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„Empirical study:
students + quadratic equations = difficulties“

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Dedication

I would like to dedicate this piece of work to my grandfather. He always wanted me to study hard and to give university top priority. He was scared of the fact that I would consider other things in life as far more important than my education. Having finished this diploma thesis, I am very close to become a teacher and to start another exciting chapter in my life. When my grandfather was young, he used to work very hard and passionately. So did I at university - so will I do at school as a teacher. I devote this paper to my grandfather as a sign of thanks and respect. I hope that he is proud of me.

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To the generous mind the heaviest debt is that of gratitude,
when it is not in our power to repay it. (Benjamin Franklin)

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1. Introduction

I almost didn't get an A in math, but then Mr. Carlo told me to stop asking "why?" all the time and just follow the formulas. So, I did. Now, I get perfect scores on all my tests. I just wish I knew what the formulas did. I honestly have no idea. *(taken from the young adult novel "The perks of being a wallflower" by Stephen Chbosky)*

Unfortunately, such a situation is not uncommon in schools. Students work with formulas and make use of strategies, without being aware of the basic components behind these tools. During the FAP (practical training at university) in mathematics, I was asked to introduce the quadratic formula in a fifth form. I noticed very soon afterwards that students had great difficulties in solving quadratic equations. The students were offered the formula as a tool/aid, but its application turned out to be very difficult. Secondly, students became kind of obsessed with the formula, because they applied it as soon as they faced a quadratic equation, neglecting any other (often quicker and more elegant) strategy. However, it was not only the formula that caused problems, but also a great number of other aspects that are strongly connected to the process of solving an equation, such as transforming an equation or checking the solutions. At this very early stage of my studies, it was clear for me: I will write my diploma thesis on quadratic equations and how students deal with them.

As the title of this paper suggests, the sum of students plus quadratic equations equals difficulties. This was found out by conducting an empirical study that will be the main part of this paper. However, also other important issues were taken into account to arrive at a decent and meaningful research. Literature, tables and figures will be used to emphasize, confirm and support ideas and statements.

I would like to give a short overview of what is to be expected in this paper. In section two, I will start off by clarifying some preliminary issues, such as some expressions, the mathematical background as well as some other organizational issues. Chapter three is concerned with the Austrian curriculum with reference to crucial aspects of the paper. I also had an insight into four different course books and the findings of this investigation will briefly be presented in chapter four. The rest of the paper is mainly dedicated to the empirical study. Chapter five offers an accurate description of

the framework, comprising a description of the participants and the setting, the questionnaire and hypotheses about the outcome. Findings, including the scoring procedure, will be discussed in chapter six. The results will be looked at from three different perspectives: the findings of the individual items, the results in connection with the curriculum and a compact summary of the three most relevant insights. Chapter seven finally combines the findings with certain issues of teaching and learning mathematics, such as for instance sustainable teaching or fostering the students' structure sense. The conclusion is meant to serve as a summary of the most significant insights, but also points out limitations and provides ideas for further research on this topic. The last two sections contain organizational aspects. The list of references can be found in section eight. The appendix features relevant documents that might be worth considering in more detail. References to the documents in the appendix will be made in the running text.

2. Preliminary issues

This section of the paper is concerned with clarifying and presenting preliminary issues, such as the format of the paper (referencing), some expressions that need to be explained – in general and with reference to mathematics and the approval of the city council for the empirical study.

2.1. Citation

Written in English, this paper follows the English style sheet for papers in linguistics as far as quotations and the use of references and tables/figures are concerned.¹ Basically, the author(s), the year of publication and the relevant pages will be given in brackets after/before a direct or indirect quotation. Some literature referred to will be indicated differently, but this will be mentioned explicitly. Tables and figures will be numbered consecutively and given a caption. At the end of the paper, an alphabetically ordered list of references provides full publication details of the literature used.

2.2. Expressions

Two terms need to be explained, as they will occur several times in the paper and can have different meanings outside of this context.

2.2.1. Secondary school

The empirical study was conducted in six different secondary schools in Vienna. In Austria, these six schools are called AHS (“Allgemeinbildende Höhere Schule” as opposed to BHS “Berufsbildende Höhere Schule” that are vocational schools). An AHS can be attended in a lower or an upper form. The study was done in 6th forms only. Therefore, if there is a reference to some sort of a secondary school (or simply to any kind of school with regard to the study), one should have in mind an AHS upper form.

¹ For the style sheet (version October 2013) that has been used for this paper, see:
http://anglistik.univie.ac.at/fileadmin/user_upload/dep_anglist/StudienServiceStelle/Formulare/Stylesheet_Oct2013.pdf

2.2.2. Error vs. mistake

The distinction between error and mistake is often crucial when it comes to teaching and learning. An error is often regarded as a competence problem. This means that the student is not yet aware of any underlying rule, whereas in case of a mistake, the student could tell a rule, but fails to apply it (performance problem). Radatz (1979: 18) even distinguishes between three kinds of errors/mistakes, namely “Irrtum”, “Fälschung” and “Fehler”. However, in this paper, no distinction will be made between error and mistake (between knowing a rule already or not). Both expressions will be used synonymously and simply describe wrong/inappropriate actions.

2.3. Mathematical background

With reference to the mathematical theory behind this paper, four aspects are worth mentioning and defining.

2.3.1. Quadratic equations as mathematical problems

The quadratic equations can be referred to as (mathematical) problems. For Bruder, Leuders & Büchter (2008: 20) a problem is subjective and means a task that is difficult and unusual. In case of the study, many items would certainly not be considered as problems, as they appear rather easy to respond to. However, in this context it is justified to talk about problems, as the three (Bruder, Leuders & Büchter 2008: 20) go on by saying:

Aber Probleme treten auch im normalen Mathematikunterricht für viele Schülerinnen und Schüler bereits dann auf, wenn *Basiswissen fehlt*, wenn wichtige *Methoden und Begriffe vergessen* wurden und nicht mehr ohne Hilfe rekonstruiert werden können. Dann werden aus für den Außenstehenden einfach erscheinenden Aufgaben subjektiv Problemaufgaben. [my emphasis]

Due to the fact that students definitely lack a certain amount of basic knowledge (this is what the findings of the survey will show), it is justified to refer to the equations as problems.

2.3.2. Null factor law

The null factor law, as Clements (2006: 54) calls it, will be mentioned in connection to certain items of the questionnaire. Different authors use different expressions. Some call it a theorem, some a law, others a method. In this paper, the expression null factor

law will be used. It states “if $ab=0$, then $a=0$, or $b=0$, or both a and b are zero” (Clements 2006: 57).

2.3.3. Completing the square

Similar to the null factor law it is the case with the process of completing the square. Mathematical insiders would exactly know what is meant by it, although there exist also numerous different terms to name it. The expression used in this paper can also be found in Clements (2006: 54) and it will be crucial in connection to the approach towards the formula and the schoolbook analysis. Figure 1 exemplifies what is meant by completing the square, if you consider a general quadratic equation of the following form:

$ax^2 + bx + c = 0$	with $a, b, c \in \mathbb{R}$ and $a \neq 0$
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	set $\frac{b}{a} = p$ and $\frac{c}{a} = q$
$x^2 + px + q = 0$	
$x^2 + px = -q$	completing the square on the left side = add $(\frac{p}{2})^2$ on both sides
$x^2 + px + (\frac{p}{2})^2 = (\frac{p}{2})^2 - q$	
$(x + \frac{p}{2})^2 = (\frac{p}{2})^2 - q$	
$x + \frac{p}{2} = \pm \sqrt{(\frac{p}{2})^2 - q}$	
$x_{1,2} = -\frac{p}{2} \pm \sqrt{(\frac{p}{2})^2 - q}$	

Figure 1 Completing the square

2.3.4. The quadratic formula

The formula derived from completing the square in Figure 1 above will be of great relevance to this paper. Most of the schoolbooks distinguish between the quadratic formula for the general form of a quadratic equation (sometimes also ‘great quadratic formula’) and the formula for the standardized form of a quadratic equation with $a=1$ as presented above (sometimes also ‘small quadratic formula’). This paper mainly concentrates on the second type and it will be referred to this formula as quadratic formula (or simply the formula) or p-q formula. The other one does only play a minor role and will be referred to as the great quadratic formula (or great formula).

2.4. Approval from the council

The empirical study was approved by the school city council of Vienna. Originally, the plan was to include also interviews with teachers. Unfortunately, these were not accredited. The following documents were required and sent to the responsible member of the council: a formal letter, an accurate description of the framework of the research (questionnaire, letter to parents/teachers, participating schools a.s.o.), an abstract of the theoretical basics, an approval of my advisor and an approval to send back some of the results and to respect privacy protection. The approval and the formal letter that were part of the proposal can be found in the appendix (section 10).

Having clarified all preliminary issues, the paper approaches the empirical study. Before that, the Austrian curriculum and an insight into some schoolbooks will be presented.

3. The Austrian curriculum

In a curriculum the needs of the learners are stated and a rough structure of the lesson content is given. The following paragraphs are concerned with the Austrian curriculum for mathematics, starting from primary school up to the sixth form in secondary school. Original passages that include relevant key concepts of the paper, such as variables, terms and equations will be presented. Going through the curriculum and considering certain aspects with respect to quadratic equations could help to identify misunderstandings and the roots of students' problems. If a teacher recognizes misunderstandings or problems, he/she could be stimulated to go back in the curriculum in order to reflect on critical stages and adjust or reconsider the way of teaching.

3.1. Primary school

Version 2003 of the primary school curriculum consists of three parts, namely general aims and objectives, lesson contents and pedagogic principles. Undoubtedly, the main focus in primary school lies on the acquisition of the natural numbers as well as the four basic arithmetical operations. However, the children also (indirectly) get to know some essential aspects that could be regarded as a preparation for much higher mathematical processes, such as solving equations. For instance, the expression 'variable' or key concepts behind solving equations, namely problem solving and abstraction are already mentioned in the primary school curriculum. In the first part of it, it is stated that rational thinking processes should be initiated and that logical thinking and problem solving behaviour should be developed² (original passage; crucial aspects in connection to the paper are written in bold):

Rationale Denkprozesse sind an geistigen Grundtätigkeiten wie [...] Abstrahieren, Verallgemeinern [...] zu schulen. Besonderes Gewicht ist auf die Entwicklung des logischen Denkens und des **Problemlöseverhaltens** zu legen.

² The curricula will not be part of the list of references at the end. If original passages are cited, this will be mentioned explicitly, for the curricula in Austria, see:

http://www.bmukk.gv.at/medienpool/3996/VS7T_Mathematik.pdf (Mathematics, primary school) and

http://www.bmukk.gv.at/medienpool/11859/lp_neu_ahs_07.pdf (Mathematics, secondary school - AHS)

This statement is crucial, as it mentions abstraction and generalization, and many experts would agree that these processes are the basis for understanding variables. It might be surprising that this is already stated in the primary school curriculum at a very early stage in the development of the learners.

The four main aspects in the second part of the curriculum are: construction of the natural numbers, operations, quantities and geometry and fractions. The relevant issues for this paper can be mainly found in the first two areas. The topics roughly stay the same, only the dimensions change along with the extension of the natural numbers. The following tables (Tables 1 and 2, original passages) roughly show which topics are dealt with at which stage. Please note that the basic arithmetic operations will not be mentioned, although they form the basis for all operations.

Table 1 Curriculum First and second form

Topic	Aims and objectives
Aufbau der natürlichen Zahlen	
Auf-und Ausbauen des Zahlenraumes bis 100	Herstellen von Relationen unter Verwendung der Symbole =, ≠, <, >
	Vergleiche (auch Termvergleiche)
Rechenoperationen	
Verstehen der Operationsstrukturen	Herausarbeiten der Operationsstrukturen
	Verwenden der entsprechenden Symbole (+, -, =, ·, :)
Rechenoperationen im additiven Bereich	Erkennen von Zusammenhängen , zB Tausch-, Nachbar-, Umkehr- und Analogieaufgaben
	Überprüfen (Abschätzen, Plausibilität, ...) der Ergebnisse von Rechenoperationen
	Vergleichen von Rechenausdrücken unter Verwendung der Relationszeichen =, ≠, <, >
Rechenoperationen im multiplikativen Bereich	Vertiefen des Verständnisses für multiplikative Beziehungen auch unter Verwendung der Null
Spielerisches Umgehen mit Zahlen und Operationen	Erkennen von Zusammenhängen und Rechenvorteilen

Table 2 Curriculum Third form

Topic	Aims and objectives
	Rechenoperationen
Mündliches Rechnen im additiven und multiplikativen Bereich	Verstehen des Operierens mit Null als Faktor
	Lösen einfacher Operationen unter Nutzung vorteilhafter Rechenwege
	Vergleichen von Rechenausdrücken unter Verwendung der Relationszeichen =, ≠, <, >
	Lösen einfacher Zahlengleichungen mit Platzhaltern (Variablen)
	Durchführen von Rechenoperationen durch Zerlegen und Notieren der einzelnen Teilschritte , Berücksichtigen der Stellenwerte, Anwenden von Rechenregeln , z.B. Verteilungsregel
Schriftliches Rechnen im additiven und multiplikativen Bereich	Begründen der Rechenschritte nach Einsicht in die den Operationen zu Grunde liegenden Rechenregeln
	Durchführen von Rechenproben
Lösen von Sachproblemen	Zuordnen von Rechenoperationen, Beschreiben von Sachverhalten mit Zahlen und Platzhaltern (Variablen)- Erstellen einfacher Gleichungen
	Kontrollieren und Verbalisieren der Ergebnisse

The last part of the primary school curriculum is concerned with some mathematic didactic principles. Relevant for the content of this paper would be the following two statements (original passages):

Für die Erkenntnisgewinnung und Denkentwicklung sind im Sinne des operativen Aufbaus und Durcharbeiten [...] die Betonung von **Problemdarstellungen**, die Grundlegung eines forschenden, experimentierenden Vorgehens, das **Aufdecken verschiedener Lösungswege**, das Herausstreichen von Zusammenhängen und das Erkennen verwandter Operationen wesentlich.

Beim mündlichen und schriftlichen Rechnen ist auf das **Verständnis** der Zusammenhänge zwischen den Operationen, auf das Erkennen zu Grunde liegender Rechenregeln und das **Finden von Lösungsstrategien** Wert zu legen.

As it can be seen from the discussion of the primary school curriculum, there are already lots of critical phases, where students could encounter difficulties that might affect upcoming mathematical challenges profoundly.

3.2. Secondary school

Some general aims and objectives that would be considered as being crucial to solving equations in the lower forms are the following (original passage):

Die Schülerinnen und Schüler sollen [...] durch Reflektieren mathematischen Handelns und Wissens Einblicke in Zusammenhänge gewinnen und Begriffe bilden; [...] durch das Benutzen entsprechender **Arbeitstechniken, Lernstrategien und heuristischer Methoden Lösungswege und -schritte bei Aufgaben und Problemstellungen planen und in der Durchführung erproben.**

Elementary algebra becomes important (original passage):

Elementare Algebra: **Variablen** als Mittel zum Beschreiben von Sachverhalten, insbesondere von Gesetzmäßigkeiten und funktionalen Beziehungen, und zum **Lösen von Problemen** verwenden können; **algebraische Ausdrücke und Formeln bzw. Gleichungen umformen können.**

Among the contributions to the different educational areas, two interesting issues can be found. Belonging to the categories language/communication and creativity, these are the following ones (expressed similar for upper and lower forms) (original passages):

Konzentrieren von Sachverhalten in **mathematische Formeln; Auflösen von Formeln** in sprachliche Formulierungen; Vermitteln und Verwenden einer Fachsprache mit spezifischen grammatikalischen Strukturen

Entwickeln verschiedener Lösungswege zu mathematischen Fragestellungen; Nutzen **heuristischer Strategien.**

The following competences should be trained in the course of different mathematic activities (among many others) (original passages):

Formal-operatives Arbeiten umfasst alle Aktivitäten, die auf **Kalkülen** bzw. Algorithmen beruhen, also das **Anwenden von Verfahren, Rechenmethoden oder Techniken**

Experimentell – **heuristisches Arbeiten** umfasst alle Aktivitäten, die etwa mit zielgerichtetem Suchen nach Gesetzmäßigkeiten [...] zu tun haben; auch das [...] Übergehen zu Verallgemeinerungen gehör[t] in der experimentellen Phase zu diesen Aktivitäten

Kritisch- argumentatives Arbeiten umfasst alle Aktivitäten, die mit Argumentieren, **Hinterfragen**, Ausloten von Grenzen und **Begründen** zu tun haben; das Beweisen heuristisch gewonnener Vermutungen ist ein Schwerpunkt dieser Tätigkeitsbereiche

The interesting content areas of the lessons can be found in the two sections ‘working with numbers’ and ‘working with variables’. An overview will be given in Table 3 (original passages):

Table 3 Curriculum Secondary school

Form/topic	Aims and objectives
1. Klasse	
Arbeiten mit Zahlen und Maßen	Kenntnisse über Umkehroperationen erweitern
	die Regeln über die Reihenfolge von Rechenoperationen, einschließlich der Klammerregeln , anwenden können
Arbeiten mit Variablen	mit Variablen allgemeine Sachverhalte beschreiben können, zB gleichartige Rechenabläufe, die sich nur durch unterschiedliche Zahlen unterscheiden, oder allgemeine Beziehungen zwischen Größen insbesondere Formeln bzw. Gleichungen aufstellen
	Lösungen zu einfachen linearen Gleichungen finden können,
	Formeln anwenden und interpretieren können
2. Klasse	
Arbeiten mit Variablen	Gleichungen und Formeln aufstellen, insbesondere auch in Sachsituationen,
	unter Verwendung von Umkehroperationen einfache lineare Gleichungen mit einer Unbekannten lösen und Formeln umformen ,
	Formeln interpretieren
3. Klasse	
Arbeiten mit Zahlen und Maßen	Verketteten der vier Grundrechnungsarten und derart entstehende Terme auch mit elektronischen Rechenhilfsmitteln berechnen können
Arbeiten mit Variablen	Formeln (bzw. Terme) umformen und durch Rechenregeln begründen können
	dabei auch Aufgaben variieren
	Lösen von linearen Gleichungen mit einer Unbekannten
4. Klasse	
Arbeiten mit Variablen	Sicherheit beim Arbeiten mit Variablen, Termen, Formeln und Gleichungen steigern
5. Klasse	
Zahlen und Rechengesetze	Aufstellen und Interpretieren von Termen und Formeln, Begründen von Umformungsschritten durch Rechengesetze

Gleichungen und Gleichungssysteme	Lösen von linearen und quadratischen Gleichungen in einer Variablen
	Anwenden der oben genannten Gleichungen und Gleichungssysteme auf inner- und außermathematische Probleme

The empirical study is directed towards a topic (quadratic equations) that is taught in the fifth form. Therefore, the curriculum was only considered until then. Some of the expressions in bold will be referred to again in the section about the findings (section 6.2.2.) of the study. One should recognize by then, why these stages in the curriculum could be argued to be highly significant.

4. An insight into schoolbooks

The schoolbook is probably the most important medium in mathematics. It is not only a collection of exercises, but the teacher relies on it by using it as a methodical manual (Dörfler & Fischer 1981: 110). Hence, investigating this medium might be interesting and revealing. Different schoolbooks of the fifth form were examined in order to gain an impression of how quadratic equations are dealt with. Teachers and students tend to consult the book a lot, not only because of the variety of exercises, but simply because it is a good resource to work with in general. The analysed books differ in so far that they use various approaches to the quadratic formula and that authors follow a different order. In total, these four books were compared with respect to five different aspects:

Bleier *et al.* (2009) – *Dimensionen Mathematik 5*

Brand *et al.* (2012) – *Thema Mathematik 5*

Götz & Reichel (2010) – *Mathematik 5*

Malle *et al.* (2010) – *Mathematik verstehen 5*

These four books are the ones that are used most frequently by the schools that participated in the empirical study. Please note that the books will be discussed in a random order that does not suggest any hierarchy of popularity.

4.1. Handling of special cases

Some authors refer to quadratic equations with ‘q equals zero’ or ‘p equals zero’ as special cases. Others do only speak of different types of quadratic equations. Also the expressions pure vs. mixed quadratic equations can be found in some older schoolbooks as well as equations of the type $(x-d)^2 = r$ that are not separately discussed in the more recent books.

Mathematik verstehen

Malle *et al.* (2010) start exactly with these special cases by mentioning three types: $ax^2=0$; $ax^2+bx = 0$ and $ax^2+c = 0$. Of all four books, this one is the only one that includes the discussion of the case $ax^2=0$. Students are advised to practice the

handling of these special cases with a variety of exercises offered in the book before the authors go on with general quadratic equations.

Dimensionen Mathematik

Special cases are dealt with in the section that is called 'algebraic solving of quadratic equations'. The following two types are mentioned: $ax^2+c = 0$ and $ax^2+bx=0$. The equation $3x^2-75=0$ is solved in two ways. For the first method, the authors start off by dividing by three, adding 25 and taking the root. The second version starts in the same way, but then the term is separated into two binomials and the null factor law is applied. Similarly, the equations of special case type two are solved.

Mathematik

Akin to *Dimensionen* (Bleier et al. 2009), the authors of this book offer two possible ways of solving an equation of the type $ax^2+c=0$. However, for the second method, they call this the "Zerfallungsmethode" (Götz & Reichel 2010: 79) and mention the null factor law afterwards. Then, an equation $ax^2+bx=0$ is solved in two different ways. For the first method, they divide by x and for the second they apply the null factor law.

Thema Mathematik

The authors broadly discuss these special cases before working on general quadratic equations. Starting out from the difference between a standardized and a general equation, equations of the type $x^2+q=0$ are discussed. They state that in this case $p=0$. Compared to other schoolbooks this early identification of p and q is rare. After the explanation of the null factor law, the second special case $x^2+px=0$ is presented. Students are told to solve such equations by using the null factor law.

4.2. Approaching the formula

The approach to the formula is a very important phase in the process of teaching quadratic equations. Students become familiar with a tool that helps them to solve quadratic equations in a convenient way.

Mathematik verstehen

After the discussion of the special cases, the authors argue that an equation $ax^2+bx+c=0$ can always be transformed into a standardized form $x^2+px+q=0$ by a division through a and the substitutions $p=\frac{b}{a}$ and $q=\frac{c}{a}$. Two concrete standardized equations are then solved by completing the square, before the whole process is described in general. Also the discriminant of a quadratic equation $x^2+px+q=0$ is introduced. The formula is then stated as a proposition. So far, equations of the type $ax^2+bx+c=0$ with $a \neq 0$ have been solved by bringing them into the standardized form and then applying the p-q formula. It is only after several exercises that the great quadratic formula is introduced in a very similar way.

Dimensionen Mathematik

The authors use a graphic approach (functions) in order to introduce general quadratic equations. It is stated in the book that such equations can be approximately solved by finding the roots of a function. However, this is only mentioned briefly. After the discussion of the special cases, it suddenly says in the book: "Um die allgemeine quadratische Gleichung $ax^2+bx+c=0$ rasch zu lösen, gibt es eine Lösungsformel" (Bleier *et al.* 2009: 133). Then, the great quadratic formula is introduced by completing the square. After this, the p-q formula is simply introduced by the substitutions $\frac{b}{a}=p$ and $\frac{c}{a}=q$.

Mathematik

A general quadratic equation is introduced at the very beginning of the chapter. After the discussion of the special cases, this type is mentioned again and solved by completing the square. The authors do also mention the distinction between the general and the standardized form of quadratic equations. Furthermore, they comment on the critical phase of approaching the formula, that is finding a suitable integer to complete the square. The great quadratic formula and the p-q formula are not introduced separately, but in one go. However, starting out from $ax^2+bx+c=0$ they do explain the relationship to the standardized version of an equation ($p=\frac{b}{a}$ and $q=\frac{c}{a}$).

Thema Mathematik

Indirectly, the students approach the formula at the very beginning of the chapter. The authors introduce quadratic equations and explain the difference between the standardized and the general form. It is already at this stage when they always identify the equation's p and q . Although the special cases follow this introduction, the students are prepared to work with q and p until the formula is actually introduced. The formula is established by an example with concrete integers and completing the square. Solving an equation of the type $ax^2+bx+c=0$ is introduced like this: "Wenn dir Normieren zu umständlich ist, kannst du die "große Lösungsformel" verwenden. Sie lässt sich aus der kleinen Lösungsformel herleiten" (Brand *et al.* 2012: 80).

4.3. Discussion of solutions

This aspect refers to the importance of the discriminant as well as the fact that an equation $x^2=a$ has always two solutions.

Mathematik verstehen

Malle *et al.* (2010) consider the amount of solutions of a quadratic equation relatively early. The authors do this during the discussion of the special cases with reference to the equation $x^2=3$. In a proposition, it is explicitly stated that an equation $x^2=a$ has two solutions, namely $x=\sqrt{a}$ and $x=-\sqrt{a}$. This is followed by the introduction of the discriminant. Also in this case, it is a proposition that states that a quadratic equation has either none, one or two solutions depending on the value of the discriminant.

Dimensionen Mathematik

This book discusses the different solutions only briefly. However, some crucial aspects are mentioned that do not get attention in other books. For instance, the authors do remind the students of the null factor law with reference to the two special cases. Furthermore, it is explicitly stated that dividing an equation by x is not a proper transformation, as one solution $x=0$ gets lost. The authors do also mention the solutions of taking the square root. Similar to the other books, the discriminant is introduced and the three different possibilities are stated.

Mathematik

Similar to the other schoolbooks, taking the square root is treated separately so that students become aware of both solutions. Of course, also the discriminant with regard to the results of a quadratic equation is mentioned. The authors do comment on the case 'one solution only', if the discriminant is zero separately by showing an example. Interestingly, the method 'dividing an equation by x ' is seen as the faster way to solve an equation: "Legt man [...] keinen Wert darauf, *alle* Lösungen zu erhalten, so kann man den schnelleren Weg 2 beschreiten [...] Die zweite Lösung $[4/3]$ geht dabei jedoch beim Quadratwurzelziehen – das keine Äquivalenzumformung ist- verloren" [original emphasis](Götz & Reichel 2010: 80).

Thema Mathematik

The authors dedicate a lot of passages to the discussion of different solutions. For each type of quadratic equations the possible solutions are explicitly stated, for instance, in the case of $x^2-25=0$ or with reference to the null factor law. There is even a short subsection called 'Wurzelziehen als Äquivalenzumformung' (Brand *et al.* 2012: 76) in which the solutions of square roots are discussed. Of course, also the discriminant is mentioned akin to the other three books. Moreover, the authors do mention the equation $x^2=-25$ and explain that this one is not solvable for the students at this stage.

4.4. Exercises

Das wohl durchgängig am häufigsten genutzte Medium für das Üben ist das Schulbuch. [...] [Der Übungsteil] versteht sich als Materialangebot an die Lehrenden und bietet Gelegenheit, die Begriffe und Verfahren in Aufgaben steigender Schwierigkeit [...] anzuwenden. (Büchter & Leuders 2005: 140)

Different authors use different types and amounts of exercises. For instance, for the questionnaire in the empirical study, it was important that students could check their solutions. In some books, students are required to do so, whereas in others this aspect is totally neglected.

Mathematik verstehen

The book offers many equations that are designed to practice the application of the formula. However, the students are often only asked to solve the equations. Sometimes they are also required to discuss the different cases. Most of the time this looks like the following: “Löse die folgende Gleichung! Beachte die verschiedenen Lösungsfälle” (Malle *et al.* 2010: 69). The students are never required to check their solutions.

Dimensionen Mathematik

Three out of twelve exercises are concerned with the graphic approach to quadratic equations. Further three exercises are textual ones, the remaining equations should be solved by using the formula. Sometimes, the students are not required to solve the equations but to come up with an equation if the solutions are given. Furthermore, the students are not asked to check their solution for any of the equations.

Mathematik

Exercises follow the theoretical part (no single exercise in between). They are structured with reference to the equation's form. This is made obvious by sub headlines, such as ‘equations of type a’, ‘equations of type b’, a.s.o.. The single word ‘Kontrolle’ marks that the students are required to check their solutions several times. The process of checking a solution is also illustrated with concrete examples in the theoretical section. Students are asked to solve many exercises by using the formula. Other task types include the use of a calculator or working with completing the square in order to find the solutions.

Thema Mathematik

The book provides a great variety of exercises for the students. Similar to the other books, most of the time the students are asked to solve certain equations with a prescribed method (‘use the null factor law’; ‘apply the formula’, etc.). Many of the exercises are textual ones. With reference to checking the solutions, no remarks are made in the whole chapter.

4.5. Additional hints and guidelines

Some authors provide guidelines to a more strategic approach towards quadratic equations. Such hints can help students to develop a feeling for quadratic equations and allow them to choose from a variety of possible ways to solve them.

Mathematik verstehen

In Malle *et al.* (2010) such guidelines can be found among ‘beachte’ statements. For instance he mentions that the p-q-formula can only be applied with the coefficient of x^2 being one. Furthermore, in all examples that are calculated through, the authors indicate p and q in order to ensure right insertion into the formula.

Dimensionen Mathematik

Besides the discussion of the solutions among ‘beachte’ no further advice or hints are given. However, the authors do demonstrate some calculations step by step. Each step is also described in words so that the ways of solving the equations can be reconstructed and students get familiar with verbalizing certain steps and results.

Mathematik

The theoretical part includes a great number of written passages. In these passages, the authors provide several guidelines that help to approach a quadratic equation. This starts at the very beginning of the chapter. The authors point out that the formula can be applied for solving every equation, however, that there are easier and quicker ways of working through these special cases. Moreover, one equation is solved by considering some logical aspects. Another one is solved by logical considerations (by simply ‘looking at it’) and using the binomial formulas. Such examples can stimulate the students to think about the structures of equations before they start calculating haphazardly.

Thema Mathematik

Hints and additional explanations are given in blue boxes with the word ‘Hinweis’ and there are a lot of these. The authors try to develop a certain feeling for structures by mentioning that the formula can be applied but that an application of the null factor law might be much more elegant in some cases. It is also stated in the book that

quadratic equations do not always have “*nice*” solutions, but can often be transformed and simplified. Also the meaning of the expression discriminant is explained. Additional historical remarks reveal interesting facts, for instance a comment on equations to a higher power other than two that sums up the chapter on quadratic equations.

Investigating schoolbooks can help to get an impression how quadratic equations are dealt with, how the formula is approached and in how far exercises support theory. So far, the paper has been concerned with organizational aspects and the mathematical as well as the didactic background. Some of these basics will be needed for a better understanding of the empirical study that will get full attention in the following sections.

5. Empirical Study

In the following paragraphs, the empirical study will be presented. As mentioned in the introduction, this section will be extensive, including an accurate description of the methods, the participants, the questionnaire and several other issues. A discussion of the findings can be found in a separate section afterwards.

5.1. Method

Investigating students' difficulties in connection with quadratic equations is a very broad topic to analyse, because many aspects are relevant to the solving process, such as algebraic calculations, basic arithmetic operations, a knowledge of the formula, a knowledge of the number of solutions and how to check them, etc. The empirical study presented in this paper was originally aimed at investigating students' difficulties in the following three areas: recognizing term structures, applying the quadratic formula and checking solutions. However, after the completed questionnaires had been analysed, other aspects turned out to be more relevant. Hence, the focus has slightly changed, but more on this in section six.

In total, 248 students (103 boys; 145 girls) completed the questionnaire. The survey was conducted in November and December 2013 in mathematics lessons in several sixth forms in six Viennese secondary schools that will be listed afterwards. After some explanations and instructions had been given (no use of technical devices, time frame of 25 minutes for the whole survey at most, use of additional papers, etc.), the questionnaires were distributed. Being present at almost all surveys in person, I paid attention that students would not work together or try to spy on their neighbours. The questionnaire was tried out on friends at roughly the same age and therefore it could be assumed that a duration of about 25 minutes would be an appropriate time frame (25 minutes including instructions and handing out/back the papers). The students asked several questions while completing the questionnaires. However, these were only answered if the response had no consequence on the outcome of the study, e.g.: students who asked for the formula clearly were not given an answer, whereas those who asked whether it was ok to use a pencil were given a response.

The questionnaire was totally anonymous. The students only had to tick female or male, although this was not taken into account for the analysis of the findings.

5.2. Participating schools

The approval from the council was given for six Viennese schools that will be presented now. The information was taken from the schools' web pages. The addresses can be found in a separate section after the list of references (section 9.2.).

5.2.1. AHS Rahlgasse

The AHS Rahlgasse is located in the sixth district and is situated close to the inner centre of the city. It hosts about 80 teachers and 30 classes. The following aspects are worth mentioning especially in connection to this school:

- It is the oldest gymnasium for girls and was founded in 1892. Boys have been allowed to attend the school since 1978. Since then, the school has put a focus on gender and co-education.
- The school has partner schools in Sweden, Romania, Germany and the Czech Republic.
- The three pillars of the schools are: social competence, gender and environment.
- The school has received some prizes for its focus on environment.
- Making the students aware of historical developments helps to counter right-wing ideologies. The school is aware of the fact that many non-native German students attend it and that therefore a variety develops.
- The school is open-minded towards social changes or pedagogic developments.
- Other important issues are: creativity, open learning and discipline.

Furthermore, the AHS Rahlgasse is one of 8000 UNESCO schools worldwide. The school clearly shows that by particularly emphasizing the aspects mentioned above, such as tolerance, intercultural learning, environmental education and sustainability.

5.2.2. AHS Theodor Kramer Straße

120 teachers, 950 students – this is the AHS Theodor Kramer Straße. The lower forms are of the type: 'Neue Wiener Mittelschule'. In the upper form, the students can decide between two types of 'Realgymnasium'. For each year, there are two bilingual classes, two 'Freiarbeitsklassen' and two 'Regelklassen'.

In the bilingual classes, an Austrian teacher works with a teacher whose mother tongue is English. The programme for upper secondary school is very challenging. Therefore, students are required to go through a special orientation procedure. In the so called 'Freiarbeitsklassen' students should learn and experience responsibility. If there is a week with 'Freiarbeit', then there is one lesson that is not bound to the teacher, but the students work on the objectives with materials on their own. Interestingly enough, students are not given feedback with grades in classes one to three but with a statement concerning their progress.

In the upper forms, there is also a project track. In the fifth form, projects are arranged with reference to the interests/needs of the classes. In the sixth forms, writing academic papers is in the foreground. In the seventh form there is a focus on projects that deal with languages (language weeks).

5.2.3. GRG 21 Schulschiff "Bertha von Suttner"

This school is probably well known for its position in the Danube, being Vienna's swimming school and unique in Europe. Originally, the intention was to construct a moveable school on the water in order to cover the need for education all along the Danube. However, the city lost interest and that is why the project was realized stationary. It was presented in 1991 to the public, before the actual teaching started in 1994. The name of the school is connected to an Austrian author, pacifist and peace researcher. The school might be distinct to other schools because of 45 minutes lessons and the obligatory subject 'KoKoKo' (communication, conflict management and cooperation). Moreover, team teaching is not uncommon. The school is a UNESCO school, which means that students are prepared to value peace, human rights, democracy and tolerance. Five principles describe the school's main pillars and guidelines:

1. The aim of the work in the school is to appreciate everybody's dignity - Individuality in the community.
2. A communal climate and open dialogue of all school partners should be in the centre.
3. The school stands for the display of individuality and diversity of teaching and learning forms.
4. The students should gather procedural and professional competences in the team and as individuals.
5. The school promotes critical thinking and personal responsibility.

80 teachers are in charge of the 900 students. The amount of students with a different first language is rather low. The school is popular for the teacher training practicum, as it hosts 22 teachers who have the required training.

5.2.4. Parhamergymnasium Grg 17

Named after an influential personality at the time of Maria Theresia's reign, the school is situated in the seventeenth district of Vienna and includes three different forms –secondary sports school for girls, secondary bilingual school or secondary economics school with project management, hosting about 750 students and 90 teachers. The school's image and attitude is depicted as a ship, the basis being the respectful treatment of others and its three masts being the development of personality, competences and knowledge/education. In addition also four principles are stated:

- Everybody is able to become what he/she is able to do – the school wants to strengthen personalities.
- It is important to take over responsibility - for oneself, for others, for anything.
- It is important to experience and recognize what makes the students the way they are.
- Mistakes in the learning process are desirable: "Because we make mistakes, we learn".

Responsibility plays an important role at this school. For some years, the school has tried to establish what they call a 'Kollegsystem', this is a newly designed set of house

rules for the students of the seventh and eighth forms that should make them more responsible citizens and attend the lessons more regularly, this means: more freedom, but also greater responsibility.

Besides the obligatory subjects, the school offers a broad range of elective subjects (sports, choir, cooking, etc.), additional programmes (reading training, job orientation, etc.) and preparation courses for extra qualifications (FCE, BEC, ECDL, peer mediation etc.).

5.2.5. Schulzentrum Friesgasse

The 'Schulzentrum' Friesgasse includes several different forms of schools, starting from nursery school up to different foci in the secondary school. The empirical study was conducted in the AHS. This is a Catholic private school and stands out due to its open-mindedness and tolerance. The exchange of cultures and the diversity of nationalities are therefore of great importance. Consequently, it is no surprise that about 40 different languages are spoken as a mother tongue among the 540 students.

The AHS includes two different modules that specialize in either languages or science. Most of the students continue their education at university. 60 teachers and some English assistant teachers are available to cater for the students' needs. Many of them do also have additional qualifications, for example in the areas German as a second language or dyslexia.

In the constitution of the SSND (School Sisters of Notre Dame) where this school belongs to, it says the following: We educate through everything we are and through everything we do. The spirituality of the school also comes to the surface in the sense of weekly prayers and worship services and also the school's focus on peace education. What goes hand in hand with this dimension of peace education is the fact that students are encouraged to strengthen their social skills. This is for example the focus of the school's buddy programme where new students are welcomed and introduced by older students.

5.2.6. Wiedner Gymnasium/Sir-Karl-Popper-Schule

Students can choose between a 'Realgymnasium' and a normal gymnasium. The school's main aim is to foster talents, because - according to the school's philosophy -

every kid has them. Aptitude should be recognized and acknowledged. The focus on these talents is especially noticeable in the upper forms. It is important that

- the personality of students is recognized.
- the learning process is individualized.
- interdisciplinary learning takes place.
- new principles of education are introduced.

In addition, especially gifted students get the opportunity to attend the Sir-Karl-Popper school. For the empirical study it will not be distinguished between the 'Popper – classes' and the "*normal*" ones. If students decide to apply for the Popper school, they need to send a written application and need to complete an intelligence test. After having done the test, the students are required to succeed in a hearing with the headmaster. Only the best 48 students are accepted each year.

Learning in the whole school means following a track, exchange, research, arguing, showing effort, enjoying achievements and being successful. Individual promotion takes place through special courses, a creative focus, a musical focus and science training workshops.

5.3. The questionnaire

In this section, the questionnaire will be presented by looking at individual items and discussing possible outcomes.

5.3.1. General structure

The questionnaire consisted of three parts (in total: 10 items) and can be found in the appendix (section 10). In the first part, the students had to find the solutions to given quadratic equations. This was followed by three quadratic equations that had already been solved. The students were asked to tick correct calculations and correct erroneous ones, by indicating the weaknesses and providing a correct version instead. Finally, the learners were required to solve two more quadratic equations, but this time they should also check their solutions.

5.3.2. Items and hypotheses

Undoubtedly, a single item often triggered more than one problem for the students. Assumptions had been made about possible answers to every item, before the study was conducted. These assumptions were based on literature and experience and will be discussed now. In addition to the hypotheses, the purposes of using the items will be stated.

In case of the items of part one, it was assumed that students are familiar with expanding products and the transformation of an equation. The critical question was if students would remember the formula and apply it correctly, in case they needed it. By the time the equations were prepared to use the formula, only minor difficulties were expected or as Heckmann & Padberg (2012: 174) state it:

Für die Bestimmung der Lösungsmenge einer Gleichung wird von den Schülern die allgemeine Strategie Klammern auflösen - Zusammenfassen - Sortieren - Isolieren *weitgehend sicher umgesetzt. Gelegentlich* isolieren die Schüler die Variable fehlerhaft, da sie beispielsweise an ungeeigneten Stellen durch die Variable teilen [item 1e] oder nicht gleichartige Glieder zusammenfassen. [my emphasis]

Item 1a: $(x - 2) \cdot (x + 3) = 6$

The purpose of including this item was to check, if students could expand the product correctly and if they could apply the formula afterwards. Hence, minor mistakes were expected in the expanding process or in the application of the formula (algebraic signs, identification of p and q).

Item 1b: $x^2 - 4x + 4 = 0$

Item 1b was included to investigate how students would deal with an equation that has only one solution. Would students indicate or comment on that differently? Similar to 1a (and all other items that include the application of the formula), no major difficulties were expected with reference to the formula.

Item 1c: $3x^2 = 147$

Item 1d: $x^2 - 81 = 0$

Items 1c and 1d were part of the questionnaire in order to show how students would handle special cases ($p=0$) and how they would treat solutions of square roots. For

both items, it was expected that students would be ignorant of the second solution: “It was expected that the most common error would be the single solution $[x=7, x=9]$ answer” (Clements 2006: 57). In addition, it was assumed that students would have problems with the division by three (item 1c) or taking the square root without a calculator. It was definitely supposed that all students would divide by three at first, as this “simplifies to a perfect square” (Star & Newton 2009: 561). The amount of students who applied the formula for these two equations was estimated rather low.

Item 1e: $(3x - 2)x = 0$

Again, the students encountered another special case ($q=0$) by working on this problem. Item 1e is connected to item 2a, as it has the same structure, but students were required to fulfil different tasks on the two similar items. It was expected that students would rather divide by x than apply the null factor law, Clements (2006: 57) also referred to this problem in connection with a different study: “It was also expected that some students would divide both sides of $2x^2 = 10x$ by x ”. Another assumption concerning this item was that only a minority of the students would apply the formula.

As mentioned above, for part two students had to go through equations that had already been solved. They had to tick correct ones and indicate mistakes of erroneous ones. Ideally, the students would also provide a correct version in the second case. Such tasks can definitely shed light on how students understand these problems themselves, or as Bürger (1991: 43) puts it:

Wenn Schüler mathematische Sachverhalte oder Handlungen, etwa den Lösungsweg einer Aufgabe [...] begründen sollen, dann müssen sie Gesetzmäßigkeiten, Zusammenhänge oder Beziehungen überdenken und ihre Überlegungen darstellen. Die Fähigkeit, die damit verbundenen Denkleistungen durchzuführen, ist ein *Zeichen für Einsicht und Verständnis*. [my emphasis]

Also Bruder, Leuders & Büchter (2008: 31) briefly comment on the usefulness of checking already solved tasks: „[D]as Untersuchen einer bereits gelösten Aufgabe auf mögliche Fehler bietet die Möglichkeit, vertiefte Einsichten über den Lerngegenstand zu gewinnen“.

Item 2a:

$$x^2 = 6x / :x$$

$$x = 6$$

The ideal answer would have been that students apply the null factor law and see that there are actually two solutions. However, this was not expected, but rather that the majority would tick this calculation as being correct.

Item 2b:

$$3x^2 - 15x + 18 = 0 / :3$$

$$x^2 - 5x + 6 = 0$$

$$x = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 6}$$

$$x = \frac{5}{2} \pm \frac{1}{2}$$

$$x_1 = 3$$

$$x_2 = 2$$

Actually, no problems were expected with this item. It was presumed that the majority of the students would tick it off as being correct.

Item 2c:

$$x^2 + 2x + 1 = 0$$

$$x = \frac{2}{2} \pm \sqrt{\frac{4}{4} - 1}$$

$$x = 1 \pm \sqrt{1 - 1}$$

$$x = 1$$

In connection with the previous item, this one was also aimed at checking the students' ability to work with the formula. It was assumed that if students would tick the previous one, that they would also tick this one (or if they would spot an error in the calculation of 2a they would also spot one for 2c), as the two items are of a similar structure from the step when the insertion in the formula takes place. Presumably, students who would know how to use and apply the formula would definitely spot the mistake and correct it. It was also assumed that some students would be uncertain about the solution and indicate the missing of a second solution as a mistake.

Item 3a: $(x - 2)^2 = 16$

Item 3a should present another different structure of quadratic equations and includes many important aspects of algebra. Basically two ways of solving this equation were expected. The first one includes the expansion of the product and the application of the formula, whereas the second way involves taking the square root and hence working through the equation in a more elegant and quicker way. In case of method one, no major problems were expected. However, students who would take the square root (minority) would certainly ignore the second solution (similar to 1c and 1d). With reference to item 3a, it was expected that students are able to check the solutions.

Item 3b: $(x - 3) \cdot (x + 11) = 0$

This item closed the circle, as it is very similar to item 1a. The only difference is that the null factor law could be applied (much more elegant way) instead of expanding the product and working with the formula. Hence, this item could be considered as rather easy. Heckmann & Padberg (2012: 176) speak of a problem with a clear structure: “die zu lösende Gleichung $[(x - 3) \cdot (x + 11) = 0]$ [weist] eine vergleichsweise einfache Struktur [auf]”. Nevertheless, it was not expected that students would choose the more sophisticated strategy, but that they would rather start transforming the equation: “Schülerinnen und Schüler können den obigen Ausdruck ohne die Antizipation von Zielen umformen (z.B. wenn ihnen nicht klar ist, dass hier eine Faktorenerlegung sinnvoll sein könnte)” (Mayer & Fischer 2013: 196).

In general, checking the solutions was thought to be a familiar process for the students. Nevertheless, there was the assumption that some students would probably insert x_1 in the first bracket and x_2 in the second, or as Heckmann & Padberg (2012: 177) put it: “Es könnte sein, dass Schüler beide Elemente der Lösungsmenge in die Gleichung einsetzen (je ein Element in einen Term)”. Asking the students to check their solutions, as in part three of the questionnaire can shed light on the understanding of equations. Firstly, if students insert the two different solutions in one equation this could suggest that they do not understand what the word variable stands for. Clements (2006: 57) mentions this in connection with a different example: “For example, with $(x-3)(x-5) = 0$ did students think that the x in $(x-3)$ stood for a

different variable from the x in $(x-5)$?”. Secondly, it was interesting to see, what students would do, in case the solution turned out to be incorrect? How would they react to that? „Durch den Vergleich eines Handlungsergebnisses mit einem gewünschten Resultat entscheidet ein Lerner, ob eine gewählte Handlungsrichtung angemessen gewesen ist“ (Meyer & Fischer 2013: 202). Soon, the actual decisions of the learners will be presented.

6. Analysis of results

The results of the survey might probably be considered as the most interesting aspect of the paper, but before these will be put forward, the scoring procedure needs to be explained.

6.1. The scoring procedure

Clements (2006: 56) used similar equations as interview questions and the interviewees were asked to solve them. He developed a rubric for measuring the understanding and assigned points to different levels of understanding (2006: 58). Unfortunately, a face to face interaction was not possible with reference to the study presented here. Consequently, the understanding cannot be traced back on the basis of the completed questionnaires, but can only be vaguely assumed.

The focus of the scoring with respect to this study was the use of strategies to solve the equations. No points were assigned to the items, but the different strategies were counted. Therefore, it was necessary to arrive at a limited range of strategies that would cover as many versions of the students' responses as possible. These categories were compiled for each and every item and they will be presented in this paragraph. If necessary, examples (original answers from the students) will be provided for a better understanding of the categories. Sometimes, the categories stay the same for more than one item, due to the structure of it. In this case, they will not be explained again.

Item 1a

For item 1a – six different categories covered the students' answers. These are the following:

- 1) Correct: The student solved the equation correctly, see Figure 2:

1. Löse die folgenden Gleichungen:

a) $(x-2) \cdot (x+3) = 6$

$$x^2 + 3x - 2x - 6 = 6$$

$$x^2 + x - 12 = 0$$

$$x_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 12}$$

$$x_1 = \frac{6}{2} = 3$$

$$x_2 = -\frac{9}{2} = -4$$

Figure 2 Item 1a Correct

- 2) Initial mistakes: The student already made a mistake while expanding, summing up or transforming the equation. The following steps were not looked at any further, although the solution might be correct by luck. For an example, see Figure 3:

1. Löse die folgenden Gleichungen:

a) $(x-2) \cdot (x+3) = 6$

$$x^2 = 6 = 6 \quad | +6$$

$$x^2 = 12$$

$$x = \sqrt{12}$$

Figure 3 Item 1a Initial mistakes

- 3) Initial steps only: The student did only expand, sum up or transform the equation (correctly), but stopped after these initial steps, see Figure 4:

1. Löse die folgenden Gleichungen:

a) $(x-2) \cdot (x+3) = 6$

$$x^2 + 3x - 2x - 6 = 6$$

$$x^2 + x - 6 = 6 \quad | +6$$

$$x^2 + x = 12$$

Figure 4 Item 1a Initial steps only

- 4) Formula recognizable: The formula can be recognized (see Figure 5), however, its application is erroneous due to mistakes with regard to the insertion of p and q, the algebraic signs or similar errors. This category does also include the case if students applied the formula, but failed to finish this process.

$a) (x-2) \cdot (x+3) = 6$
 $x^2 + 3x - 6 - 2x = 6$
 $x^2 + 3x - 12 - 2x = 0$
 $x^2 + x - 12 = 0$
 $x = \frac{-1}{2} \pm \sqrt{1 + 12}$
 $x_1 = \frac{-1}{2} + \sqrt{13}$
 $x_2 = \frac{-1}{2} - \sqrt{13}$

Figure 5 Item 1a Formula recognizable

- 5) Alternative way to “solution”: The student arrived at a “solution” in a different way. It is clear from looking at the student’s answer that he/she did not intend to use the formula, but erroneously transformed the equation until it he/she finally arrived at an expression ‘x equals anything’, as in Figure 6:

1. Löse die folgenden Gleichungen:
 $a) (x-2) \cdot (x+3) = 6$
 $x^2 + 3x - 2x - 6 = 6 \quad / +6$
 $x^2 + x = 12$
 $\frac{x^2 + x}{12} = 0 \quad \frac{x+1}{12} = 0 \quad / \cdot 12$
 $\frac{x(x+1)}{12} = 0 \quad x+1 = 0$
 $x = -1$

Figure 6 Item 1a Alternative way to “solution”

- 6) Blank: The student did not work on this equation.

Item 1b

Item 1b also comprised six different strategies:

- 1) Correct: The student did solve this equation correctly, but it was not distinguished between how he/she did that (application of the formula vs. using the binomial formula and taking the square root). Figure 7 shows the two versions:

$b) x^2 - 4x + 4 = 0$
 $x_{2,1} = \frac{4}{2} \pm \sqrt{\frac{16}{4} - 4} = 2 \pm \sqrt{\frac{0}{4}} = 2$
 $b) x^2 - 4x + 4 = 0$
 $(x-2)^2 = 0 \quad | \sqrt{}$
 $x-2 = 0$
 $x = 2$

Figure 7 Item 1b Correct

2) Alternative way to “solution”: As for example in Figure 8:

$$\begin{aligned}
 b) \quad & x^2 - 4x + 4 = 0 \\
 & x^2 - 4x = -4 \\
 & 4x^2 = 4 \quad | :4 \\
 & x^2 = 1 \\
 & x = \sqrt{1} \\
 & x = 1
 \end{aligned}$$

Figure 8 Item 1b Alternative way to “solution”

3) Initial steps only

4) Initial mistakes: see Figure 9

$$\begin{aligned}
 b) \quad & x^2 - 4x + 4 = 0 \quad | -4 \quad | :4 \\
 & x^2 - x = -1
 \end{aligned}$$

Figure 9 Item 1b Initial mistakes

5) Blank

6) Formula recognizable: see Figure 10

$$\begin{aligned}
 b) \quad & x^2 - 4x + 4 = 0 \\
 & x = 2 \pm \sqrt{\frac{-16}{4} - 4} \\
 & x = 2 \pm \sqrt{-4 - 4} = 2 \pm \sqrt{-8}
 \end{aligned}$$

Figure 10 Item 1b Formula recognizable

Item 1c

Item 1c is similar to item 1d. Hence, also the categories are akin:

1) Positive solution: The following three versions were counted towards this

category: $x=7$; $x = \sqrt{\frac{147}{3}}$ and $x = \sqrt{49}$

2) Both solutions: such as: ± 7 or $x = \sqrt{49}$ or $x = \pm \sqrt{\frac{147}{3}}$. Figure 11

illustrates how a student indicated both solutions of problems 1c and 1d.

$\begin{aligned} \text{c) } 3x^2 &= 147 \\ x^2 &= \frac{147}{3} \\ x &= \pm \sqrt{\frac{147}{3}} \end{aligned}$	$\begin{aligned} \text{d) } x^2 - 81 &= 0 \\ x^2 &= 81 \\ x &= \pm 9 \end{aligned}$
---	---

Figure 11 Items 1c and 1d Both solutions

3) Blank

4) Alternative way to "solution"

5) Formula recognizable

6) Initial mistakes: The student made a mistake during the very first step (mostly: division by 3), see Figure 12:

$$\begin{aligned} \text{c) } 3x^2 &= 147 \\ 3x^2 &= 147 : 3 \\ x^2 &= 18.6 : \sqrt{} \\ x &= \sqrt{18.6} \\ x &\approx 4.4 \end{aligned}$$

Figure 12 Item 1c Initial mistakes

Item 1d

1) Positive solution: Figure 13 provides an example of a student who indicated only one solution for each of the items 1c and 1d:

$\begin{aligned} \text{c) } 3x^2 &= 147 \quad /:3 \\ x^2 &= \frac{147}{3} \quad / \sqrt{} \\ x &= \sqrt{\frac{147}{3}} \end{aligned}$	$\begin{aligned} \text{d) } x^2 - 81 &= 0 \quad / \sqrt{} \\ x &= 9 \end{aligned}$
--	---

Figure 13 Item 1c and 1d Positive solution

2) Both solutions

- 3) Initial steps/blank: The student did either not work on that equation or started off with any steps that did not suggest a certain solution or method, as in Figure 14:

d) $x^2 - 81 = 0$ $| +81$
 $x^2 = 81$ $| \sqrt{}$
 $x =$

Figure 14 Item 1d Initial steps/blank

- 4) Alternative way to “solution”
- 5) Formula recognizable

Item 1e

- 1) Initial steps only: Figure 15 shows four examples:

e) $(3x-2)x=0$ e) $(3x-2)x=0$ e) $(3x-2)x=0$ e) $(3x-2)x=0$
 $3x^2-2x=0$ $1:3$ $3x^2-2x=0$ $3x^2-2x=0$ $3x^2-2x=0 \quad | +2x$
 $x^2-\frac{2}{3}x=0$ $x(3x-2)=0$ $3x^2-2x=0$ $3x^2=2x$

Figure 15 Item 1e Initial steps only

- 2) Alternative way to “solution”
- 3) Blank
- 4) Formula recognizable: see Figure 16:

e) $(3x-2)x=0$
 $3x^2-2x=0 \quad | :3$
 $x^2-\frac{2}{3}x=0$
 $x = \frac{2}{6} \pm \sqrt{\left(\frac{2}{6}\right)^2} = \frac{2}{6} \pm \sqrt{\frac{1}{9}} = \frac{2}{6} \pm \frac{1}{3}$

Figure 16 Item 1e Formula recognizable

- 5) Correct: Two cases were possible. The student either used the formula (great formula, as in Figure 17 or p-q formula) or applied the null factor law, as in Figure 18.

Handwritten solution for Figure 17:

$$1e) (3x-2)x = 0$$

$$3x^2 - 2x = 0$$

$$x_{1/2} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 3 \cdot 0}}{2 \cdot 3}$$

$$x_{1/2} = \frac{2 \pm \sqrt{4}}{6}$$

$$x_{1/2} = \frac{2 \pm 2}{6}$$

$$x_1 = \frac{4}{6} = \frac{2}{3}$$

$$x_2 = 0$$

Figure 17 Item 1e Correct with formula

Handwritten solution for Figure 18:

$$e) (3x-2)x = 0$$

$$(3x-2)x = 0$$

$$x_1 = 0 \vee x_2 = \frac{2}{3}$$

Figure 18 Item 1e Correct with null factor law

- 6) Initial mistakes: The student made a mistake at the very beginning (often first step of transformation), as in Figure 19:

Handwritten solution for Figure 19:

$$e) (3x-2)x = 0$$

$$4x - 2x = 0$$

$$2x = 0 \quad | :2$$

$$x = 0$$

Figure 19 Item 1e Initial mistakes

Items 2a and 2c

Items 2a and 2c comprise exactly the same categories. These are the following ones:

- 1) Tick: The student ticked the equation and considered its solution (the way it had been solved) as correct.
- 2) Blank: The student did neither tick it nor indicate a mistake. A tick and a cross (or any indication that suggests that the student regarded the

calculation as erroneous) cancel each other out and this case also belongs to this category.

3) Mark wrong: The student indicated a possible mistake, however, did not provide a correct version or more accurate explanations. It is clear from looking at the answer that the student clearly considered it as wrong, but failed to spot the actual mistake. Students used crosses to mark this case.

4) "Mistake": The student clearly indicated (or provided any kind of explanation) an error that was actually none. Figure 20 shows a collection of examples where students spotted an erroneous "mistake":

Figure 20 shows handwritten student work for two quadratic equations. Item 2a shows the equation $x^2 = 6x$ divided by x to get $x = 6$. A note in German says "man muss die wurzel von x^2 ziehen und nicht dividieren" (one must take the root of x^2 and not divide), which is crossed out with a large X. Item 2c shows the equation $x^2 + 2x + 1 = 0$ solved using the quadratic formula, resulting in $x = 1$. A note in German says "da müsste $\frac{2}{4}$ stehen" (there should be $\frac{2}{4}$), and another note says "wurzel aus 0 geht nicht" (square root of 0 does not work), both of which are crossed out with a large X.

Figure 20 Items 2a and 2c "Mistake"

5) Spotted: The student spotted the mistake and provided a correct version or commented on the erroneous steps, see Figure 21:

Figure 21 shows handwritten student work for the quadratic equation $x^2 + 2x + 1 = 0$. The student uses the quadratic formula to solve for x , resulting in $x = 1$. A note in German says "falsch kl. Lösungsformel ist $-\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$ " (wrong quadratic formula is $-\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$), indicating they spotted the mistake in their own work.

Figure 21 Item 2c Spotted

Item 2b

In case of item 2b, similar categories could be found:

1) Tick: Some students only ticked the equation, whereas others checked the solution before, as in Figure 22:

b) $3x^2 - 15x + 18 = 0 / :3$
 $x^2 - 5x + 6 = 0$
 $x = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 6}$
 $x = \frac{5}{2} \pm \frac{1}{2}$
 $x_1 = 3$
 $x_2 = 2$

ere deine Lösung:

$x = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 6}$
 $x = \frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{24}{4}}$
 $x = \frac{5}{2} \pm \sqrt{\frac{1}{4}}$
 $x = \frac{5}{2} \pm \frac{1}{2}$
 $x_1 = \frac{6}{2} = 3$
 $x_2 = \frac{4}{2} = 2$

Figure 22 Item 2b Tick and check

2) Blank

3) Mark wrong

4) "Mistake", see Figure 23

b) $3x^2 - 15x + 18 = 0 / :3$
 $x^2 - 5x + 6 = 0$
 $x = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 6}$
 $x = \frac{5}{2} \pm \frac{1}{2}$
 $x_1 = 3$
 $x_2 = 2$

$x_1 = -\frac{4}{2} = -2$
 $x_2 = -\frac{6}{2} = -3$

b) $3x^2 - 15x + 18 = 0 / :3$
 $x^2 - 5x + 6 = 0$
 $x = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 6}$
 $x = \frac{5}{2} \pm \frac{1}{2}$
 $x_1 = 3$
 $x_2 = 2$

Fehler $\frac{25}{4} - \frac{24}{4} = \frac{1}{4}$

Figure 23 Item 2b "Mistake"

Items 3a and 3b

Finally, items 3a and 3b need to be considered. Please note that for the categorization it is not relevant if and how students checked their solutions. This will be discussed in a separate section among the findings. Items 3a and 3b were approached by counting the same strategies:

1) Initial mistakes

2) Initial steps only

3) Formula recognizable

- 4) Correct: Again, two cases could be observed. Most answers that counted towards this category included the application of the formula. Some students, however, chose a more elegant way that is taking the square root (item 3a) and applying the null factor law (item 3b). Figure 24 shows examples of these more sophisticated approaches:

3) a) $(x-2)^2 = 16 / \sqrt{}$
 $x-2 = \pm 4 / +2$
 $x_1 = 6$
 $x_2 = -2$

b) $(x-3) \cdot (x+11) = 0$
 $x-3=0 \quad x+11=0$
 $x=3 \quad x=-11$

Figure 24 Items 3a and 3b Correct elegant

- 5) Alternative way to “solution”: as in Figure 24:

b) $(x-3) \cdot (x+11) = 0$
 $x^2 + 11x - 3x - 33 = 0 \quad / + 33$
 $x^2 + 8x = 33 \quad / \sqrt{}$
 $x + 8 = \pm \sqrt{33} \quad / - 8$
 $x = \pm \sqrt{33} - 8$

Figure 25 Item 3b Alternative way to “solution”

- 6) Blank

Now that the scoring procedure has been presented, the findings can be discussed.

6.2. Findings

The findings will be approached from three different perspectives. At first, the individual items will be considered (tables with the results from the individual classes can be found in the appendix – section 10). Then, another reference will be made to the curriculum. Finally, three basic findings will be discussed in order to narrow down the findings to a general end result of the study.

6.2.1. Individual items

The examples of the categories above have already provided a hint of what students achieved (or did not achieve) in the survey. Similar to the scoring procedure and the

different assumptions concerning the outcome, each and every item will be discussed separately. Bar charts and more figures (original answers from the students) will be used as illustrations. References to the assumptions in section 5.3.2.) will be made in order to compare these to the actual results.

Item 1a

As can be seen in Figure 26, most of the students encountered difficulties within the first steps of transforming the equation. In section 5.3.2. it was mentioned that students were expected to have difficulties in transforming equations only now and then – occasionally. It was not expected that the amount of students who made initial errors would be this high (70 students = 28%³).

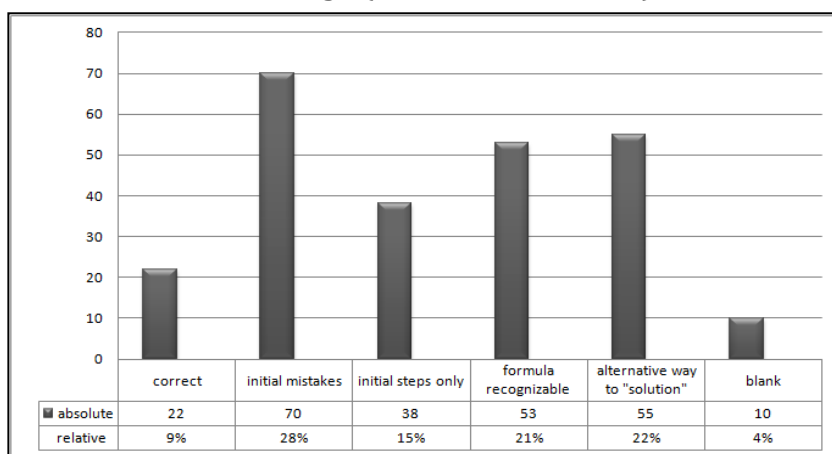


Figure 26 Item 1a Findings

Only 22 participants solved this equation correctly (by using the formula) which might be regarded as a rather low number if you consider a total amount of 248 students who completed the study. It could be assumed that students did not remember the formula accurately which is an assumption that will turn out to be justified after the findings of other items to come. Ten students did not work on this item at all. This was certainly not due to time pressure, but could hint at the fact that ten students did not have an idea of how to start at all. The strategy '*alternative way to "solution"*' was also popular for the items to come. 55 students tried to transform the equation – often neglecting any rules – until they arrived at an expression that said 'x equals anything'. Some students even tried to solve the equation by applying

³ The percentages (relative frequency) were computed by dividing the absolute values by the total amount of students. Please note that minor mistakes can occur due to rounding to the whole integer according to the basic rules (after the comma: '5 plus' means rounding up, '4 minus' means rounding down). These errors, however, were certainly kept to a minimum and do not falsify the findings.

the null factor law, see Figure 27. These absurd answers and the fact that 22% tried to use such an alternative way towards a “solution” shows that at least these 55 students did not remember the formula at all.

Figure 27 Item 1a Alternative way to “solution” Findings

Item 1b

The students were given an equation with one solution only. With reference to the application of the formula, no major difficulties were expected. Figure 28 shows that this item was indeed solved correctly by more students than problem 1a.

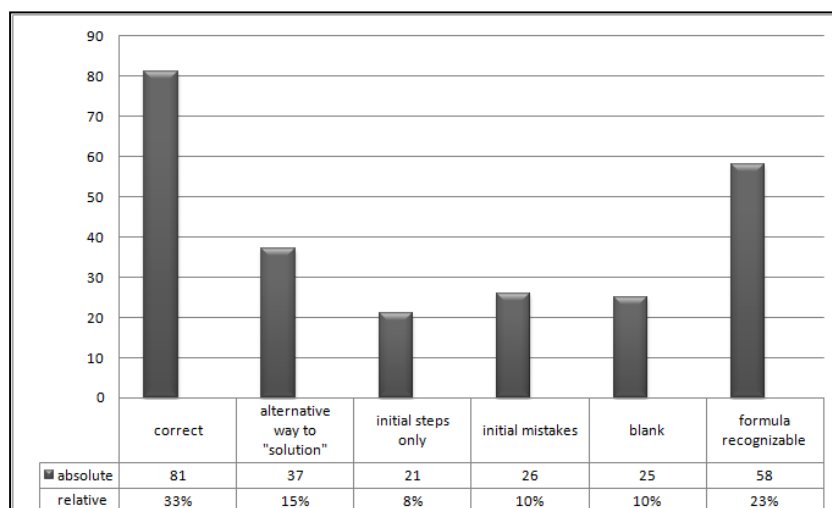


Figure 28 Item 1b Findings

Probably, this is due to the fact that the students did not have to transform the equation so that an application of the formula was possible, but that the equation was in the right form from the very beginning. Also, Fischer & Malle (1985: 60) found out that “[m]it zunehmender Anzahl der Umformungsschritte häuften sich die Fehler”. 81 students (33%) solved this equation correctly. Most of them, of course, by using the formula. Only a few learners, as shown in Figure 7 above, chose a more elegant strategy. 58 students tried to use the formula, but encountered problems. Initial

aspects of solving the equation were not as problematic as before. 15% used an alternative method to solve it and reached surprising “solutions”.

For those who mastered the equation correctly, it was interesting to see how they indicated the fact that this equation had only one solution. Different ways of notations came up, as can be seen in Figure 29:

Figure 29 Item 1b Different notations

Item 1c

The assumption that students would only indicate the positive solution was an appropriate one, as can be seen in Figures 30 and 31.

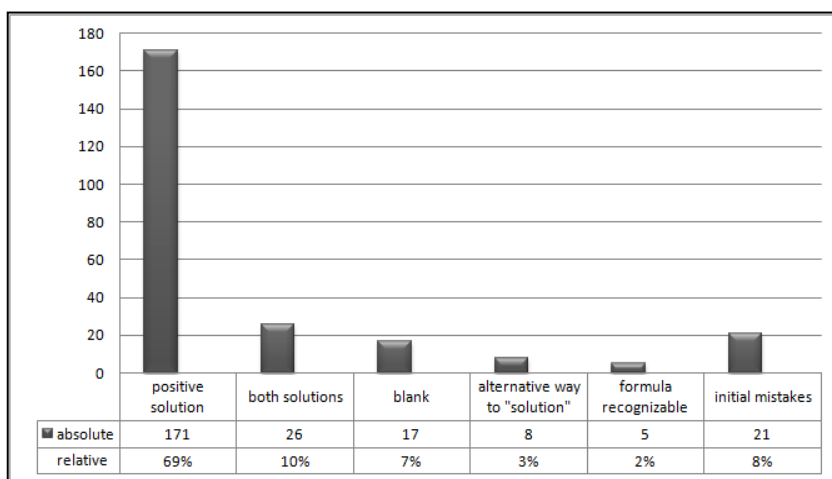


Figure 30 Item 1c Findings

171 students were ignorant of the second solution in case of item 1c. Only 26 students indicated both. Only very few students used the formula and only one student arrived at both solutions of this group. In combination with item 1d (see Figure 31), it could be argued that students felt rather confident with the two items in general, but that there seems to be an overall knowledge gap concerning the solutions of square roots.

Item 1d

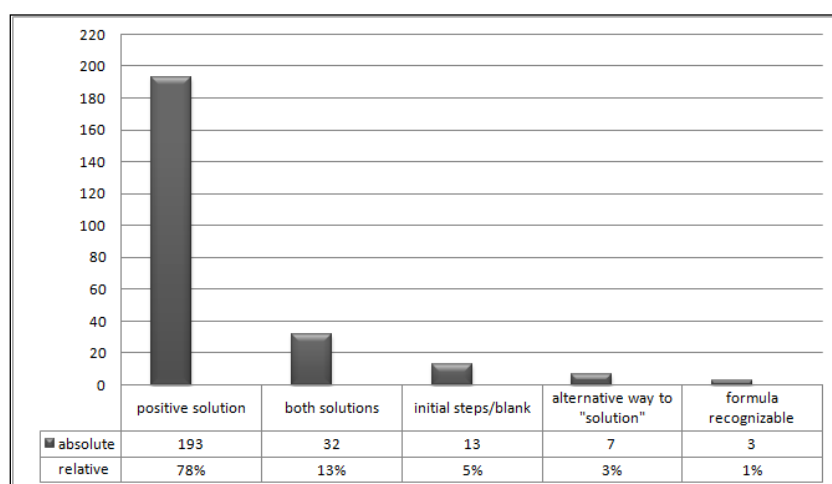


Figure 31 Item 1d Findings

Even more students (193; 78%) indicated only the positive solution in case of item 1d. Due to the structure of the given equation it was surprising that also some students used the formula (some among 'both solutions', others among 'formula recognizable'), see Figure 32:

c) $3x^2 = 147 \Rightarrow 3x^2 - 147 = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \frac{-0 \pm \sqrt{0^2 - 4 \cdot 3 \cdot (-147)}}{2 \cdot 3}$$

$$x_{1,2} = \frac{\pm \sqrt{1764}}{6} = \pm \sqrt{49} = \pm 7$$

d) $x^2 - 81 = 0$

$$x_{1,2} = \frac{-0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot (-81)}}{2 \cdot 1}$$

$$x_{1,2} = \frac{\pm \sqrt{324}}{2} = \pm \frac{18}{2} = \pm 9$$

Figure 32 Items 1c and 1d Formula

Item 1e

A brief look on Figure 33 suggests that this item was very difficult for the students. It might be very surprising that most of the students started working on the equation, but stopped after some initial steps, because one third of the students (82) did so. 31% used the already mentioned strategy 'alternative way to "solution"'. Only 22 students arrived at the correct solutions. Similar to item 2a that had already been solved (erroneously), some students divided by x. This was counted towards

'alternative way to "solution"' and definitely forms the majority of answers in this category.

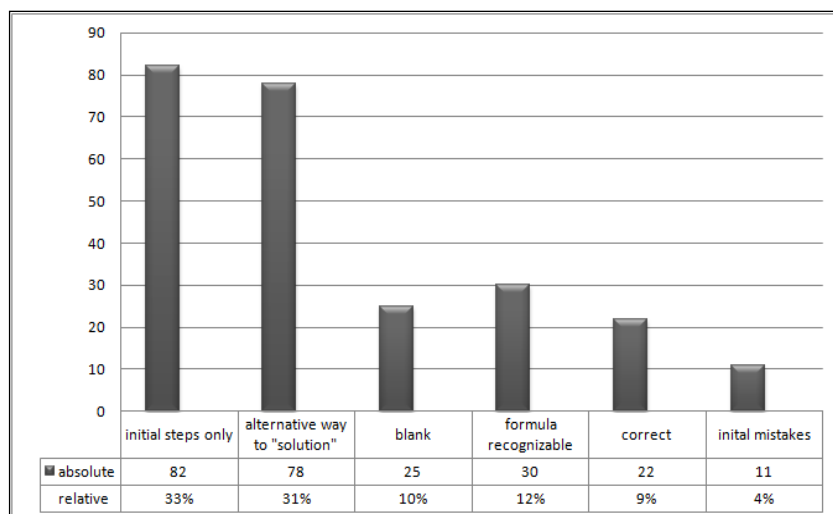


Figure 33 Item 1e Findings

The formula was only applied by a minority – more or less with success. Figure 34 shows two examples of calculations where the formula was involved. The left shows the response of a student who arrived at three "solutions" and the right example is from a student who used completing the square and got it right. In total, only six students solved the equation correctly by using the formula.

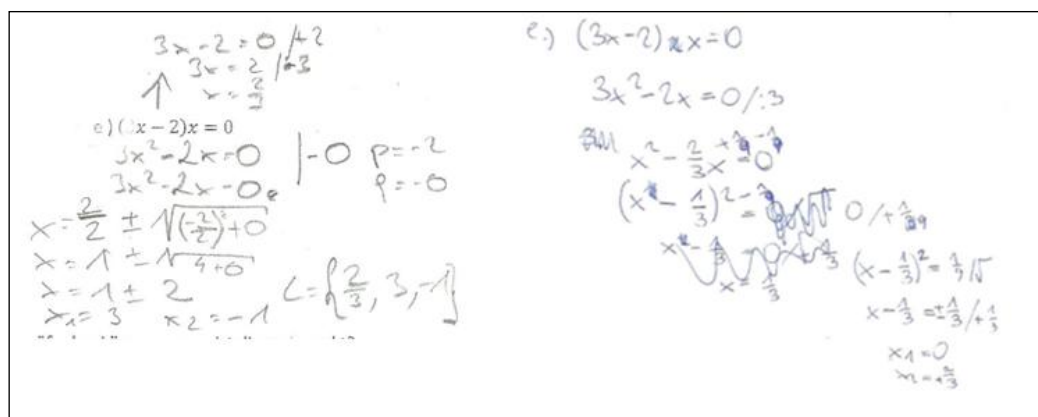


Figure 34 Item 1e Formula

Item 2a

The assumptions concerning this item were appropriate too. As can be seen from Figure 35, the majority (158 students) ticked it and hence regarded it as correct. Most of them only ticked it, whereas others (twelve students) checked the solution and then ticked it.

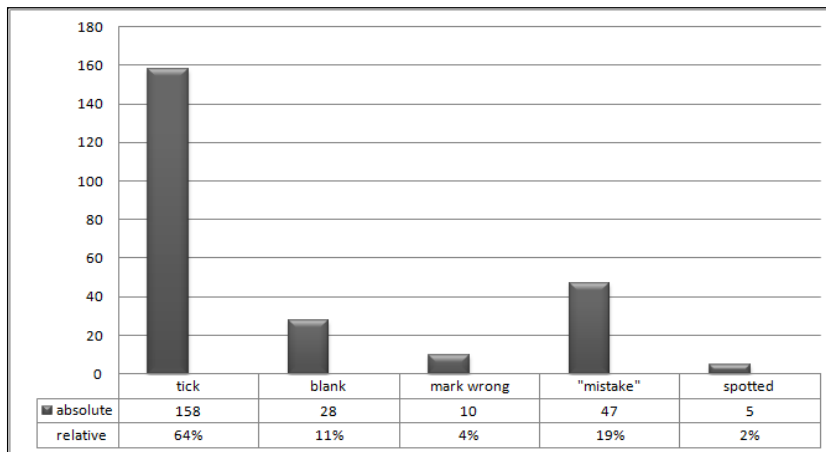


Figure 35 Item 2a Findings

The second most popular approach (47 students) and answer to this item was that students marked a “mistake” that was actually none. The most frequent response among this category was that students argued, the equation could not be solved by dividing by x , but only by taking the square root. Figure 36 shows a collection of sample answers of students who spotted “mistakes”.

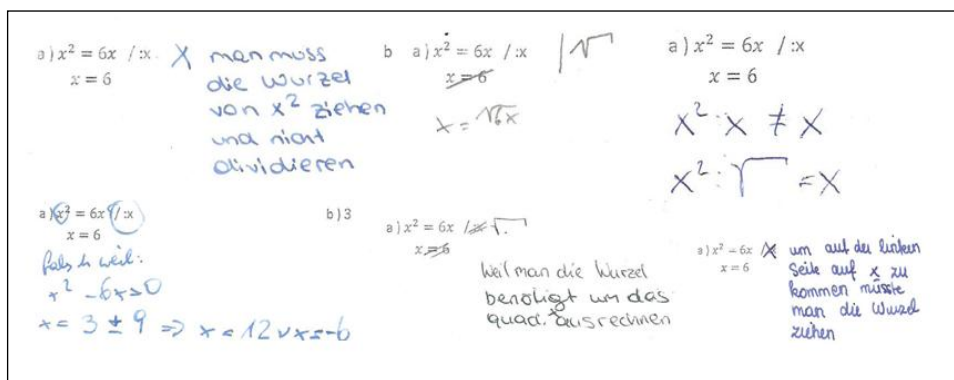


Figure 36 Item 2a “Mistakes”

Item 2b

This was the item that was supposed cause the fewest difficulties. In fact, 139 (see Figure 37) got it right by ticking it off as being correct (26 students checked it before). It was, however, surprising that nevertheless more than a fifth of the students spotted a “mistake” in this calculation. Most of the students who recognized an “error” commented on steps concerning the fractions, the square root or the division of zero by three on the right side of the equation. Interestingly enough, there seems to be kind of a connection of students’ answers between the items 2b and 2c.

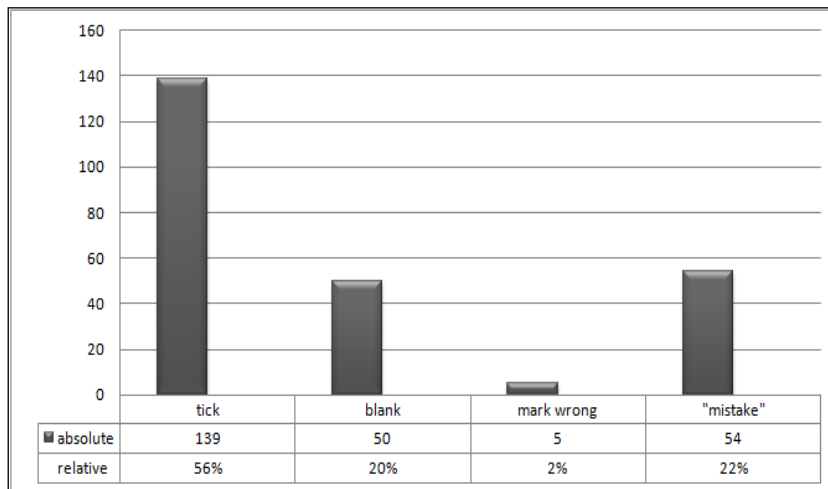


Figure 37 Item 2b Findings

Item 2c

Some students who ticked item 2b, also ticked item 2c and the other way round – students who spotted a “mistake” (frequently concerning algebraic signs of the formula) in the calculation of item 2b did also find one for item 2c. A third tendency could be recognized- that is: some students seemed to believe that the algebraic sign at the beginning of the formula must somehow correspond to the actual equation. This means that they spotted a “mistake” in item 2b, but ticked 2c, thinking that if the quadratic term follows a minus in the given equation, also the formula must start with a minus. This is illustrated in Figure 38.

b) $3x^2 - 15x + 18 = 0 :3$
 $x^2 - 5x + 6 = 0$
 $x = \frac{-5 \pm \sqrt{25 - 6}}{2}$
 $x = \frac{-5 \pm \frac{1}{2}}{2}$
 $x_1 = 3$
 $x_2 = 2$
 $x_1 = -\frac{4}{2} = -2$
 $x_2 = -\frac{6}{2} = -3$

c) $x^2 + 2x + 1 = 0$
 $x = \frac{-2 \pm \sqrt{4 - 1}}{2}$
 $x = 1 \pm \sqrt{1 - 1}$
 $x = 1$ ✓

Figure 38 Items 2b and 2c Relationship

Together with item 2b, 2c was included to check students' ability to work with the formula besides only applying it on their own. Figure 39 shows how the students' answers were distributed among the categories for item 2c:

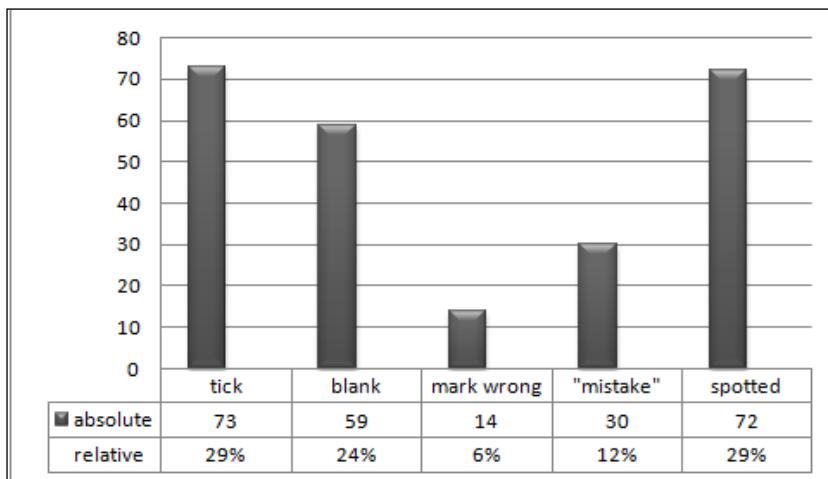


Figure 39 Item 2c Findings

The amount of students who spotted the mistake was expected to be considerably higher than 29%. The number of students who ticked this equation was almost equal to the number of students who spotted the mistake. Almost a quarter of the students did not work on this item; neither did they tick it, nor correct it. Most of the “mistakes” (12%) students found were again concerned with fractions.

Item 3a

Similar to the items of part one, a great number of students had difficulties with the first steps of transforming the equation or stopped after these initial steps, as the bar chart in Figure 40 shows for item 3a:

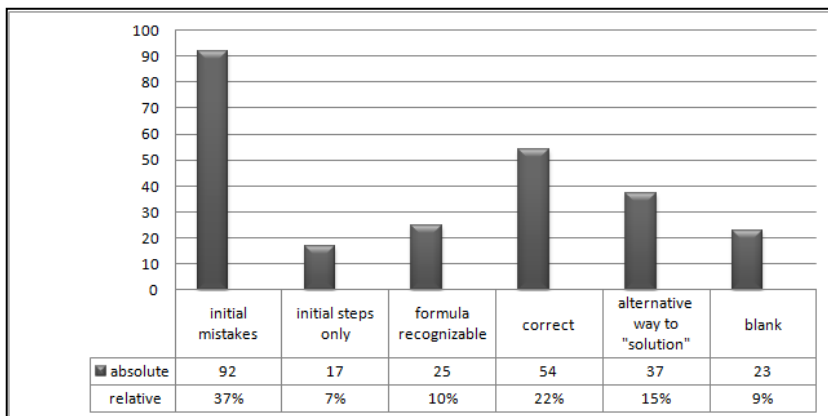


Figure 40 Item 3a Findings

Only 22% of the tested 248 students arrived at a correct solution. For this item, it was very interesting to see, how many of the 22% students who solved it correctly chose the elegant way of taking the square root over the long way. 31 students (of 54 students in the category ‘correct’) used the formula, 23 students took the square root.

Item 3b

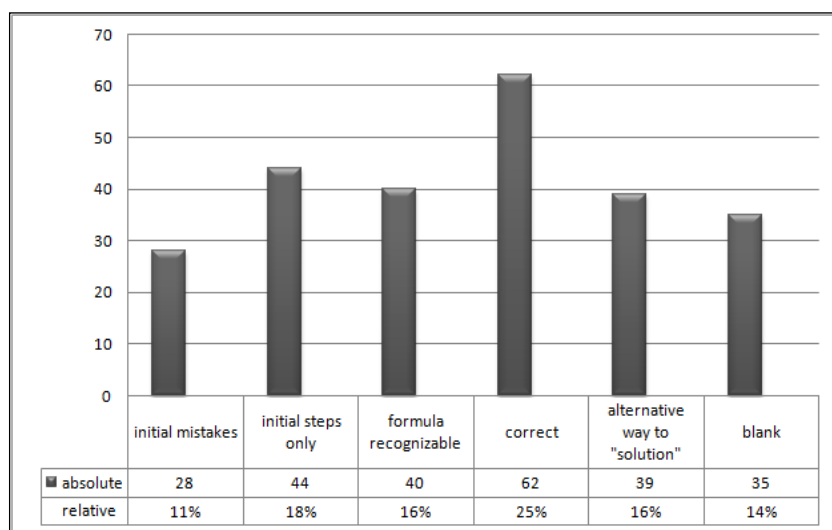


Figure 41 Item 3b Findings

Finally, Figure 41 shows the distribution of students' answers to item 3b which was similar to item 1a. However, this time – more students (25%) arrived at the correct solutions. Whereas only 12 students applied the null factor law, 50 students used the formula. All other approaches were chosen by similar amounts of students.

In part three, the students were asked to check their solutions. In general, unfortunately only a limited number of learners did so. This could be due to time pressure. Secondly, it could be that students did not understand the instructions or thirdly (and actually some students told me that when I asked them after the survey) that the students were simply too lazy. In case of the students who checked their solutions, no major problems could be noticed concerning item 3a. Those who did it, knew how to do it. Some arrived at a result, checked this, noticed that it could not be right, but then did not work over the whole equation again. This was certainly because of time issues, but basically, the students knew how to do it. Also concerning item 3b, no problems turned up. Only one student inserted the x_1 in the first bracket and x_2 in the second. Some students only checked one solution. Figure 42 illustrates the ideal way of working through part three, including the probations:

3. Löse die folgenden Gleichungen und kontrolliere deine Lösung:

a) $(x-2)^2 = 16$ b) $(x-3) \cdot (x+11) = 0$

$x^2 - 4x + 4 = 16$ $x^2 - 3x + 11x - 33 = 0$
 $x^2 - 4x - 12 = 0$ $x^2 + 8x - 33 = 0$
 $x = 2 \pm \sqrt{4+12}$ Probe: $(6-2)^2 = 16 \Rightarrow 4^2 = 16$ u. d.
 $x = 2 \pm 4$ $x = -2$ v. $x = 6$ $(-2-2)^2 = 16 \Rightarrow (-4)^2 = 16$ u. d. $\square$

$x = -4 \pm \sqrt{16+33}$
 $x = -4 \pm \sqrt{49}$
 $x = -4 \pm 7$
 $x = 3$ u. $x = -11$
 Probe: $3 \cdot 14 = 42 \neq 0$ u. d.
 $-11 \cdot 0 = 0$ u. d. $\square$

Figure 42 Items 3a and 3b Ideal

Some extreme cases revealed that students might be able to check the solutions, but that they have no clue about anything else in connection to quadratic equations. Figure 43 shows the answer of a student who neglected any rules and obviously had no idea of how to approach this equation correctly. He/she arrived at a true statement and was certainly sure that he/she succeeded. This shows that the student totally lacks an understanding of solving quadratic equations. He/she simply knows how to check a solution.

a) $(x-2)^2 = 16$

$(x-2)(x-2) = 16$
 $x^2 - 2x - 2x + 4 = 16 \quad | -4$
 $x^2 = 12 \quad | \sqrt{}$
 $x = \sqrt{12} = 6$

$(6-2)(6-2) = 16$
 $36 - 12 - 12 + 4 = 16 \quad \checkmark$

Figure 43 Checking solution

6.2.2. Critical stages in the curriculum

The analysis in the previous section was very detailed and offered an investigation of each and every item. Before the next section narrows down the findings to three main areas and statements that sum up the analysis best, this section could be regarded as link towards that. As mentioned above, the curriculum will be referred to again with reference to the results of the study. The following Tables 4, 5 and 6 do already give a hint, in which three areas students basically had the greatest difficulties. They feature some aspects from the right columns from the Tables 1,2 and 3 above (that is the left

column now, original passages from the curriculum) and comments with respect to the findings in the second columns.

Table 4 Curriculum and findings First and second form

Herstellen von Relationen unter Verwendung der Symbole =, ≠, <, >	The equality sign as such did not cause any problems. However, sometimes students forgot the rules for transforming an equation by neglecting to see an equation as a pair of scale. The study showed that "Schüler [arbeiten] vorwiegend mit [...] dem "Hinübergeben von Gliedern" [...] und nicht praktisch mit dem "Auf-beiden-Seiten-dasselbe-machen"" (Malle 1993: 200). This caused some problems with reference to the transformations.
Vergleiche (auch Termvergleiche)	Comparing terms became important for the initial steps of the transformations, where students had to sum up terms and had great difficulties doing so.
Herausarbeiten der Operationsstrukturen	The students encountered major problems with reference to the recognition of certain structures (e.g.: item 3b, where only a few students applied the null factor law)
Erkennen von Zusammenhängen , zB Tausch-, Nachbar-, Umkehr- und Analogieaufgaben	
Überprüfen (Abschätzen, Plausibilität, ...) der Ergebnisse von Rechenoperationen	Only one student (who computed three solutions) failed to check the solution with regard to plausibility.
Vergleichen von Rechenausdrücken unter Verwendung der Relationszeichen =, ≠, <, >	This was necessary to transform the equations (summing up/expanding). Many students had problems with these initial steps.
Vertiefen des Verständnisses für multiplikative Beziehungen auch unter Verwendung der Null	Problems with the null could be observed with reference to items 1e and 2e where students actually divided by zero.
Erkennen von Zusammenhängen und Rechenvorteilen	This is connected to a certain sense for structures. Students often chose the complicated strategy over the elegant way (e.g.: item 3a, where only a minority took the square root instead of expanding the product and using the formula which is much more complicated and longer)

Table 5 Curriculum and findings Third form

Verstehen des Operierens mit Null als Faktor	Items 1e, 2a and 3b – the null factor law was not applied in many cases.
Lösen einfacher Operationen unter Nutzung vorteilhafter Rechenwege	Students were often not sure how to arrive at a solution in the most elegant way.

Vergleichen von Rechenausdrücken unter Verwendung der Relationszeichen =, ≠, <, >	See above
Lösen einfacher Zahlengleichungen mit Platzhaltern (Variablen)	The notion of the variable as a place holder only became obvious when students checked their solutions and this was not problematic for them.
Durchführen von Rechenoperationen durch Zerlegen und Notieren der einzelnen Teilschritte , Berücksichtigen der Stellenwerte, Anwenden von Rechenregeln , z.B. Verteilungsregel	In general, students did note down the single steps. They also commented on several steps for part two of the questionnaire.
Begründen der Rechenschritte nach Einsicht in die den Operationen zu Grunde liegenden Rechenregeln	
Durchführen von Rechenproben	If students did check their solutions (only a few did), no major problems occurred. They did also often verbalize results for part two of the questionnaire.
Zuordnen von Rechenoperationen, Beschreiben von Sachverhalten mit Zahlen und Platzhaltern (Variablen)- Erstellen einfacher Gleichungen	Equations were only set up with reference to the monitoring process when they checked their solutions and this was therefore ok.
Kontrollieren und Verbalisieren der Ergebnisse	no major problems, see above

Table 6 Curriculum and findings Secondary school

Aims and objectives	Comments
1. Klasse	
die Regeln über die Reihenfolge von Rechenoperationen, einschließlich der Klammerregeln , anwenden können	Expanding a product and summing up terms caused great difficulties.
Lösungen zu einfachen linearen Gleichungen finden können,	Linear equations were not part of this study, but only part of some equations, e.g.: when the null factor law was applied and the brackets were treated separately. Unfortunately, the null factor law was not very popular with the students.
Formeln anwenden und interpretieren können	Frequently, the formula was not applied appropriately (insertion of p and q, algebraic signs etc.)
2. Klasse	
unter Verwendung von Umkehroperationen einfache lineare Gleichungen mit einer Unbekannten lösen und Formeln umformen , Formeln interpretieren	Transforming the equation so that the formula could be applied was often a challenge.

3. Klasse	
Verketteten der vier Grundrechnungsarten und derart entstehende Terme auch mit elektronischen Rechenhilfsmitteln berechnen können	The division of 147 by three without a calculator caused minor difficulties.
Formeln (bzw. Terme) umformen und durch Rechenregeln begründen können	Profound difficulties occurred in connection with the initial steps of the equation's transformations.
Lösen von linearen Gleichungen mit einer Unbekannten	see above
4. Klasse	
Sicherheit beim Arbeiten mit Variablen, Termen, Formeln und Gleichungen steigern	In general, the findings suggest that students did not feel comfortable with quadratic equations. Probably, students did not have enough confidence with respect to term transformations. Some might also lack knowledge in the areas of variables and equations in general.
5. Klasse	
Aufstellen und Interpretieren von Termen und Formeln, Begründen von Umformungsschritten durch Rechengesetze	Steps of the transformations were marked after each line most of the time.
Lösen von linearen und quadratischen Gleichungen in einer Variablen	In general, the study aimed at investigating students' behaviour in exactly this area. Unfortunately, there seem to be a lot of problems and misunderstandings.

6.2.3. Findings – general tendencies

The analysis of the individual items and with reference to the curriculum provided a very detailed discussion of the findings. Summing up the results from the two sections above, one could argue that the study basically revealed the following three crucial aspects:

- 1) In general, the results of the research suggests that many students lack a certain sense for the structure of quadratic equations. Only a small minority chose more elegant and definitely quicker ways of solving certain equations. Most of the students simply applied a certain scheme (the formula or alternative strategies) and stuck to that, regardless of what equation they worked on.
- 2) Secondly, students encountered major problems with respect to the initial steps of a term transformation. Whereas many had difficulties in expanding a

product or summing up the terms, others simply stopped after these initial steps.

- 3) Finally, considering the total amount of 248 participants, surprisingly few of these students remembered the formula (correctly). It is normally introduced in the fifth form and the study was conducted in the sixth form. However, this should not be an excuse. The concept and keyword that could counter this problem is sustainable teaching.

The last chapter of this paper is aimed at combining the results of the study with teaching quadratic equations and considering exactly the three aspects above.

7. Learning and teaching: quadratic equations

This last section could be regarded as a link between the study and the area of teaching and learning mathematics. It is concerned with important aspects of the three main findings of the research.

7.1. Structure sense

Especially, the recognition of term structures is of great importance in connection to the quadratic equations of the questionnaire. What is structure sense? How can it be fostered? Does everybody have it? These and more questions will be discussed in this section. Many authors talk about this crucial aspect of teaching and learning mathematics. Therefore, it will not be surprising that there are many different definitions, terms and explanations that are connected to this topic. For this paper, the two most suitable explanations would probably be the following by Meyer & Fischer (2013: 182) on the one hand: "Struktursinn bezeichnet die Fähigkeit, Strukturen in einem algebraischen Ausdruck wahrzunehmen, welche das Umformen diese Ausdrucks anleiten können" and the idea of algebraic thinking as a synonym on the other hand: „Algebraisches Denken wird hier verstanden als das mentale Umgehen mit Strukturen und Mustern“ (Mayer & Fischer 2013: 179). Going one step further, it is also useful to get an idea of what the structure of a term is. Rüede (2012: 114) defines that by saying that the structure of an expression is a "charakterisierende, objektive Eigenschaft des Ausdrucks".

The findings of the study suggest that many students lack a sense for term structures. Also authors and experts comment on this, for instance Malle (1993: 188): "das nötige Erkennen von Termstrukturen fehlt" or Rüede (2012: 114): "Die Häufigkeit von Schülerfehlern beim Erkennen und Gebrauch von Strukturen konnte in zahlreichen Studien empirisch nachgewiesen werden".

In her article "Young Student's structure sense", Lüken (2012) presents a study that investigated the structure sense of 74 children from nursery school to second grade, hinting at the fact that already at this age, children do not only display numerical competences, but also non-numerical competences, like a sense for structures (Lüken

2012:281). The problem is that structure sense cannot be learned, but only fostered. It is innate and the word 'sense' already suggests that it has something to do with possessing a feeling for mathematics and numbers rather than with a process that can be acquired. Star & Newton (2009) tried to explore the roots of structure sense by conducting a study with experts who were regarded as professionals in the field of recognizing structures. They use the expression 'flexibility' that could be regarded as another synonym for structure sense in this paper. The authors point out that flexibility can mean a lot of things – "a person's ease in switching between solution methods" or "a person's tendency to select the most appropriate method in a given situation" (Star & Newton 2009: 558). With reference to the questionnaire, it was interesting to see why students would choose one strategy over another and this fits exactly to the latter explanation of flexibility. In the course of their study it turned out that the participating experts were not pushed, but rather that flexibility rooted in own initiatives or experiences of teaching (Star & Newton 2009: 564).

As a future teacher and knowing that structure sense is implicit and based on intrinsic motivation, one would nevertheless be interested in how far it can be fostered or brought to the surface. It is not through completing tons of exercises that the students acquire structure sense (it will be talked about exercises later again in section 7.2.). Students should rather be asked more frequently to accurately look at expressions, compare terms or give reasons for certain decisions. By all means, it is desirable that students also concentrate on other aspects that do not involve calculating at all (Rüede 2012: 139). This can be achieved by working on problems like the equations of part two: "Ein anderer Weg, sich dem Problemlösen und seinen Strategien im Unterricht zu nähern, ist das Betrachten gelöster Probleme [...] [Dies] gibt damit mehr Raum für das bewusste Nachdenken über Problemlöseprozesse" (Büchter & Leuders 2005: 40).

Another critical issue is that if students lack structure sense, they often make use of a certain scheme (synonyms in this context: strategy/rules) which is problematic with respect to solving equations. Malle (1993: 197) suggests that students are guided by schemes during the solution process or the transformation process of an equation. In this context, the expression 'scheme' can have a negative or positive connotation. For

instance, a positive application of a scheme could be the process described above (quote on page 35): The student expands/adds up, sorts and isolates the variable by using the formula. Unfortunately, the study has shown that often students use their own schemes. Examples of these counted towards the category 'alternative way to "solution"'. Fischer & Malle (1985: 61) speak of 'private rules' the students have gathered in their heads, without the teacher noticing. Moreover, Malle (1993: 160) warns that "Schüler sehen den Ausdruck nur flüchtig an, wenden dann irgendwelche Schemata an (die mit Regeln nichts zu tun haben) und kommen damit zu einer fehlerhaften Umformung". Also Osta (2007: 6) notes that "students resist the use of algebra and apply their *own informal strategies*" [my emphasis].

In connection to the questionnaire, the negative connotation of schemes comes best to the surface if answers of the items 1a and 1b (also 3a and 3b) are considered. Figure 44 shows how a student applies his/her private rules, always arriving at an expression 'x equals anything'.

The figure shows four handwritten solutions for quadratic equations, each following a similar flawed pattern:

- 1a) $(x-2) \cdot (x+3) = 6$**
 $x^2 + 3x - 2x - 6 = 6 \quad | +6$
 $x^2 + x = 12$
 $2x^2 = 12 \quad | : 2$
 $x^2 = 2$
 $x = \sqrt{2}$
- 1b) $x^2 - 4x + 4 = 0$**
 $x^2 - 4x = -4$
 $4x^2 = 4 \quad | : 4$
 $x^2 = 1$
 $x = \sqrt{1}$
 $x = 1$
- 3a) $(x-2)^2 = 16$**
 ~~$x^2 - 4 = 16$~~
 ~~$x^2 = 20$~~
 ~~$x = \sqrt{20}$~~
 $(x-2)(x-2) = 16$
 $x^2 - 2x - 2x + 4 = 16$
 $x^2 - 4x = 12$
 $4x^2 = 12 \quad | : 4$
 $x^2 = 3$
 $x = \sqrt{3}$
- 3b) $(x-3) \cdot (x+11) = 0$**
 $x^2 + 11x - 3x - 33 = 0$
 $x^2 + 8x = 33$
 $8x^2 = 33 \quad | : 8$
 $x^2 = 4,1$
 $x = \sqrt{4,1}$

Figure 44 Scheme Negative

For all four items (1a,b and 3a,b) the student makes use of the same strategy. On the right side of the equation it is always a number. All expressions including an x are put to the left side. Terms with x^2 or x are erroneously added up somehow. Then the student arrives at a term ' x^2 equals anything', takes the square root and thinks he/she is finished. The learner probably thinks that his/her strategy is totally correct, because nothing tells him/her so far that it is not, or as Malle (1993: 161) puts it:

Die elementare Algebra hat nun die unangenehme Eigenschaft, daß die Grenze zwischen erlaubten und unerlaubten Anwendungen eines Schemas an der Notation oft nicht unmittelbar ersichtlich ist. [...] Die Gleichungen, die durch

eine unerlaubte Schemaanwendung entstehen, sind zwar falsch, zunächst signalisiert aber nichts, daß die Schemaanwendung unerlaubt war.

Also the formula can of course be regarded as a scheme. This is the second example of a scheme being unprofitable. As mentioned in the introduction, as soon as students are offered a certain strategy, they tend to use it and stick to it, regardless of which equation they face and this is what also the empirical study showed. Figure 45 shows an example:

Figure 45 shows three examples of handwritten student work for quadratic equations, illustrating the application of a formula as a scheme:

- c) $3x^2 = 147$ is solved by dividing by 3: $3x^2 = 147 \div 3$, resulting in $x^2 = 49$, and then taking the square root: $x = \pm \sqrt{49} = \pm 7$.
- d) $x^2 - 81 = 0$ is solved using the quadratic formula: $x = \frac{-0 \pm \sqrt{0^2 + 4 \cdot 81}}{2 \cdot 1} = \pm 9$.
- e) $(3x - 2)x = 0$ is solved by dividing by 3: $3x^2 - 2x = 0 \div 3$, resulting in $x^2 - \frac{2}{3}x = 0$, and then completing the square: $x = \frac{1}{3} \pm \sqrt{\left(\frac{2}{6}\right)^2} = \frac{1}{3} \pm \sqrt{\frac{4}{9}} = \frac{1}{3} \pm \frac{2}{3}$.

Figure 45 Formula as scheme

Obviously, the student knows exactly how to apply the formula. However, a person who possesses a certain degree of structure sense would normally definitely not use it (particularly not for items 1c and 1d). The major problem is that if students only use these offered schemes and trust them, their structure sense is not even slightly fostered, but rather inhibited. Gradually, these strategies become automated so that students no longer use their common sense at all, but fully rely on these schemes. Also Bruder, Leuders & Büchter (2008: 22-24) comment on this threat to mathematic teaching and learning on the basis of previous studies and call it a general, disadvantageous development:

[I]nhaltliches mathematisches Denken von Klassestufe 5 auf 6 [wird] durch kalkülhafte Regelanwendung ersetzt [...] Hinzu kommt, dass unseren Schülerinnen und Schülern [...] der gesunde Menschenverstand partiell abhandenkommt und schematischen Denken platzmacht.

The reason for students using rules and strategies can be easily explained. Schemes, as the formula, can be learned and students like these opportunities, true to the motto "Die Schüler möchten ja ein klares, auswendig lernbares Verfahren" (Büchter & Leuders 2005: 141). Consequently, it is more convenient for a student to use the formula (she/he knows exactly how to apply it, she/he has learned the formula by heart), than spending time on thinking of (more elegant and also quicker) strategies. Also for the teacher it is easier to introduce formal rules and schemes, because: "Die

Versuchung ist groß, sich in der Lehre auf das zurückzuziehen, was man am leichtesten in den Griff bekommt, nämlich den Formalismus. Für diesen kann man relativ klare Regeln aufstellen, die relativ *leicht lehrbar und abprüfbar* sind“ (Fischer & Malle 1985: 49) [my emphasis].

Having heard that structure sense is innate and that the use of schemes can have negative influences, the paper now concentrates on exercises in mathematics in connection to the transformation of an equation.

7.2. Transforming an equation – exercises and practice

The findings of the study showed that students had great difficulties in transforming equations, especially with regard to the initial steps of this process. Practice contributes a lot to the acquisition of mathematical proficiency and can be – depending on how the teacher incorporates practice in the lessons – motivating, supportive and groundbreaking, but also obstructive.

Oftentimes, teachers complain about the fact that students still make mistakes during term transformations in spite of hundred exercises: “Man begnügt sich oft mit einem “endlosen” Üben des Umformens” (Malle 1993: 254). This is what Malle (1993: 22) calls the practice ideology. In other words, there is the assumption of many teachers that the transformation of equations can be learned by simply working through a huge amount of exercises of similar and stereotypical character. The mixture of ‘many and similar’ is certainly not beneficial, or as Dörfler (1981: 109) put it: “[D]ie Lösung eines komplexen Anwendungsproblems kann sich auf Routinefertigkeiten reduzieren, wenn *viele gleichartige* Aufgaben eingeübt wurden” [my emphasis]. Also, Malle (1993: 50) warns of teachers who prefer “Einüben standardisierter Lösungsverfahren gegenüber einer problemhaften Unterrichtsgestaltung”. The second misbelief is connected to the choice of exercises. Teachers believe that very complex equations must be worked through in order to experience success in connection to easier ones (Malle 1993: 23), but this is not true. Unfortunately, such complicated items do rather only train concentration skills (Bruder and Leuders 2008: 159):

Typischerweise werden bei solchen Aufgaben, die eher die sichere Beherrschung von Verfahren überprüfen, die Anforderungen durch die technische Schwierigkeit [...] gesteigert. Damit werden dann eher das

Durchhalte- und Konzentrationsvermögen sowie das rechnerische Geschick überprüft als die Tragfähigkeit der zugrundeliegenden Vorstellungen und die Tiefe des Verständnisses.

Therefore, it is important that the teacher does not base the planning activities on the amount and complexity of exercises. Many experts are concerned with the choice of exercises. Taking some of them into account, it could be summed up that:

- exercises should provide insights into the understanding of students (not only into the ability to calculate or apply a scheme) (Bruder, Leuders & Büchter 2008: 159)
- exercises (especially in case of equations) should cater for different approaches towards completion, so that the students have the possibility to activate their decision and reasoning skills (Büchter & Leuders 2005: 29); exercises should stimulate their abilities to abstract, the students' heuristic abilities, their flexibility of thinking and encourage them to ask for further solving processes (more elegant, quicker...) (Malle 1996: 50)
- exercises should be discussed afterwards in form of a reflection and by asking questions such as "How was it done?" (Malle 1993: 54)

Undoubtedly, most teachers mainly consult the course book for exercises, but nevertheless there is space to adapt these to the needs of the students. As mentioned above, it is often not necessary to work through all provided tasks. The teacher should rather pick out the ones that feature the aspects mentioned above. Also, additional exercises can always be added from different sources (colleagues, internet, other books a.s.o.) or existing exercise can be easily adapted – often simply by adding the question "How did you do it?" or "Why would you approach this equation in this way" a.s.o. Moreover, if certain exercises with reference to transformations are approached, it can be very helpful to use visual representations (arrows to mark corresponding terms during the expansion of a product) as support. David, Tomaz & Ferreira (2014: 95) showed that visual displays can help to "construct and communicate mathematical meanings".

In sum, one could argue that both aspects (calculating and understanding) are of great relevance, or as Glaeser (1980: 16) puts it: "Der neuen Didaktik kommt es vor

allem auf das Begriffsverständnis an; sie vernachlässigt dabei allerdings nicht die Einübung von Mechanismen zur Anwendung verstandener Begriffe". Consequently, the intention should be a connection of form and content with regard to the choice of exercises (Fischer & Malle 1985: 49). Malle (1996: 49-50) sums up best, which attitudes students should bring towards practicing transformations. He argues that there should be a

Verhältnis zwischen disziplinierten "Abarbeiten" von Algorithmen und schöpferischer Freiheit, zwischen "mechanischem" Vollzug bekannter Handlungen und kreativem Suchen nach möglichen Lösungswegen, nicht zuletzt auch zwischen dem Gefühl relative Sicherheit der Lösungsfindung und der Furcht vor dem Erleben eines Mißerfolgs.

7.3. Sustainable teaching

The last pedagogical aspect that will be discussed is the area of sustainable teaching. Bruder, Leuders & Büchter (2008: 26) describe sustainable teaching in the following way: "Nachhaltiges Lernen erfordert einen vielseitigen vernetzenden und mehrperspektivischen, aber nicht beliebigen Umgang mit dem Lerngegenstand". Unfortunately, many students were not able to remember the formula at all, although it had been introduced approximately one year before the study was conducted. This does certainly not count positively towards a sustainable teaching and learning culture. One of the most beneficial approaches towards sustainable education might be the framework of J.S. Bruner who introduced the spiral principle/curriculum spiral. It includes the following three principles (taken from Christmann 1980: 150-151):

- 1) Der Unterricht in einem Fach darf keine fachlichen Verzerrungen enthalten.
- 2) Die Analyse komplexer Inhalte und Begriffe soll durch intuitive Erfassen und intuitive Gebrauch vorbereitet werden.
- 3) Die zentralen Begriffe [...] sollen immer wiederkehren und dabei stets weiter ausgebaut und verfeinert werden, bis sie schließlich voll erfaßt werden.

Problems in connection to quadratic equations might be solved by concentrating on the second and third principle. The intuitive aspect (second principle) is closely connected to what has been talked already about structure sense, flexibility and how the choice of exercises can stimulate the students' intuition and feeling for mathematics. The third aspect that is linked to the revision of relevant contents will be discussed in more detail now.

The major problem is that often a topic (e.g.: quadratic equations) is introduced when the course book or the curriculum suggests it. Then, necessary issues are introduced, discussed, practiced and unfortunately, often that was it – this topic is then completed. Like in a checklist, this topic is then ticked off in the minds of students and teachers. It will not be referred to it again and as soon as components of this “*ticked off topic*” turn up in connection with new issues, these are then taken as granted. The teacher carries on, although many students probably have not understood everything at first and therefore often fail to understand further issues. This leads to the sad fact that students never get the chance to fully capture the key issues. It seems that the content of the lessons is often worked through in a rush. Of course, teachers and learners often experience time pressure and going back to certain contents again and again seems to be very time consuming, but it is possible to incorporate the spiral principle as will be discussed now in connection to quadratic equations and the application of the formula.

Before the formula is introduced, many other aspects need to be revised again. For example, at this stage it is extremely important that students know the meaning of a variable or a formula/quadratic equation. The formula is normally introduced in the fifth form in secondary school. As a teacher it is important to go back in the curriculum or in the planning documents and spot previous issues that are connected to solving quadratic equations. If the teacher consults the curriculum he/she might notice that in the fourth form (see Table 3) students should become more confident with variables, terms, formula and equations. In the third form, the students are supposed to solve linear equations in one variable. These two stages might be worth mentioning again. As soon as the teacher starts thinking about introducing the formula, the following aspects could be revised quickly (this does probably not take more than five minutes) by simply asking elementary questions such as “What are linear equations?” “How did we solve them?” “Can you give me one example of a formula and explain when it is used?”. Another possibility would be to revise some exercises from the previous years. Such tasks can “Vorkenntnisse reaktivieren, die im Folgeunterricht benötigt werden” (Christmann 1980: 117). Such a revision can be very useful for new topics to come.

As soon as the formula is introduced, it is necessary that students understand how it has been deduced from completing the square. In case students do not remember it by heart, they might then be able to reproduce it on their own. Undoubtedly, it is desirable that students know the formula by heart. In order to achieve this, it is again inevitably to refer back to the formula – a week later, a month later, a topic later, a year later – as soon as it comes up it should be quickly revised again. For example, in the fifth form, it is also talked about functions (linear and simple non-linear functions). This could already be a good, first opportunity to connect the formula with finding the roots of a function and to revise its application. The formula should not only be introduced, but its usefulness should be accurately discussed by also asking questions like: “Welchen Sinn, welchen Zweck, welchen Vorteil, welchen Nutzen bringt das Aufstellen einer Formel im Rahmen der gestellten Aufgabe mit sich?” (Malle 1993: 55).

Summing up, the three problematic areas can be faced by fostering students' structure sense, by incorporating exercises in an appropriate way and by following the spiral principle for sustainable teaching.

8. Conclusion

This paper has attempted to investigate students' difficulties with quadratic equations by presenting an empirical study, a theoretical and organizational background and references to teaching and learning. A clarification of the background included explanations of mathematical aspects as well as organizational aspects that counter for the legibility of this paper. Links to teaching and learning were established by the sections on the curricula and the school books. The core of the paper was the empirical study. It was presented by providing the complete framework, participants, the scoring procedure and a discussion of the findings. In the course of this discussion, it turned out that mainly three areas in connection to quadratic equations caused the most serious problems. Attempts to overcome these difficulties and misunderstandings were presented in the section previous to this one.

Although this paper has provided some interesting insights into this complex issue of teaching mathematics, many other factors could be taken into account to arrive at an even more accurate picture. For instance, interviews with students after the completion of the questionnaire could be very beneficial and revealing, because sometimes students find a solution without even truly noticing why or how. It was impossible to reproduce the thoughts of the students when the questionnaires were looked through. In some cases, only vague assumptions could be made to explain how the student arrived at this solution or that transformation. Interviews could give more clarity, such as results of other studies show: Clements (2006: 73) concludes that "many interviewees who obtained correct solutions actually had serious misconceptions about what quadratic equations actually are" or Fischer and Malle (1985: 77) who confirm the importance of interviews:

Wir sind im Besitz von Tonbandaufnahmen, die zeigen, daß selbst richtige Lösungen von Aufgaben aus der elementaren Algebra auf völlig "falschen" Vorstellungen der Schüler von Variablen, Termen und Formeln beruhen können.

Moreover, the study was conducted in AHS only. It would also be interesting to see how students of other school types would handle the questionnaire and quadratic equations in general. Also, many other aspects that were not taken into account cater

for limitations of this study. Surely, many questions remain unanswered, but this study could trigger a more deeper and more comprehensive research project.

As a future teacher, I have already talked about my experiences during the FAP in the introduction and about my motivation to write this paper, but why would other people be interested to read about students' problems in connection to quadratic equations, although problems and misunderstandings often have a negative connotation with reference to teaching? I strongly believe that such an analysis of students' difficulties can especially help teachers to adapt or change their way of teaching with respect to this investigated area. Such a research can basically "potentially help teachers diagnose misconceptions and adjust their teaching strategies" (Osta & Labban 2007: 2) or as Radatz (1979: 71) puts it: "Die Diagnose von Lernschwierigkeiten der Schüler ist eine notwendige Voraussetzung für eine sinnvolle und effektive Differenzierung bzw. Individualisierung des Mathematikunterrichts". Moreover, since discussions have started about the standardized final exams and educational standards, it has become even more important to focus on students' competences. Problems need to be analysed. If students cannot handle quadratic equations, this needs to be transformed into 'can do' statements and the paper could provide a basis for this transformation. Of course, it is only concerned with a small area of teaching mathematics, but this project nevertheless could be seen as a starting point to improve and foster teaching and learning. In order to close the circle I would like to refer to the quotation at the very beginning of this paper. Such a situation as mentioned in the quote is not desirable in school at all, but unfortunately not uncommon, as the analysis of the survey has shown. This paper could be a motivational force in order to strive for a statement like the following: I almost didn't get an A in math, but then Mr. Carlo told me to **start** asking "why?" all the time and **not** just follow the formulas. So, I did. Now, I get perfect scores on all my tests. I **am glad I know** what the formulas **do**. I honestly **now get it**.

(19 895 words)

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9.2. Participating schools – web pages

- AHS Friesgasse - <http://www.schulefriesgasse.ac.at/ahs/>
Friesgasse 4
1150 Wien
- AHS Rahlgasse - <http://www.ahs-rahlgasse.at/>
Rahlgasse 4
1060 Wien
- AHS Theodor Kramer Straße - <http://www.thkr.at/>
Theodor Kramer Straße 3
1220 Wien
- GRG 21 Schulschiff "Bertha von Suttner" -
http://www.schulschiff.at/cms/front_content.php
Donauinselplatz
1210 Wien
- Parhamergymnasium - <http://www.parhamer.at/>
Parhamerplatz 18
1170 Wien
- Wiedner Gymnasium/Sir-Karl-Popper-Schule -
<http://www.wiednergymnasium.at/>
Wiedner Gürtel 68
1040 Wien

10. Appendix

10.1. German summary

Die vorliegende Arbeit beschäftigt sich mit den Schülerschwierigkeiten beim Lösen von quadratischen Gleichungen. Dazu wurde in sechs verschiedenen AHS in Wien eine Umfrage durchgeführt. Insgesamt füllten 248 SchülerInnen den Fragebogen aus, der aus verschiedenen quadratischen Gleichungen bestand. Einerseits sollten die SchülerInnen diese Gleichungen lösen, andererseits aber auch gelöste Gleichungen untersuchen und ihre Lösungen selbstständig kontrollieren. Das Ziel war es anhand dieser Studie heraus zu finden, wo bei den SchülerInnen Verständnisprobleme vorliegen.

In der Arbeit wurden einleitend organisatorische Grundlagen geklärt. Ein Einblick in den Lehrplan sowie eine kurze Analyse von vier häufig verwendeten Schulbüchern rundeten die Vorbereitungen auf die empirische Untersuchung ab. Die Ergebnisse wurden von drei verschiedenen Ebenen aus betrachtet. Erstens im Bezug auf die einzelnen Gleichungen des Fragebogens, zweitens im Bezug zum Lehrplan und drittens wurden drei Grundaussagen herausgearbeitet, die das Gesamtergebnis widerspiegeln. Grundsätzlich stellte sich heraus, dass SchülerInnen große Probleme hatten, die Formel abzurufen, erste Schritte beim Umformen durchzuführen, Strukturen zu erkennen und auf diese zu reagieren. Diese Bereiche bildeten dann die Basis für die abschließenden Kapitel des Projektes. Sie beschäftigten sich mit dem Zusammenhang von Lehren und Lernen. Zentrale Rollen spielten unter anderem Aspekte wie Struktursinn, nachhaltiger Unterricht und die Übungskultur im Mathematikunterricht.

Die Arbeit hat gezeigt, dass der Umgang mit quadratischen Gleichungen für SchülerInnen oft Probleme aufwirft. Obwohl dieses Resultat ein eher negatives ist, können die Erkenntnisse dieser Untersuchung eine Motivation sein, um Lösungen für diese Situation zu finden.

10.2. Proposal city council – formal letter

Lisa Grubner
Waldstraße 13/2
3204 Kirchberg/Pielach

An den Stadtschulrat Wien
Pädagogische Abteilung AHS
z.H. Herrn Mag. Roland Zielka
Wipplingerstraße 28
1010 Wien

Betrifft: Erteilung einer Bewilligung für eine wissenschaftliche Erhebung

Sehr geehrter Herr Mag. Zielka,

mein Name ist Grubner Lisa, ich bin Studierende an der Universität Wien und mit diesem Schreiben möchte ich um eine Bewilligung für eine wissenschaftliche Erhebung im Rahmen meiner Diplomarbeit ansuchen.

Ich studiere seit dem Wintersemester 2009 Englisch und Mathematik auf Lehramt und mit dem Verfassen der Diplomarbeit in Mathematik steuere ich auf das Ende meiner universitären Ausbildung zu. Im Rahmen meines Unterrichtspraktikums in Mathematik durfte ich in einer 5. Klasse die kleine Lösungsformel einführen. Ich musste schnell feststellen, dass die SchülerInnen schwerwiegende Probleme hatten, Term Strukturen zu erkennen, die Lösungsformel anzuwenden und ihre gefundenen Lösungen zu überprüfen. Nach Absprache mit Herrn Mag. Dr. Andreas Ulovec, der mich bei meiner Diplomarbeit betreut, möchte ich nun genau diese Aspekte im Rahmen meiner Abschlussarbeit untersuchen. Ich habe mich entschlossen die Arbeit auf Englisch zu verfassen, sodass wir uns auf den folgenden vorläufigen Titel festlegten: Problematic components of quadratic equations: recognising term structures, applying the quadratic formula and checking solutions- an empirical research. Im Rahmen der empirischen Untersuchung würde ich gerne Fragebögen (Lösen von verschiedenen quadratischen Gleichungen) an SchülerInnen der 6. Klassen austeilen und sie bitten, diese auszufüllen. Der zweite Teil besteht darin, dass ich sehr gerne ein paar Mathematik LehrerInnen dieser Klassen zu diesem Thema befragen möchte. Die Ergebnisse beider Teilmfragen (wobei der SchülerInnenfragebogen sicherlich der Schwerpunkt ist) werden im Zuge der Arbeit analysiert. Nachfolgend finden sie genaue Details der Erhebung, den Elternbrief, den

Fragebogen für die SchülerInnen, bzw. die Interviewleitfragen für die Lehrpersonen, sowie eine (erste) Information an die betroffenen Lehrkräfte. Weitere wichtige Dokumente, wie zum Beispiel die Bestätigung meines Betreuers oder die Verpflichtungserklärung zur Wahrung des Datenschutzes, finden Sie als zusätzliche Dateien im Anhang des Mails.

Die DirektorInnen der Schulen wurden bereits von mir per Email, bzw. telefonisch informiert. An den folgenden Schulen wurde mir eine Zusage seitens der Direktion für eine Erhebung in den 6. Klassen erteilt:

- AHS Friesgasse; Friesgasse 4; 1150 Wien
- AHS Rahlgasse; Rahlgasse 4; 1060 Wien
- AHS Theodor Kramer Straße; Theodor Kramer Straße 3; 1220 Wien
- GRG 21 Schulschiff "Bertha von Suttner"; Donauinselplatz; 1210 Wien
- Parhamergymnasium; Parhamerplatz 18; 1170 Wien
- Wiedner Gymnasium/Sir-Karl-Popper-Schule; Wiedner Gürtel 68; 1040 Wien

Ich habe mich sehr gefreut, dass diese Schulen mich durch ihre Bereitwilligkeit unterstützen. Nun hoffe ich, dass auch Sie mir eine Genehmigung für mein Projekt erteilen. Bei Fragen bin ich jederzeit für Sie per Mail erreichbar. Ich freue mich auf eine Rückmeldung und verbleibe mit freundlichen Grüßen.

Lisa Grubner

10.3. Approval city council



Frau
Lisa Grubner
Waldstraße 13/2
3204 Kirchberg/Pielach
lisa@grubners.at

Unser Zeichen/GZ
240.128/23-kanz2/2013

BearbeiterIn
Mag.^a Sabine Sommer
sabine.sommer@ssr-wien.gv.at

Tel. 525 25
DW 77213
FAX 99 77 213

Datum
17.12.2013

Antrag für eine wissenschaftliche Untersuchung an Wiener AHS

Sehr geehrte Frau Grubner!

Von Seiten der AHS-Abteilung besteht inhaltlich kein Einwand gegen die Durchführung dieser Untersuchung, sofern Sie dabei auf die Lehrer/innen-Interviews verzichten.

Sie wurden von Frau Mag.^a Sabine Sommer über die Empfehlung von Frau HR Dr.ⁱⁿ Mathilde Zeman inhaltlich informiert, nachdem der Fachausschuss die Untersuchung ablehnte.

Danach versicherten Sie in einem Telefongespräch mit Frau Mag.^a Sabine Sommer vom 6.12.2013, dass sich Ihre wissenschaftliche Untersuchung ausschließlich auf Schüler/innen beziehen wird.

Mit freundlichen Grüßen
für die Amtsführende Präsidentin:

Mag.^a Gabriele Dangi
AHS-Abteilungsleiterin

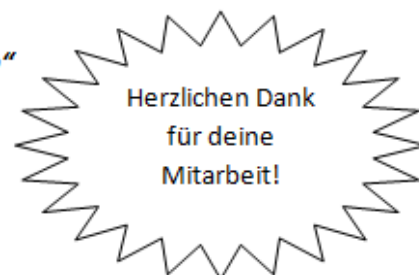
Signaturwert	6b77ecc79cab4090ac9803d30f402aee		
REPUBLIK ÖSTERREICH  stadtschulrat für wien AMTSSIGNATUR	Unterzeichner	Stadtschulrat für Wien	
	Datum/Zeit-UTC	17.12.2013 16:04:39	
	Aussteller-Zertifikat	CN=a-sign-corporate-light-02, OU=a-sign-corporate-light-02, O=A-Trust Ges. f. Sicherheitssysteme im elektr. Datenverkehr GmbH, C=AT	
	Serien-Nr.	998631622664	
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	Parameter		
Prüfinformation	Informationen zur Prüfung der elektronischen Signatur finden Sie unter: http://www.signaturpruefung.gv.at . Informationen zur Prüfung des Ausdrucks finden Sie unter: http://www.wien.gv.at/bildung/stadtschulrat/amtssignatur/index.html .		
Hinweis	Dieses Dokument wurde elektronisch signiert. Auch ein Ausdruck dieses Dokuments hat gemäß § 20 E-Government-Gesetz die Beweiskraft einer öffentlichen Urkunde.		

10.4. Questionnaire

Wissenschaftliche Erhebung zum Thema „Lösen von quadratischen Gleichungen“

Datum: _____

Geschlecht (bitte ankreuzen): m ☐ w ☐



1. Löse die folgenden Gleichungen:

a) $(x - 2) \cdot (x + 3) = 6$

b) $x^2 - 4x + 4 = 0$

c) $3x^2 = 147$

d) $x^2 - 81 = 0$

e) $(3x - 2)x = 0$

2. Die folgenden drei Gleichungen **wurden bereits gelöst. Überprüfe den Lösungsweg** – ist dieser korrekt? Wenn nicht, **kennzeichne** die fehlerhaften Stellen und **begründe** warum es sich um einen etwaigen Fehler handelt; ist der Lösungsweg korrekt, **hake** die Rechnung ab:

a) $x^2 = 6x \quad / :x$
 $x = 6$

b) $3x^2 - 15x + 18 = 0 \quad / :3$
 $x^2 - 5x + 6 = 0$
 $x = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 6}$
 $x = \frac{5}{2} \pm \frac{1}{2}$
 $x_1 = 3$
 $x_2 = 2$

c) $x^2 + 2x + 1 = 0$
 $x = \frac{-2}{2} \pm \sqrt{\frac{4}{4} - 1}$
 $x = 1 \pm \sqrt{1 - 1}$
 $x = 1$

3. Löse die folgenden Gleichungen und **kontrolliere deine Lösung**:

a) $(x - 2)^2 = 16$

b) $(x - 3) \cdot (x + 11) = 0$

10.5. Results – details

Class	A	B	C	D	E	F	G	H	I	J
ITEM 1A										
Correct	0	4	1	1	0	1	2	1	4	0
Initial mistakes	8	11	8	6	4	4	5	2	4	3
Initial steps only	7	0	6	3	1	6	1	0	1	1
Formula recognizable	2	4	1	2	5	5	4	1	9	4
Alternative way to "solution"	3	3	3	7	8	5	2	0	0	4
Blank	1	0	1	1	0	1	1	0	3	0
ITEM 1B	A	B	C	D	E	F	G	H	I	J
Correct	2	11	1	9	5	6	3	3	12	6
Alternative way to "solution"	4	1	1	0	1	3	3	0	1	3
Initial steps only	7	1	3	2	1	0	0	0	0	0
Initial mistakes	5	1	8	0	0	3	3	0	0	0
Blank	2	1	2	4	2	4	1	0	4	0
Formula recognizable	1	7	5	5	9	6	5	1	4	3
ITEM 1C	A	B	C	D	E	F	G	H	I	J
Positive solution	19	15	10	8	12	13	11	2	16	7
Both solutions	1	1	4	3	2	2	0	2	3	2
Blank	0	1	1	5	0	3	1	0	1	0
Alternative way to "solution"	1	0	1	0	3	2	0	0	0	0
Formula recognizable	0	1	1	0	0	0	1	0	0	2
Initial mistakes	0	4	3	4	1	2	2	0	1	1
ITEM 1D	A	B	C	D	E	F	G	H	I	J
Positive solution	19	18	13	13	14	14	12	2	15	9
Both solutions	1	1	4	4	2	4	1	2	3	3
Initial steps/blank	1	0	2	3	1	1	2	0	2	0
Alternative way to "solution"	0	2	0	0	0	3	0	0	0	0
Formula recognizable	0	1	1	0	0	0	0	0	1	0
ITEM 1E	A	B	C	D	E	F	G	H	I	J
Initial steps only	10	7	8	6	7	9	7	0	3	2
Alternative way to "solution"	8	2	6	5	7	4	3	2	4	2
Blank	1	4	3	3	3	3	3	0	4	0
Formula recognizable	1	9	2	2	1	1	0	0	2	5
Correct	0	0	0	3	0	4	0	1	6	3
Initial mistakes	1	0	1	1	0	1	2	1	2	0

ITEM 2A	A	B	C	D	E	F	G	H	I	J
Tick	13	12	8	10	10	11	9	4	17	7
Blank	4	3	5	1	1	2	2	0	4	1
Mark wrong	1	4	0	0	0	2	0	0	0	2
"Mistake"	3	3	7	9	6	6	4	0	0	1
Spotted	0	0	0	0	1	1	0	0	0	1

Class	K	L	M	N	SUM
ITEM 1A					
Correct	3	0	1	4	22
Initial mistakes	1	8	5	1	70
Initial steps only	1	1	10	0	38
Formula recognizable	3	3	0	10	53
Alternative way to "solution"	5	6	7	2	55
Blank	1	0	0	1	10
					248
ITEM 1B	K	L	M	N	SUM
Correct	8	2	1	12	81
Alternative way to "solution"	2	9	8	1	37
Initial steps only	0	2	5	0	21
Initial mistakes	0	1	5	0	26
Blank	1	0	4	0	25
Formula recognizable	3	4	0	5	58
					248
ITEM 1C	K	L	M	N	SUM
Positive solution	11	10	20	17	171
Both solutions	2	2	1	1	26
Blank	0	3	2	0	17
Alternative way to "solution"	1	0	0	0	8
Formula recognizable	0	0	0	0	5
Initial mistakes	0	3	0	0	21
					248
ITEM 1D	K	L	M	N	SUM
Positive solution	12	14	20	18	193
Both solutions	2	3	2	0	32
Initial steps/blank	0	1	0	0	13
Alternative way to "solution"	0	0	2	0	7
Formula recognizable	0	0	0	0	3
					248
ITEM 1E	K	L	M	N	SUM
Initial steps only	3	4	14	2	82

Alternative way to "solution"	4	12	7	12	78
Blank	0	1	0	0	25
Formula recognizable	4	0	0	3	30
Correct	3	0	1	1	22
Initial mistakes	0	1	1	0	11
					248
ITEM 2A	K	L	M	N	SUM
Tick	12	13	20	12	158
Blank	1	1	1	2	28
Mark wrong	0	1	0	0	10
"Mistake"	1	2	2	3	47
Spotted	0	1	0	1	5
					248

ITEM 2B	A	B	C	D	E	F	G	H	I	J
Tick	8	19	13	12	7	12	8	2	15	10
Blank	7	1	4	3	1	1	6	1	5	1
Mark wrong	0	0	2	0	1	0	0	0	0	0
"Mistake"	6	2	1	5	9	9	1	1	1	1
ITEM 2C	A	B	C	D	E	F	G	H	I	J
Tick	10	9	4	6	3	13	1	1	2	2
Blank	5	1	9	2	4	0	8	0	7	1
Mark wrong	1	3	2	2	0	2	0	0	0	0
"Mistake"	2	1	3	5	3	2	4	0	1	1
Spotted	3	8	2	5	8	5	2	3	11	8
ITEM 3A	A	B	C	D	E	F	G	H	I	J
Initial mistakes	6	8	9	8	13	12	3	0	0	2
Initial steps only	6	2	0	0	0	1	4	0	0	0
Formula recognizable	5	4	4	1	2	3	1	0	2	1
Correct	0	5	0	6	0	3	1	4	11	7
Alternative way to "solution"	3	2	4	4	2	3	2	0	2	2
Blank	1	1	3	1	1	0	4	0	6	0
ITEM 3B	A	B	C	D	E	F	G	H	I	J
Initial mistakes	3	4	1	0	5	3	1	1	2	0
Initial steps only	4	1	5	3	2	3	5	0	1	0
Formula recognizable	6	6	2	7	4	3	3	0	0	1
Correct	2	8	0	4	2	9	0	1	9	7
Alternative way to "solution"	3	1	4	5	2	3	2	2	1	4
Blank	3	2	8	1	3	1	4	0	8	0

ITEM 2B	K	L	M	N	SUM
Tick	6	11	7	9	139
Blank	3	4	11	2	50
Mark wrong	0	1	1	0	5
"Mistake"	5	2	4	7	54
					248
ITEM 2C	K	L	M	N	SUM
Tick	6	5	5	6	73
Blank	1	8	11	2	59
Mark wrong	0	1	3	0	14
"Mistake"	1	3	4	0	30
Spotted	6	1	0	10	72
					248
ITEM 3A	K	L	M	N	SUM
Initial mistakes	5	12	9	5	92
Initial steps only	0	0	4	0	17
Formula recognizable	1	1	0	0	25
Correct	4	0	1	12	54
Alternative way to "solution"	2	3	8	0	37
Blank	2	2	1	1	23
					248
ITEM 3B	K	L	M	N	SUM
Initial mistakes	0	2	4	2	28
Initial steps only	1	5	13	1	44
Formula recognizable	2	3	0	3	40
Correct	6	3	1	10	62
Alternative way to "solution"	3	4	5	0	39
Blank	2	1	0	2	35
					248

LEBENS LAUF

Grubner Lisa

Ausbildung

- Studium: Lehramt Englisch und Mathematik an der UNI Wien (2009 bis 2014)
- Erasmus Auslandssemester im WS 2012/13 in Edinburgh am 'College of Science and Engineering'
- BRG/BORG St. Pölten (2005-2009): ausgezeichneter Erfolg bei der Matura im Juni 2009 (Schwerpunkt: Musik)
- Hauptschule Kirchberg an der Pielach (2001 – 2005): lauter Einser im Abschlusszeugnis 2005

Zusatzqualifikationen

Musik: Dirigentenausbildung in Zeillern absolviert – seit Juni 2013 Kapellmeisterin

Sport: Absolvierung der Aerobic und Fitnesslehrerausbildung am USI Wien – Abschlussprüfung im Jänner 2013 mit ausgezeichnetem Erfolg bestanden