MASTERARBEIT

Titel der Masterarbeit

„The particle density and superfluidity in a model of interacting Bosons in a random potential"

Verfasser

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Contents

Abstract

In this thesis we discuss a one dimensional Lieb Liniger model for Bosons, with an additional random potential consisting of δ functions distributed by a Poisson process. For this model of low interacting Bose gas it was shown by Seiringer, Yngvason and Zagrebnov that for $N \to \infty$, where N denotes the number of particles, the Gross Pitaevskii energy converges to the quantum mechanical energy. Furthermore it was proven that in the many particle limit complete Bose Einstein condensation exists. For the discussion of superfluidity, a more detailed analysis of the Gross Pitaevskii density is needed and for certain limits this analysis is presented in this thesis. In particular we are able to show that for weak interaction there is no superfluidity in the considered limit, even though Bose Einstein condensation happens. On the other hand we shall show that there is a certain parameter range where there is complete superfluidity. Hence the question if superfluidity exists needs stronger assumptions than the presence of BEC in this one dimensional interacting model. In the same work by Seiringer, Yngvason and Zagrebnov it was also shown that under certain parameter constraints there exists a deterministic limit of the Gross Pitaevskii energy which is dependent on the random potential. In this thesis we extend these proofs to show explicit estimates for the convergence in probability. Moreover we prove that it is also possible to show this convergence for a Bernoulli random potential.

Zusammenfassung

In dieser Arbeit erörtern wir ein eindimensionales Lieb Liniger Modell für Bosonen mit einem zusätzlichen Zufallspotenzial. Für dieses Modell von schwach wechselwirkendem Bosegas wurde von Seiringer, Yngvason und Zagrebnov gezeigt, dass für $N \to \infty$, wobei N die Anzahl an Teilchen bezeichnet, die Gross Pitaevskii Energie zur quantenmechanischen Energie konvergiert. Weiters wurde gezeigt, dass für den Vielteilchenlimes vollständige Bose Einstein Kondensation (BEC) existiert. Für die Diskussion von Superfluidität, ist eine genauere Analyse der Gross Pitaevskii Dichte nötig und für bestimmte Limiten wird diese Analyse in dieser Arbeit durchgeführt. Im Speziellen können wir zeigen, dass für schwache Wechselwirkung keine Superfluidität im Limes existiert, auch wenn Bose Einstein Kondensation vorkommt. Andererseits können wir zeigen, dass es einen Parameterbereich gibt, in dem vollständige Superfluidität stattfindet. Daher fordert die Existenz von Superfluidität stärkere Voraussetzungen als die Existenz von BEC, in diesem eindimensionalen Modell. In derselben Arbeit von Seiringer, Yngvason und Zagrebnov wurde gezeigt, dass in einem speziellen Parameterbereich, ein deterministischer Limes für die Gross Pitaevskii Energie existiert welche von dem Zufallspotenzial abhängt. In dieser Arbeit erweitern wir diesen Beweis um Abschätzungen für die Konvergenz in Wahrscheinlichkeit. Weiters beweisen wir, dass es möglich ist, diese Konvergenz auch für ein Bernoulli Zufallspotenzial zu erreichen.

Chapter 1

Introduction

The theory of Bose Einstein condensation (BEC) was first discussed 1925 by Albert Einstein [1] who used Bose's 1924 paper [2] exploring the statistics of photons to show the phenomena of condensation for the ideal gas. For the model of a non interaction ideal gas, it is clear that the ground state of the N particle system consists of N copies of the one particle ground state. In his discussion Einstein proved that there is a finite temperature below which condensation occurs. Condensation means here that the average number of particles occupying the ground state is of order N.
During the 1930's it was discovered that 4-He acted unusually when cooled down to $\approx$ 2.17 Kelvin. Shortly after Fritz London suggested a connection with Einsteins theory of condensation for Bose gases [3]. Because helium has two electrons it should follow Bose statistic and there should also be Bose Einstein condensation, however the idealization of a non interacting gas is not valid for 4-He. In particular, even in the ground state only 9% of the gas is condensed [4].
It was Bogoliubov in 1947 who developed a theory for describing BEC in a systematic way [5]. Even though his results were of high importance for the theoretical understanding, the approximations he made were uncontrolled, hence from a mathematical point of view BEC continued to be a problem.
In the late 1930's also superfluidity for the 4-He system was discovered by Kapitza in Moscow [6] and Allen and Misener in Cambridge [7] simultaneously. This phenomena was first explained qualitatively by Laszlo Tisza [8] and later more precisely by Lev Landau by describing it with a "two fluid" model where one of the fluids behaves without friction while the other follows the rule of an ordinary fluid [9].
Since then, the problem of showing Bose Einstein condensation and superfluidity for interacting gases is an ongoing challenge for the mathematical physics community. Rigorous proofs for interacting systems are rare and limited to a few particular cases. See for example [10, 11] and for a review article on Bogoliubov model see [12]. During the 1990's the first experiments with dilute gases were possible and the topic remains of great interest.
In the last years some research was done in the behavior of Bose gases in a random potential. These problems are well understood for a non interacting gas [13–15] but there is still a lack of information for the interacting case, which was discussed for example in [16–21].

An particular intriguing problem is to understand the effects of an external random potential on BEC and superfluidity. In the paper [22], Seiringer, Yngvason and Zagrebnov studied a one-dimensional model of Bosons with contact interactions (Lieb-Liniger model) in an external potential generated by Poisson-distributed point scatterers. They showed that in a well defined limit BEC takes place in the ground state irrespective of the random potential. They also studied the particle density in a certain approximation and found that while the random potential does not destroy BEC it can have drastic effects on the density.
For the study of superfluidity, the particle density is of crucial importance and the results of [22] strongly suggests that the random potential may destroys superfluidity in a certain asymptotic parameter range. To decide this question however, precise estimates on the true density are required going beyond the approximations studied in [22]. In this thesis such an analysis will be carried out. The main result can be summarized in the statement that the model considered provides an example with complete BEC but no superfluidity in an appropriate limit.
For a more detailed review of the history about Bose Einstein condensation, see [23] and [24].

1.1 The basic N-body Hamiltonian

The model is a Lieb-Liniger model [25] with an additional external potential V. To begin with we only require that V is positive and integrable, but V may well contain δ-functions. The N particle Hamilton operator H_N in the interval $[0,1]$ and its ground state energy are defined as

$$H_N = \sum_{j}^{N} -\partial_{z_j}^2 + V(z_j) + \frac{\gamma}{N}\sum_{i<j}^{N}\delta(z_i - z_j), \qquad e_{V,\gamma}^{QM} = \inf_{\|\psi\|=1}\langle\psi|H_N\psi\rangle \tag{1.1}$$

We are interested in the ground state Ψ_0 and the corresponding energy $e_{V,\gamma}^{QM}$ of H_N with Dirichlet boundary conditions. H_N acts on the space $\otimes^N L^{2,\mathrm{sym}}([0,1])$ which we can see as a subspace of the symmetric Fock space $\mathcal{F}^{\mathrm{sym}}$.

$$\mathcal{F}^{\mathrm{sym}} = \bigoplus_{N=1}^{\infty}\mathcal{H}_N, \qquad \mathcal{H}_N = \otimes^N L^{2,\mathrm{sym}}([0,1]) \tag{1.2}$$

1.2 The Gross Pitaevskii Energy functional

We define the Gross Pitaevskii functional corresponding to the Hamiltonian defined in Section 1.1 as

$$\mathcal{E}_V^{\mathrm{GP}} = \int_0^1 \left[|\phi'(z)|^2 + V(z)|\phi(z)|^2 + \frac{\gamma}{2}|\phi(z)|^4\right]\mathrm{d}z \tag{1.3}$$

with $\gamma \geq 0$. From now on we shall abbreviate Gross Pitaevskii as GP. The GP Energy is defined by

$$e_V^{\mathrm{GP}} = \inf_{\|\phi\|_{L^2}=1} \mathcal{E}_V^{\mathrm{GP}}[\phi] \tag{1.4}$$

with $\phi \in H^1([0,1])$ and Dirichlet boundary conditions. We denote the minimizer of the GP functional with ϕ_V^{GP} and with standard methods one can show that a unique ϕ^{GP} exists. Using a trial function $\phi \otimes \cdots \otimes \phi$ for the quantum mechanical energy E^{QM}/N per particle shows that the GP energy is an upper bound. In [22] a lower bound was also derived so that the quantum mechanical energy per particle also converges to E_V^{GP} in the desired limit $N \to \infty$.

Theorem 1.1 ([22], Thm 2.1)**.**

$$E_V^{\mathrm{GP}} \geq \frac{E_0^{QM}}{N} \geq E_V^{\mathrm{GP}} \left(1 - cN^{-1/3} \min(\gamma^{1/2}, \gamma)\right) \tag{1.5}$$

1.3 Bose Einstein condensation

1.3.1 Definition

To discuss Bose Einstein Condensation we need to define the one particle reduced density matrix. For this let $a(f)$ be a annihilation operator on $\mathcal{F}^{\mathrm{sym}}$ and $a^*(f)$ the corresponding creation operator which creates a particle in the state f.

Definition 1.2. *The one particle reduced density matrix ρ_Ψ for a state $\Psi \in \mathcal{F}^{sym}$ is defined by the relation*

$$\langle f|\rho_\Psi g\rangle = \langle \Psi|a^*(f)a(g)\Psi\rangle \tag{1.6}$$

It is easy to see that for a N particle state the trace over the ρ_Ψ gives the number of particles.

$$\operatorname{tr} \rho_\Psi = N \tag{1.7}$$

Let $\psi_0 \in L^2(\mathbb{R})$ be a one particle state. We now define

$$N_0 = \operatorname{tr}(\rho_\Psi|\psi_0\rangle\langle\psi_0|) \tag{1.8}$$

This is the average number of particles in the state ψ_0. If we now consider the limit where $N \to \infty$ we say that complete Bose Einstein condensation holds for ψ_0 if

$$\frac{N_0}{N} \to 1 \tag{1.9}$$

The state ψ_0 is called the wave function of the condensate.

1.3.2 Results for the one dimensional model

Seiringer, Yngvason and Zagrebnov also discussed in [22] the existence of BEC for H_N with an arbitrary positive integrable V. The convergence involves the two first eigenvalues of the mean field Hamiltonian

$$h := -\partial_z^2 + V(z) + \gamma|\phi^{\mathrm{GP}}(z)|^2 - \frac{\gamma}{2}\int |\phi^{\mathrm{GP}}(z)|^4 \tag{1.10}$$

We denote the first two eigenvalues of this Hamiltonian by e_0 and e_1. A simple calculation shows that $e_0 = e_V^{\mathrm{GP}}$.

Theorem 1.3 ([22], Thm 2.2)**.**

$$\left(1 - \frac{N_0}{N}\right) \leq c\frac{e_0}{e_1 - e_0} N^{-1/3} \min(\gamma^{1/2}, \gamma) \tag{1.11}$$

for a constant $c > 0$*.*

Remark 1.4. Even though e_1, e_0 are not dependent on N, which means the convergence in Theorem 1.3 is always given for fixed V and γ, it is not a priori clear how (1.11) depends on γ and V. In general we would like to consider a $\gamma(N)$ but the bounds in [22] lead to an exponential decay of $e_1 - e_0$ and we are obliged to deal with strong restrictions on the growth of γ.

1.4 The random potential

The random potential we shall use in the next chapters is determined by a Poisson process, which creates points x_j^ω on $\mathbb{R}^+$ with a constant mean density d. The parameter ω defines this random sample of points. We consider the Hamiltonian for an N particle system on an interval $[0, L]$ with

$$H_N^{(L)} = \sum_j^N -\partial_{x_j}^2 + b\sum_{k=1}^{m_\omega} \delta(x_j - x_k^\omega) + g\sum_{i<j}^N \delta(x_i - x_j) \tag{1.12}$$

where b and g are arbitrary positive parameters and the number of random points in $[0, L]$ is denoted by m_ω. Further, the points x_j^ω are ordered such that $x_j^\omega < x_{j+1}^\omega$ for all j. In Figure 1.1 one can see the interval $[0, L]$ with the randomly placed points x_j^ω.
We consider this system in the limit $L \to \infty$ and $N \to \infty$ but if we perform a simple scaling $x_j = Lz_j$, $x_j^\omega = Lz_j^\omega$ the Hamiltonian (1.12) transforms into H_N/L^2 where H_N is given by (1.1) with

$$\nu = Ld, \quad \sigma = Lb, \quad V_\omega(z) = \sigma\sum_{j=1}^m \delta(z_j - z_k^\omega), \quad \gamma = LNg \tag{1.13}$$

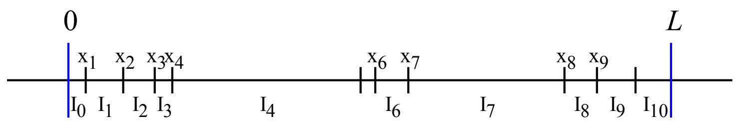

Figure 1.1: The interval $[0, L]$ and the points $x_j = x_j^\omega$ created by the Poisson process.

From now on we work with this transformed Hamiltonian on $[0, 1]$ and we consider the limit $\nu \to \infty$ as well as $N \to \infty$. The results about the GP limit, we mentioned in Section 1.2, can be expressed in terms of the parameters σ, γ and ν, and it is relevant to analyze the GP functional

$$\mathcal{E}^{\mathrm{GP}}[\phi] = \int_0^1 \left[|\phi'(z)|^2 + \frac{\gamma}{2} |\phi(z)|^4 \right] \mathrm{d}z + \sigma \sum_{j=1}^{m} |\phi(z_j^\omega)|^2 \tag{1.14}$$

We find the corresponding GP energy $e^{\mathrm{GP}} = e^{\mathrm{GP}}_{V_\omega}$ by minimizing in H^1 with Dirichlet boundary conditions. From now on ϕ^{GP} denotes the minimizer of the GP functional (1.14), and further on in this thesis we shall discuss the GP energy, its behavior in the limit and the corresponding minimizer. For simplicity we abbreviate $m = m_\omega$ and $z_j = z_j^\omega$.
The points z_j split the interval $[0, 1]$ into subintervals with lengths $l_j = z_{j+1} - z_j$. These lengths are IID distributed random variables and follow the exponential distribution

$$\mathrm{d}p_\nu(l) = \nu e^{-\nu l} \mathrm{d}l \tag{1.15}$$

with an expectation value of ν^{-1}. Because the points z_j were created by a Poisson process, m is Poisson distributed

$$P(m = k) = e^{-\nu} \frac{\nu^k}{k!} \tag{1.16}$$

We also define the mass in the interval $I_j = [z_j, z_{j+1}]$ as

$$n_j^{\mathrm{GP}} = \int_{z_j}^{z_{j+1}} |\phi^{\mathrm{GP}}(x)|^2 \mathrm{d}x \tag{1.17}$$

1.5 The interval density functional

One of the major results of [22] was the convergence of e^{GP} to an ω independent energy which minimizes the *interval density functional*

$$\mathcal{E}^{\mathrm{IDF}}_{\gamma,\nu}[n] = \nu \int \mathrm{d}p_\nu(l) \frac{n(l)}{l^2} e(\gamma l n(l), \infty) \tag{1.18}$$

with the constraint $\nu \int \mathrm{d}p_\nu(l) n(l) = 1$. This energy we call *interval density energy* and it is defined as

$$e_0(\gamma, \nu) = \inf_{\nu \int \mathrm{d}p_\nu(l) n(l) = 1} \mathcal{E}^{\mathrm{IDF}}_{\gamma,\nu}[n] \tag{1.19}$$

We used $e(\kappa, \alpha) = \inf_{\|\phi\|_{L^2}=1} \mathcal{E}_{\kappa,\alpha}[\phi]$ where $\mathcal{E}_{\kappa,\alpha}$ defined as

$$\mathcal{E}_{\kappa,\alpha}[\phi] = \int_0^1 |\phi(x)'|^2 + \frac{\kappa}{2}|\phi(x)|^4 \mathrm{d}x + \frac{\alpha}{2}\left(|\phi(0)|^2 + |\phi(1)|^2\right) \tag{1.20}$$

with Dirichlet boundary conditions in H^1, describes the energy on an interval $[0, 1]$ with delta functions at 0 and 1. With standard methods it can be shown that there exists a minimizer in H^1 and we denote it with $\phi_{\kappa,\alpha}$. If we define

$$\mathcal{E}^l_{\kappa,\alpha}[\phi] = \int_0^l |\phi(x)'|^2 + \frac{\kappa}{2}|\phi(x)|^4 \mathrm{d}x + \frac{\alpha}{2}\left(|\phi(0)|^2 + |\phi(l)|^2\right), \; e^{(n,l)}(\gamma, \nu) = \inf_{\|\phi\|_{L^2}=n} \mathcal{E}^l[\phi] \tag{1.21}$$

again with Dirichlet boundary conditions in H^1, then a simple calculation shows

$$e^{l,n}(\gamma, \nu) = \frac{n}{l^2} e(nl\gamma, l\nu) \tag{1.22}$$

Using $e^{l,n}$ we can also write the interval density functional as

$$\mathcal{E}^{\mathrm{IDF}}_{\gamma,\nu}[n(l)] = \nu \int \mathrm{d}p_\nu(l) e^{l,n(l)}(\gamma, \nu) \tag{1.23}$$

which is in some sense a weighted mean over the energies on intervals of length l, multiplied by the average interval number ν. Heuristically this is what we would expect if many intervals have nonzero mass, and hence the law of large numbers applies.

1.5.1 Convergence for the GP energy

In [22] a detailed analysis of the energy was carried out, and in the limit

$$\gamma \gg \frac{\nu}{(\log \nu)^2}, \qquad \sigma \gg \frac{\nu}{1 + \log(1 + \nu^2/\gamma)} \tag{1.24}$$

it was shown that the GP energy converges in first order to the ω independent interval density functional energy

Theorem 1.5 ([22], Theorem 3.1)**.** *In the limit* (1.24) *with* $\nu \to \infty$ *we get*

$$\frac{e^{\mathrm{GP}}(\gamma, \sigma, \nu)}{e_0(\gamma, \nu)} \overset{a.s.}{\to} 1 \tag{1.25}$$

for an appropriate subsequence of ν*.*

Here $\overset{a.s.}{\to}$ defines the almost sure limit (See Section 3.1).
In Section 2.8 we shall analyze this limit in more detail and in particular we shall learn that (1.24) ensures that infinitely many intervals have a nonzero mass.

1.5.2 The minimizer for $e_0(\gamma,\nu)$

The minimizer $n^{\mathrm{IDF}}(l)$ to the interval density functional can be found by introducing a Lagrange multiplier μ and minimizing

$$\mathcal{E}^{\mathrm{IDF}}_{\gamma,\nu}[n] - \mu\nu \int \mathrm{d}p_\nu(l) n(l). \tag{1.26}$$

The minimizer of (1.26) we denote with $n^\mu(l)$, and because of the norm constraint μ is fixed by

$$\nu \int \mathrm{d}p_\nu(l) n^\mu(l) = 1 \tag{1.27}$$

Hence we get the equality $n^{\mathrm{IDF}}(l) = n^\mu(l)$. In general we define $n^\eta(l)$ for $\eta > 0$ as the minimizer of (1.26) where we replaced μ by η.
In [22] these minimizers were analyzed and it was shown that the number of intervals with 0 mass is of order ν for the limit $\gamma \ll \nu^2$. For showing the absence of superfluidity, in the limit $\nu \to \infty$, it turns out that the density concentration plays a significant role. The results of [22] give strong indications that the minimizer of the energy e^{GP} has also similar properties, but a more detailed analysis of n_j^{GP} is needed to establish this results. In particular it was proven in [22] that

$$\int_{\{n^\mu(l)=0\}} \mathrm{d}p_\nu(l) \to \begin{cases} 1 & \gamma \ll \nu^2 \\ 0 & \gamma \gg \nu^2 \end{cases} \tag{1.28}$$

The major focus of this thesis is to carry out this detailed analysis for n_j^{GP} and to prove that there is no superfluidity in the limit $\nu \to \infty$. After discussing some of the results from [22] about $e(\kappa,\alpha)$ and the minimizer n^{IDF} in Chapter 2, and a more detailed analysis about the convergence in Theorem 1.5 in Chapter 3, we show the absence of superfluidity in the case $\gamma \ll \nu^2$ in Chapter 4. We are also able to treat some special cases for $\gamma \gg \nu^2$. In Chapter 5 we present an alternative approach to prove results from Chapter 4.

1.6 Superfluidity

As discussed in [24, 26] there is not a unique definition of the concept of Superfluidity. The one we shall use here is based on the response of the system to a small imposed velocity field, which alters the momentum operator $p \to p - v$. For the discussion of superfluidity we shall use periodic instead of Dirichlet boundary conditions.
We define an altered Hamilton operator which includes the velocity field.

$$H_N^{(v)} = \sum_j^N (i\partial_{z_j} + v)^2 + V(z) + \frac{\gamma}{N}\sum_{i<j}^N \delta(z_i - z_j) \tag{1.29}$$

on $L^2([0,1],\mathrm{d}z)$ with periodic boundary conditions. The ground state energy is denoted by E_v^{QM}. We are now interested in the question how the ground state E_v^{QM} changes compared to the energy E_0^{QM}. The part of the difference which is quadratic in v we denote with n_{SF}.

$$\frac{E_v^{\mathrm{QM}}}{N} = \frac{E_0^{\mathrm{QM}}}{N} + n_{\mathrm{SF}} v^2 + \mathcal{O}(v^4) \tag{1.30}$$

It is important that the error term $\mathcal{O}(v^4)$ can be bounded independently of N (See [11]). It is also very easy to see that $0 \leq n_{\mathrm{SF}} \leq 1$. The lower bound is trivial, and for the upper one it is sufficient to test $H_N^{(v)}$ with the ground state of H_N. Hence if the minimizer would not change we get $n_{\mathrm{SF}} = 1$, which means that there is full superfluidity. In the case that there is no change in energy, there is no superfluidity and $n_{\mathrm{SF}} = 0$.

1.6.1 Connection to the GP limit

The following argumentation follows Yngvason [27]. To connect the superfluid fraction to the GP density we analyze the to $H_N^{(v)}$ corresponding GP functional

$$\mathcal{E}^{\mathrm{GP}}[\phi] = \int_0^1 \left(|\phi'(z) + v\phi(z)|^2 + \frac{\gamma}{2}|\phi(z)|^4 + V(z)|\phi(z)|^2 \right) \mathrm{d}z \tag{1.31}$$

The minimizer is denoted by ψ_v^{GP}, and for small enough v it exists and is unique. Further the GP energy $e^{\mathrm{GP}}(v)$ is equal to the ground state energy of the mean field Hamiltonian

$$h_v = (i\partial_z + v)^2 + V(z) + \gamma|\psi_v^{\mathrm{GP}}(z)|^2 - \frac{\gamma}{2}\int |\psi_v^{\mathrm{GP}}|^4 \tag{1.32}$$

Using $\psi_v^{\mathrm{GP}} \otimes \cdots \otimes \psi_v^{\mathrm{GP}}$ as a trial function prove the upper bound

$$\frac{E_v^{\mathrm{QM}}}{N} \leq e^{\mathrm{GP}}(v) \tag{1.33}$$

For the lower bound we write $H_N^{(v)}$ equally to Eq. (7) in [22]

$$\begin{aligned} H_v = \sum_{j=1}^{N} \left[\left(1 - \frac{N-1}{2N}\varepsilon\right)(i\partial_{z_j} + v)^2 + V(z_j) \right] \\ + \frac{1}{N}\sum_{i<j}\left[\frac{\varepsilon}{2}\left((i\partial_{z_i} + v)^2 + (i\partial_{z_j} + v)^2 \right) + \gamma\delta(z_i - z_j) \right] \end{aligned} \tag{1.34}$$

Using the diamagnetic inequality we can bound expectation values of the the last term from below by expectation values of the absolute value of the wave functions

$$\frac{1}{N}\sum_{i<j}\left[\frac{\varepsilon}{2}(-\partial_{z_i}^2 - \partial_{z_j}^2) + \gamma\delta(z_i - z_j) \right] \tag{1.35}$$

Proceeding precisely like in [22] we can prove the lower bound

$$\frac{E_v^{\mathrm{QM}}}{N} \geq e_0(v)\left(1 - cN^{-1/3}\min(\gamma^{1/2}, \gamma)\right) \tag{1.36}$$

for an appropriate constant c. In the GP limit we can calculate the superfluid fraction as

$$n_{\mathrm{SF}} = \lim_{v\to 0} \frac{1}{v^2} \lim_{N\to\infty} \frac{1}{N} (E_v^{\mathrm{QM}} - E_0^{\mathrm{QM}}) \tag{1.37}$$

which is, because of (1.36) and (1.33), equal to

$$\lim_{v\to 0} \frac{1}{v^2} (e^{\mathrm{GP}}(v) - e^{\mathrm{GP}}(0)) \tag{1.38}$$

1.6.2 Twisted boundary conditions

Applying the unitary gauge transformation

$$\psi(z) \to e^{-ivz}\psi(z)$$

we can write the energy $e^{\mathrm{GP}}(v)$ as the ground state energy of $\mathcal{E}_V^{\mathrm{GP}}$ with twisted boundary conditions

$$e^{\mathrm{GP}}(v) = \inf \left\{ \mathcal{E}_V^{\mathrm{GP}}(\psi),\ \|\psi\|_{L^2} = 1, \psi(1) = e^{iv}\psi(0) \right\} \tag{1.39}$$

See for example [28]. Using the trial function $\tilde{\psi} = e^{ivg(x)}\psi^{\mathrm{GP}}(x)$ with $g(0) = 0$ and $g(1) = 1$ on $\mathcal{E}_V^{\mathrm{GP}}$ we can estimate

$$\begin{aligned} e^{\mathrm{GP}}(v) - e^{\mathrm{GP}}(0) &\leq \int \left(\left| ivg'(z)e^{-ivg(z)}\psi^{\mathrm{GP}}(z) + e^{-ivg(z)}\partial_j\psi^{\mathrm{GP}}(z) \right|^2 - |\partial_j\psi^{\mathrm{GP}}(z)|^2 \right) \mathrm{d}z \\ &= \int \left(|ivg'(z)\psi^{\mathrm{GP}}(z) + \partial_j\psi^{\mathrm{GP}}(z)|^2 - |\partial_j\psi^{\mathrm{GP}}(z)|^2 \right) \mathrm{d}z = v^2 \int |g'(z)|^2 |\psi^{\mathrm{GP}}|^2 \mathrm{d}x \end{aligned} \tag{1.40}$$

and therefore

$$n_{\mathrm{SF}} \leq \int |g'(z)|^2 |\psi^{\mathrm{GP}}|^2 \mathrm{d}x \tag{1.41}$$

for any g with $g(0) = 0$ and $g(1) = 1$. As long as the density $|\psi^{\mathrm{GP}}|^2$ is small, g can increase strongly without effecting the energy. In particular, if the density would vanish on an interval there would be no superfluidity. This is demonstrated in graph 1.2.

1.6.3 Motivation for a random potential

We have seen that superfluidity is closely related to the density of the wave function. In particular we want to prove that for a random potential in a certain limit the density is clustered on a set of measure smaller than $c < 1$, and hence we can use (1.41) to prove that there is no superfluidity. Using a random potential is a convenient way of introducing intervals with various lengths into the system. If we choose the strength of the interaction and therefore γ appropriately, we shall see that the minimizing density is concentrated in only a few large intervals. Figure 1.3 shows the case $\sigma = \infty$ for small γ and how a trial function for (1.41) can be chosen.

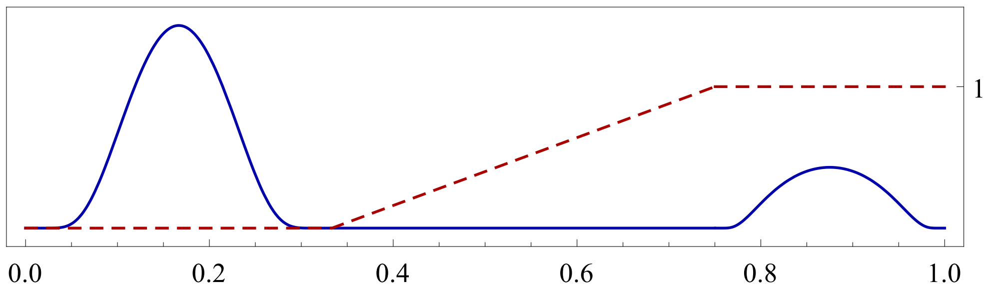

Figure 1.2: The twist function $g(z)$ (dashed) and an arbitrary GP density (solid). We twist the wave function exactly in the interval where the density is 0. Hence there is no change of energy and therefore also no superfluidity.

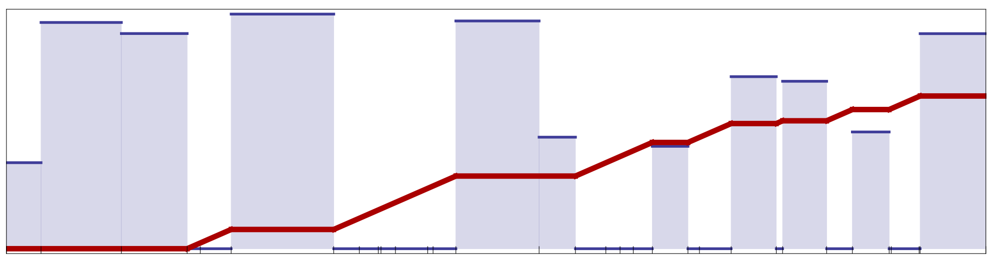

Figure 1.3: The minimizing density for $\sigma = \infty$. The twist function increases in the intervals where the density vanishes. This is an idealized picture for the case $\gamma \ll \nu^2$. For $\sigma < \infty$ the masses will be bigger than 0 on every interval.

The relation (1.41) is only useful for finding an upper bound for n_{SF}. A more detailed analysis of Seiringer [29] shows the relation

$$n_{\mathrm{SF}} = \left(\int |\phi^{\mathrm{GP}}|^{-2} \right)^{-1} \tag{1.42}$$

in the GP limit.
Let X be a set where the density $|\phi^{\mathrm{GP}}|$ is small. By Cauchy-Schwarz

$$\mu(X)^2 \coloneqq \left(\int_X \mathrm{d}x \right)^2 = \left(\int_X \frac{\phi^{\mathrm{GP}}(x)}{\phi^{\mathrm{GP}}(x)} \mathrm{d}x \right)^2 \leq n_{\mathrm{SF}}^{-1} \int_X |\phi^{\mathrm{GP}}(x)|^2 \mathrm{d}x \tag{1.43}$$

we get the same inequality as if we choose $g(x) = \mu(X)^{-1} \int_{\mu(X) \cap [0,x]} \mathrm{d}y$ for (1.41).

$$n_{\mathrm{SF}} \leq \frac{\int_X |\phi^{\mathrm{GP}}|^2}{\mu(X)^2} \tag{1.44}$$

But we also learn from (1.42) that it is sufficient to prove

$$|\phi^{\mathrm{GP}}|^{-2} \xrightarrow{L^1} 1 \tag{1.45}$$

to show the existence of superfluidity.

1.7 Organization of the thesis

1.7.1 Convergence to the interval density energy for a limit in probability

In the Chapter 2 and 3 we discuss the results of [22] in more detail. In particular we want to improve the result for the Theorem 1.5 to show a convergence in probability with an explicit estimate. A short introduction about limits in probability spaces can be found in Section 3.1. The following is a refinement of [22][Theorem 3.1]

Theorem 3.20. *We get*

$$\frac{e^{\mathrm{GP}}(\gamma,\sigma,\nu)}{e_0(\gamma,\nu)} \xrightarrow{p} 1 \tag{1.46}$$

for limits of the form (1.24)*. Further we get for* ε *small enough the probability bound*

$$P\left(\left|\frac{e^{\mathrm{GP}}}{e_0}-1\right|\geq\varepsilon\right)\leq\frac{c}{\varepsilon^2}\left(\nu^{-1/2+\delta}+\frac{\mu}{\gamma\nu}+\frac{\mu^{2/3}}{\gamma^{2/3}}\frac{\nu^{1/3}}{\sigma^{1/3}}\right)\left(1+\mathcal{O}\left(\frac{1}{\varepsilon}\frac{\mu}{\gamma\nu}+\frac{\nu^{-1/2+\delta}}{\varepsilon}\right)\right) \tag{1.47}$$

for $\gamma\gg\nu^2$*, and for* $\gamma\lesssim\nu^2$

$$P\left(\left|\frac{e^{\mathrm{GP}}}{e_0}-1\right|\geq\varepsilon\right)\leq\frac{c}{\varepsilon^2}\left(\nu^{-1/2+\delta}+\frac{\mu}{\gamma\nu}+\frac{\mu}{\sigma^2}\right)\left(1+\mathcal{O}\left(\frac{1}{\varepsilon}\frac{\mu}{\gamma\nu}+\frac{\nu^{-1/2+\delta}}{\varepsilon}\right)\right) \tag{1.48}$$

for a $c>0$*, and an arbitrary* $\delta>0$*.*

In order to prove Theorem (3.20) we shall see that the effects of the arrangement of the intervals are not important, and the theorem can be proved by using the bounds for the energy $e(\kappa,\alpha)$. Hence it will be necessary to discuss some useful estimates for $e(\kappa,\alpha)$ from [22] which we shall do in detail in Chapter 2.
We shall also analyze the properties of the minimizer n^{IDF} and we shall see that it is determined by the behavior of the ratio γ/ν^2 as ν and γ tend to infinity.

1.7.2 Properties of the minimizing density ϕ^{GP}

The essential features of the interval masses n_j^{GP} are not dependent on σ but are fully determined by the behavior of γ/ν^2. We shall distinguish the cases $\gamma\lesssim\nu^2$ in which a set of order 1 will vanish in the limit $\nu\to\infty$, and the case $\gamma\gg\nu^2$ in which the density converges to 1 in an L^1 sense.
In the first case this density concentration happens because the interaction term of the energy is small compared to the potential term. For $\gamma\gg\nu^2$ the energy contribution of the potential will be rather small and therefore the minimizing density should converge to 1 in the limit $\nu\to\infty$.

Because we deal a lot with the order of the parameters σ and γ, the following definition is helpful.

Definition 1.6. *Let $A = A_\nu, B = B_\nu$ be dependent on a parameter ν and c_1, c_2, c are constants.*

(a) We write $A \sim B$ if constants c_1, c_2 exist such that for all ν

$$c_1 \leq \frac{A}{B} \leq c_2 \tag{1.49}$$

(b) Further $A \lesssim B$ if a constant c exists such that for all ν

$$\frac{A}{B} \leq c \tag{1.50}$$

Analogously we define $A \gtrsim B \iff B \lesssim A$.

(c) We write $A \gg B$ if

$$\frac{B}{A} \to 0 \tag{1.51}$$

and analogously $A \ll B \iff B \gg A$.

While analyzing the minimizing density of the interval density functional $e_0(\gamma, \nu)$ we shall discover that there are no masses in intervals with length below $\pi/\sqrt{\mu}$. This peculiar behavior is caused by dropping all structure effects which are still present in the GP functional for finite σ because of the continuity constraint on ϕ^{GP} at the random points z_j. Hence we cannot expect the corresponding n_j^{GP} to be zero for any interval.
We are still able to show that most intervals, in the case $\gamma \ll \nu^2$, have small "occupation" in two ways. First we show that nearly all n_j^{GP} are small in the sense

Lemma 4.4. *For $\gamma \ll \nu^2$ and $\varepsilon > 0$ arbitrary we get*

$$\frac{1}{m} \sum_j g_{\varepsilon/m}(n_j^{\text{GP}}) \to 1 \tag{1.52}$$

where $g_\varepsilon(n) = 0$ for $n > \varepsilon$ and 1 otherwise and therefore we count the intervals with mass smaller than ε/m. Even though this proves that the masses are densely clustered in the limit (1.24), we can also ask if the mean density

$$\rho_j^{\text{GP}} = \frac{n_j^{\text{GP}}}{l_j} \tag{1.53}$$

is small for nearly all intervals. With the same methods as for the previous theorems we are able to show

Theorem 4.13. *For $\gamma \ll \nu^2$ the fraction of intervals where ρ_j^{GP} is smaller than δ tend to 1 for all $\delta > 0$*

$$\frac{1}{m} \sum_j^m g_\delta(\rho_j^{\text{GP}}) \xrightarrow{p} 1 \tag{1.54}$$

Our major statement of this section is an estimate for the masses in small intervals. It is based on a calculation of Seiringer and Yngvason [30] and it shows that not only for the minimizer $n^{\mathrm{IDF}}(l)$, but also for n_j^{GP}, the point $\pi/\sqrt{\mu}$ describes an important change in the form of the minimizer.

Theorem 4.5. *For the case $\sigma \gg \sqrt{\mu}$ we get the inequality*

$$N_s := \sum_{l_j < s/\sqrt{\mu}} n_j^{\mathrm{GP}} \overset{p}{\leq} C\sqrt{\frac{\sqrt{\mu}}{\sigma}} \tag{1.55}$$

for an s dependent constant C. Hence $N_s \xrightarrow{p} 0$ in the case $\gamma \lesssim \nu^2$.

where $\overset{p}{\leq}$ means that the relation $\leq$ holds with probability 1 in the limit (Def 4.2). Analogously we define the relation $\overset{p}{\lesssim}$.
From this theorem we can easily conclude that the density converges to 0 on a set with measure of order 1. Hence we can use (4.52) to show that there is no superfluidity for $\gamma \lesssim \nu^2$.

Theorem 4.9. *For $\gamma \lesssim \nu^2$ the superfluity fraction n_{SF} converges to 0 in probability.*

In the case $\gamma \gg \nu^2$ we can prove the statement

Theorem 4.11. *For $\gamma \gg \nu^2$ we get the convergence*

$$\sum_{j=1}^{m} |n_j^{\mathrm{GP}} - l_j| \overset{p}{\leq} c\sqrt{\frac{\nu}{\sqrt{\gamma}}} \tag{1.56}$$

for $\nu/\sqrt{\gamma} \ll 1$ and a appropriate constant c.

In order to prove the above theorem we find a lower bound for $\mathcal{E}^{\mathrm{GP}}_{\gamma,\sigma,\nu}$ by setting $\sigma = 0$. We shall see that for the case $\gamma \gg \nu^2$ this still matches the upper bound in the limit $\nu \to \infty$. We will be also able to find an upper bound to the energy e^{GP} which behaves like $\gamma/2$ in first order. This means that $\gamma e^{\mathrm{GP}}/2 \xrightarrow{p} 1$. This allows us to prove the Theorem 4.11.
Using this theorem we can prove a statement about the number of intervals with small n_j.

Theorem 4.12. *For $\gamma \gg \nu^2$ and $\varepsilon > 0$ arbitrary we get the relation*

$$\frac{1}{m}\sum_{j}^{m} g_{\varepsilon/m}(n_j^{\mathrm{GP}}) \overset{p}{\leq} 9\varepsilon \tag{1.57}$$

Although this bound is far from optimal, the important result is to prove that for $\varepsilon \to 0$ the fraction in 4.12 vanishes. We can also prove a similar statement using the mean density ρ_j^{GP} of an interval I_j.

Theorem 4.13. *The fraction of intervals for which $\rho_j^{\mathrm{GP}} \to 1$ tends to 1 in probability for $\gamma \gg \nu^2$*

$$\frac{1}{m}\sum_{j=1}^{m} g_\delta(|\rho_j^{\mathrm{GP}} - 1|) \xrightarrow{p} 1 \tag{1.58}$$

From this it follows that

$$\frac{1}{m}\sum_{j=1}^{m} g_\delta(\rho_j^{\mathrm{GP}}) \xrightarrow{p} 0 \tag{1.59}$$

for all $0 \leq \delta < 1$.

Further we can prove the convergence of $|\phi^{\mathrm{GP}}|^2$ to 1 in L^2.

Theorem 4.14. *For the minimizer* ϕ^{GP} *of* $e^{\mathrm{GP}}(\gamma, \sigma, \nu)$ *follows*

$$|\phi^{\mathrm{GP}}|^2 \overset{L^2([0,1])}{\to} 1 \tag{1.60}$$

for the case $\gamma \gg \nu^2$. *This also implies the convergence on* $L^1([0,1])$.

Even though Lemma 4.14 shows the convergence of the density to 1 in L^1 we know from (1.42) that we need to analyze the convergence of $|\phi^{GP}|^{-2}$ in L^1. We shall restrict our analysis to two special cases.
For the case $\sigma \gg \sqrt{\mu} \sim \sqrt{\gamma}$ we can still apply theorem 4.5 and a short calculation shows that under the condition $\sigma \gg \sqrt{\gamma}$ and $\gamma \ll \nu^4$ no superfluidity exists. The last constraint is needed because otherwise we cannot find a lower bound for the total length of “small” intervals $\sum_{l_j<1/\sqrt{\mu}} l_j$.

Theorem 4.18. *For the case* $\sigma \gg (\gamma/\nu^2)^4\gamma^{1/2}$ *and* $\gamma \ll \nu^4$ *there is no superfluidity in the limit* $\nu \to \infty$.

Furthermore we are able to discuss the case $\gamma \gg \nu^2\sigma^2$ in which the minimizer converges in $\|\cdot\|_\infty$ to 1 and therefore superfluidity exists.

Theorem 4.17. *In the case* $\gamma \gg \nu^2\sigma^2$ *we get* $n_{\mathrm{SF}} \to 1$ *in the limit* $\nu \to \infty$.

1.7.3 A mass dependent potential

In Chapter 5 we present a different approach to show properties of the GP density. For this we add a potential term $\beta(\mu/\nu)g_{\varepsilon/\nu}(n_j)$ for $|\beta| < \beta_0$ to the functional and analyze it. The additional factor μ/ν ensures that the contribution of the perturbation term is of the same order as the unperturbed energy. The analysis in Chapter 5 is restricted to the case $\gamma \ll \nu^2$. We define

$$e_*^{\mathrm{GP}}(\gamma, \sigma, \nu, \beta, \varepsilon) \coloneqq \inf_{\|\phi\|_{L^2}=1} \sum_j^m \frac{n_j}{l_j^2}\mathcal{E}_{n_j l_j\gamma, l_j\sigma}[\phi_j] + \beta\frac{\mu}{\nu} g_{\delta/\nu}(n_j) \tag{1.61}$$

and

$$e_0^*(\gamma, \nu, \beta, \varepsilon) \coloneqq \inf_{\nu\|n\|_{L^1}=1} \nu \int \mathrm{d}p_\nu(l)\frac{n(l)}{l^2} e(n(l)l\gamma, \infty) + \beta\frac{\mu}{\nu} g_{\delta/\nu}(n(l)) \tag{1.62}$$

We shall be able to show in Lemma 5.9

$$\frac{e_*^{\mathrm{GP}}(\gamma, \sigma, \nu, \beta, \varepsilon)}{e_0^*(\gamma, \nu, \beta, \varepsilon)} \xrightarrow{p} 1$$

To prove this we shall perform a more detailed analysis of the minimizers for the perturbed functionals (1.61), (1.62), and shall be able to find proper bounds which become exact in the limit $\nu \to \infty$.
We focus the discussion on the Lagrange multipliers for the different functionals, and we show that they are of the same order. In the end we shall be able to prove a statement about the n_j^{GP} by differentiating the energy term by β.

Theorem 5.10. *For all $\delta > 0$ and $\gamma \ll \nu^2$ we get*

$$\sum_{j=1}^{m} g_\delta(\nu n_j^{\mathrm{GP}}) \xrightarrow{p} 1 \tag{1.63}$$

You can see this chapter as an alternative approach to the methods of Chapter 4 for showing results about the minimizing density.

Chapter 2

Some auxiliary estimates

As preparation for proving probability bounds for the convergence of e^{GP} to the interval density energy, we discuss in this chapter estimates for the energy $e(\kappa, \alpha)$ and later also for e^{GP}. Most of these estimates where proven already by Seiringer, Yngvason and Zagrebnov [22], but we shall extend them at various points.
We shall prove that $e(\kappa, \alpha)$ is jointly concave in κ, α and that the function $n \mapsto ne(n, \alpha)$ is convex. Further we discuss inequalities for $e(\kappa, \alpha)$ which will form an essential part of our argumentation in Chapter 3 and 4.
We shall learn that we can estimate e^{GP} by upper and lower bounds which are completely characterised by the function $e(\kappa, \alpha)$ and by using estimates for $e(\kappa, \alpha)$, we shall be able to discuss the minimizer of $e_0(\gamma, \nu)$.
A key part for the proof of a deterministic limit (Thm. 3.20), will be Lemma 2.10 which shows that for $\alpha \to \infty$, the energy for finite α, $e(\kappa, \alpha)$ converges in first order to $e(\kappa, \infty)$. Convergence in first order means $e(\kappa, \infty)/e(\kappa, \alpha) \to 1$ for $\alpha \to \infty$.
One could raise the question if it is possible to find an analytic expression for the minimizer ϕ^{GP}. We can express the minimizer using special functions up to a parameter p which is fixed by the norm constraint. Even though p cannot be expressed by κ analytically we shall be able to prove a statement about $\partial_\kappa e(\kappa, \alpha)$.

2.1 Bounds on $e(\kappa, \alpha)$

For some special values we can determine $e(\kappa, \alpha)$ without problems. The minimizer of $\mathcal{E}_{\kappa,0}$ is the constant function 1 and therefore we get $e(\kappa, 0) = \kappa/2$. For $\kappa = 0$ and $\alpha = \infty$ the energy is the ground state of the laplace operator on the interval $[0, 1]$ which is π^2.
By ignoring the $|\phi'(x)|^2$ term in $e(\kappa, \alpha)$ we get a useful lower bound for the energy

$$\begin{aligned} e(\kappa, \alpha) &= \inf_{\|\phi\|=1} \int_0^1 |\phi'(x)|^2 + \frac{\kappa}{2}|\phi(x)|^4 \mathrm{d}x + \frac{\alpha}{2}\left(|\phi(0)|^2 + |\phi(1)|^2\right) \\ &\geq \frac{\kappa}{2}\int_0^1 |\phi_{\alpha,\kappa}(x)|^4 \mathrm{d}x + e(0, \alpha) \geq \frac{\kappa}{2} + e(0, \alpha) \end{aligned} \tag{2.1}$$

using the Hölder inequality in the last step

$$1 = \left(\int_0^1 |\phi(x)|^2 \right)^2 \mathrm{d}x \leq \int_0^1 \mathrm{d}x \int_0^1 |\phi(x)|^4 \mathrm{d}x = \int_0^1 |\phi(x)|^4 \tag{2.2}$$

For the upper bound we use $\phi_{0,\alpha}$ as a trial function. We get

$$\begin{aligned} e(\kappa,\alpha) \leq \int_0^1 |\phi'_{0,\alpha}(x)|^2 + \frac{\kappa}{2}|\phi_{0,\alpha}(x)|^4 \mathrm{d}x + \frac{\alpha}{2}\left(|\phi_{0,\alpha}(0)|^2 + |\phi_{0,\alpha}(1)|^2\right) \\ = e(0,\alpha) + \frac{\kappa}{2}\int_0^1 |\phi_{0,\alpha}(x)|^4 \mathrm{d}x \end{aligned} \tag{2.3}$$

We have to analyze the last term of (2.3) in more detail. Because of symmetry and $\kappa = 0$ we can determine $\phi_{0,\alpha}$

$$\phi_{0,\alpha} = b\cos(a(x-1/2)), \qquad b = \sqrt{\frac{2a}{a+\sin a}} \tag{2.4}$$

where we used $\|\phi_{0,\alpha}\| = 1$ to determine b. Because $\phi_{0,\alpha}(x) \neq 0$ on $(0,1)$ we get $a \in [0,\pi]$. A simple integration gives

$$\int |\phi_{0,\alpha}(x)|^4 \mathrm{d}x = \frac{a(3a + \sin(a)(\cos(a)+4))}{2(a+\sin(a))^2} \tag{2.5}$$

The maximum of this function on $[0,\pi]$ is $3/2$ and can be found at $a = \pi$, which means $\alpha = \infty$. Hence we get an upper estimate

$$e(\kappa,\alpha) = e(0,\alpha) + \frac{\kappa}{2}\int_0^1 |\phi_{0,\alpha}(x)|^4 \mathrm{d}x \leq e(0,\alpha) + \frac{3}{4}\kappa \tag{2.6}$$

Together we can formulate the bounds

Lemma 2.1 ([22], Eq. (31))**.**

$$\frac{1}{2} \leq \frac{e(\kappa,\alpha) - e(0,\alpha)}{\kappa} \leq \frac{3}{4} \tag{2.7}$$

These bounds grow with the same strength but still there is a gap of order κ between them. We expect $\phi_{\kappa,\alpha}$ to get very flat for $\kappa \to \infty$ and therefore $e(\kappa,\alpha) \approx \kappa/2$.

Lemma 2.2 ([22], Lemma 3.1)**.**

$$\lim_{k\to\infty} \frac{e(\kappa,\alpha)}{\kappa/2} \to 1 \tag{2.8}$$

with

$$e(\kappa,\alpha) \leq e(\kappa,\infty) \leq \frac{\kappa}{2} + 11\sqrt{\kappa} \tag{2.9}$$

for $\kappa > 16$

Proof. We use the trial function $f_{h,\varepsilon}(x)$ for $\mathcal{E}_{\kappa,\infty}$ which is defined as

$$f_{h,\varepsilon}(x) = \begin{cases} x/\varepsilon h & x < \varepsilon \\ h & \varepsilon \le x < 1-\varepsilon \\ h(1-x)/\varepsilon & 1-\varepsilon \le x \end{cases} \tag{2.10}$$

We determine h by normalizing the function $\|f(x)\| = 1$

$$\frac{1}{3}\left(3h^2 - 4h^2\epsilon\right) = 1 \;\Rightarrow\; h = \sqrt{\frac{3}{3-4\varepsilon}} \;\Rightarrow\; h \le \sqrt{3} \tag{2.11}$$

For h we also get the estimate

$$h^2 = 1 + \frac{4\varepsilon/3}{1-4\varepsilon/3} \le 1 + \frac{4\varepsilon/3}{1-4/6} = 1 + 4\varepsilon \tag{2.12}$$

Using $f_{h,\varepsilon}$ as a trial function for $\mathcal{E}_{\kappa,\infty}$ we obtain

$$\int f'_{h,\varepsilon}(x)^2 \mathrm{d}x + \frac{\kappa}{2}\int f(x)^4 \mathrm{d}x = 2\frac{h^2}{\varepsilon} + \frac{\kappa}{10}(5h^4 - 8h^4\varepsilon) \tag{2.13}$$

To minimize this we choose $\varepsilon = 1/\sqrt{\kappa}, \kappa \ge 2$ and we get

$$e(\kappa,\infty) \le \frac{h^4\kappa}{2} - \frac{4h^4\sqrt{\kappa}}{5} + 2h^2\sqrt{\kappa} \le \frac{\kappa}{2} + 4\sqrt{\kappa} + 16 - \frac{4}{5}\sqrt{\kappa} + 6\sqrt{\kappa} \le \frac{\kappa}{2} + 10\sqrt{\kappa} + 16 \tag{2.14}$$

Because we assumed $\kappa \ge 16$ we proved the lemma. □

It is not a problem that the last lemma is restricted to large κ because we can always use the bound (2.7) for small κ.
We can also prove important convexity and concavity properties for $e(\kappa,\alpha)$.

Lemma 2.3.

(a) $e(\kappa,\alpha)$ is jointly concave

(b) $\kappa e(\kappa,\alpha)$ is convex in κ for fixed α

Proof. Assume in general a functional $E_\kappa[\phi] = A[\phi] + \kappa B[\phi]$ and $e(\kappa) = \inf_{\|\phi\|=1} E_\kappa[\phi]$. Using ϕ_k, being the minimizer of E_κ, as a trial function we define the line $G_\kappa(\alpha) = A[\phi_\kappa] + \alpha B[\phi_\kappa]$. All these lines lie above $e(\alpha)$ and are tangents in the point $e(\kappa)$. Therefore all tangents are above the function which proves (a).
For (b) we transform the expression

$$\begin{aligned} \kappa e(\kappa,\alpha) &= \inf_{\|\phi\|=1} \int \kappa|\phi'(x)|^2 + \kappa^2|\phi(x)|^4 \mathrm{d}x + \frac{\kappa\alpha}{2}(|\phi(0)|^2 + |\phi(1)|^2) \\ &= \inf_{\|\phi\|=\sqrt{\kappa}} \int |\phi'(x)|^2 + |\phi(x)|^4 \mathrm{d}x + \frac{\alpha}{2}(|\phi(0)|^2 + |\phi(1)|^2) = \inf_{\|\phi\|=\sqrt{\kappa}} \mathcal{E}_{1,\alpha}[\phi] \end{aligned} \tag{2.15}$$

Since $\mathcal{E}_{1,\alpha}$ is convex in ϕ and

$$\|\lambda\phi_1 + (1-\lambda)\phi_2\| \leq \sqrt{\lambda\kappa_1 + (1-\lambda)\kappa_2} \tag{2.16}$$

if $\|\phi_1\| \leq \sqrt{\kappa_1}$ and $\|\phi_2\| \leq \sqrt{\kappa_2}$, we can conclude that $\kappa e(\kappa, \alpha)$ is convex. □

It is also easy to see that $e(\kappa, \alpha)$ is differentiable and because of Lemma 2.1 the derivative takes values in the interval $[1/2, 3/4]$. Further the derivative is monotone decreasing because of the concavity of $e(\kappa, \alpha)$.
For the convex function $\kappa e(\kappa, \alpha)$ we can do a Legendre transformation and define

$$g(\mu, \alpha) = \inf_n ne(n, \alpha) - n\mu \tag{2.17}$$

The minimizing n of (2.17) we denote by $\bar{n}(\mu, \alpha)$ but for short we abbreviate this with $\bar{n}$. If it is not 0 it fulfills the equation

$$e(\bar{n}, \alpha) + \bar{n}e'(\bar{n}, \alpha) = \mu \tag{2.18}$$

We can use the energy estimates (2.7) to find bounds for $\bar{n}(\mu, \alpha)$

Lemma 2.4 ([22], Eq. (39))**.**

$$\frac{2}{3}[\mu - e(0, \alpha)]_+ \leq \bar{n}(\mu, \alpha) \leq [\mu - e(0, \alpha)]_+ \tag{2.19}$$

Proof. We shall write $\bar{n} = \bar{n}(\mu, \alpha)$. Subtracting $e(0, \alpha)$ from (2.18) we get for $\bar{n} > 0$

$$\begin{aligned} \mu - e(0, \alpha) &= \bar{n}\frac{e(\bar{n}, \alpha) - e(0, \alpha)}{\bar{n}} + \bar{n}e'(\bar{n}, \alpha) \\ \bar{n} &= (\mu - e(0, \alpha))\frac{1}{e'(\bar{n}, \alpha) + (e(\bar{n}, \alpha) - e(0, \alpha))/\bar{n}} \end{aligned} \tag{2.20}$$

Using the estimates $e'(\kappa, \alpha) \in [1/2, 3/4]$ and $1/2\kappa \leq e(\kappa, \alpha) - e(0, \alpha) \leq 3/4\kappa$ from Lemma 2.1 we get the desired result for $\bar{n} > 0$.
Further we notice from (2.17) that in the case of $\bar{n} = 0$ we also get $\mu \leq e(0, \alpha)$ which proves the result. □

Figure 2.1 shows $e'(\kappa, \infty)$ and the derivatives of the bounds from Lemma 2.1 and Lemma 2.2.

2.2 The interval density functional and its minimizer

One of the main statements shown in [22] is that the limit of the energy is not depending on the specific choice of ω. Further it was proven that the energy limit can be described

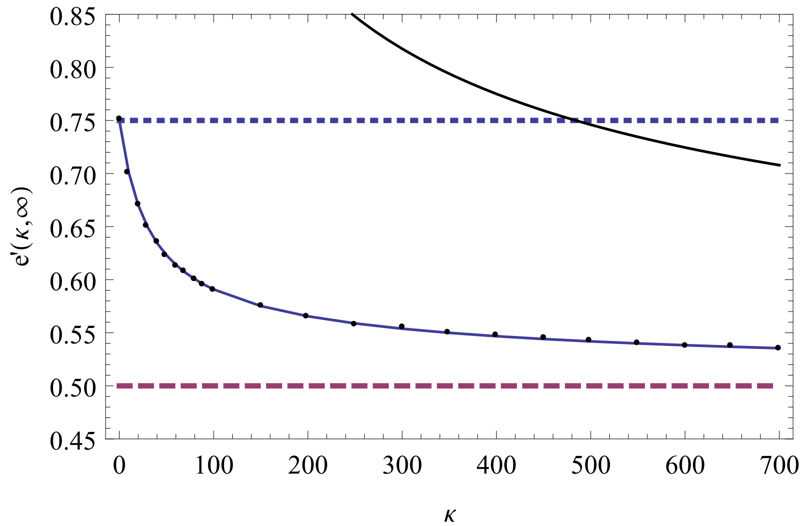

Figure 2.1: $e'(\kappa,\alpha) = \partial_\kappa e(\kappa,\alpha)$ numerical calculated and the bounds $1/2 < e'(\kappa,\alpha) < 3/4$ we got from 2.1 dotted and dashed. The solid black function is the derivative of the estimate 2.2

by the interval density functional. This functional is also independent of σ (it corresponds to $\sigma = \infty$). The interval density functional was defined by

$$\mathcal{E}^{\mathrm{IDF}}_{\gamma,\nu}[n] = \nu \int \mathrm{d}p_\nu(l) \frac{n(l)}{l^2} e(\gamma l n(l), \infty) \tag{2.21}$$

and the interval density energy by

$$e_0(\gamma,\nu) = \inf_{\nu \int \mathrm{d}p_\nu(l) n(l) = 1} \mathcal{E}^{\mathrm{IDF}}_{\gamma,\nu}[n] \tag{2.22}$$

with $\mathrm{d}p_\nu(l) = \nu e^{-l\nu}\mathrm{d}l$. $n(l)$ is now an L^1 function with a norm of ν^{-1}, and should not be confused with the n^{GP}_j. We denote the minimizer of the interval density functional by $n^{\mathrm{IDF}}(l)$ and we can determine it by introducing the lagrange multiplier μ and using the Euler Lagrange equation.

$$\begin{aligned} &\frac{\delta}{\delta n}\left(\mathcal{E}^{\mathrm{IDF}}_{\gamma,\nu}[n] - \mu\nu \int n\right)\Big|_{n(l)=n^{\mathrm{IDF}}(l)} \\ &\qquad = \nu\left(\frac{e(n^{\mathrm{IDF}}(l)l\gamma,\infty)}{l^2} + \frac{n^{\mathrm{IDF}}(l)l\gamma e'(n^{\mathrm{IDF}}(l)l\gamma,\infty)}{l^2}\right) - \nu\mu = 0 \end{aligned} \tag{2.23}$$

Together with (2.18)

$$e(\bar{n}(\mu,\infty),\infty) + \bar{n}(\mu,\infty) e'(\bar{n}(\mu,\infty),\infty) = \mu \tag{2.24}$$

we get

$$n^{\mathrm{IDF}}(l)l\gamma = \bar{n}(l^2\mu, \infty) \;\Rightarrow\; n^{\mathrm{IDF}}(l) = \frac{1}{l\gamma}\bar{n}(l^2\mu, \infty) \tag{2.25}$$

for the Lagrange multiplier μ. In general we shall denote the function $n^\eta(l)$ by

$$n^\eta(l) = \frac{1}{l\gamma}\overline{n}(l^2\eta, \infty) \tag{2.26}$$

which is equivalent to its earlier definition. Therefore we can also write $n^{\mathrm{IDF}}(l) = n^\mu(l)$ which we shall do at various points.

Even though we do not know an explicit form of $\bar{n}$ we proved some important bounds in Lemma 2.4. These bounds show that $\bar{n}(\mu, \alpha) = 0$ exactly when $\mu \leq e(0, \alpha)$. Applying this to $n^{\mathrm{IDF}}(l)$ we get

$$n^\mu(l) = 0 \iff \mu l^2 \leq e(0, \infty) = \pi^2 \iff l \leq \frac{\pi}{\sqrt{\mu}} \tag{2.27}$$

So far we have an estimate for $n^{\mathrm{IDF}}(l)$ in terms of μ and γ and we know that μ is fixed by the norm constraint.

For later application we prove an estimates for $\bar{n}'(\mu, \sigma) = \partial_\mu \bar{n}(\mu, \sigma)$.

Lemma 2.5.

$$\bar{n}'(\mu, \sigma) \geq \frac{2}{3} \quad \text{a.e.} \tag{2.28}$$

for $\bar{n}(\mu, \sigma) > 0$. $\bar{n}'$ is meant as $\partial_\mu \bar{n}$. Further we get

$$\partial_l n^\mu(l, \infty) \geq \frac{1}{3}\frac{\mu}{\gamma} \quad \text{a.e.} \tag{2.29}$$

if $\mu l^2 > e(0, l\sigma)$.

Proof. Recall the definition for $\bar{n}$ for $\bar{n} > 0$

$$e(\bar{n}(\mu, \sigma), \infty) + \bar{n}e'(\bar{n}(\mu, \sigma), \infty) = \mu \tag{2.30}$$

we differentiate this equation by $\bar{n}$ and by using the concavity of $e(n, \infty)$ we obtain

$$\frac{\mathrm{d}\mu}{\mathrm{d}\bar{n}} = 2e'(\bar{n}, \infty) + \bar{n}e''(\bar{n}, \infty) \leq 2e'(\bar{n}, \infty) \leq \frac{3}{2} \tag{2.31}$$

The derivative $e'(\kappa, \infty)$ exists everywhere and is monotone decreasing. Therefore also the second derivative $e''(\kappa, \infty)$ exists almost everywhere and because of the concavity of $e(\kappa, \alpha)$ we get $e''(\kappa, \infty) < 0$ almost everywhere. The same argument works for $\sigma < \infty$ too, which proves the relation for n' almost everywhere.

For the second claim we calculate

$$\partial_l n^\mu(l) = \partial_l \left[\frac{1}{l\gamma}\bar{n}(\mu l^2, \infty)\right] \geq -\frac{1}{l^2\gamma}\bar{n}(\mu l^2, \infty) + \frac{1}{l\gamma}\frac{2}{3}2l\mu \geq -\frac{1}{l^2\gamma}\mu l^2 + \frac{4}{3}\frac{\mu}{\gamma} \geq \frac{1}{3}\frac{\mu}{\gamma} \tag{2.32}$$

$\square$

To extract information about μ from the norm constraint we need the following inequality.

Lemma 2.6.

$$k! \leq e^x \int_x^\infty e^{-t} \left(t - \frac{x^2}{t}\right)^k \mathrm{d}t \leq 2^k k! \tag{2.33}$$

Proof. We substitute $t \to t + x$

$$e^x \int_x^\infty e^{-t} \left(t - \frac{x^2}{t}\right)^k \mathrm{d}t = \int_0^\infty e^{-t} \left(t + x - \frac{x^2}{t+x}\right)^k \mathrm{d}t = \int_0^\infty e^{-t} \left(t + \frac{tx}{t+x}\right)^k \mathrm{d}t \tag{2.34}$$

For a lower bound we just estimate the second term with zero to get

$$(2.34) \geq \int_0^\infty e^{-t} t^k = (-1)^k \left.\partial_\alpha^k\right|_{\alpha=1} \int_0^\infty e^{-t\alpha} = (-1)^k \left.\partial_\alpha^k\right|_{\alpha=1} \frac{1}{\alpha} = k! \tag{2.35}$$

The upper bound leads to the same expression if we estimate $tx/(t+x) \leq t$, which explains the factor 2^k. □

From the norm constraint and (2.7) we can prove important bounds for μ.

Lemma 2.7.

$$\frac{1}{2} \leq \frac{\mu}{\gamma} e^{-\pi\nu/\sqrt{\mu}} \leq \frac{3}{2} \;\Rightarrow\; \frac{\mu}{\gamma} e^{-\pi\nu/\sqrt{\mu}} \sim 1 \tag{2.36}$$

Proof.

$$\begin{aligned} 1 = \nu \int n^\mu(l) \mathrm{d}p_\nu &\leq \frac{\nu}{\gamma} \int_{\pi/\sqrt{\mu}}^\infty \left(\mu l - \frac{\pi^2}{l}\right) \mathrm{d}p_\nu(l) \\ &= \frac{\nu^2}{\gamma} \int_{\pi/\sqrt{\mu}}^\infty \left(\mu l - \frac{\pi^2}{l}\right) e^{-l\nu} \mathrm{d}l = \frac{\nu}{\gamma} \int_{\pi/\sqrt{\mu}}^\infty \left(\frac{\mu t}{\nu} - \frac{\pi^2 \nu}{t}\right) e^{-l\nu} \mathrm{d}l \leq 2 \frac{\mu}{\gamma} e^{-\pi\nu/\sqrt{\mu}} \end{aligned}$$

Where we used (2.33) for $k = 1$. Analogously, we can calculate the upper bound using the estimate

$$n^\mu(l) \geq \frac{2}{3} \frac{1}{\gamma} \left(\mu l - \frac{\pi^2}{l}\right) \tag{2.37}$$

□

With the estimates of Lemma 2.4 we can also estimate the expectation values of $n^\mu(l)^k$ using the measure $\mathrm{d}p_\nu(l) = \nu \exp(-\nu l) \mathrm{d}l$

Lemma 2.8.

$$\left(\frac{2}{3}\right)^k \frac{k!}{2} \frac{1}{\nu} \frac{\mu^{k-1}}{\gamma^{k-1} \nu^{k-1}} \leq \left\langle n^\mu(l)^k \right\rangle \leq 3k! 2^{k-1} \frac{1}{\nu} \frac{\mu^{k-1}}{\gamma^{k-1} \nu^{k-1}} \tag{2.38}$$

Remark 2.9. We shall prove later that $\mu/(\gamma\nu) \to 0$ under the constraint (1.24) and therefore $\nu \left\langle n^\mu(l)^k \right\rangle \to 0$ for $k > 1$.This will be of great importance for the deterministic behavior of the energy. Later in corollary 3.11 we shall use this to prove the convergence of

$$\sum_j n^{\mathrm{IDF}}(l_j) \xrightarrow{p} 1 \tag{2.39}$$

which allows us to connect the norm constraint from the interval density energy with the norm constraint of the GP energy.

Proof.

$$\begin{aligned}\left\langle n^\mu(l)^k \right\rangle \leq \int_{\pi/\sqrt{\mu}}^{\infty} \mathrm{d}p_\nu(l) \left(\frac{1}{\gamma l}(\mu l^2 - \pi^2)\right)^k \underset{l\nu = t}{=} \frac{\mu^k}{\gamma^k \nu^k} \int_{\pi\nu/\sqrt{\mu}}^{\infty} e^{-t} \left(t - \frac{\pi^2\nu^2}{\mu}\frac{1}{t}\right)^k \mathrm{d}t \\ \leq k! 2^k \frac{\mu^k}{\gamma^k \nu^k} e^{-\pi\nu/\sqrt{\mu}} \overset{2.7}{\lesssim} 3k! 2^{k-1} \frac{1}{\nu} \frac{\mu^{k-1}}{\gamma^{k-1}\nu^{k-1}}\end{aligned} \tag{2.40}$$

We used here the inequalities (2.33) and Lemma 2.7. The lower bound works analogously using the same inequalities. □

From (2.36) we can determine an upper and lower bound for μ. At these bounds μ/γ is only dependent on ν^2/γ. Lets choose the ansatz

$$\mu/\gamma = h(\nu^2/\gamma)^2 f(\nu^2/\gamma), \quad f(x) = \begin{cases} x/(1+\log(x))^2 & x > 1 \\ 1 & \text{else} \end{cases} \tag{2.41}$$

and we shall show that h is bounded. Plugging this ansatz into Lemma 2.7 we get

$$1 \sim h(x)^2 f(x) \exp\left(-\pi\sqrt{x} h(x)^{-1} f(x)^{-1/2}\right) \tag{2.42}$$

For short we used $x = \nu^2/\gamma$. If $x > 1$ we get

$$1 \sim h(x)^2 \frac{x}{(1+\log(x))^2} \exp\left(-\pi h(x)^{-1}(1+\log(x))\right) \sim h(x)^2 \frac{x}{(1+\log(x))^2} x^{-\pi h(x)^{-1}} \tag{2.43}$$

This relation can only hold if $h(x) \sim 1$. In the case $x \leq 1$ we get

$$1 \sim h(x)^2 \exp\left(-\pi\sqrt{x} h(x)^{-1}\right) \tag{2.44}$$

For $h(x) \to 0$ is a contradiction because the exponential term is bounded by 1, and for $h(x) \to \infty$ the right side is divergent which is also a contradiction.
Therefore we get

$$\mu \sim \begin{cases} \dfrac{\nu^2}{(1+\log(\nu^2/\gamma))^2} & \nu^2/\gamma \geq 1 \\ \gamma & \text{else} \end{cases} \tag{2.45}$$

2.3 Connection between $e(\kappa, \alpha)$ and $e(\kappa, \infty)$

For proving the deterministic limit we have to reduce the complex case with finite σ to the easier case with $\sigma = \infty$. For this we need a way to connect $e(\kappa, \alpha)$ with $e(\kappa, \infty)$ which was done in [22][Lemma 3.1]

Lemma 2.10 ([22], Lemma 3.1)**.**

$$e(\kappa, \infty) \geq e(\kappa, \alpha) \geq (1 - K\alpha^{-1/2}) e(\kappa, \infty) \tag{2.46}$$

where $K > 1$ is independent of κ, α.

Proof. $e(\kappa, \alpha)$ is monotone increasing in α and so we get the first inequality. For $\kappa > \alpha$ we use Lemma 2.2 to get

$$e(\kappa, \infty) \leq \frac{\kappa}{2}(1 + C\kappa^{-1/2}) \leq e(\kappa, \alpha)(1 + C\alpha^{-1/2}) \tag{2.47}$$

where C is the constant from Lemma 2.2. If we use $(1 + x)^{-1} \geq 1 - x$ we get the result for $\kappa \geq \alpha$.

For $\kappa \leq \alpha$ we define $A \coloneqq \phi_{\kappa,\alpha}(0)$ and use the symmetry of $\phi_{\kappa,\alpha}$ to get the estimate

$$e(\kappa, \alpha) \geq \frac{\kappa}{2} + \alpha A^2 \tag{2.48}$$

Because of 2.2 we get an estimate for A^2

$$A^2 \leq C\frac{\kappa^{1/2}}{\alpha} \leq C\alpha^{-1/2} \tag{2.49}$$

We define now the function $\psi(x) = \mathcal{N}(\phi_{\kappa,\alpha}(x) - A)$ where $\mathcal{N}$ is an appropriate normalization constant. It is easy to see that $\phi_{\kappa,\alpha}$ is a concave function which is bigger than 0 on $(0, 1)$ and this ensures that $\psi(x) > 0$. Using this as a trial function for $\mathcal{E}_{\kappa,\infty}$ we get

$$\begin{aligned}
e(\kappa, \infty) \leq \mathcal{E}_{\kappa,\infty}[\psi] &= \mathcal{N}^2 \int_0^1 |\phi'_{\kappa,\alpha}|^2 + \frac{\kappa}{2}(\phi_{\kappa,\alpha} - A)^4 \mathrm{d}x \\
&\leq \mathcal{N}^2 \int_0^1 |\phi'_{\kappa,\alpha}|^2 + \frac{\kappa}{2}|\phi_{\kappa,\alpha}|^4 + 6\frac{\kappa}{2}|\phi_{\kappa,\alpha}|^2|A|^2 + \frac{\kappa}{2}A^4 \mathrm{d}x \\
&\leq \mathcal{N}^2(e(\kappa, \alpha) + 3e(\kappa, \alpha)A^2 + e(\kappa, \alpha)A^4) \\
&\leq \mathcal{N}^2(e(\kappa, \alpha) + 3Ce(\kappa, \infty)\alpha^{-1} + C^2 e(\kappa, \infty)\alpha^{-2})
\end{aligned}$$

For $\mathcal{N}^2$ we can get a relation from $\|\psi\| = 1$.

$$1 = \|\psi\|^2 \geq \mathcal{N}^2 \left(\|\phi_{\kappa,\alpha}\|^2 - 2A\int \phi_{\kappa,\alpha} + A^2\right) \geq \mathcal{N}^2(1 - A)^2 \;\Rightarrow\; \mathcal{N} \leq \frac{1}{1 - A} \tag{2.50}$$

Using this we obtain

$$e(\kappa, \infty)(1 - C\alpha^{-1/2} - 3\alpha^{-1} - \alpha^{-2}) \leq e(\kappa, \alpha) \tag{2.51}$$

We can assume that $\alpha \geq K^2 \geq 1$ because otherwise $(1 - K\alpha^{-1/2})$ is negative and the statement is trivial. Hence

$$e(\kappa, \infty)(1 - (C + 4)\alpha^{-1/2}) \leq e(\kappa, \alpha) \tag{2.52}$$

□

2.4 Estimates for $e^{\mathrm{GP}}(\gamma, \sigma, \nu)$

Even though it turns out to be a difficult problem to calculate the minimizer n_j^{GP}, it will be rather simple to prove bounds for the energy. A quick calculation shows

$$e^{\mathrm{GP}}(\gamma,\sigma,\nu) = \inf_{\|\phi\|_{L^2}=1} \mathcal{E}_{V_\omega}^{\mathrm{GP}}[\phi] = \inf_{\|\phi\|_{L^2}=1} \sum_{j=1}^{m} \frac{n_j}{l_j^2} \mathcal{E}_{n_j l_j \gamma, \sigma l_j}[\phi_j] \tag{2.53}$$

with

$$n_j = \int_{I_j} |\phi|^2, \qquad \phi_j(x) = \sqrt{\frac{l_j}{n_j}}\phi(z_j + l_j x) \ \text{ for } x \in [0,1]. \tag{2.54}$$

For the upper estimate we limit the set of allowed functions of the minimization process by demanding $\phi(z_j) = 0$. On the other side we can increase the set of allowed functions by dropping continuity at the points z_j for the minimizer.

$$\inf_{\substack{\sum n_j = 1 \\ n_j \geq 0}} \sum_{j}^{m} \frac{n_j}{l_j^2} e(n_j l_j \gamma, l_j \sigma) \leq \inf_{\|\phi\|_{L^2}=1} \sum_{j=1}^{m} \frac{n_j}{l_j^2} \mathcal{E}_{n_j l_j \gamma, \sigma l_j}[\phi_j] \leq \inf_{\substack{\sum n_j = 1 \\ n_j \geq 0}} \sum_{j}^{m} \frac{n_j}{l_j^2} e(n_j l_j \gamma, \infty) \tag{2.55}$$

We shall denote the lower energy bound with $e^{\mathrm{GP},\sigma}$ and the upper estimate with $e^{\mathrm{GP},\infty}$. For the lower bound we introduce a Lagrange multiplier $\zeta^\sigma(l)$ and for the upper bound a parameter ζ. The minimizers can be found in the same way as $n^{\mathrm{IDF}}(l) = n^\mu(l)$. Using the new Lagrange multiplier we for the minimizer n_j^ζ of the upper bound

$$n_j^\zeta = n^\zeta(l_j) = \frac{1}{\gamma l_j} \bar{n}(\zeta l_j^2, \infty) \tag{2.56}$$

and for the minimizer $n_j^{\zeta^\sigma,\sigma}$ which we shall abbreviate as n_j^σ

$$n_j^\sigma = n^\sigma(l_j) = \frac{1}{\gamma l_j} \bar{n}(\zeta^\sigma l_j^2, l_j \sigma) \tag{2.57}$$

Both of these minimizers share the property with $n^{\mathrm{IDF}}(l)$ that $n^\zeta(l), n^\sigma(l) = 0$ for $l < l_0$ with an appropriate l_0 which is, for the case of n_j^σ depending on σ. For the case $\gamma \gg \nu^2$ the interval $[0, l_0]$ is shrinking very fast but for $\gamma \ll \nu^2$ nearly all $l_j \in [0, l_0]$. The fact that both n_j^ζ and n_j^σ vanish for small lengths is caused by the fact that we set $\sigma = \infty$ in the first case and that we dropped continuity in the second, and because of this the bounds are completely independent of the arrangement of the intervals.

2.5 The Legendre representation

Using the Legendre transform $g(\mu, \alpha)$ we can rewrite the energies in terms of $g(\mu, \alpha)$. The minimizer is as we proved before $n^\mu(l) = (l\gamma)^{-1}\bar{n}(\mu l^2, \infty)$ and we also use that

$$n e(n l \gamma, \alpha) = \sup_{\mu \geq 0} \left[\frac{1}{l\gamma} g(\mu l^2, \alpha) + \mu l^2 n \right] \tag{2.58}$$

which follows directly from the definition of $g(\mu,\alpha)$. Hence we get

$$\begin{aligned}
e_0(\gamma,\nu) &= \inf\left\{\nu\int_0^\infty dp_\nu(l)\frac{n(l)}{l^2}e(n(l)l\gamma,\infty), \nu\int_0^\infty dp_\nu(l)n(l)=1\right\}\\
&= \inf\left\{\nu\int_0^\infty dp_\nu(l)\frac{1}{l^2}\sup_{\mu\geq 0}\left(\frac{1}{l\gamma}g(\mu l^2,\infty)+\mu n l^2\right), \nu\int_0^\infty dp_\nu(l)n(l)=1\right\}\\
&\geq \inf\left\{\sup_{\mu\geq 0}\nu\int_0^\infty dp_\nu(l)\frac{1}{l^2}\left(\frac{1}{l\gamma}g(\mu l^2,\infty)+\mu l^2 n\right), \nu\int_0^\infty dp_\nu(l)n(l)=1\right\}\\
&= \inf\left\{\sup_{\mu\geq 0}\left[\mu+\nu\int_0^\infty dp_\nu(l)\frac{1}{l^3\gamma}g(\mu l^2,\infty)\right], \nu\int_0^\infty dp_\nu(l)n(l)=1\right\}\\
&= \sup_{\mu\geq 0}\left(\mu+\nu\int_0^\infty dp_\nu(l)\frac{1}{l^3\gamma}g(\mu l^2,\infty)\right) \qquad (2.59)
\end{aligned}$$

Furthermore a similar calculation using $n^{\mathrm{IDF}}(l)$ shows the equality. Written as an supremum, this form is helpful to find a lower estimate. The same calculation can be repeated for $e^{\mathrm{GP},\infty}$ and $e^{\mathrm{GP},\sigma}$ because we did not use any properties of the integral.
From the definition of $g(\mu,\alpha)$ in (2.17) we conclude that $g(\mu,\alpha)\leq 0$ and therefore

$$e_0(\gamma,\nu)\leq\mu \qquad (2.60)$$

In the next lemma we shall find a fitting lower bound.

Lemma 2.11.

$$\mu\geq e_0(\gamma,\nu)\geq\frac{2}{5}\mu \qquad (2.61)$$

Proof. Applying the Euler Lagrange equation we get

$$\nu\frac{1}{l^2}e(n^\mu(l)l\gamma,\infty)+\nu\frac{n^\mu(l)}{l^2}e'(n^\mu(l)l\gamma,\infty)l\gamma=\mu \qquad (2.62)$$

if we now multiply by $n^\mu(l)$ and integrate with respect to the measure p_ν we obtain

$$\nu\int \mathrm{d}p_\nu(l)\frac{n^\mu(l)}{l^2}e(n^\mu(l)l\gamma,\infty)+\nu\int\mathrm{d}p_\nu(l)\underbrace{\frac{n^\mu(l)^2}{l^2}e'(n^\mu(l)l\gamma,\infty)l\gamma}_{=:A}=\mu \qquad (2.63)$$

Because $A>0$ we get $\mu\geq e_0(\gamma,\sigma,\nu)$. For the upper bound we estimate

$$A\leq\frac{1}{l^2}\frac{3}{4}n^\mu(l)^2l\gamma\leq\frac{3}{2}\frac{n^\mu(l)}{l^2}e(n^\mu(l)l\gamma,\infty) \qquad (2.64)$$

Together we get

$$\mu\leq\frac{5}{2}e_0(\gamma,\sigma,\nu) \qquad (2.65)$$

□

For the case $\gamma \ll \nu^2$ we can further state that the upper bound is optimal in the limit $\nu \to \infty$.

Lemma 2.12.
$$\frac{e_0(\gamma,\nu)}{\mu} \to 1 \tag{2.66}$$

Proof. From Lemma 2.11 the inequality $e_0(\gamma,\nu) \leq 1$ follows. For the other direction we use the relation
$$\frac{1}{l^2} e(n^\mu(l) l\gamma, \infty) + \frac{1}{l^2} n^\mu(l) e'(n^\mu(l) l\gamma, \infty) l\gamma = \mu \tag{2.67}$$
Multiplying with $\nu n^\mu(l)$ and integrating over this equation we get
$$\mu = e_0(\gamma,\nu) + \nu\gamma \int \frac{n^\mu(l)^2}{l} \mathrm{d}p_\nu(l) \tag{2.68}$$
We prove the result with the following upper bound
$$\frac{\nu\gamma}{\mu} \int \frac{n^\mu(l)^2}{l} \mathrm{d}p_\nu(l) \leq \frac{\nu\gamma}{\mu} \frac{\sqrt{\mu}}{\pi} \int n^\mu(l)^2 \mathrm{d}p_\nu(l) \leq c \frac{\nu\gamma}{\mu} \frac{\sqrt{\mu}}{\pi} \frac{\mu}{\gamma\nu^2} = c\frac{\sqrt{\mu}}{\nu} \to 0 \tag{2.69}$$
for a constant $c > 0$. We used Lemma 2.8 to find an upper bound on $\int \mathrm{d}p_\nu(l) n^\mu(l)^2$. □

Even though the result of the last lemma shows the connection between μ and e_0, we are still not able to determine μ and therefore the energy precisely. For the case $\gamma \gg \nu^2$ on the other side we can determine the energy in the limit exactly in terms of γ.

Lemma 2.13. *For $\gamma \gg \nu^2$ we get*
$$\frac{2}{\gamma} e_0(\gamma,\nu) \to 1 \tag{2.70}$$
and further we find for $\gamma \sim \nu^2$ the upper bound
$$\frac{\gamma}{2} \leq e_0(\gamma,\nu) \leq \frac{\gamma}{2} e^{8k} \frac{1}{(4k+1)^2}(1+11k) \sim \gamma \tag{2.71}$$
with $\nu/\sqrt{\gamma} \to k$.

Proof. First we deal with the case $\gamma \gg \nu^2$ and the upper bound. We want to use Lemma 2.2 which is only valid for $\kappa > 16$. To achieve this we shall take a trial function of the form $n(l) = 0$ for $l < t$ and $n(l) = cl$ for $l \geq t$ for an appropriate $t > 0$. Because we want to apply it to $e(n(l)l\gamma, l\sigma) \geq e(t^2\gamma, t\sigma)$ for $l \geq t$ we have to demand $t \geq 4/\sqrt{\gamma}$. We determine c by normalizing $n(l)$.
$$1 = \nu \int \mathrm{d}p_\nu(l) n(l) = \nu^2 \int_{4/\sqrt{\gamma}}^\infty e^{-\nu l} cl = ce^{-4\nu/\sqrt{\gamma}} \left(\frac{4\nu}{\sqrt{\gamma}} + 1\right) \;\Rightarrow\; c \to 1,\; c > 1 \tag{2.72}$$

Here $\gamma \gg \nu^2$ is crucial. Applying this function we get an upper limit

$$e_0(\gamma,\nu) \overset{(2.7)}{\leq} \nu \int_{4/\sqrt{\gamma}}^{\infty} \mathrm{d}p_\nu \frac{c}{l} e(cl^2\gamma,\infty) \overset{2.2}{\leq} \nu c \int \mathrm{d}p_\nu \,(cl\gamma + 11\sqrt{c\gamma})$$
$$= c^2\frac{\gamma}{2} + c^{3/2} 11\sqrt{\gamma}\nu \leq c^2\frac{\gamma}{2}(1 + 11\nu/\sqrt{\gamma}) \quad (2.73)$$

which proves the upper estimate because $\sqrt{\gamma}\nu \ll \gamma$ and $c \to 1$. For the lower bound we just set $\sigma = 0$ to get

$$e_0(\gamma,\nu) \geq \inf_{\nu\|n\|_{L^1}=1} \nu \int \mathrm{d}p_\nu \frac{n(l)}{l^2} e(n(l)l\gamma, 0) \geq \inf_{\nu\|n\|_{L^1}=1} \nu \int \mathrm{d}p_\nu \frac{n(l)^2\gamma}{2l} = \frac{\gamma}{2} \quad (2.74)$$

For the case $\gamma \sim \nu^2$ we just use equation 2.73. □

2.6 Estimates for $e(0,\alpha)$

Later we shall work with cases where $\sigma < \infty$. We have already seen in Lemma 2.4 that $e(0,\alpha)$ appears in the estimate of $\bar{n}$. To find a proper upper estimate we shall need to discuss the function $e(0,\alpha)$.

Lemma 2.14.

$$\pi^2\left(1 - \frac{12}{4+\alpha}\right) \leq \pi^2 \frac{\alpha^3}{(4+\alpha)^3} \leq e(0,\alpha) \leq \pi^2 \quad (2.75)$$

for $\alpha > \pi$ *and*

$$\frac{\pi\alpha}{\alpha+\pi} \leq e(0,\alpha) \leq \pi^2 \quad (2.76)$$

for $\alpha \leq \pi$

Proof. We know that for the ground state of $e(0,\alpha)$ we have

$$\phi^b_{0,\alpha} = a\cos(b(x-1/2)) \quad \text{with} \quad a = \sqrt{\frac{2b}{b+\sin b}} \quad (2.77)$$

where we determined a by the normalization constraint. We now find b by minimizing the energy.

$$\mathcal{E}_{0,\alpha}(\phi^b_{0,\alpha}) = \alpha\phi(0)^2 + \int_0^1 \mathrm{d}x|\phi'(x)|^2 = \alpha a^2\cos^2\left(\frac{b}{2}\right) + \frac{1}{2}a^2 b(b - \sin(b))$$
$$= \frac{b\,(\alpha + b^2 + \alpha\cos(b) - b\sin(b))}{b+\sin(b)}$$

Taking the derivative by b we get

$$\partial_b \mathcal{E}_{0,\alpha}(\phi^b_{0,\alpha}) = \left(2\sin\left(\frac{b}{2}\right)\left(-b^2+\cos(b)+1\right) - 2b\cos\left(\frac{b}{2}\right)\right)$$
$$\left(\alpha\cos\left(\frac{b}{2}\right) - 2b\sin\left(\frac{b}{2}\right)\right) \quad (2.78)$$

Only the first bracket is of interest because the second is independent of α and a quick calculation shows that it is < 0 for $b \in (0,\pi)$. We get the equation

$$b = \frac{\alpha}{2}\cot(b/2) \quad (2.79)$$

On $[0,\pi]$ we know that $\cot(b/2)$ is convex and therefore we can find a lower border for b by approximating $\cot(b/2)$ linearly in π.

$$\cot(b/2) \geq -\frac{b-\pi}{2} \;\Rightarrow\; b_{low} = -\alpha\frac{b-\pi}{4} \quad (2.80)$$

which has the solution $b_{low} = \pi\alpha/(4+\alpha)$. What we actually want to know is a lower estimate of the energy.

$$\frac{b^3}{b+\sin(b)} - \frac{b^2\sin(b)}{b+\sin(b)} + \frac{\alpha b}{b+\sin(b)} + \frac{\alpha b\cos(b)}{b+\sin(b)} \geq \frac{b^3}{b+\sin(b)} \geq \frac{b^3}{b+\pi-b} = \frac{b^3}{\pi} \quad (2.81)$$

Here we used $\alpha \geq \pi$ so that the second and the last term of the sum are together greater than 0. The estimate is monotone increasing and so we get an lower estimate by plugging in b_{low}. Together we get

$$e(0,\alpha) \geq \pi^2\frac{\alpha^3}{(4+\alpha)^3} \geq \pi^2\left(1 - \frac{12}{4+\alpha}\right) \quad (2.82)$$

The last inequality can be confirmed by a simple calculation. For the case $\alpha < \pi$ we shall use the last two terms for an estimate. Further we can estimate $b + \sin b < 2b$ and $1 - \cos b \geq 1 - b^2/2$.

$$\alpha\frac{b(1-\cos b)}{b+\sin b} \geq \alpha\frac{2b - b^3/2}{2b} \geq \alpha\left(1 - \frac{b^2}{4}\right) \quad (2.83)$$

As a lower estimate for the energy we get therefore

$$\alpha - \frac{\pi^2\alpha^3}{4(\alpha+4)^2} \geq \frac{\pi\alpha}{\alpha+\pi} \quad (2.84)$$

for $\alpha < \pi$. $\square$

To make later calculations easier it is helpful to combine these bounds into one.

Corollary 2.15.

$$e(0,\alpha) \geq \pi^2 \frac{\alpha}{34+\alpha} \tag{2.85}$$

Proof. We take the ansatz $f(x) = x/(k+x)$. Taking the derivative shows

$$f'(x) = \frac{k}{(k+x)^2} \leq \frac{12}{(4+x)^2} = \partial_x \left(1 - \frac{12}{4+x}\right) \tag{2.86}$$

for $k > 4$. Hence we only need to choose a k high enough such that both lower bounds are satisfied in on a finite interval. The last equation shows that the bound will stay below $e(0,\alpha)$ in infinity. A possible choice would be $k = 34$. This bound has the advantage that it converges to π^2 for $\alpha \to \infty$ which will be useful in Chapter 4. □

2.7 Analytic approach

We could ask the question if we can calculate $e(\kappa,\alpha)$ analytically. We will not be able to do that, but we can express the minimizers in terms of special functions and we can reduce the problem of calculating $e(\kappa,\alpha)$ to the problem of solving the normalization constraint. Even though this approach gets complicated very fast, it still provides us with possible solutions to the equation which will be helpful for understanding the minimizer. This calculation is based on [31].

2.7.1 Jacobi $\mathrm{sn}(x;k)$

There are different ways how to introduce the function sn. We define $\mathrm{sn}(x;k)$ as the inverse function

$$z = \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad \mathrm{sn}(z;k) := \sin(\phi) \tag{2.87}$$

For a more detailed discussion of sn see [32].

2.7.2 Minimizer of the $e(\kappa,\alpha)$

The Euler-Lagrange equation for the functional $\mathcal{E}_{\kappa,\alpha}$ is

$$\phi'' = \kappa|\phi|^2\phi - \eta\phi \tag{2.88}$$

We are looking for real solutions and therefore we assume $|\phi|^2 = \phi^2$. Multiplying with ϕ' and integrating lead to

$$(\phi'(x))^2 = \frac{\kappa}{2}\phi^4(x) - \eta\phi^2(x) + \underbrace{(\phi'(x_0))^2 - \frac{\kappa}{2}\phi^4(x_0) + \eta\phi^2(x_0)}_{=:E} \tag{2.89}$$

Another integration leads to

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} \frac{1}{\sqrt{\frac{\kappa}{2}\phi^4 - \eta\phi^2 + E}} \tag{2.90}$$

This expression can be easily reduced to the canonical form using [32][Section 3.1]. The result is very much depended on the initial conditions but in our case we are only interested in the non-divergent solution. The solution is

$$\phi(x) = A\mathrm{sn}(4K(p)\frac{x + x_0}{L}; p) \tag{2.91}$$

where $K(p)$ is the complete elliptic integral of the first kind and

$$A^2 = \frac{2\eta p}{\kappa(p+1)}, \qquad L^2 = \frac{8(p+1)K(p)^2}{\eta} \tag{2.92}$$

η, p are free parameters which are in theory fixed by the normalization constraint and the optimal boundary condition $\phi(0) = \phi(1) = \text{const}$. To find η for finite α, we need to optimize the energy which will be to complicated. Hence we restrict to $\alpha = \infty$ which is equivalent to the BVP $\phi(0) = 0, \phi(1) = 0$. In this case we can easily fix η knowing that $\mathrm{sn}(x; p)$ is a $4K(p)$ periodic function and $\mathrm{sn}(0; p) = 0$.

$$\phi_0(x) = 2\sqrt{\frac{2p}{\kappa}} K(p)\mathrm{sn}(2xK(p); p) \tag{2.93}$$

where $\phi_0(x)$ is the minimizer to $e(\kappa, \infty)$. To determine p we use the normalization constraint

$$\int_0^1 |\phi_0(x)|^2 \mathrm{d}x = 1 \tag{2.94}$$

Even though the integral can still be expressed by special functions, it is to complicated to find an explicit inverse function to express p by κ. Nevertheless we can easily conclude that $p(\kappa)$ is C^1 using the inverse function theorem and that both $K(p)$ and $\mathrm{sn}(2xK(p); p)$ are in C^1 in p. On the other side it is easy to express κ in terms of p with a function $f(p) = \kappa$ because of the specific form of ϕ_0. With this we can calculate

$$g(p) \coloneqq \int_0^1 |\phi_0(x)|^4 \mathrm{d}x \tag{2.95}$$

analytically in terms of p. If we define the helper function $h(p) \coloneqq p/(1 + p)$, a more detailed analysis shows that $f \circ h(x)$ is convex and monotone increasing. The inverse function $h^{-1} \circ f^{-1}$ is therefore concave and monotone increasing. Further one can see that $g \circ h(x)$ is monotone decreasing and convex. This shows that

$$g \circ h \circ h^{-1} \circ f^{-1}(\kappa) = g \circ f^{-1}(\kappa) \tag{2.96}$$

is a convex function if we apply standard rules for the composition of convex/concave functions.

Theorem 2.16. *$e'(\kappa, \infty) = \partial_\kappa e(\kappa, \infty)$ is a convex, monotone decreasing C^1 function.*

Proof. The derivative

$$e'(\kappa,\infty) = \frac{1}{2}\int |\phi_{\kappa,\infty}|^4 = g \circ f(\kappa) \tag{2.97}$$

is convex and monotone decreasing. Because $g(p)$ is C^1 in p and $p(\kappa)$ is C^1 in κ the result follows. □

2.8 Discussion of the limit

In this section we want to discuss why we have to choose the constraints (1.24) for a deterministic limit. Because these arguments are based on statements about the minimizer of the interval density functional, they are of heuristic nature. Nevertheless they give the right impression of what could go wrong in the limit if we do not restrict γ, σ. In Chapter 3 we shall see that these demands are sufficient. Throughout this section we shall discover very useful relations between the parameters ν, γ, σ, μ.

2.8.1 The limit constraint for γ

For the interval density functional minimizer we can define the number of intervals with nonzero mass by

$$\lambda\nu := \nu \int_{\{n^\mu(l)>0\}} \mathrm{d}p_\nu(l) = \nu \int_{\pi/\sqrt{\mu}}^{\infty} \mathrm{d}p_\nu(l) = \nu e^{-\pi\nu/\sqrt{\mu}} \tag{2.98}$$

From this we can express μ by lambda

$$\mu = \frac{\pi^2\nu^2}{(\log\lambda)^2} \tag{2.99}$$

Using the relation from Lemma 2.7 we get

$$\lambda \sim \frac{\gamma}{\mu} \;\Rightarrow\; \gamma \sim \frac{\lambda\nu^2}{(\log\lambda)^2} \tag{2.100}$$

Let us discuss the following example

Example 2.17. *Let us assume $\gamma = 0$ and $\sigma = \infty$. Which means that there is no interaction and therefore all particles occupy the interval with the maximal length. The energy is therefore π^2/L^2 with $L = \max_j l_j$ which is depended on the specific choice of ω.*

The problem which occurs, and which prevents an deterministic limit is that only finitely many intervals have non zero mass. If we want to apply the law of large numbers we need particles in infinitely many intervals and hence we demand $\lambda \gg \nu^{-1}$. This gives the condition (1.24) for γ

$$\gamma \gg \frac{\nu}{(\log\nu)^2} \tag{2.101}$$

We used that $x/\log(x)^2$ is an increasing function on $[0,1]$.

Lemma 2.18. *The following relations are equivalent*

$$(a)\quad \frac{\mu}{\gamma\nu} \ll 1 \qquad\qquad (b)\quad \lambda \gg \nu^{-1} \qquad\qquad (c)\quad \gamma \gg \frac{\nu}{(\log\nu)^2}$$

Proof. We have already discussed $(b) \Rightarrow (c)$.
For $(a) \iff (b)$ we use

$$1 \gg \frac{\mu}{\gamma\nu} \sim \frac{1}{\lambda\nu} \iff \lambda \gg \nu^{-1} \tag{2.102}$$

and for $(c) \Rightarrow (b)$ we show that from $\lambda \lesssim \nu^{-1}$ it follows that

$$\gamma \sim \frac{\lambda\nu^2}{(\log\lambda)^2} \lesssim \frac{\nu}{(\log\nu)^2} \tag{2.103}$$

using again the monotonicity of $x/\log(x)^2$.

□

In Chapter 3 we shall learn that the convergence $\mu/(\gamma\nu) \to 0$ is needed for the existence of a deterministic limit. Because of Lemma lemEqDefLim this shows that the constraint on γ in (1.24) is also a necessary condition for Corollary 3.11.

2.8.2 The limit constraint for σ

The first approach for finding a necessary constraint for σ could be to assume $\sigma\langle l\rangle \sim \sigma\nu^{-1} \gg 1$ which turns out to be to conservative. It is sufficient to assume that the weighted length $\bar{l}$

$$\begin{aligned}
\bar{l} = \nu\int \mathrm{d}p_\nu(l) l n^{\mathrm{IDF}}(l) &\le \int_{\pi/\sqrt{\mu}}^{\infty} \nu e^{-\nu l}\left(\frac{\mu}{\gamma}l^2 - \frac{\pi^2}{\gamma}\right)\mathrm{d}l \\
&\le e^{-\pi\nu/\sqrt{\mu}}\nu\int_0^\infty \nu e^{-\nu y}\left(\frac{\mu}{\gamma}y^2 + \frac{2\pi\sqrt{\mu}}{\gamma}y\right)\mathrm{d}y \\
&\le e^{-\pi\nu/\sqrt{\mu}}\nu\int_0^\infty e^{-x}\left(\frac{\mu}{\nu^2\gamma}x^2 + \frac{2\pi\sqrt{\mu}}{\gamma\nu}x\right) \\
&\le 2e^{-\pi\nu/\sqrt{\mu}}\left(\frac{\mu}{\nu\gamma} + \frac{\pi\sqrt{\mu}}{\gamma}\right) \sim \left(\frac{1}{\nu} + \frac{1}{\sqrt{\mu}}\right)
\end{aligned} \tag{2.104}$$

times σ converges to infinity. We used the substitutions $y = l + \pi/\sqrt{\mu}$ and $x = \nu y$. A similar calculation, using the lower bound of $n^{\mathrm{IDF}}(l)$, shows up to a constant the same lower bound. A necessary condition for σ would therefore be $\sigma \gg \min(\nu, \sqrt{\mu})$ which we can formulate as

$$\sigma \gg \frac{\nu}{1 + \log(1 + \nu^2/\gamma)} \tag{2.105}$$

In the case $\gamma \lesssim \nu^2$ the right side of (2.105) is of the same order as $\sqrt{\mu}$ and further $\nu \gg \sqrt{\mu}$. Otherwise we get $(1+\log(1+\nu^2/\gamma))^{-1} \sim 1$ and $\nu \ll \sqrt{\mu}$. Hence this constraint is equivalent to

$$\sigma \gg \min(\nu, \sqrt{\mu}) \tag{2.106}$$

Chapter 3

Explicit estimate for the deterministic limit

In this chapter we discuss the proof of Theorem 3.20 and extend the results of [22]. In particular we shall not restrict ourselves to almost sure convergence in a subsequence, but we shall show explicit estimates for convergence in probability. In Section 3.1 and in more detail in [33] we show that this is a stronger statement. The proofs very explicitly use the properties of the distribution of the l_j. In particular we make use of the Poisson distribution of the number of random points m:

$$P(m=k) = e^{-\nu}\frac{\nu^k}{k!} \tag{3.1}$$

It is easy to see that this distribution is centered around ν, which is also the expectation value of m. In Lemma 3.18 we show that the fraction $m/\nu \xrightarrow{p} 1$ which implies that in the probability limit they are exchangeable. More precisely we discuss this in Lemma 3.8.
Beside the distribution of m we also make excessive use of the IID variables l_j following an exponential distribution

$$\mathrm{d}p_\nu(l) = \nu e^{-\nu l}\mathrm{d}l \tag{3.2}$$

The first step to show the convergence will be to connect the sum to an expectation value

$$\sum_j w(l_j) \to \nu \int \mathrm{d}p_\nu(l) w(l) \tag{3.3}$$

in first order. This is proven for specific ω in Theorem 3.9. In particular this will show that

$$N^\mu \xrightarrow{p} 1 \tag{3.4}$$

where

Definition 3.1.

$$N^\eta := \sum n^\eta(l_j) \tag{3.5}$$

In particular we use $N^\mu = \sum_{j=1}^m n^{\mathrm{IDF}}(l_j)$ *where* μ *is the Lagrange multiplier corresponding to* $e_0(\gamma,\nu)$.

To show the upper bound we first restrict to show the convergence $\mu/\zeta \stackrel{p}{\to} 1$ which will imply the convergence of

$$\frac{e^{\mathrm{GP},\infty}}{e_0} \stackrel{p}{\to} 1$$

In the last step we need to relate the energy $e^{\mathrm{GP},\sigma}$ with $e^{\mathrm{GP},\infty}$ to prove the lower bound. For this we make use of Lemma 2.10.

3.1 Limits in probability theory

Because the potential was created by a Poisson process, the GP energy is a random variable. For random variables it is not a priori clear how to define a limit and it will turn out that there are several nonequivalent methods. For a more detailed introduction see [33].
One possibility, which shall be heavily utilized, is the limit in probability.

Definition 3.2. *Let X_n be a sequence of random variables on a probability space Ω. We define $X_n \stackrel{p}{\to} Y$ if*

$$P(|X_n - Y| > \varepsilon) \to 0 \tag{3.6}$$

for all $\varepsilon > 0$

A stronger convergence is defined by the almost sure limit.

Definition 3.3. *Let X_n be a sequence of random variables on a probability space Ω. We define $X_n \stackrel{a.s.}{\to} Y$ if*

$$\lim_{n\to\infty} P(\omega \in \Omega \mid \sup_{m\geq n} |X_m(\omega) - X(\omega)| \geq \varepsilon) = 0 \tag{3.7}$$

for all $\varepsilon > 0$

This limit is strictly stronger than the convergence in probability, since there can still be a limit in probability, even if for all ω $|X_n(\omega) - X(\omega)| > \varepsilon$ for infinitely many n. The typical example would be a moving step function on the one dimensional torus with width $1/n$ and height 1. For the almost sure convergence this would be obviously a contradiction because $\sup |X_n(\omega) - X(\omega)| = 1$.
Another limit we shall encounter is the limit in L^p.

Definition 3.4. *Let X_n be a sequence of random variables on a probability space Ω. We define $X_n \stackrel{L^p}{\to} Y$ if*

$$\lim \langle |X_n(\omega) - X(\omega)|^p \rangle \to 0 \tag{3.8}$$

The L^p limit implies the convergence in probability. This result we shall discuss in more detail in Lemma 3.6.
Even though convergence in probability does not imply almost sure convergence, we can establish almost sure convergence for a subsequence using the Borel-Cantelli-Lemma.

Lemma 3.5. *Let $X_n \stackrel{p}{\to} X$ then there exists a subsequence X_{n_k} such that $X_{n_k} \stackrel{a.s.}{\to} X$*

Proof. Let us choose the subsequence in a way that

$$P(|X_{n_k} - X| > \varepsilon) < \frac{1}{k^2} \tag{3.9}$$

We now estimate

$$P(\omega \in \Omega \mid \sup_{m \geq n} |X_m(\omega) - X(\omega)| \geq \varepsilon) \leq \sum_{k=m}^{\infty} P(|X_{n_k} - X| > \varepsilon) \leq \sum_{k=m}^{\infty} \frac{1}{k^2} \to 0 \tag{3.10}$$

□

In Theorem 3.9 we shall show the convergence in an L^2 sense which implies the convergence in probability, as the next lemma shows.

Lemma 3.6. *Let X be a random distributed variable. We obtain*

$$P(|X - c| > a) \leq \frac{\mathrm{Var}\,(X)}{(a - |\langle X \rangle - c|)^2} \tag{3.11}$$

in the case that

$$a \gg |\langle X \rangle - c| \tag{3.12}$$

Further if $\langle x \rangle - c \to 0$ and $\mathrm{Var}\,(X) \to 0$ we get the first order approximation

$$P(|X - c| > a) \leq \frac{\mathrm{Var}\,(X)}{a^2} + \mathcal{O}(a^{-3}(\langle X \rangle - c)\,\mathrm{Var}\,(X)) \tag{3.13}$$

Proof. By the lemma of Tschebyschew we get

$$\begin{aligned}\frac{\mathrm{Var}\,(X)}{(a - |\langle X \rangle - c|)^2} &\geq P(|(X - \langle X \rangle)| > a - |\langle X \rangle - c|) \\ &\geq P(|X - c| - |\langle X \rangle - c| > a - |\langle X \rangle - c|) = P(|X - c| > a)\end{aligned} \tag{3.14}$$

□

Later we shall have to deal with inverse of random variables and their convergence. From $A \xrightarrow{L^2} c$ we can't conclude that $A^{-1} \xrightarrow{L^2} c^{-1}$, but if we replace the limit with a limit in probability, the statement holds.

Lemma 3.7. *Let A be a probability variable we get*

$$P\left(\left|\frac{1}{A} - \frac{1}{c}\right| > \varepsilon\right) \leq P(|A - c| > \frac{c^2 \varepsilon}{2}) \tag{3.15}$$

for small ε.

Proof.

$$P(A > c + \varepsilon) = P\left(\frac{1}{A} < \frac{1}{c + \varepsilon}\right) = P\left(\frac{1}{A} < \frac{1}{c}\frac{1}{1 + \varepsilon/c}\right) \geq P\left(\frac{1}{A} < \frac{1}{c} - \frac{\varepsilon}{c^2}\right) \tag{3.16}$$

with $1 - \varepsilon/c < 1/(1 + \varepsilon/c)$

$$P(A < c - \varepsilon) = P\left(\frac{1}{A} > \frac{1}{c - \varepsilon}\right) = P\left(\frac{1}{A} > \frac{1}{c}\frac{1}{1 - \varepsilon/c}\right) \geq P\left(\frac{1}{A} > \frac{1}{c} + \frac{2\varepsilon}{c^2}\right) \tag{3.17}$$

for $\varepsilon < c/2$.

□

3.2 Convergence of the norm constraint

Due to the fact m is not fixed but in fact a random variable, the proofs are more complex. We shall still be able to show that the probabilities for large and small m decreases exponentially which is enough to establish the results. The characteristic function describing the event for m intervals we shall denote by χ_m. This function has a expectation value of

$$\langle \chi_m \rangle = \frac{\nu^m}{m!} e^{-\nu} \tag{3.18}$$

The next lemma shows that the parts of the sum where m is not close to ν are decreasing exponentially. Heuristically we can therefore calculate

$$\left\langle \sum_j^m w(l_j) \right\rangle \approx \left\langle \sum_j^\nu w(l_j) \right\rangle = \nu \langle w(l) \rangle = \nu \int \mathrm{d}p_\nu(l) w(l) \tag{3.19}$$

Lemma 3.8.

(a)

$$\sum_{m \leq \lambda\nu} \langle \chi_m \rangle \leq \exp\left(-\nu(\lambda-1)^2/3\right) \qquad 0 < \lambda \leq 1 \tag{3.20}$$

for $\lambda \in [0, 1)$. An equal statement holds for $m \geq \lambda\nu$ with $\lambda \in [2, 1)$.

(b) If $\lambda = 1 + \nu^{-1/2+\varepsilon} + c\nu^{-1}$ with $\varepsilon > 0$ then the probability still decay exponentially.

$$\exp\left(-\nu(\lambda-1)^2/3\right) \lesssim \exp\left(-\nu^{2\varepsilon}/3\right) \tag{3.21}$$

Proof. For $\lambda < 1$ we can show

$$\sum_{m \leq \lambda\nu} \langle \chi_m \rangle = e^{-\nu} \sum_{m \leq \lambda\nu} \frac{\nu^m}{m!} \leq e^{-\nu} \sum_{m \geq 0} \frac{\nu^m}{m!} \lambda^{m-\lambda\nu} \leq \exp\left(-\nu(1 - \lambda + \lambda \log \lambda)\right) \tag{3.22}$$

and for $\lambda > 1$ we get

$$\sum_{m \geq \lambda\nu} \langle \chi_m \rangle = e^{-\nu} \sum_{m \geq \lambda\nu} \frac{\nu^m}{m!} \leq e^{-\nu} \sum_{m \geq 0} \frac{\nu^m}{m!} \lambda^{\lambda\nu-m} \leq \exp\left(-\nu(1 - \lambda + \lambda \log \lambda)\right) \tag{3.23}$$

To prove the theorem we need to analyze the function $h(\lambda) := 1 - \lambda + \lambda \log \lambda$ in more detail. A simple analysis shows the lower bound on $\lambda \in [0, 2]$

$$h(\lambda) \geq \frac{(\lambda-1)^2}{3} \tag{3.24}$$

which proves (a). For (b) we calculate with $\lambda = 1 - \nu^{-1/2+\varepsilon} + c\nu^{-1}$

$$-\nu\frac{1}{2}(\lambda-1)^2 = -\nu\frac{1}{2}(\nu^{-1+2\varepsilon} - 2c\nu^{-3/2+\varepsilon} + c^2\nu^{-2}) \leq -\frac{\nu^{2\varepsilon}}{2} + c\nu^{-1/2+\varepsilon} \tag{3.25}$$

As long as $\varepsilon > 0$ we get still an exponential decay for the sum. We can further estimate the last term if we assume that $\nu \gg 1$. □

From the definition of the *GP* energy e^{GP} it is clear that we somehow have to deal with a sum of the form

$$\sum_j^m w(l_j) \tag{3.26}$$

where w is an arbitrary function, and in particular we are interested in $w(l) = n^{\mathrm{IDF}}(l)$. Because the l_j are distributed by a exponential distribution we hope that

$$\left\langle \sum_j^m w(l_j) \right\rangle \to \nu \int \mathrm{d}p_\nu(l) w(l) \tag{3.27}$$

It will turn out that for this we only need very few assumptions on w. We are also interested on the convergence of the variance for which we need stronger assumptions.

Theorem 3.9 (Convergence of the sum)**.** *Let us denote $N = \sum_j^m w(l_j)$ and assume $w(l)$ has the following properties*

(i)

$$\nu \int \mathrm{d}p_\nu(l) w(l) = 1 \tag{3.28}$$

(ii)

$$\nu \left\langle w^2 \right\rangle \to 0 \tag{3.29}$$

(iii)

$$e^{-c\nu} \left\langle w^4 \right\rangle \to 0 \quad \forall c > 0 \tag{3.30}$$

then we get $N \overset{L^2}{\to} 1$ for $\nu \to \infty$ and for the convergence in probability

$$P(|N-1| > \varepsilon) \leq \frac{1}{\varepsilon^2} \left(4\nu^{-1/2+\delta} + 2\nu \left\langle w^2 \right\rangle \right) \left(1 + \mathcal{O}(\nu^{-1/2+\delta}/\varepsilon)\right) \tag{3.31}$$

with $\nu > 5$ and $\delta > 0$.

Remark 3.10. For the convergence of the expectation value we only need to assume (i) which is easily deduced in all of the previous cases. (iii) is only excluding some pathological cases and therefore (ii) ensures that a probability limit exists. For cases where $\nu^2 \left\langle w^4 \right\rangle \to 0$ we could simplify the proof because it implies (ii) and (iii).

Proof. To show this we show $\langle N \rangle \to 1$ and $\langle N^2 \rangle \to 1$. First, we see that

$$\nu \sum_{m \geq 1} \langle \chi_m w(l) \rangle = \nu \langle w(l) \rangle = 1 \tag{3.32}$$

Claim: $\langle N \rangle \to 1$

We rewrite the expectation value with the help of χ_m. Remember that the l_j are IID.

$$\langle N \rangle = \left\langle \sum_{m \geq 2} \sum_{j=1}^{m-1} \chi_m w(l_j) \right\rangle = \sum_{m \geq 2} \left\langle \sum_{j=1}^{m-1} \chi_m w(l_j) \right\rangle = \sum_{m \geq 2} (m-1) \langle \chi_m w(l_j) \rangle \tag{3.33}$$

Now we estimate a lower bound for $\langle N\rangle$. For this we restrict the sum to $m \geq 1+(1-h)\nu$ with $h>0$ which we choose later.

$$\langle N\rangle \geq (1-h)\nu \sum_{m\geq 1+(1-h)\nu} \langle \chi_m w(l)\rangle \overset{(3.32)}{=} (1-h)\left(1-\nu \sum_{m<1+(1-h)\nu} \langle \chi_m w(l)\rangle\right) \tag{3.34}$$

We now obtain

$$\langle N\rangle \geq (1-h)\left(1-\nu\left\langle Pw\right\rangle\right), \quad P(w)=\sum_{m<1+(1-h)\nu} \chi_m \tag{3.35}$$

In the next step we want to use Lemma 3.8 to estimate $\langle P\rangle$. This gives us a restriction to how fast $h\to 0$. We set $h=\nu^{-1/2+\delta}$ with $\delta>0$, which allows us to still have exponential decay in $\langle P\rangle$. Using the relation (3.22) with $\lambda = 1/\nu+(1-h)\leq 1$ we obtain

$$\langle P\rangle \lesssim \exp\left(-\nu^{2\delta}/3\right) \tag{3.36}$$

W.l.o.g we can assume that $\nu>1$. We now estimate with Cauchy-Schwartz

$$\nu\left\langle Pw\right\rangle \leq \nu\left\langle P\right\rangle^{1/2}\left\langle w^2\right\rangle^{1/2} \lesssim \underbrace{\nu\exp\left(-\frac{\nu^{2\delta}}{6}\right)\left\langle w^2\right\rangle^{1/2}}_{\to 0} \tag{3.37}$$

Ignoring the terms which decay exponentially we proved the lower bound

$$\langle N\rangle \geq 1-\nu^{-1/2+\delta} \tag{3.38}$$

For the upper estimate we now use

$$\begin{aligned}\sum_{m\geq 2}(m-1)\left\langle \chi_m w(l_1)\right\rangle &= \sum_{m\leq 1+(1+h)\nu}(m-1)\left\langle \chi_m w(l_1)\right\rangle + \sum_{m>1+(1+h)\nu}(m-1)\left\langle \chi_m w(l_1)\right\rangle \\ &\leq (1+h)+\sum_{m>1+(1+h)\nu}(m-1)\left\langle \chi_m w(l_1)\right\rangle \leq (1+h)+\left\langle P_2 w(l_1)\right\rangle \end{aligned} \tag{3.39}$$

with

$$P_2=\sum_{m>1+(1+h)\nu}(m-1)\chi_m \tag{3.40}$$

and we can estimate again with $h=\nu^{-1/2+\delta}$

$$\langle P_2\rangle \leq \sum_{m>1+(1+h)\nu}(m-1)^2 e^{-\nu}\frac{\nu^m}{m!} \leq \nu^2 \sum_{m>1+(1+h)\nu} e^{-\nu}\frac{\nu^m}{m!} \lesssim \nu^2\exp\left(-\frac{\nu^{2\delta}}{3}\right) \tag{3.41}$$

This proves the claim

$$|1-\langle N\rangle| \leq 1+\nu^{-1/2+\delta}+\mathcal{O}(\nu\exp\left(-\nu^{-2\delta}/6\right)) \tag{3.42}$$

Claim: $\langle N^2 \rangle \to 1$

First split N^2 into two sums

$$\left\langle N^2 \right\rangle = \underbrace{\sum_{m\geq 3}(m-1)(m-2)\left\langle \chi_m w(l_1)w(l_2)\right\rangle}_{=:A} + \underbrace{\sum_{m\geq 2}(m-1)\left\langle \chi_m w(l_1)^2\right\rangle}_{=:B} \tag{3.43}$$

Proving $A \to 1$ works similar to the above by spitting it into two sums.

$$A = \sum_{m\leq(1+h)\nu}(m-1)(m-2)\left\langle \chi_m w(l_1)w(l_2)\right\rangle + \underbrace{\sum_{m\geq(1+h)\nu}(m-1)(m-2)\left\langle \chi_m w(l_1)w(l_2)\right\rangle}_{=:\langle P_3 w(l_1)w(l_2)\rangle}$$

We apply the same estimates to the projectors as in the convergence $\langle N \rangle \to 1$ and get

$$\left\langle P_3^2 \right\rangle \leq \sum_{m\geq(1+h)\nu}(m-1)^2(m-2)^2\left\langle \chi_m\right\rangle \lesssim \nu^4 \exp\left(-\frac{\nu^{2\delta}}{3}\right) \tag{3.44}$$

For the first part of A we obtain

$$\sum_{m\leq(1+h)\nu}(m-1)(m-2)\left\langle \chi_m w(l_1)w(l_2)\right\rangle \leq (1+h)^2 \leq (1+\nu^{-1/2+\delta})^2 \tag{3.45}$$

Again, after dropping the exponential terms we are left with

$$A \leq (1+\nu^{-1/2+\delta})^2 \tag{3.46}$$

We learn that $A \leq 1$ in the considered limit, if we therefore prove that B vanishes we established the result. To show this we split B into two sums.

$$B = \underbrace{\sum_{m=2}^{1+2\nu}(m-1)\left\langle \chi_m w(l_1)^2\right\rangle}_{=:B_1} + \underbrace{\sum_{m>1+2\nu}(m-1)\left\langle \chi_m w(l_1)^2\right\rangle}_{=:B_2} \tag{3.47}$$

Using $m-1 < 2\nu$ we get

$$B_1 \leq 2\nu\left\langle w(l)^2\right\rangle \to 0 \tag{3.48}$$

For the second term we use Cauchy Schwarz

$$B_2 \leq \nu\sqrt{2}e^{-\nu/3}\left\langle w(l)^4\right\rangle^{1/2} \to 0 \tag{3.49}$$

If we sum up the bounds (3.46),(3.48) and (3.49) we proved the claim.

$$\langle N \rangle^2 \leq \left\langle N^2 \right\rangle \leq (1+\nu^{-1/2+\delta})^2 + 2\nu\left\langle w(l)^2\right\rangle + \text{ exp. decay} \tag{3.50}$$

This shows the L^2 convergence but we shall continue to use Lemma 3.6 to get an estimate for the convergence in probability.

Claim: $\mathrm{Var}\,(N) \to 0$

For big ν we get an estimate for the variance for N.

$$\begin{aligned}\mathrm{Var}\,(N) \leq \left\langle N^2 \right\rangle - \langle N \rangle^2 &\leq (1+\nu^{-1/2+\delta})^2 + 2\nu \left\langle w^2 \right\rangle - (1-\nu^{-1/2+\delta})^2 + \mathcal{O}(\nu^{-1+2\delta}) \\ &\leq 4\nu^{-1/2+\delta} + 2\nu \left\langle w^2 \right\rangle + \mathcal{O}(\nu^{-1+2\delta}) \quad (3.51)\end{aligned}$$

For the probabilities we can now prove using Lemma 3.6

$$P(|N-1| > \varepsilon) \leq \frac{1}{\varepsilon^2}\left(4\nu^{-1/2+\delta} + 2\nu \left\langle w^2 \right\rangle\right)\left(1 + \mathcal{O}(\nu^{-1/2+\delta}/\varepsilon)\right) \tag{3.52}$$

□

This theorem will be very useful for proving the upper bound for e^{GP}/e_0 but we shall also use it for example to show the convergences of the sum $\sum_{l_j \leq x_\nu} l_j$ for various $x_\nu > 0$. The next corollaries show possible applications of this theorem.

Corollary 3.11. *$0 \leq w(l) \leq cn^\mu(l)$ fulfills all properties of the Theorem 3.9. And for the bounds of probability we get*

$$P(|N-1| > \varepsilon) \leq \frac{1}{\varepsilon^2}\left(4\nu^{-1/2+\delta} + 32c^2\frac{\mu}{\gamma\nu}\right)\left(1 + \mathcal{O}(\nu^{-1/2+\delta}/\varepsilon)\right) \tag{3.53}$$

Proof. For estimating this we need a bound for $\left\langle w(l_1)^k \right\rangle$ which we proved in Lemma 2.8. □

Corollary 3.12. *Let X be a set and χ_X the characteristic function for it. Further we assume that $p_\nu(X) = \int_X \mathrm{d}p_\nu(l) \gg \nu^{-1}$. We get*

$$\frac{\sum_{j=1}^m \chi_X(l_j)}{\nu \int_X \mathrm{d}p_\nu(l)} \xrightarrow{p} 1 \tag{3.54}$$

The assumptions are in particular fulfilled for the interval $[\pi/\sqrt{\mu}, \infty)$.

Proof. We use Theorem 3.9 together with $w(l) = \nu^{-1}\chi_X(l)p_\nu(X)^{-1}$.

$$\nu \left\langle w(l)^2 \right\rangle = \frac{1}{\nu p_\nu(X)^2}\int_X \mathrm{d}p_\nu(l) = (\nu p_\nu(X))^{-1} \tag{3.55}$$

The condition (iii) of Theorem 3.9 is trivial to prove. For the last point we calculate

$$\nu p_\nu([\pi/\sqrt{\mu}, \infty)) = \nu \int_{\pi\nu/\sqrt{\mu}}^\infty \mathrm{d}p_\nu(l) = \nu e^{-\pi\nu/\sqrt{\mu}} \sim \frac{\nu\gamma}{\mu} \to \infty \tag{3.56}$$

which confirms the statement. □

Corollary 3.13. *Let $I = [0, b]$ be an interval with $b \gg \nu^{-2}$ and χ_I the characteristic function for it. We get*

$$\frac{\sum_{j=1}^{m} l_j \chi_I(l_j)}{\nu \int_I l \, \mathrm{d}p_\nu(l)} \xrightarrow{p} 1 \tag{3.57}$$

Analogously we get a convergence for $I = [b, \infty]$ with $b \lesssim \nu^{-1}$.

Proof. We use Theorem 3.9 together with $w(l) = \nu^{-1} l \chi_X(l) p_\nu(X)^{-1}$.

$$\nu \left\langle w(l)^2 \right\rangle = \frac{1}{\nu} \left(\int_0^b l \mathrm{d}p_\nu(l) \right)^{-2} \int_0^b l^2 \mathrm{d}p_\nu(l) = \frac{1}{\nu} \frac{e^{-b\nu}(-b\nu(b\nu+2)-2)+2}{\left(e^{-b\nu}(b\nu+1)-1\right)^2} \tag{3.58}$$

For the case that $b\nu > c$ the convergence $v \left\langle w(l)^2 \right\rangle \to 0$ is trivial. Otherwise we estimate

$$\frac{1}{\nu} \frac{e^{-b\nu}(-b\nu(b\nu+2)-2)+2}{\left(e^{-b\nu}(b\nu+1)-1\right)^2} \leq \frac{1}{\nu} \frac{2(1-e^{-b\nu})}{(e^{-b\nu}-1)^2} \to \frac{b\nu}{\nu} \frac{2}{(b\nu)^2} = \frac{2}{b\nu^2} \tag{3.59}$$

For a convergence of this expression to 0 we need $b \gg \nu^{-2}$. □

Remark 3.14. In the probability bound of Corollary 3.11 we deal with two terms. First we get $\nu^{-1/2+\delta}$ which we obtain from the convergence m/ν. Second the term $\mu/(\gamma\nu)$ which comes from the expectation value of the minimizer $\left\langle n^{\mathrm{IDF}}(l)^2 \right\rangle$.
We have seen in 2.18 that the constraint on γ of (1.24) is equivalent to $\mu/(\gamma\nu) \ll 1$. Hence this convergence can be arbitrary slow.
On the other side we get $\mu/(\gamma\nu) \sim \nu^{-1} \ll \nu^{-1/2}$ in the case $\gamma \gtrsim \nu^2$ which shows that both terms are necessary.

3.3 The upper bound

The obvious application of Corollary 3.11 is to apply it to $w(l) = n^\mu(l)$ which shows that

$$N^\mu = \sum_{j=1}^{m} n^\mu(l_j) \xrightarrow{p} 1 \tag{3.60}$$

In the next step we apply Corollary 3.11 to

$$\sum_j^m \frac{n^\mu(l_j)}{l_j^2} e(n^\mu(l_j) l_j \mu, \infty) \tag{3.61}$$

to get a convergence of the energy.

Lemma 3.15. *We get the convergence*

$$\frac{\sum_j^m n^\mu(l_j) e(n^\mu(l_j) l_j \gamma, \infty)/l_j^2}{e_0(\gamma, \nu)} \xrightarrow{L^2} 1 \tag{3.62}$$

Proof. This problem is very much related to the last theorem. If we choose

$$w(l) = \frac{n^\mu(l)}{l^2}\frac{e(n^\mu(l)l\gamma,\infty)}{e_0(\gamma,\nu)} \tag{3.63}$$

$w(l)$ fulfills all properties we need for Theorem 3.9. Clearly $\langle w\rangle = 1$ and we can estimate with help of

$$\frac{n^\mu(l)}{l^2}e'(n^\mu(l)l\gamma,\infty) + \frac{1}{l^2}e(n^\mu(l)l\gamma,\infty) = \mu \tag{3.64}$$

$$w(l) \le n^\mu(l)\frac{\mu}{e_0(\gamma,\nu)} \le \frac{5}{2}n^\mu(l) \tag{3.65}$$

With this estimate we can apply 3.9 to $w(l)$ and get the desired result. □

The last lemma is not the actual proof for the upper bound because the n_j do not fulfill the norm constraint $\sum_j n^\mu(l_j) = 1$. We need to show the convergence for $n^\zeta(l_j)$, and for this we need to discuss the convergence of ζ/μ. We shall learn that this convergence is implied by the convergence of $N^\mu \xrightarrow{p} 1$. Remember that ζ was the Lagrange multiplier corresponding to the energy $e^{\mathrm{GP},\infty}$ and μ corresponds to interval density functional $\mathcal{E}^{\mathrm{IDF}}$.

Lemma 3.16.

$$\frac{\zeta}{\mu} \xrightarrow{p} 1 \tag{3.66}$$

where ζ is such that $\sum_k n^\zeta(l_k) = 1$. Furthermore we get

$$\frac{\nu}{\sqrt{\mu}}\left|\frac{\zeta}{\mu} - 1\right| \xrightarrow{p} 1 \tag{3.67}$$

which is a stronger condition for $\gamma \ll \nu^2$. The probabilities can be estimated by

$$P\left(\left|\frac{\zeta}{\mu} - 1\right| > \varepsilon\right) \le \frac{\hat{c}}{\varepsilon}\left(\nu^{-1/2+\delta} + \frac{\mu}{\gamma\nu}\right)(1 + \mathcal{O}(\nu^{-1/2+\delta}/\varepsilon)) \tag{3.68}$$

$$P\left(\frac{\nu}{\sqrt{\mu}}\left|\frac{\zeta}{\mu} - 1\right| > \varepsilon\right) \le \frac{\hat{c}}{\varepsilon}\left(\nu^{-1/2+\delta} + \frac{\mu}{\gamma\nu}\right)(1 + \mathcal{O}(\nu^{-1/2+\delta}/\varepsilon)) \tag{3.69}$$

Proof. We shall use the convergence $|N^\mu - 1| \xrightarrow{p} 1$ we proved in Corollary 3.11 to show this result. Further we need that $\bar{n}(\mu)$ is C^1 in μ for $\mu \ge \pi^2$ which follows from the continuous differentiability of $\kappa \mapsto e'(\kappa,\infty)$ we discussed in Section 2.7, and the inverse function theorem used on (2.18).
In the same way we showed Lemma 3.15 we can argue that

$$\frac{\sum_j^m n^\zeta(l_j)}{\nu\int \mathrm{d}p_\nu(l)n^\zeta(l)} \xrightarrow{p} 1 \tag{3.70}$$

Taking the derivative of $\bar{N}^\eta = \nu \int \mathrm{d}p_\nu(l) n^\eta(l)$ we get

$$\begin{aligned}\partial_\eta \nu \int \mathrm{d}p_\nu(l) n^\eta(l) &= \nu \int \mathrm{d}p_\nu(l) \partial_\eta \left[\frac{1}{\gamma l}\bar{n}(\eta l^2)\right] = \nu \int \mathrm{d}p_\nu(l) \partial_l \left[\frac{1}{2\gamma\eta}\bar{n}(\eta l^2)\right] \\ &= \frac{\nu^2}{2\gamma}\int \mathrm{d}p_\nu(l)\left[\bar{n}(\eta l^2)\right] \sim \frac{\nu^2}{2\gamma\eta}\int_{\pi/\sqrt{\eta}}^\infty \mathrm{d}p_\nu(l)\left[\eta l^2 - \pi^2\right] \\ &= e^{-\pi\nu/\sqrt{\eta}}\frac{\left(\pi\sqrt{\eta}\nu + \eta\right)}{\eta\gamma} \sim N^\eta \frac{1}{\eta}\left(\frac{\nu}{\sqrt{\eta}} + 1\right) \end{aligned} \tag{3.71}$$

With this estimate we can calculate

$$|\bar{N}^\mu - \bar{N}^\zeta| \geq \int_{[\zeta,\mu]} (\partial_\xi \bar{N}^\xi)(\eta)\,\mathrm{d}\eta \geq \int_{[\zeta,\mu]} \mathrm{d}\eta \left(\bar{N}^\eta \frac{1}{\eta}\left(\frac{\nu}{\sqrt{\eta}} + 1\right)\right) \tag{3.72}$$

For $\zeta < \mu$ we get

$$|\bar{N}^\mu - \bar{N}^\zeta| \geq \int_{[\zeta,\mu]} \mathrm{d}\eta \left(\bar{N}^\eta \frac{1}{\eta}\left(\frac{\nu}{\sqrt{\eta}} + 1\right)\right) \geq \bar{N}^\zeta \frac{\nu}{\sqrt{\mu}}\left|\frac{\zeta}{\mu} - 1\right| + \bar{N}^\zeta \left|\frac{\zeta}{\mu} - 1\right| \tag{3.73}$$

Because $\bar{N}^\zeta \xrightarrow{p} 1$ this proves the claim. The case $\zeta > \mu$ works analogously. The bounds for the probability follow directly from (3.72) and Corollary 3.11.

□

From the convergence of ζ/μ we can easily conclude the convergence of $e^{\mathrm{GP},\infty}$ in the limit $\nu \to \infty$.

Theorem 3.17 (The upper bound)**.** *For the upper bound $e^{\mathrm{GP},\infty}$ of e^{GP} we get*

$$\frac{e^{\mathrm{GP},\infty}}{e_0(\gamma,\nu)} \xrightarrow{p} 1 \tag{3.74}$$

This proves the upper bound

$$P\left(\frac{e^{\mathrm{GP}}}{e_0(\gamma,\nu)} > 1 + \varepsilon\right) \leq \frac{\hat{c}}{\varepsilon}\left(\nu^{-1/2+\delta} + \frac{\mu}{\gamma\nu}\right)(1 + \mathcal{O}(\nu^{-1/2+\delta}/\varepsilon)) \tag{3.75}$$

for all $\varepsilon > 0$ and a constant $\hat{c} > 0$.

Proof. From Lemma 3.16 we know that $\zeta \xrightarrow{p} \mu$ in first order. Like in Lemma 3.15 we can prove the convergence

$$\frac{e^{\mathrm{GP},\infty}}{\nu \int \mathrm{d}p_\nu(l) n^\zeta e(n^\zeta(l) l\gamma, \infty)/l^2} \xrightarrow{p} 1 \tag{3.76}$$

This proves the upper bound, as long as the convergence $\zeta/\mu \xrightarrow{p} 1$ is implying the convergence of

$$\frac{\nu \int \mathrm{d}p_\nu(l) n^\zeta e(n^\zeta(l) l\gamma, \infty)/l^2}{e_0(\gamma,\nu)} \xrightarrow{p} 1 \tag{3.77}$$

Using again the definition $\bar{N}^\eta = \nu \int \mathrm{d}p_\nu(l) n^\eta(l)$ we calculate

$$\left|\nu \int \mathrm{d}p_\nu(l) \frac{n^\mu(l)}{l^2} e(n^\mu(l) l\gamma, \infty) - \nu \int \mathrm{d}p_\nu(l) \frac{n^\zeta(l)}{l^2} e(n^\zeta(l) l\gamma, \infty)\right|$$

$$= \int_{[\zeta,\mu]} \eta \mathrm{d}\eta (\partial_\xi \bar{N}^\xi(l))(\eta) \leq \int_{[\zeta,\mu]} \mathrm{d}\eta N^\eta \left(\frac{\nu}{\sqrt{\eta}} + 1\right) \quad (3.78)$$

using (3.71). For the case $\zeta < \mu$ we get

$$(3.78) \leq \frac{\bar{N}^\mu}{\mu} \left(\frac{\nu}{\sqrt{\mu}} + 1\right) \left|\frac{\zeta}{\mu} - 1\right| \quad (3.79)$$

This shows the convergence and an estimate for the probabilities can be proven from the last line and Corollary 3.11. The case $\mu < \zeta$ works analogously using $\bar{N}^\zeta \to 1$. □

For the lower bound, we shall also need to discuss the convergence of m.

Lemma 3.18. *We get for ε small enough*

$$P\left(\left|\frac{\nu}{m} - 1\right| > \varepsilon\right) \leq \frac{4}{\varepsilon^2} \frac{1}{\nu} \quad (3.80)$$

Proof. We start looking at m/ν

$$\left\langle \frac{m}{\nu} \right\rangle = 1 \qquad \left\langle \frac{m^2}{\nu^2} \right\rangle = \frac{\nu^2 + \nu}{\nu^2} = 1 + \frac{1}{\nu} \quad \Rightarrow \quad \mathrm{Var}\left(\frac{m}{\nu}\right) \leq \frac{1}{\nu} \quad (3.81)$$

and for the probabilities

$$P\left(\left|\frac{m}{\nu} - 1\right| > \varepsilon\right) \leq \frac{1}{\varepsilon^2} \frac{1}{\nu} \quad (3.82)$$

Now we use 3.7 and get for small ε

$$P\left(\left|\frac{\nu}{m} - 1\right| > \varepsilon\right) \leq \frac{4}{\varepsilon^2} \frac{1}{\nu} \quad (3.83)$$

□

3.4 The lower bound

For proving the lower bound we shall try to connect the lower bound $e^{\mathrm{GP},\sigma}$ with the energy upper bound $e^{\mathrm{GP},\infty}$ for which we already showed the convergence to e_0. We shall split the energy into the energy coming from the contribution of intervals with $l_j < \tilde{l}$ and the contribution of $l_j > \tilde{l}$. For showing bounds for the convergence in probability we separate the cases $\gamma \gg \nu^2$ and $\gamma \ll \nu^2$ in the next proof.

Theorem 3.19 (The lower bound)**.** *For γ, σ of the form* (1.24) *we get*

$$\frac{e^{\mathrm{GP},\sigma}}{e_0} \xrightarrow{p} 1 \tag{3.84}$$

For ε small enough we get the probability bound

$$P\left(\frac{e^{\mathrm{GP}}}{e_0} \leq 1-\varepsilon\right) \leq \frac{c}{\varepsilon^2}\left(\nu^{-1/2+\delta} + \frac{\mu}{\gamma\nu} + \frac{\mu^{2/3}}{\gamma^{2/3}}\frac{\nu^{1/3}}{\sigma^{1/3}}\right)\left(1 + \mathcal{O}\left(\frac{1}{\varepsilon}\frac{\mu}{\gamma\nu} + \frac{\nu^{-1/2+\delta}}{\varepsilon}\right)\right) \tag{3.85}$$

for $\gamma \gg \nu^2$, and for $\gamma \lesssim \nu^2$

$$P\left(\frac{e^{\mathrm{GP}}}{e_0} \leq 1-\varepsilon\right) \leq \frac{c}{\varepsilon^2}\left(\nu^{-1/2+\delta} + \frac{\mu}{\gamma\nu} + \frac{\mu}{\sigma^2}\right)\left(1 + \mathcal{O}\left(\frac{1}{\varepsilon}\frac{\mu}{\gamma\nu} + \frac{\nu^{-1/2+\delta}}{\varepsilon}\right)\right) \tag{3.86}$$

with c is an appropriate constant, and $\delta > 0$.

Proof. Using the definition of $g(\mu l_j^2, l_j\sigma)$ we have

$$e_\omega^{\mathrm{GP}}(\gamma,\sigma,\nu) \geq \inf_{\{n_j\}} \sum_{j=0}^{m} \frac{n_j}{l_j^2} e(n_j l_j \gamma, l_j\sigma) \geq \eta + \sum_{j=0}^{m} \frac{1}{\gamma l_j^3} g(\eta l_j^2, l_j\sigma) \tag{3.87}$$

This is valid for all $\eta > 0$, and we shall later choose η appropriately in a way such that $\mu \sim \eta$. From the definition of g we already get $g \leq 0$. To find a lower bound we use $e(n, l\sigma) \geq n/2 + e(0, l\sigma)$

$$g(\eta l^2, l\sigma) = \inf_n (ne(n,l\sigma) - n\eta l^2) \geq \inf_n \left(\frac{n^2}{2} + ne(0,l\sigma) - n\eta l^2\right) \tag{3.88}$$

Now we minimize the last expression and get

$$n_{min} = \eta l^2 - e(0, l\sigma) \tag{3.89}$$

Because the second derivative is greater 0, this is really a minimum in the case that $n_{min} \geq 0$. If $n_{min} < 0$ we find the minimum at $n_{min} = 0$. Together with the concavity of $e(0, l\sigma)$ we get

$$g(\eta l^2, l\sigma) \geq -\frac{1}{2}[\eta l^2 - e(0,l\sigma)]_+^2 \geq -\frac{1}{2}\left[\eta l^2 - \frac{\pi^2 l\sigma}{C + l\sigma}\right]_+^2 \geq -\frac{\eta^2 l^4}{2}\theta(l - \tilde{l}) \tag{3.90}$$

For an appropriate C and with $\tilde{l}$ such that

$$\eta\tilde{l}^2 = \pi^2\frac{\tilde{l}\sigma}{C + \tilde{l}\sigma} \;\Rightarrow\; \tilde{l}^2\sigma\eta + C\tilde{l}\eta - \pi^2\sigma = 0 \;\Rightarrow\; \tilde{l}^2 + \frac{C\tilde{l}}{\sigma} - \frac{\pi^2}{\eta} = 0 \tag{3.91}$$

We can solve this quadratic equation with the restriction $\tilde{l} > 0$ and get

Estimate for the first and last interval:

$$\tilde{l} = -\frac{C}{2\sigma} + \sqrt{\frac{C^2}{4\sigma^2} + \frac{\pi^2}{\eta}} \geq \frac{\pi}{\sqrt{\eta}} - \frac{C}{2\sigma} \tag{3.92}$$

For the first and last interval $j = 0$ and $j = m$ we use the estimate

$$\frac{1}{\gamma l_j^3} g(\eta l_j^2, l_j \sigma) \geq -\frac{\eta^2 l_j}{2\gamma} \tag{3.93}$$

Divided by $e_0(\gamma, \nu)$ and under the assumption that η is the minimizing parameter of e_0 the expectation value is

$$\left\langle \frac{\eta^2 l_j}{2\gamma e_0} \right\rangle \leq \frac{5}{2} \frac{\eta \langle l_j \rangle}{2\gamma} \leq \frac{5}{4} \frac{\eta}{\gamma\nu} \to 0 \tag{3.94}$$

where we used $\langle l_j \rangle \leq \nu^{-1}$. For the variance we first calculate

$$\left\langle l_j^2 \right\rangle - \langle l_j \rangle^2 = \frac{2}{\nu^2} - \frac{1}{\nu^2} = \frac{1}{\nu^2} \tag{3.95}$$

$$\left\langle \frac{\eta^4 \left\langle l_j^2 \right\rangle}{4\gamma^2 e_0^2} \right\rangle - \left\langle \frac{\eta^4 \langle l_j \rangle^2}{4\gamma^2 e_0^2} \right\rangle = \frac{25}{16} \frac{\eta^2}{\gamma^2 \nu^2} \to 0 \tag{3.96}$$

and with Lemma 3.6 for the probabilities in first order

$$P\left(\left| \frac{\eta^2 l_j}{2\gamma e_0} \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \frac{25}{16} \frac{\eta^2}{\gamma^2 \nu^2} \left(1 + \mathcal{O}\left(\frac{1}{\varepsilon} \frac{\eta}{\gamma\nu} \right) \right) \tag{3.97}$$

Contribution of small intervals

Now we take a constant $\alpha > 0$ and consider the contribution of all intervals with $l_j < (\sigma\alpha)^{-1}$ with $j \in [1, m-1]$. For $\tilde{l} \geq (\alpha\sigma)^{-1} \geq l_j$ we have no contribution. From the condition $\tilde{l} \leq (\alpha\sigma)^{-1}$ follows

$$\frac{\pi}{\sqrt{\eta}} - \frac{C}{2\sigma} \leq \frac{1}{\alpha\sigma} \;\Rightarrow\; \sigma \leq \left(\frac{1}{\alpha} + \frac{C}{2} \right) \frac{1}{\pi} \sqrt{\eta} \tag{3.98}$$

In the case $\gamma \ll \nu^2$ we know by our assumption on σ and $\eta \sim \mu$ that $\eta \ll \sigma^2$ which is a contradiction to the above statement. Therefore we only have a nonzero contribution for large ν if $\gamma \gtrsim \nu^2$. We shall restrict now to the latter cases.

$$\frac{1}{\gamma l_j^3} g(\eta l_j^2, l_j \sigma) \geq -\frac{\eta^2}{2\gamma\alpha\sigma} \geq -\frac{1}{\alpha} \underbrace{\frac{\eta}{\gamma}}_{\sim 1} \frac{\eta}{\nu} \frac{\nu}{\sigma} \tag{3.99}$$

By the definition of σ the ν/σ tends to 0 and because in this case $\gamma \sim \eta \sim e_0$ the first term is bounded. Because we have to sum over all intervals we get an additional factor m. We use Lemma 3.18 and get for all $\varepsilon > 0$

$$P\left(\frac{m}{\nu}\frac{\nu}{\sigma} \geq \varepsilon\right) \leq P\left(m/\nu > \varepsilon\frac{\sigma}{\nu}\right) \leq \frac{4}{\varepsilon^2}\frac{\nu^2}{\sigma\nu} \leq \frac{4}{\varepsilon^2}\frac{\nu}{\sigma} \tag{3.100}$$

For the case $\varepsilon\sigma/\nu > 1$. It is enough to demand $\varepsilon < 4$.

Estimate for large intervals:

Now we have everything we need for the contributions of small intervals. For the larger intervals we get

$$\begin{aligned} g(\eta l_j^2, l_j\sigma) &= \inf_n ne(n, l_j\sigma) - n\eta l_j^2 \geq \inf_n (1 - K(l_j\sigma)^{-1/2})ne(n,\infty) - n\eta l_j^2 \\ &\geq \inf_n (1 - K\alpha^{1/2})ne(n,\infty) - n\eta l_j^2 \geq (1 - K\alpha^{1/2})g(\eta l_j^2(1 - K\alpha^{1/2})^{-1}, \infty) \end{aligned} \tag{3.101}$$

Here we used Lemma 2.10. Together we get the lower bound

$$\frac{1}{\gamma l_j^3}g(\eta l_j^2, l_j\sigma) \geq (1 - K\alpha^{1/2})\frac{1}{\gamma l_j^3}g(\eta l_j^2(1 - K\alpha^{1/2})^{-1}, \infty) - \frac{\eta}{\gamma}\frac{\eta}{\nu\alpha}\frac{\nu}{\sigma} \tag{3.102}$$

We now choose η such that $\mu = \eta(1 - K\alpha^{1/2})^{-1}$ with μ being the Lagrange multiplier for the interval density functional. We can write $g(\mu l^2, \infty)$ now as

$$g(\mu l^2, \infty) = n(\mu l^2, \infty)e(n(\mu l^2, \infty), \infty) - \mu l^2 n(\mu l^2, \infty) \tag{3.103}$$

Convergence for $\gamma \gg \nu^2$:

Altogether we get

$$e^{\mathrm{GP}}(\gamma, \sigma, \nu) \geq (1 - K\alpha^{1/2})\left(\mu(1 - N^\mu) + \sum_{j=1}^{m-1}\frac{n_j}{l_j^2}e(n_j l_j\gamma, \infty)\right) - \eta\frac{1}{\alpha}\frac{\eta}{\gamma}\frac{m}{\nu}\frac{\sqrt{\nu}}{\sigma} - \frac{\mu^2(l_0 + l_m)}{2\gamma} \tag{3.104}$$

To show convergence in probability we set $\alpha = \varepsilon^2/(2K)^2$ and calculate

$$\begin{aligned} P\left(\frac{e^{\mathrm{GP}}(\gamma,\sigma,\nu)}{e_0(\gamma,\nu)} \leq 1 - \varepsilon\right) \leq P\Bigg(&\frac{\mu}{e_0}(1 - N) + \sum_{j=1}^{m-1}\frac{n_j}{l_j^2}e(n_j l_j\gamma, \infty)/e_0(\gamma,\nu) \\ &- \frac{5}{2}\frac{(2K)^2}{\varepsilon^2}\frac{\mu}{\gamma}\frac{m}{\nu}\frac{\nu}{\sigma} - \frac{\mu(l_0 + l_m)}{2\gamma} \leq 1 - \varepsilon/2\Bigg) \end{aligned} \tag{3.105}$$

We used $\varepsilon < 1$ and $\mu \geq \eta$. Further we used that $\mu/e_0 \leq 5/2$. We split this sum now into separate events where the second term will tend to 1 and the rest will vanish in probability.

$$
\begin{aligned}
(3.105) \leq & P\left(\left|\frac{5}{2}(1-N)\right| > \varepsilon/8\right) + P\left(\left|\sum_{j=1}^{m-1}\frac{n_j}{l_j^2}e(n_j l_j\gamma,\infty)/e_0(\gamma,\nu)-1\right| > \varepsilon/8\right) \\
& + P\left(\frac{5}{2}\frac{\mu}{\gamma}\frac{m}{\nu}\frac{\nu}{\sigma}\frac{(2K)^2}{\varepsilon^2} > \varepsilon/8\right) + 2P\left(\frac{\mu l_0}{\gamma} \geq \varepsilon/16\right) \\
\leq & \frac{c}{\varepsilon^2}\left(\nu^{-1/2+\delta} + \frac{\mu}{\gamma\nu} + \frac{1}{\varepsilon^4}\frac{\mu^2}{\gamma^2}\frac{\nu}{\sigma}\right)\left(1+\mathcal{O}\left(\frac{1}{\varepsilon}\frac{\mu}{\gamma\nu} + \frac{\nu^{-1/2+\delta}}{\varepsilon}\right)\right) \\
\leq & \frac{c}{\varepsilon^2}\left(\nu^{-1/2+\delta} + \frac{\mu}{\gamma\nu} + \frac{\mu^{2/3}}{\gamma^{2/3}}\frac{\nu^{1/3}}{\sigma^{1/3}}\right)\left(1+\mathcal{O}\left(\frac{1}{\varepsilon}\frac{\mu}{\gamma\nu} + \frac{\nu^{-1/2+\delta}}{\varepsilon}\right)\right)
\end{aligned}
$$

for an appropriate c. For convenience we estimated in the last step

$$\frac{1}{\varepsilon^6}\frac{\mu^2}{\gamma^2}\frac{\nu}{\sigma} \geq \frac{\mu^{2/3}}{\gamma^{2/3}}\frac{\nu^{1/3}}{\sigma^{1/3}} \tag{3.106}$$

and so we can keep an overall factor ε^{-2} in front of the probability bound.

Convergence for $\gamma \lesssim \nu^2$:

The case $\gamma \ll \nu^2$ works similarly with the exception that the contribution of small intervals vanish if we demand that

$$\frac{\sqrt{\mu}}{\sigma}\left(\frac{1}{\alpha}+\frac{C}{2}\right)\frac{1}{\pi} \leq 1 \tag{3.107}$$

For $\alpha < 2/C$ it is enough to add the term

$$\frac{\sqrt{\mu}}{\sigma}\frac{2}{\pi\alpha} \tag{3.108}$$

Hence we estimate the probability

$$P\left(\frac{e^{\mathrm{GP}}(\gamma,\sigma,\nu)}{e_0(\gamma,\nu)} \leq 1-\varepsilon\right) \leq \frac{c}{\varepsilon^2}\left(\nu^{-1/2+\delta} + \frac{\mu}{\gamma\nu} + \frac{\sqrt{\mu}}{\sigma}\right)\left(1+\mathcal{O}\left(\frac{1}{\varepsilon}\frac{\mu}{\gamma\nu} + \frac{\nu^{-1/2+\delta}}{\varepsilon}\right)\right) \tag{3.109}$$

□

Combining the upper and the lower bound we get

Theorem 3.20. *We get*

$$\frac{e^{\mathrm{GP}}(\gamma,\sigma,\nu)}{e_0(\gamma,\nu)} \xrightarrow{p} 1 \tag{3.110}$$

for limits of the form (1.24). *Further we get for ε small enough the probability bound*

$$P\left(\left|\frac{e^{\mathrm{GP}}}{e_0}-1\right| \geq \varepsilon\right) \leq \frac{c}{\varepsilon^2}\left(\nu^{-1/2+\delta} + \frac{\mu}{\gamma\nu} + \frac{\mu^{2/3}}{\gamma^{2/3}}\frac{\nu^{1/3}}{\sigma^{1/3}}\right)\left(1+\mathcal{O}\left(\frac{1}{\varepsilon}\frac{\mu}{\gamma\nu} + \frac{\nu^{-1/2+\delta}}{\varepsilon}\right)\right) \tag{3.111}$$

for $\gamma \gg \nu^2$*, and for* $\gamma \lesssim \nu^2$

$$P\left(\left|\frac{e^{\mathrm{GP}}}{e_0}-1\right| \geq \varepsilon\right) \leq \frac{c}{\varepsilon^2}\left(\nu^{-1/2+\delta}+\frac{\mu}{\gamma\nu}+\frac{\mu}{\sigma^2}\right)\left(1+\mathcal{O}\left(\frac{1}{\varepsilon}\frac{\mu}{\gamma\nu}+\frac{\nu^{-1/2+\delta}}{\varepsilon}\right)\right) \tag{3.112}$$

for a $c > 0$*, and an arbitrary* $\delta > 0$*.*

$\sqrt{\mu}/\sigma$ and ν/σ can both decay arbitrary fast under the constraint (1.24). This shows that the three terms appearing in the upper bounds of Theorem 3.20 are independent of each other and they are all needed.

Chapter 4

The GP density and superfluidity

In the following sections we shall establish the connection between the very characteristic form of the minimizer of e_0 and $e^{\mathrm{GP},\infty}$ which we called n^{IDF} and n_j^ζ respectively, and the GP density $|\phi^{\mathrm{GP}}|^2$. More precisely we are interested in the masses n_j^{GP} which we defined as

$$n_j^{\mathrm{GP}} = \int_{I_j} |\phi^{\mathrm{GP}}|^2 \tag{4.1}$$

We shall distinguish the cases of $\gamma \lesssim \nu^2$ and $\gamma \gg \nu^2$ which causes a concentration of the density in the former case and the convergence $|\phi^{\mathrm{GP}}|^2 \xrightarrow{L^1} 1$ in the latter case. Further we shall show results for the mean density which is defined as

Definition 4.1. *The mean density ρ_j^{GP} is defined as*

$$\rho_j^{\mathrm{GP}} = \frac{n_j^{\mathrm{GP}}}{l_j} \tag{4.2}$$

We shall be able to show that

$$\frac{1}{m} \sum_{j=1}^{m} g_\varepsilon(\rho_j^{\mathrm{GP}}) \xrightarrow{p} \begin{cases} 1 & \gamma \ll \nu^2 \\ 0 & \gamma \gg \nu^2 \end{cases} \tag{4.3}$$

with $g_\varepsilon(n) = 0$ for $n > \varepsilon$ and $g_\varepsilon(n) = 1$ otherwise.
For the proofs we shall find lower bounds which will violate the energy upper bound $e^{\mathrm{GP}} \sim \mu$ under certain conditions. In particular we use the result from Lemma 2.13 for the case $\gamma \gg \nu^2$.
In the case of $\gamma \ll \nu^2$ the main energy contribution comes from the potential term, and this is why the estimate

$$e(\kappa, \alpha) \geq e(0, \alpha) \tag{4.4}$$

will be sufficient. For this purpose we shall make a lot of use of Lemma 2.14 and in particular

$$e(0, \alpha) \geq \pi^2 \frac{\alpha}{c + \alpha} \tag{4.5}$$

As in the previous chapters we denote the the Lagrange multiplier of the interval density functional with μ and the Lagrange multiplier for $e^{\mathrm{GP},\infty}$ with ζ and further we use the constraint (1.24) for γ and σ. A lot of the inequalities we prove in this chapter are only valid in a probability sense and therefore it is helpful to define relations which are correct in the probability limit.

Definition 4.2. *We define for two random variables X_ν, Y_ν the relation*

$$X_\nu \overset{p}{\leq} Y_\nu \iff P(X_\nu > Y_\nu) \to 0 \quad \text{for } \nu \to \infty \tag{4.6}$$

in the same sense we shall use

$$X_\nu \overset{p}{\lesssim} Y_\nu \iff P(X_\nu \gg Y_\nu) \to 0 \quad \text{for } \nu \to \infty \tag{4.7}$$

Analogously we define $\overset{p}{\geq}, \overset{p}{\gtrsim}$

4.1 The case $\gamma \ll \nu^2$

For the interval density functional minimizer we know that

$$\sum_{l_j < \pi/\sqrt{\mu}} n^\mu(l_j) = 0 \tag{4.8}$$

and hence we want to show that we can prove a similar result for the GP density in the limit $\nu \to \infty$. For all proofs in this section we shall use a violation of the energy bound

$$e^{\mathrm{GP}}(\gamma, \sigma, \nu) \sim \mu \sim \frac{\nu^2}{(1 + \log(\nu^2/\gamma))^2} \tag{4.9}$$

Because we restrict to the case $\gamma \ll \nu^2$ we can use the relations

$$\nu^2 \gg \mu, \quad \sigma \gg \sqrt{\mu} \tag{4.10}$$

We start by showing a simpler statement where we only sum up the masses with with length smaller than δ/ν for an arbitrary $\delta > 0$.

Lemma 4.3. *For all $k > 0$ we get*

$$\sum_{l_j < k/\nu} n_j^{\mathrm{GP}} \xrightarrow{p} 0 \tag{4.11}$$

for $\gamma \ll \nu^2$.

Recall $g_\varepsilon(n) = 0$ if $n > \varepsilon$ and 1 otherwise. With this statement we are able to conclude that nearly all intervals are empty. This result will turn out to be fairly easy to prove because we limited our analysis to the case $\gamma \ll \nu^2$, which allows us to find a contradiction to the energy bound by only considering the energy of small intervals.

Lemma 4.4. *For $\gamma \ll \nu^2$ we get*

$$\frac{1}{m}\sum_j g_\varepsilon(mn_j^{\mathrm{GP}}) \xrightarrow{p} 1 \tag{4.12}$$

Afterwards we discuss a more sophisticated bound for the sum of small intervals which also covers the case $\gamma \sim \nu^2$.

Theorem 4.5. *For the case $\sigma \gg \sqrt{\mu}$ we get the inequality*

$$N_s := \sum_{l_j < s/\sqrt{\mu}} n_j^{\mathrm{GP}} \overset{p}{\leq} C\sqrt{\frac{\sqrt{\mu}}{\sigma}} \tag{4.13}$$

for an s dependent constant C. Hence $N_s \xrightarrow{p} 0$ in the case $\gamma \lesssim \nu^2$.

In contrast to the proof of Lemma 4.3, we have to consider carefully the energy contribution of large intervals and for small intervals. Even though Lemma 4.3 would be good enough to show that $n_{\mathrm{SF}} \to 0$ for $\gamma \gg \nu^2$ we actually need 4.5 to conclude this for the case $\gamma \sim \nu^2$. A very similar argument to the one in Lemma 4.3 leads to a statement for the mean density.

Theorem 4.8. *The fraction of intervals with $\rho_j^{\mathrm{GP}} < \delta$ tend to 1 for all $\delta > 0$*

$$\frac{1}{m}\sum_j^m g_\delta(\rho_j^{\mathrm{GP}}) \xrightarrow{p} 1 \tag{4.14}$$

4.1.1 Bounds for the mass in small intervals

We start discussing the case $\gamma \ll \nu^2$ for which it will be sufficient to ignore the interaction energy completely.

Lemma 4.3. *For all $k > 0$ we get*

$$\sum_{l_j < k/\nu} n_j^{\mathrm{GP}} \xrightarrow{p} 0 \tag{4.15}$$

In probability we can also give a bound to this convergence

$$\sum_{l_j < k/\nu} n_j^{\mathrm{GP}} \overset{p}{\ll} \frac{\mu}{\nu^2 + \nu\sigma} \tag{4.16}$$

Proof. Let us assume that $l_j < k/\nu$ for a $k > 0$, then we get the lower estimate

$$\nu \sum_j \frac{n_j}{l_j^2} e(n_j l_j \gamma, l_j \sigma) \geq \sum_{l_j < k/\nu} n_j \frac{1}{l_j} \frac{\sigma}{c + \sigma l_j} \geq \underbrace{\pi^2 \frac{\nu\sigma}{kc + k^2\sigma/\nu}}_{=:A(\sigma,\nu)} \sum_{l_j < k/\nu} n_j \tag{4.17}$$

We now analyze A in the limit. For this we distinguish $\nu/\sigma \to \infty$ and $\nu/\sigma < C$. In the first case we get $A(\sigma, \nu) \sim \nu^2$ because

$$A(\sigma, \nu) = \pi^2 \nu^2 \frac{\sigma/\nu}{kc + k^2\sigma/\nu} \sim \nu^2 \gg \mu \tag{4.18}$$

In the second case we get $A(\sigma, \nu) \sim \sigma\nu \gg \mu$. We proved in Chapter 3 that in probability the GP energy is of the same order as μ. If we assume that there is a non vanishing mass for small particles $l_j < k/\nu$, this would lead to a contradiction because

$$\sum_{l_j < k/\nu} n_j^{\mathrm{GP}} > \varepsilon \ \Rightarrow \ \mu \sim e^{\mathrm{GP}} > c \min(\nu^2, \nu\sigma) \gg \mu \tag{4.19}$$

To validate the probability inequality we estimate

$$P\left(\sum_{l_j < k/\nu} n_j^{\mathrm{GP}} > \varepsilon\right) \leq P\left(\frac{e^{\mathrm{GP}}}{\mu} > c\nu \min(\sigma, \nu) \frac{\varepsilon}{\mu}\right) \tag{4.20}$$

for a $c > 0$. We know that $e^{\mathrm{GP}}/\mu \xrightarrow{p} 1$ which allows us to conclude the desired bound. □

Using this theorem we can now easily conclude that nearly all intervals are empty

Lemma 4.4. *For $\gamma \ll \nu^2$ we get*

$$\frac{1}{m} \sum_j g_\varepsilon(m n_j^{\mathrm{GP}}) \xrightarrow{p} 1 \tag{4.21}$$

Proof. Let us calculate the number of intervals which are in the range $l_j < k/\nu$.

$$\frac{1}{\nu} \sum_{l_j < k/\nu} 1 \xrightarrow{p} \int_0^{k/\nu} \mathrm{d}p_\nu(l) = 1 - e^{-k} \tag{4.22}$$

We used Corollary 3.12 for this convergence and we can choose k arbitrarily and moreover $1 - e^{-k} \to 1$ for $k \to \infty$. Let α denote the number of intervals with $n_j > \varepsilon/m$ and $l_j < k/\nu$ the we can find an upper estimate for α/ν

$$\alpha \frac{\varepsilon}{\nu} \leq \sum_{l_j < k/\nu} n_j^{\mathrm{GP}} \to 0 \tag{4.23}$$

The number of $n_j < \varepsilon/\nu$ can be estimate by

$$\frac{1}{m}\sum_j g_\varepsilon(\nu n_j^{\mathrm{GP}}) \geq \frac{\nu}{m}\left(1 - e^{-k} - \frac{\alpha}{\nu}\right) \geq \frac{\nu}{m}\left(1 - e^{-k} - \frac{1}{\varepsilon}\sum_{l_j<k/\nu} n_j^{\mathrm{GP}}\right) \tag{4.24}$$

Because k was arbitrary and $\nu/m \xrightarrow{p} 1$ was proven in 3.18 we showed

$$\frac{1}{m}\sum_j g_\varepsilon(\nu n_j^{\mathrm{GP}}) \xrightarrow{p} 1 \tag{4.25}$$

Due to ε was arbitrary we can replace ν with m and so we proved the theorem. □

This shows already that the mass in small intervals vanishes, but we want to extend the result to the case $\gamma \sim \nu^2$. Furthermore we shall extend the range of the sum of Lemma 4.3 to $l_j < \pi/\sqrt{\mu}$ in the next theorem. This is exactly the range where the interval density functional minimizer vanishes. The proof of Theorem 4.5 is based on a calculation of Yngvason and Seiringer [30].
It is important to point out that this proof also holds for the case $\gamma \sim \nu^2$ and in fact we shall apply it also for the case $\gamma \gg \nu^2$ and $\sigma \gg \sqrt{\gamma}$. In comparison to the proof of Lemma 4.3 we have to keep track of the energy contribution of large intervals.

Theorem 4.5. *For the case $\sigma \gg \sqrt{\mu}$ we get the inequality*

$$N_s := \sum_{l_j<s/\sqrt{\mu}} n_j^{\mathrm{GP}} \overset{p}{\leq} C\sqrt{\frac{\sqrt{\mu}}{\sigma}} \tag{4.26}$$

for an s dependent constant C. Hence $N_s \xrightarrow{p} 0$ in the case $\gamma \lesssim \nu^2$.

Proof. We shall proceed similarly to the proof of Lemma 4.3 but we cannot neglect the energy contribution of intervals with large lengths. We define $\tilde{l} = s/\sqrt{\mu}$ and for short we denote $h = \sigma s/\sqrt{\mu}$. As in Lemma 4.3 we get the lower bound

$$\begin{aligned} e^{\mathrm{GP}}(\gamma,\sigma,\nu) &= \sum_{l_j\geq\tilde{l}} \frac{n_j^{\mathrm{GP}}}{l_j^2} e(n_j^{\mathrm{GP}} l_j\gamma, l_j\sigma) + \sum_{l_j<\tilde{l}} \frac{n_j^{\mathrm{GP}}}{l_j^2}\pi^2\frac{\sigma l_j}{c+\sigma l_j} \\ &\geq \sum_{l_j\geq\tilde{l}} \frac{n_j^{\mathrm{GP}}}{l_j^2} e(n_j^{\mathrm{GP}} l_j\gamma, l_j\sigma) + \frac{\pi^2}{s^2}\mu\frac{h}{c+h}\sum_{l_j<\tilde{l}} n_j^{\mathrm{GP}} \end{aligned} \tag{4.27}$$

We define $N_s = \sum_{l_j<\tilde{l}} n_j^{\mathrm{GP}}$ and obtain

$$\begin{aligned} \sum_{l_j>\tilde{l}}^{m} \frac{n_j^{\mathrm{GP}}}{l_j^2} e(n_j^{\mathrm{GP}} l_j\gamma, l_j\sigma) &\geq \inf_{\sum n_j = 1-N_s} \sum_{j=1}^{m} \frac{n_j}{l_j^2} e(n_j l_j\gamma, l_j\sigma) \\ &\geq (1 - Ch^{-1/2}) \inf_{\sum n_j=1-N_s} \sum_{j=1}^{m} \frac{n_j}{l_j^2} e(n_j l_j \gamma, \infty) \end{aligned} \tag{4.28}$$

It is easy to see that this estimate is convex in N_s and therefore we obtain a lower bound if we calculate the linear approximation at $N_s = 0$. For this we rewrite $n_j \to (1 - N_s)n_j$ to remove the N_s from the infimum. Further, ζ denotes the Lagrange multiplier which corresponds to the energy $e^{\mathrm{GP},\infty}$ with the normalization constraint $\sum_j n_j = 1$.

$$\begin{aligned}\partial_{N_s}\big|_{N_s=0} &\inf_{\sum n_j=1} \sum_{j=1}^{m} \frac{n_j(1-N_s)}{l_j^2} e(n_j(1-N_s)l_j\gamma, \infty) \\ &= -\sum_{j=1}^{m} \frac{n^\zeta(l_j)}{l_j^2} e(n^\zeta(l_j)l_j\gamma,\infty) - \sum_{j=1}^{m} \gamma \frac{n^\zeta(l_j)^2}{l_j} e'(n^\zeta(l_j)l_j\gamma,\infty) \end{aligned} \tag{4.29}$$

Now we use $e(n^\zeta(l_j)l_j\gamma,\infty)l_j^{-2} + \gamma n^\zeta(l_j)e'(n^\zeta(l_j)l_j\gamma,\infty)l_j^{-1} = \zeta$.

$$-\sum_{j=1}^{m} \left(\frac{n^\zeta(l_j)}{l_j^2} e(n^\zeta(l_j)l_j\gamma,\infty) + \gamma \frac{n^\zeta(l_j)^2}{l_j} e'(n^\zeta(l_j)l_j\gamma,\infty) \right) = -\zeta \tag{4.30}$$

Lemma 3.16 showed that $\zeta/\mu \xrightarrow{p} 1$. Dividing (4.27) by μ and using inequality (4.30) we obtain

$$\frac{e^{\mathrm{GP}}(\gamma,\sigma,\nu)}{\mu} \geq \left(\frac{e^{\mathrm{GP},\infty}(\gamma,\nu)}{\mu} - N_s \frac{\zeta}{\mu} \right) \left(1 - Ch^{-1/2}\right) + \frac{\pi^2}{s^2} N_s \frac{h}{c+h} \tag{4.31}$$

Using $e^{\mathrm{GP}} < e^{\mathrm{GP},\infty}$ and $-\left(1 - Ch^{-1/2}\right) \geq -1$ we get

$$\underbrace{\frac{e^{\mathrm{GP},\infty}(\gamma,\nu)}{\mu} \left(Ch^{-1/2}\right)}_{\to 0} \geq N_s \underbrace{\left(\frac{\pi^2}{s^2} \frac{h}{c+h} - \frac{\zeta}{\mu} \right)}_{\to \pi^2/s^2 - 1 > 0} \tag{4.32}$$

This proves the convergence $N_s \xrightarrow{p} 0$. To find an upper bound for N_s we use $e^{\mathrm{GP},\infty} < \zeta$ and we assume $h/(c+h) > s/\pi \;\Rightarrow\; h > sc/(\pi - s)$. In the case that $\mu/\zeta < \sqrt{s/\pi}$ we obtain

$$Ch^{-1/2} \geq N_s \left(\sqrt{\frac{s}{\pi}} - 1 \right) \;\Rightarrow\; N_s \leq \frac{Ch^{-1/2}}{\sqrt{s/\pi} - 1} \tag{4.33}$$

Hence we can determine the probability that (4.33) holds

$$P\left(N_s > \frac{Ch^{-1/2}}{\sqrt{s/\pi} - 1} \right) \leq P(\mu/\zeta < \sqrt{s/\pi}) \to 0 \tag{4.34}$$

In the case that $\gamma \lesssim \nu$ we get from (1.24) that $\sigma \gg \sqrt{\mu}$ which proves the theorem. $\square$

4.1.2 The case $\gamma \sim \nu^2$

Theorem 4.5 also covers the case where $\gamma \sim \nu^2$. For simplicity we shall assume that $k^2 := \lim \nu^2/\gamma$ exists but a similar analysis can be carried out using $\limsup$ and $\liminf$.

Using (2.36) we get the relation

$$\mu \geq m e^{-\pi\nu/\sqrt{\mu}} \geq \frac{\gamma}{2} \tag{4.35}$$

and for the upper bound

$$\frac{3}{2}\gamma \geq \mu e^{-\pi\nu/\sqrt{\mu}} \overset{(4.35)}{\geq} \mu e^{-\pi\sqrt{2}\nu/\sqrt{\gamma}} \Rightarrow \mu \leq \frac{3}{2}\gamma e^{\pi\sqrt{2}\nu/\sqrt{\gamma}} \tag{4.36}$$

Applying this to Theorem 4.5 we get

Corollary 4.6. *Let $\nu/\sqrt{\gamma} \to k > 0$ then*

$$\sum_{l_j < \delta p_k/\nu} n_j^{\mathrm{GP}} \xrightarrow{p} 0 \tag{4.37}$$

for $\delta \in [0, \pi)$ and

$$p_k = \sqrt{\frac{2}{3}}\delta k e^{-\pi k/\sqrt{2}}. \tag{4.38}$$

Proof. We get the estimate

$$\frac{\delta\nu}{\sqrt{\mu}} \geq \sqrt{\frac{2}{3}}\delta\frac{\nu}{\sqrt{\gamma}}e^{-\pi\nu/\sqrt{2\gamma}} \to \sqrt{\frac{2}{3}}\delta k e^{-\pi k/\sqrt{2}} \tag{4.39}$$

□

Using corollary 3.12 for $[0, x/\nu]$ we get that

$$\sum_{l_j < x/\nu} 1 \xrightarrow{p} \int_0^{x/\nu} \mathrm{d}p_\nu(l) = 1 - e^{-x} \tag{4.40}$$

If we choose $x = \delta p_k$ we see that the mass of a finite fraction of intervals vanishes in the limit.

4.1.3 Bounds for the mean density

The result that n_j tends to 0 for most of the intervals is an important result but it does not state that the ground state wave function ϕ^{GP} is small for nearly all intervals. For a fixed ν the mass n_j^{GP} can be arbitrary large even though $|\phi^{\mathrm{GP}}|_{I_j}$ is bounded. To prove that nearly all intervals have a small wave function we define the mean density.

Definition 4.7. *The average GP density in the interval j is defined as*

$$\rho_j^{\mathrm{GP}} = n_j^{\mathrm{GP}}/l_j \tag{4.41}$$

We show that $\rho_j^{\mathrm{GP}} > \delta$ for a finite fraction of small intervals is a contradiction to the upper energy bound μ. Rewriting the lower estimate $e^{\mathrm{GP},\sigma}$ to e^{GP} in terms of ρ_j^{GP} we get

$$e^{\mathrm{GP}}(\gamma,\sigma,\nu) \geq \sum_j \frac{\rho_j^{\mathrm{GP}}}{l_j} e(\rho_j^{\mathrm{GP}} l_j^2 \gamma, l_j \sigma) \tag{4.42}$$

This illustrates well, that for a fixed ρ_j^{GP} we get for the interaction energy $\rho_j^{\mathrm{GP}} l_j^2 \gamma \sim \nu^{-2}\gamma$, which furthermore explains why it is useful to distinguish between the cases $\gamma \gg \nu^2, \gamma \ll \nu^2$ and $\gamma \sim \nu^2$.

Theorem 4.8. *The fraction of intervals with $\rho_j^{\mathrm{GP}} < \delta$ tend to 1 for all $\delta > 0$*

$$\frac{1}{m}\sum_j^m g_\delta(\rho_j^{\mathrm{GP}}) \xrightarrow{p} 1 \tag{4.43}$$

Proof. For a single interval with $\rho_j^{\mathrm{GP}} > \delta$ we get the estimate

$$\frac{\rho_j^{\mathrm{GP}}}{l_j} e(\rho_j^{\mathrm{GP}} l_j^2 \gamma, l_j\sigma) \geq \frac{\delta}{l_j} e(0, l_j\sigma) \geq \frac{\rho_j^{\mathrm{GP}}}{l_j} \pi^2 \frac{l_j\sigma}{c + l_j\sigma} \sim \frac{\sigma}{c + l_j\sigma} \tag{4.44}$$

If we further restrict $l_j \ll \sigma^{-1}$ then

$$\frac{\sigma}{c + l_j\sigma} \sim \sigma \tag{4.45}$$

and for the case $l_j \gg \sigma^{-1}$ we get

$$\frac{\sigma}{c + l_j\sigma} \sim \frac{1}{l_j} \tag{4.46}$$

Let us now assume that for a fraction of α intervals with $l_j < k/\nu$ with $k > 0$ arbitrary we have $\rho_j^{\mathrm{GP}} \geq \delta$. This is a contradiction because

$$\alpha m \min\left(\frac{1}{l_j}, \sigma\right) \geq \alpha m \min\left(\frac{\nu}{k}, \sigma\right) \gg \mu \tag{4.47}$$

which violates $\mu \sim e^{\mathrm{GP}}$ and hence we proved that nearly all intervals with $l_j < k/\nu$ fulfill $\rho_j^{\mathrm{GP}} < \delta$. The fraction of intervals with $l_j > k/\nu$ converges to 0 if $k \to \infty$ and because k was arbitrary we proved the statement for all $\delta > 0$. □

4.2 Absence of superfluidity for $\gamma \lesssim \nu^2$

We shall see that there is no superfluidity in the limit $\gamma \lesssim \nu^2$ because of the randomness of the potential. In particular we showed that the masses are concentrated in large intervals with $l_j \gtrsim \nu^{-1}$ and we can find a set with measure of order 1, in which the GP density vanishes in the limit $\nu \to \infty$. This follows simply from Theorem 4.5 and Corollary 3.13.

$$\sum_{l_j < 1/\sqrt{\mu}} l_j \xrightarrow{p} \int_0^{1/\sqrt{\mu}} l \, \mathrm{d}p_\nu(l) = 1 - e^{-\nu/\sqrt{\mu}}(1 + \nu/\sqrt{\mu}) \sim 1 \tag{4.48}$$

where we needed the results from Section 4.1.2 to show that the total length is bounded from below for $\gamma \sim \nu^2$. Using estimate (1.41) we get an upper bound for n_{SF}.

$$n_{\mathrm{SF}} \leq \int_0^1 |g'(z)|^2 |\phi^{\mathrm{GP}}(z)|^2 \, \mathrm{d}z \tag{4.49}$$

for a g with $g(0) = 0$ and $g(1) = 1$. Let X be the set

$$X = \bigcup_{l_j < 1/\sqrt{\mu}} I_j \tag{4.50}$$

then we define

$$g(l) = \mu(X)^{-1} \int_{X \cap [0,l]} \mathrm{d}x \tag{4.51}$$

This g fulfills the required boundary conditions and it is partially linear on all I_j. In particular we increase it whenever we are in an interval which belongs to the set of intervals with small lengths. Plugging it into the inequality 4.49.

$$n_{\mathrm{SF}} \leq \int_X \mu(X)^{-2} |\phi^{\mathrm{GP}}|^2 = \frac{1}{\mu(X)^2} \sum_{l_j < 1/\sqrt{\mu}} n_j^{\mathrm{GP}} \xrightarrow{p} 0 \tag{4.52}$$

We used that $\mu(X) = \sum_{l_j < 1/\sqrt{\mu}} l_j \sim 1$ and Theorem 4.5. In the case $\gamma \sim \nu^2$ we showed in Section 4.1.2, that there is still a finite fraction of intervals with vanishing density and therefore $\mu(X) \xrightarrow{p} c > 0$.

Theorem 4.9. *For $\gamma \lesssim \nu^2$ the superfluity fraction n_{SF} converges to 0 in probability.*

4.3 The case $\gamma \gg \nu^2$

Because we know the first order growth of the energy e^{GP} precisely from Lemma 2.13 and Theorem 3.20 we can deal with this case in a different way than with the case $\gamma \lesssim \nu^2$. Combining the results we get

$$\frac{2}{\gamma} e^{\mathrm{GP}}(\gamma, \sigma, \nu) \xrightarrow{p} 1 \tag{4.53}$$

This is an advantage to the case $\gamma \ll \nu^2$ where we only knew $e^{\mathrm{GP}} \sim \mu$. One of the main goals of this section is to show the convergence of $|\phi^{\mathrm{GP}}|^2 \xrightarrow{L^1} 1$ in Lemma 4.14. Furthermore we shall discuss a bound for the convergence of the masses n_j^{GP}.

Theorem 4.11. *For $\gamma \gg \nu^2$ we get the convergence*

$$\sum_{j=1}^m |n_j^{\mathrm{GP}} - l_j| \overset{p}{\leq} c \sqrt{\frac{\nu}{\sqrt{\gamma}}} \tag{4.54}$$

for $\nu/\sqrt{\gamma} \ll 1$ and a appropriate constant c.

4.3.1 Convergence of the masses

We shall proceed similarly to Lemma 2.13 where we showed the limit of $e_0(\gamma,\nu)$.

$$\sum_j \frac{n_j}{l_j^2} e(n_j l_j \gamma, \infty) \geq \sum_j \frac{n_j}{l_j^2} \mathcal{E}_{n_j l_j \gamma, l_j \sigma}[\phi] \geq \sum_j \frac{n_j}{l_j^2} e(n_j l_j \gamma, l_j \sigma)$$
$$\geq \sum_j \frac{n_j}{l_j^2} e(n_j l_j \gamma, 0) = \sum_j \frac{n_j^2 l_j \gamma}{2 l_j^2} = \frac{\gamma}{2} \sum_j \frac{n_j^2}{l_j} \quad (4.55)$$

for ϕ minimizing $\mathcal{E}^{\mathrm{GP}}[\phi]$ under the assumption $n_j = \int_{x_j}^{x_{j+1}} \phi(x)$ for arbitrary masses $(n_j)_j$. We discussed the first two inequalities before using $\phi = \phi^{\mathrm{GP}}$ but they are in fact true for all possible choices of n_j. The lowest bound totally ignores the potential and we can easily determine the minimum $n_j = l_j$. For the energy at this minimizer we obtain

$$\frac{\gamma}{2} \sum_j \frac{l_j^2}{l_j} = \frac{\gamma}{2} \sum_j l_j = \frac{\gamma}{2} \quad (4.56)$$

Similarly to Lemma 2.13 we show the result for $e^{\mathrm{GP},\infty}$.

Lemma 4.10. *For $\sqrt{\gamma}/\nu \gg 1$ we get in probability*

$$e^{\mathrm{GP},\infty}(\gamma,\sigma,\nu) \overset{p}{\leq} \frac{\gamma}{2}\left(1 + \frac{\nu}{\sqrt{\gamma}}\right) \quad (4.57)$$

Proof. Using the trial function $h_j = c[\ell_j - 4/\sqrt{\gamma}]_+$.

$$1 = \sum_{\ell_j > 4/\sqrt{\gamma}} h_j \geq c - 4c\frac{m}{\sqrt{\gamma}} \;\Rightarrow\; c \to 1$$

Therefore we get the estimate

$$1 \leq c \leq \left(1 - 4\frac{m}{\sqrt{\gamma}}\right)^{-1} \quad (4.58)$$

Using now Lemma 2.2 we get an upper estimate

$$e^{\mathrm{GP},\infty} \leq \sum_{\ell_j > 4/\sqrt{\gamma}} \frac{c}{\ell_j} e(c\ell_j^2 \gamma, \infty) \leq c \sum_j c\ell_k \gamma + 11\sqrt{c\gamma}$$
$$\leq c^2 \frac{\gamma}{2}(1 + 11m/\sqrt{\gamma}) \leq \frac{\gamma}{2}\left(1 + \frac{19m}{(\sqrt{\gamma} - 4m)^2}\right) \quad (4.59)$$

For $\sqrt{\gamma}/\nu \gg 1$ we get in probability

$$e^{\mathrm{GP},\infty} \overset{p}{\leq} \frac{\gamma}{2} + \frac{\gamma}{2} k \frac{\nu}{\sqrt{\gamma}} \tag{4.60}$$

with k being an appropriate constant. □

In the following it will be helpful to work with the finite sequences $(n_j^{\mathrm{GP}})_j$ and $(l_j)_j$. For this we set $l_j = 0$ and $n_j^{\mathrm{GP}} = 0$ for $j > m$. Let us now assume that the n_j^{GP} are not converging to l_j in an ℓ^1 sense which means

$$\sum_{j=1}^{m} |n_j^{\mathrm{GP}} - l_j| > \varepsilon \tag{4.61}$$

for all ν. Then for every $\kappa \in \ell^1$, n_j and $h \in \mathbb{R}^+$ with $\int \kappa = 0$ we get for the lower bound

$$e^{\mathrm{GP},\infty} \geq \frac{\gamma}{2} \sum_j \frac{l_j + \kappa_j}{l_j^2} e((l_j + h\kappa_j) l_j \gamma, 0) = \frac{\gamma}{2} \sum_j \frac{(l_j + h\kappa_j)^2}{l_k} \geq \frac{\gamma}{2} \underbrace{\sum_j l_k}_{=1} + 2 \underbrace{\sum_j \kappa_j}_{=0} + \sum_j \frac{\kappa_j^2}{l_j} \tag{4.62}$$

using

$$\sum_j |\kappa_i| = \sum_j \sqrt{l_j} \frac{|\kappa_j|}{\sqrt{l_j}} \leq \left(\sum_j \frac{\kappa_j^2}{l_j} \right)^{1/2} \left(\sum_j l_j \right)^{1/2} = \left(\sum_j \frac{\kappa_j^2}{l_j} \right)^{1/2} \tag{4.63}$$

we can now estimate the energies with help of (4.60) and (4.63)

$$\frac{\gamma}{2} + \frac{\gamma}{2} k \frac{\nu}{\sqrt{\gamma}} \geq e^{\mathrm{GP},\infty} \geq \frac{\gamma}{2} + \frac{\gamma}{2} \left(\sum_j |\kappa_j| \right)^2 \tag{4.64}$$

From this we conclude

$$k \frac{\nu}{\sqrt{\gamma}} \overset{p}{\geq} \left(\sum_j |\kappa_j| \right)^2 \tag{4.65}$$

This gives an upper bound to the convergence of $(n_j^{\mathrm{GP}})_j \to (l_j)_j$ in the sense

$$\sum_{j=1}^{m} |n_j^{\mathrm{GP}} - l_j| \overset{p}{\leq} \sqrt{k} \sqrt{\frac{\nu}{\sqrt{\gamma}}} \tag{4.66}$$

for $\nu / \sqrt{\gamma} \ll 1$.

Theorem 4.11. *For $\gamma \gg \nu^2$ we get the convergence*

$$\sum_{j=1}^{m} |n_j^{\mathrm{GP}} - l_j| \overset{p}{\leq} c \sqrt{\frac{\nu}{\sqrt{\gamma}}} \tag{4.67}$$

for $\nu / \sqrt{\gamma} \ll 1$ and a appropriate constant c.

If we compare the regime $\gamma \ll \nu^2$ to the case $\gamma \gg \nu^2$, we expect in the latter case, that nearly all intervals have a non vanishing mass in the limit. Particularly we calculate the fraction of intervals with $n_j^{\mathrm{GP}} \geq \varepsilon/m$ and we shall be able to prove that it converges to 0 if $\varepsilon \to 0$. Because $n_j^{\mathrm{GP}} > 0$ for all intervals we cannot proceed like we did for $n^{\mathrm{IDF}}(l)$ and calculate the fraction of intervals where $n_j^{\mathrm{GP}} = 0$.

Theorem 4.12. *For $\gamma \gg \nu^2$ and $\varepsilon > 0$ arbitrary we get the relation*

$$\frac{1}{m}\sum_j^m g_{\varepsilon/m}(n_j^{\mathrm{GP}}) \overset{p}{\leq} 9\varepsilon \tag{4.68}$$

Proof. There is a constant C such that

$$\sum_{j=1}^m |n_j^{\mathrm{GP}} - l_j| < \frac{C}{m}\sqrt{\frac{m}{\sqrt{\gamma}}} \tag{4.69}$$

is only violated by a fraction of intervals smaller than ε. k is further bounded by $2\sqrt{C}/\varepsilon$ in probability, and therefore we can choose ν in such a way that

$$C\sqrt{\frac{m}{\sqrt{\gamma}}} \leq \varepsilon \tag{4.70}$$

We now estimate

$$\frac{1}{m}\sum_j^m g_{\varepsilon/m}(n_j^{\mathrm{GP}}) \leq \frac{1}{m}\sum_j^m g_\varepsilon(ml_j - \varepsilon) + \varepsilon \leq \frac{1}{m}\sum_j^m g_{2\varepsilon}(ml_j) \leq \frac{2}{\nu}\sum_j^m g_{4\varepsilon}(\nu l_j) + \varepsilon \tag{4.71}$$

In the first step we include all intervals which violate 4.69 and proceeded to use $m \overset{p}{\geq} \nu/2$. An easy application of Theorem 3.9 shows

$$\frac{1}{\nu}\sum_j^m g_{4\varepsilon}(\nu l_j) \to \int_0^{4\varepsilon/\nu} \mathrm{d}p_\nu = 1 - e^{-4\varepsilon} \leq 4\varepsilon \tag{4.72}$$

Adding up the terms proves the theorem. □

4.3.2 Convergence for the mean density

In principle we are also interested in the convergence of ρ_j^{GP} but because the l_j are of order ν^{-1} we only get

$$\sum_{j=1}^m |\rho_j^{\mathrm{GP}} - 1| = \sum_{j=1}^m \frac{1}{|l_j|}|n_j^{\mathrm{GP}} - l_j| \tag{4.73}$$

Hence we cannot show the convergence for the above sum but we can show it for almost all intervals. Let us fix an $1 > \varepsilon > 0$, then there exists a $c > 0$ such that $l_j > c/\nu$ for more than $(1-\varepsilon)m$ intervals in probability. Further there exists also a $k > 0$ such that $(1-\varepsilon)m$ intervals fulfill the estimate

$$|n_k^{\mathrm{GP}} - l_k| \le \frac{k}{\nu}\sum_{j=1}^{m} |n_j^{\mathrm{GP}} - l_j| \tag{4.74}$$

This statement is true because if there would not be a $k > 0$ then in probability $\varepsilon\nu$ intervals would grow faster than $\nu^{-1}\sum_{j=1}^{m} |n_j^{\mathrm{GP}} - l_j|$ which is a contradiction to Theorem 4.11. Combining these two constraints there are at least $(1-2\varepsilon)m$ intervals left which fulfill both properties.
Heuristically, if the convergence is not necessarily equally spread in the way

$$|n_k - l_k| \approx \frac{1}{\nu}\sum_{j=1}^{m} |n_j^{\mathrm{GP}} - l_j| \tag{4.75}$$

then the divergence is concentrated in only a view terms. If we are only interested in the convergence for "nearly all" intervals we can exclude them.

$$|\rho_j^{\mathrm{GP}} - 1| = |n_j^{\mathrm{GP}} - l_j| l_j^{-1} \le c^{-1}\nu |n_j^{\mathrm{GP}} - l_j| \le c^{-1} k \sum_{j=1}^{m} |n_j^{\mathrm{GP}} - l_j| \tag{4.76}$$

for all $\varepsilon > 0$.

Theorem 4.13. *The fraction of intervals for which $\rho_j^{\mathrm{GP}} \to 1$ tends to 1 in probability for $\gamma \gg \nu^2$*

$$\frac{1}{m}\sum_{j=1}^{m} g_\varepsilon(|\rho_j^{\mathrm{GP}} - 1|) \xrightarrow{p} 1 \tag{4.77}$$

From this it follows that

$$\frac{1}{m}\sum_{j=1}^{m} g_\delta(\rho_j^{\mathrm{GP}}) \xrightarrow{p} 0 \tag{4.78}$$

for all $0 \le \delta < 1$.

Further we can prove in an easy way the convergence $|\phi^{GP}|^2 \xrightarrow{L^2} 1$

Theorem 4.14. *For the minimizer ϕ^{GP} of $e^{\mathrm{GP}}(\gamma,\sigma,\nu)$ follows*

$$|\phi^{\mathrm{GP}}|^2 \xrightarrow{L^2([0,1])} 1 \tag{4.79}$$

for the case $\gamma \gg \nu^2$. This also implies the convergence on $L^1([0,1])$.

Proof. We get

$$\frac{\gamma}{2}\int |\phi^{\mathrm{GP}}|^4 \le e^{\mathrm{GP}}(\gamma,\sigma,\nu) \overset{p}{\le} \frac{\gamma}{2} + \frac{\gamma}{2} k \frac{\nu}{\sqrt{\gamma}} \tag{4.80}$$

for a constant k and therefore

$$\int |\phi^{\mathrm{GP}}|^4 \overset{p}{\lesssim} 1 + k\frac{\nu}{\sqrt{\gamma}} \tag{4.81}$$

For the convergence in L^2 we get

$$\int (|\phi^{\mathrm{GP}}(x)|^2 - 1)^2 \mathrm{d}x \leq \int |\phi^{\mathrm{GP}}(x)|^4 \mathrm{d}x - 1 \overset{p}{\lesssim} k\frac{\nu}{\sqrt{\gamma}} \tag{4.82}$$

□

Remark 4.15. Even though we know $|\phi^{\mathrm{GP}}|^2 \xrightarrow{L^1} 1$ we cannot conclude that there is superfluidity in the limit. It would be interesting to discuss the convergence of $|\phi^{\mathrm{GP}}|^{-1}$ in L^1 but we shall be only able to to this for stronger assumptions and we shall learn in the next section that there is also a case where the superfluid fraction converges to 0.

4.4 Discussion of superfluidity for $\gamma \gg \nu^2$

We will not be able to treat all possible cases for $\gamma \gg \nu^2$ but we can prove that there exists superfluidity for the case

$$\gamma \gg \nu^2\sigma^2 \tag{4.83}$$

and we can treat the case where there is no superfluidity for

$$\sigma \gg (\gamma/\nu^2)^4\gamma^{1/2}, \qquad \gamma \ll \nu^4 \tag{4.84}$$

4.4.1 Full superfluidity for $\gamma \gg \nu^2\sigma^2$

In this limit we are able to prove the convergence of $|\psi^{\mathrm{GP}}|^2 \to 1$ in $\|\cdot\|_\infty$. The next lemma shows the connection between the random potential and the convergence of the density $|\psi^{\mathrm{GP}}|^2$

Theorem 4.16.

$$\frac{\left\||\psi^{\mathrm{GP}}|^2 - 1\right\|_\infty^2}{\sqrt{1 + \||\psi^{\mathrm{GP}}|^2 - 1\|_\infty}} \leq \frac{1}{\sqrt{\gamma}} \int V_\omega \tag{4.85}$$

Proof. This proof follows [27]. For an arbitrary function $f \in H^1([0,1])$ continuity implies that from $\int_0^1 f = 0$ we can conclude $f(z) = 0$ for $z \in [0,1]$. Hence we can write

$$f(x)^2 = 2\int_z^x f'(y)f(y)\mathrm{d}y \tag{4.86}$$

From this we get an estimate for the sup norm

$$\|f\|_\infty^2 \leq 2\|f'\|_2\|f\|_2 \tag{4.87}$$

Applying this to the function $f(x) = |\phi(x)|^2 - 1$ we obtain

$$\left\||\phi|^2 - 1\right\|_\infty^2 \leq 4\|\phi'\phi\|_2 \left\||\phi|^2 - 1\right\|_2 \leq 4\|\phi'\|_2\|\phi\|_\infty \left\||\phi|^2 - 1\right\|_2 \tag{4.88}$$

Bounding $\|\phi\|_\infty^2 \leq 1 + \||\phi|^2 - 1\|_\infty$ we get

$$\|\phi'\|_2 \left\||\phi|^2 - 1\right\|_2 \geq \frac{1}{4} \frac{\||\phi|^2 - 1\|_\infty^2}{\sqrt{1 + \||\phi|^2 - 1\|_\infty}} \tag{4.89}$$

Introducing the parameter γ we can calculate

$$\|\phi'\|_2^2 + \frac{\gamma}{2}\|\phi\|_4^4 - \frac{\gamma}{2} = \|\phi'\|_2^2 + \frac{\gamma}{2}\left\||\phi|^2 - 1\right\|_2^2 \geq \sqrt{\gamma}\|\phi'\|_2 \left\||\phi|^2 - 1\right\|_2 \tag{4.90}$$

On the other side we get an upper estimate if we fix $\phi = \phi^{\mathrm{GP}}$.

$$\left\|\phi^{\mathrm{GP}\prime}\right\|_2^2 + \frac{\gamma}{2}\|\phi\|_4^4 - \frac{\gamma}{2} \leq \left\|\phi^{\mathrm{GP}\prime}\right\|_2^2 + \frac{\gamma}{2}\|\phi\|_4^4 + \int_0^1 V_\omega(z)|\phi^{\mathrm{GP}}(z)|^2 \mathrm{d}z - \frac{\gamma}{2} \leq \int V_\omega \tag{4.91}$$

In the last step we used the constant function 1 as a trial function. Combining these bounds we proved the lemma. □

The last lemma worked for any positive potential V but in the case of V_ω we can simplify the result to

$$\frac{\left\||\psi^{\mathrm{GP}}|^2 - 1\right\|_\infty^2}{\sqrt{1 + \||\psi^{\mathrm{GP}}|^2 - 1\|_\infty}} \leq \frac{m\sigma}{\sqrt{\gamma}} \xrightarrow{p} 0 \tag{4.92}$$

in the case $\gamma \gg \nu^2\sigma^2 \sim m^2\sigma^2$ we obtain uniform convergence for the GP density to 1. Because the density $|\psi^{\mathrm{GP}}|^2$ converges to 1 uniformly on the compact interval $[0, 1]$ we also get $|\psi^{\mathrm{GP}}|^{-2} \xrightarrow{L^1} 1$. Hence we showed that there is full superfluidity in the limit $\nu \to \infty$ using (1.42).

Theorem 4.17. *In the case $\gamma \gg \nu^2\sigma^2$ we get $n_{\mathrm{SF}} \to 1$ in the limit $\nu \to \infty$.*

4.4.2 Absence of superfluidity for $\sigma \gg \sqrt{\gamma}$ and $\gamma \ll \nu^4$

In the case $\sigma^2 \gg \gamma \sim \mu$ we can apply Theorem 4.5 and get

$$N_s \overset{p}{\lesssim} C\sqrt{\frac{\sqrt{\gamma}}{\sigma}} \tag{4.93}$$

On the other side we get for the length of intervals with $l_j < 1/\sqrt{\mu} \sim 1/\sqrt{\gamma}$ with help of Corollary 3.13

$$\sum_{j=1}^m l_j \xrightarrow{p} \nu \int_0^{1/\sqrt{\mu}} l \mathrm{d}p_\nu(l) = 1 - e^{-\frac{\nu}{\sqrt{\mu}}}\left(\frac{\nu}{\sqrt{\mu}} + 1\right) \tag{4.94}$$

Analyzing the function $x \mapsto 1 - e^{-x}(x+1)$ we learn that for $\nu/\sqrt{\mu} \ll 1$ this behaves like ν^2/μ. Using (4.52) we find an upper bound for the superfluid fraction.

$$n_{SF} = \frac{C\sqrt{\sqrt{\gamma}/\sigma}}{\nu^4/\mu^2} \sim \frac{\gamma^2}{\nu^4}\frac{\gamma^{1/4}}{\sqrt{\sigma}} \tag{4.95}$$

As long as $\sigma \gg (\gamma/\nu^2)^4\gamma^{1/2}$ there is no superfluidity in the limit $\nu \to \infty$.

Theorem 4.18. *For the case $\sigma \gg (\gamma/\nu^2)^4\gamma^{1/2}$ and $\gamma \ll \nu^4$ there is no superfluidity in the limit $\nu \to \infty$.*

4.5 Different limit regimes

In section 2.8 we discussed the behavior of the minimizer n^{IDF} of the interval density functional in different limits. It is a priori not clear that these statements about $n^{\mathrm{IDF}}(l)$ can be repeated for ϕ^{GP}. In this section summarize the results from this chapter to show that equal statements hold for the GP density.
n_j^{GP} will not vanish for any interval, and therefore we have to alter the definition of λ from section 2.8. We shall introduce $\lambda_\varepsilon^{\mathrm{GP}}$, the fraction of the intervals with mass bigger than ε.

$$\lambda_\varepsilon^{\mathrm{GP}} := \frac{1}{m}\sum_{j=1}^{m}(1 - g_{\varepsilon/m}(n_j^{\mathrm{GP}})) \tag{4.96}$$

We showed in Theorem 4.12 and Lemma 4.4 how this expression behaves in the limit $\nu \to \infty$. Further we define the fraction of intervals with a mean density bigger than ε

$$\zeta_\varepsilon^{\mathrm{GP}} := \frac{1}{m}\sum_{j=1}^{m}(1 - g_\varepsilon(\rho_j^{\mathrm{GP}})) \tag{4.97}$$

For $\zeta_\varepsilon^{\mathrm{GP}}$ we use Theorem 4.13 and Theorem 4.8.
Therefore we can distinguish the following cases

- If $\gamma \gg \nu^2$ we get $\lambda_\varepsilon^{\mathrm{GP}} \overset{p}{\geq} 1 - 9\varepsilon$, $\zeta_\varepsilon^{\mathrm{GP}} \overset{p}{\to} 1$ and $e^{\mathrm{GP}} \to \gamma/2$. Therefore, the interaction is strong enough to force nearly all intervals to be "occupied". As we showed in Theorem 4.14, the density converges in L^2 to 1.
    (a) In the case $\gamma \gg \nu^2\sigma^2$ we get complete superfluidity in the limit $\nu \to \infty$ (See Theorem 4.17). Further we can deduce $\phi^{\mathrm{GP}} \to 1$ uniformly which is, because we are on the interval $[0,1]$ stronger than L^1 convergence.
    (b) In the case $\sigma \gg (\gamma/\nu^2)^4\gamma^{1/2}$ and $\gamma \ll \nu^4$ the convergence we can apply Theorem 4.5 to show the absence of superfluidity.
- If $\gamma \ll \nu^2$ then $\lambda_\varepsilon^{\mathrm{GP}} \to 0$, $\zeta_\varepsilon^{\mathrm{GP}} \overset{p}{\to} 0$ and $e^{\mathrm{GP}} \to \mu \sim \nu^2/(1+\log(\nu^2/\gamma))^2$. In this case all the mass is concentrated in only a few intervals and we showed in Section 4.2 that there is no superfluidity in the limit $\nu \to \infty$.

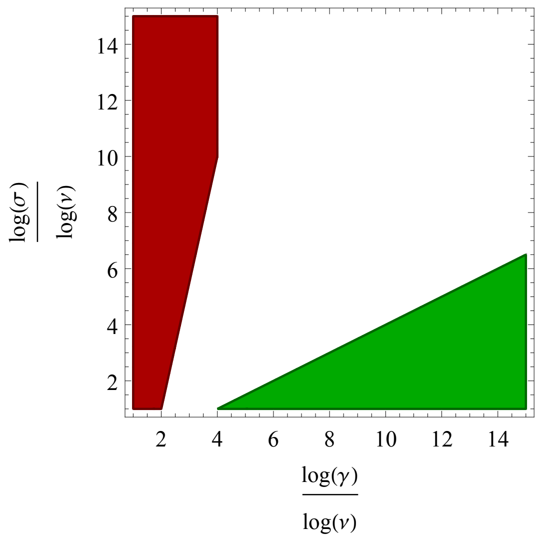

Figure 4.1: Different parameter regimes for γ, σ and the parts where we proved superfluidity (green) and where there is for sure no superfluidity (red) in the limit $\nu \to \infty$.

- For $\gamma \sim \nu^2$ the the total mass of a finite fraction of intervals vanishes and there is also no superfluidity in the limit $\nu \to \infty$

Figure 4.1 shows gives an overview for which parameters regimes we proved statements about superfluidity.

Chapter 5

A mass dependent potential

In this chapter we shall generalize the functional $\mathcal{E}_\omega^{\mathrm{GP}}$ by adding a $n_j = \int_{I_j} |\phi|^2$ depended perturbation term for the case $\gamma \ll \nu^2$. We shall show that there is still a deterministic limit, which can be described by a perturbed interval density functional similar to $e_0(\gamma,\nu)$ in Chapter 3. By differentiating for the perturbation term we are able to show an alternative approach to show a bound on n_j^{GP}.

5.1 Basic definitions

We introduce the parameter $\beta > 0$ and $1 > \varepsilon > 0$ which define the height and the strength of the perturbation term.

$$e_*^{\mathrm{GP}}(\gamma,\sigma,\nu,\varepsilon,\beta) \coloneqq \inf_{\|\phi\|_{L^2}=1} \sum_j^m \frac{n_j}{l_j^2} \mathcal{E}^{\mathrm{GP}}_{\gamma n_j l_j, l_j\sigma}[\phi_j] + \beta \frac{\mu}{m} g_\varepsilon(n_j) \tag{5.1}$$

where $g_\varepsilon(n) = 0$ for $n > \varepsilon$ and $g_\varepsilon(n) = 1$ otherwise. For simplicity we frequently write $e_*^{\mathrm{GP}} = e_*^{\mathrm{GP}}(\gamma,\sigma,\nu,\varepsilon,\beta)$. Because we want to discuss an interesting case, the additional term should be of the same magnitude as $\mu \sim e^{\mathrm{GP}}(\gamma,\sigma,\nu)$ which is ensured by the factor μ/m. Because $n_j^{\mathrm{GP}} \approx \nu^{-1}$ we set $\varepsilon = \delta/\nu$ with $\delta > 0$. The minimizer which corresponds to this functional we denote $\phi^{\mathrm{GP}}_{\gamma,\sigma,\nu,\varepsilon,\beta}$ and its mass in an interval I_j by $u_j^{\mathrm{GP}} = \int_{I_j} |\phi^{\mathrm{GP}}_{\gamma,\sigma,\nu,\varepsilon,\beta}|^2$. Further we recall that the unperturbed energy we defined in Chapter 1 corresponds to the case $\beta = 0$.

$$e^{\mathrm{GP}}(\gamma,\sigma,\nu) = e_*^{\mathrm{GP}}(\gamma,\sigma,\nu,0,0) \tag{5.2}$$

In the same way we perturb the interval density functional

$$e^*(\gamma,\nu,\varepsilon,\beta) \coloneqq \inf_{\nu\int n(l)\mathrm{d}p_\nu(l)=1} \nu \int \mathrm{d}p_\nu(l) \frac{n(l)}{l^2} e(n(l)l\gamma,\infty) + \beta\mu \int \mathrm{d}p_\nu(l) g_\varepsilon(n(l)) \tag{5.3}$$

The minimizer is denoted by $u^{\mathrm{IDF}}(l)$ but we shall see that we can write the minimizer in the form $u^{\mu^*}(l)$ where μ^* is the associated Lagrange multiplier and $u^\eta(l)$ a function we shall define later.

Energy	Multiplier
$e_*^{\mathrm{GP},\infty}$	ζ^*
$e_*^{\mathrm{GP},\sigma}$	$\zeta^{*,\sigma}$
e_0^*	μ^*
$e_0^{*,\sigma}$	$\mu^{*,\sigma}$
$e^{\mathrm{GP},\infty}$	ζ
$e^{\mathrm{GP},\sigma}$	ζ^σ
e_0	μ
e_0	μ^σ

Table 5.1: The energies and the corresponding Lagrange multiplier.

Additionally we introduce the interval density functional for a finite σ. We denote this with

$$e_0^{*,\sigma}(\gamma,\sigma,\nu,\varepsilon,\beta) = \inf_{\nu\int n(l)\mathrm{d}p_\nu(l)=1} \nu\int \mathrm{d}p_\nu(l)\frac{n(l)}{l^2}e(n(l)l\gamma, l\sigma) + \beta\mu\int \mathrm{d}p_\nu(l)g_\varepsilon(n(l)) \tag{5.4}$$

Analogously to the unperturbed case we can find an upper bound for e_*^{GP} by fixing $\phi(z_j) = 0$.

$$e_*^{\mathrm{GP}}(\gamma,\sigma,\nu,\varepsilon,\beta) \leq \inf_{\substack{\sum n_j=1\\ n_j>0}} \sum_j^m \frac{n_j}{l_j^2}e(n_jl_j\gamma,\infty) + \beta\frac{\mu}{\nu}g_\varepsilon(n_j) =: e_*^{\mathrm{GP},\infty}(\gamma,\nu,\varepsilon,\beta) \tag{5.5}$$

We shall denote the minimizer of $e_*^{\mathrm{GP},\infty}$ by $u_j^{\zeta^*} = u^{\zeta^*}(l_j)$ and the corresponding Lagrange multiplier is denoted by ζ^*. The notation for the minimizer is chosen because the minimizer of the perturbed interval density theory and the corresponding minimizer for $e^{\mathrm{GP},\infty}$ can written in terms of the function $u^\eta(l)$ which we shall define later. A lower bound is achieved by dropping continuity for the minimizer at the points z_j. We denote this with

$$e_*^{\mathrm{GP},\sigma}(\gamma,\sigma,\nu,\varepsilon,\beta) := \inf_{\substack{\sum n_j=1\\ n_j>0}} \sum_j^m \frac{n_j}{l_j^2}e(n_jl_j\gamma,l_j\sigma) + \beta\frac{\mu}{\nu}g_\varepsilon(n_j) \leq e_*^{\mathrm{GP}}(\gamma,\sigma,\nu,\varepsilon,\beta) \tag{5.6}$$

The minimizer is $u_j^{\zeta^{*,\sigma},\sigma} = u^{\zeta^{*,\sigma},\sigma}(l_j)$ and $\zeta^{*,\sigma}$ we denote the corresponding Lagrange multiplier.
We shall see that the most important part are the Lagrange multipliers to the corresponding energies. The definitions can be found in table 5.1.

5.2 Outline of the proof

In the end we want to take the derivative of the functional to show that

$$\frac{1}{\nu}\sum_{j=1}^m g_\delta(\nu n_j^{\mathrm{GP}}) \to 1 \tag{5.7}$$

To do this we carry out a detailed analysis about the minimizer u^{μ^*} and we shall prove that it is not changing much for $|\beta| < \beta_0$. We find the same type of minimizer for the interval density functional as for the upper bound functional $e^{\mathrm{GP},\infty}$. The differences are completely characterized by the new Lagrange multipliers, which we shall denote with μ^* for the interval density functional and ζ^* for the minimization of $e^{\mathrm{GP},\infty}$. Both of these cases have $\sigma = \infty$ but for the lower bound we also need to work with finite σ. Even though the minimizer will change slightly, the most important variables are again the Lagrange multiplier which we denote with $\mu^{*,\sigma}$ for and $\zeta^{*,\sigma}$.
Even though it is possible to show that these variables converge to each other in first order, for our purposes it is good enough to show that

$$\mu^* \sim \mu \sim \mu^{*,\sigma} \sim \zeta \sim \zeta^* \sim \zeta^{*,\sigma} \tag{5.8}$$

Using these relations we are able to show that the energy contribution from the added term converges to $\beta\mu$ in the limit in first order. We can therefore conclude, by taking a derivative, that nearly all intervals have vanishing mass.

5.3 The minimizer

First we restrict to the case $\beta > 0$. We split the domain of the integral of functional $\mathcal{E}_{\gamma,\nu,\beta,\varepsilon}$ in three parts and minimize each of them to find the minimizer $u^{\mu^*}(l)$. This is possible because the functional doesn't contain derivatives of n. For $u^{\mathrm{IDF}}(l) > \varepsilon$ or $u^{\mathrm{IDF}}(l) < \varepsilon$ the perturbation term $g_\varepsilon(\nu n(l))$ is constant, hence the minimizer has the same form as in the unperturbed case with the new Lagrange multiplier μ^*

$$n^{\mu^*}(l) = \frac{1}{l\gamma}\bar{n}(\mu^* l^2) \tag{5.9}$$

The range where $u^{\mathrm{IDF}}(l) = \varepsilon$ is now not limited to a point but it is constant on an interval $[L^{\mu^*}, S^{\mu^*}]$. To abbreviate this we write $L = L^{\mu^*}, S = S^{\mu^*}$. S can be determined by the relation

$$n^{\mu^*}(S) = \varepsilon \tag{5.10}$$

and L can be found by solving

$$\frac{n^{\mu^*}(L)}{L^2} e(n^{\mu^*}(L)L\gamma, \infty) + \beta\frac{\mu}{\nu} = \frac{\varepsilon}{L^2} e(\varepsilon L\gamma, \infty) \tag{5.11}$$

This describes the point where it is equally efficient to choose $u^{\mu^*}(L) = \varepsilon$ or $u^{\mu^*}(L) = n^{\mu^*}(L)$. A point like this has to exist because the right side of (5.11) goes to ∞ for $L \to 0$, while the left side of (5.11) goes to 0 in this limit and n^{μ^*} is monotone increasing.
With this we get the minimizer $u^{\mu^*}(l)$ of the perturbed interval density functional

$$u^{\mathrm{IDF}}(l) = \begin{cases} \varepsilon & l \in [L, S] \\ n^{\mu^*}(l) & \text{else} \end{cases}$$

In general we denote the function $u^\eta(l)$ as

$$u^\eta(l) = \begin{cases} \varepsilon & l \in [L^\eta, S^\eta] \\ n^{\mu^\eta}(l) & \text{else} \end{cases}$$

and therefore we can write $u^{\mathrm{IDF}}(l) = u^{\mu^*}(l)$.
Like in the case for the unperturbed functional we can repeat the procedure to find the minimizer of $e_*^{\mathrm{GP},\infty}$ and $e_*^{\mathrm{GP},\sigma}$. The difference is that we now need to work with the lagrange multipliers ζ^* and $\zeta^{*,\sigma}$. The interval where the minimizer is exactly ε we denote with $[S^{\zeta^*}, L^{\zeta,*}]$ for $e_*^{\mathrm{GP},\infty}$ and $[S^{\zeta^{*,\sigma}}, L^{\zeta^{*,\sigma},\sigma}]$ for $e_\omega^{\mathrm{GP},\sigma}$.
Because all this minimizers are completely characterized by the corresponding Lagrange multipliers, it will be important to determine their relation to each other. To do this we shall discuss bounds on S and L.

Remark 5.1. The case $\beta < 0$ works analogously to the case discussed above. The only difference is that we have to redefine the functional because otherwise no minimizer exists. In particular we define $g_\varepsilon^<(n) = 0$ if $n < \varepsilon$ and 1 otherwise and replace g_ε with $g_\varepsilon^<$.

5.3.1 Determine S

The two parameters L, S of the minimizer are determined by μ^*. We can calculate a lower and an upper bound for S by using the estimates for $n^{\mu^*}(l)$. We get for the lower bound

$$\frac{\mu^*}{\gamma}S - \frac{\pi^2}{S\gamma} = \varepsilon \;\Rightarrow\; S = \frac{\sqrt{\gamma^2\epsilon^2 + 4\pi^2\mu^*} + \gamma\epsilon}{2\mu^*} \tag{5.12}$$

For the upper bound we get an additional 2/3.

$$\frac{\gamma}{2\mu^*}\varepsilon + \frac{\pi}{\sqrt{\mu^*}} \leq \frac{\sqrt{\gamma^2\varepsilon^2 + 4\pi^2\mu^*} + \gamma\epsilon}{2\mu^*} \leq S \leq \frac{3\gamma}{2\mu^*}\varepsilon + \frac{\pi}{\sqrt{\mu^*}} \tag{5.13}$$

Because of the measure $p_\nu(l)$ the interesting size is νS which behaves like

$$\nu S \sim \frac{\gamma}{\mu^*}\delta + \frac{\pi\nu}{\sqrt{\mu^*}} \tag{5.14}$$

Later we argue $\mu \sim \mu^*$ which shows that the dominating term in the case $\gamma \ll \nu^2$ is $\nu/\sqrt{\mu^*}$.

5.3.2 Determine L

For the case $\gamma \ll \nu^2$ we need to show two results about the parameters $\zeta^{*,\sigma}, \zeta^*$. First it is important that

$$\zeta^* \sim \zeta^{*,\sigma} \sim \mu \tag{5.15}$$

Further it will be necessary to show that

$$\nu L \to \infty \tag{5.16}$$

The same should be true for $L^{\zeta^*,\sigma,\sigma}$ and L^{ζ^*}. To discuss the case where $\sigma < \infty$ we need to define $l^{0,\eta}$ as the point where

$$n^{\eta,\sigma}(l) = 0 \ \forall l < l^{0,\eta} \iff \eta(l^{0,\eta})^2 = e(0, l^{0,\eta}\sigma) \tag{5.17}$$

A simple calculation using Corollary 2.15 shows

$$l^{0,\eta} \geq \frac{\pi}{\sqrt{\eta}} - \frac{c}{\sigma} \tag{5.18}$$

for an appropriate constant c. For short we abbreviate $l^0 := l^{0,\zeta^*,\sigma}$.

Lemma 5.2. *We get for $\gamma \ll \nu^2$*

$$\sigma L \to \infty \tag{5.19}$$

The same is valid for $L^{\zeta^*,\sigma,\sigma}$ and L^{ζ^*} in probability. Because $\gamma \ll \nu^2$ this implies

$$\nu L \to \infty \tag{5.20}$$

Proof. For the proof we shall define $L^\sigma = L^{\zeta^*,\sigma,\sigma}$. If we know that $L^\sigma \geq l_0$ we are done using (5.18). Otherwise the equation (5.11) simplifies to

$$\beta\frac{\mu}{\nu} = \frac{\delta}{(\nu L^\sigma)^2} e(L^\sigma \varepsilon\gamma, \sigma L^\sigma) \geq \sigma^2 \frac{\delta}{\nu L^\sigma \sigma} \frac{1}{L^\sigma \sigma + c} \tag{5.21}$$

Because we know $\mu/\sigma^2 \to 0$ we get

$$\frac{\delta}{L^\sigma \sigma} \frac{1}{L^\sigma \sigma + c} \leq \beta \frac{\mu}{\sigma^2} \to 0 \tag{5.22}$$

If $\sigma L^\sigma < k$ for $k > 0$ this condition would be violated. The limit $\nu L \to \infty$ works analogously. □

To prove that μ^* is not growing too fast for $\beta < 0$ we need to study the interval $[L, S]$. The next lemma proves a bound of its length.

Lemma 5.3. *For $\beta < 0$ we get the estimate*

$$-\beta\frac{\mu\gamma}{\nu\eta^2} \frac{1}{1 + c\sqrt{\eta}/\sigma} \gtrsim L - S \tag{5.23}$$

Proof. We use $n^\eta(L) \geq n^\eta(S) = \varepsilon$ to find an upper bound for $L - S$.

$$-\beta\frac{\mu}{\nu} = \frac{n^\eta(L)}{L^2} e(n^\eta(L)L\gamma, \sigma L) - \frac{\varepsilon}{L^2} e(\varepsilon L\gamma, \sigma L) \geq \frac{n^\eta(L) - n^\eta(S)}{L^2} e(\varepsilon L\gamma, \sigma L) \tag{5.24}$$

Because $n^\eta(L) \geq \varepsilon$ we get $\partial_L n^\eta(L) > 2\eta/(3\gamma)$ from Lemma 2.5. It follows

$$-\beta\frac{\mu}{\nu} \geq (L-S)\frac{\eta}{L^2\gamma}\frac{\sigma L}{\sigma L + c} \gtrsim (L-S)\frac{\eta^2}{\gamma}\frac{1}{1+c\sqrt{\eta}/\sigma} \tag{5.25}$$

and so we proved the result

$$-\beta\frac{\mu\gamma}{\nu\eta^2}\frac{1}{1+c\sqrt{\eta}/\sigma} \gtrsim L-S \tag{5.26}$$

□

To exclude the cases $\mu^* \gg \mu$ and $\mu^* \ll \mu$ we need to find a contradiction of the norm constraint. We define $\bar{N}^\eta$ and $\bar{N}^{\eta,\sigma}$ as

$$\bar{N}^\eta := \int n^\eta(l), \qquad \bar{N}^{\eta,\sigma} := \int n^{\eta,\sigma}(l) \tag{5.27}$$

The next proof shows that it is enough to show that $\bar{N}^\eta$ is bounded for $\eta \sim \mu$.

Lemma 5.4. *We get*

$$\begin{cases} \bar{N}^\eta \to 0 & \eta \ll \mu \\ \bar{N}^\eta \to \infty & \eta \gg \mu \end{cases} \tag{5.28}$$

and

$$\begin{cases} \bar{N}^{\eta,\sigma} \to 0 & \eta \ll \mu^\sigma \\ \bar{N}^{\eta,\sigma} \to \infty & \eta \gg \mu^\sigma \end{cases} \tag{5.29}$$

Proof. A simple calculation shows

$$N^\eta \sim \frac{\eta}{\gamma}e^{-\pi\nu/\sqrt{\eta}} \tag{5.30}$$

which converges to 0 if $\eta \ll \mu$ and to ∞ if $\eta \gg \mu$. The same procedure can be used for $\sigma < \infty$. □

Lemma 5.5. *We get the relations*

$$\mu^* \sim \mu \sim \zeta \sim \zeta^*, \qquad \zeta^{*,\sigma} \sim \zeta^\sigma \tag{5.31}$$

Proof. In the case $\beta > 0$ we learn from the specific form of the minimizer $u^{\mu^*}(l)$ that $\mu^* \leq \mu$. In the case that $\mu^* \ll \mu$ we would get for the norm of $u^{\mu^*}(l)$

$$\nu\int_L^S u^{\mu^*}(l)\mathrm{d}p_\nu(l) = \delta p_\nu([L,S]) \leq \delta < 1 \tag{5.32}$$

Hence we have to demand

$$\nu\int n^{\mu^*}(l)\mathrm{d}p_\nu(l) \geq \nu\int_{\mathbb{R}\setminus[L,S]} u^{\mu^*}(l)\mathrm{d}p_\nu(l) \geq 1-\delta \tag{5.33}$$

$\mu \gg \mu^*$ would therefore be a contradiction because of Lemma 5.4. The same argument can be repeated in probability for ζ, ζ^* and ζ^σ, $\zeta^{\sigma,*}$.
Let us consider now $\beta < 0$ in which $\mu^* \geq \mu$. In the case $\mu^* \gg \mu$ we get $\nu(L-S) \to 0$ from Lemma 5.3 which shows that

$$\nu \int_S^L u^{\mu^*}(l)\mathrm{d}p_\nu(l) = \delta p_\nu([L,S]) \to 0 \tag{5.34}$$

On the other side the contribution of $n^\mu(l)$ goes to infinity

$$\int_{\mathbb{R}\setminus[S,L]} n^{\mu^*}(l)\mathrm{d}p_\nu(l) \to \bar{N}^{\mu^*} \to \infty \tag{5.35}$$

which is a contradiction because the difference term is small in comparison

$$\int_S^L n^{\mu^*}(l)\,\mathrm{d}p_\nu(l) \leq \frac{\mu^*}{\gamma}\int_S^L l\,\mathrm{d}p_\nu(l) \ll \frac{\mu^*}{\gamma}e^{-\pi\nu/\sqrt{\mu^*}} \tag{5.36}$$

Hence we would violate the norm constraint if $\mu^* \gg \mu$. We can repeat the same argument in probability for ζ, ζ^* and $\zeta^\sigma, \zeta^{*,\sigma}$. □

Lemma 5.6.

$$\zeta^\sigma \sim \mu^\sigma, \qquad \mu^\sigma \sim \mu \tag{5.37}$$

Proof. For this we just have to repeat the argument from Lemma 3.16. This time we use Theorem 3.9 for the sum

$$N^{\mu^\sigma,\sigma} = \sum_{j=1}^m n^{\mu^\sigma,\sigma}(l_j) \tag{5.38}$$

and from the convergence $N^{\mu^\sigma,\sigma} \to 1$ we can conclude $\zeta^\sigma \sim \mu^\sigma$. A calculation similar to the lower bound in Chapter 3 shows $\mu^\sigma \sim \mu$. □

5.3.3 Convergence of $e_*^{\mathrm{GP},\infty}$

We can write the energy $e_*^{\mathrm{GP},\infty}$ in terms of the minimizer $u_j^{\zeta^*}$ to get a lower bound

$$\begin{aligned} e_*^{\mathrm{GP},\infty} &= \sum \frac{u_j^{\zeta^*}}{l_j^2} e(u_j^{\zeta^*} l_j\gamma,\infty) + \beta\frac{\mu}{m} g_\varepsilon(u_j^{\zeta^*}) \\ &\geq \sum_{j=1}^m \frac{n_j^\zeta}{l_j^2} e(n_j^\zeta l_j\gamma,\infty) + \beta\mu\underbrace{\frac{1}{m}\sum_{j=1}^m g_\varepsilon(u_j^{\zeta^*})}_{=:T^\zeta} \end{aligned} \tag{5.39}$$

In the same way we calculate a lower bound for e_0^*

$$e_0^* \geq e_0 + \beta\mu\underbrace{\int \mathrm{d}p_\nu(l) g_\varepsilon(u^{\mu^*}(l))}_{=:T} \tag{5.40}$$

Because of our analysis above and with Corollary 3.12 we get

$$T^{\zeta} \xrightarrow{p} \int_0^L \mathrm{d}p_\nu(l) = 1 - e^{-L\nu} \geq 1 - e^{-\pi\nu/\sqrt{\zeta^*}} \overset{\mu\sim\zeta^*}{\rightarrow} 1 \tag{5.41}$$

in probability. For the upper bound we use $n^{\zeta}(l)$ as a trial function

$$e_*^{\mathrm{GP},\infty} \leq \sum \frac{n_j^{\zeta}}{l_j^2} e(n_j^{\zeta} l_j \gamma, \infty) + \beta \frac{\mu}{m} g_\varepsilon(n_j^{\zeta}) \leq \sum_{j=1}^m \frac{n_j^{\zeta}}{l_j^2} e(n_j^{\zeta} l_j \gamma, \infty) + \beta\mu \tag{5.42}$$

A similar calculation shows the upper bound for e_0^*

$$e_0^* \leq e_0 + \beta\mu \tag{5.43}$$

Combining these bounds we get for the limit $\nu \to \infty$

Lemma 5.7. *In the case $\gamma \ll \nu^2$ we get*

$$\frac{e_*^{\mathrm{GP},\infty}}{e_0^*} \xrightarrow{p} 1 \tag{5.44}$$

Proof. We calculate in probability

$$\frac{e_*^{\mathrm{GP},\infty}}{e_0^*} \leq \frac{e^{\mathrm{GP},\infty} + \beta\mu}{e_0 + \mu\beta T} \overset{3.20}{\rightarrow} 1 \tag{5.45}$$

and in the same way we prove the lower bound

$$\frac{e_*^{\mathrm{GP},\infty}}{e_0^*} \geq \frac{e^{\mathrm{GP},\infty} + \beta\mu T^{\zeta}}{e_0 + \mu\beta} \overset{3.20}{\rightarrow} 1 \tag{5.46}$$

□

5.3.4 The lower bound $e_*^{\mathrm{GP},\sigma}$

We can repeat the argument from Chapter 3

Lemma 5.8. *For $\gamma \ll \nu^2$ we get*

$$\frac{e_*^{\mathrm{GP},\sigma}}{e_*^{\mathrm{GP},\infty}} \xrightarrow{p} 1 \tag{5.47}$$

Proof. We use Lemma 2.10 to show

$$\begin{aligned} e_*^{\mathrm{GP},\sigma} &= \sum \frac{u_j^{\zeta^{*,\sigma},\sigma}}{l_j^2} e(u_j^{\zeta^{*,\sigma},\sigma} l_j \gamma, \sigma l_j) + \beta \frac{\mu}{m} g_\varepsilon(u_j^{\zeta^{*,\sigma},\sigma}) \\ &\geq \left(1 - \left(\frac{\sigma}{\sqrt{\zeta^{*,\sigma}}}\right)^{-1/2}\right) e_*^{\mathrm{GP},\infty} + \beta\mu \left(\frac{\sigma}{\sqrt{\zeta^{*,\sigma}}}\right)^{-1/2} \underbrace{\frac{1}{m}\sum_{j=1}^m g_\varepsilon(u_j^{\zeta^{*,\sigma},\sigma})}_{=:T^\sigma} \end{aligned} \tag{5.48}$$

T^σ is bounded with $0 \le T^\sigma \le 1$. Because of the relation $\zeta^{*,\sigma} \sim \mu$ the term

$$\left(\frac{\sigma}{\sqrt{\zeta^{*,\sigma}}}\right)^{-1/2} \to 0 \tag{5.49}$$

and hence we proved the desired convergence. □

5.4 Convergence of the perturbation term

Combining the upper and the lower bound we get the convergence

Lemma 5.9. *For $\gamma \ll \nu^2$ we get*

$$\frac{e_*^{\mathrm{GP}}}{e_0^*} \xrightarrow{p} 1 \tag{5.50}$$

With help of Lemma 5.9 we can show

Theorem 5.10. *For $\gamma \ll \nu^2$ we get for $\delta > 0$ arbitrary*

$$\frac{1}{m}\sum g_\delta(\nu n_j^{\mathrm{GP}}) \to 1 \tag{5.51}$$

Proof. From the bounds of e_*^{GP} we get

$$\frac{e_*^{\mathrm{GP}}}{\mu} \xrightarrow{p} \frac{e_0}{\mu} + \beta \tag{5.52}$$

Because e_*^{GP} is concave in β in an open interval around 0, the derivatives of both sides converge to each other.

$$\partial_\beta|_{\beta=0}\,\frac{e_*^{\mathrm{GP}}}{\mu} = \frac{1}{m}\sum_{j=1}^{m} g_\delta(\nu n_j^{\mathrm{GP}}) \xrightarrow{p} 1 \tag{5.53}$$

□

The difficulties with this argumentation were that we had to prove that $\zeta^{*,\sigma}/\mu$, ζ^*/μ and μ^*/μ are not divergent. After all this should be considered as an alternative approach because the method used in Chapter 4 was shorter and equally efficient.

Chapter 6

Extension to a Bernoulli type potential

In the previous chapters we discussed a Poisson distributed random potential with the length distribution $\mathrm{d}p_\nu(l) = \nu \exp(-\nu l)\mathrm{d}l$ and a Poisson distribution for the number of random points m. Up to this point all proofs were based on properties of these distributions. We shall discuss in this section that our argumentation from Chapter 2 and 3 can be repeated choosing a Bernoulli random potential.
First we summarize in which properties of the length distribution and the distribution of m were needed, and where they were used. Afterwards we shall show that all these properties are also fulfilled using a Bernoulli distributed potential. One should notice that the form of the minimizer $n^{\mathrm{IDF}}(l)$ is not dependent on the distribution but the norm constraint, and hence only the Lagrange multipliers will change.

6.1 Dependencies on the distribution

The first step to prove a deterministic limit was to discuss the interval density theory and its properties in Chapter 2. In particular we determined μ which was the Lagrange multiplier for the energy $e_0(\gamma, \nu)$. The parameter μ was strongly dependent on the underlying length distribution and therefore we have to rethink the relations between μ, γ, ν, σ for a different distribution. Additionally the limit (1.24) will change and it is also not clear if a deterministic limit exists. To talk about a distribution for the l_j we should at least assume that the lengths l_j are IID random variables.
For the following discussions we shall rename the Lagrange multipliers to $\bar{\mu}, \bar{\zeta}$ and the lengths to $\bar{l}_j$. The number of random points is denoted by $\bar{m}$.
In Chapter 3 we discussed the convergence of sums of the form

$$\sum_j w(\bar{l}_j) \xrightarrow{L^2} 1 \tag{6.1}$$

We never used any specific properties of the distribution of the $\bar{l}_j$ in Theorem 3.9 but the proof was based on the distribution of $\bar{m}$. In particular we used that

$$\left\langle \sum_{|1-\bar{m}/\nu|>\varepsilon} \chi_m \right\rangle \leq Ce^{-k\nu} \tag{6.2}$$

with appropriate constants C, k, and χ_m being the projector onto the case of m intervals. In other words we used that $\bar{m}$ is very localized around ν. In order to apply again Theorem 3.9 we need to show the convergence

$$\nu \left\langle n^{\bar{\mu}}(l)^2 \right\rangle \sim \frac{\bar{\mu}}{\gamma\nu} \to 0 \tag{6.3}$$

Applying the theorem 3.9 we get the convergence $N^{\bar{\mu}} \xrightarrow{p} 1$ and with this we can discuss the convergence of

$$\left| \frac{\bar{\zeta}}{\bar{\mu}} - 1 \right| \to 0 \tag{6.4}$$

For Lemma 3.16 we used specific properties of $\bar{\mu}$ and in particular

$$\frac{\bar{\mu}}{\gamma} e^{-\pi\nu/\sqrt{\bar{\mu}}} \sim 1 \tag{6.5}$$

Hence we have to act very carefully if we are not able to establish this relation for $\bar{\mu}$. If we obtain the relation (6.5) we can continue with the proof of the upper bound analogously to Chapter 3.
The lower bound needed, beside the convergence of $\bar{m}/\nu$, also various demands on $\bar{\mu}$ and σ.

6.2 The Bernoulli potential

We shall see that the Bernoulli potential behaves in a lot of ways equivalently to the Poisson distributed potential, with the exception that there are no "small" intervals. Because we know that in the limit $\gamma \ll \nu^2$ the mass in small intervals vanished, it is reasonable that the Bernoulli case will work similarly.
We define $x_k = \alpha k$ and for all x_k we define the IID random variables $X_k = X_k^\omega$ as Bernoulli distributed with parameter p and ω being the state in the probability space.

$$P(X_k^\omega = 1) = p, \qquad P(X_k^\omega = 0) = 1 - p \tag{6.6}$$

Analogously to the argumentation in Chapter 1 we rescale the points $z_j = x_j/\nu$ to get a Hamiltonian on $[0, 1]$. The points z_k where $X_k = 1$ divide $\mathbb{R}$ into intervals I_j with lengths $\bar{l}_j$. We define the potential as

$$\bar{V}_\omega(z) = \sum_{z_k \in [0,1]} X_k \delta(z - z_k) \tag{6.7}$$

The probability distribution of the number of points in $[0,1]$ where $X_j = 1$ is denoted by m is

$$P(m = k+1) = \binom{[\nu/\alpha]-1}{k} p^k (1-k)^{[\nu/\alpha]-(k+1)} \tag{6.8}$$

where $[\nu]$ is the next small integer for $\nu \in \mathbb{R}$. The expectation value of this binomial distribution is

$$\langle \bar{m} \rangle = p[\nu/\alpha] \tag{6.9}$$

Because we want to fulfill again $\langle \bar{m} \rangle = \nu$, we shall choose $p = \alpha$ and we restrict ν to values in $\alpha\mathbb{N}$. In the same way as in the Poisson case we can recover all values of α by scaling the length of the system.
The length of an interval $\bar{l}_j$ follows the distribution

$$P(l = pk/\nu) = (1-p)^{k-1} p \tag{6.10}$$

6.2.1 The convergence of $\bar{m}$

With help of X_k we can write $\bar{m}$ as

$$\bar{m} = \sum_{k=1}^{\nu/p-1} X_k \tag{6.11}$$

We define the random Variable

$$\bar{X} = \frac{p}{\nu} \sum_{k=1}^{\nu/p} X_k \tag{6.12}$$

We see $\left\langle \bar{X} \right\rangle = p$ and with Hoeffding's inequality (see [34] and [33][chapter 7]) we get

$$P\left(\left| \sum_{k=1}^{\nu/p} X_k - 1 \right| > \varepsilon\nu \right) = P\left(|\bar{X} - p| > \varepsilon p \right) \leq 2\exp\left(-2\frac{\nu^2\varepsilon^2 p^2}{p^2}\frac{p}{\nu} \right) = 2e^{-2\nu\varepsilon^2/p} \tag{6.13}$$

Because

$$\left| \bar{m} - \sum_{k=1}^{\nu/p} \right| \leq 1 \tag{6.14}$$

we established Lemma 3.8 which formed the essential part of Theorem 3.9.

6.2.2 Determine $\bar{\mu}$

The interval density energy for a Bernoulli distribution is defined as

$$\bar{e}_0(\gamma,\nu) = \nu \sum_{\substack{j=1 \\ \bar{l}_j = pj/\nu}}^{\infty} P_j \frac{n_j}{\bar{l}_j^2} e(n_j \bar{l}_j \gamma, \bar{l}_j \sigma), \qquad P_j = (1-p)^{j-1} p \tag{6.15}$$

Because the distribution of possible lengths is discrete we get a minimizer $(\bar{n}_j^{\mathrm{IDF}})_j$ in ℓ^1. We can determine the minimizer in the same way as in Chapter 2

$$\bar{n}_j^{\mathrm{IDF}} = n^{\bar{\mu}}(\bar{l}_j) = \frac{1}{\bar{l}_j \gamma} \bar{n}(\bar{\mu}\bar{l}_j^2, \infty) \tag{6.16}$$

but this time $\bar{\mu}$ is fixed by the normalization constraint

$$\nu \sum_{\substack{j=1 \\ \bar{l}_j = pj/\nu}}^{\infty} P_j n_j^{\bar{\mu}} = 1. \tag{6.17}$$

We can write this constraint in the form

$$\nu \frac{p}{1-p} \sum_{\substack{j=1 \\ \bar{l}_j = pj/\nu}}^{\infty} e^{\log(1-p)\bar{l}_j \nu / p} n_j^{\bar{\mu}} = 1 \tag{6.18}$$

Lemma 6.1.

$$\nu \sum_{j=1}^{\infty} e^{-c\bar{l}_j \nu} n_j^{\bar{\mu}} \sim \nu \int_0^\infty \nu e^{-cl\nu} n^{\bar{\mu}}(l) \mathrm{d}l \tag{6.19}$$

Proof. We use the fact that $e^{-c\bar{l}_j\nu}$ is monotone decreasing and that $n^{\bar{\mu}}(\bar{l}_j)$, with $\bar{l}_j = pj/\nu$, is monotone increasing. If we view $e^{-c(l-p/\nu)\nu}$ as a function in the interval $[\bar{l}_j, \bar{l}_j + p/\nu]$ then $e^{-cl\nu + cp} > e^{-c\bar{l}_j\nu}$. Because $n^{\bar{\mu}}(\bar{l}_j)$ is monotone increasing we can approximate the sum from above.

$$\nu \sum_{j=1}^{\infty} e^{-c\bar{l}_j\nu} n_j^{\bar{\mu}} \leq \nu \frac{1}{p} e^{cp} \int_{p/\nu}^\infty \nu e^{-cl\nu} n_j^{\bar{\mu}}(l) \mathrm{d}l \leq \nu \frac{1}{p} e^{cp} \int_0^\infty \nu e^{-cl\nu} n_j^{\bar{\mu}}(l) \mathrm{d}l \tag{6.20}$$

The lower bound works analogously if we shift the integral.

$$\nu \sum_{j=1}^{\infty} e^{-c\bar{l}_j\nu} n_j^{\bar{\mu}} \geq \frac{\nu}{p} \int_{p/\nu}^\infty \nu e^{-cl\nu} n^{\bar{\mu}}(l - p/\nu) \geq \frac{\nu}{p} e^{-cp} \int_0^\infty \nu e^{-cy\nu} n^{\bar{\mu}}(y) \mathrm{d}y \tag{6.21}$$

□

From this proof we get the relation $e^{c\nu\pi/\sqrt{\bar{\mu}}} \sim \bar{\mu}/\gamma$ and therefore

$$\bar{\mu} \sim \frac{c^2\nu^2}{(1 + \log(c^2\nu^2/\gamma))} \sim c^2 \frac{\nu^2}{(1 + \log(\nu^2/\gamma))^2} \sim \mu \tag{6.22}$$

for $\gamma \ll \nu^2$ and

$$\bar{\mu} \sim \gamma \sim \mu \tag{6.23}$$

otherwise. We used the same argument as in the Poisson potential case, replacing ν with $c\nu$. We therefore collected all necessary relations we used in Chapter 3 to prove the convergence to the interval density energy.

Theorem 6.2. *Let $e_0(\gamma, \nu)$ be the interval density functional for a Bernoulli distributed random potential 6.15 and $e^{\mathrm{GP}}(\gamma, \sigma, \nu)$ the GP energy for the same potential, then*

$$\frac{e_0(\gamma, \nu)}{e^{\mathrm{GP}}(\gamma, \sigma, \nu)} \xrightarrow{p} 1 \tag{6.24}$$

To discuss superfluidity in detail we need to repeat the arguments of Chapter 4 for this potential. Because $\mu \sim \bar{\mu}$ the behavior of $\bar{n}_j^{\mathrm{IDF}} = n^{\bar{\mu}}(\bar{l}_j)$ is similar to $n^{\mathrm{IDF}}(l)$ we can conclude that in the case $\gamma \ll \nu^2$ nearly all intervals fulfill $\bar{n}_j^{\mathrm{IDF}} = 0$. On the other side we get for $\gamma \gg \nu^2$ that the mean density will converge to 1 on all intervals because there is a lower bound on the length of possible intervals.

Chapter 7

Summary

In the first two chapters we gave an introduction to the current state of research to Bose gas in a random potential. In particular we discussed the results from [22] which allowed us to describe the ground state of a one dimensional system with an arbitrary potential using the GP functional and the corresponding GP energy in the many particle limit.
In Chapter 2 and 3 we discussed the existence of a deterministic limit for the GP energy in more detail and also proved bounds for the convergence in probability. These bounds show that there are three reoccurring terms which determine the speed of convergence. First there is the term $\nu^{-1/2}$ which is a consequence of the convergence of $m/\nu \to 1$ which we need to prove Theorem 3.9. Second we get $\mu/(\gamma\nu)$ which ensures that the limit is deterministic and last we need $\sqrt{\mu}/\sigma \to 0$ for $\gamma \lesssim \nu^2$ or $\nu/\sigma \to 0$ otherwise to ensure the convergence of the lower bound.
Using these these results we discussed in Chapter 4 the properties of n_j^{GP} and we were able to show the convergence

$$\frac{1}{m}\sum_{j=1}^{m} g_\varepsilon(mn_j^{\mathrm{GP}}) \xrightarrow{p} 1 \tag{7.1}$$

in Lemma 4.4 for $\gamma \ll \nu^2$ which allowed us to conclude that there is no superfluidity in this limit. A more detailed analysis in Theorem 4.5 showed that there is also no superfluidity for the case $\gamma \sim \nu^2$. For the remaining case $\gamma \gg \nu^2$ we were able to show

$$|\phi^{\mathrm{GP}}|^2 \xrightarrow{L^2} 1 \tag{7.2}$$

in Theorem 4.14.
In Section 4.4 we discussed superfluidity for the case $\gamma \gg \nu^2$ and even though we were not able to answer the question if superfluidity exists for all parameters, we showed that for the case $\gamma \gg \nu^2\sigma^2$ there is complete superfluidity in the limit $\nu \to \infty$. Further we analyzed the case $\sigma \gg (\gamma/\nu^2)^4\sqrt{\gamma}$ with $\gamma \ll \nu^4$ using Theorem 4.5 to show the absence ob superfluidity.
In Chapter 5 we showed a different approach for analyzing the GP density and with this approach we showed the concentration of the density for $\gamma \ll \nu^2$. The method we used,

was to add a perturbation term to the GP functional and showed the convergence to deterministic energy. By differentiation the energy we were able to conclude the result.
In the last Chapter 6 we argued that it is possible to extend the results from Chapter 3 to a Bernoulli random potential. The biggest change using this potential was that the lengths of the intervals are bounded from below, which altered the case $\gamma \gg \nu^2$ in so far, that the mean density of all intervals converged to 1.

Appendices

Appendix A

German summary

In den ersten beiden Kapiteln haben wir frühere Forschungsergebnisse zum Thema eindimensionale Bose Gase in einem Zufallspotential diskutiert. Insbesondere die Resultate von [22] welche einem ermöglichen den Grundzustand eines eindimensionalen Systems im Vielteilchenlimes mit dem Gross Pitaevskii Funktional zu beschreiben.
Anschließend haben wir ein Poisson-verteiltes Zufallspotential eingeführt und das Lieb Liniger Modell für Bose Gas mit externen Potential durch die Gross Pitaevskii (GP) Energie beschrieben. Die GP Energie ist fixiert durch die Parameter γ, σ und ν wobei γ die Stärke der Wechselwirkung und σ die Stärke des Potentials beschreibt.
In Kapitel 2 und 3 haben wir die Existenz eines deterministischen Limes für die GP Energie in Wahrscheinlichkeit gezeigt und zusätzlich zu den Resultaten von [22] auch noch explizite Abschätzungen für die Wahrscheinlichkeiten bewiesen. Im Speziellen sieht man, dass ein deterministischer Limes existiert wenn

$$\nu \gg 1, \qquad \gamma \gg \frac{\nu}{\log \nu}, \qquad \sigma \gg \frac{\nu}{1 + \log(1 + \nu^2/\gamma)} \tag{A.1}$$

Diese Resultate zeigen die Konvergenz der Energien, aber für den Beweis, dass es keine Superfluidität gibt, braucht man Resultate über den Minimierer des GP Funktionals. In Kapitel 4 wird gezeigt dass man die Eigenschaften des Minimierers enes Dichtefunktionals, was von der Verteilung der Intervalle verschiedener Längen abhängt und die Energie des Systems im Limes beschrebt, zum Teil auch auf den Minimierer des GP Funktionals übertragen kann. Es zeigt sich unter der Bedingung (A.1), dass die Struktureffekte, die durch die Anordnung der Intervalle entstehen, vernachlässigbar sind und wir somit die Eigenschaften über die Konzentration der Dichte übertragen können. Im Speziellen können wir zeigen, dass im Limes für $\gamma \ll \nu^2$ fast alle Intervalle "leer" sind und für den Fall $\gamma \gg \nu^2$ die Dichte des Minimierers gegen die konstante Funktion 1 in L^1 konvergiert. Mit etwas mehr Aufwand kann auch gezeigt werden, dass im Fall $\gamma \sim \nu$ die Dichte auf Intervallen mit kleinen Längen beliebig klein wird im Limes und deshalb auch in diesem Fall keine Superfluidität existiert.
Weiters haben wir für den Fall $\gamma \gg \nu^2\sigma^2$ gezeigt, dass Superfluidität existiert und die GP Dichte gleichmäßig gegen 1 konvergiert.

In Kapitel 5 wurde eine alternative Methode diskutiert um die Konzentration der Dichte, welche wir in Kapitel 4 gezeigt haben, zu beweisen. Dafür wurde ein Störterm in das Funktional eingefügt und mittels Ableiten der Energie nach diesem Störterm wurde das Resultat gezeigt.
Durch Analyse der Beweise aus Kapitel 3 können wir in Kapitel 6 zeigen, dass wir uns nicht zwangsläufig auf ein System mit Poisson verteilten Punkten beschränken müssen sondern, dass auch ein Bernoulli Potential die nötigen Eigenschaften erfüllt.

List of Notations

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Allgemein

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Bibliography

[1] Einstein A. Quantentheorie des einatomigen idealen Gaseszweite Abhandlung. *Preussischen Akademie der Wissenschaften*, 1925.

[2] Satyendranath Bose. Plancks Gesetz und Lichtquantenhypothese. *Zeitschrift für Physik. 26 (1924)*, 1924.

[3] Fritz London. On the bose-einstein condensation. *Physical Review*, 54(11):947, 1938.

[4] Allan Griffin, David W Snoke, and Sandro Stringari. *Bose-Einstein Condensation*. Cambridge University Press, 1996.

[5] N. N. Bogoliubov. On the theory of superfluidity. *Journal of Physics 11 (1)*, page 232, 1947.

[6] P. Kapitsa. *Nature*, 141:74, 1938.

[7] A. D. Misener J. F. Allen. Flow of liquid helium II. *Nature*, 141:75–75, 1938.

[8] Lászlo Tisza. Transport phenomena in helium II. *Nature*, 141:913, 1938.

[9] L. D. Landau. The theory of superfluidity of helium II. *J. Phys*, 5(1):71–90, 1941.

[10] Elliott H. Lieb and Jakob Yngvason. The ground state energy of a dilute two-dimensional bose gas. *Journal of Statistical Physics*, 103(3-4):509–526, 2001.

[11] Elliott H. Lieb, Robert Seiringer, and Jakob Yngvason. Superfluidity in dilute trapped Bose gases. *Phys. Rev. B*, 66:134529, Oct 2002.

[12] Valentin A Zagrebnov and Jean-Bernard Bru. The Bogoliubov model of weakly imperfect Bose gas. *Physics Reports*, 350(5):291–434, 2001.

[13] V.A. Zagrebnov O. Lenoble, L.A. Pastur. Bose-Einstein condensation in random potentials. *C. R. Physique*, 5:129–142, 2004.

[14] V.A. Zagrebnov O. Lenoble. Bose-Einstein condensation in the Luttinger-Sy model. *Markov Proc.Rel.Fields*, 13:441–468, 2007.

[15] V.A. Zagrebnov Th. Jaeck, J.V.Pulé. On the nature of Bose-Einstein condensation enhanced by localization. *J.Math.Phys.*, 51, 2010.

[16] H. Gimperlein, S. Wessel, J. Schmiedmayer, and L. Santos. Ultracold atoms in optical lattices with random on-site interactions. *Physical Review Letters*, 95(17):170401, 2005.

[17] V. I. Yukalov and R. Graham. Bose-Einstein-condensed systems in random potentials. *Physical Review A*, 75(2):023619, 2007.

[18] Leonardo Fallani, Chiara Fort, and Massimo Inguscio. Bose–einstein condensates in disordered potentials. *Advances In Atomic, Molecular, and Optical Physics*, 56:119–160, 2008.

[19] Laurent Sanchez-Palencia, David Clément, Pierre Lugan, Philippe Bouyer, and Alain Aspect. Disorder-induced trapping versus anderson localization in Bose-Einstein condensates expanding in disordered potentials. *New Journal of Physics*, 10(4):045019, 2008.

[20] Pierre Lugan, Alain Aspect, Laurent Sanchez-Palencia, Dominique Delande, Benoît Grémaud, Cord A Müller, and Christian Miniatura. One-dimensional anderson localization in certain correlated random potentials. *Physical Review A*, 80(2):023605, 2009.

[21] Laurent Sanchez-Palencia and Maciej Lewenstein. Disordered quantum gases under control. *Nature Physics*, 6(2):87–95, 2010.

[22] Robert Seiringer, Jakob Yngvason, and Valentin A Zagrebnov. Disordered Bose Einstein condensates with interaction in one dimension. *Journal of Statistical Mechanics: Theory and Experiment*, 2012(11):P11007, 2012.

[23] E.H. Lieb, R. Seiringer, J.P. Solovej, and J. Yngvason. *The Mathematics of the Bose Gas and its Condensation*. Oberwolfach Seminars. Birkhauser Verlag GmbH, 2006.

[24] R.C. Clark and G.H. Derrick. *Mathematical methods in solid state and superfluid theory*. SUSSP publications. Plenum Press, 1968.

[25] Elliott H. Lieb and Werner Liniger. Exact analysis of an interacting bose gas. i. the general solution and the ground state. *Phys. Rev.*, 130:1605–1616, 1963.

[26] A. J. Leggett. Superfluidity. *Rev. Mod. Phys.*, 71:S318–S323, 1999.

[27] J. Yngvason. private communication.

[28] J. Yngvason. BEC, interactions and disorder. In *Disordered quantum many-body systems*. BIRS, Banff, 2013, (https://www.birs.ca/events/2013/5-day-workshops/13w5058/videos).

[29] R. Seiringer. private communication.

[30] R. Seiringer and J. Yngvason. private communication.

[31] D. Witthaut, K. Rapedius, and H.J. Korsch. The nonlinear Schrger equation for the delta-comb potential: quasi-classical chaos and bifurcations of periodic stationary solutions. *J. Nonlinear Math. Phys.*, 16(2):207–233, 2009.

[32] Derek F Lawden. Jacobis elliptic functions. In *Elliptic Functions and Applications*, pages 24–49. Springer, 1989.

[33] A. Shiryaev and S.S. Wilson. *Probability*. Graduate Texts in Mathematics. Springer, 1995.

[34] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.