathcal{F} : \cdots \to F_n \overset{\phi_n}{\to} F_{n-1} \to \cdots \to F_1 \overset{\phi_1}{\to} F_0 \to 0,$$

where the ϕ_i are homomorphisms and $\phi_i \circ \phi_{i+1} = 0$ for all $i \in \mathbb{N}$. We set ϕ_0 to be the zero map $F_0 \to 0$. The collection of maps $\phi = \{\phi_i, i \in \mathbb{N}_0\}$ is called the *differential* of $\mathcal{F}$. We write $(\mathcal{F}, \phi)$ for the complex $\mathcal{F}$ together with the differential ϕ. If M is an R-module, the complex $(\mathcal{F}, \phi)$ is a *complex over* M if there is a homomorphism $F_0 \to M$ given. An element of F_i is said to have *homological degree* i.

Remark 4.1. It is common to define complexes to be infinite in both directions. *Left complexes*, as above are sufficient for our studies (we set $F_i = 0$ for $i < 0$).

Definition 4.2. The complex $\mathcal{F}$ is called *graded* if the modules F_i are graded R-modules and each homomorphism is graded of degree 0.

Definition 4.3. Let $(\mathcal{F}, \phi)$ be a complex and $i \in \mathbb{N}_0$. The *i-th homology* $\mathrm{H}_i(\mathcal{F})$ of $\mathcal{F}$ is the R-module $\mathrm{Ker}(\phi_i)/\mathrm{Im}(\phi_{i+1})$. An element of $\mathrm{Ker}(\phi_i)$ is called a *cycle*, an element of $\mathrm{Im}(\phi_{i+1})$ a *boundary*. If M is an R-module, the complex $\mathcal{F}$ is called *acyclic over* M if $\mathrm{H}_i(\mathcal{F}) = 0$ for $i > 0$ and $\mathrm{H}_0(\mathcal{F}) = M$. A complex $\mathcal{F}$ is *exact* if $\mathrm{H}_i(\mathcal{F}) = 0$ for all $i \in \mathbb{N}_0$.

Definition 4.4. Let $(\mathcal{F}, \phi)$ and $(\mathcal{G}, \psi)$ be complexes of R-modules. A *homomorphism of complexes* $\alpha : \mathcal{F} \to \mathcal{G}$ is a family of homomorphisms $\alpha_i : F_i \to G_i$ (for $i \in \mathbb{N}_0$) satisfying $\alpha_{i-1} \circ \phi_i = \psi_i \circ \alpha_i$ for all $i \in \mathbb{N}_0$, making the following diagram is commutative for all n.

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & F_n & \xrightarrow{\phi_n} & F_{n-1} & \xrightarrow{\phi_{n-1}} & F_{n-2} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_{n-2}} & & \\
\ldots & \longrightarrow & G_n & \xrightarrow{\psi_n} & G_{n-1} & \xrightarrow{\psi_{n-1}} & G_{n-2} & \xrightarrow{\psi_{n-2}} & \ldots
\end{array}$$

For graded complexes, the homomorphism α is called *graded* if all maps $\alpha : F_i \to G_i$ are graded of some degree m.

Proposition 4.1. *Let $(\mathcal{F}, \phi)$ and $(\mathcal{G}, \psi)$ be complexes of R-modules and $\alpha : \mathcal{F} \to \mathcal{G}$ be a homomorphism. Then $\alpha_i(Ker(\phi_i)) \subseteq Ker(\psi_i)$ and $\alpha_i(Im(\phi_{i+1})) \subseteq Im(\psi_{i+1})$ hold for all $i \in \mathbb{N}_0$.*

Proof. Suppose that $g \in \alpha_i(\operatorname{Ker}(\phi_i)) \subseteq G_i$, so that there is a $f \in \operatorname{Ker}(\phi_i) \subseteq F_i$ with $g = \alpha_i(f)$. Hence

$$\psi_i(g) = \psi(\alpha_i(f)) = \alpha_{i-1}(\phi_i(f)) = \alpha_{i-1}(0) = 0,$$

which proves the first inclusion. For the second inclusion, let $g \in \alpha_i(\operatorname{Im}(\phi_{i+1})) \subseteq G_i$. Then there exists an $f \in \operatorname{Im}(\phi_{i+1})$ with $g = \alpha_i(f)$ and an $h \in F_{i+1}$ satisfying $\phi_{i+1}(h) = f$. Hence

$$g = \alpha_i(\phi_{i+1}(h)) = \psi_{i+1}(\alpha_{i+1}(h)) \in \operatorname{Im}(\psi_{i+1}),$$

which proves the second inclusion. □

Therefore, every homomorphism of complexes $\alpha : \mathcal{F} \to \mathcal{G}$ induces a collection of maps $\widetilde{\alpha}_i : \mathrm{H}_i(\mathcal{F}) \to \mathrm{H}_i(\mathcal{G})$. It is called the *induced map on homology.*

Remark 4.2. Given three complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and two homomorphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$, let $\overline{f}$ and $\overline{g}$ be the maps induced on homology. Then the induced maps on homology $\overline{g \circ f}_i : \mathrm{H}_i(\mathcal{A}) \to \mathrm{H}_i(\mathcal{C})$ for $i \in \mathbb{N}$ fulfil $\overline{g \circ f} = \overline{g} \circ \overline{f}$.

Theorem 4.1. *Let $(\mathcal{A}, \alpha), (\mathcal{B}, \beta), (\mathcal{C}, \gamma)$ be complexes, $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ homomorphisms of complexes. Suppose that $B_2 \to C_2$ is an epimorphism, $A_0 \to B_0$ is a monomorphism and $A_1 \to B_1 \to C_1$ is exact. Then the sequence of induced maps on homology*

$$H_1(\mathcal{A}) \xrightarrow{\overline{f}} H_1(\mathcal{B}) \xrightarrow{\overline{g}} H_1(\mathcal{C})$$

is exact, where $\overline{f}$ and $\overline{g}$ are the induced maps on homology.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
A_2 & \xrightarrow{\alpha_2} & A_1 & \xrightarrow{\alpha_1} & A_0 & \xrightarrow{\alpha_0} & 0 \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\
B_2 & \xrightarrow{\beta_2} & B_1 & \xrightarrow{\beta_1} & B_0 & \xrightarrow{\beta_0} & 0 \\
\downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \\
C_2 & \xrightarrow{\gamma_2} & C_1 & \xrightarrow{\gamma_1} & C_0 & \xrightarrow{\gamma_0} & 0
\end{array}$$

By Remark 4.2, $\overline{g \circ f} = \overline{g} \circ \overline{f}$. Since $g_1 \circ f_1 : A_1 \to C_1$ is the zero map, $\operatorname{Ker}(\alpha_1)$ is mapped to $\operatorname{Im}(\gamma_2)$ by $g \circ f$, so that $\overline{g \circ f}_1 : \mathrm{H}_1(\mathcal{A}) \to \mathrm{H}_1(\mathcal{C})$ is the zero map. Thus $\overline{g} \circ \overline{f}$ is the zero map and hence $\operatorname{Im}(\overline{f_1}) \subseteq \operatorname{Ker}(\overline{g_1})$.
For the other inclusion, let $b \in \mathrm{H}_1(\mathcal{B})$ such that $\overline{g_1}(b) = 0$. Choose an element $b_1 \in \operatorname{Ker}(\beta_1)$ representing b. Hence, $\overline{g_1}(b)$ is represented by $g_1(b_1)$ as an element in $H_1(\mathcal{C})$. Since $\overline{g_1}(b) = 0$, $g(b_1) \in \operatorname{Im}(\gamma_2)$, so that there exists a $c_2 \in C_2$ such that $g_1(b_1) = \gamma_2(c_2)$. Since g_2 is an epimorphism, there is a $b_2 \in B_2$ with $c_2 = g_2(b_2)$. Hence

$$g_1(b_1) = \gamma_2(c_2) = \gamma_2(g_2(b_2)) = g_1(\beta_2(b_2))$$

and $b_1 - \beta_2(b_2) \in \operatorname{Ker}(g_1) = \operatorname{Im}(f_1)$, by the exactness of $A_1 \to B_1 \to C_1$. Therefore, there is an $a_1 \in A_1$ such that $b_1 - \beta_2(b_2) = f_1(a_1)$. We have

$$f_0(\alpha_1(a_1)) = \beta_1(f_1(a_1)) = \beta_1(b_1) - \beta_1(\beta_2(b_2)) = 0,$$

and since f_0 is a monomorphism, $a_1 \in \operatorname{Ker}(\alpha_1)$. Hence, a_1 represents an element a of $\mathrm{H}_1(\mathcal{A})$ and $\overline{f}_1(a)$ is represented by $f(a_1) = b_1 - \beta_2(b_2)$. However, b_1 and $b_1 - \beta_2(b_2)$ represent the same element of $\mathrm{H}_i(\mathcal{B})$, since $\beta_2(b_2) \in \operatorname{Im}(\beta_2)$. Hence $\overline{f}_1(a) = b \in \operatorname{Im}(\overline{f}_1)$, such that $\operatorname{Ker}(\overline{g}_1) \subseteq \operatorname{Im}(\overline{f}_1)$. □

Definition 4.5. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be complexes and $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{B} \to \mathcal{C}$ be homomorphisms. The sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is called *exact* if $0 \to A_i \to B_i \to C_i \to 0$ is an exact sequence for each $i \in \mathbb{N}_0$, where 0 is the zero complex, i.e. $F_j = 0$ for all j.

Corollary 4.1. *Given complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$, where $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is exact, and homomorphisms as above, f and g induce a family of short exact sequences of homology $H_i(\mathcal{A}) \to H_i(\mathcal{B}) \to H_i(\mathcal{C})$ for each $i \in \mathbb{N}_0$.*

The connecting homomorphism

As described in Corollary 4.1, exact sequences of complexes induce exact sequences in homology. Surprisingly, one can construct a natural homomorphism $\Delta_i : \mathrm{H}_i(\mathcal{C}) \to \mathrm{H}_{i-1}(\mathcal{A})$, called the *connecting homomorphism*.

Construction 4.1. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & A_2 & \xrightarrow{\alpha_2} & A_1 & \xrightarrow{\alpha_1} & A_0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
B_3 & \xrightarrow{\beta_3} & B_2 & \xrightarrow{\beta_2} & B_1 & \xrightarrow{\beta_1} & B_0 \\
\downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \\
C_3 & \xrightarrow{\gamma_3} & C_2 & \xrightarrow{\gamma_2} & C_1 & &
\end{array}$$

We construct a homomorphism $\Delta_2 : \mathrm{H}_2(\mathcal{C}) \to \mathrm{H}_1(\mathcal{A})$. Let c be an element of $\mathrm{H}_2(\mathcal{C})$. Choose a representative $c_2 \in \mathrm{Ker}(\gamma_2)$ for c. Since $B_2 \to C_2$ is surjective, we can choose an element $b_2 \in B_2$ such that $g_2(b_2) = c_2$. Let $b_1 = \beta_2(b_2)$, then

$$g_1(b_1) = g_1(\beta_2(b_2)) = \gamma_2(g_2(b_2)) = \gamma_2(c_2) = 0.$$

Hence $b_1 \in \mathrm{Ker}(g_1) = \mathrm{Im}(f_1)$, by exactness. Choose an element $a_1 \in A_1$ such that $f_1(a_1) = b_1$. We have that

$$f_0(\alpha_1(a_1)) = \beta_1(f_1(a_1)) = \beta_1(b_1) = \beta_1(\beta_2(b_2)) = 0.$$

Since f_0 is injective, it follows that $a_1 \in \mathrm{Ker}(\alpha_1)$. Hence, a_1 represents an element a of $\mathrm{H}_1(\mathcal{A})$. Define $\Delta(c) = a$. This construction can be applied to find a homomorphism Δ_i for all $i \in \mathbb{N}$.

Theorem 4.2. *The above construction is well-defined and defines a homomorphism $H_i(\mathcal{C}) \to H_{i-1}(\mathcal{A})$.*

Proof. Suppose that in selecting a_1, b_1, b_2, c_2 in the above construction, we had made different choices in selecting a_1', b_1', b_2', c_2'. Since c_2 and c_2' both represent c, $c_2 - c_2' \in \mathrm{Im}(\gamma_3)$. Hence, there exists a $c_3 \in C_3$ with $\gamma_3(c_3) = c_2 - c_2'$. Since g_3 is surjective, there is a $b_3 \in B_3$ with $g_3(b_3) = c_3$. Then

$$g_2\left(b_2 - b_2' - \beta_3(b_3)\right) = c_2 - c_2' - g_2(\beta_3(b_3)) = c_2 - c_2' - \gamma_3(c_3) = 0,$$

so $b_2 - b_2' - \beta_3(b_3) \in \mathrm{Ker}(g_2) = \mathrm{Im}(f_2)$. Choose $a_2 \in A_2$ with $f_2(a_2) = b_2 - b_2' - \beta_3(b_3)$. Then

$$f_1\left(a_1 - a_1' - \alpha_2(a_2)\right) = b_1 - b_1' - f_1(\alpha_2(a_2)) = \beta_2\left(b_2 - b_2' - f_2(a_2)\right) = \beta_2(\beta_3(b_3)) = 0.$$

The map $f_1 : A_1 \to B_1$ is injective, hence $a_1 - a_1' - \alpha_2(a_2) = 0$. Hence, a_1 and a_1' belong to the same coset modulo $\mathrm{Im}(\alpha_2)$ and therefore represent the same element in homology. It is trivial that Δ is an homomorphism. □

Definition 4.6. Given an exact sequence of complexes, the *connecting homomorphism* is the set of homomorphisms

$$\Delta = \{\Delta_i : \mathrm{H}_i(\mathcal{C}) \to \mathrm{H}_{i-1}(\mathcal{A})\},$$

where the Δ_i are given as in Construction 4.1.

Theorem 4.3. *With the notation from above, the sequence*

$$H_i(\mathcal{B}) \overset{\overline{g}_i}{\to} H_i(\mathcal{C}) \overset{\Delta_i}{\to} H_{i-1}(\mathcal{A}) \overset{\overline{f}_{i-1}}{\to} H_{i-1}(\mathcal{B})$$

is a short exact sequence for all $i \in \mathbb{N}$, where Δ is the connecting homomorphism.

Proof. If $b \in \mathrm{H}_i(\mathcal{B})$, there is an element $b_i \in \mathrm{Ker}(\beta_i)$ that represents b, such that $\overline{g}_i(b)$ is represented by $g_i(b_i)$. Since $\beta_i(b_i) = 0 = f_{i-1}(0)$, $\Delta_i(\overline{g}_i(b)) \in \mathrm{H}_{i-1}(\mathcal{A})$ is represented by the zero element in A_{i-1}, so that it is zero itself. Hence $\mathrm{Im}(\overline{g}_i) \subseteq \mathrm{Ker}(\Delta_i)$.
For the other inclusion, let $c \in \mathrm{H}_i(\mathcal{C})$ with $\Delta_i(c) = 0$ and define $c_i, b_i, b_{i-1}, a_{i-1}$ as it was done in Construction 4.1 for $i = 2$. Then $a_{i-1} \in \mathrm{Im}(\alpha_i)$. Choose an element $a_i \in A_i$ with $\alpha_i(a_i) = a_{i-1}$. Then

$$\beta_i(b_i - f_i(a_i)) = b_{i-1} - \beta_i(f_i(a_i)) = b_i - f_i(a_i) = 0.$$

Hence $b_i - f_i(a_i) \in \mathrm{Ker}(\beta_i)$ and $b_i - f_i(a_i)$ represents an element $b \in \mathrm{H}_i(\mathcal{B})$. Then the element $\overline{g}_i(b)$ is represented by the element

$$g_i(b_i - f_i(a_i)) = g_i(b_i) = c_i.$$

Threfore $\overline{g}_i(b) = c$, meaning that $c \in \mathrm{Im}(\overline{g}_i)$ and $\mathrm{Ker}(\Delta_i) \subseteq \mathrm{Im}(\overline{g}_i)$. Hence the sequence is exact at $\mathrm{H}_i(\mathcal{C})$.
Let $c \in \mathrm{H}_i(\mathcal{C})$ and construct $c_i, b_i, b_{i-1}, a_{i-1}$ in a similar way as in Construction 4.1. Then $\overline{f}_{i-1}(\Delta_i(c))$ is represented by $f_{i-1}(a_{i-1}) = b_{i-1} = \beta_i(b_i) \in \mathrm{Im}(\beta_i)$, by Construction 4.1. Hence $\overline{f}_{i-1}(\Delta_i(c)) = 0$ and therefore $\mathrm{Im}(\Delta_i) \subseteq \mathrm{Ker}\left(\overline{f}_{i-1}\right)$.
On the other hand, let $a \in \mathrm{H}_{i-1}(\mathcal{A})$ with $\overline{f}_{i-1}(a) = 0$. Let $a_{i-1} \in \mathrm{Ker}(\alpha_{i-1})$ represent a, then $\overline{f}_{i-1}(a)$ is represented by $f_{i-1}(a_{i-1})$. Since $\overline{f}_{i-1}(a) = 0$, $f_{i-1}(a_{i-1}) := b_{i-1} \in \mathrm{Im}(\beta_i)$. Hence there is a $b_i \in B_i$ with $\beta(b_i) = b_{i-1}$. Set $c_i = g_i(b_i)$. Then

$$\gamma_i(c_i) = \gamma_i(g_i(b_i)) = g_{i-1}(\beta_i(b_i)) = g_{i-1}(b_1) = g_{i-1}(f_{i-1}(a_{i-1})) = 0.$$

Hence c_2 represents an element c of $\mathrm{H}_i(\mathcal{C})$. By Construction 4.1, $\Delta_i(c)$ is represented by a_{i-1}. Therefore, $a = \Delta_i(c) \in \mathrm{Im}(\Delta_i)$, showing that the sequence is exact at $H_{i-1}(\mathcal{A})$. □

Theorem 4.4. *Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a exact sequence of complexes. There is an exact sequence*

$$\cdots \to H_i(\mathcal{A}) \to H_i(\mathcal{B}) \to H_i(\mathcal{C}) \overset{\Delta}{\to} H_{i-1}(\mathcal{A}) \to \cdots \to H_0(\mathcal{C}) \to 0,$$

called the long exact sequence in homology.

Proof. This follows immediately from Construction 4.1 of the connecting homomorphism and Theorems 4.1 and 4.3. □

The Koszul complex

We closely follow [10] and [16] for the rest of this section.

Definition 4.7. Let M be an R-module. The exterior algebra $\wedge M$ over M is the free algebra $\bigoplus_{i\geq 0} \underbrace{M \otimes M \otimes \cdots \otimes M}_{i \text{ times}}$ modulo the ideal generated by all relations $m \otimes m = 0$ and $m \otimes n + n \otimes m = 0$ for all $m, n \in M$. We write $m \wedge n$ for the product of two elements $m, n \in \wedge M$. The exterior algebra is a graded algebra, where the graded part $\wedge^k M$ of degree k is the R-module generated by the products of exactly k elements of M. In this case $\wedge^0 = R$.

Theorem 4.5. *Let $f_1, \ldots, f_n \in R$, and M be a free module of rank n. In this case, $\wedge^{n+1} M = 0$. Let $e_1, \ldots, e_n$ be a basis of M. Then the sequence of homomorphisms*

$$0 \to \wedge^n M \xrightarrow{d_n} \wedge^{n-1} M \to \cdots \to \wedge^1 M \xrightarrow{d_1} \wedge^0 M \to 0$$

is a complex, where the differential d is given by

$$d\left(e_{j_1} \wedge \cdots \wedge e_{j_k}\right) = \sum_{i=1}^{k} (-1)^{i+1} f_{j_i} \left(e_{j_1} \wedge \cdots \wedge \widehat{e_{j_i}} \wedge \cdots \wedge e_{j_k}\right),$$

where $\widehat{e_{j_i}}$ means the omission of e_{j_i}.

Proof. It is easy to see that d is well-defined. We have to prove that applying the differential d twice is the zero map, i.e. $d^2 = 0$. For all k with $1 \leq k \leq n$, we have

$$\begin{aligned}
d^2\left(e_{j_1} \wedge \cdots \wedge e_{j_k}\right) &= d_{k-1}\left(\sum_{r=1}^{k} (-1)^r f_{j_r} \left(e_{j_1} \wedge \cdots \wedge \widehat{e_{j_r}} \wedge \cdots \wedge e_{j_k}\right)\right) \\
&= \sum_{r=1}^{k} (-1)^r f_{j_r} d_{k-1}\left(\left(e_{j_1} \wedge \cdots \wedge \widehat{e_{j_r}} \wedge \cdots \wedge e_{j_k}\right)\right) \\
&= \sum_{r=1}^{k} (-1)^r f_{j_r} \left(\sum_{s=1, s\neq r}^{k} (-1)^s f_{j_s} \left(e_{j_1} \wedge \cdots \wedge \widehat{e_{j_r}} \wedge \cdots \wedge \widehat{e_{j_s}} \wedge \ldots, j_k\right)\right),
\end{aligned}$$

where $e_{j_1} \wedge \cdots \wedge \widehat{e_{j_r}} \wedge \cdots \wedge \widehat{e_{j_s}} \wedge \cdots \wedge e_{j_k}$ means that both indices j_r and j_s are deleted, whether $r < s$ or $s < r$. After expanding the above expression we will encounter the term $f_{j_r} f_{j_s} \left(e_{j_1} \wedge \cdots \wedge \widehat{e_{j_r}} \wedge \cdots \wedge \widehat{e_{j_s}} \cdots \wedge e_{j_k}\right)$ twice. If j_s is deleted first and j_r afterwards, the coefficient will be $(-1)^{r+s}$. Otherwise, if j_r is deleted first, then j_s drops one place, yielding the coefficient $(-1)^{r+s-1}$. Therefore, these terms cancel and the above is indeed a complex. □

Definition 4.8. The complex constructed in Theorem 4.5 is called the *Koszul complex* of $f_1, \ldots, f_n$ and denoted by $K\left(f_1, \ldots, f_n\right)$.

Remark 4.3. If $e_1, \ldots, e_n$ is a basis of R^n, then the i-th graded part $\wedge^i R^n$ of $\wedge R^n$ is generated by the elements $e_{j_1} \wedge \cdots \wedge e_{j_i}$, where $j_1 < \cdots < j_i$. Hence, the rank of $\wedge^i R^n$ is $\binom{n}{i}$. We can therefore write the Koszul complex as

$$K\left(f_1, \ldots, f_n\right) : 0 \to R \to R^n \to R^{\binom{n}{n-2}} \to \cdots \to R^{\binom{n}{2}} \to R^n \to R \to 0.$$

The Koszul complex as a graded free resolution

Given elements $f_1, \ldots, f_n$ of R, the Koszul complex $K(f_1, \ldots, f_n)$ is in general not an exact sequence. The question we now ask is which conditions on $f_1, \ldots, f_n$ have to be imposed in order that

$$0 \to R \to R^n \to \cdots \to R^n \to R \to {}^{R}\!/_{(f)} \to 0$$

is a free resolution of $R/(f)$. In general, the Koszul complex $K(f_1, \ldots, f_n)$ is not a free resolution of $R/(f)$.

Lemma 4.1. *Let $(K(f_1, \ldots, f_n), d)$ be the Koszul complex of $f_1, \ldots, f_n \in R$ and let $1 \leq j_1 < j_2 < \cdots < j_k \leq n$ be natural numbers. Then*

$$d\left(e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \wedge e_{j_k}\right) = d\left(e_{j_1} \wedge \cdots \wedge e_{j_{k-1}}\right) \wedge e_{j_k} + (-1)^{k+1} f_{j_k}\left(e_{j_1} \wedge \cdots \wedge e_{j_{k-1}}\right).$$

Proof. Let $e = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{k-1}}$. Then

$$\begin{aligned} d(e \wedge e_{j_k}) =& \sum_{i=1}^{k} (-1)^{i+1} f_{j_i}(e_{j_1} \wedge \cdots \wedge \widehat{e_{j_i}} \wedge \cdots \wedge e_{j_k}) \\ =& \sum_{i=1}^{k-1} (-1)^{i+1} f_{j_i}(e_{j_1} \wedge \cdots \wedge \widehat{e_{j_i}} \wedge \cdots \wedge e_{j_k}) + (-1)^{k+1} f_{j_k}\left(e_{j_1} \wedge \cdots \wedge e_{j_{k-1}}\right) \\ =& d(e) \wedge e_{j_k} + (-1)^{k+1} f_{j_k} e. \end{aligned}$$

□

Lemma 4.2. *Let $f_1, \ldots, f_n$ be elements of R. Then there is an exact sequence of complexes*

$$0 \to K(f_1, \ldots, f_{n-1}) \to K(f_1, \ldots, f_n) \to K(f_1, \ldots, f_{n-1})[-1] \to 0,$$

where $K(f_1, \ldots, f_{n-1})[-1]_i = K(f_1, \ldots, f_{n-1})_{i-1}$ is the Koszul complex shifted by degree 1. The long sequence in homology induced by this exact sequence is

$$\begin{aligned} \cdots \to H_i(K(f_1, \ldots, f_{n-1})) \to H_i(K(f_1, \ldots, f_n)) \to \\ \to H_{i-1}(K(f_1, \ldots, f_{n-1})) \overset{\Delta_i}{\to} H_{i-1}(K(f_1, \ldots, f_{n-1})) \to \ldots. \end{aligned}$$

The connecting homomorphism Δ_i is given by multiplication with $(-1)^{i+1} f_n$.

Proof. By Remark 4.3, $K(f_1, \ldots, f_n)_i = R^{\binom{n}{i}}$, $K(f_1, \ldots, f_{n-1})_i = R^{\binom{n-1}{i}}$ and for the shifted complex $K(f_1, \ldots, f_{n-1})[-1]_i = R^{\binom{n-1}{i-1}}$. Since $R^{\binom{n}{i}} = R^{\binom{n-1}{i}} \oplus R^{\binom{n-1}{i-1}}$, the sequence is exact at each i, which therefore yields a long sequence of homology by Theorem 4.3. To compute the connecting homomorphism, we follow the construction from Theorem 4.2. Let $a \in \mathrm{H}_{i-1}(K(f_1, \ldots, f_{n-1}))$ and choose a representative $r \in K(f_1, \ldots, f_{n-1})_{i-1}$ for a. Let $s = r \wedge e_n \in K(f_1, \ldots, f_n)_i$. By Lemma 4.1, $d(s) = d(r) \wedge e_n + (-1)^{i+1} f_n r$. Note that $(-1)^{i+1} f_n r \in K(f_1, \ldots, f_{n-1})_{i-1}$ and $d(r) \wedge e_n \in K(f_1, \ldots, f_{n-1})_{i-2} \wedge e_n$. Therefore, the connecting homomorphism is given by multiplication with $(-1)^{i+1} f_n$. □

Definition 4.9. Let M be an R-module. Elements $f_1, \ldots, f_n$ of R are called an *M-regular sequence*, if

1. f_1 is not a zerodivisor for M,
2. The image of f_{i+1} under the canonical map $M \to M/(f_1, \ldots, f_i) M$ is not a zerodivisor in $M/(f_1, \ldots, f_i) M$ for all $1 \leq i \leq n-1$,
3. $M \neq (f_1, \ldots, f_n) M$.

The ideal $I = (f_1, \ldots, f_n)$ of R is then called a *complete intersection ideal.*

Theorem 4.6. *Let R be a graded ring, and let $f_1, \ldots, f_n$ be homogeneous elements of positive degree. The following are equivalent.*

1. *The elements $f_1, \ldots, f_n$ are R-regular sequence.*
2. *$H_i(K(f_1, \ldots, f_n)) = 0$ if $i > 0$ and $H_0(K(f_1, \ldots, f_n)) = R/(f_1 \ldots, f_n)$.*

Proof. We first prove that (1) implies (2). By construction, we have

$$\mathrm{H}_0(K(f_1, \ldots, f_n)) = R/(f_1, \ldots, f_n).$$

For the other homology modules, we use induction on n. If $n = 1$, the Koszul complex of $K(f_1)$ is $0 \to R \to R \to 0$, where the map $R \to R$ is given by multiplication with f_1. Then

$$\mathrm{H}_1(K(f_1)) = \mathrm{Ker}(R \to R) = 0,$$

since f_1 is not a zerodivisor of R. Suppose now that the claim holds for the sequence $f_1, \ldots, f_{n-1}$. By Lemma 4.2 (with $i = 1$), we get an exact sequence

$$0 \to \mathrm{H}_1(K(f_1 \ldots, f_n)) \to R/(f_1, \ldots, f_{n-1}) \xrightarrow{\Delta_1} R/(f_1, \ldots, f_{n-1}),$$

where Δ_1 is given by multiplication with f_n. Since f_n is not a zerodivisor of $R/(f_1, \ldots, f_{n-1})$, it follows that $\mathrm{Ker}(\Delta_1) = 0$. By the exactness of this sequence, $\mathrm{H}_1(K(f_1 \ldots, f_n)) = 0$. If $i > 1$, there is an exact sequence

$$\mathrm{H}_i(K(f_1, \ldots, f_{n-1})) \to \mathrm{H}_i(K(f_1, \ldots, f_n)) \to \mathrm{H}_{i-1}(K(f_1, \ldots, f_{n-1})),$$

so that $\mathrm{H}_i(K(f_1, \ldots, f_n)) = 0$.
We still have to prove that (2) implies (1). Assume that $R/(f_1, \ldots, f_n) = 0$, then $R = (f_1, \ldots, f_n) R$ and by Nakayama's lemma we have $R = 0$, which is a contradiction. Hence, $R/(f_1, \ldots, f_n) \neq 0$. Furthermore, using again Lemma 4.2, there is an exact sequence

$$\mathrm{H}_1(K(f_1, \ldots, f_{n-1})) \xrightarrow{f_n} \mathrm{H}_1(\mathrm{K}(f_1, \ldots, f_{n-1})) \to \mathrm{H}_1(K(f_1, \ldots, f_n)).$$

By assumption, $\mathrm{H}_1(K(f_1, \ldots, f_n)) = 0$, so that

$$\mathrm{H}_1(\mathrm{K}(f_1, \ldots, f_{n-1})) = f_n \mathrm{H}_1(\mathrm{K}(f_1, \ldots, f_{n-1})).$$

Applying Nakayama's lemma yields that $\mathrm{H}_1(\mathrm{K}(f_1, \ldots, f_{n-1})) = 0$. Hence, the induction hypotheses shows that the elements $f_1, \ldots, f_{n-1}$ are a R-regular sequence. Using once again Lemma 4.2,

$$\mathrm{H}_1(K(f_1, \ldots, f_n)) \to \mathrm{H}_0(K(f_1, \ldots, f_{n-1})) \xrightarrow{f_n} \mathrm{H}_0(K(f_1, \ldots, f_{n-1}))$$

is an exact sequence. However, $\mathrm{H}_1\left(K\left(f_1,\ldots,f_n\right)\right)=0$ and

$$\mathrm{H}_0\left(K\left(f_1,\ldots,f_n\right)\right)=R/\left(f_1,\ldots,f_{n-1}\right),$$

so that f_n is a regular $R/\left(f_1,\ldots,f_{n-1}\right)$-sequence, showing that $f_1,\ldots,f_n$ are a R-regular sequence. □

Corollary 4.2. *Let R be a ring and $f_1,\ldots,f_n$ be an R-regular sequence of elements of R. Then the Koszul complex*

$$0\to R\to R^n\to\cdots\to R^{\binom{n}{2}}\to R^n\to R\to {}^{R}\!/_{(f_1,\ldots,f_n)}\to 0$$

is a finite free resolution of $R/\left(f_1,\ldots,f_n\right)$.

Definition 4.10. The above free resolution of $R/\left(f_1,\ldots,f_n\right)$ is called the *Koszul resolution.*

Remark 4.4. One can prove that the construction of the Koszul complex is equivalent to choosing a minimal generating set of the kernels as in Theorem 3.2. Therefore, a Koszul resolution is a minimal free resolution in the sense of Definition 3.6.

Corollary 4.3. *Let R be a graded ring and $f_1,\ldots,f_n$ be a R-regular sequence of elements of R, with $deg\left(f_i\right)=d_i$. A graded free resolution of $R/\left(f_1,\ldots,f_n\right)$ is given by*

$$0\to F_n\to\cdots\to F_1\to F_0\to {}^{R}\!/_{(f_1,\ldots,f_n)}\to 0,$$

where

$$F_i=\bigoplus_{1\le j_1<\cdots<j_i\le n} R\left(-\sum_{k=1}^{i} d_{j_k}\right)$$

for $1\le i\le n$.

Proof. This follows immediately from Theorem 3.3 and Corollary 4.2. □

4.2 Chain conditions and additive functions

For the remainder of this section, we closely follow [4].

Length of a module

Definition 4.11. Let M be an R-module. A *chain* of submodules of M is a sequence of submodules

$$0=M_n\subsetneq M_{n-1}\subsetneq\cdots\subsetneq M_0=M.$$

A *composition series* of M is a chain for which every factor M_{i-1}/M_i is simple (a module that has no proper submodules). This is equivalent to the chain being maximal, so that no more submodules can be inserted into the chains.

The reader is assumed to be familiar with chain conditions as well as the concepts of noetherian rings and artinian rings. An introduction to these topics can be found in [4].

Definition 4.12. Let M be an R-module and C be a chain $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n$ of submodules of M. Then n is called the *length of* C. The *length of* M, denoted by $l(M)$ or $l_R(M)$ is defined to be the maximum length of any of its chains.

Remark 4.5. If M possesses a composition series, every composition series is of the same length, and this length is equal to the length of the module. A proof is given in [4].

Theorem 4.7. *Let M be a R-module. Then $l_R(M) < \infty$ if and only if M is artinian and noetherian.*

Proof. If $l_R(M) < \infty$, M has a composition series. Hence all chains of submodules of M are of finite length. Therefore, M fulfils the ascending chain condition, and also the descending chain condition.
On the other hand, if M is noetherian, there exists a maximal submodule of M. Call it $M_1 \subsetneq M$. Similarly, M_1 has a maximal submodule. Continuing in this fashion, we get a chain of submodules, which must become stationary, since M is noetherian. Since we choose maximal submodules in every step, this gives a composition series of finite length. □

Lemma 4.3. *Let*

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules. Then

1. *M_2 is noetherian if and only if M_1 and M_3 are noetherian.*
2. *M_2 is artinian if and only if M_1 and M_3 are artinian.*

Proof. The map f embeds M_1 into M_2, so that M_1 is noetherian/artinian. Since factor modules of noetherian/artinian modules are noetherian/artinian again, $M_3 \cong M_2/M_1$ is noetherian/artinian.
For the other direction, let $N_1 \subseteq N_2 \subseteq M_2$ be submodules of M_2. Suppose that $f^{-1}(N_1) = f^{-1}(N_2)$ and $g(N_1) = g(N_2)$. We shall prove that $N_1 = N_2$. Let $n_2 \in N_2$. There exists an element $n_1 \in N_1$ such that $g(n_1) = g(n_2)$. Therefore, $n_1 - n_2 \in \operatorname{Ker}(g) = \operatorname{Im}(f)$ by exactness. Hence there is an element $m \in M_1$ with $f(m) = n_1 - n_2$. Therefore $m \in f^{-1}(N_2) = f^{-1}(N_1)$, which implies $n_1 - n_2 \in N_1$, so that $n_2 \in N_1$, which proves the claim.
Since M_1 and M_3 are noetherian/artinian, ascending/descending chains of M_2 become stationary by the above consideration. □

Corollary 4.4. *Let $M_1, \ldots, M_n$ be R-modules.*

1. *If $M_1, \ldots, M_n$ are noetherian, so is $\bigoplus_{i=1}^{n} M_i$.*
2. *If $M_1, \ldots, M_n$ are artinian, so is $\bigoplus_{i=1}^{n} M_i$.*

Proof. Proof by induction. If $n = 1$, this is trivial. Otherwise, observe that the following is an exact sequence,

$$0 \to M_n \to \bigoplus_{i=1}^{n} M_i \to \bigoplus_{i=1}^{n-1} M_i \to 0,$$

and use Lemma 4.3.

□

Proposition 4.2. *Let M be a finitely generated R-module.*

1. *If R is noetherian, so is M.*
2. *If R is artinian, so is M.*

Proof. Since M is finitely generated, say by n elements, M is isomorphic to some factor module R^n/N, where N is a submodule of R^n. Consider the exact sequence

$$0 \to N \to R^n \to M \to 0.$$

Since R is noetherian/artinian, so is R^n. By Lemma 4.3, M is noetherian/artinian. □

Corollary 4.5. *Let R be an artinian ring and let M be a finitely generated R-module. Then $l_R(M) < \infty$.*

Proof. By Proposition 4.2, M is artinian. Since every artinian ring is noetherian, R is noetherian, so that M is noetherian. By Theorem 4.7, $l(M) < \infty$. □

Lemma 4.4. *Let $M = \bigoplus_{j\in\mathbb{Z}} M_j$ be a finitely generated graded R-module where $R = \bigoplus_{i\geq 0} R_i$ is an $\mathbb{N}_0$-graded noetherian ring. Then M_n is a finitely generated R_0-module for all $n \in \mathbb{Z}$.*

Proof. Since M is a finitely generated module over a noetherian ring, M is noetherian itself. Consider the submodule of M generated by M_n for some $n \in \mathbb{Z}$. Since M is noetherian, this submodule is finitely generated, say by elements $m_1, \ldots, m_k \in M_n$. Thus, for any $m \in M_n$, there exist $r_1, \ldots, r_k \in R$ such that $m = r_1 m_1 + \cdots + r_k m_k$. Let $r_i = \sum_{l\geq 0} r_{il}$ be the unique decomposition of r_i into homogeneous elements of R. If $l > 0$, $r_{il} \cdot m_1 \in M_{n+l}$. Hence $m = r_{10} m_1 + \cdots + r_{k0} m_k$. □

Corollary 4.6. *Let $M = \bigoplus_{j\in\mathbb{Z}} M_j$ be a finitely generated graded R-module where $R = \bigoplus_{i\geq 0} R_i$ is a noetherian graded ring and suppose that R_0 is artinian. Then $l_{R_0}(M_n) < \infty$ holds for all $n \in \mathbb{Z}$.*

Proof. The graded part M_n of M is a finitely generated module over an artinian ring so that its length is finite by Corollary 4.5. □

Example 4.1.

1. Consider the polynomial ring $R = k[x_1, \ldots, x_n]$ with $\deg(x_i) = d_i > 0$ for some d_i. Then $R_0 = k$ is a field and hence artinian. Therefore, if M is a finitely generated graded module over the polynomial ring $k[x_1, \ldots, x_n]$, the length of any homogeneous part of M is finite.

2. If $R = (R, \mathfrak{m})$ is a local ring, the *associated graded ring*, defined as

$$G(R) = \bigoplus_{n\in\mathbb{N}_0} {\mathfrak{m}^n}\big/{\mathfrak{m}^{n+1}}.$$

is itself a (nonnegatively) graded ring. In this case, $G(R)_0 = R/\mathfrak{m}$ is a field, so that the above considerations also apply.

Additive functions

We want to prove that the length of a module is an additive function on the category of finitely generated modules over artinian rings.

Definition 4.13. Let R_0 be an artinian ring and let $\mathfrak{M}$ be the category of finitely generated R_0-modules. A function $\lambda : \mathfrak{M} \to \mathbb{Z}$ is called *additive*, if it is additive over exact sequences, that is, for any exact sequence

$$0 \to L \to M \to N \to 0,$$

we have $\lambda(L) + \lambda(N) = \lambda(M)$.

Proposition 4.3. *Let λ be an additive function on $\mathfrak{M}$, and $M_1, \ldots, M_n \in \mathfrak{M}$. Then*

$$\sum_{i=1}^{n} \lambda(M_i) = \lambda\left(\bigoplus_{i=1}^{n} M_i\right).$$

Proof. Proof by induction. If $n = 1$, this is trivial. Otherwise, consider the short exact sequence

$$0 \to M_n \to \bigoplus_{i=1}^{n} M_i \to \bigoplus_{i=1}^{n-1} M_i \to 0.$$

Since λ is additive,

$$\lambda\left(\bigoplus_{i=1}^{n} M_i\right) = \lambda\left(\bigoplus_{i=1}^{n-1} M_i\right) + \lambda(M_n) = \sum_{i=1}^{n} \lambda(M_i).$$

□

Proposition 4.4. *Suppose that*

$$0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$$

is an exact sequence of elements in $\mathfrak{M}$. Then

$$\sum_{i=1}^{n} (-1)^i \lambda(M_i) = 0.$$

Proof. Let the maps be given by $f_i : M_i \to M_{i+1}$ for $1 \leq i \leq n-1$, $f_0 : 0 \to M_1$, and $f_n : M_n \to 0$. By the exactness, $\operatorname{Ker}(f_i) = \operatorname{Im}(f_{i-1}) := M_i'$ for $1 \leq i \leq n$. For $1 \leq i \leq n$, consider the short exact sequences

$$0 \to M_i' \to M_i \to M_{i+1}' \to 0.$$

Since λ is additive, $\lambda(M_i) = \lambda(M_i') + \lambda\left(M_{i+1}'\right)$. Therefore, $\sum_{i=1}^{n} (-1)^i \lambda(M_i)$ is a telescopic sum equal to 0. □

Theorem 4.8. *Let R_0 be an artinian ring. Then the length $l : \mathfrak{M} \to \mathbb{Z}$ is an additive function.*

Proof. We have already shown in Corolarry 4.6 that l is well defined. Suppose that

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is a short exact sequence of finitely generated R_0-modules. We have to show that $l(M) = l(L) + l(N)$. Suppose that the composition series of L and N are given by

$$L = L_0 \supset L_1 \supset \cdots \supset L_{l(L)} = 0,$$

$$N = N_0 \supset N_1 \supset \cdots \supset N_{l(N)} = 0.$$

Note that L_i/L_{i+1} and N_j/N_{j+1} are simple. The map $f : L \to M$ is injective, therefore

$$f(L) = f(L_0) \supset f(L_1) \supset \cdots \supset f\left(L_{l(L)}\right) = 0$$

is the composition series of $f(L) = \operatorname{Im}(f) = \operatorname{Ker}(g)$. Since $N \cong M/\operatorname{Ker}(g)$, we have

$$N = {g^{-1}(N_0)}\big/{\operatorname{Ker}(g)} \supset \cdots \supset {g^{-1}(0)}\big/{\operatorname{Ker}(g)} = \operatorname{Ker}(g)$$

is a composition series for N (all composition series are of the same length). We therefore get a longer chain of submodules

$$\begin{aligned} M = g^{-1}(N_0) \supset g^{-1}(N_1) \supset \cdots \supset g^{-1}\left(N_{,l(N)}\right) &= \operatorname{Ker}(g) \\ &= \operatorname{Im}(f) = f(L_0) \supset f(L_1) \supset \cdots \supset f\left(L_{l(L)}\right) = 0. \end{aligned}$$

We only have to prove that $g^{-1}(N_i)/g^{-1}(N_{i+1})$ is simple for $1 \leq i \leq l(N) - 1$. But

$${g^{-1}(N_i)}\big/{g^{-1}(N_{i+1})} \cong \left({g^{-1}(N_i)}\big/{\operatorname{Ker}(g)}\right)\Big/\left({g^{-1}(N_{i+1})}\big/{\operatorname{Ker}(g)}\right).$$

Factor modules of simple modules are simple, thus proving that the above series is indeed a composition series of length $l(L) + l(N)$. □

Example 4.2. Let $M = \bigoplus_{j \in \mathbb{Z}} M_j$ be a finitely generated graded $k[x_1, \ldots, x_n]$-module. By Theorem 3.3, there exists a finite graded free resolution

$$0 \to F_n \to \cdots \to F_0 \to M \to 0$$

of M. The homomorphisms are graded, so there is for each j an exact sequence

$$0 \to (F_n)_j \to \cdots \to (F_0)_j \to M_j \to 0.$$

Since $R_0 = k$ is a field, all these graded pieces are finite dimensional k-vector spaces by Corollary 4.6 and therefore

$$\dim(M_j) = \sum_{i=0}^{n} (-1)^i \dim_{R_0}\left((F_i)_j\right)$$

by Proposition 4.4. Since $F_0, \ldots, F_n$ are only shifted copies of R^{k_i}, the dimension of M_j can be computed by computing dimensions of graded pieces of R.

Rationality of the Poincaré series

The following can be found in every book covering Hilbert series (see for example [17] or [4]). Assume for this section that R is an $\mathbb{N}_0$-graded ring and M is a $\mathbb{N}_0$-graded R-module.
We saw that the graded pieces M_j of some finitely generated graded R-module M are of finite length if M is finitely generated and R_0 is artinian. It is therefore natural to keep note of the growth of the numbers $l(M_j)$ as j increases.

Definition 4.14. Let M be a finitely generated, nonnegatively graded R-module, where R is noetherian and R_0 is artinian. The *Poincaré series* of M is

$$\mathrm{HP}_M(t) = \sum_{j\geq 0} l(M_j)\, t^j \in \mathbb{Q}[|t|],$$

where t is a variable. If R_0 is a field, the M_j are R_0-vector spaces, and $l(M_j) = \dim_{R_0}(M_j)$. Then the above series is called *Hilbert series* of M. It is denoted by $\mathrm{HS}_M(t)$. The well-defined map

$$\begin{aligned} \mathrm{HF}: \mathbb{N}_0 \to &\quad \mathbb{Z} \\ i \to &\quad \dim_k(M_i) \end{aligned}$$

is called *Hilbert function* of M.

Theorem 4.9. *(Hilbert-Serre) Let M be a finitely generated graded $R_0[x_1,\ldots,x_n]$-module, where the elements $x_1,\ldots,x_n$ are homogeneoeus elements of degree $d_1,\ldots,d_n > 0$ and R_0 is artinian. Then there exists a polynomial $p(t) \in \mathbb{Z}[t]$ satisfying*

$$HP_M(t) = \frac{p(t)}{\prod_{i=1}^{n}(1-t^{d_i})}.$$

Proof. The ring $R_0[x_1,\ldots,x_n]$ is $\mathbb{N}_0$-graded. Since R_0 is artinian, it is noetherian, and every submodule of $R_0[x_1,\ldots,x_n]$ is finitely generated by Hilbert's basis theorem. Note that $R_+ = R_0[x_1,\ldots,x_n]/R_0$ is a submodule. We prove the theorem by induction on the number l of generators of R_+.
If $l = 0$, $R = R_0$, so that each M_j is an R-submodule of M, so only finitely many M_j are not 0.
If $l \geq 1$, consider the multiplication with the last generator $x_l \in R_{d_l}$. This is a graded homomorphism $f: M \xrightarrow{\cdot x_l} M$ of degree d_l. We obtain an exact sequence

$$0 \to (\mathrm{Ker}(f))_{n-d_k} \to M_{n-d_k} \to M_n \to \mathrm{Im}(f)_n \to 0.$$

By Proposition 4.4,

$$l\left(\mathrm{Ker}(f)_{n-d_k}\right) - l(M_{n-d_k}) + l(M_n) - l(\mathrm{Im}(f)_n) = 0.$$

Multiplication by t^n and summing over all n gives the identity

$$t^{d_k}\mathrm{HP}_{\mathrm{Ker}(f)}(t) - t^{d_k}\mathrm{HP}_M(t) + \mathrm{HP}_M(t) - \mathrm{HP}_{\mathrm{Im}(f)}(t) = 0.$$

Both $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are annihilated by x_k. Hence, $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are finitely generated $R_0[x_1,\ldots,x_{k-1}]$-modules. Applying the induction hypothesis to the above formula yields the desired result. □

Corollary 4.7. *Let $R = k[x_1, \ldots, x_n]$ be graded via $deg(x_i) = d_i > 0$ for some d_i. Then*

$$HS_R(t) = \frac{1}{\prod_{i=1}^{n}(1 - t^{d_i})}.$$

Proof. The l-th graded part R_l of R is given by

$$R_l = \left\langle x_1^{\alpha_1} \ldots x_l^{\alpha_l}, \alpha_1 d_1 + \cdots + \alpha_l d_l = l \right\rangle_k.$$

Let $p(t)$ be the numerator of the Hilbert series of R. Then

$$\mathrm{HS}_R(t) = \prod_{i=1}^{k} \sum_{j \geq 0} t^{jd_i} = \sum_{\alpha \in \mathbb{N}^k} t^{\alpha_1 d_1 + \cdots + \alpha_k d_k} = \frac{1}{\prod_{i=1}^{k}(1 - t^{d_i})}.$$

Therefore, $p(t) = 1$. □

5 Integer partitions and identities

5.1 Introduction

Integer partitions are a well-studied object in combinatorics and representation theory. We will establish an algebraic description of integer partitions and use graded free resolutions to prove identities on partition functions. Furthermore, recursion formulas for partition functions can be conjectured with this machinery. For a more detailed introduction to the topic, the reader is advised to consult [2].

Definition 5.1. Let n be a positive integer. A *partition* of n is a weakly increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers satisfying $\sum_{i=1}^{k} \lambda_i = n$. In this case, we write $\lambda \vdash n$. We call λ_i a *part* of λ. Denote by $p(n)$ the number of partitions of n. If $n = 0$, set $p(0) = 1$.

We will want to restrict ourselves to partitions, where special conditions on the parts are imposed.

Definition 5.2. Let I be a subset of $\mathbb{N}$. By $\texttt{Part}(I)$ we denote the set of integer partitions, where all parts are in I, and we will write $\texttt{Part}$ for $\texttt{Part}(\mathbb{N})$. By $p(I, n)$ we denote the number of partitions λ of n, where all parts of λ are in I. Note that $p(\mathbb{N}, n) = p(n)$.

Lemma 5.1. *Let n be a nonnegative integer, I be a subset of $\mathbb{N}$, and suppose that $i_1, \ldots, i_k$ are the elements of I that are smaller than or equal to n, where $i_1 < ... < i_k$. The number of solutions $(a_1, a_2, \ldots, a_n) \in \mathbb{N}_0^n$ of $\sum_{l=1}^{k} i_l \cdot a_l = n$ is $p(I, n)$.*

Proof. Consider the map

$$\left\{ (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k, \sum_{l=1}^{k} i_l \cdot a_l = n \right\} \to \texttt{Part}(I),$$

given by

$$(a_1, a_2, \ldots, a_k) \mapsto \underbrace{i_1 + \cdots + i_1}_{a_1} + \underbrace{i_2 + \cdots + i_2}_{a_2} + \cdots + \underbrace{i_k + \cdots + i_k}_{a_k}.$$

It is easy to see that this map is well-defined, and the image is the set of all partitions of n with parts in I . The map to the image is clearly a bijection. □

Corollary 5.1. *Let n be a positive integer. The number of solutions $(a_1, a_2, \ldots, a_n) \in \mathbb{N}_0^n$, of $\sum_{i=1}^{n} i \cdot a_i = n$ is $p(n)$.*

Proof. Set $I = [n]$ in Lemma 5.1. □

Theorem 5.1. *Let $I \subseteq \mathbb{N}$ be any subset. The generating function of* `Part` (I) *is given by*

$$GF_{\texttt{Part}(I)}(t) = \prod_{i \in I} \frac{1}{(1-t^i)}.$$

Proof. We have

$$\begin{aligned}\prod_{i \in I} \frac{1}{(1-t^i)} &= \prod_{i \in I} \left(1 + t^i + t^{2i} + \dots\right) \\ &= \sum_{\alpha \in \mathbb{N}^{(|I|)}} t^{\sum\limits_{i \in I} \alpha_i \cdot i} \\ &= \sum_{n \geq 0} p(I, n)\, t^n,\end{aligned}$$

as an identity of formal power series. The last equality holds because of Lemma 5.1. □

Corollary 5.2. *The generating function of* `Part` *is given by*

$$GF_{\texttt{Part}}(t) = \prod_{i > 0} \frac{1}{(1-t^i)}.$$

Proof. Set $I = \mathbb{N}$ in Theorem 5.1. □

Associating polynomial rings to integer partitions

Remark 5.1. The fact that the Hilbert series of $R = k[x_1, \dots, x_n]$, graded via $\deg(x_i) = d_i$, is given by

$$HS_R(t) = \frac{1}{(1-t^{d_1}) \cdot \dots \cdot (1-t^{d_n})},$$

shows that there is a connection between the polynomial ring R in n variables graded via $\deg(x_i) = i$ and integer partitions of n. Indeed, the *truncated* generating function of `Part`,

$$\frac{1}{(1-t) \dots (1-t^n)},$$

satisfies

$$\left\langle t^k \right\rangle \frac{1}{(1-t) \dots (1-t^n)} = p(k)$$

for $k \leq n$. It is therefore natural to consider R, factor rings, subrings and R-modules for a description of partitions.

Remark 5.2. If $J \subset \mathbb{N}$ is an infinite subset, $k[x_j, j \in J]$ is the polynomial ring in infinitely many variables $x_j, j \in J$. Elements of this ring are polynomials, where each term is a finite product of variables.

Proposition 5.1. *Let $J \subseteq \mathbb{N}$ be some subset and $R = k[x_j, j \in J]$, graded via $deg(x_j) = j$ for $j \in J$. Then the graded parts of R are finitely generated k-vector spaces, and*

$$HS_R(t) = GF_{\texttt{Part}(J)}(t).$$

Proof. Monomials $x^\alpha \in R$ of degree n are products of variables x_j with $j \leq n, j \in J$. Clearly, there are only finitely many monomials of a fixed degree n. If J is a finite subset, the claim holds by Corollary 4.7 and Theorem 5.1. However, if J is not finite, there are $p(I,n)$ monomials of degree n in R, so the n-th graded part of R has dimension $p(I,n)$. This proves the claim. □

Remark 5.3. Note that, when $R = k[x_1, \ldots, x_n]$ is graded via $\deg(x_i) = i$ and $R = \bigoplus_{j \geq 0} R_j$, we have

$$p(j) = \dim_k(R_j)$$

for $j \leq n$, but in general,

$$p(j) \neq \dim_k(R_j) \text{ for } j > n.$$

Corollary 5.3. *Let $R = k[x_i, i \in \mathbb{N}]$, graded via $deg(x_i) = i$. Then,*

$$HS_R(t) = GF_{Part}(t),$$

and there is a bijection

$$\{\lambda \vdash n\} \longleftrightarrow \{x^\alpha, \alpha \in \mathbb{N}^n, deg(x^\alpha) = n\}.$$

Proof. The first part is evident by setting $J = \mathbb{N}$ in Proposition 5.1. The bijection is given by

$$\{x^\alpha, \alpha \in \mathbb{N}^n, \deg(x^\alpha) = n\} \longleftrightarrow \left\{x^\alpha, \alpha \in \mathbb{N}^n, \sum_{i=1}^{n} i \cdot \alpha_i = n\right\} \longleftrightarrow \{\lambda \vdash n\},$$

because of Corollary 5.1. □

Example 5.1. Under the above bijection, partitions of $n = 5$ correspond to the monomials

$$x_1^5, x_1^3 x_2, x_1 x_2^2, x_1^2 x_3, x_2 x_3, x_1 x_4, x_5 \in k[x_1, \ldots, x_5]$$

where the polynomial ring is graded via $\deg(x_i) = i$. The dimension of the fifth graded piece of $k[x_1, x_2, x_3, x_4, x_5]$ is therefore 7. It is generated as a k-vector space by the above monomials. Partitions of 5 with at most one part equal to 1 correspond under the above bijection to the monomials

$$x_1 x_2^2, x_2 x_3, x_1 x_4, x_5 \in k[x_1, \ldots, x_5].$$

These are the monomials of degree 5 that are not divisible by x_1^2. Hence, the dimension of the fifth graded piece of $k[x_1, \ldots, x_5] / (x_1^2)$ is 4. It is generated (as k-vector space) by the classes of the above monomials.
Partitions of 5, where consecutive parts differ by at least 2, correspond under the above bijection to the monomials

$$x_1 x_4, x_5 \in k[x_1, \ldots, x_5].$$

These are the monomials of degree 5 that are not divisible by any of $x_1^2, x_1 x_2, x_2^2, x_2 x_3$. Hence,

$$\dim_k \left(k[x_1, \ldots, x_5] \Big/ \left(x_1^2, x_1 x_2, x_2^2, x_2 x_3\right) \right)_5 = 2.$$

5.2 Partition identities

Imposing additional properties on the parts of a partition, one may consider various partition functions like the ones in Example 5.1. The goal is to use the algebraic interpretation of partition functions and graded rings to detect a similiarity of these objects.

Graded free resolutions and partition identities

As already seen in Chapter 4, given a graded R-module M, a graded free resolution is suited to compute the Hilbert series of M. Furthermore, if $R = k[x_1, \ldots, x_n]$ with $\deg(x_i) = i$, every graded and finitely generated R-module has a finite free resolution by the Hilbert Syzygy theorem (see Theorem 3.2). The next proposition shows how we can use this fact to prove identities for partition functions.

Proposition 5.2. *Let $R = k[x_1, \ldots, x_n]$ be graded via $deg(x_i) > 0$. Suppose that M_1, M_2 are two graded R-modules, and let $\mathcal{F}$ and $\mathcal{G}$ be graded free resolutions of M_1 and M_2. Furthermore, suppose that both $\mathcal{F}$ and $\mathcal{G}$ have the same finite length l and satisfy*

$$F_i \cong G_i$$

as graded R-modules. Then

$$HS_{M_1}(t) = HS_{M_2}(t).$$

Proof. Using the formula for computing the Hilbert series via a graded free resolution, we get

$$\begin{aligned} HS_{M_1}(t) =& \sum_{i=0}^{l} (-1)^i HS_{F_i}(t) \\ =& \sum_{i=0}^{l} (-1)^i HS_{G_i}(t) \\ =& HS_{M_2}(t). \end{aligned}$$

□

Remark 5.4. Two graded $k[x_1, \ldots, x_n]$-modules M_1, M_2 may have the same Hilbert series without being isomorphic. For example, the factor rings $k[x] / (x^3)$ with $\deg(x) = 1$ and $k[y, z] / (y^2, yz, z^2)$ with $\deg(y) = 1$ and $\deg(z) = 2$ share the same Hilbert series $(1 + t + t^2)$, but are not isomorphic, since the first is a principal ideal domain, while the second is not.

5.2.1 Examples for complete intersections

Corollary 5.4. *Suppose that $R = k[x_1, \ldots, x_n]$ is graded via $deg(x_i) = d_i > 0$. Let I and J be ideals of R, both generated by l homogeneous elements $\{f_1, \ldots, f_l\}$ and $\{g_1, \ldots, g_l\}$ that form regular sequences and satisfy $deg(f_k) = deg(g_{\sigma(k)})$ for some permutation σ. Then*

$$HS_{R/I}(t) = HS_{R/J}(t).$$

Proof. Since both I and J are generated by a R-regular sequence, the graded Koszul complexes of R/I and R/J are already graded free resolutions. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are minimal graded free resolutions of R/I and R/J respectively. Then, for $m \leq l$,

$$F_m \cong \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_m \leq l} R\left(-\sum_{j=1}^{m} \deg\left(f_{i_j}\right)\right).$$

Since $\deg\left(f_k\right) = \deg\left(g_{\sigma(k)}\right)$ for some $\sigma \in \sigma_l$,

$$\bigoplus_{1 \leq i_1 < i_2 < \cdots < i_m \leq l} R\left(-\sum_{j=1}^{m} \deg\left(f_{i_j}\right)\right) \cong \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_m \leq l} R\left(-\sum_{j=1}^{m} \deg\left(g_{i_j}\right)\right).$$

Since $\{g_1, \ldots, g_l\}$ is R-regular, the right hand side is isomorphic (as a graded R-module) to G_m, so $F_m \cong G_m$ for all $0 \leq m \leq l$. By Proposition 5.2, R/I and R/J share the same Hilbert series. □

Proving partition identities with Corollary 5.4

As seen above, by describing partition functions via factor rings, where the ideals are generated by regular sequences, we can compare their Hilbert series (and therefore sometimes show equality of objects).

Corollary 5.5. *Let n be a positive integer. The number of partitions λ of n where at most one part is 1 (i.e., $\lambda_2 > 1$) is equal to the number of partitions of n where all parts are different from 2.*

Proof. Let $R = k\left[x_1, \ldots, x_n\right]$. First, partitions with at most one part equal to 1 correspond via the above bijection to monomials that are not divisible by x_1^2. Therefore, classes of monomials that are not divisible by x_1^2 generate the graded parts of $R/\left(x_1^2\right)$.
Second, partitions where all parts are different from 2 correspond to monomials that are not divisible by x_2. The monomials not divisible by x_2 generate the graded parts of $R/\left(x_2\right)$. Both ideals are generated by one element, which are R-regular sequences, and the degrees of the generators are equal to 2. By Corollary 5.4, the Hilbert series coincide, proving the identity. □

Corollary 5.6. *(Euler) Let n be a positive integer. The number of partitions of n into odd parts is equal to the number of partitions of n into pairwise distinct parts.*

Proof. Let $R = k\left[x_1, \ldots, x_n\right]$ graded via $\deg\left(x_i\right) = i$. The factor ring corresponding to partitions with odd parts only is

$$R\big/_I = R\Big/_{\left(x_2, x_4, \ldots, x_{2\lfloor \frac{n}{2} \rfloor}\right)}.$$

Furthermore, partitions into distinct parts correspond to monomials that are square-free in the variables $x_1, \ldots, x_{\lfloor \frac{n}{2} \rfloor}$ (since $\deg\left(x^2_{\lfloor \frac{n}{2} \rfloor + 1}\right) > n$), therefore the corresponding factor ring is given by

$$R\big/_J = R\Big/_{\left(x_1^2, x_2^2, \ldots, x^2_{\lfloor \frac{n}{2} \rfloor}\right)}.$$

The ideals I and J are both generated by R-regular sequences, that are equal in length and degrees. By Corollary 5.4, the two partition functions are equal. □

The last identity we will prove in this section is a part of Schur's partition identity. The proof is slightly more subtle than that of Euler's identity.

Corollary 5.7. *Let n be a positive integer. Denote by*

$A(n) =$ *number of partitions of n where all parts are congruent to 1 or 5 modulo 6,*

$B(n) =$ *number of partitions of n where all parts are distinct and are congruent to 1 or 2 modulo 3*

$C(n) =$ *number of partitions of n where all parts are odd and no three parts are equal.*

Then, $A(n) = B(n) = C(n)$.

Proof. Let $R = k[x_1, \ldots, x_n]$, be graded via $\deg(x_i) = i$. The factor ring corresponding to partitions counted by $A(n)$ is given by

$$R/I_1 = R/(x_i \text{ for } i \equiv 0, 2, 3, 4\,(6)\,, i \leq n)\cdot$$

The degrees of the generators of the ideal are exactly the numbers $6m + l$ with $l \in \{0, 2, 3, 4\}$ and $6m + l \leq n$. The factor ring corresponding to partitions counted by $B(n)$ is given by

$$R/I_2 = R/\left(x_{3i}, x_j^2 \text{ for } 3i \leq n, 2j \leq n, j \equiv 1, 2\,(3)\right)\cdot$$

The degrees of the generators of I_2 are

- the numbers $3i$ for $3i \leq n$, i.e., the numbers smaller than or equal to n that are of the form $6m$ or $6m + 3$, and
- the numbers $2j$ with $2j \leq n$ and $j \equiv 1, 2\,(3)$, i.e., the numbers smaller than or equal to n that are of the form $6m + 2$ or $6m + 4$.

This proves $A(n) = B(n)$. The factor ring corresponding to partitions counted by $C(n)$ is given by

$$R/I_3 = R/\left(x_{2i+1}^3, x_{2j} \text{ for } 3(2i + 1) \leq n, 2j \leq n\right)\cdot$$

The degrees of the generators of I_3 are

- the numbers $3 \cdot (2i + 1)$ for $3(2i + 1) \leq n$, i.e., the numbers smaller than or equal to n of the form $6m + 3$, and
- the numbers $2j$ for $2j \leq n$, i.e., the numbers smaller than or equal to n of the form $6m$ or $6m + 2$ or $6m + 4$.

This now proves that $A(n) = B(n) = C(n)$ by Corollary 5.4.

□

Remark 5.5.

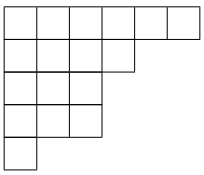
1. The Schur partition identity from above is only a part of what Schur proved. We will get back to this identity and look at more partition functions that are counted by $A(n)$.
2. Corollary 5.4 is only suited to prove *easy* partition identities. We will soon see examples where proofs are not as obvious. This is the case if the generators of the corresponding ideals are not regular sequences.

Encoding the number of parts in ideals

Not all partition identities can be described via factor rings of the polynomial ring as seen above. For example, there is no way to encode the number of parts of a partition by an ideal of the polynomial ring. However, it turns out that there is an interpretation of the number of parts which uses two ideals.

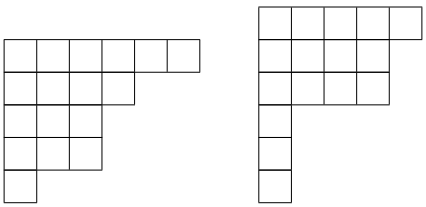
Definition 5.3. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n. A *Ferrers diagram* for λ is a graphical representation of λ as a pattern of boxes with k rows, where the $(k-i+1)$-th row consists of λ_i boxes.

Example 5.2. The Ferrers diagram for $\lambda = (1, 3, 3, 4, 6)$ is given by the following diagram.



Definition 5.4. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n. The *conjugate partition* $\mu = (\mu_1, \ldots, \mu_{\lambda_k})$ to λ is the partition corresponding to the *reflected* Ferrers diagram of λ.

Example 5.3. Consider the following Ferrers diagrams.



The transformation is given by the reflection about the line $x+y=0$. The conjugate partition to $(1, 3, 3, 4, 6)$ is therefore $(1, 1, 1, 4, 4, 5)$.

Remark 5.6. The transition to the conjugate partition is clearly a bijection of the set of partitions of n to itself.

Lemma 5.2. *Let n be a positive integer. The number of partitions of n with exactly l parts is equal to the number of partitions of n where the biggest part is l.*

Proof. The conjugate partition to a partition with exactly l parts has biggest part l, and vice versa. By the above remark we are finished. □

We will now show how the number of parts of a partition can be encoded as ideals. Partitions of n where the biggest part is l correspond to monomials of $k[x_1, \ldots, x_n]$ graded via $\deg(x_i) = i$ which are divisible by x_l, but not divisible by $x_{l+1}, \ldots, x_n$. The dimension of the n-th graded piece of

$$k[x_1, \ldots, x_n] \big/ (x_{l+1}, \ldots, x_n) \cong k[x_1, \ldots, x_l]$$

is equal to the number of partitions of n with the biggest part being less or equal to l. The dimension of the n-th graded piece of

$$k[x_1, \ldots, x_n] \big/ (x_l, \ldots, x_n) \cong k[x_1, \ldots, x_{l-1}]$$

is equal to the number of partitions of n with biggest part less than l. Therefore, the difference

$$\dim_k \left(k\left[x_1,\ldots,x_n\right]/(x_{l+1},\ldots,x_n)\right) - \dim_k \left(k\left[x_1,\ldots,x_n\right]/(x_l,\ldots,x_n)\right)$$

is equal to the number of partitions of n with the biggest part being l. This is equivalent to saying that

$$\left(k\left[x_1,\ldots,x_n\right]/(x_{l+1},\ldots,x_n)\right)/\left(k\left[x_1,\ldots,x_n\right]/(x_l,\ldots,x_n)\right)$$
$$\cong (x_l,\ldots,x_n)/(x_{l+1},\ldots,x_n)$$

encodes the partitions of n where the biggest part is equal to l. This also describes the number of partitions of n with exactly l parts by Lemma 5.2.

5.2.2 Examples for non-complete intersections

As already seen in previous chapters, if ideals I corresponding to partition functions are complete intersections, the graded free resolution of $K\left[x_1,\ldots,x_n\right]/I$ is easy to describe.
If the ideal is not a complete intersection, the Koszul complex is not a graded free resolution - and the computation of a free resolution is much more tedious (Gröbner basis computations are needed). We will consider some examples where partition functions do not correspond to complete intersections.

The Rogers-Ramanujan identities

The Rogers-Ramanujan identities are two identities on hypergeometric series which can be interpreted as partition identities. The identities were discovered by Rogers in 1894. Like many of the formulas he conjectured, Ramanujan found this identity by chance and discovered Rogers proof in 1917. In 1919, they published together a second proof of the identities.

Theorem 5.2. *(First Rogers-Ramanujan identity) Let n be a positive integer. Denote by $a_1(n)$ the number of partitions of n where all parts are congruent to 1 or 4 modulo 5 and by $b_1(n)$ the number of partitions of n where consecutive parts differ by at least 2. Then $a_1(n) = b_1(n)$.*

Theorem 5.3. *(Second Rogers-Ramanujan identity) Let n be a positive integer. Denote by $a_2(n)$ the number of partitions of n where all parts are congruent to 2 or 3 modulo 5 and by $b_2(n)$ the number of partitions of n, where consecutive parts differ by at least 2 and no part is equal to 1. Then $a_2(n) = b_2(n)$.*

Proof. See for example [3]. □

We already saw that using graded free resolutions may result in almost trivial proofs for some partition identities. However, this ansatz does not work as easily in the case of the Rogers-Ramanujan identities. We will not be able to prove the Rogers-Ramanujan identities (yet) with our method. However, we will see some applications of our method to the first identity.

The first Rogers-Ramanujan identity

As shown in Corollary 5.3, partitions of n where all parts are congruent to 1 or 4 modulo 5 correspond to monomials without any variable x_i with $i \equiv 0, 2, 3\,(5)$. We are hence interested in the factor ring

$$k\left[x_1, \ldots, x_n\right] \Big/ (x_i \text{ for } i \leq n, i \equiv 0, 2, 3\,(5))\cdot$$

Note that the numbers of the form $2i + 3j$ for $i, j \in \mathbb{N}_0, i + j \geq 1, |i - j| \leq 1$ are exactly the numbers that are $0, 2, 3$ mod 5.We can therefore rewrite the ideal as

$$\left(x_{2i+3j}, i, j \geq 0, i + j \neq 0, |i - j| \leq 1\right).$$

This ideal is generated by a regular sequence, which will be helpful to find a recursion formula for $a_1\,(n)$ in Theorem 5.11.
On the other hand, partitions of n where all parts differ by at least 2 correspond to squarefree monomials without any occurence of $x_i x_{i+1}$ (see also Example 5.1). This corresponds (via the bijection in Corollary 5.3 to the polynomial ring

$$k\left[x_1, \ldots, x_n\right] \Big/ (x_i x_j \text{ for } i + j \leq n, |i - j| \leq 1)\cdot$$

The ideal in this case is not a complete intersection, therefore the Koszul complex is not a free resolution. Indeed, we know no way to give a precise explicit description of a graded free resolution for all n (for fixed n, one can compute the graded free resolution with any computer algebra program). However, note that in both ideals describing the different kinds of partitions, we have the condition $|i - j| \leq 1$, which might help in proving the identities by using a free resolution.

The second Rogers-Ramanujan identity

For the second identity, the corresponding ideals are given by

$$(x_i \text{ for } i \leq n, i \equiv 0, 1, 4\,(5)) \text{ and } (x_1, x_i x_j \text{ for } i + j \leq n, |i - j| \leq 1).$$

Again, the first ideal is generated by a regular sequence, while the other is not.

Gordon's partition identity

Gordon published a generalization of the Rogers-Ramanujan identities (see [1] for a proof). We will give the algebraic interpretation of it.

Theorem 5.4. *(Gordon's partition identity) Let n, i, k be positive integers. Denote by $A_{k,i}\,(n)$ the number of partitions of n where all parts are not congruent to 0, i or $-i$ modulo $2k + 1$, and denote by $B_{k,i}\,(n)$ the number of partitions of n where*

1. *at most $i - 1$ parts are equal to 1,*
2. *two consecutive integers appear at most $k - 1$ times among the parts*

Then $A_{k,i}\,(n) = B_{k,i}\,(n)$.

By taking $k = 2$ and $i = 2$, the first Rogers-Ramanujan identity is a special case. For $k = 2, i = 1$, the second follows. The corresponding ideals for the Gordon identities are less nice than the ideals for the Rogers-Ramanujan identities. They are given by

$$(x_j, \text{ where } j \leq n, j \equiv 0, i, -i\,(2k+1))$$

and

$$\left(x_1^i, x_j^p x_{j+1}^q, 1 \leq j \leq n-1, p+q=k\right).$$

The first ideal is again a complete intersection, while the second is not.

Schur's Partition Theorem

Theorem 5.5. *(Schur's Partition Theorem) Define $A(n), B(n), C(n)$ as in Corollary 5.7 and denote by $D(n)$ the number of partitions of n, imposing the condition that any two parts differ by at least 3, and the difference between multiples of three parts is at least 6. Then*

$$A(n) = B(n) = C(n) = D(n).$$

The corresponding ideal for $D(n)$ is given by

$$\left(x_i x_j, x_{3k} x_{3(k+1)}, |i-j| \leq 2, i+j \leq n, 3k+1 \leq n\right).$$

Again, this ideal is not generated by a regular sequence, so that we cannot use the techniques established above to prove this theorem.

5.2.3 Treating the case of non-complete intersections

Ideals that are not generated by a regular sequence are not as easy to handle as complete intersections, as seen in the examples above. It might be that using graded free resolutions in this case is not the appropriate way to prove partition identities. It may be necessary to think of other graded structures than the polynomial ring which give a description for partition identities in order to deal with this case. Chapter 6 serves the purpose of showing two further strategies to work with these non-complete intersections without graded free resolutions.

Example 5.4. Consider the ring of power series $k[|x_1, \ldots, x_n|]$, graded via $\deg(x_i) = d_i > 0$. The power series ring is not a graded ring as the polynomial ring, but we can write it as

$$k[|x_1, \ldots, x_n|] = \prod_{i \in \mathbb{N}_0} R_i,$$

where R_i is the k-vector space spanned by all monomials of degree i. We therefore do not lose information on the dimensions by replacing the polynomial ring with the power series ring.

Remark 5.7. A Noetherian local ring R is called a *complete intersection*, if its completion is of the form $S/(f_1, \ldots, f_n)$, where S is a regular local ring and $f_1, \ldots, f_n$ are a regular sequence. It can be proven that localizations of complete intersections are complete intersections again. Hence, it might also be helpful to use localizations to prove partition identities.

5.3 Conjecturing and proving recursion formulas

Euler proved, as a direct consequence of his pentagonal number theorem, the formula

$$0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) + \dots.$$

for the number $p(n)$ of partitions of n, see also Theorem 5.11. For a translation of his original proof, see for example [11]. We will show how graded free resolutions are suitable to conjecture and detect such kind of formulas and show how graded free resolutions of ideals generated by regular sequences are related to the inclusion-exclusion principle.

Prerequisities

Proposition 5.3. *Let $R = k[x_1, \dots, x_n]$ be the graded via $deg(x_i) = i$, and I be a homogeneous ideal. Let*

$$\mathcal{F} : 0 \to F_n \to \cdots \to F_1 \to F_0 \to {}^{R}\!/_{I} \to 0$$

be a minimal graded free resolution of R/I, meaning that $F_i \cong \bigoplus_{p\in\mathbb{Z}} R(-p)^{c_{i,p}}$, where the $c_{i,p}$ are the Betti numbers of I. Then

$$dim_k\left(\left({}^{k[x_1,\dots,x_n]}\!/_{I}\right)_n\right) = \sum_{l=0}^{n} p(n-l) \sum_{i=0}^{n} (-1)^i c_{i,l}.$$

Proof. By the additivity of dimension with respect to exact sequences,

$$\begin{aligned}
\dim_k\left(\left({}^{R}\!/_{I}\right)_n\right) &= \langle t^n \rangle \operatorname{HS}_{{}^{R}\!/_{I}}(t) \\
&= \langle t^n \rangle \sum_{i=0}^{n} (-1)^i \operatorname{HS}_{F_i}(t) \\
&= \langle t^n \rangle \sum_{i=0}^{n} (-1)^i \operatorname{HS}_{\bigoplus_{l\in\mathbb{Z}} R(-l)^{c_{i,l}}}(t) \\
&= \langle t^n \rangle \sum_{i=0}^{n} (-1)^i \left(\sum_{l\in\mathbb{Z}} c_{i,l} \cdot \operatorname{HS}_{R(-l)}(t)\right) \\
&= \langle t^n \rangle \sum_{i=0}^{n} (-1)^i \left(\sum_{l\in\mathbb{Z}} c_{i,l} \cdot \operatorname{HS}_{R}(t)\, t^l\right) \\
&= \sum_{i=0}^{n} \sum_{l\in\mathbb{Z}} (-1)^i c_{i,l} p(n-l) \\
&= \sum_{l=0}^{n} p(n-l) \sum_{i=0}^{n} (-1)^i c_{i,l}.
\end{aligned}$$

The last step holds since $c_{i,l} = 0$ if $l < 0$ and $p(n-l) = 0$ if $l > n$. □

Therefore, the Betti numbers carry a lot of information that will lead to recursion formulas.

Example 5.5. Suppose we are interested in $a_1(6)$ (see also 5.2). Let $R = k[x_1, \ldots, x_6]$ be graded via $\deg(x_i) = i$. We have $I = (x_2, x_3, x_5)$ and the graded free resolution of R/I is given by

$$\begin{aligned} 0 \to R(-10) \to R(-5) \oplus R(-7) \oplus R(-8) \to \\ \to R(-2) \oplus R(-3) \oplus R(-5) \to R \to R/I \to 0, \end{aligned}$$

where we omit the description of the maps. Therefore $c_{0,0} = c_{1,2} = c_{1,3} = c_{1,5} = c_{2,5} = c_{2,7} = c_{2,8} = c_{3,10} = 1$ and $c_{i,j} = 0$ otherwise. Applying the above formula gives

$$\begin{aligned} a_1(6) =& p(6) \cdot \left((-1)^0\right) + p(4) \cdot \left((-1)^1\right) + p(3) \cdot \left((-1)^1\right) + p(1)\left((-1)^1 + (-1)^2\right) \\ =& p(6) - p(4) - p(3) = 11 - 5 - 3 = 3. \end{aligned}$$

Lemma 5.3. *Suppose that, in the setting from above, I is a homogeneous ideal generated by a regular sequence $f_1, \ldots, f_m$ with $deg(f_j) = d_j > 0$. Then $c_{i,l}$ can be interpreted as a number of partitions λ in exactly i parts, where the multiset of parts $\{\lambda_1, \ldots, \lambda_i\}$ of λ is a subset of the multiset $\{d_1, \ldots, d_m\}$.*

Proof. The generators form a regular sequence. Hence,

$$F_i \cong \bigoplus_{1 \le j_1 < \cdots < j_i \le m} R\left(-\left(\sum_{r=1}^{i} d_{j_r}\right)\right) \cong \bigoplus_{l \in \mathbb{Z}} R(-l)^{c_{i,l}},$$

that is, $c_{i,l}$ is the number of ways to write l as an ordered sum of degrees of the f_i. These ordered sums are simply partitions of l into exactly i parts, where the multiset of parts is a subset of the multiset of degrees of $f_1, \ldots, f_m$. □

Example 5.6. Let $I = (x_1, \ldots, x_n)$ be an ideal of the polynomial ring $R = k[x_1, \ldots, x_n]$ graded via $\deg(x_i) = i$. By 5.3, the graded Betti number $c_{i,l}$ is equal to the number of partitions of l into exactly i parts, where all parts are pairwise distinct.

5.3.1 A bijective proof of the pentagonal number theorem

In order to use Proposition 5.3 and Lemma 5.3 together with Example 5.6 to prove the partition formula at the beginning of this chapter, we need Euler's pentagonal number theorem. We will give a combinatorial proof, which will also hint at a recurrence formula for the Rogers-Ramanujan identities. This proof is due to Franklin (1881) (see [2] and [3].

Theorem 5.6. *(Euler pentagonal number theorem) Let n be a positive integer. Denote by $X(n)$ the number of partitions of n into an even number of distinct parts, and denote by $Y(n)$ the number of partitions of n into an odd number of distinct parts. Then*

$$X(n) - Y(n) = \begin{cases} 1, & \text{if there is an even } i \text{ such that } n \in \left\{\frac{i(3i+1)}{2}, \frac{i(3i-1)}{2}\right\} \\ -1, & \text{if there is an odd } i \text{ such that } n \in \left\{\frac{i(3i+1)}{2}, \frac{i(3i-1)}{2}\right\} \\ 0, & \text{otherwise.} \end{cases}$$

Remark 5.8. We will prove Theorem 5.6 by giving a sum-preserving function from the set of partitions with even number of distinct parts to the set of partitions with odd number of distinct parts which is *almost* a bijection. The partitions that are not targeted correspond to the special cases described in Theorem 5.6.

Example 5.7. Let us try to find a *natural* bijective map for the case $n = 10$. Partitions of 10 with an even number of pairwise distinct parts are $1+2+3+4, 4+6, 3+7, 2+8, 1+9$. For partitions of 10 with an odd number of pairwise distinct parts, we get $1+4+5, 2+3+5, 1+3+6, 7+2+1, 10$. As we wish to give a bijection, we might naturally think of either combining two numbers of a partitions (therefore lowering the number of parts by 1), or separating a part into two smaller parts (increasing the number of parts by 1). In both cases, we will change from an odd number of parts to an even number of parts, and vice versa.
Let us start with the trivial partition 10. Since it has only one part, we will definitely want to separate 10 into two parts. The primary candidate for this procedure is clearly $1+9$.
On the other hand, given any partition λ with k parts and $\lambda_1 = 1$, we might be tempted to add 1 to the biggest part λ_k of λ, giving a partition λ', where 1 and λ_k are omitted, but the biggest part λ'_{k-1} is $\lambda_k + 1$. This prescription is clearly sum preserving. By this description, $1+2+7$ corresponds to $2+8$, $1+3+6$ corresponds to $3+7$ and $1+4+5$ corresponds to $4+6$ and $1+2+3+4$ corresponds to $2+3+5$. For this case, the above considerations apply. The proof below will show that this was the correct intuition.

Example 5.8. If $n = 12$, using the same prescription as in 5.7 also works, but the partition $3, 4, 5$ of 12 has no partition that corresponds to it under the above observations. Recall from 5.6 that $n = 12$ is a special case, since it is of the form $i(3i-1)/2$.

Example 5.9. Let us finally consider the case $n = 13$. After using the prescription given in Example 5.7, two partitions of 13 are left, they are $2+5+6$ and $6+7$. We want the prescription to assign these two partitions together, which happens by merging $2+5$ to 7, or vice versa, by splitting 7 into $2+5$. Note that, in the partition $\lambda = (2, 5, 6)$ of 13, the biggest $\lambda_1 = 2$ parts are a arithmetic sequence of difference 1. This will be enough to give the bijection, as shown in the proof below.

Proof. Consider the following sets:

$$S_{\text{even}}(n) = \left\{(\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ even}, \lambda_1 < \cdots < \lambda_k, \lambda \vdash n\right\},$$

$$S_{\text{odd}}(n) = \left\{(\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ odd}, \lambda_1 < \cdots < \lambda_k, \lambda \vdash n\right\},$$

$$S(n) = S_{\text{even}}(n) \dot{\cup} S_{\text{odd}}(n), S = \dot{\bigcup_{i \geq 1}} S(n).$$

Clearly, S is a well-defined disjoint union of the $S(n)$. A element of $S(n)$ corresponds to a partition of n with only distinct parts in the natural way via

$$(\lambda_1, \ldots, \lambda_k) \leftrightarrow \lambda_1 + \cdots + \lambda_k.$$

By the above notation, we have $|S_{\text{even}}(n)| = X(n)$ and $|S_{\text{odd}}(n)| = Y(n)$.
In order to prove the similarity of these two numbers, we define the following sets:

$$
\begin{aligned}
F_{\text{even}}(n) = \Big\{ & (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ even}, \lambda_1 < \cdots < \lambda_k, \\
& \lambda_1 < k, \lambda_{k-\lambda_1+1} = \lambda_k - \lambda_1 + 1, \lambda \vdash n \big\}, \\
F_{\text{odd}}(n) = \Big\{ & (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ odd}, \lambda_1 < \cdots < \lambda_k, \\
& \lambda_1 < k, \lambda_{k-\lambda_1+1} = \lambda_k - \lambda_1 + 1, \lambda \vdash n \big\}, \\
F(n) = F_{\text{even}}(n) \dot{\cup} F_{\text{odd}}(n), F = & \dot{\bigcup_{i \geq 1}} F(n).
\end{aligned}
$$

Elements λ of F are strictly increasing sequences, where the smallest part λ_1 is smaller than the number of elements and the biggest λ_1 parts form an arithmetic sequence with common difference 1. For example, $(3,6,7,8) \in F$, since the biggest $\lambda_1 = 3$ parts are an arithmetic sequence with common difference 1. Note that every element with $\lambda_1 = 1$ is an element of F (see also Example 5.7). Furthermore set

$$
\begin{aligned}
G_{\text{even}}(n) = \Big\{ & (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ even}, \lambda_1 < \cdots < \lambda_k, \\
& \lambda_1 > k+1 \text{ or } \lambda_{k-\min(k,\lambda_1)+1} = \lambda_k - \min(k, \lambda_1) + 1, \lambda \vdash n \big\}, \\
G_{\text{odd}}(n) = \Big\{ & (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ odd}, \lambda_1 < \cdots < \lambda_k, \\
& \lambda_1 > k+1 \text{ or } \lambda_{k-\min(k,\lambda_1)+1} = \lambda_k - \min(k, \lambda_1) + 1, \lambda \vdash n \big\}, \\
G(n) = G_{\text{even}}(n) \dot{\cup} G_{\text{odd}}(n), G = & \dot{\bigcup_{i \geq 1}} G(n).
\end{aligned}
$$

Elements of G are strictly increasing sequencens, where the smallest part λ_1 is either bigger than the number of parts plus 1, or the biggest $\min(k, \lambda_1)$ parts are not an arithmetic sequence with common difference 1. For example, $(4,5,6,7)$ is neither in G, nor in F, but $(5,6,7) \in G$. Set

$$
\begin{aligned}
V_{\text{even}}(n) &= \Big\{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ even}, \lambda_1 < \cdots < \lambda_k, \lambda_1 = k, \lambda_k = 2k-1, \lambda \vdash n \Big\} \\
&= \left\{ (k, k+1, \ldots, 2k-2, 2k-1), k \text{ even}, \sum_{i=0}^{k-1} k+i = n \right\} \\
V_{\text{odd}}(n) &= \Big\{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ odd}, \lambda_1 < \cdots < \lambda_k, \lambda_1 = k, \lambda_k = 2k-1, \lambda \vdash n \Big\} \\
&= \left\{ (k, k+1, \ldots, 2k-2, 2k-1), k \text{ odd}, \sum_{i=0}^{k-1} k+i = n \right\} \\
V(n) &= V_{\text{even}}(n) \dot{\cup} V_{\text{odd}}(n), V = \dot{\bigcup_{i \geq 1}} V(n).
\end{aligned}
$$

$$
\begin{aligned}
W_{\text{even}}(n) &= \Big\{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ even}, \lambda_1 < \cdots < \lambda_k, \lambda_1 = k+1, \lambda_k = 2k, \lambda \vdash n \Big\} \\
&= \left\{ (k+1, k+2, \ldots, 2k-1, 2k), k \text{ even}, \sum_{i=0}^{k-1} k+1+i = n \right\} \\
W_{\text{odd}}(n) &= \Big\{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k, k \text{ odd}, \lambda_1 < \cdots < \lambda_k, \lambda_1 = k+1, \lambda_k = 2k, \lambda \vdash n \Big\}
\end{aligned}
$$

$$= \left\{ (k+1, k+2, \ldots, 2k-1, 2k), k \text{ odd}, \sum_{i=0}^{k-1} k+i = n \right\}$$

$$W(n) = W_{\text{even}}(n) \dot{\cup} W_{\text{odd}}(n), W = \dot{\bigcup}_{i \geq 1} W(n).$$

Elements of V and W are arithmetic sequences with common difference 1, where the smallest part is either equal to the number of parts or the number of parts plus 1. These sets will correspond to the special cases, see also Example 5.8.

Step 1. We will first prove that

$$S(n) = F(n) \dot{\cup} G(n) \dot{\cup} V(n) \dot{\cup} W(n).$$

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in S(n)$. We have to consider 4 cases:

1. If $\lambda_1 > k+1$, $\lambda \in G$.
2. If $\lambda_1 = k+1$, we have two cases to consider.
   - If $\lambda_k = 2k$, $\lambda \in W$.
   - If $\lambda_k \neq 2k$, the biggest k parts $\lambda_{n-k+1}, \ldots, \lambda_n$ are not an arithmetic sequence, so that $\lambda \in G$.
3. If $\lambda_1 = k$, we also have two cases to consider.
   - If $\lambda_k = 2k-1$, $\lambda \in V$.
   - If $\lambda_k \neq 2k-1$, the biggest k parts $\lambda_{n-k+1}, \ldots, \lambda_n$ are not an arithmetic sequence, so that $\lambda \in G$.
4. If $\lambda_1 < k$, we have either $\lambda \in F$ (if the biggest λ_1 parts are an arithmetic sequence with difference 1) or $\lambda \in G$ (if they are not).

It is easy to see that the above union is disjoint. Therefore,

$$S_{\text{even}}(n) = F_{\text{even}}(n) \dot{\cup} G_{\text{even}}(n) \dot{\cup} V_{\text{even}}(n) \dot{\cup} W_{\text{even}}(n)$$

and

$$S_{\text{odd}}(n) = F_{\text{odd}}(n) \dot{\cup} G_{\text{odd}}(n) \dot{\cup} V_{\text{odd}}(n) \dot{\cup} W_{\text{odd}}(n).$$

Furthermore,

$$X(n) = |F_{\text{even}}(n)| + |G_{\text{even}}(n)| + |V_{\text{even}}(n)| + |W_{\text{even}}(n)|$$

and

$$Y(n) = |F_{\text{odd}}(n)| + |G_{\text{odd}}(n)| + |V_{\text{odd}}(n)| + |W_{\text{odd}}(n)|.$$

Step 2. We will now prove that $|F_{\text{even}}(n)| = |G_{\text{odd}}(n)|$ and $|F_{\text{odd}}(n)| = |G_{\text{even}}(n)|$.
Consider the maps $\alpha : F \to G$ and $\beta : G \to F$, defined as follows.
If $\lambda = (\lambda_1, \ldots, \lambda_k) \in F$,

$$\alpha(\lambda) = \left(\lambda_2, \ldots, \lambda_{k-\lambda_1}, \widehat{\lambda_{k-\lambda_1+1}}, \lambda_{k-\lambda_1+2}, \ldots, \lambda_k, \lambda_k + 1\right).$$

If $\mu = (\mu_1, \ldots, \mu_m) \in G$,

$$\beta(\mu) = (r, \mu_1, \ldots, \mu_m - r, \ldots, \mu_{m-1}),$$

where $r = \min_i \{\mu_{m-i} < \mu_m - i\}$.

Step 2a. The maps α and β are well-defined.
If $\lambda_2 > k$, α is clearly well-defined. Otherwise, we have to prove that the last λ_2 parts of $\alpha(\lambda)$ are not in arithmetic progression with common difference 1. By definition, the last λ_1 parts are a arithmetic progression with common difference 1, but

$$\lambda_{k-\lambda_1} + 1 < \lambda_{k-\lambda_1+2},$$

so that these λ_1 parts are a maximal progression. Since $\lambda_2 > \lambda_1$, we are finished.
Proving that β is well-defined is more difficult. Let us first prove that $r < m+1$. If $\mu_1 < m+1$, $r < m+1$, since $r \leq \mu_1$ by the definition of G. Otherwise, we have to consider two cases. First, consider the case $\mu_1 \geq m+1$ and the $m+1$ parts are in arithmetic progression with common difference 1. Then

$$\beta(\mu_1, \ldots, \mu_m) = (m, \mu_1 - 1, \mu_1, \ldots, \mu_{m-1}),$$

so that $r = m$. In the second case, the elements of $\beta(\mu)$ are not in arithmetic progression, so that $r < m+1$ holds trivially.
By the minimality of r, the last r parts of $\beta(\mu)$ are in arithmetic progression with common difference 1, so that $\beta(\mu) \in F$.

Step 2b. $\alpha|_{F_{\text{even}}(n)}$: $F_{\text{even}}(n) \to G_{\text{odd}}(n)$ is well-defined.
Note that α and β are sum-preserving. $\alpha(\lambda)$ omits $\lambda_{k-\lambda_1+1} = \lambda_k - \lambda_1 + 1$ and λ_1, but adds $\lambda_{k-\lambda_1+1} + \lambda_1 = \lambda_k + 1$, so that α is sum-preserving. β splits μ_m into two summands $\mu_m - r$ and r and is therefore sum-preserving.
Since α and β either add or subtract a part, $\alpha|_{F_{\text{even}}(n)}$ is well-defined, as well as $\alpha|_{F_{\text{odd}}(n)}$, $\beta|_{G_{\text{even}}(n)}$ and $\beta|_{G_{\text{odd}}(n)}$.

Step 2c. We prove that $\alpha \circ \beta = \text{id}_F$ and $\beta \circ \alpha = \text{id}_G$.
We first prove that $\alpha \circ \beta = \text{id}_F$. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in F$ and

$$\alpha(\lambda) = \left(\lambda_2, \ldots, \lambda_{k-\lambda_1}, \widehat{\lambda_{k-\lambda_1+1}}, \lambda_{k-\lambda_1+2}, \ldots, \lambda_k, \lambda_k + 1\right).$$

The last λ_1 parts of $\alpha(\lambda)$ are in arithmetic progression with common difference 1, but the last $\lambda_1 + 1$ parts are not. Therefore, $r = \lambda_1$, and $\beta(\alpha(\lambda)) = (\lambda_1, \ldots, \lambda_k)$, so that $\alpha \circ \beta = \text{id}_F$.
For $\beta \circ \alpha$, let $\mu = (\mu_1, \ldots, \mu_m) \in G$ and

$$\beta(\mu) = (r, \mu_1, \ldots, \mu_m - r, \mu_{m-r+1}, \ldots, \mu_{m-1}).$$

By the minimality of r, $\mu_m - r, \mu_{m-r+1}, \ldots, \mu_{m-1}$ are r elements in arithmetic progression of with common difference 1, such that the first entry of $\beta(\mu)$, r, will be added to $\mu_m - r$. This shows that $\beta \circ \alpha = \text{id}_G$.
Therefore, $\alpha = \beta^{-1}$ and α and β are bijections.

Step 3. We compute $|V_{\text{even}}(n)|$, $|V_{\text{odd}}(n)|$, $|W_{\text{even}}(n)|$ and $|W_{\text{odd}}(n)|$.

We have

$$\begin{aligned}|V_{\text{even}}(n)| &= \left|\left\{(k, k+1, \ldots, 2k-2, 2k-1), k \text{ even}, \sum_{i=0}^{k-1} k+i = n\right\}\right| \\ &= \left|\left\{k \text{ even}, \sum_{i=0}^{k-1} k+i = n\right\}\right| \\ &= \left|\left\{k \text{ even}, \frac{k(3k-1)}{2} = n\right\}\right|.\end{aligned}$$

In a similar manner, we get

$$\begin{aligned}|W_{\text{even}}(n)| &= \left|\left\{k \text{ even}, \frac{k(3k+1)}{2} = n\right\}\right|, \\ |V_{\text{odd}}(n)| &= \left|\left\{k \text{ odd}, \frac{k(3k-1)}{2} = n\right\}\right|, \\ |W_{\text{odd}}(n)| &= \left|\left\{k \text{ odd}, \frac{k(3k+1)}{2} = n\right\}\right|.\end{aligned}$$

Step 4. Summarizing steps 1 to 3, we get

$$\begin{aligned}X(n) - Y(n) &= |F_{\text{even}}(n)| + |G_{\text{even}}(n)| + |V_{\text{even}}(n)| + |W_{\text{even}}(n)| - \\ &\quad - |F_{\text{odd}}(n)| - |G_{\text{odd}}(n)| - |V_{\text{odd}}(n)| - |W_{\text{odd}}(n)| \\ &= |V_{\text{even}}(n)| + |W_{\text{even}}(n)| - |V_{\text{odd}}(n)| - |W_{\text{odd}}(n)| \\ &= \begin{cases} 1, & \text{if } \exists i \in \mathbb{N}, i \equiv 0\,(2) : n \in \left\{\frac{i(3i+1)}{2}, \frac{i(3i-1)}{2}\right\} \\ -1, & \text{if } \exists i \in \mathbb{N}, i \equiv 1\,(2) : n \in \left\{\frac{i(3i+1)}{2}, \frac{i(3i-1)}{2}\right\} \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

□

Remark 5.9. Note that the generating function of the sequence of numbers $(X(n) - Y(n))_{n\in\mathbb{N}}$ is given by

$$\prod_{i\geq 1}\left(1 - t^i\right).$$

The above theorem is therefore equivalent to the following identity of power series:

$$\prod_{i>0}\left(1 - t^i\right) = \sum_{i=-\infty}^{\infty} (-1)^i\, t^{\frac{i(3i-1)}{2}}.$$

This is a special case of the Jacobi triple product identity (see also Theorem 5.9).

5.3.2 Examples of recursion formulas

Recursion formula for partitions

Theorem 5.7. *Let n be a positive integer and $p(n)$ the number of partitions of n. Then*

$$0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \ldots,$$

where the general term is given by the numbers $i(3i \pm 1)/2$.

Proof. For a fixed number $n \in \mathbb{N}$ consider the graded polynomial ring $R = k[x_1, \ldots, x_n]$ with $\deg(x_i) = i$. The factor ring $R/I = R/(x_1, \ldots, x_n) \cong k$ is again graded, and the n-th graded part is of dimension $\delta_{0,n}$.
Consider now the minimal graded free resolution of R/I that is given by the Koszul complex. The degrees of the generators of I are exactly the numbers $1, 2, \ldots, n$. Therefore, by the construction of the Koszul complex, the i-th R-module in the graded free resolution is isomorphic to

$$\bigoplus_{1 \leq j_1 < \cdots < j_i \leq n} R\left(-\sum_{r=1}^{i} j_r\right) \cong \bigoplus_{l \in \mathbb{Z}} R(-l)^{c_{i,l}},$$

where the $c_{i,l}$ are the graded Betti numbers of I. By Lemma 5.3, $c_{i,l}$ is the number of ways to write l as a sum of i distinct positive numbers, that is, the number of partitions of l into exactly i pairwise distinct parts (see also Example 5.6).
Consider now the sum $\sum_{i=0}^{n} (-1)^i c_{i,l}$ which we want to compute. By the above argument, this is for $l \leq n$ the number of partitions of l into an even number of distinct parts, minus the number of partitions of l into an odd number of distinct parts meaning that $\sum_{i=0}^{n} (-1)^i c_{i,l} = Y(l) - X(l)$.
Now, using Proposition 5.3 with $I = (x_1, \ldots, x_n)$ and Theorem 5.6, we get

$$\begin{aligned} 0 =& \sum_{l=0}^{n} p(n-l) \sum_{i=0}^{n} (-1)^i c_{i,l} \\ =& \sum_{l=0}^{n} p(n-l)\left(Y(l) - X(l)\right) \\ =& p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \ldots \end{aligned}$$

□

Recursion formula for the Euler identity

Remark 5.10. We already saw in Corollary 5.6 that the number $d(n)$ of partitions of n into distinct parts is equal to the number of partitions of n into odd parts only.

Theorem 5.8. *Denote by $d(n)$ the number of partitions of n into distinct parts. Then*

$$d(n) = p(n) - p(n-2) - p(n-4) + p(n-10) + p(n-14) - \ldots,$$

where the numbers in parentheses are exactly the numbers of the form $i(3i \pm 1)$.

Proof. Fix n and consider the polynomial ring $R = k[x_1, \ldots, x_n]$, graded via $\deg(x_i) = i$, and the ideal $I = \left(x_j^2, 2j \leq n\right)$, as in Corollary 5.6. The degrees of the generators of I are exactly the even numbers smaller than or equal to n.
The i-th graded R-module of the minimal graded free resolution of R/I is by the construction of the Koszul complex isomorphic to

$$\bigoplus_{1 \leq j_1 < \cdots < j_i \leq \frac{n}{2}} R\left(-2\sum_{r=0}^{i} j_r\right) \cong \bigoplus_{l \in \mathbb{Z}} R(-l)^{d_{i,l}},$$

where the $d_{i,l}$ are the graded Betti numbers of I. By Lemma 5.3, $d_{i,l}$ is the number of ways to write l as a sum of i distinct even numbers. If l is odd, $d_{i,l} = 0$. Otherwise, $d_{i,l}$ is also the number of ways to write $\frac{l}{2}$ as a sum of distinct positive integers. Using the notation from above, as well as Theorem 5.6, we get

$$\begin{aligned} d(n) =& \sum_{l=0}^{n} p(n-l) \sum_{i=0}^{n} (-1)^i d_{i,l} \\ =& \sum_{m=0}^{\frac{n}{2}} p(n-2m) \sum_{i=0}^{n} (-1)^i d_{i,2m} \\ =& \sum_{m=0}^{\frac{n}{2}} p(n-2m) (Y(m) - X(m)) \\ =& p(n) - p(n-2) - p(n-4) + p(n-10) + p(n-14) - \dots . \end{aligned}$$

□

Rogers-Ramanujan identities

To prove a similar result for the Rogers-Ramanujan identities, we will not use a bijective correspondence as in Theorem 5.6, but instead power series computations.

Theorem 5.9. *(Jacobi Triple Product identity) Let z, q be variables. Then we have the following identity of power series:*

$$\prod \left(1 - q^{2n}\right) \left(1 + zq^{2n-1}\right) \left(1 + z^{-1}q^{2n-1}\right) = \sum_{n \in \mathbb{Z}} z^n q^{n^2}.$$

Proof. See [3]. □

Corollary 5.8. *If t is a variable, the following identity of power series holds:*

$$\prod_{n \geq 1} \left(1 - t^{5n}\right) \left(1 - t^{5n-2}\right) \left(1 - t^{5n-3}\right) = \sum_{n \in \mathbb{Z}} (-1)^n t^{\frac{n(5n+1)}{2}}.$$

Proof. Set $q = t^{\frac{5}{2}}$ and $z = -t^{\frac{1}{2}}$ in Theorem 5.9. □

Theorem 5.10. *Let n be a positive integer. Denote by $W_1(n)$ the number of partitions of n into an even number of distinct parts that are all congruent to 0, 2 or 3 modulo 5, and denote by $W_2(n)$ the number of partitions of n into an odd number of distinct parts that are all congruent to 0, 2 or 3 modulo 5. Then*

$$W_1(n) - W_2(n) = \begin{cases} 1, & \text{if there is an even } i \text{ such that } n \in \left\{ \frac{i(5i+1)}{2}, \frac{i(5i-1)}{2} \right\} \\ -1, & \text{if there is an odd } i \text{ such that } n \in \left\{ \frac{i(5i+1)}{2}, \frac{i(5i-1)}{2} \right\} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Note that the generating function of the sequence $(W_1(n) - W_2(n))_{n \in \mathbb{N}}$ is given by

$$\prod_{n \geq 1} \left(1 - t^{5n}\right) \left(1 - t^{5n-2}\right) \left(1 - t^{5n-3}\right).$$

The rest follows from Corollary 5.8. □

Theorem 5.11. *Let n be a positive integer. Then*

$$a_1(n) = p(n) - p(n-2) - p(n-3) + p(n-9) + p(n-11) - \ldots,$$

where the numbers in parentheses are exactly the numbers of the form $\frac{i(5i\pm1)}{2}$.

Proof. Fix n and consider the polynomial ring $R = k[x_1, \ldots, x_n]$, graded via $\deg(x_i) = i$, and the ideal $I = (x_i, i \leq n, i \equiv 0, 2, 3\,(5))$. The i-th graded R-module of the minimal graded free resolution of R/I is by the construction of the Koszul complex isomorphic to

$$\bigoplus_{\substack{1 \leq j_1 < \cdots < j_i \leq n \\ j_1, \ldots, j_i \equiv 0,2,3(5)}} R\left(-\sum_{r=0}^{i} j_r\right) \cong \bigoplus_{l \in \mathbb{Z}} R(-l)^{e_{i,l}}.$$

Therefore, $e_{i,l}$ is the number of ways to write l as a sum of i distinct numbers that are congruent to 0,2 or 3 modulo 5. By Proposition 5.3 and Theorem 5.10,

$$\begin{aligned} a_1(n) =& \sum_{l=0}^{n} p(n-l) \sum_{i=0}^{n} (-1)^i e_{i,l} \\ =& \sum_{l=0}^{n} p(n-l)\,(W_2(l) - W_1(l)) \\ =& p(n) - p(n-2) - p(n-3) + p(n-9) + p(n-11) - \ldots. \end{aligned}$$

□

Remark 5.11. A similar argument also gives a recursion formula for the second identity of Rogers and Ramanujan.

$$a_2(n) = p(n) - p(n-1) - p(n-4) + p(n-7) + p(n-13) - \ldots.$$

The numbers in parentheses are exactly the numbers of the form $\frac{i(5i\pm3)}{2}$

Remark 5.12. Computing the free resolution for the $b_1(n)$ also gives a recursion formula - but it is much more tedious to compute it. We cannot conclude that the outcome of the computation of the graded free resolution gives the same recursion formula, although we abstractly know that the $b_1(n)$ also fulfil the formula.

Remark 5.13. The examples above give enough insight to conjecture the following identity. Let $A_{k,i}(n)$ be defined as in Theorem 5.4. Then

$$\begin{aligned} A_{k,i}(n) = p(n) &+ \sum_{m \geq 1} (-1)^m p\left(n - \frac{m((2k+1)m - (2k+1-2i))}{2}\right) \\ &+ \sum_{m \geq 1} (-1)^m p\left(n - \frac{m((2k+1)m + (2k+1-2i))}{2}\right). \end{aligned}$$

5.4 Two further strategies to prove partition identities

Our goal is to show two more strategies that can be used to prove partition identities.

5.4.1 Strategy 2 - Hilbert series of leading term ideals

Our goal is to show that, under some assumptions, Hilbert series computations of ideals in the polynomial ring can be reduced to computations of the Hilbert series on the leading terms of the ideal, with respect to a monomial order. In general, if we choose two different monomial orders, the ideal generated by the leading terms will not be equal, but they will have the same Hilbert series.

Definition 5.5. Let $>$ be a monomial order on the monomials of $k[x_1, \ldots, x_n]$, graded via $\deg(x_i) = d_i > 0$. We say that $>$ is a *graded* monomial order if

$$|\alpha| >_{\mathbb{N}} |\beta| \Rightarrow x^\alpha > x^\beta.$$

Theorem 5.12. *Suppose that $k[x_1, \ldots, x_n]$ is graded via $deg(x_i) = d_i > 0$ and let I be a homogeneous ideal of $k[x_1, \ldots, x_n]$. Furthermore, let $>$ be a graded monomial order on the monomials $k[x_1, \ldots, x_n]$. Then*

$$HS_I(t) = HS_{LT_>(I)}(t).$$

Proof. Since I is homogeneous, it is graded, and the m-th graded piece of I, which consists of the elements of I of degree m, is a k-vector space. Also, the m-th graded piece of $\langle \mathrm{LT}_>(I)\rangle_k$ is a k-vector space. We have to prove that these two vector spaces are of the same dimension for all $m \in \mathbb{Z}$. This is trivial for $m \leq 0$, so assume $m \geq 1$.
Fix m. Consider $I_{\leq m}$, the k-vector space of all elements of I of degree less than or equal to m. The leading terms of elements of $I_{\leq m}$ are all of degree m or less. Since there are only finitely many monomials of degree m or less, there exist $f_1, \ldots, f_t \in I_{\leq m}$ that satisfy

$$\{\mathrm{LM}_>(f), f \in I_{\leq m}\} = \{\mathrm{LM}_>(f_1), \ldots, \mathrm{LM}_>(f_t)\}.$$

We can assume that there are no multiples in the latter and that they are ordered by $\mathrm{LM}_>(f_i) > \mathrm{LM}_>(f_j)$ for $1 \leq i < j \leq t$. We claim that $f_1, \ldots, f_t$ are a k-vector space basis for $I_{\leq m}$.
First, consider a linear combination $a_1 f_1 + \cdots + a_t f_t = 0$ with $a_1, \ldots, a_n \in k$. Suppose that there is a $a_j \neq 0$, and let i be the smallest i satisfying $a_i \neq 0$. Then $a_i \cdot \mathrm{LT}_>(f_i)$ will not cancel by the minimality of i, so that the linear combination is not 0. Therefore, $f_1, \ldots, f_t$ are k-linearly independent.
Next, suppose that $\langle f_1, \ldots, f_n\rangle \subsetneq I_{\leq m}$. Pick $f \in I_{\leq m} \setminus \langle f_1, \ldots, f_n\rangle$ with $\mathrm{LM}_>(f)$ minimal. Then there is an index i, $1 \leq i \leq n$, such that $\mathrm{LM}_>(f) = \mathrm{LM}_>(f_i)$. Hence, there is an element $\lambda \in k$ such that $\mathrm{LT}_>(f) = \lambda \mathrm{LT}_>(f_i)$, so $\mathrm{LM}_>(f - \lambda f_i) < \mathrm{LM}_>(f)$. By the minimality of $\mathrm{LM}_>(f)$, we get $f - \lambda f_i \in \langle f_1, \ldots, f_n\rangle$ and therefore $f \in \langle f_1, \ldots, f_n\rangle$, contradiction. Therefore, $f_1, \ldots, f_n$ are a k-basis of $I_{\leq m}$.
We still have to show that $\dim \left\langle (\mathrm{LT}_>(I))_{\leq m} \right\rangle_k = t$. We claim that $\mathrm{LM}_>(f_1), \ldots, \mathrm{LM}_>(f_t)$ are a vector space basis for $\langle \mathrm{LT}(I)\rangle_{\leq m}$. By the same argumentation as above, they are k-linearly independent. Furthermore, notice that

$$\begin{aligned}
\langle \mathrm{LM}_>(f_1), \ldots, \mathrm{LM}_>(f_t)\rangle &= \langle \mathrm{LM}_>(f), f \in I_{\leq m}\rangle \\
&= \langle \mathrm{LM}_>(f), f \in I, \deg(\mathrm{LM}(f)) \leq m\rangle \\
&= \langle \mathrm{LT}(I)\rangle_{\leq m}.
\end{aligned}$$

The second equality holds since $>$ is graded and therefore $\deg(f) = \deg(\mathrm{LT}(f))$. The last equality holds since an element of a monomial ideal is in the ideal if and only if all terms of the element are in the ideal. Therefore, $\mathrm{LM}_{>}(f_1), \ldots, \mathrm{LM}_{>}(f_n)$ are a k-vector space basis. Since

$$\dim_k I_{\leq m} = \sum_{j=0}^{m} \dim_k I_j,$$

the claim follows by induction. □

Strategy 2. The strategy for proving partition identities A_1 and A_2 with this theorem is as follows.

1. For some partition identity and a fixed integer n, find the ideals J_1, J_2, corresponding to the partitions A_1 and A_2 (as shown in Section 5.1).
2. Find an ideal $I \subseteq k[x_1, \ldots, x_n]$ and two graded monomial orders $<_1, <_2$, so that
$$\langle \mathrm{LT}_{<_1}(I) \rangle = J_1 \text{ and } \langle \mathrm{LT}_{<_2}(I) \rangle = J_2.$$
3. Applying Theorem 5.12 yields that the n-th graded piece of J_1 is of the same dimension as the n-th graded piece of J_2, which shows the partition identity for n.

Revisiting Eulers identity

We will use the strategy discussed above to give a second proof of Euler's identity (see Corollary 5.6).

Proof. Fix $n \in \mathbb{N}$. Consider the ideal $I = \left(x_i^2 + x_{2i}, 2i \leq n\right)$ of $k[x_1, \ldots, x_n]$, graded via $\deg(x_i) = i$. Let $<_1$ be the deglex order and $<_2$ be the degrevlex order. Then

$$\mathrm{LT}_{<_1}(I) = \left\{x_i^2, 2i \leq n\right\}$$

and

$$\mathrm{LT}_{<_2}(I) = \{x_{2i}, 2i \leq n\}.$$

It is easy to prove that these equalities are true. Clearly, these two sets are minimal generating sets of the ideal of leading terms they generate, and those ideals are exactly the ideals in Corollary 5.6. □

Revisiting the Rogers-Ramanujan identities

One may ask whether this strategy gives an obvious proof for the Rogers-Ramanujan identities. Unfortunately, the author has not found any suitable ideal and monomial orders up to date.

Question. For a fixed n, are there an ideal I of $k[x_1, \ldots, x_n]$ and graded monomial orders $<_1, <_2$, so that

$$\langle \mathrm{LT}_{<_1}(I) \rangle = \langle x_i, i \leq n, i \equiv 0, 2, 3\,(5) \rangle$$

and

$$\langle \mathrm{LT}_{<_2}(I) \rangle = \langle x_i x_j, i + j \leq n, |i - j| \leq 1 \rangle?$$

If yes, this proves that $a_1(n) = b_1(n)$ by the above consideration.

5.4.2 Strategy 3 - Formula of Stillman and Bayer

Stillman and Bayer gave in [5] a formula to compute the Hilbert series of any monomial ideal in a recursive manner. This is the algorithm that the computer algebra system Macaulay uses to compute Hilbert series for monomial ideals. We review their result:

Lemma 5.4. *Let J be a monomial ideal of $R = k[x_1, \ldots, x_n]$ and let x^α be a monomial in R. Then*

$$J \cap (x^\alpha) = (x^\alpha) \cdot (J : (x^\alpha)),$$

where $(J : (x^\alpha))$ is the ideal quotient.

Proof. If $f \in J \cap (x^\alpha)$, there is an element $r \in R$ with $f = r \cdot x^\alpha$. Since $f \in J$, $r \in (J : (x^\alpha))$ by definition. Therefore, $f \in (x^\alpha) \cdot (J : (x^\alpha))$. If $f \in (x^\alpha) \cdot (J : (x^\alpha))$, $f = x^\alpha s$ for some $s \in (J : (x^\alpha))$. Therefore, $f \in J$, so that $f \in J \cap (x^\alpha)$. □

Remark 5.14. Recall that if R is a graded polynomial ring over a field k and I is a homogeneous ideal, the Hilbert series of R/I is rational by Theorem 4.9.

Definition 5.6. Let I be a homogeneous ideal of $R = k[x_1, \ldots, x_n]$. We denote by $n(I)$ the numerator of the Hilbert series of R/I (before the cancelling of fractions).

Theorem 5.13. *Let I, J be monomial ideals of $R = k[x_1, \ldots, x_n]$, graded via $deg(x_i) = d_i > 0$, and let $x^\alpha \notin J$ be such that $I = (J, x^\alpha)$. Then*

1. $n(x^\alpha) = 1 - t^{|\alpha|}$,
2. $n(J \cap (x^\alpha)) = 1 - t^{|\alpha|} + t^{|\alpha|} n(J : (x^\alpha))$,
3. $n(I) = n(J) - t^{|\alpha|} n(J : (x^\alpha))$.

Proof. For the first formula, we have to count the monomials in $R/(x^\alpha)$, that is, the monomials in R that don't belong to $\langle x^\alpha \rangle$. The generating function for the degrees of monomials of R is given by

$$\frac{1}{\prod_{i=1}^{n} (1 - t^{d_i})}$$

by Corollary 4.7. The dimension of the m-th graded piece of $\langle x^\alpha \rangle$ is equal to the dimension of $R_{m-|\alpha|}$ (x^α is responsible for the shift), so that

$$\frac{t^{|\alpha|}}{\prod_{i=1}^{n} (1 - t^{d_i})}$$

counts monomials in (x^α). Subtracting now gives the desired equality.
For the second formula, we are consider the monomials of ${}^{R}\!/_{J \cap (x^\alpha)}$. These can be split into two groups: Those, that are outside of (x^α) (which was counted in (1)), and those that are in (x^α), but not in J. By Lemma 5.4,

$${}^{(x^\alpha)}\!/_{J \cap (x^\alpha)} = {}^{(x^\alpha)}\!/_{(x^\alpha)(J : (x^\alpha))} = \left({}^{R}\!/_{(J : (x^\alpha))}\right)_{-|\alpha|}.$$

Monomials in the latter are counted by $t^{|\alpha|} \cdot n\left(J : \left(x^{\alpha}\right)\right)$. Adding the numerators of the two groups, the identity is proven.
By (1) and (2), we have

$$\begin{aligned} n(I) =& n(J) + n\left(x^{\alpha}\right) - n\left(J \cap \left(x^{\alpha}\right)\right) \\ =& n(J) - t^{|\alpha|} n\left(J : \left(x^{\alpha}\right)\right). \end{aligned}$$

This proves the last formula. □

For our purposes, we will be interested in quotients of monomial ideals. Computing these is an easy exercise.

Proposition 5.4. *Let $I = (m_1, \ldots, m_k)$ be a monomial ideal and m be a monomial in $k[x_1, \ldots, x_n]$. Then*

$$I : m = \left(\frac{m_1}{m}, \ldots, \frac{m_k}{m}\right),$$

where $\frac{m_i}{m} = \frac{m_i}{gcd(m_i, m)}$ for $1 \leq i \leq k$.

Proof. We have

$$\begin{aligned} I : m =& \{f \in k[x_1, \ldots, x_n], f \cdot g \in I \quad \text{for all } g \in (m)\} \\ =& \{f \in k[x_1, \ldots, x_n], f \cdot m \in I\} \\ =& \{f \in k[x_1, \ldots, x_n], f \cdot m \in (m_1, \ldots, m_k)\} \\ =& \left\{f \in k[x_1, \ldots, x_n], f \in \left(\frac{m_1}{m}, \ldots, \frac{m_k}{m}\right)\right\} \\ =& \left(\frac{m_1}{m}, \ldots, \frac{m_k}{m}\right). \end{aligned}$$

The second to last equality holds since

$$\begin{aligned} fm \in (m_1, \ldots, m_k) \Leftrightarrow& \exists r_1, \ldots, r_k \in R : fm = \sum_{i=1}^{s} r_i m_i = \sum_{i=1}^{k} r_i \cdot \gcd(m, m_i) \frac{m_i}{m} \\ \Leftrightarrow& \exists t_1, \ldots, t_k \in R : f = \sum_{i=1}^{s} t_i \frac{m_i}{m} \\ \Leftrightarrow& f \in \left(\frac{m_1}{m}, \ldots, \frac{m_k}{m}\right). \end{aligned}$$

Note that m divides $r_i m_i$ for $1 \leq i \leq k$ in the above argumentation, since m is monomial. □

We finally want to prove a statement on the numerator of the Hilbert series.

Proposition 5.5. *Let m, n be positive integers, $m < n$, let I be a homogeneous ideal of $R = k[x_1, \ldots, x_m]$, graded via $deg(x_i) = d_i > 0$ for some d_i.Suppose that*

$$HS_{R/I}(t) = \frac{p(t)}{\prod_{j=1}^{m} \left(1 - t^{d_j}\right)}$$

for some polynomial $p(t) \in \mathbb{Z}[t]$. *Let* S *be the polynomial ring* $k[x_1, \ldots, x_m, \ldots, x_n]$, *graded via* $deg(x_i) = d_i$ *for* $1 \leq i \leq m$ *and* $deg(x_i) = d_i > 0$ *for some* d_i *with* $m < i \leq n$. *Then*

$$HS_{S/IS}(t) = \frac{p(t)}{\prod_{j=1}^{n}\left(1 - t^{d_j}\right)}.$$

In particular, the numerator of the Hilbert series of R/I *is the same as the numerator of* S/IS, *so it does not depend on the number of variables of the polynomial ring.*

Proof. It suffices to prove the assertion for $n = m + 1$. The rest follows by induction.
We have to count monomials in IS. Monomials in IS of degree d correspond to monomials in I that are of degree d, $d - d_{m+1}$,$d - 2d_{m+1}$,.... Therefore,

$$\begin{aligned}\mathrm{HS}_{S/IS}(t) =& \mathrm{HS}_{R/I}(t) + t^{d_{m+1}} \cdot \mathrm{HS}_{R/I}(t) + t^{2d_{m+1}} \cdot \mathrm{HS}_{R/I}(t) + \ldots \\ =& \mathrm{HS}_{R/I}(t) \cdot \frac{1}{1 - t^{d_{m+1}}} \\ =& \frac{p(t)}{\prod_{i=1}^{m+1}(1 - t^{d_i})}.\end{aligned}$$

□

Revisiting Euler's identity again

The above considerations give another easy proof of the Euler partition identity.

Proof. Consider for some fixed $m \in \mathbb{N}$ the polynomial ring $R = k[x_1, \ldots, x_m]$ graded via $\deg(x_i) = i$ and let $I = (x_{2i}, 2i \leq m)$. Monomials of degree n in R/I correspond to partitions of n without even parts, for $1 \leq n \leq m$. Therefore,

$$\mathrm{HS}_{R/I}(t) = \frac{1}{\prod_{1 \leq 2i \leq m-1}(1 - t^{2i+1})} = \frac{\prod_{1 \leq 2i \leq m}\left(1 - t^{2i}\right)}{\prod_{1 \leq i \leq m}(1 - t^{i})}$$

by Theorem 5.1. Hence,

$$n(I) = \prod_{1 \leq 2i \leq m}\left(1 - t^{2i}\right).$$

□

Revisiting the Rogers-Ramanujan identities

Our goal is to compute $n(I)$ and $n(J)$ for the ideals corresponding to the first Rogers-Ramanujan identity. Fix some integer m and suppose that the Rogers-Ramanujan identities hold for all numbers smaller than m. Recall that we used the ideals

$$I_m = (x_i, i \equiv 0, 2, 3\,(5), i \leq m)$$

and

$$J_m = (x_i x_j, 0 < i + j \leq m, |i - j| \leq 1),$$

considered in the polynomial ring $k[x_1, \ldots, x_m]$ graded via $\deg(x_i) = i$. We need to show that

$$\langle t^m \rangle n(I_m) = \langle t^m \rangle n(J_m).$$

In Theorem 5.1, we proved that $n(I) = \prod_{i \equiv 0,2,3(5), i \leq m} \left(1 - t^i\right)$. It is not as easy to give a formula for $n(J_m)$, but the formula of Stillman and Bayer (see Theorem 5.13) gives the following recursive formulas, by using also Proposition 5.5 and Proposition 5.4:
If $m = 2l + 1$ is odd and division by $x_l x_{l+1}$, we get

$$\begin{aligned} n(J_m) =& n\left(x_1^2, \ldots, x_l^2, x_l x_{l+1}\right) \\ =& n\left(x_1^2, \ldots, x_l^2\right) - t^m \cdot n\left(x_1^2, \ldots, x_{l-2}^2, x_{l-1}, x_l\right) \\ =& n(J_{m-1}) - t^m \left(1 - t^l\right)\left(1 - t^{l-1}\right) \cdot n(J_{m-3}). \end{aligned}$$

If $m = 2l$ is even, we get two different formulas, either by dividing by $x_{l-1}x_l$ or by x_l^2. Dividing by $x_{l-1}x_l$, we get

$$\begin{aligned} n(J_m) =& n\left(x_1^2, \ldots, x_l^2\right) \\ =& n\left(x_1^2, \ldots, x_{l-1}^2, x_l^2\right) - t^{m-1} n\left(x_1^2, \ldots, x_{l-3}^2, x_{l-2}, x_{l-1}, x_l\right) \\ =& \left(1 - t^m\right) n(J_{m-2}) - t^{m-1}\left(1 - t^{l-2}\right)\left(1 - t^{l-1}\right)\left(1 - t^l\right) \cdot n(J_{m-6}). \end{aligned}$$

Dividing by x_l^2 leads another formula, which can be computed similarly:

$$\begin{aligned} n(J_m) =& n\left(x_1^2, \ldots, x_l^2\right) \\ =& n\left(x_1^2, \ldots, x_{l-1}x_l\right) - t^m \cdot n\left(x_1^2, \ldots, x_{l-2}^2, x_{l-1}\right) \\ =& n(J_{m-1}) - t^m\left(1 - t^{l-1}\right) \cdot n(J_{m-4}). \end{aligned}$$

Although these recursion formulas look promising, the author was not able to prove that $\langle t^m \rangle n(I_m) = \langle t^m \rangle n(J_m)$ using these recursive formulas. However, they might at least give some insight on a (future) combinatorial proof of the Rogers-Ramanujan identities.

Summary

In this thesis, we study the concept of graded free resolutions, which will be used to give algebraic proofs of identities for integer partitions.
Chapters 1-4 serve the purpose to settle all algebraic preparation which allows us to work with integer partitions in Chapter 5.
In Chapter 1, we will recall the concept of a Gröbner basis on $k[x_1, \ldots, x_n]^t$ and state all important theorems.
In Chapter 2, we introduce graded rings, graded modules, graded submodules and homomorphisms of graded modules. We show some elementary properties of these objects. Finally, we prove the structure theorem for finitely generated graded modules.
In Chapter 3, we give the definition of a syzygy module of a generating system of a module and find (under certain conditions) a generating system for the syzygy module by using Gröbner bases and the Buchberger algorithm. Afterwards, we introduce free resolutions of modules, which are exact sequences

$$\cdots \to F_1 \to F_0 \to M \to 0,$$

where the F_i are free R-modules and prove Hilbert's Syzygy Theorem with the help of Gröbner bases, which states that, under certain conditions, there is a free resolution of M of finite length. We then consider free resolutions of graded modules and find out in which sense a free resolution can be called minimal.
In Chapter 4, we continue with our preparations and concentrate on homological algebra and chain conditions. We first consider complexes and homologies and construct the connecting homomorphism and introduce the long exact sequence in homology. We then construct the Koszul complex and show that the Koszul complex is a free resolution of a module in certain cases. Lastly, we will give conditions on which the graded parts of a module M are finitely dimensional, which allows us to define the Hilbert series of certain modules. We then prove the rationality of the Hilbert series.
In Chapter 5, we use the preparations from Chapters 1-4 and apply them to integer partitions. We will first introduce the reader to the topic and then give a bijection, showing the connecting of integer partitions to monomials in a polynomial ring. We then prove some partition identities by computing graded free resolutions and consider cases where the computation of a graded free resolution is not suited as proof. Then we show how graded free resolutions can be used to find recursion formulas for partition numbers. We will prove Euler's pentagonal number theorem combinatorially and deduce Euler's formula from this. We then also prove a recursion formula for the Rogers-Ramanujan identities.
Lastly, we show two more strategies to prove partition identities. We first show that Hilbert series of ideals are equal to the Hilbert series of the leading term ideal, and finally see a recursive formula to compute the Hilbert series as introduced by Stillman and Bayer.

Zusammenfassung

In dieser Arbeit besprechen wir graduierte freie Auflösungen, mit welchen es uns möglich sein wird, algebraische Beweise für Identitäten zu Zahlpartitionen zu geben.
Die Kapitel 1-4 dienen zur Vorbereitung und Zusammenfassung der algebraischen Voraussetzungen, die wir schlussendlich in Kapitel 5 verwenden werden.
In Kapitel 1 wiederholen wir das Konzept von Gröbner-Basen für $k[x_1, \ldots, x_n]^t$ und besprechen alle wichtigen Sätze und Informationen.
In Kapitel 2 werden graduierte Ringe, graduierte Moduln, graduierte Untermoduln und graduierte Homomorphismen eingeführt. Wir beweisen elementare Eigenschaften dieser Objekte. Letztlich beweisen wir den Struktursatz für endlich erzeugte graduierte Moduln.
In Kapitel 3 beginnen wir mit der Definition des Relationsmoduls eines Erzeugendensystems eines Moduls und geben (unter bestimmten Voraussetzungen an den Modul) ein Erzeugendensystem für den Relationenmodul an, indem wir die Theorie der Gröbner-Basen und den Buchberger-Algorithmus verwenden. Danach besprechen wir freie Auflösungen von Moduln, also exakte Sequenzen

$$\cdots \to F_1 \to F_0 \to M \to 0,$$

wobei the F_i freie R-Moduln sind, und beweisen Hilbert's Syzygiensatz mit der Hilfe von Gröbner-Basen. Danach betrachten wir freie Auflösungen von graduierten Moduln und besprechen, in welchem Sinn man von einer minimalen freien Auflösung sprechen kann.
In Kapitel 4 besprechen wir weitere algebraische Vorbereitungen und besprechen vor allem wichtige Sätze der homologischen Algebra und Kettenbedingungen. WIr betrachten zuerst Komplexe und Homologien und konstruieren den Verbindungshomomorphismus und zeigen the lange exakte Folge der Homologien. Danach konstruieren wir den Koszul-Komplex und beweisen, dass der Koszul-Komplex unter bestimmten Bedingungen als freie Auflösung verwendet werden kann. Letztlich werden wir Bedingungen angeben, unter welchen die graduierten Teile eines graduierten Modulss endlichdimensional sind, womit wir Hilbertreihen definieren können. Wir beweisen danach die Rationalität der Hilbertreihe.
In Kapitel 5 verwenden wir alle Vorbereitungen der Kapitel 1-4 und wenden sie auf Zahlpartitionen an. Zuerst geben wir eine kurze Einführung in das Gebiet der Zahlpartitionen und geben eine Bijektion an, die eine Verbindung zwischen Zahlpartitionen und Monomen in einem Polynomring aufzeigt. Wir beweisen danach Identitäten von Zahlpartitionen, indem wir graduierte freie Auflösungen berechnen und zeigen auch Identitäten, bei denen freie Auflösungen nicht reichen, um sie zu beweisen. Danach zeigen wir, dass man mit graduierten freien Auflösungen auch rekursive Formeln für das Berechnen von Partitionszahlen verwenden kann. Wir beweisen den Pentagonalsatz von Euler und werden daraus die Formel von Euler folgern. Danach beweisen wir auch noch eine Rekursion für die Rogers-Ramanujan Identitäten.
Schlussendlich besprechen wir zwei weitere Strategien, um Partitionsidentitäten zu beweisen. Wir zeigen zuerst dass die Hilbertreihe eines Ideals der Hilbertreihe des Ideals der Leitterme entspricht, und besprechen danach noch eine rekursive Formel von Stillman und Bayer, welche das Berechnen der Hilbertreihe erlaubt.

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