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„Manipulation in Preference Aggregation Situations: An Optimality Approach for the Walrasian Pure Exchange Economy"

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MANIPULATION IN PREFERENCE AGGREGATION SITUATIONS:
AN OPTIMALITY APPROACH FOR THE WALRASIAN PURE EXCHANGE ECONOMY

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Abstract. This article gives an overview of the most important theoretical results in the field of manipulability of voting procedures. Furthermore it reconsiders the topic applied to a two agent, two good pure exchange economy. Parameterized functions are introduced to examine possible optimality of manipulation.
Keywords. Manipulability of Voting Procedures, Arrow's Possibility Theorem, Gibbard-Satterthwaite Theorem, Optimality of Manipulation

Contents

List of Figures

1. Preface

In a democratic environment there are many situations in which a decision is not made by a single individual. But rather several individuals are involved in the decision process to achieve an outcome which is accepted by the whole participating society. Examples can be political elections, committees, decisions in an executive board in a firm and many others where individuals interact with each other to achieve a decision. Therefore voting procedures are important instruments in collective decision finding. A voting procedure aggregates the preferences of the individuals and determines the social decision. There are different aggregation methods but each method determines the social decision based on individual preferences.

There exists a comprehensive literature in this field gathered under the term social choice theory. A part of this literature is concerned with the analysis of an important feature of voting procedures called manipulability. Usually individuals which participate in a vote, should cast their vote in accordance with their preferences. But if it is possible for some individual to improve the outcome of the vote in its favor by casting a vote not in line with its preferences then it is highly likely that it will do so. Thus this individual manipulates the vote. The research which has been done on this topic is either strengthening already existing results or extending them to more general forms. Optimality aspects are, at least to my knowledge, not really considered. But they should be. If an individual can manipulate a vote then it will not only manipulate in an arbitrary way but it will also try to somehow make its improvement as large as possible. This individual will try to manipulate the voting procedure optimally.

This article will give a summary of the essential findings on the topic of manipulability of voting procedures. Furthermore it will review a specific economic application of these results. A two agent, two commodity pure exchange economy with Walrasian allocation mechanism will be considered. Building up on this application, the focus will be on optimality of manipulation. By introducing parameterized functions it will be investigated whether it is possible to manipulate the allocation procedure in an optimal way.

This article is organized in the following way. Section 2 will provide the necessary definitions and formalism for comprehension of the key terms and the main theoretical results on collective decision making. Section 3 will present and review *Arrow's General Possibility Theorem*, an essential theorem in the field of social choice theory. Many of the findings which followed over time are using the logical structure of this theorem or are based on it. Section 4 will summarize the most important literature on the topic of manipulation of voting procedures. While section 5 will present the mentioned economic application to the Walrasian Pure Exchange Economy with two agents and two commodities. Finally, as already pointed out, Section 6 will investigate possible optimality of manipulation introducing parameterized utility functions. At the end section 7 will provide some conclusions.

2. Notation and Definitions

2.1. Preliminaries.

Let $N = \{1, 2, 3, \ldots, n\}$ denote the finite set of individual voters. The set of alternatives for all individual voters will be denoted by $X = \{x, y, z, \ldots\}$. Each voter has preferences over the set of alternatives X.

2.2. Binary Relations.

These preferences will be displayed by binary relations R on the set X such that R is a subset of $X \times X$. We can say that $(x, y) \in R \Leftrightarrow xRy$.

Let I be the *symmetric part* of a relation R denoting an indifference relation: $xIy \Leftrightarrow (xRy \wedge yRx)$ and let P be the *asymmetric part* of a relation R indicating a strict preference relation: $xPy \Leftrightarrow (xRy \wedge \neg xIy)$.
So we can say that an alternative x is *at least as good as* an alternative y if xRy. Furthermore an alternative x is *strictly preferred to* an alternative y if xPy and there is *indifference* between the alternatives x and y if xIy.

2.2.1. *Properties of Binary Relations.*

The following properties of binary relations can now be defined:

Definition 1. *R is* reflexive, *if $\forall\ x \in X$ it holds that xRx.*

Definition 2. *R is* transitive, *if $\forall\ x, y, z \in X$ it holds that $xRy \wedge yRz \Rightarrow xRz$.*

Definition 3. *R is* complete, *if $\forall\ x, y \in X$ it holds that $xRy \vee yRx$.*

Definition 4. *R is* symmetric, *if $\forall\ x, y \in X$ it holds that $xRy \Rightarrow yRx$.*

With the notion of properties of binary relations the following terms can be defined:

Definition 5 (Quasi Ordering)**.** *A binary relation R is a* quasi ordering, *if R is reflexive and transitive.*

Definition 6 (Ordering)**.** *A binary relation R is an* ordering, *if R is reflexive, transitive and complete.*

Definition 7 (Equivalence Statement)**.** *A binary relation R is an* equivalence statement, *if R is a symmetric quasi ordering.*

Definition 8. *A binary relation R is* quasi transitive, *if its asymmetric part P is transitive.*

Definition 9. *A binary relation R is* acyclical, *if there does not exist any finite sequence $(x_1, x_2, \ldots, x_k)$ in the set of alternatives such that $x_1 \succ x_2 \wedge x_2 \succ x_3 \wedge \ldots \wedge x_{k-1} \succ x_k \wedge x_k \succ x_1$, i.e. there do not exist strict preference cycles.*

2.3. **Choice Functions.**

As a next step certain essential subsets of the set of alternatives X will be defined, using the definitions of maximal and best elements in a particular set:

Definition 10 (Maximal Element)**.** *A* maximal element *of the set $S \subseteq X$ is any $x \in S$ such that $\nexists\, y \in S \; yPx$.*

Let the *set of all maximal elements* of a set $S \subseteq X$ with respect to the underlying binary relation R be denoted by $M_R(S) = \{x \in S \mid \nexists\, y \in S : yPx\}$.

Definition 11 (Best Element)**.** *A* best element *of a set $S \subseteq X$ is any $x \in S$ such that $xRy \;\forall\, y \in S$.*

Following the above definition, let the *set of all best elements* of a set $S \subseteq X$ with respect to the underlying binary relation R be denoted by
$C_R(S) = \{x \in S \mid \forall\, y \in S : xRy\}$.

The set of best elements $C_R(S)$ will denote the *choice set*.

From the above definitions of best elements and maximal elements we can conclude that a best element in a set S is also a maximal element in the set S . In other words the set of best elements is a subset of the set of maximal elements, i.e. $C_R(S) \subseteq M_R(S)$. This results holds for all $S \subseteq X$.

Proof. Let $x \in C_R(S)$ and suppose that $x \notin M_R(S)$. $x \notin M_R(S) \Rightarrow \exists\, y \in S$ such that yPx. $yPx \Rightarrow (yRx) \wedge (\neg yIx)$ ↯.

□

If the binary relation R is in addition an ordering then it holds that the set of best elements equals the set of maximal elements, i.e. $C_R(S) = M_R(S)$.

Proof. Let $x \in M_R(S)$. $x \in M_R(S) \Rightarrow \neg(yPx) \;\forall\, y \in S$. Using the property of completeness of an ordering $\neg(yPx) \Rightarrow xRy \;\forall y \in S$. □

Given these definitions and used properties it is still possible that the choice set is an empty set, unless we define that the set $S \subseteq X$ must be non-empty and finite and require that the binary relation R is an ordering. Then it follows that: $C_R(S) = M_R(S) \neq \emptyset$.

Definition 12 (Choice Function)**.** *Let X be the set of alternatives and let $P_0(X)$ be the set of all non-empty subsets of X. Then the choice function is defined as mapping from $P_0(X)$ into $P_0(X)$ assigning a non-empty subset C(S) to every set S in the set of all subsets $P_0(X)$. Formally: $C:\ P_0(X) \longrightarrow P_0(X)$ sucht that $C(S) \neq \emptyset \, \forall \, S \subseteq P_0(X)$.*

Let the binary relation R_C denote the binary relation derived from a choice function C such that $xR_Cy \Leftrightarrow x \in C(\{x,y\})$.

From the definition of a choice function we conclude that, if the function C is a choice function then the derived binary relation from this choice function is reflexive and complete.

Proof. Let the set of alternatives X be the set $\{x, y\}$. Then, by definition, the choice set is a non-empty subset of $\{x, y\}$, i.e. $C(\{x, y\}) \subseteq \{x, y\}$. The possible non-empty subsets of $\{x, y\}$ are $\{x\}, \{y\}$ and $\{x, y\}$.
$C(\{x, y\}) = \{x\} \Rightarrow xR_Cy$
$C(\{x, y\}) = \{y\} \Rightarrow yR_Cx$ and
$C(\{x, y\}) = \{x, y\} \Rightarrow xR_Cy \wedge yR_Cx$
From the definition of R_C it follows immediately that:
$C(\{x, y\}) = \{x\} \Rightarrow xR_Cx$
$C(\{x, y\}) = \{y\} \Rightarrow yR_Cy$
$C(\{x, y\}) = \{x, y\} \Rightarrow xR_Cx \wedge yR_Cy.$
The binary relation R_C is therefore reflexive and complete. □

In the literature such a binary relation R_C is called a *base relation* corresponding to the choice function.

2.3.1. *Properties of Choice Functions.*

Definition 13 (Property α)**.** *Let X be a finite set of alternatives. Then $\forall\ S,T$ with $S \subseteq T \subseteq X$ and $\forall\, x \in S$ it holds that: $x \in C(T) \Rightarrow x \in C(S)$.*

This property of a choice function is a so called *consistency requirement for set contraction.* Let us consider an alternative x being in the choice set of a set T, which means that x is a best element in the set T. If we reduce the set T now to an arbitrary subset S and x is still available in this set then it must also be a best element in this subset.

Definition 14 (Property β)**.** *Let X be a finite set of alternatives. Then $\forall\, S,T$ with $S \subseteq T \subseteq X$ and $\forall\, \{x, y\} \subseteq C(S)$ it holds: $y \in C(T) \Leftrightarrow x \in C(T)$.*

This is a *consistency requirement for set expansion* for a choice function. Consider the elements x and y in the choice set of a smaller set S . They are both best elements in this set. If the set is now expanded to an arbitrary superset T then it must follow that either both elements are best elements in this superset or none of them.

Definition 15 (Property γ)**.** *Let X be a finite set of alternatives. Then for any family M of subsets of $S \subseteq X$ it holds that:* $[x \in \bigcap_{S \in M} C(S)] \Rightarrow [x \in C(\bigcup_{S \in M} S)]$.

Property γ is also a *consistency requirement for set expansion.* Consider an element x being a best element in the intersection of some subsets of the set of alternatives. If x is in this choice set and the set will be expanded to the union of the subsets then the alternative x must still be in the choice set of this larger set.

2.4. **Preference Aggregation Mechanisms.**

The following subsection deals with the question how preferences and choices of a group of individuals can be aggregated to a social choice, giving an overview and definitions of aggregation mechanism.

Let $i = 1, \ldots, n$ be a finite number of individuals and let $(R_1, \ldots R_n)$ be a *preference profile* on the set of alternatives $X$where R_i is an ordering of individual i.

2.4.1. *Plurality Rules.*

Definition 16 (Simple Majority Rule)**.** $\forall\, (R_1, \ldots, R_n)$ *and for any* $x, y \in X$:
$xRy \Leftrightarrow [N(xR_iy) \geq N(yR_ix)]$.[1]

A society prefers the alternative x over y if the number of individuals weakly preferring x over y is at least as large as the number of individuals weakly preferring y over x.
It is equivalent to define the Simple Majority Rule with strict preferences in the following way:
$xRy \Leftrightarrow [N(xP_iy) \geq N(yP_ix)]$.[2]

Definition 17 (α-Majority Rule)**.** $\forall\, (R_1, \ldots, R_n)$ *and for any* $x, y \in X$:
$xPy \Leftrightarrow [N(xP_iy) \geq \alpha \times |N|]$, *where* $|N|$ *is the absolute number of individuals* $i = 1, \ldots, n$ *and* $\alpha \in [0, 1]$

Aggregating preferences according to the α-Majority Rule implies a social preference for x over y if and only if the number of individuals preferring x over y is at least as large as α times the absolute number of individuals in the society.
A special case of the α-Majority Rule is the so called *Absolute Majority Rule* where the factor $\alpha = \frac{1}{2}$.

Definition 18 (Absolute Majority Rule)**.** $\forall\, (R_1, \ldots, R_n)$ *and for any* $x, y \in X$:
$xPy \Leftrightarrow [N(xP_iy) > \frac{1}{2} \times |N|]$, $yPx \Leftrightarrow [N(yP_ix) > \frac{1}{2} \times |N|]$ *and* xIy *otherwise.*

[1] $N(xR_iy)$ gives the number of voters with the preference xR_iy.
[2] $N(xP_iy)$ gives the number of voters with the preference xP_iy.

2.4.2. *Scoring Rules.*

Another possibility to aggregate preferences of a society are so called *scoring rules* or *positional procedures* where the decision is not made through plurality, like in the case of majority rules, but instead points are distributed to the available alternatives with respect to the individual preferences of the individuals. An example for a scoring rule is the so called *Borda Rank Order Rule*.
Each voter assign the score $s_l \in \{s_1, \ldots, s_k\} \subseteq \mathbb{R}$ to his l^{th}-most-preferred alternative and the choice set consists of the alternatives with the highest total score.

Definition 19 (Borda Rank-Order Rule)**.** *Let X be a finite set of alternatives containing k elements and let there be a finite number of n individual voters. Each voter $i = 1, \ldots, n$ distributes $k - 1$ points to his most preferred alternative, $k - 2$ points to his second preferred alternative, up to $k - k = 0$ points to his least preferred alternative. This results in a point distribution of each voter i for each alternative x.*
If a voter is indifferent between alternatives then the average of points, which would have been given under strict preference, will be assigned to the indifferent alternatives.
The social preference is then computed by the sum of total points of an alternative from all individual voters. Formally, let $g(x)$ the number of total points of alternative x from all voters i be defined as: $g(x) = \sum\limits_{i=1}^{n} g_i(x)$.
Then $xPy \Leftrightarrow g(x) > g(y)$ and $xIy \Leftrightarrow g(x) = g(y)$.

2.4.3. *Utility Rules.*

Another aggregation approach is the *Utilitarian Approach* where the concept of utility is used to define the aggregate social outcome of voting on different alternatives.

Definition 20 (Utilitarian Approach)**.** *Let X be a finite set of alternatives and let the number of individual voters $i = 1, \ldots, n$ be finite and let further the aggregate utility of an alternative $x \in X$ be defined as: $U(x) = \sum\limits_{i=1}^{n} u_i(x)$*
where $u_i(x)$ is the utility for individual i from alternative x.
Then $xRy \Leftrightarrow U(x) \geq U(y)$ and $xPy \Leftrightarrow U(x) > U(y)$ and $xIy \Leftrightarrow U(x) = U(y)$.

Using the Utilitarian Approach as a voting procedure the society will prefer the alternative x over the alternative y if and only if the aggregate utility from alternative x is larger than from alternative y.

Another utility based voting procedure is the so called *Rawl's Criterion* where the the socially preferred alternative is that alternative which maximizes the utility of the individual which is worst off.

The *Utility Gain* approach considers the utility of the participating voters before and after the decision. It chooses that alternative as a winner of the vote which ensures the highest utility gain for the individuals in a before-after comparison by multiplying the utilities of the individuals for each alternative available.

Definition 21 (Utility Gain Approach)**.** *Let X be a finite set of alternatives and let the number of voters be finite and given by $i = 1, \ldots, n$.*
Further let $G(x) = \prod_{i=1}^{n}[u_i(x) - u_i(\bar{x})] \; \forall \, x \in X$ define the aggregate gain in utilities where $u_i(\bar{x})$ is the utility for individual i at status quo and $u_i(\bar{x}) = 0$. Then $xPy \Leftrightarrow G(x) > G(y), xRy \Leftrightarrow G(x) \geq G(y)$ and $xIy \Leftrightarrow G(x) = G(y)$.

2.5. **Social Welfare Function.**

The following subsection gives a definition of a *Social Welfare Function* and of properties or conditions that can be associated with social welfare functions.

Definition 22 (Social Welfare Function)**.** *A* Social Welfare Function *is a mapping of the set of preference profiles into the set of orderings:*
$R = f(R_1, \ldots, R_n)$.

Definition 23 (Condition U)**.** (Unrestricted Domain)
The domain of f consists of all logically possible n-tuples of the orderings of the set of alternatives X.

Definition 24 (Condition P)**.** (Weak Pareto Principle)
$\forall \, x, y \in X : [xP_iy \, \forall \, i = 1, \ldots, n] \Rightarrow xPy$

If an alternative x if strictly preferred to an other alternative y by all individuals. Then this implies that the society prefers alternative x over y.

Definition 25 (Condition I)**.** (Independence of Irrelevant Alternatives)
Let $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$ be two possible preference profiles, such that $R = f(R_1, \ldots, R_n)$ and $R' = f(R'_1, \ldots, R'_n)$.
If $(xR_iy \Leftrightarrow xR'_iy) \wedge (yR_ix \Leftrightarrow yR'_ix) \, \forall i = 1, \ldots, n \Rightarrow (xRy \Leftrightarrow xR'y) \wedge (yRx \Leftrightarrow yR'x)$.

Consider two different preference profiles which coincide in at least one relation x and y for all individuals. Then it is implied that the society has the same preference in this respect.

Definition 26 (Condition D)**.** (Non-Dictatorship)
$\nexists \, i \in N$ *such that* $\forall \, (R_1, \ldots, R_n)$ *and* $\forall \, (x, y)$ *it holds that* $: xP_iy \Rightarrow xPy$.

There is no individual in the society which preference over any pair of alternatives determine the social preference over that pair.

Definition 27 (Condition A)**.** (Anonymity)
Let $(R_1, \ldots, R_n)$ *and* $(R'_1, \ldots, R'_n)$ *be two different preference profiles related by a permutation then it holds that:* $\forall\, (x, y) \in X : xRy \Leftrightarrow xR'y$*, where* $R = f(R_1, \ldots, R_n)$ *and* $R' = f(R'_1, \ldots, R'_n)$*.*

A voting procedure is said to be anonymous, if it does not make any difference which individual in the society expresses a certain preference. If the sequence of voters gets permuted then the social preference should not be affected.
It should be noticed that Condition Anonymity is a stronger requirement than Condition Non-Dictatorship. In other words Condition A implies Condition D.

Definition 28 (Condition N)**.** (Neutrality)
Let $(R_1, \ldots, R_n)$ *and* $(R'_1, \ldots, R'_n)$ *be two preference profiles with corresponding social preferences denoted by* R *and* R'*.*
Then $\forall\, x, y, z, w \in X : [(xR_iy \Leftrightarrow zR'_iw) \wedge (yR_ix \Leftrightarrow wR'_iz)]\ \forall\, i \in N \Rightarrow [(xRy \Leftrightarrow zR'w) \wedge (yRx \Leftrightarrow wR'z)]$*.*

Note that Condition N and Condition I are strongly related to each other. Condition N is a stronger requirement than Condition I. Neutrality implies Independence of Irrelevant Alternatives. Neutrality requires that different issues, which can be voted on, are treated in the same way. The aggregation mechanism is independent on the identity of alternatives.

Definition 29 (Condition PR)**.** (Positive Responsiveness)
Let $(R_1, \ldots, R_n)$ *and* $(R'_1, \ldots, R'_n)$ *be two preference profiles with corresponding social preferences denoted by* R *and* R'*.*
Then $\forall\, (x, y) \in X : [s_1(x, y) \wedge s_2(x, y)] \Rightarrow (xRy \Rightarrow xP'y)$*, where*
$s_1(x, y) \Leftrightarrow [\forall\, i \in N : (xP_iy \Rightarrow xP'_iy \wedge xI_iy \Rightarrow xR'_iy)]$ *and*
$s_2(x, y) \Leftrightarrow [\exists\, j \in N : (xI_jy \Rightarrow xP'_jy \vee yP_jx \Rightarrow xR'_jy)]$*.*

The Condition Positive Responsiveness requires an aggregation mechanism to react to changes in the individual preferences. The property $s_1(x, y)$ means that no voter has changed his preference from alternative x to y and property $s_2(x, y)$ holds, if at least one voter changed his preference preferring x over y.
An aggregation mechanism fulfills Positive Responsiveness, if the properties s_1 and s_2 imply that the social preference will react to such a change in the way that a weak social preference for x will turn into a strict social preference for x.

Definition 30 (Condition CONS)**.** Consistency
Let f *be a social choice function and let* R' *and* R'' *be preference profiles for disjoint subsets of voters* N' *and* N''*. Further let* R *be the preference profile of the population of voters* $N' \cup N''$ *that agrees with* R' *on* N' *and with* R'' *on* N''*. Then the function* f *is* consistent *if it holds that:*
$[f_{[R']}(S) \cap f_{[R'']}(S) \neq \emptyset] \Rightarrow f_{[R']}(S) \cap f_{[R'']}(S) = f_{[R]}(S)$*.*

Definition 31 (Condition F)**.** (Faithfulness)
A social choice function f is faithful, if it holds that:
$f_{R_i}(S) \Leftrightarrow M_{R_i}(S)\, \forall\, i \in N.$

Definition 32 (Balanced Profile)**.** *The preference profile $R = (R_1, \ldots, R_n)$ is balanced if:*
$\forall\, (x, y) \in X : N(xR_iy) = N(yR_ix).$

Definition 33 (CP)**.** (Cancelation Property)
Let X be a finite set of alternatives and let $S \subseteq X$ be non-empty. A social choice function f satisfies CP if $f_{[R]}(S) = S$ for all balanced profiles.

Definition 34 (Condorcet Winner)**.** *Let X be a finite set of alternatives. A* Condorcet Winner *is an alternative $x \in X$ that is not beaten by any other alternative $y \in X$ in a simple majority vote. Formally:*
$N(xR_iy) \geq N(yR_ix)\, \forall\, y \in X.$

Definition 35 (Betweenness)**.** *Let X be a finite set of alternatives, and let S be an arbitrary strict ordering on X. Then $B(x, y, z) \Leftrightarrow [(xSy \wedge ySz) \vee (zSy \wedge ySx)]\, \forall\, x, y, z \in X.$*

Betweenness considers three distinct alternatives x, y and z. An arbitrary alternative y is said to be between x and z if and only if it holds that x is less than y and y is less than z or z is less than y and y is less than x with regard to the strict ordering S.

From this definition it follows directly that, if there exists a strict ordering S on the set of alternatives X then for all distinct alternatives x, y and z in this set it must either be the case that $B(x, y, z)$ or $B(y, x, z)$ or $B(y, z, x)$.

With the stated definition of Betweenness it is possible to define now the concept of *Single Peaked Preferences*.

Definition 36 (Single Peaked Preferences)**.** *Let $R = (R_1, \ldots, R_n)$ be a preference profile. R is* Single Peaked*, if there exists an ordering S on X such that*
$\forall\, x, y, z \in X$ it holds that $[xR_iy \wedge B(x, y, z)] \Rightarrow yP_iz\, \forall\, i \in N.$

The shape of such a preference profile is single peaked and monotonically decreasing in the directions away from the peak.

3. Arrow's General Possibility Theorem

In order to value and compare different choice rules or aggregation mechanisms it is necessary to introduce certain characteristics or properties such an aggregation mechanism should fulfill. These characteristics are based on commonly agreed rationality criteria and fairness concerns.

In the following Arrow [1] formulated the conditions *Unrestricted Domain (U), Independence of Irrelevant Alternatives (I), the Weak Pareto Principle (P)* and *Non-Dictatorship (D)* requiring that a rational, fair and socially acceptable aggregation mechanism should meet these conditions.

1951 Arrow formulated one of the most important results in social choice theory. The so called *General Possibility Theorem* [1] which became more famous known as *Arrow's Impossibility Theorem.*

Theorem 1 (Arrow's Impossibility Theorem)**.** *For a finite number of individuals and at least three distinct alternatives there is no Social Welfare Function satisfying the conditions U, P, I and D.*

The Proof needs the definition of decisiveness of a group and furthermore two lemmata for group contraction and field expansion. The presented proof strategy was introduced by Sen [34] and needs less definitions than the original version provided by Arrow.

Definition 37 (Decisiveness)**.** *Let* $N_0 \subseteq N$ *and let* (x, y) *be any pair of alternatives in X. Then* N_0 *is* decisive over the pair (x,y) *if and only if* $xP_iy \,\forall\, i \in N_0 \Rightarrow xPy$*.* N_0 *is* decisive *if it is decisive over all possible pairs of alternatives.*

Lemma 1 (Field Expansion)**.** *If a set* $N_0 \subseteq N$ *is decisive over any pair of alternatives* (x, y)*, then it is decisive.*

Lemma 2 (Group Contraction)**.** *If a set* $N_0 \subseteq N$ *contains more than 1 individual and is decisive, then there exists a proper subset* $N_1 \subset N_0$ *that is also decisive.*

The proof uses the following logical steps:
If the Weak Pareto Principle holds then this implies that the whole society N is decisive under application of the Field Expansion Lemma. If the Group Contraction Lemma is then repeatedly used then this yields that there is a subset of N with only one element which is decisive. Such a single individual would be a dictator which implies that the Non-Dictatorship Condition is violated.

Schick [31] reconsiders the assumptions which Arrow has imposed in this influential work. He focuses on the assumption of transitivity of preferences. Arrow requires that the preferences of the individuals should fulfill the properties asymmetry, transitivity of preferences and transitivity of indifference in order to be internally consistent. Schick [31] argues that intransitivity of indifference relations must not necessarily lead to inconsistence. It is possible that an individual can have a rational incentive for a strict preference which means that $(xIy \wedge yIz) \Rightarrow xIz$ should not be generally required. This possibility would be excluded, if transitivity of preferences and indifference is required. Requiring transitivity of indifference can therefore lead to inconsistency but it need not be the case. This is shown in a game theoretical approach using $2x2$ and $3x2$ normal form games. It can exist a pure strategy which strictly dominates one of the remaining strategies. In this sense full consistency is given even if the preferences are quasi-transitive instead of transitive.

Sen [32] argues that Arrow's General Possibility Theorem (Impossibility Theorem) holds if and only if the preferences are transitive. If this requirement is relaxed to quasi-transitivity then the result does not hold. Sen introduces the term of *Social Decision Function*.

Definition 38 (Social Decision Function[32])**.** *A* Social Decision Function *is a mapping from the set of preference profiles into the set of choice functions:*
$C = f(R_1, \ldots, R_n)$.

In contrast to Arrow's [2] Social Welfare Function, which is a mapping into the set of orderings, the Social Decision Function defined by Sen [32] is a mapping into the set of choice functions. The difference is that Sen does not require the underlying preference profile to fulfill transitivity. This allows Sen to formulate the following theorem:

Theorem 2. *[32, p. 387] There is a Social Decision Function satisfying Conditions U, P, I and D, for any finite set X.*

This result has a crucial drawback. Mas-Colell and Sonnenschein [21] prove that for any Social Decision Function and a set of alternatives containing at least 3 distinct alternatives the function is either dictatorial or at least one individual has a veto right.

Definition 39 (Veto Right)**.** *An individual $i \in N$ has a* veto right *if and only if it holds that: $xP_iy \Rightarrow xRy$.*

Satterthwaite [30] establishes a relation between voting procedures and social welfare functions defined by Arrow [2]. It is shown that there exits a one-to-one correspondence between strategy-proof voting procedures and social welfare functions in the sense of Arrow.[3] As a second result it is also proved that a voting procedure which is derived from a social welfare function can not be strategy-proof, if Arrow's conditions are not met. These two results are then used to prove Arrow's General Possibility Theorem in a new way.

[3] For definition and discussion of strategy-proofness and strategy-proof voting procedures see section 4.

Reny [26] provides a specific proof showing the connection between Arrow's Theorem and the Gibbard-Satterthwaite Theorem. This leads to the conclusion that both results have a strong logical relation to each other.[4]

4. Manipulability

As discussed in the last section a desirable aggregation mechanism, representing the preferences of the individuals in a given society, should be based on fairness and rationality as important cornerstones. Another feature which should be handled with equal importance is that an aggregation mechanism should guarantee that the participating individuals reveal their preferences truthfully. This means that they vote in accordance with their actual real preferences. This will be the topic of the following section and is discussed in the literature under the expression *manipulability*.
In every situation where individuals cast a vote on an arbitrary topic there are two possibilities in revealing the preferences. Either individuals vote in accordance with their real preferences or they vote not in accordance with their real preferences. A rational individual will reveal its true preferences unless there is some incentive for not doing so. In a voting situation there is an incentive for the individuals to misrepresent their true preferences if, by using a strategy, it is possible to ensure some more desired outcome compared to the outcome when representing the preferences truthfully. Farquharson [12] introduced the corresponding terms *sincere voting* and *sophisticated voting*.

4.1. **Motivation.**

The following example, taken from Taylor [35], illustrates the described problem: Let there be four voters and four alternatives, x, y, z and w and let the aggregation mechanism be the Borda Rank-Order Rule. Formally: $N = \{1, 2, 3, 4\}$,
$X = \{x, y, z, w\}$. Furthermore let the preferences of the individuals be given by $xP_1yP_1zP_1w$, $yP_2wP_2zP_2x$, $wP_3zP_3xP_3y$ and $zP_4xP_4yP_4w$.

1	2	3	4
x	y	w	z
y	w	z	x
z	z	x	y
w	x	y	w

The given preferences of the individuals can be displayed as in the table above.

[4] The Gibbard-Satterthwaite Theorem will be discussed in section 4.

Using the Borda Rank-Order Rule for aggregation the social outcome will be: $zPyIxPw$. Alternative z would be the winner of this vote.
Now let the same example be reconsidered with the difference that individual 1 states the following preferences instead of its real preferences: $yP_1'xP_1'wP_1'z$. The other individuals do not state different preferences. The preferences are displayed in the following table:

1'	2	3	4
y	y	w	z
x	w	z	x
w	z	x	y
z	x	y	w

The social outcome, using again the Borda Rank-Order Rule, has changed to $yPzIwPx$. The winner of this vote is alternative y. As individual 1 has the true preference yP_1z, it has improved the outcome by misrepresenting its preferences. Therefore individual 1 has an incentive for sophisticated voting or voting strategically to improve the outcome in its favor . Thus the aggregation mechanism is manipulable.

An aggregation mechanism can be manipulable. If it is not manipulable then the mechanism is called strategy-proof. In the following the respective definitions are stated in terms of choice functions.

4.2. **Definition of Manipulability.**

Let $N = \{1, \dots, n\}$ be a finite set of individual voters, let $X = \{x, y, z, \dots\}$ be a set of alternatives. Furthermore let the set $S \subseteq X$ be an agenda which can be voted on and let $C(S) \subseteq S$ be a choice function determining the social outcome of the vote on the agenda S and let $R = (R_1, \dots, R_n)$ be the underlying preference profile so that the choice function can be written as $C(S, R)$.

Definition 40 (Manipulability). *A choice function C(S,R) is* manipulable *by an individual i at the preference profile $R = (R_1, \dots, R_{i-1}, R_i, R_{i+1}, \dots, R_n)$ and agenda S, if there exists an i-variant preference profile*
$R' = (R_1, \dots, R_{i-1}, R_i', R_{i+1}, \dots, R_n)$ which may differ only with respect to the preferences of individual i from R, such that
$C(S, R')\ P_i\ C(S, R)$.

Definition 41 (Strategy-Proofness). *A choice function is* strategy-proof*, if it is not manipulable by any individual under any preference profile and any agenda.*

4.3. **Review of Theory.**

In the literature the technical setup for the definitions of manipulability and strategy-proofness of voting procedures is used in different shades, but the key statement is everywhere the same. Strategy-proofness is always defined via manipulability of a voting procedure.

4.3.1. *Conjectures.*

Vickrey [36] already conjectures the key problem concerning manipulability and the theoretical work which follows subsequently.
If a voting procedure satisfies Arrow's [1] conditions of Independence of Irrelevant Alternatives, Non-Negative Response and the function is limited to rankings then it must be strategy-proof.[5]The provided statement is originally formulated for social welfare functions, but is influential for the theoretical work on manipulability. Another Conjecture by Vickrey [36] is revisited in the work of Sen [32]. If the assumption of transitivity is relaxed to quasi transitivity then the required conditions for a social welfare function in the sense of Arrow [1] are met. Vickrey [36] consideres the process of social decision making as a game in the technical sense of the word. He identifies a connection between Arrow's General Possibility Theorem [1] and manipulability of preferences. If the conditions of Arrow [1] are met then the voting procedure is also strategy-proof. Furthermore he argues that strategy-proofness and the condition of Independence of Irrelevant alternatives are at least logically very similar requirements, if not equivalent.
Vickrey [36] makes another important point. The strategic behavior of individuals in the process of voting is due to the available information. If an individual has absolutely no information about the preferences of the others or about the possible outcomes of the voting process then it is also not possible to set up an applicable strategy. Therefore the beliefs of the individuals are important for the evaluation of their behavior.
With these statements Vickrey [36] anticipates already the most important results in this area of research.

Dummet and Farquharson [11] consider the voting process in the form of a n-person game with ordinal preferences. The considered aggregation mechanisms are functions, which select one alternative as the socially preferred winner of a vote out of the set of alternatives. This class of aggregation mechanisms is discussed in the literature under the the term *voting procedure*. In contrast Arrow [1] considers strong orderings of all possible outcomes. Dummet and Farquharson [11] introduce the notion stable situation and stable outcome. If a majority of voters can secure any alternative as outcome of the vote then this outcome is stable. A situation is

[5]Rankings mean that the function is limited to strict preferences as arguments.

defined as stable, if no voter or group of voters can improve the outcome of the vote in their favor by voting strategically.
Dummet and Farquharson [11] distinguish between stable situations and stable outcomes. It is possible that an outcome is stable, but the situation is not. A provided example illustrates the problem:

Consider individuals $i = \{1, 2, 3\}$ and the set of alternatives be $X = \{a, b, c\}$ with the following preferences displayed in the table below. Aggregation mechanism is the Majority Rule with a managing order of $\{a, c, b\}$. This means that first there is a vote between alternatives a and c and then a vote between the winner of this vote and alternative b to determine the overall winner of the vote.

1	2	3
a	b	c
b	a	a
c	c	b

The overall winner of this vote will be alternative a. There is no majority of voters who would prefer another outcome. The outcome can be called stable. If individual 2 would instead of its real preferences report the preference $c > a$ then the overall winner would be alternative b. Therefore the situation can not be called stable.

Dummett and Farquharson [11] show that requiring the preferences to be single peaked is a sufficient condition to guarantee that a stable situation exists. The required condition of single peakedness is conceptually weaker than required by Black [5]. Dummett and Farquharson [11] allow for indifferences in the peak of individual preferences while Black [5] does not. The result of Dummett and Farquharson [11] relies strongly on the work of Black [5]. They show that if it would be possible to restrict preferences to be single peaked, it would be possible to construct a voting procedure which is strategy-proof.
Simultaneously Dummett and Farquharson [11] suspect that generally there is no voting procedure that is immune to strategical voting and conjecture the results of the theoretical work to follow in this field just as Vickrey [36] did.

4.3.2. *Theorems.*

The most important result for the topic of manipulability of voting procedures is known as the *Gibbard-Satterthwaite Theorem*. Gibbard [15] and Satterthwaite [29, 30] independently prove the same result.
On a logical level this result is closely related to the work of Arrow [1] on Social Welfare Functions. In fact it is also an impossibility theorem.
Gibbard [15] and Satterthwaite [29, 30] state that a strategy proof voting procedure which satisfies Arrow's [1] conditions and is not dictatorial does not exist.
The following paragraphs will give an overview of the these results.

4.3.3. *Satterthwaite's Theorem.*

Definition 42 (Weak Alternative Excluding)**.** *A voting procedure is weak alternative excluding, if:*
$\exists\, x \in X$ *such that* $x \notin C_P(X)\ \forall\, P = (P_1, \ldots, P_n)$.

Theorem 3. *[29, p. 10] Consider a situation with $n \geq 1$ individuals and $m \geq 3$ alternatives. A voting procedure is strategy proof if and only if it is either dictatorial or weak alternative excluding.*

Satterthwaite [29] distinguishes between a voting rule and a voting procedure. A voting rule is the general term for a specific allocation type of corresponding voting procedures. A voting procedure determines the choice for a particular set of voters and alternatives. Satterthwaite [29] also notes that a social decision may not only depend on individual preferences but also on beliefs on the preferences of the other individuals.
Satterthwaite [29] argues that a voting rule is strategy proof if and only if all corresponding voting procedures are strategy proof. Consequently a voting procedure is strategy proof, if the set of sincere strategies $P = (P_1, \ldots, P_n)$ is a Nash equilibrium [24].[6]Thus a voting procedure is not strategy proof, if there exists a set of sincere preferences $P = (P_1, \ldots, P_n)$ which is not a Nash equilibrium. It is shown that for strict preference orderings every voting procedure is either dictatorial or weak alternative excluding.

A dictatorial voting procedure obviously violates the Non-Dictatorship requirement. A weak alternative excluding voting procedure restricts the range of possible outcomes to a subset $S \in X$ which violates the Weak Pareto Principle.

The stated conditions are necessary, not sufficient conditions for strategy proofness of a voting procedure. The theorem is formulated as a possibility theorem, but its consequence is an impossibility theorem. No desirable, strategy proof, democratic voting procedure for strict preference orderings exists.

[6]For definition, discussion and results see: Nash, J.: "Non-Cooperative Games". Annals of Mathematics, 54:286−295, 1951

Satterthwaite [30] extends the result by allowing for indifferences in the individual preferences. The conclusion is that no non-dictatorial strategy proof voting procedure exists. The technical framework and argumentation is similar to Satterthwaite [29].

Theorem 4. *[30, p. 193] Consider a situation with $n \geq 1$ individuals, corresponding (strict) preferences and $m \geq 3$ alternatives. A voting procedure is strategy proof if and only if it is dictatorial.*

Furthermore Satterthwaite [30] investigates the relationship between a strategy proof voting procedure and a social welfare function, which satisfies Arrow's [1] conditions. It is shown that the requirement of strategy proofness for voting procedures and Arrow's [1] requirements for social welfare functions can be linked by a one-to-one correspondence. This leads to the insight that a strategy proof voting procedure derived from a social welfare function must satisfy Arrow's [1] conditions. If the conditions are not satisfied then the voting procedure is not strategy proof.

4.3.4. *Gibbard's Theorem.*

Definition 43 (Straightforward Game)**.** *A game is called* straightforward*, if there exists a dominant strategy for every player and every preference ordering.*

Theorem 5. *[15] Every straightforward game with at least three possible outcomes is dictatorial.*

Corollary 6. *[15] Every voting scheme with at least three outcomes is either dictatorial or manipulable.*

Gibbard [15] formulates and proves the theorem independently from Satterthwaite. The conjectures by Vickrey [36] are influential for the result of Gibbard.
The theorem is formulated for game forms. The result for voting procedures is a logical consequence of the theorem.[7]The definition of a straightforward game is essential for the theorem.
Gibbard [15] shows that there is no non-trivial game which is straightforward. A trivial game is either dictatorial per definition or there are at most two possible outcomes. Thus every game with more than two possible outcomes is either dictatorial or manipulable as no equilibrium in dominant strategies exist.

Voting procedures are a special class of game forms. The strategies of the players in a voting procedures are preferences orderings of the alternatives.[8]The result for voting procedures follows immediately from the theorem on game forms. Every

[7]In contrast to Arrow's social welfare function [1], Gibbard [15] considers the class of single-valued functions as voting procedures.

voting procedure is either dictatorial or manipulable. Honest voting can not be a dominant strategy in a non-dictatorial voting procedure.

Gibbard [15] proves that any game form which is not dictatorial violates Arrow's conditions [1]. Subsequently the same follows for voting procedures. A dictator for the general class of game forms must also be a dictator in the class of voting procedures.

4.3.5. *Further Proofs of the Gibbard-Satterthwaite Theorem.*

Alternatively to the original proof by Gibbard [15], Benoît [4] provides a straightforward proof for the theorems of Gibbard [15] and Satterthwaite [29, 30].

Reny [26] shows that it is possible to construct a proof, such that both results, the Gibbard-Satterthwaite Theorem and Arrow's General Possibility Theorem, are reached. This approach displays the strong logical connection of the two different results.

[8] Gibbard [15] allows for indifferences in the preference orderings of the individuals.

4.3.6. *The Effect of Single Peakedness.*

Following the results of Gibbard [15] and Satterthwaite [29, 30], Blin and Satterthwaite [6] consider restrictive conditions on the set of preferences and the voting mechanism to guarantee strategy proofness of the voting procedure. They consider the effect of restricting preferences to be single peaked, as defined by Black [5] using a generalized majority rule as voting mechanism. The voting procedure selects the Condorcet winner, if it exists. If it does not exist then the voting process is carried out using the Borda rank order rule. In the case there is still a tie, one of the individuals is decisive.
Blin and Satterthwaite [6] find, by providing examples, that restricting the preferences to be single peaked is not sufficient to guarantee strategy proofness of the voting procedure. But the condition of single peakedness of preferences is sufficient to establish a transitive social welfare function in the sense of Arrow [1]. To guarantee strategy proofness of the voting procedure, the possible ballots which the individuals can vote on, must also be restricted to be single peaked in addition to the possible preferences. These two restrictions together are sufficient to ensure that the voting procedure is strategy proof under the described aggregation mechanism.

In relation to this result Penn et al. [25] confirm the findings and show that the condition of single peakedness of preferences is insufficient to ensure strategy proofness, regardless which voting procedure is used. They generalize the result of Blind and Satterthwaite [6] to all choice rules.

4.3.7. *Beliefs.*

As already mentioned, the involvement with manipulability can not be done without taking into account that individuals can only manipulate a voting procedure, if they have information about the preferences of other individuals. If the information about the preferences of the others is not certain, which is very likely, then the individuals must form beliefs on the preferences of the other individuals involved in the voting situation.
Following this reasoning Blin and Satterthwaite [7] show that it is not possible to construct a voting procedure, such that the participating individuals take only into account their own preferences, regardless what the preferences of the others might be.
They consider voting procedures as single valued functions and beliefs of individuals based on their past experience. The importance of uncertainty of beliefs is pointed out. Their result strengthens the results of Gibbard [15] and Satterthwaite [29, 30] on manipulability of voting procedures.

4.3.8. *Chance, Probabilities and Lottery.*

Gibbard [16] considers voting situations, or voting procedures in which the outcome is not uniquely determined, but instead probabilities play a role. He considers rank order rules where he does not allow for indifferences in the underlying preferences of the individuals. Gibbard [16] defines a situation as a decision scheme in which either a single alternative is chosen by the procedure or if there is more than one alternative chosen then the decision is made by a lottery over the chosen alternatives. Manipulability is then due to the preferences over the defined lotteries. The concluding result is that such a decision scheme will be strategy proof if and only if it either assigns heavy weight to one of the participating voters, which is in conflict with the condition of non-dictatorship, or if the outcome is restricted to certain alternatives from the alternative set, which violates the condition of Pareto optimality. Thus Gibbard [16] concludes that a decision scheme which is not violating the condition of Pareto optimality must be a mixture of dictatorial voting schemes with certain probabilities assigned. If the decision scheme is Pareto optimal itself then it must be dictatorial. This result also strengthens the prior findings on manipulability of Gibbard [15] and Satterwaith [29, 30, 6, 7]. It can be seen as a generalization of these results.

This result of Gibbard [16] is further enhanced by Duggan and Schwartz [10]. In contrast to the formulation by Gibbard [15, 16], Duggan and Schwartz [10] require that the property of manipulation holds for all pairs of alternatives. While Gibbard [16] considers the same assignment of probability for all individuals, Duggan and Schwartz [10] generalize this result and consider different assignments of probability on individual level. The result of Gibbard [16] is confirmed and provided in a more general formulation.

4.3.9. *Another Point of View.*

The presented work is philosophically in the same line, in the sense that, manipulability of a voting procedure is considered as a property that should be avoided in a desirable, democratic preference aggregation mechanism. In contrast to this mindset Miller [23] shows that manipulability of a voting procedure may not always be undesirable. If cooperation in voting is allowed it is possible to improve the outcome of the voting procedure in contrast to honest voting.[9]This is shown for majority voting with three individuals and three alternatives, yielding the result that in majority voting procedures cooperation in voting can belong to the Condorcet set, while honest voting does not.

[9]Cooperation in voting means that individuals know the preferences of the others and can negotiate in order achieve agreements over the vote.

4.3.10. *Summary.*

The following paragraph will try to sum up the presented findings. The main result regarding the topic of manipulability of voting procedures is that every desirable and possible voting procedure is vulnerable to manipulation. This result is confirmed and generalized by constructive and independent research. It does not seem to be possible to find a mechanism which ensures that the participating individuals will always honestly reveal their preferences to achieve a decision on a social level. In every decision finding procedure, where a plurality of individuals are involved, it can be assumed that individuals will try to gain an advantage by misrepresenting their preferences. This circumstance causes serious implications for every social decision. Thus it is not safe to assume that the outcome of a democratic voting procedure represents truthfully the real preferences of the individuals, which should be the case as it is the reason why a democratic voting procedure is used after all. Furthermore this circumstance can not be avoided in democratic decision finding. The only way out of this dilemma, which is known so far, is in fact a more theoretical approach with premises and restrictions which can not be guaranteed in real life situations.

Another fact that should be pointed out concerns the impact of beliefs and information. By manipulating a voting procedure to achieve an individually more preferred outcome individuals behave rational in the economic sense. The individual decision is based on the own preferences which are internal and not known to others just as the individual utilities generated by possible outcomes. It is a logical fact that manipulation is not possible without knowledge of the preferences of the other individuals involved in a decision situation. As the individual preferences are typically not known as long as individuals decide to reveal them, this lack of knowledge must be offset by forming beliefs on the preferences of the other individuals. These beliefs must be based on information collected in the past or some other reasonable assumptions on the others. Thus the distribution of information among the individuals and belief formation of the individuals is of high relevance in every decision situation.

5. Manipulability of the Walrasian Allocation Mechanism in a pure Exchange Economy

5.1. **Introduction.**

Manipulability has been considered so far in typical voting situations. First individuals cast a vote on a certain topic. Then their preferences are aggregated by a voting procedure to determine the social outcome or decision. Although manipulability is a circumstance which can hardly be avoided in real life voting situations, it is also a property which is immanent in every other situation where individuals have to interact with each other. Consider for example the provision of a public good. A public good is provided to the society and the costs will be distributed among the individuals in the society. The distribution depends on the individual willingness to pay for this good. A public good is by its definition non-excludable and non-rival.[10]If the individuals should share the costs of the provision by reporting their willingness to pay for this good every individual in the society has an incentive to understate its preference. Thus manipulability is immanent in this example. This is just one example for an economic application of the discussed results and there are many others where individuals interact with each other strategically in an economic environment.

This section deals with a specific economic application of the problem of manipulability. Object of consideration is a pure exchange economy with two individuals and two private goods. Each individual has an initial endowment and a private utility function. The individuals trade on the market in order to maximize their individual utilities from the two available goods. For the process of trade an imaginary person, the so called *auctioneer*, is assumed, who knows the utility functions of the individuals. The auctioneer decides on basis of the individual utility functions on the distribution of the goods to the individuals and implements the Pareto optimal market equilibrium. The consequences of manipulation and the process of manipulation itself will be investigated in the following.

The provided example was initially considered by Hurwicz [17], but can also be found in Gaertner [14], Jackson [18] as well as in many other textbooks and papers on the topic of manipulability.

[10]Non-Excludability means that no one in the society can be excluded from consuming the good. Non-Rivalness of a good is given, if the consumption of one individual does not affect the availability of this good for others. One possible example for a non-excludable and non-rival good is the provision of a pavement.

5.2. **Formalization.**

Let there be two individuals, $i = \{1, 2\}$ and let there be two different private goods x_j, $j = \{1, 2\}$. The consumption of individual i of good j will be displayed by: x^i_j for $i = 1, 2$ and $j = 1, 2$. So that $x^i_j = \{(x^1_1, x^1_2), (x^2_1, x^2_2)\}$.

Both individuals have the same Cobb-Douglas utility function representing their true preferences: $u^i(x^i_j) = x^i_1 x^i_2$.

Let the prices of the goods be given by p_j, $j = 1, 2$.
Without loss of generality, assume that the price of good 2 is normalized to 1, i.e. $p_2 = 1$.

Individual 1 is initially endowed with one unit of good 1, nothing of good 2 and individual 2 with one unit of good 2 and nothing of good 1. Formally the endowment can be written as $e^1_j = (1, 0)$ and $e^2_j = (0, 1)$.
The given initial endowments define the feasible allocations as the set
$A = \{(x^1, x^2) \in \mathbb{R}^4_+ \mid x^1_1 + x^2_1 = 1 \wedge x^1_2 + x^2_2 = 1\}$.

The initial wealth of individual 1 is then

$$w^1 = \sum_{j=1}^{2} p_j e^1_j = p_1 e^1_1 + p_2 e^1_2 = p_1$$

and the initial wealth of individual 2 is

$$w^2 = \sum_{j=1}^{2} p_j e^2_j = p_1 e^2_1 + p_2 e^2_2 = p_2 = 1$$

.
The individuals state their preferences to the introduced auctioneer by revealing their utility functions to him. The auctioneer collects the information, solves the optimization problem of the individuals, maximize their utilities and implements the equilibrium.

Consider first the case where the individuals state their preferences truthfully to the auctioneer.

The optimization problem for individual 1 can be formulated as maximizing the utility subject to the budget constraint, which is:

$$\begin{array}{ll} \max & u(x^1_j) = x^1_1 x^1_2 \\ \text{subject to} & p_1 x^1_1 + p_2 x^1_2 = p_1 \end{array}$$

The corresponding maximization problem for individual 2 has the form:

$$\begin{array}{ll} \max & u(x^2_j) = x^2_1 x^2_2 \\ \text{subject to} & p_1 x^2_1 + p_2 x^2_2 = 1 \end{array}$$

5.3. **Equilibrium derived from honest revelation of preferences.**

First Order Condition for optimization requires that the marginal rate of substitution equals the price ratio. Formally:

$$MRS^1 = \frac{p_1}{p_2} = MRS^2$$

Where the MRS^i is defined as:

$$MRS^i = \frac{\frac{\partial u^i}{\partial x_1^i}}{\frac{\partial u^i}{\partial x_2^i}}$$

So the marginal rate of substitution for individual 1 is:

$$MRS^1 = \frac{x_2^1}{x_1^1} = \frac{p_1}{p_2} \tag{1}$$

Corresponding for individual 2:

$$MRS^2 = \frac{x_2^2}{x_1^2} = \frac{p_1}{p_2} \tag{2}$$

From $p_2 = 1$ it follows that:

$$\frac{x_2^2}{x_1^2} = p_1$$

Rewriting the the budget constraint and (2) gives:

$$x_2^2 = 1 - p_1 x_1^2 \tag{3}$$

and:

$$x_2^2 = p_1 x_1^2 \tag{4}$$

Combining equations (3) and (4) yields:

$$2x_2^2 = 1 \tag{5}$$

Rewriting equation (5) results in:

$$x_2^2 = \frac{1}{2} \tag{6}$$

As $\sum_{i=1}^{2} x_2^i = 1$, equation (6) implies:

$$x_2^1 = \frac{1}{2} \tag{7}$$

Now lets turn focus on the optimization problem of individual 1.

From $p_2 = 1$ it follow that the MRS^1 is given by:

$$\frac{x_2^1}{x_1^1} = p_1$$

Rewriting the budget constraint and the MRS^1 leads to:

(8) $$x_2^1 = p_1 - p_1 x_1^1$$

and:

(9) $$x_2^1 = p_1 x_1^1$$

Combining equation (8) and (9) yields:

(10) $$2x_2^1 = p_1$$

Rewriting (10) gives:

(11) $$x_2^1 = \frac{p_1}{2}$$

From equation (7) it is known that $x_2^1 = \frac{1}{2}$. Combining equations (7) and (11) results in:

(12) $$\frac{1}{2} = \frac{p_1}{2}$$

Solving (12) for p_1 gives $p_1 = 1$.

Now plugging equation (7) and $p_1 = 1$ into equation (9) yields:

(13) $$x_1^1 = \frac{1}{2}$$

Knowing that $\sum_{i=1}^{2} x_1^i = 1$ gives the result:

(14) $$x_1^2 = \frac{1}{2}$$

Thus the unique equilibrium is given by:

$$x_j^i = \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$$

Plugging the results into the utility function of the individuals gives the corresponding utility levels of:

$$u^i = (\frac{1}{4}, \frac{1}{4})$$

Under the condition that the individuals state their preferences truthfully, the equilibrium is uniquely defined by an equal distribution of the available goods and equal levels of utility for both individuals.

5.4. **Equilibrium derived from strategic revelation of preferences.**

What happens if one of the individuals uses a sophisticated strategy, trying to improve the outcome in its favor? Consider the same situation, but individual 2 manipulates by misrepresenting its preferences and stating a different utility function. Let individual 2 state the following utility function:
$u^{2\prime}(x_j^2) = x_2^2 - \frac{1}{(1+x_1^2)}$.

The initial endowments are again given by $e_j^1 = (1,0)$ and $e_j^2 = (0,1)$. The set of feasible allocations is again the set
$A = \{(x^1, x^2) \in \mathbb{R}_+^4 \mid x_1^1 + x_1^2 = 1 \wedge x_2^1 + x_2^2 = 1\}$.
Without loss of generality let again be assumed that the price of good 2 is normalized to one, i.e. $p_2 = 1$.

The maximization problem for individual 1 has again the form:

$$\begin{array}{ll} \max & u(x_j^1) = x_1^1 x_2^1 \\ \text{subject to} & p_1 x_1^1 + p_2 x_2^1 = p_1 \end{array}$$

The corresponding maximization problem for individual 2 is now given by:

$$\begin{array}{ll} \max & u(x_j^2) = x_2^2 - \dfrac{1}{(1+x_1^2)} \\ \text{subject to} & p_1 x_1^2 + p_2 x_2^2 = 1 \end{array}$$

The First order condition for optimization is given by:

$$MRS^1 = \frac{p_1}{p_2} = MRS^2$$

where the MRS^i is defined as stated above.

5.4.1. *Optimization problem of the honest individual.*

Consider first the optimization problem for individual 1. The problem is similar to the previous one, where both individuals honestly represented their preferences. The result from this optimization problem is therefore the same. So it follows that:

$$\frac{p_1}{2} = p_1 x_1^1 \Rightarrow x_1^1 = \frac{1}{2} \tag{15}$$

From (15) it follows immediately that:

$$x_1^2 = \frac{1}{2} \quad \text{as} \sum_{i=1}^{2} x_1^i = 1 \text{ must hold.} \tag{16}$$

5.4.2. *Optimization problem of the sophisticated individual.*

Consider now the optimization problem for individual 2.

First Order Condition for optimization for individual 2 yields:

$$\frac{1}{(1+x_1^2)^2} = p_1 \tag{17}$$

As $x_1^1 + x_1^2 = 1$ holds, it follows that:

$$x_1^2 = 1 - x_1^1 \tag{18}$$

Using (18) in (17) it follows that:[11]

$$\frac{1}{(2-x_1^1)^2} = p_1 \tag{19}$$

From (15) it is already known that $x_1^1 = \frac{1}{2}$. So (15) can be used in (19) to get:

$$p_1 = \frac{1}{(2-\frac{1}{2})^2} = \frac{4}{9} \tag{20}$$

Using (20) and the initial endowment $e_j^2 = (0,1)$, the budget constraint of individual 2 can be formulated as:

$$x_2^2 = 1 - \frac{4}{9} \times \frac{1}{2} = \frac{7}{9} \tag{21}$$

As $\sum_{i=1}^{2} x_2^i = 1$ must hold, equation (21) immediately defines:

$$x_2^1 = 1 - \frac{7}{9} = \frac{2}{9} \tag{22}$$

Under the premise that individual 2 reports the utility function $u^{2\prime}$ as stated above, the unique equilibrium is given by:

$$x_j^i = \{(\frac{1}{2}, \frac{2}{9}), (\frac{1}{2}, \frac{7}{9})\}$$

The real utility function for both individuals is given by $u^i(x_j^i) = x_1^i x_2^i$. Plugging in the derived results gives then corresponding utility levels of:

$$u^i = (\frac{1}{9}, \frac{7}{18})$$

[11]It would have also been possible to use the result of (16) to calculate p_1.

Comparing the results from the case of honest revelation of preferences with the case of strategic revelation of preferences two results are obvious. Both the distribution of commodities and the utility levels have changed. As individual 2 uses a sophisticated strategy it ensures an increase in commodity 2 ($\frac{7}{9} > \frac{1}{2}$) and a gain in utility ($\frac{7}{18} > \frac{1}{4}$) compared to the case where both individuals reveal their preferences truthfully. As a logical consequence in this setup, individual 1 suffers from the sophisticated strategy of individual 2.

This example illustrates that the Walrasian allocation mechanism is vulnerable to manipulation and both individuals have an incentive to manipulate the allocation mechanism in this setup. It is possible for the individuals to improve the outcome in their own favor by misrepresenting their preferences.
A crucial assumption has to be made in order to allow for manipulation. The manipulating individual must have knowledge of the utility function of the other individual to improve the outcome in its favor. Following Gibbard [16] it would also be possible to manipulate the outcome, if the individuals have beliefs on the utility functions of the others.[12]
Nevertheless a sophisticated individual must anticipate the behavior of the other and have to understand the procedure in which he is involved to successfully manipulate the allocation mechanism.[13]

[12] For further research it would be interesting to investigate and answer the question: is there a Nash equilibrium in pure strategies or mixed strategies, if both individuals have beliefs of the utility functions of the others in the Setup Hurwicz [17] used? The answer of this question is beyond the scope of this study.

[13] There is an alternative to the assumption that an individual must have information about the process and the utility function of the other individual. It is also possible that an individual is learning over time from the previous own outcomes and tries different strategies to improve the outcome. This approach would require that an individual is repeatedly involved in the same scenario.

6. Optimality of Manipulation

Hurwicz's [17] shows as an application of the *Gibbard-Sattherthwaite Theorem* [15, 30] that the Walrasian allocation mechanism can be manipulated.
In the following section this application will be used as starting point for further investigations on the manipulability of the Walrasian allocation mechanism.
The idea of this section is the following: a rational individual will try to use a sophisticated strategy and manipulate the outcome of the Walrasian allocation mechanism in its favor. But if individual rationality is assumed then an individual should not only try to gain an advantage by manipulating arbitrarily. It should also try to maximize its advantage by using the appropriate strategy.
Following this reasoning this section considers two different parameterized functions to investigate possible optimal manipulation. The basic setup which will be applied will be the same as considered by Hurwicz [17]. Using the Walrasian allocation mechanism in a two agent pure exchange economy it will be analyzed whether it is possible to maximize the advantage of the manipulating agent for two different cases. In the first case the sophisticated agent will use a Cobb-Douglas utility function. In the second case a quasi-linear utility function.

6.1. **Formal Setup.**

The formal setup corresponds to Hurwicz [17], but will be stated for completeness.

Let there be two individuals $i = \{1, 2\} \in \mathbb{N}$ and let there be two commodities $x^i = (x^i_1, x^i_2) \forall i = \{1, 2\}$ which can be traded. Furthermore let the the commodities satisfy $\sum\limits_{i=1}^{2} x^i_1 = 1$ and $\sum\limits_{i=1}^{2} x^i_2 = 1$. The set of feasible allocations therefore will be $A = \{(x^1, x^2) \in \mathbb{R}^4_+ | \sum\limits_{i=1}^{2} x^i_1 = 1 \wedge \sum\limits_{i=1}^{2} x^i_2 = 1\}$.
Let the true utility functions of both individuals be given by
$u^i(x^i) = x^i_1 x^i_2 \,\forall\, i = \{1, 2\}$. The initial endowment of the individuals is denoted by $e^i = (x^i_1, x^i_2)$ and given by $e^1 = (1, 0)$ and $e^2 = (0, 1)$. Without loss of generality assume that the price of commodity 2 is normalized to 1.
Finally assume throughout the whole section that individual 2 has knowledge of the utility function of individual 1.[14]

[14] This assumption is not mandatory. It is also possible that individual 2 manipulates by trying different strategies to find the best possible one. As already mentioned in the last section, this approach would imply that the two individuals in this setup are competing against each other repeatedly. The problem could then be formulated in a dynamic setup taking into account the past outcomes of the individuals. This study does not consider this approach.

6.2. **The Cobb-Douglas Case.**

First consider the case where individual 2 manipulates the allocation mechanism by stating a Cobb-Douglas of the form $u^2(x^2) = (x_1^2)^\alpha (x_2^2)^{(1-\alpha)}$.
So the optimization problem for both individuals can be written down as follows.

For individual 1, who honestly represents his preferences, the optimization problem has the form:

$$\begin{aligned} \max \quad & u^1(x^1) = x_1^1 x_2^1 \\ \text{subject to} \quad & p_1 x_1^1 + p_2 x_2^1 = \sum_{j=1}^{2} p_j e^1 = p_1 \end{aligned}$$

Individual 2 tries to manipulate the procedure by using a Cobb-Douglas utility function. Its maximization problem has the form:

$$\begin{aligned} \max \quad & u^2(x^2) = (x_1^2)^\alpha (x_2^2)^{1-\alpha} \quad \alpha \in (0,1) \\ \text{subject to} \quad & p_1 x_1^2 + p_2 x_2^2 = \sum_{j=1}^{2} p_j e^2 = 1 \end{aligned}$$

First Order Condition for optimization requires for both individuals that the marginal rate of substitution equals the price ratio.

$$MRS^i = \frac{p_1}{p_2} \quad i = 1, 2$$

where the marginal rate of substitution is is defined as:

$$MRS^i = \frac{\frac{\partial u^i}{\partial x_1^i}}{\frac{\partial u^i}{\partial x_2^i}}$$

6.2.1. *Optimization problem of the honest individual.*

Lets consider first the optimization problem of individual 1. The calculation is similar to the case presented in the last section. Therefore it follows that:

$$x_1^1 = \frac{1}{2} \tag{23}$$

(23) and $\sum_{i=1}^{2} x_1^i = 1$ immediately imply that:

$$x_1^2 = \frac{1}{2} \tag{24}$$

So from the optimization problem of individual 1 it follows that $x_1^i = (\frac{1}{2}, \frac{1}{2})$.

6.2.2. *Optimization problem of the sophisticated individual.*

Lets turn focus on the optimization problem of individual 2. Taking partial derivatives of the utility function of individual 2 leads to:

$$MRS^2 = \frac{\alpha (x_1^2)^{\alpha-1}(x_2^2)^{1-\alpha}}{(1-\alpha)(x_1^2)^{\alpha}(x_2^2)^{1-\alpha-1}}$$

This equation can be simplified to:

$$\Rightarrow MRS^2 = \frac{\alpha}{1-\alpha}\frac{x_2^2}{x_1^2} \tag{25}$$

First order condition for optimization of individual 2 is:

$$\frac{\alpha}{1-\alpha}\frac{x_2^2}{x_1^2} = \frac{p_1}{p_2}$$

Using $p_2 = 1$, this expression can be rewritten into:

$$x_2^2 = (p_1 x_1^2)\frac{1-\alpha}{\alpha} \tag{26}$$

The budget constraint of individual 2 can be rearranged to:

$$x_2^2 = 1 - p_1 x_1^2 \tag{27}$$

Equation (27) can now be extended by the factor $\frac{1-\alpha}{\alpha}$ to get:

$$\frac{1-\alpha}{\alpha}x_2^2 = \frac{1-\alpha}{\alpha} - \frac{1-\alpha}{\alpha}p_1 x_1^2 \tag{28}$$

The next step is to combine the equations (26) and (28) by adding them up. This leads to:

$$(1 + \frac{1-\alpha}{\alpha})x_2^2 = \frac{1-\alpha}{\alpha} \tag{29}$$

Simplifying (29) gives:

$$\Rightarrow x_2^2 = 1 - \alpha \tag{30}$$

As the condition $\sum_{i=1}^{2} x_2^i = 1$ must hold, it follows immediately from (30) that:

$$x_2^1 = \alpha \tag{31}$$

From the optimization problem of individual 2 it follows that $x_2^i = (\alpha, 1-\alpha)$.

The equilibrium allocation can now be defined and is given by:

$$x_j^i = \{(\frac{1}{2}, \alpha), (\frac{1}{2}, 1-\alpha)\}$$

6.2.3. *Results from Cobb-Douglas Manipulation.*

The calculation shows that the distribution of commodities and the equilibrium allocation depend on the choice of α of individual 2.

Moreover the true utility function of individual 2, which is $u^2(x_j^2) = x_1^2 x_2^2$, can be written as function in α and is given by:

$$u^2(x^2) = f(\alpha) = [(\frac{1}{2})(1-\alpha)] \quad \alpha \in (0,1) \tag{32}$$

The utility function (32) can now be investigated to check whether an optimal solution for individual 2 exists. An optimal solution would maximize the utility from using a Cobb-Douglas for manipulation. This can be checked by setting the first derivative of the utility function (32) equal to zero and solve for α.

$$\frac{\partial f(\alpha)}{\partial \alpha} = -\frac{1}{2} \tag{33}$$

From (33) it can be concluded that unfortunately no optimal value for $\alpha \in (0,1)$ exists. A maximum of the true utility function of individual 2 does not exits, because $-\frac{1}{2} \neq 0 \ \forall \alpha \in \mathbb{R}$.

The utility of individual 2 would be $u^2(x^2) = \frac{1}{4}$, if it honestly represents its preferences. Manipulation is beneficial for individual 2, if the following inequality holds:

$$(\frac{1}{2})(1-\alpha) > \frac{1}{4} \tag{34}$$

The utility from manipulating the procedure must be larger than the utility from honest representation. Solving (34) leads to the result that this is the case if:

$$\alpha < \frac{1}{2} \tag{35}$$

To strengthen the result the limits of (32) can be calculated when approaching the boundaries:

$$\begin{aligned} \lim_{\alpha \to 1} [(\frac{1}{2})(1-\alpha)] &= 0 \\ \lim_{\alpha \to 0} [(\frac{1}{2})(1-\alpha)] &= \frac{1}{2} \end{aligned} \tag{36}$$

Figure 1 illustrates the calculated results. Clearly the function $[(\frac{1}{2})(1-\alpha)]$ has no maximum and intersects the constant of $\frac{1}{4}$ at $\alpha = \frac{1}{2}$. If $\alpha < \frac{1}{2}$ then the value of the function is larger than $\frac{1}{4}$ and if $\alpha > \frac{1}{2}$ then the value of the function is below $\frac{1}{4}$.

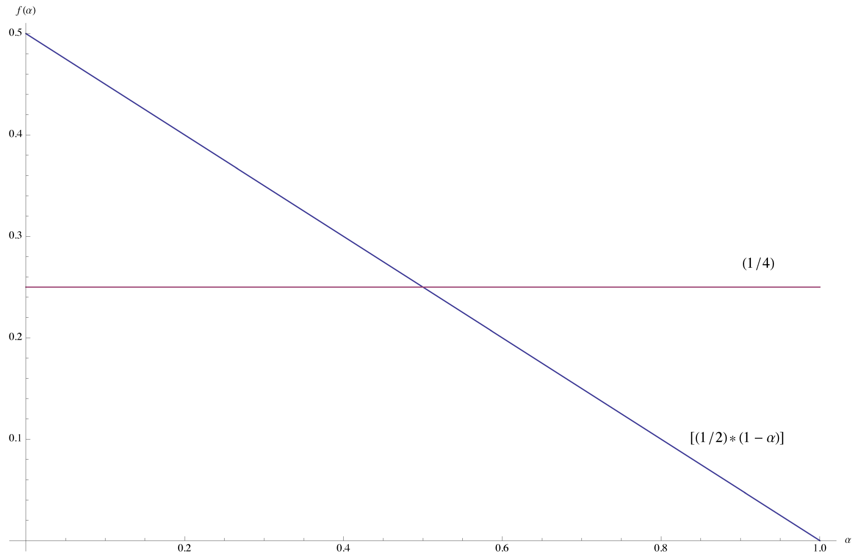

FIGURE 1. Cobb-Douglas Utility Function

The results from this section will now be summed up. A sophisticated individual can manipulate the Walrasian allocation mechanism using a Cobb-Douglas utility function. The individuals also have an incentive to do so. The equilibrium allocation depends on the choice of the parameter α. There is no optimal solution for the choice of the parameter α. The manipulating individual can not maximize its advantage from misrepresenting its preferences. However it does make a difference how the sophisticated agent chooses the parameter α. If it chooses α smaller than $\frac{1}{2}$ it can secure a greater share of commodity 2 and a gain in utilities. The smaller the chosen α the greater the advantage for the manipulating individual.

6.3. **The Quasi-Linear Case.**

This section considers the second case where the manipulating agent, which will be again individual 2, uses a quasi linear utility function to manipulate the allocation mechanism. The used function will be of the form: $u^2(x^2) = x_2^2 - (x_1^2)^\alpha \ \ \alpha < 0$.

Consider the optimization problems of the individuals. Individual 1 honestly represents its preferences. Its optimization problem will be the same as in the previous case:

$$\begin{array}{ll} \max & u^1(x^1) = x_1^1 x_2^1 \\ \text{subject to} & p_1 x_1^1 + p_2 x_2^1 = p_1 \end{array}$$

Individual 2 tries to gain an advantage by manipulating the allocation mechanism. Its has a maximization problem of the form:

$$\begin{array}{ll} \max & u^2(x_j^2) = x_2^2 - (x_1^2)^\alpha \\ \text{subject to} & p_1 x_1^2 + p_2 x_2^2 = 1 \end{array}$$

The first order condition for optimization is again given by:

$$MRS^i = \frac{\frac{\partial u^i}{\partial x_1^i}}{\frac{\partial u^i}{\partial x_2^i}} = \frac{p_1}{p_2}$$

6.3.1. *Optimization problem of the honest individual.*

The optimization problem of individual 1 has not changed. The results of this problem are same as in the Cobb-Douglas case before. From the optimization problem of the honest individual it follows that:

$$x_1^1 = \frac{1}{2} \tag{37}$$

The condition $\sum_{i=1}^{2} x_1^i = 1$ must hold. It can be immediately concluded from (37) that:

$$x_1^2 = \frac{1}{2} \tag{38}$$

Hence the optimization problem of individual 1 determines the distribution of commodity 1, which is[15]: $x_1^i = (\frac{1}{2}, \frac{1}{2})$.

[15]For details of the calculation of x_1^i see section 5.3

6.3.2. *Optimization problem of the sophisticated individual.*

Lets turn the focus on the manipulating individual 2. First the partial derivatives of the postulated utility function will be calculated to determine the the MRS^2.

$$\frac{\partial u^2}{\partial x_1^2} = -\alpha(x_1^2)^{\alpha-1} \tag{39}$$

$$\frac{\partial u^2}{\partial x_2^2} = 1 \tag{40}$$

Which leads to:

$$MRS^2 = -\alpha(x_1^2)^{\alpha-1} = \frac{p_1}{p_2} \tag{41}$$

Taking into account that p_2 is normalized to 1 allows to formulate:

$$-\alpha(x_1^2)^{\alpha-1} = p_1 \tag{42}$$

The budget constraint of individual 2 can be reformulated, using the fact that p_2 is normalized to 1 and taking into account the initial endowment of individual 2 $e_j^2 = (0, 1)$.

$$x_2^2 = 1 - p_1 x_1^2 \tag{43}$$

p_1 is known from equation (42), so x_2^2 can be determined as:

$$x_2^2 = 1 - [-\alpha(x_1^2)^{\alpha-1} x_1^2] \tag{44}$$

Equation (44) can be simplified to:

$$x_2^2 = 1 + \alpha(x_1^2)^{\alpha} \tag{45}$$

6.3.3. *Results from Quasi-Linear Manipulation.*

With the results from (38) and (45) it is possible to rewrite the true utility function of individual 2 as a function in α.

$$u^2(x_j^2) = x_1^2 x_2^2 = f(\alpha) = (\frac{1}{2})[1 + \alpha(\frac{1}{2})^\alpha] \tag{46}$$

By construction the condition $x_2^2 \in [0,1]$ has to hold. Therefore it has to be verified for which values of α this condition is is fulfilled. Formally:

$$x_2^2 \in [0,1] \Leftrightarrow [1 + \alpha(\frac{1}{2})^\alpha > 0 \ \wedge \ 1 + \alpha(\frac{1}{2})^\alpha < 1] \tag{47}$$

First the condition $1 + \alpha(\frac{1}{2})^\alpha > 0$ will be verified.[16]

$$1 + \alpha(\frac{1}{2})^\alpha > 0$$

$$\Leftrightarrow \alpha(\frac{1}{2})^\alpha > -1$$

$$\Leftrightarrow \alpha > -\frac{ProductLog[ln(2)]}{ln(2)}$$

$$\Leftrightarrow \alpha > -0.641186 \tag{48}$$

Now the second condition $1 + \alpha(\frac{1}{2})^\alpha < 1$ will be checked:

$$1 + \alpha(\frac{1}{2})^\alpha < 1$$

$$\Leftrightarrow \alpha(\frac{1}{2})^\alpha < 0$$

$$\Leftrightarrow \alpha < 0 \tag{49}$$

From these calculations it follows the condition that $\alpha \in (-0.641186, 0)$.

Figure 2 illustrates the conditions (48) and (49).

[16] The numerical values for α in section has been calculated using *Mathematica 9*.
The *ProductLog-Function* corresponds to the so called *Lambert-W-Function* or *Omega-Function*.
ProductLog[z] gives the principal solution for w in $z = we^w$ or also $Y = We^x \Leftrightarrow X = W(Y)$.

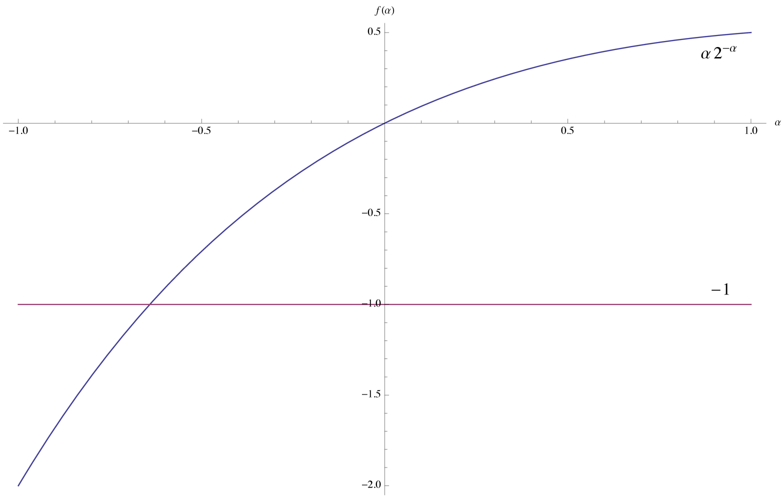

Figure 2. Conditions on the Parameter for the Quasi Linear Case.

To investigate whether there is a local maximum of $u^2(x^2) = f(\alpha) = (\frac{1}{2})[1+\alpha(\frac{1}{2})^\alpha]$ the first derivate will be calculated and checked if the derivate is equal to 0 for $\alpha \in (-0.641186, 0)$.

$$u^2(x^2) = f(\alpha) = (\frac{1}{2})[1 + \alpha(\frac{1}{2})^\alpha]$$

$$\frac{\partial f(\alpha)}{\partial \alpha} = (\frac{1}{2})[2^{-\alpha}(1 - \alpha ln(2))] \tag{50}$$

$$(\frac{1}{2})[2^{-\alpha}(1 - \alpha ln(2))] = 0$$

$$\Leftrightarrow \alpha = \frac{1}{ln(2)} \approx 1.443 \tag{51}$$

$$\Rightarrow (\frac{1}{2})[2^{-\alpha}(1 - \alpha ln(2))] \neq 0 \quad \forall\, \alpha \in (-0.641186, 0) \tag{52}$$

The maximum of the function $u^2(x^2)$ is outside of the required interval for α. There is no value for α which maximizes the utility of individual 2 and is feasible. Nevertheless the investigation continues to make some statements on the possibility of manipulation using a quasi-linear utility function of the presented form.

Manipulation is beneficial for individual 2, if the utility from manipulation is larger than the utility from honest representation of preferences. Formally this can be written as:

$$(\frac{1}{2})[1 + \alpha(\frac{1}{2})^\alpha] > \frac{1}{4} \quad \alpha \in (-0.641186, 0) \tag{53}$$

The following calculation will verify the stated inequality for α.

$$(\frac{1}{2})[1+\alpha(\frac{1}{2})^{\alpha}] > \frac{1}{4}$$

$$\Leftrightarrow \alpha > \frac{ProductLog[\frac{ln(2)}{2}]}{ln(2)}$$

$$\Leftrightarrow \alpha > -0.383332 \tag{54}$$

Manipulation is possible and beneficial for individual 2, if he chooses parameter α correctly for the quasi-linear utility function $u^2(x^2) = x_2^2 - (x_1^2)^{\alpha}$.
This is the case if: $\alpha \in (-0.383332, 0)$.

For further insights the limits of the utility function will be analyzed when approaching the feasible boundaries.

$$\lim_{\alpha \to 0}(\frac{1}{2})[1+\alpha(\frac{1}{2})^{\alpha}] = \frac{1}{2}$$
$$\lim_{\alpha \to -0.641186}(\frac{1}{2})[1+\alpha(\frac{1}{2})^{\alpha}] = 0$$

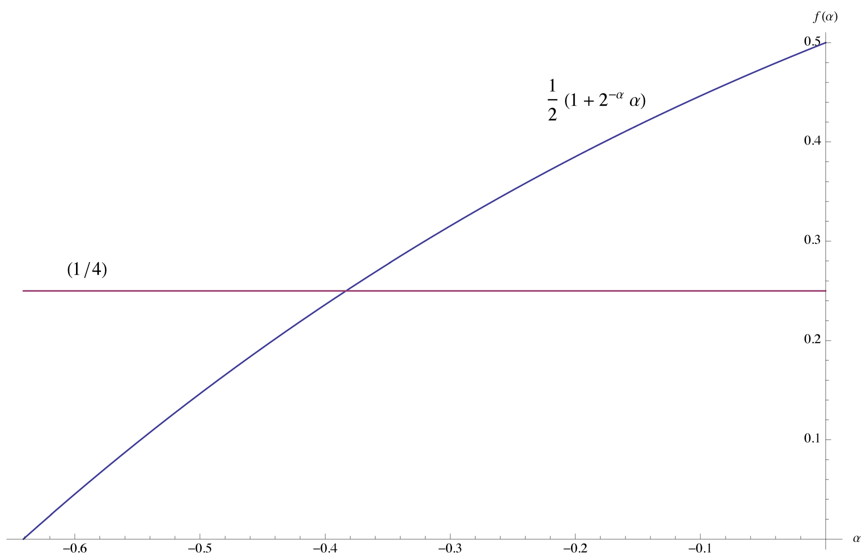

FIGURE 3. Quasi-Linear Utility Function

Figure 3 illustrates the case using a quasi-linear utility function of the presented form for manipulation.

It can be seen that the utility from manipulation using the function $u^2(x_j^2) = x_2^2 - (x_1^2)^\alpha$ depends on the choice of α. Unfortunately no value of α in the feasible interval maximizes the utility. However there is a certain range for the parameter α in which manipulation is beneficial. This is case if $\alpha \in (-0.383332, 0)$.

It is possible to manipulate the allocation mechanism using a quasi-linear utility function of the presented form. A rational individual should manipulate the procedure by stating the fictive utility function and choosing the correct value for the parameter α such that there is a gain in utility. Manipulating the procedure and choosing α in a way such that there is a loss in utilities can not be considered as rational and does not make any sense.

6.4. **Remark on the presented Cases.**

6.4.1. *A Special Case.*

The solutions of the presented cases suggest the consideration of a special case. In section 6.2 Individual 2 uses a Cobb-Douglas function of the form $u^2(x_j) = (x_1^2)^\alpha (x_2^2)^{1-\alpha}$ for manipulation of the allocation procedure. It is shown that the utility of individual 2 increases, the smaller it chooses parameter $\alpha \in (0, 1)$. This solution suggests to investigate the special case $\alpha = 0$.

Let the formal setup of section 6.2 be unchanged and assume that individual 2 states the utility function $u^2(x_j) = x_2^2$ instead of its real utility function.

The optimization problem of individual 2 is therefore:

$$\begin{array}{ll} \max & u^2(x_j^2) = x_2^2 \\ \text{subject to} & p_1 x_1^2 + p_2 x_2^2 = 1 \end{array}$$

Individual 2 has an initial endowment of $x_2^2 = 1$. As individual 2 states that its utility is only dependent on x_2^2, commodity x_2 will not be shared. Therefore the distribution of x_2 will be: $(0, 1)$.
From (11) it is known that $x_2^1 = \frac{p_1}{2}$. This equation holds if and only if $p_1 = 0$.

$$0 = \frac{p_1}{2} \Leftrightarrow p_1 = 0$$

If the price of commodity 1 is zero, the initial endowment of individual 1 is worthless and its initial wealth is zero. Individual 1 can not afford anything of commodity 2 and is indifferent between any share of commodity 1, i.e. $u^1(x_j) = 0 \ \forall x_1^1 \in [0, 1]$.

As $p_1 = 0$ individual 2 has an unbounded demand for commodity 1.

The unique Pareto optimal and feasible allocation in case will be $x_j^i = \{(0, 1), (0, 1)\}$ with corresponding utility levels $u^i(x_j) = (0, 1)$.

In this scenario individual 2 would have maximized its utility by manipulating the allocation procedure. Manipulation would be optimal.
However this allocation is not a Walrasian equilibrium. An allocation is a Walrasian equilibrium, if the following necessary conditions are fulfilled. The allocation must be feasible, Pareto optimal and the excess demand must be equal to zero for all goods which can be traded.[17]

Definition 44 (Excess Demand)**.** *For given wealth and prices the excess demand for a commodity x is given by the difference between the demand for that commodity and the available quantity of that commodity. Let there be $i = 1, \dots, n$ individuals and $j = 1, \dots, m$ commodities. The excess demand of individual i for commodity x_j is given by the function:*
$z^i(p_j) = d^i[x^i_j(p_j, pe^i)] - e^i$
where p_j is the given price of commodity x_j, $d^i[x^i_j(p, pe^i)]$ is the demand of individual i for commodity x_j given its price p_j, the initial wealth of individual i pe^i and the initial endowment or available quantity e^i.

If the price of commodity 1 equals zero, the demand of individual 2 for commodity 1 is unbounded and can not be satisfied by the available quantity, i.e.

$$\lim_{p_1 \to 0} z^2(p_1) = d^2[x^2_1(p_1, pe^2)] = \infty$$

This violates the necessary equilibrium condition. The allocation $\{(0,1),(0,1)\}$ with prices $p_j = (0,1)$ can not be an equilibrium.

The same argumentation holds for the special case of section 6.3 if it is assumed that individual 2 states the utility function $u^2(x_j) = x^2_2 - (x^2_1)^\alpha \; \alpha = 0$.

[17] To require the excess demand to be zero is the same as to require that demand equals supply.

7. Conclusion

The examples presented in section 6.2 and section 6.3 confirm that it is possible to manipulate the Walrasian allocation mechanism using either a Cobb-Douglas utility function or a quasi-linear utility function. It turns out that whereas manipulation is possible, an optimal strategy does not exist. This means that the manipulating individual can not choose a specific form of the function such that its utility will be maximized. Nevertheless it is possible to draw inferences from the actual choice of the manipulating individual. An individual could manipulate the allocation mechanism by stating a fictive utility function and still end up worse than in the case of honest representation. This would immediately imply that the behavior of this individual is not rational. Furthermore it does make a difference which specific function is chosen by the manipulating individual. In both situations considered the gain in utility increases in one direction if the choice is made adequately. So optimality is an issue which deserves to be considered in the allocation procedure.

The considered situations help to understand the underlying mechanism of optimal manipulation. But they must be considered as a first approach to the topic. There are various possibilities to carry on with the investigations on this topic. Two classes of functions are found which allow a sophisticated individual to manipulate the allocation mechanism in its favor. The next logical step would be to generalize the result and investigate which properties a utility function must exhibit in order to guarantee the possibility of beneficial manipulation. Moreover it should be analyzed which properties of utility functions guarantee manipulation in an optimal way. If the utility functions has the right properties then a rational individual should choose exactly this type of function to manipulate the procedure optimally and maximize its utility.

Another extension can be made by assuming that not only one individual has the knowledge of the utility functions of the participating individuals but both of them. It would be interesting to analyze how an individual reacts if it knows that the other individual is manipulating the allocation mechanism in its favor. How will the response look like? Will there be some optimal strategy and maybe finally some equilibrium? Can individuals manipulate optimally if the utility functions are not known with certainty but instead they have beliefs?

Aside from these ideas it would also be very interesting to conduct laboratory experiments with participants to examine the behavior and achieve applicable data and further insights on this topic.

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Zusammenfassung

Diese Arbeit gibt einen Überblick über die wichtigsten theoretischen Resultate aus dem Bereich der Manipulierbarkeit von Präferenz-Aggregationsmechanismen. Weiters wird diese Thematik in einer Walrasianischen zwei Individuen, zwei Güter Tauschökonomie angewendet. Parametrisierte Funktionen werden eingeführt, um mögliche Optimalität in der Manipulation zu untersuchen.
In Kapitel 2 werden die verständnishalber notwendigen formalen Definitionen und Fachbegriffe eingeführt. Kapitel 3 und Kapitel 4 geben eine Zusammenfassung der Literatur aus dem Bereich der Manipulation von Abstimmungsverfahren. In Kapitel 5 werden die beschriebenen Erkenntnisse in einer Walrasianischen zwei Individuen, zwei Güter Tauschökonomie angewendet. In Kapitel 6 wird anhand von parametrisierten Funktionen die Möglichkeit einer optimalen Manipulation des Walrasianischen Allokationsmechanismus untersucht. Kapitel 7 enthält abschließend folgende Schlussfolgerungen:
Den Individuen ist es möglich den Walrasianischen Allokationsmechanismus durch Bekanntgabe einer fiktiven Nutzenfunktion zu manipulieren. In dieser Arbeit wurden als Beispiele eine Cobb-Douglas Nutzenfunktion und eine quasi-lineare Nutzenfunktion gewählt. Die Individuen können jedoch den Nutzen aus der Manipulation nicht maximieren. Das bedeutet, dass eine optimale Manipulation hier nicht möglich ist. Es konnte jedoch gezeigt werden, dass die konkrete Wahl des eingeführten Parameters sehr wohl einen Unterschied macht. Folglich ist es möglich anhand der Parameterwahl Aussagen über die Rationalität der Individuen zu tätigen. Untersuchungen über die Optimalität der Manipulation sind sinnvoll und sollten erweitert durchgeführt werden.

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