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DIPLOMARBEIT

Linear Hyperbolic Second Order Partial Differential Equations on Space Time

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Zusammenfassung

Diese Arbeit beschäftigt sich mit Existenz- und Eindeutigkeitsresultaten für lineare hyperbolische partielle Differentialgleichungen zweiter Ordnung auf Lorentzmannigfaltigkeiten. Der Hauptteil eines strikt hyperbolischen Differentialoperators zweiter Ordnung kann als Laplace-Beltrami-Operator einer Lorentzmetrik geschrieben werden. Daher ist die Verwendung von Methoden der Lorentzgeometrie in der Existenztheorie für diese Art von Differentialgleichungen naheliegend. Der geometrische Standpunkt gestattet eine elegante Formulierung der Energieabschätzungen im Rahmen der Sobolevräume, indem Energietensoren verwendet werden. Interesse für diese Verfahren kam in jüngster Zeit auf, da Problemstellungen aus der allgemeinen Relativitätstheorie und der mathematischen Geophysik eine Verallgemeinerung für den Fall von Koeffizienten niedriger Regularität erfordern.

Abstract

This thesis deals with existence and uniqueness results for linear hyperbolic partial differential equations of second order on Lorentzian manifolds. The principal part of a linear strictly hyperbolic operator of second order may be written as the Laplace-Beltrami operator of some Lorentzian metric, thus it is natural to use methods of Lorentzian geometry in the existence theory for this class of PDEs. This geometric viewpoint allows an elegant formulation of the energy estimates in the framework of Sobolev spaces by the use of energy tensors. Recent interest in these techniques arises from generalizations to the case of coefficients of low regularity motivated by applications in general relativity and mathematical geophysics.

There is a theory which states
that if ever anyone discovers exactly
what the Universe is for and why it is here,
it will instantly disappear and be replaced
by something even more bizarre and inexplicable.
There is another theory which states
that this has already happened.

Douglas Adams

Vorwort

Nachdem im Sommersemester 2004 sich mein Mathematik- und Physikstudium langsam dem Ende nähern sollte, begann ich mir Gedanken ber Gebiet und Betreuer einer künftigen Diplomarbeit zu machen. Damals – eigentlich noch mehrheitlich der theoretischen Physik zugewandt – wollte mir aber nicht so recht einfallen, in welche Richtung ich mich orientieren soll. So war es ein glücklicher Zufall, daß ich in diesem Semester bei Roland Steinbauer das Seminar „*Tensoren in Mathematik und Physik*“ besuchte. Im Laufe des Semesters kamen wir ins Gespräch über meine Diplomarbeitspläne. Sein Angebot eine Diplomarbeit bei der Forschungsgruppe DIANA (Differential Algebras and Nonlinear Analysis) am Institut für Mathematik zu schreiben war einerseits verlockend, andererseits war ich wieder skeptisch, wollte ich doch „eigentlich Physiker werden“. Jedoch besuchte ich auf seinen Hinweis im nächsten Wintersemester das DIANA-Seminar und entschloß mich meine Diplomarbeit unter seiner Betreuung zu beginnen. Das Themengebiet der Distributionen und verallgemeinerten Funktionen erwies sich als interessant und bot zudem die Möglichkeit Verknüpfungen mit physikalischen Problemen herzustellen. Auch die Perspektive eventuell in Folge an einer Dissertation auf diesem Gebiet weiterzuarbeiten war sicherlich ein Anreiz. Nicht unwesentlich zu meiner Entscheidung hat aber die nette Aufnahme bei der DIANA-Forschungsgruppe beigetragen.

Nun ist einige Zeit vergangen und viele Definitionen, Sätze und Beweise später kann ich diese Arbeit als abgeschlossen betrachten. Allen voran möchte ich hier meinen Eltern, Annemarie und Gerhard Hanel, danken, die mich ber Jahre hinweg untersttzt haben und damit meine Studien ermöglicht haben. Ebenso gilt mein besonderer Dank meiner Taufpatin Marianne Hainisch und meiner Firmpatin Cornelia Hainisch für ihre langjährige Unterstützung und ihr Vertrauen in meine Fähigkeiten.

Meiner Physik- und Mathematiklehrerin Dr. Margarethe Obiditsch und meinem Chemielehrer Dipl.-Ing. Dr. Gerhard Grünzweig will ich meinen herzlichen Dank aussprechen, sie haben in meiner Mittelschulzeit meine naturwissenschaftliche Ausbildung gefördert und dadurch das Interesse erst richtig geweckt – so weit, daß ich bis nach der Matura noch immer nicht wußte, ob ich nun Mathematik, Physik oder Chemie studieren sollte.

Auch haben mich durch die Zeit viele Freunde begleitet: Seit der Volksschule Oswald Massiczek und Paul Haberfehlner, mit denen ich viele Stunden verbracht habe und auch noch verbringe, die ich nicht vermissen will. Seit der ersten Analysis-Vorlesung Andreas Németh; wie oft mußten wir doch gemeinsam die eine oder andere Prüfung überstehen. Vielen Dank an dieser Stelle auch für das Korrekturlesen der vorliegenden Arbeit. Überaus dankbar bin ich auch Evelina Erlacher, meiner ehemaligen Zimmerkollegin am Institut, die mittlerweile zu einer guten Freundin geworden ist. Sehr häufig „mußte“ sie sich meine (nicht nur) mathematischen Probleme anhören, was in vielen Fällen im wahrsten Sinne des Wortes in einer Diskussion über Gott und die Welt endete – natürlich nicht ohne, daß sie einen Lösungsvorschlag für das ursprüngliche Problem parat hatte.

Den Kollegen von der DIANA-Forschungsgruppe vielen Dank für ihre Untersttzung; vor allem Michael Kunzinger und Michael Grosser, die auf fast alle Fragen eine Antwort haben oder wenigstens wissen, wo diese zu finden ist. Ebenso danke ich Eberhard Mayerhofer für seine Anregungen und die interessante Zusammenarbeit.

Diese Diplomarbeit wurde vom Fonds zur Förderung der wissenschaftlichen Forschung (FWF), Projekt Y237 (Nonlinear Distributional Geometry), unterstützt.

Zu guter letzt aber sei mein Betreuer Roland Steinbauer genannt, der mit viel Geduld, Einsatz und Wissen dazu beigetragen hat, daß aus dieser Diplomarbeit das geworden, was nun vorliegt. Vielen herzlichen Dank für die exzellente Betreuung!

Wien, im November 2006 Clemens Hanel

Preface

In the summer term of 2004 my mathematics and physics studies approached their completion and I started to think about the field and supervisor of my masters thesis. My main focus at that time was theoretical physics, but I had no idea of any suitable topic. Fortunately I attended Roland Steinbauer's seminar on "*Tensors in mathematics and physics*". During this term Roland Steinbauer asked me about my plans concerning a master thesis and offered me the possibility of writing a thesis at the department of mathematics in collaboration with the DIANA (Differential Algebras and Nonlinear Analysis) research group. On one hand, this proposal was very tempting. On the other hand, I was a little bit sceptical since up to that point I had wanted to become a physicist. However, in the next winter term I visited the DIANA-seminar and decided to start with a diploma thesis under Roland Steinbauer's supervision. The field of distributions and generalized functions proved to be attractive and provided many possible links to issues from physics. In the end, the prospect of continuing my work by writing a doctoral thesis in this field of mathematics as well as the really nice people of the DIANA group made my decision easy.

Since then several months passed and many definitions, theorems and proofs later this work has been finished. Above all I want to thank my parents, Annemarie and Gerhard Hanel, for their support over the years and for making possible my extensive studies. Equal thanks go to my godmothers Marianne and Cornelia Hainisch for their support and confidence in my abilities.

I owe my high school teachers Margarethe Obiditsch, PhD, (in physics and mathematics) and Dipl.-Ing. Gerhard Grünzweig, PhD, (in chemistry) many thanks. They promoted my education and my interest in natural sciences. In the end it was really hard to decide if I should study mathematics, physics or chemistry.

Many friends accompanied me through the past times: Since the elementary school Oswald Massiczek and Paul Haberfehlner with whom I spent much time that I would not have liked to miss. Since the first calculus lecture Andreas Németh who faced several exams together with me. At this point I would like to thank him for proofreading this thesis. Moreover, I am truly grateful to Evelina Erlacher, my former office mate, who has become a good friend of mine. Quite frequently she "was obliged" to listen to my (not only) mathematical problems, which in many cases ended in various discussions about nearly any topic. Needless to say that in every case she proposed at least a solution to the initial problem.

Many thanks to the colleagues from the DIANA group for their support; in particular Michael Kunzinger and Michael Grosser who had an immediate answer to nearly any question (or at least knew where to find the answer). Also thanks to Eberhard Mayerhofer for his suggestions and our interesting collaboration.

The writing of this masters thesis has been supported by the Austrian Science Foundation (FWF), project Y237 (Nonlinear Distributional Geometry).

Finally, I owe my supervisor Roland Steinbauer special thanks for contributing much endurance, commitment and knowledge in order to support my work on this diploma thesis. I would like to express my gratefulness for his excellent supervision of this work.

Vienna, November 2006 Clemens Hanel

Contents

1 Introduction

This thesis is concerned with existence and uniqueness results for linear hyperbolic partial differential equations of second order on Lorentzian manifolds.

The prototype of a second order hyperbolic equation is the wave equation, i.e.,

$$\Box u = (-\partial_t^2 + \partial_{x_1}^2 + \cdots + \partial_{x_n})u = 0$$

for a function u on $\mathbb{R}^{1+n}$. The wave operator $\Box$ can be seen as the Laplace-Beltrami operator of the Minkowski metric $g_{ik} = \mathrm{diag}(-1, 1, \ldots, 1)$ on $\mathbb{R}^{1+n}$, that is we may write

$$\Box = \sum_{i,k=0}^{n} g^{ik} \partial_i \partial_k,$$

where we follow the usual convention that $\partial_0 = \partial_t$ and $\partial_k = \partial_{x_k}$ for $1 \leq k \leq n$ and g^{ik} denotes the inverse metric. More general, every linear strictly hyperbolic operator of second order may be written as the Laplace-Beltrami operator for some Lorentzian metric plus lower order terms (for details, see [Hör94], section 24.1).

Having said this, it seems natural to use methods of Lorentzian geometry and also general relativity in the existence theory of this class of PDEs and indeed this has been done in the literature (e.g. [Fri75], [HE73], chapter 7) and this will also be the viewpoint of this work.

On the other hand the (classical) existence theory of hyperbolic PDEs to a large extend is formulated in terms of Sobolev spaces and based upon energy estimates; see e.g. [Hör94], [Hör97], [Sog95].

The advantage of the geometric view point is that energy estimates—especially in the case of higher order energies and tensorial as opposed to scalar equations—can be written in an elegant way using the notion of energy-tensors (cf. subsection 5.2.1); their key property being the dominant energy condition (cf. definition 5.2.2) which implies the key estimate—the divergence theorem (proposition 5.2.5).

The aim of this work is to provide a self-contained account on the existence theory of linear second order hyperbolic PDEs on Lorentzian manifolds from a geometrical viewpoint. In some more detail we provide a full proof of two existence and uniqueness theorems in Sobolev spaces based upon the exposition in [HE73], chapter 7. Moreover, we also provide a self-contained discussion of the prerequisites used throughout this work, in particular, basic notions from pseudo-Riemannian geometry, Sobolev spaces on manifolds and general relativity. Before giving a more detailed description of the content of this work we will, however, discuss some sources for the recent strong interest in the topic of this thesis which goes beyond the mere fact that it is an interesting piece of classical mathematics.

If we want to use a single catch phrase for our sources of interest this should be more or less “the wave equations on singular space times”, meaning the study of the Laplace-Beltrami operator on Lorentzian manifolds with a metric of low differentiability. Technically speaking the resulting equations have coefficients of low regularity and recalling the fact that we are interested not only in classical (i.e., twice differentiable) solutions we find ourselves immediately in the realm of multiplication of distributions, which of course is a delicate matter. The interest into this kind of equations itself has at least two roots in applications; the first one being general relativity the second one mathematical geophysics.

In general relativity the notion of a singularity is a very sophisticated one (cf. [Wal84], section 9.1). In short, the famous singularity theorems by Stephen Hawking and George Ellis ([HE73], section 8.2) classify many physically reasonable space times as singular in the sense that they are geodesically incomplete. A long standing problem is to relate singularities to a disruption of the evolution of Einstein’s equations. One approach to this problem—put forward by Chris Clarke in [Cla96]—leads to the concept of generalized hyperbolicity. A space time is called generalized hyperbolic (referring to the classical concept of global hyperbolicity, cf. [Wal84], section 8.3) if the Cauchy problem for the wave equation is well posed in a certain weak sense (see [Cla98]).

In mathematical geophysics one is interested in modelling the travel of sound waves through the earth where again one ends up with hyperbolic equations with coefficients of low regularity, modelling the varying sound speed in different layers within the earth's upper crust, see e.g. [BS95] and literature cited therein.

Recent progress in dealing with such equations has been achieved using the theory of algebras of generalized functions in the sense of Jean-François Colombeau (see [Col92]) by James Vickers and Jonathan Wilson in [VW00]. In particular, they proved existence and uniqueness of solutions to the wave equation on conic space times in a suitable space of generalized functions. These space times possess a metric which is locally bounded but not continuous and were studied in the context of non-linear generalized functions in [CVW96]. Even more recently the result of Vickers and Wilson has been generalized to a class of static singular metrics (modelled again by generalized functions in the sense of Colombeau) in the course of a Ph.D. thesis at the Faculty of Mathematics, University of Vienna, by Eberhard Mayerhofer ([May06]).

For an overview on the recent development in the context of geophysics see [HdH04].

With this thesis we hope to provide a solid introduction into the techniques which were used and generalized in the above mentioned recent results.

This work is organized in the following way: Chapters two and three are designed to provide the mathematical prerequisites for this thesis. In chapter two we give an introduction into aspects of differential geometry used later, in particular pseudo-Riemannian geometry. In the third chapter we develop some aspects of the theory of Sobolev spaces, especially the Sobolev imbedding theorem (compare [Ada75], chapter 5) on manifolds. Chapter four gives a short introduction into a few concepts from general relativity.

The main part of this work is chapter five where we prove existence and uniqueness results for hyperbolic PDEs based on [HE73], proposition 7.4.7. We define the energy tensor and prove the dominant energy condition, which is vital for the divergence theorem. Then we prove the energy estimates using this theorem and an application of the Sobolev imbedding theorem. Uniqueness of solutions is a direct consequence of the energy estimates. Finally, we use these estimates to prove the existence result via an approximation procedure using the weak compactness theorem. For convenience of the reader several lengthy calculations are shifted to appendices A and B.

2 Elements of Differential Geometry

In this work we are dealing with hyperbolic second order PDEs on Lorentzian manifolds. To set up a coherent notation we review some basic concepts from differential geometry, in particular *manifolds*, *vector bundles*, *tensor fields*, *differential forms*, and *Riemannian* resp. *Lorentzian* geometry. Our main references on differential geometry are [Kun06], [Kri05], [O'N83], and [Spi99]. On the contrary to most books we consider the case of $\mathcal{C}^m$-functions and tensor fields.

2.1 Manifolds and Charts

2.1.1. Definition (Charts): Let M be a set. A *chart* or *coordinate system* (φ, V) on M is a bijective map from $V \subseteq M$ to an open subset $U \subseteq \mathbb{R}^n$. Two charts (φ_1, V_1) and (φ_2, V_2) are called $\mathcal{C}^m$*-related* if $\varphi_1(V_1 \cap V_2)$ and $\varphi_2(V_1 \cap V_2)$ are open sets in $\mathbb{R}^n$ and if

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(V_1 \cap V_2) \to \varphi_2(V_1 \cap V_2)$$

is a $\mathcal{C}^m$-diffeomorphism.

2.1.2. Definition (Atlases): A $\mathcal{C}^m$*-atlas* of a set M is a family of pairwise related charts $\mathcal{A} = \{(\varphi_\alpha, V_\alpha) | \alpha \in A\}$ such that $M = \bigcup_{\alpha \in A} V_\alpha$. Two atlases $\mathcal{A}_1$, $\mathcal{A}_2$ are called equivalent if $\mathcal{A}_1 \cup \mathcal{A}_2$ is again an atlas. We call an equivalence class of such atlases a $\mathcal{C}^m$-structure on M

2.1.3. Definition (Manifolds): A $\mathcal{C}^m$-manifold is a set M together with a $\mathcal{C}^m$-structure on M.

2.1.4. Definition (Paracompact spaces): A topological space is *paracompact* if every open cover admits an open locally finite refinement.

Most of the time we consider $\mathcal{C}^\infty$-manifolds and additionally suppose them to be Hausdorff and paracompact (where we use the topology induced by the charts). Whenever we call M a manifold these assumptions are in effect.

2.1.5. DEFINITION: Let M be a $\mathcal{C}^m$-manifold ($m \in \mathbb{N}_0 \cup \{\infty\}$).

(i) A function $f : M \to \mathbb{R}$ is said to be in $\mathcal{C}^k(M)$, $(k \leq m)$ if $f \circ \varphi^{-1}$ is k-times continuously differentiable for all charts (φ, U).

(ii) Analogously we say a function $\phi : M \to N$ with N a $\mathcal{C}^l$-manifold is in $\mathcal{C}^k(M, N)$ $(k \leq \min\{m, l\})$ if $\psi \circ \phi \circ \varphi^{-1}$ is k-times continuously differentiable for all charts φ in M and all charts ψ in N.

2.2 Vectors and Tangent Spaces

We will generally be interested not only in scalar functions but also in vector valued or tensorial objects. The basic notion underlying these concepts is the tangent space to M.

2.2.1. DEFINITION: Let M be a manifold, $p \in M$ and $c_1, c_2 : I \to M$ continuously differentiable curves with $c_1(0) = c_2(0) = p$. Those curves are called *tangential* at p if $(\psi \circ c_1)'(0) = (\psi \circ c_2)'(0)$ for a chart (ψ, V) around p. An equivalence relation on $\mathcal{C}^1(I, M)$ is defined by $c_1 \sim c_2$ if c_1 is tangential to c_2 at p. We denote the equivalence class by $[c]_p$.

Indeed this definition is independent of the chart.

2.2.2. DEFINITION (Tangent space): The *tangent space* T_pM of a manifold M at p is defined by

$$\mathrm{T}_pM := \{[c]_p | c \in \mathcal{C}^1(I, M) \text{ with } c(0) = p\}.$$

The elements $[c]_p \in \mathrm{T}_pM$ are called *tangent vectors*.

For a tangent vector we also write for short v or X_p instead of $[c]_p$.

2.2.3. DEFINITION (Differential map): Let M, N be m- resp. n-dimensional manifolds and let $\phi : M \to N$ be at least $\mathcal{C}^1$. For each $p \in M$ the function $\mathrm{T}_p\phi : \mathrm{T}_pM \to \mathrm{T}_{\phi(p)}N$ (or sometimes denoted as $\mathrm{d}\phi_p$) sending $[c]_p$ to

$[\phi \circ c]_{\phi(p)}$ is called the *differential* or *tangent map* of ϕ at p. In local coordinates we have for a chart $\varphi = (x_1, \dots, x_m)$ of M and a chart $\psi = (y_1, \dots, y_n)$ of N with transition functions $\phi^{\nu}_{\psi\varphi} := \psi^{\nu} \circ \phi \circ \varphi^{-1}$

$$\mathrm{T}_p\phi(v) = \sum_{\mu,\nu} \frac{\partial \phi^{\nu}_{\psi\varphi}}{\partial x^{\mu}} v^{\mu} \left.\frac{\partial}{\partial y^{\nu}}\right|_{\phi(p)} .$$

One now easily shows that T_pM is a vector space and that it is isomorphic to the space of derivations on $\mathcal{C}^m(M)$ at p, i.e. there exists a linear bijective map identifying vectors and derivations:

2.2.4. PROPOSITION: *Let M be an n-dimensional manifold, $p \in M$, and (φ, V) a chart around p. The linear structure on T_pM induced by the bijective map $\mathrm{T}_p\varphi : \mathrm{T}_pM \to \mathrm{T}_p\varphi(V) \cong \mathbb{R}^n$ is independent of the chosen chart.*

For a proof see e.g. 2.4.11 in [Kun06].

2.2.5. DEFINITION (Derivations at a point): For any natural number m a mapping $\partial : \mathcal{C}^m(M) \to \mathbb{R}$ is called *$\mathcal{C}^m$-derivation at $p \in M$* if

(i) $\partial(f + \alpha g) = \partial(f) + \alpha\partial(g)$ (Linearity),

(ii) $\partial(fg) = \partial(f)g(p) + f(p)\partial(g)$ (Leibniz rule),

for all $\alpha \in \mathbb{R}$ and $f, g \in \mathcal{C}^m(M)$. The vector space of $\mathcal{C}^m$-derivations at p is denoted by $\mathrm{Der}_p(\mathcal{C}^m(M), \mathbb{R})$.

2.2.6. THEOREM: *Writing $\partial_v(f) := \mathrm{T}_pf(v)$ the map*

$$\begin{aligned} A : \mathrm{T}_pM &\to \mathrm{Der}_p(\mathcal{C}^m(M), \mathbb{R}) \\ v &\mapsto \partial_v \end{aligned}$$

is a linear isomorphism.

We follow exactly the proof for the smooth case given in 2.4.13 of [Kun06].

Thus a tangent vector defines a map from $\mathcal{C}^m(M)$ into the real numbers, denoted by X_pf for $X_p \in \mathrm{T}_pM$ and $f \in \mathcal{C}^m(M)$. In local coordinates (φ, V) with $\varphi = (x^1, \dots, x^n)$ we have

$$X_pf = \sum_{\mu=1}^{n} X^{\mu}_p \left.\frac{\partial f}{\partial x^{\mu}}\right|_p .$$

2.3 Vector Bundles and Sections

2.3.1. DEFINITION (Vector bundles): A *vector bundle* of fibre dimension $n' \in \mathbb{N}$ is a triple (E, M, π), where E and M are manifolds such that

(i) $\pi : E \to M$ is a smooth surjective map,

(ii) $\forall p \in M$ the *fibres* $E_p := \pi^{-1}(p)$ are n'-dimensional vector spaces, and

(iii) $\forall p \in M$ there exists an open neighbourhood $U \subseteq M$ and a diffeomorphism $\Phi : U \times \mathbb{R}^{n'} \to \pi^{-1}(U)$, which is fibrewise linear for all points $p \in U$, i. e. $\Phi_p := \Phi(p, \cdot) \in \mathrm{L}(\mathbb{R}^{n'}, E_p)$, such that the following diagram commutes:

$$\begin{tikzcd}
U \times \mathbb{R}^{n'} \arrow[rr, "\Phi"] \arrow[dr, "\mathrm{pr}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\
& U &
\end{tikzcd}$$

The diffeomorphism Φ is called a *local trivialization*, E is called *total space*, and M *base space*. For any chart (ψ, U) of M we call the mapping

$$\begin{aligned}
\Psi := (\psi \times \mathrm{id}_{\mathbb{R}^{n'}}) \circ \Phi^{-1} : \pi^{-1}(U) &\to \psi(U) \times \mathbb{R}^{n'} \\
z &\mapsto (\psi(p), v) \text{ with } p = \pi(z)
\end{aligned}$$

a *vector bundle chart* over ψ.

2.3.2. DEFINITION (Tangent bundle): We define the *tangent bundle* of a manifold to be the disjoint union of the tangent spaces, i.e.

$$\mathrm{T}M := \bigsqcup_{p \in M} \mathrm{T}_p M := \bigcup_{p \in M} \{p\} \times \mathrm{T}_p M. \tag{2.1}$$

The following theorem from [Kri05], section 25.6 shows that by choosing the tangent spaces $\mathrm{T}_p M$ as fibres we obtain a vector bundle.

2.3.3. THEOREM: *The tangent bundle* $(\mathrm{T}M, M, \pi)$ *of a manifold* M *is indeed a vector bundle.*

2.3.4. DEFINITION (Sections): A *section* of a vector bundle (E, M, π) is a map $s : M \to E$ that satisfies $\pi \circ s = \mathrm{id}_M$. The space of $\mathcal{C}^m$-sections is denoted by $\Gamma^m(M, E)$, for smooth sections we drop the index m. One can easily show that $\Gamma^m(M, E)$ is a module over $\mathcal{C}^m(M)$ and a vector space over $\mathbb{R}$. The elements of $\Gamma^m(M, \mathrm{T}M)$ are called *$\mathcal{C}^m$-vector fields* and assign to each point $p \in M$ a tangent vector. The subspace of smooth vector fields, $\Gamma(M, \mathrm{T}M)$, will be denoted by $\mathfrak{X}(M)$.

By replacing the tangent spaces with their duals T_p^*M, called *cotangent spaces*, one obtains the *cotangent bundle* T^*M. The sections $\Gamma^m(M, \mathrm{T}^*M)$ are called *one-forms* or *covector fields* and assign to each point $p \in M$ a linear functional on T_pM. Similarly as above we denote by $\mathfrak{X}^*(M)$ the space of smooth one-forms.

2.3.5. Definition (Tensor product): Let (E, M, π_1) and (F, M, π_2) be vector bundles. We define the *tensor product* of E and F to be

$$E \otimes F := \bigsqcup_{p \in M} (E_p \otimes F_p)$$

with the following local trivializations

$$\Phi_p^{E \otimes F} := \Phi_p^E \otimes \Phi_p^F : \mathbb{R}^{n'n''} \to E_p \otimes F_p,$$

where $\mathbb{R}^{n'n''} \cong \mathbb{R}^{n'} \otimes \mathbb{R}^{n''}$. Here n' and n'' denote the dimensions of the corresponding fibres.

We can now define tensor fields of order (l, k).

2.3.6. Definition (Tensor fields): The spaces $\Gamma^m(M, \mathrm{T}_k^l M)$, where

$$\mathrm{T}_k^l M := \left(\bigotimes_{i=1}^{l} \mathrm{T}M \right) \otimes \left(\bigotimes_{i=1}^{k} \mathrm{T}^*M \right),$$

are defined to be the *l-times contravariant* and *k-times covariant tensor fields* over M or for short (l, k)-tensor fields. Again these spaces are modules over $\mathcal{C}^m(M)$ and vector spaces over $\mathbb{R}$. The space of smooth fields will be denoted by $\mathcal{T}_k^l(M)$.

Hence an (l, k)-tensor field is a map that assigns to each point $p \in M$ a multilinear map from l-times T_p^*M and k-times T_pM into the real numbers.

The well known pointwise operations $+$ and $\otimes$ extend obviously to operations on the tensor fields. Additionally there is the *contraction mapping* from $\Gamma^m(M, \mathrm{T}^l_k M)$ to $\Gamma^m(M, \mathrm{T}^{l-1}_{k-1} M)$. More precisely we have by section 2.6 in [O'N83]:

2.3.7. LEMMA (Contraction): *There exists a unique $\mathcal{C}^m(M)$-linear function $\mathrm{C} : \Gamma^m(M, \mathrm{T}^1_1 M) \to \mathcal{C}^m(M)$, called $(1,1)$-contraction, such that we have $\mathrm{C}(X \otimes \omega) = \omega(X)$ for all $X \in \Gamma^m(M, \mathrm{T}^1_0 M)$ and $\omega \in \Gamma^m(M, \mathrm{T}^0_1 M)$.*

This operation can be generalized to tensors of higher order by shifting the operation of contraction to the arguments of the tensor field. Suppose $T \in \Gamma^m(M, \mathrm{T}^l_k M)$. For arbitrary fixed one-forms $\omega^1, \ldots, \omega^{l-1}$ and vector fields $X_1, \ldots, X_{k-1}$ the function

$$(\omega, X) \mapsto T(\omega^1, \ldots, \omega, \ldots, \omega^{l-1}, X_1, \ldots, X, \ldots, X_{k-1}),$$

where ω is inserted at the j^{th} position and X is inserted at the i^{th} position, is a $(1,1)$ tensor field. Applying the $(1,1)$ contraction to this field yields a real-valued function denoted by

$$(\mathrm{C}^j_i T)(\omega^1, \ldots, \omega^{l-1}, X_1, \ldots, X_{k-1}).$$

Obviously $\mathrm{C}^j_i T$ is $\mathcal{C}^m(M)$-multilinear in its arguments, hence a field in $\Gamma^m(M, \mathrm{T}^{l-1}_{k-1})$ called the contraction of T over j and i.

2.3.8. DEFINITION (Differential map): For $m \in \mathbb{N}$ and $\phi \in \mathcal{C}^m(M, N)$ we define the *differential map* by

$$\begin{aligned} \mathrm{T}\phi : \mathrm{T}M &\to \mathrm{T}N \\ (p, v) &\mapsto (\phi(p), \mathrm{T}_p\phi(v)), \end{aligned}$$

where $\mathrm{T}_p\phi$ was defined in 2.2.3.

Among all tensor fields the totally antisymmetric covariant fields or *differential forms* are of particular interest. Therefore we define the so called *exterior product* of a vector bundle.

2.3.9. DEFINITION (Exterior product): Let (E, M, π) be a vector bundle. We define the k-fold *exterior product* of E to be

$$\Lambda^k E := \bigsqcup_{p \in M} \Lambda^k E_p = \mathcal{L}^k_{\mathrm{alt}}(E^*, \mathbb{R})$$

with local trivializations

$$\Phi_p^{\Lambda^k E} := \bigwedge_{j=1}^{k} (\Phi_p^E) : \Lambda^k \mathbb{R}^n \cong \mathbb{R}^{\binom{n}{k}} \to \Lambda^k E_p,$$

where $\mathcal{L}^k_{\mathrm{alt}}(E^*, \mathbb{R})$ denotes the space of k-linear alternating functionals on the space E^*.

2.3.10. DEFINITION (Differential forms): Elements of $\Gamma^m(M, \Lambda^k \mathrm{T}^* M)$ are called *m-times differentiable k-forms*. These spaces are $\mathcal{C}^m(M)$-modules and $\mathbb{R}$-vector spaces. By $\Omega^k(M) := \Gamma(M, \Lambda^k \mathrm{T}^* M)$ we denote the smooth sections of $\Lambda^k \mathrm{T}^* M$.

Thus a k-form is a map that assigns to each point $p \in M$ an alternating k-linear map from $\mathrm{T}_p M$ into the real numbers.

2.4 Differentiation on Manifolds

Like in Euclidean space we want to differentiate functions and tensor fields on a manifold. We therefore introduce $\mathcal{C}^m$-derivations, exterior derivatives and tensor derivations. Most properties will be similar to the smooth case.

The following proposition characterizes vector fields of class $\mathcal{C}^{m-1}$.

2.4.1. PROPOSITION: *For a vector field X on M the following are equivalent:*

1. *$X : M \to \mathrm{T}M$ is $\mathcal{C}^{m-1}$, i.e. X is a $\mathcal{C}^{m-1}$-section of $\mathrm{T}M$.*
2. *For all $f \in \mathcal{C}^m(M)$ the mapping $Xf : p \mapsto X_p f : M \to \mathbb{R}$ is $\mathcal{C}^{m-1}$.*
3. *For all charts $(\psi, V), \psi = (x^1, \ldots, x^n)$ we have that for the local representation*
$$X(p) = \sum_{i=1}^{n} X^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$
all X^i are in $\mathcal{C}^{m-1}(V)$.

The proof is a straight-forward generalization of 2.5.9 in [Kun06].

2.4.2. DEFINITION (Derivations): Let V be a module over the commutative A-Algebra B. A mapping $\delta : B \to V$ is called an A-Derivation of B in V if

(i) δ is A-linear

(ii) $\forall b, c \in B$ we have $\delta(bc) = c\delta(b) + b\delta(c)$.

We are now interested in the case $A = \mathbb{R}$, $B = \mathcal{C}^m(M)$, and $V = \mathcal{C}^{m-1}(M)$.

2.4.3. DEFINITION: We call an $\mathbb{R}$-linear map $\mathrm{D} : \mathcal{C}^m(M) \to \mathcal{C}^{m-1}(M)$ a $\mathcal{C}^m$-derivation if it satisfies the Leibniz rule

$$\mathrm{D}(fg) = f\mathrm{D}(g) + g\mathrm{D}(f).$$

The space of $\mathcal{C}^m$-derivations is denoted by $\mathrm{Der}(\mathcal{C}^m(M), \mathcal{C}^{m-1}(M))$.

In theorem 2.2.6 we proved isomorphy between $\mathrm{Der}_p(\mathcal{C}^m(M), \mathbb{R})$ and T_pM. For $X \in \Gamma^{m-1}(M, \mathrm{T}M)$ and each $p \in M$ the vector X_p is a $\mathcal{C}^m$-derivation at p. The mapping $f \mapsto Xf$ where Xf is $p \mapsto X_p(f)$ is therefore linear in f and obeys the Leibniz rule, since

$$(X(fg))(p) = X_p(fg) = X_p(f)g(p) + f(p)X_p(g) = (X(f)g + fX(g))(p).$$

Hence X is a derivation on $\mathcal{C}^m(M)$.
The reverse direction will be concluded from the next theorem.

2.4.4. THEOREM:

$$\mathrm{Der}(\mathcal{C}^m(M), \mathcal{C}^{m-1}(M)) = \Gamma^{m-1}(M, \mathrm{T}M)$$

Proof: $\Gamma^{m-1}(M, \mathrm{T}M) \subseteq \mathrm{Der}(\mathcal{C}^m(M), \mathcal{C}^{m-1}(M))$ by proposition 2.4.1.2 and the observations above. Conversely, let $\mathrm{D} \in \mathrm{Der}(\mathcal{C}^m(M), \mathcal{C}^{m-1}(M))$ be given. For $p \in M$ the mapping

$$\begin{aligned} \mathcal{C}^m(M) &\to \mathbb{R} \\ f &\mapsto (\mathrm{D}(f))(p) \end{aligned}$$

is a derivation at p. Linearity is obvious, Leibniz' rule follows by

$$\mathrm{D}(fg)(p) = (\mathrm{D}(f)g + f\mathrm{D}(g))(p) = (\mathrm{D}(f)(p)g(p) + f(p)\mathrm{D}(g)(p)).$$

By theorem 2.2.6 there exists exactly one $X_p \in \mathrm{T}_pM$ with $X_p(f) = \mathrm{D}(f)(p)$. Thus $p \mapsto X_p$ is a vector field on M with $X(f) = \mathrm{D}(f)\, \forall f \in \mathcal{C}^m(M)$. By 2.4.1.2 it is $\mathcal{C}^{m-1}$.

q. e. d.

2.4.5. DEFINITION (Tensor derivation): A $\mathcal{C}^m$*-tensor derivation* D on a $\mathcal{C}^\infty$-manifold M is a set of $\mathbb{R}$-linear functions

$$\mathrm{D} = \mathrm{D}^l_k : \Gamma^m(M, \mathrm{T}^l_k M) \to \Gamma^{m-1}(M, \mathrm{T}^l_k M)$$

such that for any $\mathcal{C}^m$-tensor fields S and T:

(1) $\mathrm{D}(S \otimes T) = \mathrm{D}S \otimes T + S \otimes \mathrm{D}T$,

(2) for any contraction C^j_i the following diagram commutes:

$$\begin{tikzcd}
\Gamma^m(M, \mathrm{T}^l_k M) \arrow[r, "\mathrm{D}^l_k"] \arrow[d, "\mathrm{C}^j_i"] & \Gamma^{m-1}(M, \mathrm{T}^l_k M) \arrow[d, "\mathrm{C}^j_i"] \\
\Gamma^m(M, \mathrm{T}^{l-1}_{k-1} M) \arrow[r, "\mathrm{D}^{l-1}_{k-1}"] & \Gamma^{m-1}(M, \mathrm{T}^{l-1}_{k-1} M)
\end{tikzcd}$$

i.e. differentiation and contraction commute. For short we write $[\mathrm{D}, \mathrm{C}]S = 0\,(S \in \Gamma^m(M, \mathrm{T}^l_k M)$.

For the special case $l = k = 0$ the mapping D^0_0 is a derivation on $\Gamma^m(M, \mathrm{T}^0_0 M) = \mathcal{C}^m(M)$, so there exists a unique vector field V such that $\mathrm{D}f = Vf$ for all $f \in \mathcal{C}^m(M)$.

Since contraction and derivation commute by definition, one can achieve a formula for D of an arbitrary tensor field in terms of D applied only to functions and vector fields (see e.g. [O'N83], sec. 2.13). Thus we have the following corollary:

2.4.6. COROLLARY: *If tensor derivations* D_1 *and* D_2 *agree on functions in* $\mathcal{C}^m(M)$ *and vector fields in* $\Gamma^m(M, \mathrm{T}^l_k M)$*, then* $\mathrm{D}_1 = \mathrm{D}_2$.

This can be used to construct tensor derivations just by giving D^0_0 and D^1_0 (see also [O'N83], sec. 2.15).

2.4.7. THEOREM: *Given a vector field* $Y \in \Gamma^{m-1}(M, \mathrm{T}M)$ *and an* $\mathbb{R}$*-linear function* $\delta : \Gamma^m(M, \mathrm{T}M) \to \Gamma^{m-1}(M, \mathrm{T}M)$ *with*

$$\delta(fX) = (Yf)X + f\delta(X) \quad \forall f \in \mathcal{C}^m(M), X \in \Gamma^m(M, \mathrm{T}M).$$

Then there exists a unique tensor derivation D *on* M *and such that* $\mathrm{D}^0_0 = Y : \mathcal{C}^m(M) \to \mathcal{C}^{m-1}(M)$ *and* $\mathrm{D}^1_0 = \delta$.

2.4.8. DEFINITION (Exterior derivative): Let $f \in \mathcal{C}^m(M)$ ($m \in \mathbb{N}$). Then

$$\begin{aligned} \mathrm{d}f : M &\to \mathrm{T}^*M \\ p &\mapsto \mathrm{T}_p f \end{aligned}$$

is called *exterior derivative* or for short *differential* of f.

2.4.9. REMARK: Obviously $\mathrm{d}f$ is an $(m-1)$-times differentiable one-form. Moreover, for every vector field $X \in \Gamma^{m-1}(M, \mathrm{T}M)$ one has $\mathrm{d}f(X) = Xf$.

The exterior derivative can be extended to operate on any differentiable k-form. This situation is covered by the following theorem.

2.4.10. THEOREM: *On a manifold M for every $U \subseteq M$ there exists a unique family of maps*

$$\mathrm{d}^k(U) : \Gamma^m(M, \Lambda^k \mathrm{T}^*M) \to \Gamma^{m-1}(M, \Lambda^{k+1}\mathrm{T}^*M) \ \forall m \in \mathbb{N},$$

denoted by d*, such that*

(i) d *is $\mathbb{R}$-linear and satisfies the Leibniz rule for the exterior product, i.e. for differential forms $\omega \in \Gamma^m(M, \Lambda^k \mathrm{T}^*M)$ and $\nu \in \Gamma^m(M, \Lambda^l \mathrm{T}^*M)$ we have*

$$\mathrm{d}(\omega \wedge \nu) = \mathrm{d}\omega \wedge \nu + (-1)^k \omega \wedge \mathrm{d}\nu,$$

(ii) for $f \in \mathcal{C}^m(M)$ we have that $\mathrm{d}f$ agrees with the differential from definition 2.4.8,

(iii) for $m \geq 2$ we have $\mathrm{d}^2 := \mathrm{d} \circ \mathrm{d} = 0$, and

(iv) for U, V open in M, $U \subseteq V$ and $\omega \in \Gamma^m(V, \Lambda^k \mathrm{T}^*V)$ the following diagram commutes:

$$\begin{tikzcd}
\Gamma^m(V, \Lambda^k \mathrm{T}^*V) \arrow[r, "|_U"] \arrow[d, "\mathrm{d}"] & \Gamma^m(U, \Lambda^k \mathrm{T}^*U) \arrow[d, "\mathrm{d}"] \\
\Gamma^{m-1}(V, \Lambda^{k+1} \mathrm{T}^*V) \arrow[r, "|_U"] & \Gamma^{m-1}(U, \Lambda^{k+1} \mathrm{T}^*U)
\end{tikzcd}$$

A proof for the $\mathcal{C}^\infty$-case can be found in [Kun06], 2.7.22. The $\mathcal{C}^m$-case can easily be derived.

2.5 Riemannian and Lorentzian Manifolds

We now define Riemannian and Lorentzian manifolds, thus we need to introduce metric tensor fields.

To begin with, we recall that a *symmetric bilinear form* on a vector space V over $\mathbb{R}$ is an $\mathbb{R}$-bilinear function $b : V \times V \to \mathbb{R}$ such that for all $v, w \in V : b(v,w) = b(w,v)$. The *index* ν of b is the dimension of the largest subspace $W \subseteq V$ such that $b|_{W \times W}$ is negative definite. It is easy to see that $0 \leq \nu \leq \dim V$ and $\nu = 0$ if and only if b is positive definite.

2.5.1. DEFINITION (Metric tensor fields): A *metric tensor field* g on a manifold M is a symmetric nondegenerate $(0,2)$-tensor field of constant index. Here the expression *nondegenerate* means that $X \mapsto g(X,\cdot)$ is a bijective map from $\Gamma(M, \mathrm{T}M)$ to $\Gamma(M, \mathrm{T}^*M)$. Such a tensor field is called a *Riemannian* metric if g is positive definite. For $\dim M \geq 2$ and index $\nu = 1$ we call the metric *Lorentzian*.

The pair (M, g) of a manifold and a Riemannian metric is a Riemannian manifold, if g is a Lorentzian metric we call (M, g) a Lorentzian manifold. In coordinates we denote g by $g_{\alpha\beta}$ and its inverse by $g^{\alpha\beta}$.

We now introduce the concept of linear connections. Our definition is slightly different from [O'N83], sec. 3.9 and 3.11, since we consider the $\mathcal{C}^\infty$-case.

2.5.2. DEFINITION (Connections): A *linear connection* of order $m \in \mathbb{N}$ on a smooth manifold M is a function

$$\nabla : \Gamma^{m-1}(M, \mathrm{T}M) \times \Gamma^{m}(M, \mathrm{T}M) \to \Gamma^{m-1}(M, \mathrm{T}M)$$

such that

(∇1) $\nabla_X Y$ is $\mathcal{C}^{m-1}(M)$-linear in X,

(∇2) $\nabla_X Y$ is $\mathbb{R}$-linear in Y,

(∇3) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$ for $f \in \mathcal{C}^m(M)$.

For the connection ∇ we call $\nabla_X Y$ the *covariant derivative* of Y with respect to X.

2.5.3. THEOREM (Levi-Cività connection): *Let (M, g) be a Riemannian or Lorentzian manifold where g is at least $\mathcal{C}^1$. Then there exists a unique connection ∇ such that*

(∇4) In any coordinate system (V, ψ) the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ defined by $\nabla_{\partial_\beta}(\partial_\gamma) = \sum_\alpha \Gamma^{\alpha}_{\beta\gamma}\partial_\alpha$ satisfy $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$.

(∇5) $Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$,

for all $Z \in \Gamma^0(M, \mathrm{T}M)$ and $X, Y \in \Gamma^1(M, \mathrm{T}M)$. ∇ is called the Levi-Cività connection of M and it is locally given by

$$\nabla_X Y := \sum_{\alpha,\beta,\gamma} \left(X^\beta \frac{\partial Y^\alpha}{\partial x^\beta} + \Gamma^{\alpha}_{\beta\gamma} X^\beta Y^\gamma \right) \frac{\partial}{\partial x^\alpha} \tag{2.2}$$

where

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\sum_\delta g^{\alpha\delta} \left\{ \frac{\partial g_{\gamma\delta}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right\}. \tag{2.3}$$

Note that in the $\mathcal{C}^\infty$-case one would write for (∇4)

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$

where $[X, Y]$ denotes the Lie bracket of two vector fields. This is not possible here since the bracket operation would reduce the differentiation order of any involved vector field ($\mathcal{C}^m$-derivations are not an algebra!), i.e. X and Y had to be at least $\mathcal{C}^1$, in contrary to definition 2.5.2 which was chosen such that the covariant derivatives on functions agree with the derivations.

Proof: We split the proof into two parts: At first we prove local existence and uniqueness, then we extend the construction onto the whole manifold.

1. Let (V, ψ) be a chart on M with $\psi = (x^1, \dots, x^n)$. Obviously, (∇1) is satisfied by (2.2), so is (∇2). To prove (∇3) we write

$$\begin{aligned}
\nabla_X(fY) &= \sum_{\alpha,\beta,\gamma} \left(X^\beta \frac{\partial}{\partial x^\beta}(fY^\alpha) + \Gamma^{\alpha}_{\beta\gamma} X^\beta (fY^\gamma) \right) \frac{\partial}{\partial x^\alpha} \\
&= \sum_{\alpha,\beta,\gamma} \left(X^\beta \frac{\partial f}{\partial x^\beta} Y^\alpha + f X^\beta \frac{\partial Y^\alpha}{\partial x^\beta} + f\Gamma^{\alpha}_{\beta\gamma} X^\beta Y^\gamma \right) \frac{\partial}{\partial x^\alpha} \\
&= \sum_{\alpha,\beta} X^\beta \frac{\partial f}{\partial x^\beta} Y^\alpha \frac{\partial}{\partial x^\alpha} + f \sum_{\alpha,\beta,\gamma} \left(X^\beta \frac{\partial Y^\alpha}{\partial x^\beta} + \Gamma^{\alpha}_{\beta\gamma} X^\beta Y^\gamma \right) \frac{\partial}{\partial x^\alpha} \\
&= (Xf)Y + f\nabla_X Y.
\end{aligned}$$

Hence ∇ is locally a connection.

Assuming (∇4) and (∇5) to be true, we now derive formula (2.3), thus deriving a local expression for the Levi-Cività connection. Indeed the left hand side of (∇5) reads

$$\begin{aligned} Zg(X,Y) &= \sum_{\alpha,\beta,\gamma} Z^\alpha \frac{\partial}{\partial x^\alpha}(g_{\beta\gamma}X^\beta X^\gamma) \\ &= \sum_{\alpha,\beta,\gamma} \left(\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} Z^\alpha X^\beta Y^\gamma + g_{\beta\gamma} Z^\alpha \frac{\partial X^\beta}{\partial x^\alpha} Y^\gamma + g_{\beta\gamma} Z^\alpha X^\beta \frac{\partial Y^\gamma}{\partial x^\alpha} \right). \end{aligned} \tag{2.4}$$

Similarly the right hand side is

$$\begin{aligned} g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = &\sum_{\alpha,\beta,\gamma,\delta} g_{\delta\gamma} \left(Z^\alpha \frac{\partial X^\delta}{\partial x^\alpha} + \Gamma^\delta_{\alpha\beta} Z^\alpha X^\beta \right) Y^\gamma \\ &+ \sum_{\alpha,\beta,\gamma,\delta} g_{\delta\beta} \left(Z^\alpha \frac{\partial Y^\delta}{\partial x^\alpha} + \Gamma^\delta_{\alpha\gamma} Z^\alpha Y^\gamma \right) X^\beta. \end{aligned} \tag{2.5}$$

By combining (2.4) and (2.5) one observes that the terms containing partial derivatives of X and Y cancel. Hence (∇5) yields

$$\sum_{\alpha,\beta,\gamma} \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} Z^\alpha X^\beta Y^\gamma = \sum_{\alpha,\beta,\gamma,\delta} (g_{\delta\gamma}\Gamma^\delta_{\alpha\beta} Z^\alpha X^\beta Y^\gamma + g_{\delta\beta}\Gamma^\delta_{\alpha\gamma} Z^\alpha X^\beta Y^\gamma). \tag{2.6}$$

Since (2.6) holds for arbitrary continuous Z and differentiable X, Y one has

$$\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} = \sum_\delta (g_{\delta\gamma}\Gamma^\delta_{\alpha\beta} + g_{\delta\beta}\Gamma^\delta_{\alpha\gamma}) = \Gamma_{\gamma\alpha\beta} + \Gamma_{\beta\alpha\gamma}.$$

Taking a cyclic sum we find using (∇4)

$$\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = 2\Gamma_{\gamma\alpha\beta},$$

from which we obtain (2.3) by applying the inverse metric.

2. We show that the locally defined connection gives a global object. We explicitly check the behaviour under coordinate transformations. We have the relations

$$X = \sum_\alpha X^\alpha(x^\beta) \frac{\partial}{\partial x^\alpha} = \sum_\lambda \bar{X}^\lambda(\bar{x}^\mu) \frac{\partial}{\partial \bar{x}^\lambda}.$$

and

$$\bar{X}^\lambda(\bar{x}^\mu) = \sum_\alpha \frac{\partial \bar{x}^\lambda}{\partial x^\alpha}(x^\beta(\bar{x}^\mu))X^\alpha(x^\beta(\bar{x}^\mu)).$$

For convenience of the reader from now on we drop the coordinate dependence in the component functions. Thus we set

$$\nabla_X \bar{Y}^\lambda = \sum_\alpha \frac{\partial \bar{x}^\lambda}{\partial x^\alpha}\nabla_X Y^\alpha = \sum_{\alpha,\beta} \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} X^\beta \frac{\partial Y^\alpha}{\partial x^\beta} + \sum_{\alpha,\beta,\gamma} \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} \Gamma^\alpha_{\beta\gamma} X^\beta Y^\gamma.$$

Observe that

$$\sum_\alpha \frac{\partial}{\partial x^\beta}\bar{Y}^\lambda = \sum_\alpha \frac{\partial}{\partial x^\beta}\left(\frac{\partial \bar{x}^\lambda}{\partial x^\alpha} Y^\alpha\right) = \sum_\alpha \left(\frac{\partial^2 \bar{x}^\lambda}{\partial x^\beta \partial x^\alpha} Y^\alpha + \frac{\partial \bar{x}^\lambda}{\partial x^\alpha}\frac{\partial Y^\alpha}{\partial x^\beta}\right).$$

Hence we obtain

$$\begin{aligned}\nabla_X \bar{Y}^\lambda =& \sum_{\alpha,\beta}\left(X^\beta \frac{\partial \bar{Y}^\lambda}{\partial x^\beta} - X^\beta Y^\alpha \frac{\partial^2 \bar{x}^\lambda}{\partial x^\beta \partial x^\alpha}\right) + \sum_{\alpha,\beta,\gamma} \frac{\partial \bar{x}^\lambda}{\partial x^\alpha}\Gamma^\alpha_{\beta\gamma} X^\beta Y^\gamma \\ =& \sum_\mu \bar{X}^\mu \frac{\partial \bar{Y}^\lambda}{\partial \bar{x}^\mu} - \sum_{\alpha,\beta} X^\beta Y^\alpha \frac{\partial^2 \bar{x}^\lambda}{\partial x^\beta \partial x^\alpha} + \sum_{\alpha,\beta,\gamma} \frac{\partial \bar{x}^\lambda}{\partial x^\alpha}\Gamma^\alpha_{\beta\gamma} X^\beta Y^\gamma. \qquad (2.7)\end{aligned}$$

Inserting the expression for the usual transformation law of the Christoffel symbols (see e.g. [Ibe95], section 4.2.4) into (2.7) yields

$$\begin{aligned}\nabla_X \bar{Y}^\lambda =& \sum_\mu \bar{X}^\mu \frac{\partial \bar{Y}^\lambda}{\partial \bar{x}^\mu} - \sum_{\alpha,\beta} X^\beta Y^\alpha \frac{\partial^2 \bar{x}^\lambda}{\partial x^\beta \partial x^\alpha} \\ &+ \sum_{\beta,\gamma,\mu,\nu} \frac{\partial \bar{x}^\mu}{\partial x^\beta}\frac{\partial \bar{x}^\nu}{\partial x^\gamma}\bar{\Gamma}^\lambda_{\mu\nu} X^\beta Y^\gamma + \sum_{\beta,\gamma}\frac{\partial^2 \bar{x}^\lambda}{\partial x^\beta \partial x^\gamma} X^\beta Y^\gamma \\ =& \sum_\mu \bar{X}^\mu \frac{\partial \bar{Y}^\lambda}{\partial \bar{x}^\mu} + \sum_{\mu\nu}\bar{\Gamma}^\lambda_{\mu\nu}\bar{X}^\mu \bar{Y}^\nu\end{aligned}$$

and we are done.

q. e. d.

Theorem 2.4.7 justifies the following definition.

2.5.4. Definition (Covariant derivative): Let $Y \in \Gamma^{m-1}(M, \mathrm{T}M)$ on a Riemannian or Lorentzian manifold M. The *Levi-Cività covariant derivative* ∇_Y is the unique tensor derivation on M such that

$$\nabla_Y f = Y f \quad \forall f \in \mathcal{C}^m(M)$$

and $\nabla_Y X$ is the Levi-Cività covariant derivative for all $X \in \Gamma^m(M, \mathrm{T}M)$.

2.5.5. Definition (Covariant differential): The *covariant differential* of a tensor field $A \in \Gamma^m(M, \mathrm{T}^l_k M)$ is the tensor field $\nabla A \in \Gamma^{m-1}(M, \mathrm{T}^l_{k+1} M)$ such that

$$(\nabla A)(\omega^1, \dots, \omega^l, X_1, \dots, X_k, V) = (\nabla_V A)(\omega^1, \dots, \omega^l, X_1, \dots, X_k)$$

for all $V, X_i \in \Gamma^{m-1}(M, \mathrm{T}M)$ and $\omega_i \in \Gamma^{m-1}(M, \mathrm{T}^* M)$.

In the case $l = k = 0$ we have that the covariant differential coincides with the total differential $\mathrm{d}f$.

2.5.6. Notation (Abstract indices): When convenient we will denote tensor fields on M using the abstract index notation by Roger Penrose (see e.g. [PR84], chapter 2.2, pp. 76–91): A tensor field will be written $T^{i_1 \dots i_l}{}_{j_1 \dots j_k}$ with Latin indices not to be understood as components but just indicating the order of the field, e.g. a contraction mapping of a tensor field is denoted by

$$T^{i_1 \dots a \dots i_l}{}_{j_1 \dots a \dots j_k} .$$

For calculations in a given coordinate system, we use Greek indices denoting the tensor components, i.e.

$$\sum_{\alpha=1}^{n} T^{\mu_1 \dots \alpha \dots \mu_l}{}_{\nu_1 \dots \alpha \dots \mu_k} .$$

2.6 Integration on Manifolds

In this section we give some basic definitions concerning integration on manifolds. Let us start with the definition of orientability.

2.6.1. Definition (Orientability): A smooth manifold of dimension n will be called orientable if there exists an orientable atlas $\mathcal{A}$, i.e. for $\mathcal{A} = \{(\varphi_\alpha, V_\alpha)\}$ we have that $\det(\mathrm{D}(\varphi_\beta \circ \varphi_\alpha^{-1}))(x)$ is positive for all $x \in \varphi_\alpha(V_\alpha \cap V_\beta)\ \forall \alpha, \beta$.

Note that not every manifold is orientable (e.g. the Möbius strip). For an oriented manifold M and a positively oriented chart let ω be a compactly supported n-form, such that $\operatorname{supp} \omega \subseteq U$, where (φ, U) is a chart on M. We set

$$\int_U \omega := \int_{\varphi(U)} \varphi_* \omega(x) \,\mathrm{d}^n x.$$

Here $\varphi_* \omega$ denotes the *pull-back* of ω.

2.6.2. DEFINITION (Integral): Let ω be a compactly supported n-form and let χ_α be a locally finite partition of unity subordinate to the charts $(\varphi_\alpha, V_\alpha)$. Then

$$\int_M \omega := \sum_\alpha \int_{V_\alpha} \chi_\alpha \omega.$$

Moreover, $\int_M \omega$ is independent of the charts and the partition of unity.

Our next task is the integration of vector fields on submanifolds $N \subseteq M$ of codimension 1. It is required that M is a Riemannian or Lorentzian manifold a with metric. Its action is denoted by $\langle \cdot, \cdot \rangle$. Now, let $\mathrm{d}v$ be the metric volume form on M and let $\mathrm{d}\sigma$ be the volume form on N. Moreover, denote by ν the positively oriented unit normal vector field on N, i.e. if $e_1, \dots, e_{n-1}$ is a positively oriented basis of $\mathrm{T}_p N$, we have $\det(\nu_p, e_1, \dots, e_{n-1}) > 0$ for all $p \in N$. Then one has

$$\mathrm{d}\sigma = \mathrm{d}v|_N = \iota_\nu \,\mathrm{d}v,$$

where $\iota_X \omega = \omega(X, \cdot, \dots)$. For any vector field X on M one obtains (see [Kri05], chapter 48)

$$\iota_X \,\mathrm{d}v = \langle X, \nu \rangle \,\mathrm{d}\sigma \text{ on } N.$$

We set $\mathrm{d}\sigma_b := \nu_b \,\mathrm{d}\sigma$ and call this the *vector valued* volume element on N. Now we give the following definition.

2.6.3. DEFINITION (Integration over vector fields): Let ν, N, $\mathrm{d}v$, and $\mathrm{d}\sigma$ as above. Then for any vector field X we define

$$\int_N X := \int_N \iota_X \,\mathrm{d}v = \int_N \langle X, \nu \rangle \,\mathrm{d}\sigma = \int_N X^b \,\mathrm{d}\sigma_b.$$

3 Sobolev Spaces

3.1 Basics of Sobolev Spaces

Since we are interested in differential equations whose coefficient functions and solutions are of low regularity, we have to define appropriate spaces of functions. Our approach uses the theory of Sobolev spaces. On Euclidean spaces a comprehensive introduction into this topic is given in [Ada75]. The situation on manifolds is covered by the books of Aubin and Hebey ([Aub82],[Heb99]).

3.1.1. REMARK: Note that on any manifold M we may construct a complete Riemannian metric. Indeed we cover M by charts and "glue" the pullback of the Euclidean metric by means of a subordinate $\mathcal{C}^\infty$-partition of unity to obtain a Riemannian metric, of course depending on the covering and the partition. Then by the theorem of Nomizu and Ozeki [NO61] there exists a conformally equivalent complete Riemannian metric on M.

3.1.2. DEFINITION: Let (M, e) be a Riemannian manifold. For a real valued function $f \in \mathcal{C}^m(M)$ we define

$$\begin{aligned}|\mathrm{D}^m f| :=& (\nabla_{p_1} \dots \nabla_{p_m} f \nabla_{q_1} \dots \nabla_{q_m} f e^{p_1 q_1} \dots e^{p_m q_m})^{1/2}\\ =& (e^{0,m^*}(\mathrm{D}^m f, \mathrm{D}^m f))^{1/2},\end{aligned}$$

where ∇ denotes covariant differentiation with respect to e, D^m denotes $\nabla_{p_1} \dots \nabla_{p_m}$ and e^{l,k^*} denotes $\underbrace{e \otimes \dots \otimes e}_{l\text{-times}} \otimes \underbrace{e^* \otimes \dots \otimes e^*}_{k\text{-times}}$.

In particular, we have

$$|\mathrm{D}^0 f| = |f|, |\mathrm{D}^1 f| = |\nabla f| = \sqrt{e(\nabla f, \nabla f)} = \sqrt{\nabla^i f \nabla_i f}.$$

3.1.3. Definition (Sobolev Norms I): Let a Riemannian manifold (M,e) be given with $m \in \mathbb{N}_0$ and $\mathbb{R} \ni p \geq 1$. We set

(i) $\|f\|_{\mathrm{L}^p(M)} := \left(\int\limits_M |f|^p \,\mathrm{d}v(e)\right)^{1/p}$ for $f \in \mathcal{C}(M)$,

(ii) $\mathcal{C}^{m,p}(M) := \{f \in \mathcal{C}^m(M) | \forall\, 1 \leq j \leq m, \|\mathrm{D}^j f\|_{\mathrm{L}^p(M)} < \infty\}$, and

(iii)

$$\|f\|_{\mathcal{W}^{m,p}(M)} := \left(\sum_{j=0}^{m} \|\mathrm{D}^j f\|^p_{\mathrm{L}^p(M)}\right)^{1/p} \tag{3.1}$$

for $f \in \mathcal{C}^{m,p}(M)$.

3.1.4. Remark: Alternatively, one could use the equivalent norms

$$\|f\|'_{\mathcal{W}^{m,p}(M)} := \sum_{j=0}^{m} \|\mathrm{D}^j f\|_{\mathrm{L}^p(M)},$$

or

$$\|f\|''_{\mathcal{W}^{m,p}(M)} := \max_{j \leq m} \|\mathrm{D}^j f\|_{\mathrm{L}^p(M)}.$$

We may regard $\mathcal{C}^{m,p}(M)$ as the space of functions, where f and its derivatives until m^{th}-order are in $\mathrm{L}^p(M)$.

Now we are ready to define the Sobolev spaces $\mathcal{W}^{m,p}(M)$ as follows:

3.1.5. Definition (Sobolev spaces I): Let (M,e) be a Riemannian manifold, the *Sobolev spaces* $\mathcal{W}^{m,p}(M)$ are the completion of $\mathcal{C}^{m,p}(M)$ with respect to the norms defined in (3.1). For $p = 2$ we write $\mathcal{H}^m(M)$.

The spaces $\mathcal{W}^{m,p}_{\mathrm{loc}}(M)$ consist of all distributions $u \in \mathcal{D}'(M)$ such that $fu \in \mathcal{W}^{m,p}(M)$ for all $f \in \mathcal{D}(M)$. We equip $\mathcal{W}^{m,p}(M)$ with the topology defined by the family of semi-norms $u \mapsto \|fu\|_{\mathcal{W}^{m,p}(M)}$. See also [CP82], pp. 98–104.

Since the $\mathcal{W}^{m,p}$-spaces are complete and normed by definition, they are Banach spaces. $\mathcal{W}^{m,p}_{\mathrm{loc}}(M)$ are Fréchet spaces, see remark 3.1.10. $\mathcal{H}^m(M)$ are Hilbert spaces equipped with the scalar product

$$(f,g)_m = \sum_{j=0}^{m} \int\limits_M e^{0,j^*}(\mathrm{D}^j f, \mathrm{D}^j g)\,\mathrm{d}v(e)$$

This construction may be easily generalized to tensor fields on Lorentzian manifolds in the following way:

3.1.6. DEFINITION: Let (M, g) be a Lorentzian manifold. Denote by e a Riemannian metric on M (see remark 3.1.1) and let $T \in \Gamma^m(M, \mathrm{T}^l_k(M))$. Then for $m \in \mathbb{N}_0$ we define the norm

$$\begin{aligned}|\mathrm{D}^m T| :=& (\nabla_{r_1} \dots \nabla_{r_m} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_{\bar{r}_1} \dots \nabla_{\bar{r}_m} T^{\bar{\imath}_1 \dots \bar{\imath}_l}{}_{\bar{\jmath}_1 \dots \bar{\jmath}_k} \\ & e_{i_1 \bar{\imath}_1} \dots e_{i_l \bar{\imath}_l} e^{j_1 \bar{\jmath}_1} \dots e^{j_k \bar{\jmath}_k} e^{r_1 \bar{r}_1} \dots e^{r_m \bar{r}_m})^{\frac{1}{2}} \\ =& (e^{l,(m+k)^*}(\mathrm{D}^m T, \mathrm{D}^m T))^{1/2}\end{aligned}$$

where ∇ denotes covariant differentiation with respect to g.

3.1.7. DEFINITION (Sobolev norms II): Let $M, g, e,$ and T be as above. For non-negative integers m and real $p \geq 1$ we define

(i) $\|T\|_{\mathrm{L}^p(l,k,M)} := \left(\int_M |T|^p \, \mathrm{d}v(e) \right)^{1/p}$ for $T \in \Gamma^0(M, \mathrm{T}^l_k M)$,

(ii) $\mathcal{T}^{m,p}(l,k,M) := \{T \in \Gamma^m(M, \mathrm{T}^l_k M) | \forall 1 \leq j \leq m, \|\mathrm{D}^j T\|_{\mathrm{L}^p(l,k,M)} < \infty\}$, and

(iii)

$$\|T\|_{\mathcal{W}^{m,p}(l,k,M)} := \left(\sum_{j=0}^{m} \|\mathrm{D}^j T\|^p_{\mathrm{L}^p(l,k,M)} \right)^{1/p} \tag{3.2}$$

for $T \in \mathcal{T}^{m,p}(l,k,M)$.

3.1.8. REMARK: Recall that $|\mathrm{D}^j T|$ in this context does refer to the covariant derivative with respect to g not e. On compact manifolds this does not affect the following definition of Sobolev spaces, since two derivatives on M only differ by the Christoffel symbols Γ^a_{bc} which are bounded on compact sets. Moreover, we achieve independence of e for compact manifolds ([Aub82], theorem 2.20).

This motivates

3.1.9. DEFINITION (Sobolev spaces II): Let (M, g) be a Lorentzian manifold and let e be a Riemannian metric. The Sobolev spaces $\mathcal{W}^{m,p}(l,k,M)$ are the completion of $\mathcal{T}^{m,p}(l,k,M)$ with respect to the norms defined in (3.2). For $p = 2$ we shall write $\mathcal{H}^m(l,k,M)$.

The spaces $\mathcal{W}^{m,p}_{\mathrm{loc}}(l,k,M)$ consist of all distributional tensor fields such that $fT \in \mathcal{W}^{m,p}(l,k,M)$ for all $f \in \mathcal{D}(M)$.

Again the $\mathcal{W}^{m,p}$-spaces are Banach spaces and $\mathcal{H}^m$ is a Hilbert space. See the following remark concerning the $\mathcal{W}^{m,p}_{\mathrm{loc}}$-spaces.

3.1.10. REMARK ($\mathcal{W}^{m,p}_{\mathrm{loc}}$ via projective limit): It is also possible to characterize the $\mathcal{W}^{m,p}_{\mathrm{loc}}$-spaces as a projective limit. Choose $(K_i)_i$ a countable, exhausting sequence of compact sets for M, i. e. $K_i \subseteq K^\circ_{i+1}$ and $M \subseteq \bigcup_{i \in \mathbb{N}} K_i$. Then let $\mathcal{W}^{m,p}(K_i^\circ)$ be the Sobolev space of type (m,p) over K_i°. We define $\mathcal{W}^{m,p}_{\mathrm{loc}}(M)$ together with the projections $\pi_i : \mathcal{W}^{m,p}_{\mathrm{loc}}(M) \to \mathcal{W}^{m,p}(K_i^\circ)$ to be the projective limit of $(\mathcal{W}^{m,p}(K_i^\circ), \pi_{ij})$, where π_{ij} denote the restrictions $\pi_{ij} : \mathcal{W}^{m,p}(K_j^\circ) \to \mathcal{W}^{m,p}(K_i^\circ)$ for $i \leq j$, i. e. the following diagram commutes for all $i, j \in \mathbb{N}, i \leq j$:

$$\begin{tikzcd}
 & \mathcal{W}' \arrow[d, "\psi"] \arrow[ddl, "\psi_j"'] \arrow[ddr, "\psi_i"] & \\
 & \mathcal{W}^{m,p}_{\mathrm{loc}}(M) \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & \\
\mathcal{W}^{m,p}(K_j^\circ) \arrow[rr, "\pi_{ij}"] & & \mathcal{W}^{m,p}(K_i^\circ)
\end{tikzcd}$$

In fact $\mathcal{W}^{m,p}_{\mathrm{loc}}(M)$ is therefore a subspace of $\prod_{i \in I} \mathcal{W}^{m,p}(K_i^\circ)$ such that

$$\mathcal{W}^{m,p}_{\mathrm{loc}}(M) := \left\{ (f_i)_i \in \prod_{i \in I} \mathcal{W}^{m,p}(K_i^\circ) \,\middle|\, \forall l \leq k, \pi_{lk}(f_k) = f_l \right\}.$$

One can show since the sequence $(K_i)_i$ is countable that $\mathcal{W}^{m,p}_{\mathrm{loc}}(M)$ is a metric space. Furthermore the space $\mathcal{W}^{m,p}_{\mathrm{loc}}(M)$ is a Fréchet space since we have each $\mathcal{W}^{m,p}(K_i^\circ)$ complete.

3.1.11. DEFINITION: For N an imbedded submanifold with compact closure in M we denote the differential operator tangential to N with $\tilde{\nabla}$ resp.

$\tilde{\mathrm{D}}^j$ for $j \in \mathbb{N}_0$. Therefore we have

$$\|T\|_{\mathcal{W}^{m,p}(l,k,N)} = \left(\sum_{j=0}^{m} \|\tilde{\mathrm{D}}^j T\|^p_{\mathrm{L}^p(l,k,N)} \right)^{1/p}.$$

Additionally we define for elements in $\mathcal{W}^{m,p}(l,k,M)$ with existing restriction in N, denoting this space by $\mathcal{W}^{m,p}(l,k,N \prec M)$,

$$\|T\|_{\mathcal{W}^{m,p}(l,k,N \prec M)} = \left(\sum_{j=0}^{m} \|\mathrm{D}^j T\|^p_{\mathrm{L}^p(l,k,N)} \right)^{1/p}.$$

Clearly we obtain

$$\|T\|_{\mathcal{W}^{m,p}(l,k,N)} \leq \|T\|_{\mathcal{W}^{m,p}(l,k,N \prec M)} \,. \tag{3.3}$$

It is obvious that for any $N \subseteq M$ submanifold of M with $\dim N = \dim M$ we have $\nabla = \tilde{\nabla}$.

3.2 Some Fundamental Results on Sobolev Spaces

3.2.1. REMARK: Note that it suffices to prove most results in this section for functions instead of tensor fields, since we have that $\mathcal{W}^{m,p}$-sections are smooth sections with $\mathcal{W}^{m,p}$-coefficients (see e.g. [Sim90], ex. 2.1.11), i.e.

$$\mathcal{H}^m(l,k,N) = \mathcal{H}^m(N) \otimes_{\mathcal{C}^\infty} \mathcal{T}^l_k(N).$$

In the following we prove three auxiliary results which will be essential in the proof of the main result; they can be found in [HE73], sec. 7.4, unfortunately without proof.

From now on let $N \subseteq M$ be a relatively compact submanifold of dimension n with differentiable boundary, where M is equipped with a Lorentzian metric g and a Riemannian metric e.

3.2.2. LEMMA: *Let U_i $(1 \leq i \leq L)$ be a finite covering of N and let χ_i be a partition of unity subordinate to U_i. Then we have for any function f in $\mathcal{H}^m(N)$—without loss of generality the same applies to $\mathcal{H}^m(N \prec M)$—that*

$$\sum_i \|\chi_i f\|_{\mathcal{H}^m(N)} \leq CL(m+1)\, \|f\|_{\mathcal{H}^m(N)} \,.$$

Proof: The statement follows from a short calculation. We start with

$$\begin{aligned}\sum_i \|\chi_i f\|_{\mathcal{H}^m(N)} =& \sum_i \Big(\sum_{l=0}^m \int_N |\tilde{\mathrm{D}}(\chi_i f)|^2 \,\mathrm{d}\sigma(e)\Big)^{\frac{1}{2}} \\ \le& \sum_i \Big(\sum_{l=0}^m \int_N \Big(\sum_{k=0}^l \binom{l}{k} |\tilde{\mathrm{D}}^{l-k}\chi_i||\tilde{\mathrm{D}}^k f|\Big)^2 \mathrm{d}\sigma(e)\Big)^{\frac{1}{2}}.\end{aligned}$$

Now since $\{\chi_i\}$ is a partition of unity $|\chi_i| \le 1$ and $\binom{l}{k}|\tilde{\mathrm{D}}^{l-k}\chi_i|$ is bounded by a constant C greater than 1, therefore we have

$$\begin{aligned}\sum_i \|\chi_i f\|_{\mathcal{H}^m(N)} \le& \sum_i \Big(\sum_{l=0}^m \int_N C^2 \Big(\sum_{k=0}^l |\tilde{\mathrm{D}}^k f|\Big)^2 \mathrm{d}\sigma(e)\Big)^{\frac{1}{2}} \\ =& CL \Big(\sum_{l=0}^m \int_N \Big(\sum_{k=0}^l |\tilde{\mathrm{D}}^k f|\Big)^2 \mathrm{d}\sigma(e)\Big)^{\frac{1}{2}} \\ \le& CL \sum_{l=0}^m \Big(\int_N (l+1) \sum_{k=0}^l |\tilde{\mathrm{D}}^k f|^2 \,\mathrm{d}\sigma(e)\Big)^{\frac{1}{2}} \\ & \text{by } \Big(\sum_{k=0}^l a_k\Big)^2 \le (l+1) \sum_{k=0}^l a_k^2 \\ \le& CL \Big(\sum_{l=0}^m (l+1) \sum_{k=0}^l \int_N |\tilde{\mathrm{D}}^k f|^2 \,\mathrm{d}\sigma(e)\Big)^{\frac{1}{2}} \\ =& CL \Big(\sum_{l=0}^m (l+1) \, \|f\|^2_{\mathcal{H}^l(N)}\Big)^{\frac{1}{2}} \\ =& CL \Big(\sum_{l=0}^m (m+1) \, \|f\|^2_{\mathcal{H}^m(N)}\Big)^{\frac{1}{2}} \\ \le& CL(m+1) \, \|f\|_{\mathcal{H}^m(N)} .\end{aligned}$$

Hence the claim follows. *q. e. d.*

3.2.3. LEMMA: *Let $T \in \mathcal{H}^m(l,k,N)$, for $2m > n$ there exists some constant $\tilde{P}_1$ depending on N, e, and g such that*

$$|T| \le \tilde{P}_1 \, \|T\|_{\mathcal{H}^m(l,k,N)} ,$$

i.e.

$$\mathcal{H}^m(l,k,N) \hookrightarrow \mathcal{C}_{\mathrm{b}}(l,k,N).$$

Proof: Let U_i be a finite covering of N with appropriate charts (φ_i, U_i) for $(1 \leq i \leq L)$. Let χ_i be a compactly supported partition of unity subordinate to U_i. By remark 3.2.1 it is sufficient to consider the scalar case. Hence let f be a function in $\mathcal{H}^m(N)$. By the Rellich-Kondrachov theorem ([Ada75], theorem 6.2) we have on $\varphi_i(U_i)$ that

$$|(\chi_i f) \circ \varphi_i^{-1})| \leq P(i) \|(\chi_i f) \circ \varphi_i^{-1})\|_{\mathcal{H}^m(\varphi_i(U_i))}.$$

Since e, g and ∇g are bounded on $\operatorname{supp} \chi_i$, we obtain

$$\begin{aligned} |\chi_i f| \leq& P'(i)\ \|\chi_i f\|_{\mathcal{H}^m(U_i)}\ \forall i \\ =& P'(i)\ \|\chi_i f\|_{\mathcal{H}^m(N)}\ \forall i. \end{aligned}$$

Since $\{\chi_i\}$ is a partition of unity it follows that

$$\begin{aligned} |f| =& |\sum_i \chi_i f| \\ \leq& \sum_i |\chi_i f| \\ \leq& (\max_i P'(i)) \sum_i \|\chi_i f\|_{\mathcal{H}^m(N)}. \end{aligned}$$

Application of lemma 3.2.2 immediately gives the final result

$$|f| \leq \tilde{P}_1\ \|f\|_{\mathcal{H}^m(N)}$$

with $\tilde{P}_1 := CL(m+1) \max_i P'(i)$. *q. e. d.*

3.2.4. COROLLARY: *For $2m > n$ as in lemma 3.2.3, and $j \in \mathbb{N}$ we have*

$$\mathcal{H}^{m+j}(l,k,N) \subseteq \mathcal{C}^j_{\mathrm{b}}(l,k,N)$$

or

$$|\tilde{\mathrm{D}}^j T| \leq \tilde{P}_1(j)\ \|T\|_{\mathcal{H}^{m+j}(l,k,N)}.$$

This is a direct consequence of lemma 3.2.3 applied to $\tilde{\mathrm{D}}^j T$.

3.2.5. COROLLARY: *If $\|T\|_{\mathcal{H}^m(l,k,N \prec M)}$ exists for a T in lemma 3.2.3 we obviously have a constant $P_1(j)$ such that*

$$|\mathrm{D}^j T| \leq P_1(j)\ \|T\|_{\mathcal{H}^{m+j}(l,k,N \prec M)}.$$

Apply lemma 3.2.3 to $\mathrm{D}^j T$. The result follows from inequality (3.3).

3.2.6. LEMMA: *Let $T \in \mathcal{H}^m(l,k,N)$ and suppose $2m \leq n$ and $2 \leq r \leq \frac{2n}{n-2m}$ or $2 \leq r < \infty$ if $2m = n$. Then there exists a constant K depending on N, e, and g such that*

$$\int\limits_N |T|^r \, \mathrm{d}v(e) \leq K \, \|T\|^r_{\mathcal{H}^m(l,k,N)} \, .$$

Proof: It is sufficient to prove the scalar case, hence $f \in \mathcal{H}^m(N)$. So let U_i be a finite covering of N with appropriate charts (φ_i, U_i), $(1 \leq i \leq L)$. Let χ_i be a compactly supported partition of unity subordinate to U_i. By the Sobolev imbedding theorem ([Ada75], thm. 5.4, cases A and B), one has on $\varphi_i(U_i)$

$$\|\chi_i f \circ \varphi_i^{-1}\|_{\mathrm{L}^r(\varphi_i(U_i))} \leq K(i) \|\chi_i f \circ \varphi_i^{-1}\|_{\mathcal{H}^m(\varphi_i(U_i))}.$$

By the boundedness of e, g, and ∇g on $\operatorname{supp} \chi_i$ we obtain

$$\begin{aligned} \|\chi_i f\|_{\mathrm{L}^r(U_i)} \leq& K'(i) \|\chi_i f\|_{\mathcal{H}^m(U_i)} \, \forall i \\ =& K'(i) \|\chi_i f\|_{\mathcal{H}^m(N)} \, \forall i. \end{aligned}$$

Moreover, since $\{\chi_i\}$ is a partition of unity we get

$$\begin{aligned} \|f\|_{\mathrm{L}^r(N)} =& \|\sum_i \chi_i f\|_{\mathrm{L}^r(N)} \\ \leq& \sum_i \|\chi_i f\|_{\mathrm{L}^r(N)} \\ =& \sum_i \|\chi_i f\|_{\mathrm{L}^r(U_i)} \\ \leq& (\max_i K'(i)) \sum_i \|\chi_i f\|_{\mathcal{H}^m(N)}. \end{aligned}$$

Using lemma 3.2.2 the final result

$$\|f\|^r_{\mathrm{L}^r(N)} \leq K \, \|f\|^r_{\mathcal{H}^m(N)}$$

with $K := (CL(m+1) \max_i K'(i))^r$ follows and we are done. *q. e. d.*

3.2.7. COROLLARY: *Assuming the conditions of lemma 3.2.6 for any $j \in \mathbb{N}$ we obtain*

$$\int\limits_N |\tilde{\mathrm{D}}^j T|^r \, \mathrm{d}v(e) \leq \tilde{K}(j) \, \|T\|^r_{\mathcal{H}^{m+j}(l,k,N)} \, .$$

Proof: Apply lemma 3.2.6 to the tensor field $\tilde{\mathrm{D}}^j T$. This yields

$$\begin{aligned}\int\limits_N |\tilde{\mathrm{D}}^j T|^r \,\mathrm{d}v(e) \leq& \tilde{K}(j) \|\tilde{\mathrm{D}}^j T\|^r_{\mathcal{H}^m(l,k,N)} \\ \leq& \tilde{K}(j) \|T\|^r_{\mathcal{H}^{m+j}(l,k,N)}.\end{aligned}$$

q. e. d.

3.2.8. COROLLARY: *If $\|T\|_{\mathcal{H}^{m+j}(l,k,N\prec M)}$ exists for a T in lemma 3.2.6 we have constants $K(j)$ such that for all $j \in \mathbb{N}$*

$$\int\limits_N |\mathrm{D}^j T|^r \,\mathrm{d}v(e) \leq K(j) \, \|T\|^r_{\mathcal{H}^{m+j}(l,k,N\prec M)} \,.$$

Proof: Application of lemma 3.2.6 to $\mathrm{D}^j T$ and inequality (3.3) gives

$$\begin{aligned}\int\limits_N |\mathrm{D}^j T|^r \,\mathrm{d}v(e) \leq& K(j) \|\mathrm{D}^j T\|^r_{\mathcal{H}^m(l,k,N)} \\ \leq& K(j) \|T\|^r_{\mathcal{H}^{m+j}(l,k,N\prec M)}\end{aligned}$$

and we are done. *q. e. d.*

3.2.9. LEMMA:

(i) Let $S \in \mathcal{H}^t(l,k,N)$ and $T \in \mathcal{H}^u(l',k',N)$. If $s < t+u-\frac{n}{2}$, $s \leq t$, and $s \leq u$ then there exists a constant P_2 depending on N, e, and g such that

$$\|S \otimes T\|_{\mathcal{H}^s(l+l',k+k',N)} \leq P_2 \, \|S\|_{\mathcal{H}^t(l,k,N)} \, \|T\|_{\mathcal{H}^u(l',k',N)}$$

holds.

(ii) Moreover, if $2m > n$, we have

$$\|S \otimes T\|_{\mathcal{H}^m(l+l',k+k',N)} \leq \tilde{P}_2' \, \|S\|_{\mathcal{H}^m(l,k,N)} \, \|T\|_{\mathcal{H}^m(l',k',N)} \,. \tag{3.4}$$

Proof:

(i) Again we only have to consider the scalar case. Thus the statement follows from a theorem in [Cla93], section 4.3.1.

(ii) The proof is a manifold version of theorem 5.23 in [Ada75]. To show (3.4), it is sufficient to prove

$$\int\limits_N |\tilde{\mathrm{D}}^j (S \otimes T)|^2 \,\mathrm{d}v(e) \leq K_j \, \|S\|^2_{\mathcal{H}^m(l,k,N)} \, \|T\|^2_{\mathcal{H}^m(l',k',N)} \tag{3.5}$$

for $j \leq m$ and K_j depending on N, e, and g, since then

$$\begin{aligned}\|S \otimes T\|^2_{\mathcal{H}^m(l+l',k+k',N)} &= \sum_{j=0}^{m} \int_N |\tilde{\mathrm{D}}^j (S \otimes T)|^2 \,\mathrm{d}v(e) \\ &\leq \sum_{j=0}^{m} K_j \, \|S\|^2_{\mathcal{H}^m(l,k,N)} \, \|T\|^2_{\mathcal{H}^m(l',k',N)} \,.\end{aligned}$$

Let us assume for the moment that $S \in \mathcal{T}^l_k(N)$. By a similar argument it follows using Leibniz' product rule that it is sufficient to have for $i \leq j \leq m$

$$\int_N |\tilde{\mathrm{D}}^i S \otimes \tilde{\mathrm{D}}^{j-i} T|^2 \,\mathrm{d}v(e) \leq K_{i,j} \, \|S\|^2_{\mathcal{H}^m(l,k,N)} \, \|T\|^2_{\mathcal{H}^m(l',k',N)} \,, \tag{3.6}$$

where $K_{i,j}$ depends on N, e, and g.

Corollary 3.2.7 entails for each natural number $i \leq m$ a positive constant $K(i) = K(i, N, g, e)$ such that for any $W \in \mathcal{H}^m(l, k, N)$

$$\int_N |\tilde{\mathrm{D}}^i W|^r \,\mathrm{d}v(e) \leq K(i) \, \|W\|^r_{\mathcal{H}^m(l,k,N)} \tag{3.7}$$

holds, assumed that $2(m-i) \leq n$ and $2 \leq r \leq \frac{2n}{n-2(m-i)}$ (or $2 \leq r < \infty$ if $2(m-i) = n$), or

$$|\tilde{\mathrm{D}}^i W| \leq K(i) \, \|W\|_{\mathcal{H}^m(l,k,N)} \tag{3.8}$$

assuming $2(m-i) > n$ as in corollary 3.2.4.

To prove (3.6) let i' be the largest integer such that $2(m-i') > n$. Since $2m > n$ by assumption, we have that $i' \geq 0$. Now we distinguish several cases.

(a) i. If $i \leq i'$, then $2(m-i) > n$, so

$$\begin{aligned}\int_N |\tilde{\mathrm{D}}^i S \otimes \tilde{\mathrm{D}}^{j-i} T|^2 \,\mathrm{d}v(e) &\overset{(3.8)}{\leq} \int_N K(i)^2 \, \|S\|^2_{\mathcal{H}^m(l,k,N)} \, |\tilde{\mathrm{D}}^{j-i} T|^2 \,\mathrm{d}v(e) \\ &\leq K(i)^2 \, \|S\|^2_{\mathcal{H}^m(l,k,N)} \, \|\tilde{\mathrm{D}}^{j-i} T, N\|^2_0 \\ &\leq K(i)^2 \, \|S\|^2_{\mathcal{H}^m(l,k,N)} \, \|T\|^2_{\mathcal{H}^m(l',k',N)} \,.\end{aligned}$$

ii. Similarly if $j - i \leq i'$, we have

$$\int_N |\tilde{\mathrm{D}}^i S \otimes \tilde{\mathrm{D}}^{j-i} T|^2 \,\mathrm{d}v(e) \overset{(3.8)}{\leq} K(j-i)^2 \, \|S\|^2_{\mathcal{H}^m(l,k,N)} \, \|T\|^2_{\mathcal{H}^m(l',k',N)} \,.$$

(b) Now if $i > i'$ and $j - i > i'$, then we clearly have $i \geq i' + 1$ and $j - i \geq i' + 1$ such that $n \geq 2(m-i)$ and $n \geq 2(m-j+i)$. Thus

$$\frac{n-2(m-i)}{n} + \frac{n-2(m-j+i)}{n} = 2 - \frac{2(2m-j)}{n} < 2 - \frac{2m}{n} < 1.$$

Hence we find $r, r' \in \mathbb{R}^+$ with ${}^1\!/_r + {}^1\!/_{r'} = 1$ so that

$$2 \leq 2r < \frac{2n}{n-2(m-i)}, \quad 2 \leq 2r' < \frac{2n}{n-2(m-j+i)}.$$

Applying Hölder's inequality and (3.7) we get

$$\begin{aligned}\int\limits_N &|\tilde{\mathrm{D}}^i S \otimes \tilde{\mathrm{D}}^{j-i} T|^2 \, \mathrm{d}v(e) \\ &\leq \Big(\int\limits_N |\tilde{\mathrm{D}}^i S|^{2r} \, \mathrm{d}v(e)\Big)^{{}^1\!/_r} \Big(\int\limits_N |\tilde{\mathrm{D}}^{j-i} T|^{2r'} \, \mathrm{d}v(e)\Big)^{{}^1\!/_{r'}} \\ &\leq [K(i)]^{{}^1\!/_r} [K(j-i)]^{{}^1\!/_{r'}} \, \|S\|^2_{\mathcal{H}^m(l,k,N)} \, \|T\|^2_{\mathcal{H}^m(l',k',N)},\end{aligned}$$

which completes the proof of inequality (3.5) for $S \in \mathcal{T}^l_k(N)$ and $T \in \mathcal{H}^m(l', k', N)$.

If $S \in \mathcal{H}^m(l,k,N)$, then by the definition of the Sobolev spaces there exists a sequence S_n of $\mathcal{T}^l_k(N)$ tensor fields converging to S in $\mathcal{H}^m(l,k,N)$. Then by the above argument $S_n \otimes T$ is a Cauchy sequence in $\mathcal{H}^m(l+l', k+k', N)$ and therefore converging to an element W of that space. Since $2m > n$ we may assume that S and T are continuous and bounded on N, hence

$$\begin{aligned}\|W - S \otimes T\|_{\mathrm{L}^2(l+l',k+k',N)} &\leq \|W - S_n \otimes T\|_{\mathrm{L}^2(l+l',k+k',N)} \\ &\quad + \|(S_n - S) \otimes T\|_{\mathrm{L}^2(l+l',k+k',N)} \\ &\leq \|W - S_n \otimes T\|_{\mathrm{L}^2(l+l',k+k',N)} \\ &\quad + \|T\|_{\mathrm{L}^\infty(l',k',N)} \, \|S_n - S\|_{\mathrm{L}^2(l,k,N)} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned}$$

So we have $W = S \otimes T$ is L^2-integrable and $W = S \otimes T$ in the sense of distributions. Therefore $W = S \otimes T$ in $\mathcal{H}^m(l+l', k+k', N)$ and

$$\begin{aligned}\|S \otimes T\|_{\mathcal{H}^m(l+l',k+k',N)} &= \|W\|_{\mathcal{H}^m(l+l',k+k',N)} \\ &\leq \limsup_{n\rightarrow\infty} \|S_n \otimes T\|_{\mathcal{H}^m(l+l',k+k',N)} \\ &= \|S, N\|_{\mathcal{H}^m(l,k,N)} \, \|T, N\|_{\mathcal{H}^m(l',k',N)} .\end{aligned}$$

This completes the proof of part (ii).

q. e. d.

3.2.10. COROLLARY: *If $S \in \mathcal{H}^m(l,k,N \prec M)$ and $T \in \mathcal{H}^m(l',k',N \prec M)$, we have for $2m > n$ a constant P_2' depending on N, e, and g such that*

$$\|S \otimes T\|_{\mathcal{H}^m(l+l',k+k',N\prec M)} \leq P_2' \, \|S\|_{\mathcal{H}^m(l,k,N\prec M)} \, \|T\|_{\mathcal{H}^m(l',k',N\prec M)} \, .$$

This is proven analogously to lemma 3.2.9. Instead of the corollaries 3.2.7 and 3.2.4 use 3.2.8 resp. 3.2.5.

3.2.11. LEMMA: *Let $T \in \mathcal{H}^{m+j}(l,k,N)$, $N' \subseteq N$ be a smooth submanifold of dimension n'. Suppose either $2m < n$, $n - 2m < n' < n$, and $2 \leq \frac{2n'}{n-2m}$ or $2m = n$ and $1 \leq n' < n$. Then for each n' and j there exists a positive constant $\tilde{P}_3(j,n')$ such that*

$$\|T\|_{\mathcal{H}^j(l,k,N')} \leq \tilde{P}_3(j,n') \, \|T\|_{\mathcal{H}^{j+m}(l,k,N)} \, .$$

Proof: It is sufficient to prove the scalar case, hence let $f \in \mathcal{H}^{j+m}(N)$. Let U_i be a finite covering of N with appropriate charts (ϕ_i, U_i), $(1 \leq i \leq L)$. Let χ_i be a compactly supported partition of unity subordinate to U_i. By [Ada75], theorem 5.4, cases A and B we have on $\phi_i(U_i)$

$$\|\chi_i f \circ \varphi_i^{-1}\|_{\mathcal{H}^j(\varphi_i(U_i)|_{N'})} \leq P_3(i,j,n') \|\chi_i f \circ \varphi_i^{-1}\|_{\mathcal{H}^{m+j}(\varphi_i(U_i))}.$$

Since e, g, and ∇g are bounded on supp χ_i we have

$$\begin{aligned}\|\chi_i f\|_{\mathcal{H}^j(U_i \cap N')} &\leq P_3'(i,j,n') \|\chi_i f\|_{\mathcal{H}^{m+j}(U_i)} \\ &= P_3'(i,j,n') \|\chi_i f\|_{\mathcal{H}^{m+j}(N)}.\end{aligned}$$

Furthermore, we have that $\{\chi_i\}$ is a partition of unity. Thus we obtain

$$\begin{aligned}\|f\|_{\mathcal{H}^j(N')} &= \| \sum_i \chi_i f\|_{\mathcal{H}^j(N')} \\ &\leq \sum_i \|\chi_i f\|_{\mathcal{H}^j(N')} \\ &= \sum_i \|\chi_i f\|_{\mathcal{H}^j(U_i \cap N')} \\ &\leq (\max_i P_3'(i,j,n')) \sum_i \|\chi_i f\|_{\mathcal{H}^{m+j}(N)}.\end{aligned}$$

By application of lemma 3.2.2 this yields

$$\|f\|_{\mathcal{H}^j(N')} \leq \tilde{P}_3(j,n')\|f\|_{\mathcal{H}^{m+j}(N)}$$

with $\tilde{P}_3(j,n') := CL(m+j+1)\max_i P_3'(i,j,n')$. *q. e. d.*

3.2.12. COROLLARY: *In fact in lemma 3.2.11—if the restriction exists—for each $i \in \mathbb{N}_0$ we have the stronger estimate*

$$\|T\|_{\mathcal{H}^i(l,k,N' \prec N)} \leq P_3(i,n')\ \|T\|_{\mathcal{H}^{i+m}(l,k,N)}\,.$$

Proof: For convenience of the reader, in this proof we write D^i for the differentials on N. Set $j=0$ in lemma 3.2.11. This yields for the field $\mathrm{D}^{i'}T$

$$\|\mathrm{D}^{i'}T\|_{\mathrm{L}^2(l,k,N')} \leq \tilde{P}_3(0,n')\|\mathrm{D}^{i'}T\|_{\mathcal{H}^m(l,k,N)} \leq \tilde{P}_3(0,n')\|T\|_{\mathcal{H}^{i'+m}(l,k,N)}.$$

Summing up, we obtain the final result

$$\|T\|_{\mathcal{H}^{i+m}(l,k,N' \prec N)} = \sum_{i'=0}^{i} \|\mathrm{D}^{i'}T\|_{\mathrm{L}^2(l,k,N')} \leq |i|\tilde{P}_3(0,n')\|T\|_{\mathcal{H}^{i+m}(l,k,N)}$$

by defining $P_3(i,n') := |i|\tilde{P}_3(0,n')$. *q. e. d.*

4 Some Concepts from General Relativity

The differential equations we are interested in are hyperbolic ones, i.e. where the differential operator is derived from a Lorentzian metric. Since Lorentz geometry is deeply connected to the theory of relativity, we have to introduce some relativistic ideas concerning causality. For further study we refer to the books of O'Neill [O'N83] and Wald [Wal84].

A main concept in relativistic physics is the trichotomy of a vector space with a Lorentzian scalar product. Here we consider the tangent space of a Lorentzian manifold.

4.1.1. DEFINITION: On a Lorentzian manifold (M, g) a tangent vector $v \in T_pM$ is called

- *spacelike* if $g(v, v) > 0$ or $v = 0$,
- *null* if $g(v, v) = 0$ and $v \neq 0$,
- *timelike* if $g(v, v) < 0$.

The terms *non-timelike* and *non-spacelike/causal* refer to the corresponding relations with $\geq$ and $\leq$.

This definition can be extended to vector fields requiring that it holds for any point p.

4.1.2. DEFINITION: On a Lorentzian manifold (M, g) a curve $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an arbitrary open interval, is spacelike, timelike or null if for its tangent vector field $\dot{\gamma}(s)$ the relations of definition 4.1.1 hold for all $s \in I$.

In a similar way we can define those properties for hypersurfaces.

4.1.3. DEFINITION: On a Lorentzian manifold (M, g) a *hypersurface* is *spacelike*, *timelike* or *null* if its normal vector field ν is timelike, spacelike or null respectively.

For Lorentzian manifolds the concept of time-orientability is important. Thus let us start with the following definition.

4.1.4. DEFINITION (Time cone): Let $p \in M$, where (M, g) is a Lorentzian manifold. We define for timelike $u \in \mathrm{T}_pM$ the set

$$\mathrm{C}_p(u) := \{v \in \mathrm{T}_pM \text{ timelike} | g(u, v) < 0\},$$

called *time cone* of T_pM containing u.

The opposite time cone is

$$\mathrm{C}_p(-u) = -\mathrm{C}(u) = \{v \in \mathrm{T}_pM \text{ timelike} | g(u, v) > 0\}.$$

We call the set $\overline{\mathrm{C}}(u)$ of all causal vectors v with $g(u, v) < 0$ the *causal cone* containing u.

Let τ be a function on M assigning to each point a time cone C_p. The map τ is smooth if for each $p \in M$ there exists a smooth vector field X on a neighbourhood U of p with $X_q \in \mathrm{C}_p \; \forall q \in U$.

4.1.5. DEFINITION (Time orientability): If for a Lorentzian manifold M there exists a smooth function τ as above, called *time orientation*, then M is said to be *time-orientable*.

4.1.6. LEMMA: *A Lorentzian manifold M is time-orientable if and only if there exists a smooth timelike vector field X.*

For a proof see section 5.32 in [O'N83].

The time orientation of M is called the *future* and its negative is called the *past*. Thus we have

4.1.7. DEFINITION (Future-directed):

1. A vector $v \in \mathrm{T}_pM$ is called *future-directed* if it is in the future causal cone.

2. A curve is called future-directed if all its tangent vectors are future-directed.

By replacing future with past in this definition one obtains past-directed vectors and curves.

4.1.8. DEFINITION (Causality relations): We define the following causality relations on M. If $p, q \in M$, then

(1) $p \ll q$ if there is a future-directed timelike curve in M from p to q,

(2) $p < q$ if there is a future-directed non-spacelike curve in M from p to q.

(3) $p \leq q$ if either $p < q$ or $p = q$.

4.1.9. DEFINITION (Chronological and causal future): For $H \subseteq M$ we set

$$I^+(H) := \{q \in M | \exists p \in H : p \ll q\}, \text{ the } \textit{chronological future},$$

and

$$J^+(H) := \{q \in M | \exists p \in H : p \leq q\}, \text{ the } \textit{causal future}.$$

Furthermore, the chronological resp. causal future of H with respect to $U \subseteq M$ is defined by

$$I^+(H, U) := \{q \in U | \exists p \in H : p \ll q\}$$

and

$$J^+(H, U) := \{q \in U | \exists p \in H : p \leq q\}$$

The *chronological* and *causal past* I^- resp. J^- are defined analogously by replacing $p \ll q$ and $p \leq q$ by $q \ll p$ resp. $q \leq p$.

4.1.10. DEFINITION: A set $H \subseteq M$ is said to be *achronal* if for all $p, q \in H$, $p \not\ll q$.

4.1.11. DEFINITION (Extendible curves): We call a continuous curve γ from (s_0, s_1) to M *extendible* provided that there exists a continuous extension $\tilde{\gamma} : (s_0, s_1] \to M$, otherwise γ will be called *inextendible*. Then $q = \gamma(s_1)$ is called an *endpoint* of γ. Obviously one may cover the left end of the interval in the same way.

4.1.12. DEFINITION (Cauchy development): For an achronal set H we call a region $D^+(H)$ the *future Cauchy development* or *domain of dependence* of H if it is the set of all points $p \in M$ such that every past-inextendible non-spacelike curve through p intersects H. The *past* Cauchy development is defined analogously.

Here the expression *past-inextendible* means past-directed and inextendible.

The last proposition in this section covers the construction of a Riemannian metric from a Lorentzian one.

4.1.13. PROPOSITION: *Let (M, g) be a Lorentzian manifold, X be a timelike unit vector field, and let X^* denote the corresponding one-form with respect to g. Then*

$$e := X^* \otimes X^* - \frac{1}{2} g(X, X) g \tag{4.1}$$

is a Riemannian metric.

For a proof see [O'N83], sec. 5.36.

5 Existence and Uniqueness Theorems

5.1 The Main Theorems

We are now in a position to state two theorems on existence and uniqueness of solutions to hyperbolic second order partial differential equations on Lorentzian manifolds. The first theorem uses less strict conditions on the tensor fields but holds only for manifolds of dimension $n \leq 4$, whereas the second theorem due to more restrictive conditions is valid for any dimension.

5.1.1. REMARK: Note that if not stated otherwise throughout the following sections covariant derivatives are taken with respect to the Lorentzian metric g, scalar products are to be understood with respect to the Lorentzian metric A and norms of tensor fields refer to the Riemannian metric e. Moreover, the concepts of timelike, spacelike and null vectors generally refer to the metric A. We shall write *g-derivative*, *e-norm*, or *A-timelike*.

We consider the following initial value problem: Let (M, g) be a Lorentzian manifold, let A be a Lorentz metric and furthermore, let B, C, and F be tensor fields on M. We are interested in solutions T of

$$\begin{aligned} L(T) :=& A^{ab}\nabla_a\nabla_b T^{i_1\ldots i_l}{}_{j_1\ldots j_k} + B^{aq_1\ldots q_k i_1\ldots i_l}{}_{p_1\ldots p_l j_1\ldots j_k}\nabla_a T^{p_1\ldots p_l}{}_{q_1\ldots q_k} \\ &+ C^{q_1\ldots q_k i_1\ldots i_l}{}_{p_1\ldots p_l j_1\ldots j_k} T^{p_1\ldots p_l}{}_{q_1\ldots q_k} = F^{i_1\ldots i_l}{}_{j_1\ldots j_k}. \end{aligned} \tag{5.1}$$

Let $U \subseteq M$ open and relatively compact with $\mathcal{C}^1$-boundary and let h be a smooth function on a neighbourhood of $\overline{U}$ with timelike gradient. We set

$$H(t) := \{p | h(p) = t\}$$

and

$$U_+ := U \cap \bigcup_{t \geq 0} H(t).$$

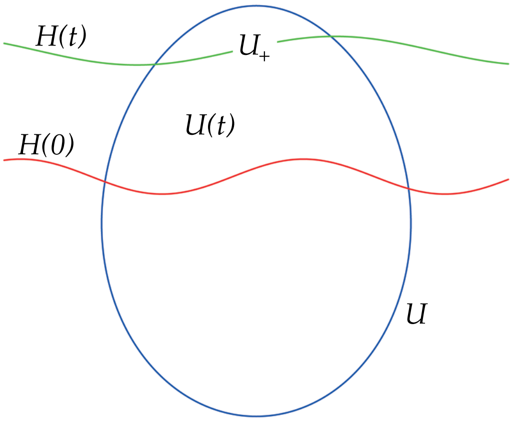

Figure 5.1: The set U

Since U is relatively compact, one has that t is bounded by some constant t_1. We specify initial conditions as follows

$$T^{i_1\ldots i_l}{}_{j_1\ldots j_k}|_{H(0)\cap U_+} = T_0{}^{i_1\ldots i_l}{}_{j_1\ldots j_k} \qquad \nabla_a T^{i_1\ldots i_l}{}_{j_1\ldots j_k} u^a|_{H(0)\cap U_+} = T_1{}^{i_1\ldots i_l}{}_{j_1\ldots j_k} \tag{5.2}$$

where T_0, T_1 are tensor fields of indicated type and u is any $\mathcal{C}^{3+\alpha}$-vector field non-tangential to $H(0)$. This scenario is illustrated in figure 5.1.

5.1.2. NOTATION: We denote functions and tensor fields on a set U with Lipschitz continuous $(m-1)^{\text{th}}$ derivative by $\mathcal{C}^{m-}(U)$ resp. $\Gamma^{m-}(U, \mathrm{T}^l_k U)$.

5.1.3. THEOREM: *Let M, U, $H(t)$ as above, $\alpha \in \mathbb{N}_0$, $\dim M \leq 4$ and the metric $g \in \Gamma^{5+\alpha}(M, \mathrm{T}^0_2 M)$. Assume that*

(1) $\partial U \cap \overline{U}_+$ is achronal with respect to A (causality),

(2) $A \in \mathcal{H}^{4+\alpha}(2, 0, U_+)$,
$B \in \mathcal{H}^{3+\alpha}(k+l+1, l+k, U_+)$,
$C \in \mathcal{H}^{3+\alpha}(l+k, l+k, U_+)$ (regularity of the coefficients),

(3) $T_0 \in \mathcal{H}^{4+\alpha}(l, k, H(0) \cap \overline{U})$,
$T_1 \in \mathcal{H}^{3+\alpha}(l, k, H(0) \cap \overline{U})$,
$u \in \Gamma^{3+\alpha}(M, \mathrm{T}M)$ (regularity of the initial data), and

(4) $F \in \mathcal{H}^{3+\alpha}(l, k, U_+)$ (regularity of the right hand side).

Then there exists a unique solution $T \in \mathcal{H}^{4+\alpha}(l,k,U_+)$ *of* (5.1) *with initial data* (5.2).

Here the norms $\|\cdot\|_{\mathcal{H}^m(l,k,U)}$ *on the Sobolev spaces* $\mathcal{H}^m(l,k,U)$ *are defined via a Riemannian metric* $e \in \Gamma^{1-}(M, \mathrm{T}^0_2 M)$.

Since theorem 5.1.3 deals only with manifolds of dimension less or equal to 4, we phrase a second theorem for higher dimensions. In order to establish the necessary estimates we require the tensor fields to be in appropriate Sobolev spaces of higher order.

5.1.4. THEOREM: *Let* M, U, $H(t)$, $\alpha \in \mathbb{N}$ *as above,* $\dim M = n$, *and* $g \in \Gamma^{2+\alpha}(M, \mathrm{T}^0_2 M)$. *Assume that*

(1) $\partial U \cap \overline{U}_+$ *is achronal with respect to* A *(causality).*

(2) $A \in \mathcal{H}^{\lfloor \frac{n}{2} \rfloor + 1 + \alpha}(2, 0, U_+)$,
$B \in \mathcal{H}^{\lfloor \frac{n}{2} \rfloor + 1 + \alpha}(k+l+1, l+k, U_+)$,
$C \in \mathcal{H}^{\lfloor \frac{n}{2} \rfloor + 1 + \alpha}(l+k, l+k, U_+)$ *(regularity of the coefficients),*

(3) $T_0 \in \mathcal{H}^{1+\alpha}(l,k,H(0) \cap \overline{U})$,
$T_1 \in \mathcal{H}^{\alpha}(l,k,H(0) \cap \overline{U})$,
$u \in \Gamma^{\alpha}(M, \mathrm{T}M)$ *(regularity of the initial data), and*

(4) $F \in \mathcal{H}^{\alpha}(l,k,U_+)$ *(regularity of the right hand side).*

Then there exists a unique solution $T \in \mathcal{H}^{1+\alpha}(l,k,U_+)$ *of* (5.1) *with initial data* (5.2). *Again the norms* $\|\cdot\|_{\mathcal{H}^m(l,k,U)}$ *on the Sobolev spaces* $\mathcal{H}^m(l,k,U)$ *are defined via a Riemannian metric* $e \in \Gamma^{1-}(M, \mathrm{T}^0_2 M)$.

5.2 Energy Estimates

The proof of theorem 5.1.3 will be split up into several independent results. In the beginning we show some energy inequalities as well as the uniqueness part. The final result then follows from approximating the coefficients and initial data by analytic tensor fields.

5.2.1 Zero-Order Energy Estimate

We start with a proposition covering the zero-order energy inequality. By abuse of notation (see definition 3.1.11) we write $\mathcal{H}^m(l,k,H(t) \prec U_+)$ instead of $\mathcal{H}^m(l,k,H(t)\cap U_+ \prec U_+)$

5.2.1. PROPOSITION: *Let M, U, $H(t)$, e as in theorem 5.1.3, let the Lorentzian metric $g \in \Gamma^{2-}(M, \mathrm{T}^0_2 M)$, and h a $\mathcal{C}^2$-function on a neighbourhood of $\overline{U}$ with A-timelike g-gradient. If condition (1) from 5.1.3 and additionally*

(2′) there exists some $Q_1 > 0$ such that on $\overline{U}_+$

$$A^{ab}\nabla_a h \nabla_b h \leq -Q_1$$

and

$$A^{ab}\omega_a\omega_b \geq Q_1 e^{ab}\omega_a\omega_b$$

almost everywhere for any continuous one form ω which satisfies $A^{ab}\nabla_a h\omega_b = 0$, and

(2″) there exists some Q_2 such that on $\overline{U}_+$

$$|A| \leq Q_2 \quad |\mathrm{D}A| \leq \mathrm{Q}_2 \quad |\mathrm{B}| \leq \mathrm{Q}_2 \quad |\mathrm{C}| \leq \mathrm{Q}_2 \text{ almost everywhere,}$$

hold, then we have some positive constant P_4 (depending on U, e, g, Q_1, and Q_2) such that for all solutions T of (5.1) *and $\forall t \leq t_1$ we obtain*

$$\|T\|_{\mathcal{H}^1(l,k,H(t)\prec U_+)} \leq P_4\{\|T\|_{\mathcal{H}^1(l,k,H(0)\prec U_+)} + \|F\|_{\mathrm{L}^2(l,k,U(t))}\}, \tag{5.3}$$

where

$$U(t) = U \cap \bigcup_{0\leq t'\leq t} H(t').$$

To prove proposition 5.2.1 we show a result on the *energy tensor*

$$\begin{aligned} S^{ab} := \Bigg\{ & \left(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd}\right)\nabla_c T^{i_1\ldots i_l}{}_{j_1\ldots j_k}\nabla_d T^{p_1\ldots p_l}{}_{q_1\ldots q_k} \\ & -\frac{1}{2}A^{ab}T^{i_1\ldots i_l}{}_{j_1\ldots j_k}T^{p_1\ldots p_l}{}_{q_1\ldots q_k}\Bigg\} e^{j_1 q_1\ldots j_k q_k}{}_{i_1 p_1\ldots i_l p_l} \end{aligned} \tag{5.4}$$

which helps to establish an estimate for $\|T\|_{\mathcal{H}^1(l,k,H(t)\prec U_+)}$ in terms of its values on the initial surface, some integral over the derivative of the energy tensor $\nabla_b S^{ab}$, and the mean value of $\|T\|_{\mathcal{H}^1(l,k,H(t')\prec U_+)}$ over all $t' \in [0,t]$. Furthermore, we prove an estimate for $\nabla_b S^{ab}$ which allows us to rephrase this inequality in terms of the tensor field F. By a Gronwall argument we will finally conclude (5.3).

Note that in this context $e^{j_1 q_1 \dots j_k \dots q_k}{}_{i_1 p_1 \dots i_l p_l}$ has to be understood as $\underbrace{e^* \otimes \cdots \otimes e^*}_{k\text{-times}} \otimes \underbrace{e \otimes \cdots \otimes e}_{l\text{-times}}$.

5.2.2. DEFINITION (The dominant energy condition): A tensor field S on a Lorentzian manifold (M, A) is said to satisfy the *dominant energy condition*, if

(i) $S^{ab}\omega_a\omega_b \geq 0$,

(ii) $S^{ab}\omega_a$ is non-spacelike.

for all timelike one-forms ω.

5.2.3. LEMMA: *Define the* energy tensor *on U_+ by* (5.4) *with A, g, e, and T as in proposition 5.2.1. Then S satisfies the dominant energy condition 5.2.2 (with respect to A).*

5.2.4. DEFINITION: We will divide S^{ab} into the quantities

$$S_0^{ab} := \left(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd}\right)\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\,\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}\,e^{j_1 q_1\dots j_k q_k}{}_{i_1 p_1\dots i_l p_l}$$

and

$$\tilde{S}^{ab} := -\frac{1}{2}A^{ab}T^{i_1\dots i_l}{}_{j_1\dots j_k}\,T^{p_1\dots p_l}{}_{q_1\dots q_k}\,e^{j_1 q_1\dots j_k q_k}{}_{i_1 p_1\dots i_l p_l}.$$

We split the proof into five parts. Part 1 will show that $\tilde{S}^{ab}\omega_a$ is non-spacelike, part 2 will prove that $\tilde{S}^{ab}\omega_a\omega_b \geq 0$, which means—assuming that part 1 is proven—that $\tilde{S}^{ab}\omega_a$ has the same orientation as $-\omega^b$. Part 3 and 4 prove the respective statements for S_0^{ab}. In the fifth part we will gather the results of parts 1-4 and prove the lemma.

Proof of lemma 5.2.3:

1. CLAIM: $\tilde{S}^{ab}\omega_a$ is non-spacelike, i. e. $\tilde{S}^{ab}\omega_a\tilde{S}^{cd}\omega_c A_{bd} \leq 0$.

   As one can easily see

$$\begin{aligned}\tilde{S}^{ab}\omega_a\tilde{S}^{cd}\omega_c A_{bd} =& \frac{1}{4}A^{ab}\omega_a T^{i_1\dots i_l}{}_{j_1\dots j_k}\,T^{p_1\dots p_l}{}_{q_1\dots q_k}\,e^{j_1 q_1\dots j_k q_k}{}_{i_1 p_1\dots i_l p_l}\\ &\cdot A^{cd}\omega_c T^{\bar{i}_1\dots\bar{i}_l}{}_{\bar{j}_1\dots\bar{j}_k}\,T^{\bar{p}_1\dots\bar{p}_l}{}_{\bar{q}_1\dots\bar{q}_k}\,e^{\bar{j}_1\bar{q}_1\dots\bar{j}_k\bar{q}_k}{}_{\bar{i}_1\bar{p}_1\dots\bar{i}_l\bar{p}_l}A_{bd}\\ =&\frac{1}{4}A^{ab}A^{cd}A_{bd}\omega_a\omega_c|T|^2\\ =&\frac{1}{4}A^{ac}\omega_a\omega_c|T|^2.\end{aligned}$$

The above expression is even strictly less than zero for $T \neq 0$, since ω_a is timelike. ☐

2. CLAIM: $\tilde{S}^{ab}\omega_a\omega_b \geq 0$.

   A straight forward calculation leads to

$$\begin{aligned}\tilde{S}^{ab}\omega_a\omega_b &= -\frac{1}{2}A^{ab}\omega_a\omega_b T^{i_1\ldots i_l}{}_{j_1\ldots j_k}\, T^{p_1\ldots p_l}{}_{q_1\ldots q_k}\, e^{j_1 q_1 \ldots j_k q_k}{}_{i_1 p_1 \ldots i_l p_l}\\ &= -\frac{1}{2}A^{ab}\omega_a\omega_b|T|^2 \geq 0,\end{aligned}$$

   since ω_a is timelike. ☐

3. CLAIM: $S_0^{ab}\omega_a$ is non-spacelike, i.e. $S_0^{ab}\omega_a S_0^{\bar{a}\bar{b}}\omega_{\bar{a}}A_{b\bar{b}} \leq 0$.

   Similar to part 1 we find for scalar T that

$$\begin{aligned}S_0^{ab}\omega_a =& A^{ac}A^{bd}\nabla_c T\nabla_d T\omega_a - \frac{1}{2}A^{ab}A^{cd}\nabla_c T\nabla_d T\omega_a\\ =& \omega^c\nabla_c TA^{bd}\nabla_d T - \frac{1}{2}\omega^b A^{cd}\nabla_c T\nabla_d T\end{aligned}$$

$$\begin{aligned}A_{b\bar{b}}S_0^{ab}\omega_a =& \omega^c\nabla_c TA_{b\bar{b}}A^{bd}\nabla_d T - \frac{1}{2}\omega^b A_{b\bar{b}}A^{cd}\nabla_c T\nabla_d T\\ =& \omega^c\nabla_c T\nabla_{\bar{b}}T - \frac{1}{2}\omega_{\bar{b}}A^{cd}\nabla_c T\nabla_d T\\ A_{b\bar{b}}S_0^{ab}\omega_a S_0^{\bar{a}\bar{b}}\omega_{\bar{a}} =& \left(\omega^c\nabla_c T\nabla_{\bar{b}}T - \frac{1}{2}\omega_{\bar{b}}A^{cd}\nabla_c T\nabla_d T\right)\\ & \cdot\left(\omega^{\bar{c}}\nabla_{\bar{c}}TA^{\bar{b}\bar{d}}\nabla_{\bar{d}}T - \frac{1}{2}\omega^{\bar{b}}A^{\bar{c}\bar{d}}\nabla_{\bar{c}}T\nabla_{\bar{d}}T\right)\\ =& \omega^c\nabla_c T\omega^{\bar{c}}\nabla_{\bar{c}}TA^{\bar{b}\bar{d}}\nabla_{\bar{b}}T\nabla_{\bar{d}}T\\ & -\frac{1}{2}\omega^{\bar{d}}\nabla_{\bar{d}}T\omega^{\bar{c}}\nabla_{\bar{c}}TA^{cd}\nabla_c T\nabla_d T\\ & -\frac{1}{2}\omega^c\nabla_c TW^{\bar{b}}\nabla_{\bar{b}}TA^{\bar{c}\bar{d}}\nabla_{\bar{c}}T\nabla_{\bar{d}}T\\ & +\frac{1}{4}\omega_{\bar{b}}\omega^{\bar{b}}A^{cd}\nabla_c T\nabla_d TA^{\bar{c}\bar{d}}\nabla_{\bar{c}}T\nabla_{\bar{d}}T\\ =& \frac{1}{4}\omega_{\bar{b}}\omega^{\bar{b}}(A^{cd}\nabla_c T\nabla_d T)^2 \geq 0, \qquad (5.5)\end{aligned}$$

   because the sum of the first three terms of (5.5) is zero and the last one is greater or equal to zero, since ω is timelike.

To show the general case where T is a tensor field, we choose an orthonormal basis with respect to the metric e. Therefore in such coordinates

$$S_0^{\alpha\beta} = \sum_{\substack{\mu_1,\dots,\mu_l \\ \nu_1,\dots,\nu_k}} \left(A^{\alpha\gamma}A^{\beta\delta} - \frac{1}{2}A^{\alpha\beta}A^{\gamma\delta} \right) \nabla_\gamma T^{\mu_1\dots\mu_l}{}_{\nu_1\dots\nu_k} \nabla_\delta T^{\mu_1\dots\mu_l}{}_{\nu_1\dots\nu_k}$$
$$- \sum_{\substack{\mu_1,\dots,\mu_l \\ \nu_1,\dots,\nu_k}} \frac{1}{2}A^{\alpha\beta} T^{\mu_1\dots\mu_l}{}_{\nu_1\dots\nu_k} T^{\mu_1\dots\mu_l}{}_{\nu_1\dots\nu_k}.$$

We apply equation (5.5) on $S_0{}^{\alpha\beta\mu_1\dots\mu_l}{}_{\nu_1\dots\nu_k}\omega_\alpha\omega_\beta$ for all $\mu_1,\dots,\mu_l,\nu_1,\dots,\nu_k$ fixed. By (5.5) we know that all the $S_0{}^{\alpha\beta\mu_1\dots\mu_l}{}_{\nu_1\dots\nu_l}\omega_\alpha$ have the same time orientation as $-\omega^\beta$. Therefore we finally obtain that

$$S_0^{\alpha\beta}\omega_\alpha = \sum_{\substack{\mu_1,\dots,\mu_l \\ \nu_1,\dots,\nu_k}} S_0{}^{\alpha\beta\mu_1\dots\mu_l}{}_{\nu_1\dots\nu_l}\omega_\alpha$$

is non-spacelike, since the sum of non-spacelike vectors of the same time orientation is indeed non-spacelike. ☐

4. CLAIM: $S_0^{ab}\omega_a\omega_b \geq 0$.

   We conclude that

   $$S_0^{ab}\omega_a\omega_b = \tilde{A}^{cd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} \nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k} e^{j_1 q_1\dots j_k q_k}{}_{i_1 p_1\dots i_l p_l},$$

   where $\tilde{A}^{cd} = (A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd})\omega_a\omega_b$. This expression is greater or equal to zero because $\tilde{A}^{cd}e^{j_1\dots j_k q_1\dots q_k}{}_{i_1\dots i_l p_1\dots p_l}$ is Riemannian by proposition 4.1.13. ☐

5. We know from part 1-4 that $\tilde{S}^{ab}\omega$ and $S_0^{ab}\omega_a$ have the same time orientation (more precisely the time orientation of $-\omega^b$) since they are non-spacelike and their scalar product with ω_a (timelike) is non-negative. Again by an argument similar to that one in the third part the tensor $S^{ab}\omega_a = S_0^{ab}\omega_a + \tilde{S}^{ab}\omega_a$ has the same orientation too and as a matter of fact $S^{ab}\omega_a$ must be non-spacelike.

This completes the proof and thus we know that S^{ab} satisfies the dominant energy condition. *q. e. d.*

Within the proof of proposition 5.2.1 we need a divergence theorem to estimate the energy tensor. Thus let us consider a situation with $h \in \mathcal{C}^2(M)$,

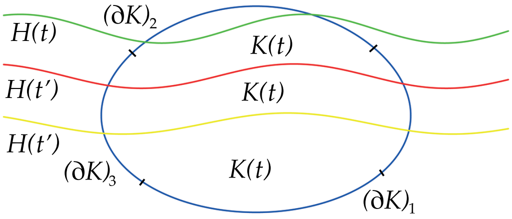

Figure 5.2: The set K

whose gradient with respect to g is everywhere A-timelike. Such a function does indeed exist provided that the space time M does not violate causality. Consider a connected compact set K, whose boundary consists of three parts. A part $(\partial K)_1$ where the normal form ν is non-spacelike with respect to A and $A(\nu, \nabla h) > 0$, a part $(\partial K)_2$ where the normal form is non-spacelike with respect to A and $A(\nu, \nabla h) < 0$, and a part $(\partial K)_3$ which may be empty. Here we have chosen ν such that $A(\nu, X)$ is positive for all vector fields X pointing out of K. Let $H(t')$ denote the surface $h(p) = t'$ and $K(t')$ the part of K for which $h(p) < t'$. Note that the derivative operator ∇ is taken with respect to g. For an illustration see figure 5.2.
For $t_0 < t$ such that $K \cap H(\tau) = \emptyset \, \forall \tau < t_0$ we now establish the inequality

$$\begin{aligned}\int\limits_{H(t)\cap K} S^{ab}\nabla_a h \, \mathrm{d}\sigma_b(g) \leq & - \int\limits_{K(t)\cap(\partial K)_1} S^{ab}\nabla_a h \, \mathrm{d}\sigma_b(g) \\ & + P \int\limits_{t_0}^{t} \int\limits_{H(t')\cap K} S^{ab}\nabla_a h \, \mathrm{d}\sigma_b(g) \, \mathrm{d}t' \\ & + \int\limits_{t_0}^{t} \int\limits_{H(t')\cap K} \nabla_a S^{ab} \, \mathrm{d}\sigma_b(g) \, \mathrm{d}t' \end{aligned} \tag{5.6}$$

(see [HE73], lemma 4.7) which holds for any symmetric (2,0) tensor field S vanishing on $(\partial K)_3$. As an application we will show that any symmetric and divergence free $(2,0)$-tensor field vanishes everywhere on K if it vanishes on $(\partial K)_3$ and the initial surface $(\partial K)_1$.

5.2.5. Proposition (Divergence Theorem): *Let K be as above then there exists some positive constant P such that for any energy tensor S satisfying the dominant energy condition 5.2.2 and vanishing on $(\partial K)_3$ inequality* (5.6) *holds, where $\mathrm{d}\sigma_a(g)$ denotes the surface element induced by g.*

Proof: We consider the volume integral ($\mathrm{d}v(g)$ is the volume element induced by g)

$$I(t) = \int\limits_{K(t)} \nabla_b(S^{ab}\nabla_a h)\,\mathrm{d}v(g) = \int\limits_{K(t)} S^{ab}\nabla_b\nabla_a h\,\mathrm{d}v(g) + \int\limits_{K(t)} \nabla_b S^{ab}\nabla_a h\,\mathrm{d}v(g). \tag{5.7}$$

We can transform (5.7) into a surface integral over $\partial K(t)$ by using Stokes' theorem. Hence

$$I(t) = \int\limits_{\partial K(t)} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g).$$

The boundary of $K(t)$ consists of $K(t) \cap \partial K$ and $K \cap H(t)$. Since S^{ab} is zero on $(\partial K)_3$ the integral is given by

$$\begin{aligned} I(t) = &\int\limits_{K(t)\cap(\partial K)_1} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g) + \int\limits_{K(t)\cap(\partial K)_2} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g) \\ &+ \int\limits_{K\cap H(t)} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g). \end{aligned}$$

By the dominant energy condition, definition 5.2.2, $S^{ab}\nabla_a h$ is a non-spacelike vector with respect to A and $S^{ab}\nabla_a h\nabla_b h \geq 0$. On (∂K_2) we have $\nu_a\nabla_b h A^{ab} < 0$. Since the scalar product of two non-spacelike vectors with opposite direction is non negative $S^{ab}\nabla_a h\nu_b \geq 0$ on (∂K_2) holds. We conclude that

$$\int\limits_{K(t)\cap(\partial K)_2} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g) = \int\limits_{K(t)\cap(\partial K)_2} S^{ab}\nabla_a h\nu_b\,\mathrm{d}\sigma(g) \geq 0.$$

Thus

$$\begin{aligned} \int\limits_{K\cap H(t)} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g) \leq &- \int\limits_{K(t)\cap(\partial K)_1} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g) \\ &+ \int\limits_{K(t)} (S^{ab}\nabla_b\nabla_a h + \nabla_b S^{ab}\nabla_a h)\,\mathrm{d}v(g). \end{aligned} \tag{5.8}$$

Since K is a compact set there exists an upper bound to the components of $\nabla_b\nabla_a h$ in any orthonormal basis whose timelike vector is in the direction of $\nabla_a h$. Hence by the dominant energy condition we have some $P > 0$ such that on K

$$S^{ab}\nabla_b\nabla_a h \leq PS^{ab}\nabla_a h\nabla_b h.$$

We now decompose the volume integral over $K(t)$ into a surface integral over $H(t') \cap K$ followed by an integral with respect to t':

$$\int\limits_{K(t)} (PS^{ab}\nabla_a h \nabla_b h + \nabla_b S^{ab} \nabla_a h)\,\mathrm{d}v(g) = \int\limits_{t_0}^{t} \int\limits_{H(t')\cap K} (PS^{ab}\nabla_b h + \nabla_b S^{ab})\,\mathrm{d}\sigma_a(g)\,\mathrm{d}t'. \qquad (5.9)$$

Combining (5.8) and (5.9) and since S^{ab} is symmetric, we have

$$\int\limits_{K\cap H(t)} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g) \leq - \int\limits_{K(t)\cap(\partial K)_1} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g) + P\int\limits_{t_0}^{t} \int\limits_{H(t')\cap K} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g)\,\mathrm{d}t' + \int\limits_{t_0}^{t} \int\limits_{H(t')\cap K} \nabla_a S^{ab}\,\mathrm{d}\sigma_b(g)\,\mathrm{d}t'$$

which proves the lemma stated above. *q. e. d.*

5.2.6. COROLLARY (The conservation theorem): *Let the tensor S (for S and K as in the lemma above) be zero on $(\partial K)_3$ as well as on the initial surface $(\partial K)_1$ and let its divergence be zero (i. e. $\nabla_a S^{ab} = 0$), then S is zero anywhere on K.*

Proof: Let

$$x(t) := \int\limits_{K(t)} S^{ab}\nabla_a h \nabla_b h\,\mathrm{d}v(g) = \int\limits_{t_0}^{t} \int\limits_{H(t')\cap K} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g)\,\mathrm{d}t' \geq 0. \qquad (5.10)$$

Hence

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \int\limits_{H(t)\cap K} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g). \qquad (5.11)$$

If we insert (5.10) and (5.11) into inequality (5.6), we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} \leq - \int\limits_{(\partial K)_1} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g) + Px + \int\limits_{t_0}^{t} \int\limits_{H(t')\cap K} \nabla_a S^{ab}\,\mathrm{d}\sigma_b(g)\,\mathrm{d}t'.$$

The first term on the right hand side vanishes because $S = 0$ on $(\partial K)_1$ and the last term vanishes because $\nabla_a S^{ab} = 0$. Thus $\frac{\mathrm{d}x}{\mathrm{d}t} \leq Px$ and as a consequence

$$x(t) \leq P \int\limits_{t_0}^{t} x(s)\,\mathrm{d}s.$$

By Gronwall's lemma (see corollary 6.2 in [Ama83]) we have

$$x(t) \leq 0$$

which proves the assertion since S satisfies the dominant energy condition

q. e. d.

For the remaining part of this chapter we shall omit the tensor order when denoting Sobolev norms, e.g. $\mathcal{H}^m(N) = \mathcal{H}^m(l,k,N)$.

Proof of proposition 5.2.1: To begin with, we form the energy tensor S (see equation (5.4)). By applying lemma 5.2.3 we know that S obeys the dominant energy condition with respect to the metric A. We can easily see that there exists a constant Q_4 such that

$$\begin{aligned} S^{ab}\nabla_a h \nabla_b h \leq & |A|^2|\mathrm{D}h|^2|\mathrm{D}T|^2 + |A||\mathrm{D}h|^2|T|^2 \\ \leq & Q_4(|T|^2 + |\mathrm{D}T|^2) \end{aligned}$$

holds on $\overline{U}_+$, since $|\mathrm{D}h|$ is bounded on compact sets and since A is bounded by assumption. Moreover we have that

$$\tilde{A}^{cd} := \left(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd}\right)\nabla_a h \nabla_b h$$

is a Riemannian metric by proposition 4.1.13. Now it follows that

$$S^{ab}\nabla_a h \nabla_b h \geq \tilde{A}^{cd}\nabla_c T^{i_1\ldots i_l}{}_{j_1\ldots j_k}\nabla_d T^{p_1\ldots p_l}{}_{q_1\ldots q_k} e^{j_1 q_1\ldots j_k q_k}{}_{i_1 p_1\ldots i_l p_l} + \frac{Q_1}{2}|T|^2$$

by assumption (2′). Furthermore,

$$S^{ab}\nabla_a h \nabla_b h \geq C_1|\mathrm{D}T|^2 + \frac{Q_1}{2}|T|^2$$

since on $\overline{U}_+$ we have $C_1 e(\xi,\xi) \leq \tilde{A}(\xi,\xi) \leq C_2 e(\xi,\xi)$ for any positive constants C_1, C_2 and any vector field ξ. Summing up we have that

$$Q_4'(|T|^2 + |\mathrm{D}T|^2) \leq S^{ab}\nabla_a h \nabla_b h \leq Q_4(|T|^2 + |\mathrm{D}T|^2). \qquad (5.12)$$

We are now ready to apply lemma 5.2.5, taking $\overline{U}_+$ as the compact region K and using the volume element resp. the surface element induced by g as well as the covariant derivative with respect to g. Thus

$$\begin{aligned}\int\limits_{H(t)\cap\overline{U}_+} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g) \leq{}& -\int\limits_{H(0)\cap\overline{U}_+} S^{ab}\nabla_a h\,\mathrm{d}\sigma_b(g)\\ &+\int\limits_0^t\int\limits_{H(t')\cap\overline{U}_+}(PS^{ab}\nabla_a h+\nabla_a S^{ab})\,\mathrm{d}\sigma_b(g)\,\mathrm{d}t',\end{aligned}\tag{5.13}$$

where P is a positive constant. To exploit (5.13) we rewrite the volume on $H(t)$ resp. on $H(0)$ in order to produce a term proportional to $\nabla_a h$. Note first that $\nabla_a h$ is parallel to ν_a on $H(t)$ but antiparallel to ν_a on $H(0)$, since in lemma 5.2.5 we have used the outward pointing unit vector field. Hence we write $\nabla h = f\nu_a$ with $0 < f \in \mathcal{C}^\infty(H(t))$ resp. $0 > f \in \mathcal{C}^\infty(H(0))$. Now we introduce

$$\mathrm{d}\tilde{\sigma} := \frac{1}{|f|}\,\mathrm{d}\sigma.$$

So we obtain

$$\begin{aligned}\int\limits_{H(t)\cap\overline{U}_+} S^{ab}\nabla_a h\nabla_b h\,\mathrm{d}\tilde{\sigma}(g) \leq{}& \int\limits_{H(0)\cap\overline{U}_+} S^{ab}\nabla_a h\nabla_b h\,\mathrm{d}\tilde{\sigma}(g)\\ &+\int\limits_0^t\int\limits_{H(t')\cap\overline{U}_+}(PS^{ab}\nabla_a h+\nabla_a S^{ab})\nabla_b h\,\mathrm{d}\tilde{\sigma}(g)\,\mathrm{d}t',\end{aligned}\tag{5.14}$$

Furthermore, we have that e and g are continuous, thus there exist positive constants Q_5 and Q_6 such that on $\overline{U}_+$

$$Q_5\,\mathrm{d}\sigma(e) \leq \mathrm{d}\tilde{\sigma}(g) \leq Q_6\,\mathrm{d}\sigma(e) \tag{5.15}$$

holds. Then by inserting (5.12) into (5.14) and using $\mathrm{d}\sigma(e)$ instead of $\mathrm{d}\tilde{\sigma}(g)$, we have, since S^{ab} is symmetric, some Q_7 such that

$$\begin{aligned}\|T\|^2_{\mathcal{H}^1(H(t)\prec U_+)} \leq{}& Q_7\Bigg\{\|T\|^2_{\mathcal{H}^1(H(0)\prec U_+)}+\int\limits_0^t\|T\|^2_{\mathcal{H}^1(H(t)\prec U_+)}\,\mathrm{d}t'\\ &+\int\limits_0^t\int\limits_{H(t')\cap\overline{U}_+}\nabla_b S^{ab}\nabla_a h\,\mathrm{d}\sigma(e)\,\mathrm{d}t'\Bigg\}.\end{aligned}\tag{5.16}$$

Now we would like to obtain an estimate for the divergence term $\nabla_a S^{ab}$. Therefore we calculate

$$\begin{align}
\nabla_a S^{ab} =& \nabla_a \Bigg\{ \left(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd} \right) \nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} \nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k} \nonumber\\
& - \frac{1}{2}A^{ab} T^{i_1\dots i_l}{}_{j_1\dots j_k} T^{p_1\dots p_l}{}_{q_1\dots q_k} \Bigg\} e^{j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l} \tag{5.17a}\\
& + \Bigg\{ \left(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd} \right) \nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} \nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k} \nonumber\\
& - \frac{1}{2}A^{ab} T^{i_1\dots i_l}{}_{j_1\dots j_k} T^{p_1\dots p_l}{}_{q_1\dots q_k} \Bigg\} \nabla_a e^{j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l}. \tag{5.17b}
\end{align}$$

Since we aim at an estimate in $\|\cdot\|_{\mathcal{H}^1}$, we are not interested to improve expressions containing T and ∇T. Moreover, A and the derivative of e are essentially bounded on $\overline{U}_+$. As a consequence we may skip further evaluation of line (5.17b) and only deal with line (5.17a). Our goal is then to get rid of $\mathrm{D}^2 T$ by using the partial differential equation (5.1). We have that

$$\begin{align}
\text{(5.17a)} =& \Bigg\{ \nabla_a \left(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd} \right) \nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} \nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k} \tag{5.18a}\\
& + \left(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd} \right) \nabla_a (\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} \nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}) \tag{5.18b}\\
& - \frac{1}{2} \nabla_a A^{ab} T^{i_1\dots i_l}{}_{j_1\dots j_k} T^{p_1\dots p_l}{}_{q_1\dots q_k} \tag{5.18c}\\
& - \frac{1}{2} A^{ab} \nabla_a (T^{i_1\dots i_l}{}_{j_1\dots j_k} T^{p_1\dots p_l}{}_{q_1\dots q_k}) \Bigg\} \tag{5.18d}\\
& e^{j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l}. \tag{5.18e}
\end{align}$$

Again we just need to consider line (5.18b), since it is the only term with second order derivatives of T. All expressions containing ∇A are bounded almost everywhere on $\overline{U}_+$.

$$\begin{aligned}
\text{(5.18b)} =& (A^{ac}A^{bd} \nabla_a \nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} \nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k} \\
& + A^{ac}A^{bd} \nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} \nabla_a \nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}) e^{j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l} \\
& - \frac{1}{2} (A^{ab}A^{cd} \nabla_a \nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} \nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k} \\
& + A^{ab}A^{cd} \nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} \nabla_a \nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}) e^{j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l}.
\end{aligned}$$

Hence by the symmetry of A^{cd} we obtain

$$\begin{aligned}
(5.18\text{b}) =&(A^{ac}A^{bd}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&+A^{ac}A^{bd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_a\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k})e^{j_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}\\
&-\frac{1}{2}(A^{ab}A^{cd}\nabla_c T^{p_1\dots p_l}{}_{q_1\dots q_k}\nabla_a\nabla_d T^{i_1\dots i_l}{}_{j_1\dots j_k}\\
&+A^{ab}A^{cd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_a\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k})e^{j_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}.
\end{aligned}$$

Furthermore we know that $e^{j_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}$ is pairwise symmetric in j_r, q_r, $(1\le r\le k)$ resp. $i_s, p_s, (1\le s\le l)$, such that we have

$$\begin{aligned}
(5.18\text{b}) =&(A^{ac}A^{bd}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&+A^{ac}A^{bd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_a\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&-A^{ab}A^{cd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_a\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k})e^{j_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}\\
=&(A^{ac}A^{bd}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&+A^{ac}A^{bd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_a\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&-A^{ac}A^{bd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d\nabla_a T^{p_1\dots p_l}{}_{q_1\dots q_k})e^{j_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}
\end{aligned}$$

and by Ricci's identity

$$\begin{aligned}
[\nabla_d,\nabla_a]T^{p_1\dots p_l}{}_{q_1\dots q_k} =&\sum_{r=1}^{k}T^{p_1\dots p_l}{}_{q_1\dots q_{r-1}uq_{r+1}\dots q_k}R^u{}_{q_rda}\\
&-\sum_{s=1}^{l}T^{p_1\dots p_{s-1}up_{s+1}\dots p_l}{}_{q_1\dots q_k}R^{p_s}{}_{uda}
\end{aligned}$$

we obtain

$$\begin{aligned}
A^{ac}&A^{bd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d\nabla_a T^{p_1\dots p_l}{}_{q_1\dots q_k}e^{j_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}\\
=&A^{ac}A^{bd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_a\nabla_d T^{p_1\dots p_l}{}_{q_1\dots q_k}e^{j_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}\\
&+A^{ac}A^{bd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\sum_{r=1}^{k}T^{p_1\dots p_l}{}_{q_1\dots q_{r-1}uq_{r+1}\dots q_k}R^u{}_{q_rda}e^{j_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}\\
&-A^{ac}A^{bd}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\sum_{s=1}^{l}T^{p_1\dots p_{s-1}up_{s+1}\dots p_l}{}_{q_1\dots q_k}R^{p_s}{}_{uda}e^{j_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l},
\end{aligned}$$

where R denotes the Riemannian curvature tensor. Finally, this yields

$$\begin{aligned}
(5.18\text{b}) = & A^{bd} e^{j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l} \Big(A^{ac} \nabla_a \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_d T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\
& - A^{ac} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \sum_{r=1}^{k} T^{p_1 \dots p_l}{}_{q_1 \dots q_{r-1} u q_{r+1} \dots q_k} R^{u}{}_{q_r da} \\
& + A^{ac} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \sum_{s=1}^{l} T^{p_1 \dots p_{s-1} u p_{s+1} \dots p_l}{}_{q_1 \dots q_k} R^{p_s}{}_{uda} \Big).
\end{aligned}$$

By the partial differential equation (5.1) we have

$$\begin{aligned}
A^{ac} \nabla_a \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k} = & - B^{a \bar{j}_1 \dots \bar{j}_k i_1 \dots i_l}{}_{\bar{i}_1 \dots \bar{i}_l j_1 \dots j_k} \nabla_a T^{\bar{i}_1 \dots \bar{i}_l}{}_{\bar{j}_1 \dots \bar{j}_k} \\
& - C^{\bar{j}_1 \dots \bar{j}_k i_1 \dots i_l}{}_{\bar{i}_1 \dots \bar{i}_l j_1 \dots j_k} T^{\bar{i}_1 \dots \bar{i}_l}{}_{\bar{j}_1 \dots \bar{j}_k} + F^{i_1 \dots i_l}{}_{j_1 \dots j_k}.
\end{aligned} \tag{5.19}$$

We substitute (5.19) into line (5.18b) and obtain

$$\begin{aligned}
(5.18\text{b}) = & e^{j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l} \nabla_d T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\
& \Big(- A^{bd} B^{a \bar{j}_1 \dots \bar{j}_k i_1 \dots i_l}{}_{\bar{i}_1 \dots \bar{i}_l j_1 \dots j_k} \nabla_a T^{\bar{i}_1 \dots \bar{i}_l}{}_{\bar{j}_1 \dots \bar{j}_k} \\
& - A^{bd} C^{\bar{j}_1 \dots \bar{j}_k i_1 \dots i_l}{}_{\bar{i}_1 \dots \bar{i}_l j_1 \dots j_k} T^{\bar{i}_1 \dots \bar{i}_l}{}_{\bar{j}_1 \dots \bar{j}_k} + A^{bd} F^{i_1 \dots i_l}{}_{j_1 \dots j_k} \\
& - A^{bd} A^{ac} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \sum_{r=1}^{k} T^{p_1 \dots p_l}{}_{q_1 \dots q_{r-1} u q_{r+1} \dots q_k} R^{u}{}_{q_r da} \\
& + A^{bd} A^{ac} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \sum_{s=1}^{l} T^{p_1 \dots p_{s-1} u p_{s+1} \dots p_l}{}_{q_1 \dots q_k} R^{p_s}{}_{uda} \Big).
\end{aligned}$$

The coefficients A, B, C in line (5.18b) are all essentially bounded on $\overline{U}_+$ by assumption (2″) in proposition 5.2.1. Furthermore, the curvature tensor R consisting of first and second order derivatives of g is essentially bounded on $\overline{U}_+$ since g is $\mathcal{C}^{2-}$ and ∇h is bounded on $\overline{U}_+$ since h is $\mathcal{C}^2$ on a neighbourhood of $\overline{U}$.

Therefore we have some Q_8 such that

$$\nabla_b S^{ab} \nabla_a h \leq Q_8 (|F|^2 + |T|^2 + |\mathrm{D}T|^2) \quad \text{on } \overline{U}_+. \tag{5.20}$$

Recall that the expression $|ab|$ can always be estimated by $\frac{1}{2}(a^2 + b^2)$. Using estimate (5.20) we see that in (5.16) we can absorb everything but the term involving F into $\int_0^t \|T\|^2_{\mathcal{H}^1(H(t')\prec U_+)}\,\mathrm{d}t'$, hence we obtain

$$\begin{aligned}\|T\|^2_{\mathcal{H}^1(H(t)\prec U_+)} \leq & Q_9\Bigg(\|T\|^2_{\mathcal{H}^1(H(0)\prec U_+)} \\ & + \int\limits_0^t \|T\|^2_{\mathcal{H}^1(H(t')\prec U_+)}\,\mathrm{d}t' + \|F\|^2_{\mathrm{L}^2(U(t))}\Bigg)\end{aligned} \tag{5.21}$$

with a positive constant Q_9. We now define

$$x(t) := \|T\|^2_{\mathcal{H}^1(H(t)\prec U_+)}.$$

Thus

$$\begin{aligned} x(t) &\overset{(5.21)}{\leq} Q_9\Bigg(\|T\|^2_{\mathcal{H}^1(H(0)\prec U_+)} + \int\limits_0^t x(t')\,\mathrm{d}t' + \|F\|^2_{\mathrm{L}^2(U(t))}\Bigg) \\ &\leq Q_9\Bigg\{y(t) + \int\limits_0^t x(t')\,\mathrm{d}t'\Bigg\}, \end{aligned}$$

where

$$y(t) = \|T\|^2_{\mathcal{H}^1(H(0)\prec U_+)} + \|F\|^2_{\mathrm{L}^2(U(t))}. \tag{5.22}$$

By Gronwall's lemma (see e. g. corollary 6.2 in [Ama83]) we obtain

$$x(t) \leq Q_9\, y(t)\, e^{\int_0^t Q_9\,\mathrm{d}t'}.$$

Then for all $t \leq t_1$ it follows that $x(t) \leq Q_{10} y(t)$, i. e.

$$\begin{aligned}\|T\|^2_{\mathcal{H}^1(H(t)\prec U_+)} \leq & Q_{10}\{\|T\|^2_{\mathcal{H}^1(H(0)\prec U_+)} + \|F\|^2_{\mathrm{L}^2(U(t))}\} \\ \leq & Q_{10}\{\|T\|^2_{\mathcal{H}^1(H(0)\prec U_+)} + \|F\|^2_{\mathrm{L}^2(U(t))} \\ & + 2\|T\|_{\mathcal{H}^1(H(0)\prec U_+)}\|F\|_{\mathrm{L}^2(U(t))}\}.\end{aligned}$$

Extracting the root of the last inequality (both sides are positive) yields some P_4 such that

$$\|T\|_{\mathcal{H}^1(H(t)\prec U_+)} \leq P_4\{\|T\|_{\mathcal{H}^1(H(0)\prec U_+)} + \|F\|_{\mathrm{L}^2(U(t))}\}, \tag{5.23}$$

thereby concluding the proof of proposition 5.2.1. *q. e. d.*

5.2.7. THEOREM (Uniqueness of solutions): *Let A be a $\mathcal{C}^{1-}$ Lorentz metric on M and let B, C, and F be locally bounded. Furthermore, let $H \subseteq M$ be* *a spacelike, acausal hypersurface with respect to A. Then if $V \subseteq D^+(H, A)$, the solution on V of the linear equation* (5.1) *is uniquely determined by its values and the values of its first derivatives on $H \cap J^-(V, A)$.*

Proof: Suppose T_1 and T_2 were solutions of (5.1), i. e. $L(T_i) = F$ ($i \in \{1, 2\}$), with the same initial values and first derivatives on $H(0) \cap U$. Since the differential equation is linear we have $L(T_1 - T_2) = 0$ and by inequality (5.3) we obtain

$$\|T_1 - T_2\|_{\mathcal{H}^1(H(t) \prec U_+)} \leq P_4 \|T_1 - T_2\|_{\mathcal{H}^1(H(0) \prec U_+)}.$$

Now $\|T_1 - T_2\|_{\mathcal{H}^1(H(0) \prec U_+)} = 0$ by assumption and the left hand side is non-negative. Therefore $T_1 = T_2$ on U_+ almost everywhere. By proposition 6.6.7 in [HE73] $D^+(H, A)$ is of the form $H \times \mathbb{R}$. For $q \in V$ we obtain by proposition 6.6.6 in [HE73] that $J^-(q) \cap J^+(H)$ is compact and thus may be taken for $\overline{U}_+$. *q. e. d.*

5.2.2 Higher Order Energy Estimates

What we have left out until now are the energy estimates of higher order that we need to prove the main theorem. Those shall be covered in this section.

5.2.8. Proposition: *Let M, U, $H(t)$, and h be defined as in theorem 5.1.3 such that $H(0) \cap \overline{U}$ has a smooth boundary. Let g be a Lorentzian $C^{5+\alpha}$ metric on U_+, where $\alpha \in \mathbb{N}_0$, and let e be a Riemannian C^{1-} metric. Then if conditions (1) and (2') of proposition 5.2.1 hold and additionally,*

(3) there is some Q_3 such that

$$\|A\|_{\mathcal{H}^{4+\alpha}(U_+)} \leq Q_3 \quad \|B\|_{\mathcal{H}^{3+\alpha}(U_+)} \leq Q_3 \quad \|C\|_{\mathcal{H}^{3+\alpha}(U_+)} \leq Q_3,$$

where A, B, and C are tensor fields in an appropriate space on U_+, we have

$$\|T\|_{\mathcal{H}^{4+\alpha}(H(t) \prec U_+)} \leq P_{5,\alpha} \{ \|T\|_{\mathcal{H}^{4+\alpha}(H(t) \prec U_+)} + \|F\|_{\mathcal{H}^{3+\alpha}(U(t))} \} \tag{5.24}$$

for any solution T of (5.1).

Proof: For now let $\alpha = 0$. In proposition 5.2.1 we proved an estimate for $\|T, H(0) \cap U_+\|_1$. Now we will prove an analogous result for the next higher differentiation order of T. Therefore we form the energy tensor for ∇T, i.e.

$$\begin{aligned} S_1^{ab} := \Big\{ & \Big(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd} \Big) \nabla_c \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_d \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\ & - \frac{1}{2} A^{ab} \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \Big\} e^{e_1 f_1 j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l}. \end{aligned}$$

One can see that we aim at a similar result to equation (5.12). From a longer calculation, which can be found in appendix A.1, it follows that

$$\begin{aligned} \nabla_a S_1^{ab} = \Big\{ & \nabla_a \nabla_c \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_d \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\ & - \sum_{r=1}^{k} \nabla_{f_1} T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^u{}_{j_r da} \nabla_c \nabla_{e_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\ & - \nabla_u T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^u{}_{f_1 da} \nabla_c \nabla_{e_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\ & + \sum_{r=1}^{l} \nabla_{f_1} T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_r}{}_{uda} \nabla_c \nabla_{e_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \Big\} \\ & A^{ac} A^{bd} e^{e_1 f_1 j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l} + \tilde{S}_1^b, \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} \tilde{S}_1^b = \Big\{ & \nabla_a \Big(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd} \Big) \nabla_c \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_d \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\ & - \frac{1}{2} \nabla_a (A^{ab} \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k}) \Big\} e^{e_1 f_1 j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l} \\ + \Big\{ & \Big(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd} \Big) \nabla_c \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_d \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\ & - \frac{1}{2} A^{ab} \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \Big\} \nabla_a e^{e_1 f_1 j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l}. \end{aligned}$$

Our goal is to remove third order differentials of T in $\nabla_a S_1^{ab}$ by using the partial differential equation (5.1). Therefore we interchange the highest order derivatives in (5.25), which yields

$$\begin{aligned} \nabla_a \nabla_c \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} = {} & \nabla_{e_1} \nabla_a \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k} - \sum_{r=1}^{k} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^u{}_{j_r e_1 a} \\ & - \nabla_u T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^u{}_{c e_1 a} + \sum_{r=1}^{l} \nabla_c T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_r}{}_{u e_1 a} \\ & - \sum_{r=1}^{k} \nabla_a (T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^u{}_{j_r e_1 c}) + \sum_{r=1}^{l} \nabla_a (T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_r}{}_{u e_1 c}). \end{aligned} \tag{5.26}$$

Calculating the derivative of equation (5.1) gives

$$\begin{aligned}
A^{ac}\nabla_{e_1}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} =&\nabla_{e_1}F^{i_1\dots i_l}{}_{j_1\dots j_k} - \nabla_{e_1}A^{ac}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\\
&- B^{a\bar{\jmath}_1\dots\bar{\jmath}_k i_1\dots i_l}{}_{\bar{\imath}_1\dots\bar{\imath}_l j_1\dots j_k}\nabla_{e_1}\nabla_a T^{\bar{\imath}_1\dots\bar{\imath}_l}{}_{\bar{\jmath}_1\dots\bar{\jmath}_k}\\
&- \nabla_{e_1}B^{a\bar{\jmath}_1\dots\bar{\jmath}_k i_1\dots i_l}{}_{\bar{\imath}_1\dots\bar{\imath}_l j_1\dots j_k}\nabla_a T^{\bar{\imath}_1\dots\bar{\imath}_l}{}_{\bar{\jmath}_1\dots\bar{\jmath}_k}\\
&- C^{\bar{\jmath}_1\dots\bar{\jmath}_k i_1\dots i_l}{}_{\bar{\imath}_1\dots\bar{\imath}_l j_1\dots j_k}\nabla_{e_1} T^{\bar{\imath}_1\dots\bar{\imath}_l}{}_{\bar{\jmath}_1\dots\bar{\jmath}_k}\\
&- \nabla_{e_1}C^{\bar{\jmath}_1\dots\bar{\jmath}_k i_1\dots i_l}{}_{\bar{\imath}_1\dots\bar{\imath}_l j_1\dots j_k} T^{\bar{\imath}_1\dots\bar{\imath}_l}{}_{\bar{\jmath}_1\dots\bar{\jmath}_k}.
\end{aligned} \tag{5.27}$$

On combining equations (5.25), (5.27), and (5.26), $\nabla_a S_1^{ab}$ reads

$$\begin{aligned}
\nabla_a S_1^{ab} =&\Bigg\{\nabla_d\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\Bigg(\nabla_{e_1}F^{i_1\dots i_l}{}_{j_1\dots j_k} - \nabla_{e_1}A^{ac}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\\
&- B^{a\bar{\jmath}_1\dots\bar{\jmath}_k i_1\dots i_l}{}_{\bar{\imath}_1\dots\bar{\imath}_l j_1\dots j_k}\nabla_{e_1}\nabla_a T^{\bar{\imath}_1\dots\bar{\imath}_l}{}_{\bar{\jmath}_1\dots\bar{\jmath}_k}\\
&- \nabla_{e_1}B^{a\bar{\jmath}_1\dots\bar{\jmath}_k i_1\dots i_l}{}_{\bar{\imath}_1\dots\bar{\imath}_l j_1\dots j_k}\nabla_a T^{\bar{\imath}_1\dots\bar{\imath}_l}{}_{\bar{\jmath}_1\dots\bar{\jmath}_k} && \text{(i)}\\
&- C^{\bar{\jmath}_1\dots\bar{\jmath}_k i_1\dots i_l}{}_{\bar{\imath}_1\dots\bar{\imath}_l j_1\dots j_k}\nabla_{e_1} T^{\bar{\imath}_1\dots\bar{\imath}_l}{}_{\bar{\jmath}_1\dots\bar{\jmath}_k}\\
&- \nabla_{e_1}C^{\bar{\jmath}_1\dots\bar{\jmath}_k i_1\dots i_l}{}_{\bar{\imath}_1\dots\bar{\imath}_l j_1\dots j_k} T^{\bar{\imath}_1\dots\bar{\imath}_l}{}_{\bar{\jmath}_1\dots\bar{\jmath}_k}\Bigg) && \text{(ii)}\\
&+ A^{ac}\Bigg(-\sum_{r=1}^{k}\nabla_{f_1}T^{i_1\dots i_l}{}_{j_1\dots u\dots j_k}R^u{}_{j_r da}\nabla_c\nabla_{e_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&- \nabla_u T^{i_1\dots i_l}{}_{j_1\dots j_k}R^u{}_{f_1 da}\nabla_c\nabla_{e_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&+ \sum_{r=1}^{l}\nabla_{f_1}T^{i_1\dots u\dots i_l}{}_{j_1\dots j_k}R^{i_r}{}_{uda}\nabla_c\nabla_{e_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\Bigg)\\
&+ A^{ac}\nabla_d\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\Bigg(-\sum_{r=1}^{k}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots u\dots j_k}R^u{}_{j_r e_1 a}\\
&- \nabla_u T^{i_1\dots i_l}{}_{j_1\dots j_k}R^u{}_{c e_1 a}\\
&+ \sum_{r=1}^{l}\nabla_c T^{i_1\dots u\dots i_l}{}_{j_1\dots j_k}R^{i_r}{}_{u e_1 a} - \sum_{r=1}^{k}\nabla_a(T^{i_1\dots i_l}{}_{j_1\dots u\dots j_k}R^u{}_{j_r e_1 c})\\
&+ \sum_{r=1}^{l}\nabla_a(T^{i_1\dots u\dots i_l}{}_{j_1\dots j_k}R^{i_r}{}_{u e_1 c})\Bigg)\Bigg\}\\
&A^{bd}e^{e_1 f_1 j_1 q_1\dots j_k q_k}{}_{i_1 p_1\dots i_l p_l} + \tilde{S}_1^b.
\end{aligned} \tag{5.28}$$

By assumption (3) of proposition 5.2.8 and corollary 3.2.5 we have that A, ∇A, B, and C are bounded in L^2 on U_+ as well as R, ∇R, e, and ∇e. Thus we have to look more closely at expressions involving ∇B and ∇C. We define by $I(\nabla B)$ the integral of (i) over $H(t') \cap \overline{U}_+$. Now

$$|I(\nabla B)| \leq \int\limits_{H(t')\cap\overline{U}_+} |A^{bd} \nabla_{e_1} B^{a\bar{\jmath}_1\ldots\bar{\jmath}_k i_1\ldots i_l}{}_{\bar{\imath}_1\ldots\bar{\imath}_l j_1\ldots j_k} \nabla_a T^{\bar{\imath}_1\ldots\bar{\imath}_l}{}_{\bar{\jmath}_1\ldots\bar{\jmath}_k} \nabla_d \nabla_{f_1} T^{p_1\ldots p_l}{}_{q_1\ldots q_k} e^{e_1 f_1 j_1 q_1\ldots j_k q_k}{}_{i_1 p_1\ldots i_l p_l}| \, \mathrm{d}\sigma(e).$$

Estimating the bounded expressions A and e it is apperent that there exists a constant Q_4, such that

$$\begin{aligned} |I(\nabla B)| \leq & Q_4 \int\limits_{H(t')\cap\overline{U}_+} |\mathrm{D}B||\mathrm{D}T||\mathrm{D}^2 T| \, \mathrm{d}\sigma(e) \\ \leq & \frac{Q_4}{2} \int\limits_{H(t')\cap\overline{U}_+} (|\mathrm{D}^2 T|^2 + |\mathrm{D}B|^2 |\mathrm{D}T|^2) \, \mathrm{d}\sigma(e). \end{aligned}$$

Furthermore, we have, setting $s = 0$ and $t = u = 1$ in corollary 3.2.10(i),

$$\begin{aligned} \int\limits_{H(t')\cap\overline{U}_+} |\mathrm{D}B|^2 |\mathrm{D}T|^2 \, \mathrm{d}\sigma(e) = & \|\mathrm{D}B \otimes \mathrm{D}T\|^2_{\mathrm{L}^2(H(t')\cap U_+)} \\ \leq & \tilde{P}_2^2 \, \|\mathrm{D}B\|^2_{\mathcal{H}^1(H(t')\prec U_+)} \, \|\mathrm{D}T\|^2_{\mathcal{H}^1(H(t')\prec U_+)} \\ \leq & \tilde{P}_2^2 \, \|B\|^2_{\mathcal{H}^2(H(t')\prec U_+)} \, \|T\|^2_{\mathcal{H}^2(H(t')\prec U_+)} \, . \end{aligned}$$

Corollary 3.2.12 and assumption (3) entail

$$\|B\|_{\mathcal{H}^2(H(t')\prec U_+)} \leq \tilde{P}_3 \, \|B\|_{\mathcal{H}^3(U_+)} \leq \tilde{P}_3 Q_3. \tag{5.29}$$

Now we perform the same calculation for (ii). Thus

$$|I(\nabla C)| \leq \int\limits_{H(t')\cap\overline{U}_+} |A^{bd} \nabla_{e_1} C^{\bar{\jmath}_1\ldots\bar{\jmath}_k i_1\ldots i_l}{}_{\bar{\imath}_1\ldots\bar{\imath}_l j_1\ldots j_k} T^{\bar{\imath}_1\ldots\bar{\imath}_l}{}_{\bar{\jmath}_1\ldots\bar{\jmath}_k} \nabla_d \nabla_{f_1} T^{p_1\ldots p_l}{}_{q_1\ldots q_k} e^{e_1 f_1 j_1 q_1\ldots j_k q_k}{}_{i_1 p_1\ldots i_l p_l}| \, \mathrm{d}\sigma(e). \tag{5.30}$$

For a constant Q_4' the expression $|I(\nabla C)|$ is less or equal to

$$\begin{aligned} |I(\nabla C)| \leq & Q_4' \int\limits_{H(t')\cap\overline{U}_+} |\mathrm{D}C||T||\mathrm{D}^2 T| \, \mathrm{d}\sigma(e) \\ \leq & \frac{Q_4'}{2} \int\limits_{H(t')\cap\overline{U}_+} (|\mathrm{D}^2 T|^2 + |\mathrm{D}C|^2 |T|^2) \, \mathrm{d}\sigma(e) \end{aligned}$$

and by corollary 3.2.10(i)

$$\begin{aligned}\int\limits_{H(t')\cap\overline{U}_+} |\mathrm{D}C|^2|T|^2\,\mathrm{d}\sigma(e) &= \left\|\mathrm{D}C\otimes T, H(t')\cap U_+\right\|_0^2\\ &\leq \tilde{P}'^2_2\,\|\mathrm{D}C\|^2_{\mathcal{H}^1(H(t')\prec U_+)}\,\|\mathrm{T}\|^2_{\mathcal{H}^1(H(t')\prec U_+)}\\ &\leq \tilde{P}'^2_2\,\|C\|^2_{\mathcal{H}^2(H(t')\prec U_+)}\,\|T\|^2_{\mathcal{H}^2(H(t')\prec U_+)}\,.\end{aligned}$$

Hence by application of corollary 3.2.12 under assertion (3)

$$\|C\|_{\mathcal{H}^2(H(t')\prec U_+)} \leq \tilde{P}'_3\,\|C\|_{\mathcal{H}^3(U_+)} \leq \tilde{P}'_3 Q_3. \tag{5.31}$$

follows. Finally we obtained bounds for ∇B and ∇C.

We now continue formulating (5.12) for S_1, i.e. for constants Q_4' and Q_4

$$Q_4'(|\mathrm{D}^2T|^2+|\mathrm{D}T|^2) \leq S_1^{ab}\nabla_a h\nabla_b h \leq Q_4(|\mathrm{D}^2T|^2+|\mathrm{D}T|^2). \tag{5.32}$$

This last estimate is in fact true since $S(T)$ was defined for arbitrary tensor fields T of any order. Thus $S_m(T) = S(T')$ where $T' := \nabla^m T$ for all $m\in\mathbb{N}$. Hence statements previously made for S are also true for S_m. Application of lemma 5.2.5 on (5.32) yields

$$\begin{aligned}\int\limits_{H(t)\cap\overline{U}_+} (|\mathrm{D}^2T|^2+|\mathrm{D}T|^2)\,\mathrm{d}\sigma(e) \leq{}& Q_5'\Bigg(\int\limits_{H(0)\cap\overline{U}_+} (|\mathrm{D}^2T|^2+|\mathrm{D}T|^2)\,\mathrm{d}\sigma(e)\\ &+\int\limits_0^t\int\limits_{H(t')\cap\overline{U}_+} (|\mathrm{D}^2T|^2+|\mathrm{D}T|^2)\,\mathrm{d}\sigma(e)\,\mathrm{d}t'\\ &+\int\limits_0^t\int\limits_{H(t')\cap\overline{U}_+} \nabla_b S_1^{ab}\nabla_a h\,\mathrm{d}\sigma(e)\,\mathrm{d}t'\Bigg).\end{aligned} \tag{5.33}$$

From what has been previously said about $\nabla_a S_1^{ab}$ (boundedness of coefficients, inequalities (5.29), (5.31), symmetry of S_1), one has

$$\begin{aligned}\int\limits_0^t\int\limits_{H(t')\cap\overline{U}_+} \nabla_b S_1^{ab}\nabla_a h\,\mathrm{d}\sigma(e)\,\mathrm{d}t' \leq{}& \tilde{Q}_5\Bigg\{\int\limits_0^t\Bigg(\|T\|^2_{\mathcal{H}^2(H(t')\prec U_+)}\\ &+\int\limits_{H(t')\cap\overline{U}_+} |\mathrm{D}F|^2\,\mathrm{d}\sigma(e)\Bigg)\,\mathrm{d}t'\Bigg\}.\end{aligned}$$

Together with (5.33) we have

$$\int\limits_{H(t)\cap\overline{U}_+} (|\mathrm{D}^2T|^2 + |\mathrm{D}T|^2)\,\mathrm{d}\sigma(e) \leq Q_5\Big\{ \int\limits_{H(0)\cap\overline{U}_+} (|\mathrm{D}^2T|^2 + |\mathrm{D}T|^2)\,\mathrm{d}\sigma(e) + \int\limits_0^t \|T\|^2_{\mathcal{H}^2(H(t')\prec U_+)}\,\mathrm{d}t' + \int\limits_{U(t)} |\mathrm{D}F|^2\,\mathrm{d}v(e)\Big). \tag{5.34}$$

Since $\int_{H(t)\cap\overline{U}_+} |T|^2\,\mathrm{d}\sigma(e) = \|T\|^2_{\mathrm{L}^2(H(t)\cap U_+)}$ we have

$$\int\limits_{H(t)\cap\overline{U}_+} |T|^2\,\mathrm{d}\sigma(e) \leq \|T\|^2_{\mathcal{H}^1(H(t)\prec U_+)}$$

and by proposition 5.2.1

$$\int\limits_{H(t)\cap\overline{U}_+} |T|^2\,\mathrm{d}\sigma(e) \leq 2P_4^2\{\|T\|^2_{\mathcal{H}^1(H(0)\prec U_+)} + \|F\|^2_{\mathrm{L}^2(U(t))}\}.$$

Adding the last estimate to inequality (5.34), we obtain

$$\|T\|^2_{\mathcal{H}^2(H(t)\prec U_+)} \leq Q_6\Big(\|T\|^2_{\mathcal{H}^2(H(0)\prec U_+)} + \int\limits_0^t \|T\|^2_{\mathcal{H}^2(H(t')\prec U_+)}\,\mathrm{d}t' + \|F\|^2_{\mathcal{H}^1(U(t))}\Big). \tag{5.35}$$

Similarly as in 5.21 we obtain by Gronwall's lemma and by extracting the root that there exists a $Q_7 > 0$ such that

$$\|T\|_{\mathcal{H}^2(H(t)\prec U_+)} \leq Q_7(\|T\|_{\mathcal{H}^2(H(0)\prec U_+)} + \|F\|_{\mathcal{H}^1(U(t))}). \tag{5.36}$$

We now calculate the divergence of the energy tensor S_2 for the second order derivatives of T. This yields (by appendix A.1)

$$\begin{aligned}\nabla_a S_2^{ab} = \Big\{ & A^{ac}\nabla_a\nabla_c\nabla_{e_2}\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d\nabla_{f_2}\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k} \\ & A^{ac}\Big(-\sum_{r=1}^{k}\nabla_{f_2}\nabla_{f_1}T^{i_1\dots i_l}{}_{j_1\dots u\dots j_k}R^u{}_{j_r da}\nabla_c\nabla_{e_2}\nabla_{e_1}T^{p_1\dots p_l}{}_{q_1\dots q_k} \\ & -\nabla_u\nabla_{f_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}R^u{}_{f_2 da}\nabla_c\nabla_{e_2}\nabla_{e_1}T^{p_1\dots p_l}{}_{q_1\dots q_k} \\ & -\nabla_{f_2}\nabla_u T^{i_1\dots i_l}{}_{j_1\dots j_k}R^u{}_{f_1 da}\nabla_c\nabla_{e_2}\nabla_{e_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\end{aligned}$$

$$+\sum_{r=1}^{l} \nabla_{f_2}\nabla_{f_1} T^{i_1\ldots u\ldots i_l}{}_{j_1\ldots j_k} R^{i_r}{}_{uda} \nabla_c \nabla_{e_2}\nabla_{e_1} T^{p_1\ldots p_l}{}_{q_1\ldots q_k}\Big)\Big\}$$
$$A^{bd}{}_{e}{}^{e_1 f_1 e_2 f_2 j_1 q_1 \ldots j_k q_k}{}_{i_1 p_1 \ldots i_l p_l} + \tilde{S}_2^b.$$

Moreover, by interchanging the indices of $\nabla_a\nabla_c\nabla_{e_2}\nabla_{e_1} T^{i_1\ldots i_l}{}_{j_1\ldots j_k}$ according to (A.4) and then by inserting the result of equation (A.3) for $m = 2$, we obtain an analogous expression to (5.28), i.e. where the order of derivatives on T has been reduced by 1. Note that the curvature tensor R and its derivatives up to order 2 are bounded by assumption, as well as A, ∇A, B, and C. Integrating over $\nabla_a S_2^{ab}$ yields (where Q form now on denotes an arbitrary positive constant)

$$\begin{aligned}
&\int_0^t \int_{H(t')\cap\overline{U}_+} \nabla_a S_2^{ab} \nabla_b t \,\mathrm{d}\sigma(e)\,\mathrm{d}t' \\
&\quad \le Q \int_0^t \int_{H(t')\cap\overline{U}_+} \{|\mathrm{D}^2 F||\mathrm{D}^3 T| + |\mathrm{D}^2 A||\mathrm{D}^2 T||\mathrm{D}^3 T| \\
&\quad + |\mathrm{D}^2 B||\mathrm{D}T||\mathrm{D}^3 T| + |\mathrm{D}^2 C||T||\mathrm{D}^3 T| + |\mathrm{D}A||\mathrm{D}^3 T||\mathrm{D}^3 T| \\
&\quad + |\mathrm{D}B||\mathrm{D}^2 T||\mathrm{D}^3 T| + ||\mathrm{D}C||\mathrm{D}T||\mathrm{D}^3 T| \\
&\quad + (\text{bd. expr. in } |A|, |\mathrm{D}A|, |B|, |C|, |R|, |\mathrm{D}R|, |\mathrm{D}^2 R|)\}\,\mathrm{d}\sigma(e)\,\mathrm{d}t' \\
&\quad \le Q \int_0^t \int_{H(t')\cap\overline{U}_+} \{|\mathrm{D}^3 T|^2 + |\mathrm{D}^2 F|^2 + |\mathrm{D}^2 A|^2|\mathrm{D}^2 T|^2 + |\mathrm{D}^2 B|^2|\mathrm{D}T|^2 \\
&\quad + |\mathrm{D}^2 C|^2|T|^2 + |\mathrm{D}B|^2|\mathrm{D}^2 T|^2 + |\mathrm{D}C|^2|\mathrm{D}T|^2 \\
&\quad + (\text{bd. expr. in } |A|, |\mathrm{D}A|, |B|, |C|, |R|, |\mathrm{D}R|, |\mathrm{D}^2 R|)\}\,\mathrm{d}\sigma(e)\,\mathrm{d}t'.
\end{aligned}$$

In appendix B.1 one can find the estimates for each non bounded term. Thus we obtain in total

$$\int_0^t \int_{H(t')\cap\overline{U}_+} \nabla_a S_2^{ab} \nabla_b t \,\mathrm{d}\sigma(e)\,\mathrm{d}t' \le Q \int_0^t \|T\|^2_{\mathcal{H}^3(H(t')\prec U_+)}\,\mathrm{d}t'.$$

Following the proof for S_1 we easily get analogously to (5.12) for positive constants Q and Q'

$$Q'(|\mathrm{D}^3 T|^2 + |\mathrm{D}^2 T|^2) \le S_2^{ab}\nabla_a h \nabla_b h \le Q(|\mathrm{D}^3 T|^2 + |\mathrm{D}^2 T|^2),$$

and

$$\begin{aligned}\int\limits_{H(t)\cap U_+} (|\mathrm{D}^3T|^2 + |\mathrm{D}^2T|^2)\,\mathrm{d}\sigma(e) \le Q\Bigg(& \int\limits_{H(0)\cap U_+} (|\mathrm{D}^3T|^2 + |\mathrm{D}^2T|^2)\,\mathrm{d}\sigma(e) \\ & + \int\limits_0^t \|T\|^2_{\mathcal{H}^3(H(t')\prec U_+)}\,\mathrm{d}t' \\ & + \int\limits_{U(t)} |\mathrm{D}^2F|^2\,\mathrm{d}v(e)\Bigg).\end{aligned}$$

Adding inequality (5.23) to the last result and some other positive expressions on the right side we have

$$\begin{aligned}\|T\|^2_{\mathcal{H}^3(H(t)\prec U_+)} \le Q\Bigg(& \|T\|^2_{\mathcal{H}^3(H(0)\prec U_+)} \\ & + \int\limits_0^t \|T\|^2_{\mathcal{H}^3(H(t')\prec U_+)}\,\mathrm{d}t' + \|F\|_{\mathcal{H}^2(U(t))}\Bigg).\end{aligned}$$

Using again the Gronwall argument one obtains

$$\|T\|_{\mathcal{H}^3(H(t)\prec U_+)} \le Q(\|T\|_{\mathcal{H}^3(H(0)\prec U_+)} + \|F\|_{\mathcal{H}^2(U(t))}) \tag{5.37}$$

Finally, we proof the result for S_3. Similarly to the previous case

$$\begin{aligned}&\int\limits_0^t \int\limits_{H(t')\cap\overline{U}_+} \nabla_a S_3^{ab} \nabla_b t\,\mathrm{d}\sigma(e)\,\mathrm{d}t' \\ &\qquad \le Q\int\limits_0^t \int\limits_{H(t')\cap\overline{U}_+} \{|\mathrm{D}^4T|^2 + |\mathrm{D}^3F|^2 + |\mathrm{D}^3A|^2|\mathrm{D}^2T|^2 \\ &\qquad + |\mathrm{D}^3B|^2|\mathrm{D}T|^2 + |\mathrm{D}^3C|^2|T|^2 + |\mathrm{D}^2A|^2|\mathrm{D}^3T|^2 + |\mathrm{D}^2B|^2|\mathrm{D}^2T|^2 \\ &\qquad + |\mathrm{D}^2C|^2|\mathrm{D}T|^2 + |\mathrm{D}B|^2|\mathrm{D}^3T|^2 + |\mathrm{D}C|^2|\mathrm{D}^2T|^2 \\ &\qquad + (\text{bd. expr. in } |A|, |\mathrm{D}A|, |B|, |C|, |R|, |\mathrm{D}R|, |\mathrm{D}^2R|, |\mathrm{D}^3R|)\} \\ &\qquad \mathrm{d}\sigma(e)\,\mathrm{d}t'\end{aligned}$$

holds. According to the calculations in appendix B.2 we have

$$\int\limits_0^t \int\limits_{H(t')\cap\overline{U}_+} \nabla_a S_3^{ab} \nabla_b t\,\mathrm{d}\sigma(e)\,\mathrm{d}t' \le Q\int\limits_0^t (1 + l^2(t'))\,\|T\|^2_{\mathcal{H}^4(H(t')\prec U_+)}\,\mathrm{d}t',$$

where $l(t) = (\|A\|^2_{\mathcal{H}^4(H(t')\prec U_+)} + \|B\|^2_{\mathcal{H}^3(H(t')\prec U_+)} + \|C\|^2_{\mathcal{H}^3(H(t')\prec U_+)})^{\frac{1}{2}}$. Now since all the components of l are square integrable l itself is square integrable. Hence we have analogously to S_1^{ab}

$$\begin{aligned}\int\limits_{H(t)\cap U_+} (|\mathrm{D}^4 T|^2 + |\mathrm{D}^3 T|^2)\,\mathrm{d}\sigma(e) \leq & Q\Big(\int\limits_{H(0)\cap U_+} (|\mathrm{D}^4 T|^2 + |\mathrm{D}^3 T|^2)\,\mathrm{d}\sigma(e) \\ & + \int\limits_0^t (1 + l^2(t'))\,\|T\|^2_{\mathcal{H}^4(H(t')\prec U_+)}\,\mathrm{d}t' \\ & + \int\limits_{U(t)} |\mathrm{D}^3 F|^2\,\mathrm{d}v(e)\Big).\end{aligned}$$

Using inequality (5.36) this yields

$$\begin{aligned}\|T\|^2_{\mathcal{H}^4(H(t)\prec U_+)} \leq & Q\Big(\|T\|^2_{\mathcal{H}^4(H(0)\prec U_+)} + \|F\|^2_{\mathcal{H}^3(U(t))} \\ & + \int\limits_0^t (1 + l^2(t'))\,\|T\|^2_{\mathcal{H}^4(H(t')\prec U_+)}\,\mathrm{d}t'\Big).\end{aligned}$$

By Gronwall's lemma we obtain the result

$$\|T\|^2_{\mathcal{H}^4(H(t)\prec U_+)} \leq Q(\|T\|^2_{\mathcal{H}^4(H(0)\prec U_+)} + \|F\|^2_{\mathcal{H}^3(U(t))}) e^{\int_0^t Q(1+l^2(t'))\,\mathrm{d}t'}.$$

Since the function l is square integrable and 1 is locally square integrable the exponent can be estimated by a constant depending only on the relatively compact set U_+. Therefore

$$\|T\|^2_{\mathcal{H}^4(H(t)\prec U_+)} \leq Q(\|T\|^2_{\mathcal{H}^4(H(0)\prec U_+)} + \|F\|^2_{\mathcal{H}^3(U(t))})$$

and as before

$$\|T\|_{\mathcal{H}^4(H(t)\prec U_+)} \leq Q(\|T\|_{\mathcal{H}^4(H(0)\prec U_+)} + \|F\|_{\mathcal{H}^3(U(t))}). \tag{5.38}$$

The final result

$$\|T\|_{\mathcal{H}^{4+\alpha}(H(t)\prec U_+)} \leq Q(\|T\|_{\mathcal{H}^{4+\alpha}(H(0)\prec U_+)} + \|F\|_{\mathcal{H}^{3+\alpha}(U(t))}).$$

for $\alpha > 0$ follows immediately by applying the proof of (5.38) to higher orders of differentiation. The occurring integrals contain expressions with higher derivatives but can be estimated in the same way according to the more restrictive assumptions. *q. e. d.*

5.2.9. COROLLARY: *For any $\mathcal{C}^{3+\alpha}$ vector field u on $H(0)$ which is non-tangential to $H(0)$ there exist constants $P_{6,\alpha}$ and $P_{7,\alpha}$ such that*

$$\|T\|_{\mathcal{H}^{4+\alpha}(H(t)\prec U_+)} \leq P_{6,\alpha}\{\|T\|_{\mathcal{H}^{4+\alpha}(H(0))} + \|\nabla_u T\|_{\mathcal{H}^{3+\alpha}(H(0))} + \|F\|_{\mathcal{H}^{3+\alpha}(U_+)}\} \tag{5.39}$$

and

$$\|T\|_{\mathcal{H}^{4+\alpha}(U_+)} \leq P_{7,\alpha}\{\|T\|_{\mathcal{H}^{4+\alpha}(H(0))} + \|\nabla_u T\|_{\mathcal{H}^{3+\alpha}(H(0))} + \|F\|_{\mathcal{H}^{3+\alpha}(U_+)}\} \tag{5.40}$$

Proof: Let—without loss of generality—U_+ be covered by a chart (φ, V) with $\varphi = (t, x^{\beta'}) = (x^0, x^{\beta'}) = (x^\beta)$. Then equation (5.1) reads

$$\sum_{\beta,\gamma=0}^{n-1} A^{\beta\gamma}\frac{\partial^2}{\partial x^\beta \partial x^\gamma} T^{\mu_1\ldots\mu_l}{}_{\nu_1\ldots\nu_k} + \sum_{\substack{\beta,\bar\nu_1,\ldots,\bar\nu_k\\ \bar\mu_1,\ldots,\bar\mu_l=0}}^{n-1} B^{\beta\bar\nu_1\ldots\bar\nu_k\mu_1\ldots\mu_l}{}_{\bar\mu_1\ldots\bar\mu_l\nu_1\ldots\nu_k}\frac{\partial}{\partial x^\beta} T^{\bar\mu_1\ldots\bar\mu_l}{}_{\bar\nu_1\ldots\bar\nu_k}$$
$$+ \sum_{\substack{\bar\nu_1,\ldots,\bar\nu_k\\ \bar\mu_1,\ldots,\bar\mu_l=0}}^{n-1} C^{\bar\nu_1\ldots\bar\nu_k\mu_1\ldots\mu_l}{}_{\bar\mu_1\ldots\bar\mu_l\nu_1\ldots\nu_k} T^{\bar\mu_1\ldots\bar\mu_l}{}_{\bar\nu_1\ldots\bar\nu_k} = F^{\mu_1\ldots\mu_l}{}_{\nu_1\ldots\nu_k}$$

and after separating the time derivatives and using Schwartz' theorem we have

$$A^{00}\frac{\partial^2}{\partial t^2} T^{\mu_1\ldots\mu_l}{}_{\nu_1\ldots\nu_k} = F^{\mu_1\ldots\mu_l}{}_{\nu_1\ldots\nu_k} - \sum_{\beta,\gamma=1}^{n-1} A^{\beta\gamma}\frac{\partial^2}{\partial x^\beta \partial x^\gamma} T^{\mu_1\ldots\mu_l}{}_{\nu_1\ldots\nu_k}$$
$$- \sum_{\beta=1}^{n-1} A^{\beta 0}\frac{\partial^2}{\partial x^\beta \partial x^0} T^{\mu_1\ldots\mu_l}{}_{\nu_1\ldots\nu_k}$$
$$+ \sum_{\substack{\beta,\bar\nu_1,\ldots,\bar\nu_k\\ \bar\mu_1,\ldots,\bar\mu_l=0}}^{n-1} B^{\beta\bar\nu_1\ldots\bar\nu_k\mu_1\ldots\mu_l}{}_{\bar\mu_1\ldots\bar\mu_l\nu_1\ldots\nu_k}\frac{\partial}{\partial x^\beta} T^{\bar\mu_1\ldots\bar\mu_l}{}_{\bar\nu_1\ldots\bar\nu_k}$$
$$+ \sum_{\substack{\bar\nu_1,\ldots,\bar\nu_k\\ \bar\mu_1,\ldots,\bar\mu_l=0}}^{n-1} C^{\bar\nu_1\ldots\bar\nu_k\mu_1\ldots\mu_l}{}_{\bar\mu_1\ldots\bar\mu_l\nu_1\ldots\nu_k} T^{\bar\mu_1\ldots\bar\mu_l}{}_{\bar\nu_1\ldots\bar\nu_k}.$$

Thus iteratively we may express second and higher order derivatives of T out of the surface $H(0)$ by F and its derivatives, by $\nabla_u T$, and by derivatives of T in $H(0)$. Condition (3) in proposition 5.2.8 and corollary 3.2.12 give

$$\|A\|_{\mathcal{H}^3(H(0)\prec U_+)} < P_3 Q_3$$
$$\|B\|_{\mathcal{H}^2(H(0)\prec U_+)} < P_3 Q_3$$
$$\|C\|_{\mathcal{H}^2(H(0)\prec U_+)} < P_3 Q_3$$
$$\|F\|_{\mathcal{H}^2(H(0)\prec U_+)} < P_3 \|F\|_{\mathcal{H}^3(U_+)}.$$

Hence, we come to the conclusion that there exists a constant $\tilde{Q}$ such that

$$\begin{aligned}\|T\|_{\mathcal{H}^{4+\alpha}(H(0)\prec U_+)} \leq &\tilde{Q}\|T\|_{\mathcal{H}^{4+\alpha}(H(0)\cap U_+)} \\ &+ \|\nabla_u T\|_{\mathcal{H}^{3+\alpha}(H(0)\cap U_+)} + \|F\|_{\mathcal{H}^{3+\alpha}(U_+)},\end{aligned}$$

which proves inequality (5.39). Since $t \leq t_1 < \infty$ by the compactness of $\overline{U}_+$ (5.40) immediately follows and we are done.

q. e. d.

5.3 Proofs of The Theorems

In order to prove the main theorem we need two more theorems—the *Cauchy-Kovalevskaya* theorem and the *weak compactness* theorem)—which can be found in [Eva98].

5.3.1. THEOREM (Cauchy-Kovalevskaya): *Let us assume some real analytic functions $b_j : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{M}^{m\times m}$ for $(1 \leq j \leq n-1)$ and $c : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}^m$. Then there exists $r > 0$ and a real analytic function*

$$u = \sum_{\alpha} \frac{\partial^\alpha u(0)}{\alpha!} x^\alpha,$$

where $\alpha \in \mathbb{N}^n$ and $x \in \mathbb{R}^n$, solving the boundary value problem

$$\partial_n u = \sum_{j=1}^{n-1} b_j(u, x')\partial_j u + c(u, x') \text{ for } \|x\| < r$$

$$u = 0 \text{ for } \|x'\| < r, x_n = 0.$$

where x' denotes $(x_1, \dots, x_{n-1})$.

REMARK: In fact we only need this theorem on Euclidean spaces, since it will be applied only locally on analytic charts of the manifold M.

5.3.2. THEOREM (Weak compactness): *Let X be a reflexive Banach space and suppose the sequence $(u_k)_{k=1}^\infty \subseteq X$ is bounded. Then there exists a subsequence $(u_{k_j})_{j=1}^\infty$ of $(u_k)_{k=1}^\infty$ and $u \in X$ such that $u_{k_j} \rightharpoonup u$. Here $\rightharpoonup$ denotes weak convergence.*

Proof of theorem 5.1.3: In a first step we prove the theorem for analytic functions. Therefore suppose that A, B, C, F, u, and g are analytic functions in local coordinates on a chart $(V, (t = x^0, x^i))$ $(1 \leq i \leq n-1)$. Furthermore take the initial data $T = T_0$ and $\nabla_u T = T_1$ to be analytic functions on $(H(0) \cap V, x^i)$. Using equation (5.1) we can calculate the derivatives $\partial_0^2 T^{\mu_1 \ldots \mu_l}{}_{\nu_1 \ldots \nu_k}$, $\partial_0^2 \partial_i T^{\mu_1 \ldots \mu_l}{}_{\nu_1 \ldots \nu_k}$, $\partial_0^3 T^{\mu_1 \ldots \mu_l}{}_{\nu_1 \ldots \nu_k}$ etc. out of $H(0)$ in terms of T_0 and T_1 in $H(0)$. As a consequence we are allowed to expand T into a power series of the coordinates about the origin $p \in V$. By the Cauchy-Kovalevskaya theorem 5.3.1 this series will converge in a ball $B_r(p) \subseteq V$ $(r > 0)$ and gives a solution to the initial value problem. From the $\mathcal{C}^\infty$-atlas $\mathcal{A}$ on M we now select an analytic subatlas $\mathcal{A}'$, cover $H(0) \cap \overline{U}$ with coordinate neighbourhoods $B_r(p_j)$ $(j \in \mathbb{N})$ from $\mathcal{A}'$ (this is always possible since $\overline{U}$ is compact), and in each $B_r(p_j)$ construct a solution as before. Thus we obtain a solution on a set $U(\tau_1)$ for some $\tau_1 > 0$ and repeat the process using $H(\tau_1)$ as initial surface. Theorem 5.3.1 guarantees the independence of the convergence time intervals from the initial data and the solution can be extended to the whole of U_+ using only a finite number of steps. The existence for the analytic case is now proven.

In the second step we shall obtain solutions for the function spaces stated in the theorem, thus let $(A_m)_m$ be a sequence of analytic fields converging strongly to A in $\mathcal{H}^{4+\alpha}(2, 0, U_+)$, i.e. $\lim_{m\to\infty} \|A - A_m\|_{4+\alpha} \to 0$. Let (for all sequences assumed analytic) $(B_m)_m$ converge strongly to B in $\mathcal{H}^{3+\alpha}(1+k+l, l+k, U_+)$, let $(C_m)_m$ converge strongly to the field C in $\mathcal{H}^{3+\alpha}(k+l, l+k, U_+)$ and let finally $(F_m)_m$ converge strongly to F in $\mathcal{H}^{3+\alpha}(l, k, U_+)$. Moreover, the sequences $(T_{0,m})_m$ and $(T_{1,m})_m$ on $H(0) \cap \overline{U}$ shall converge strongly to the fields T_0 and T_1 in $\mathcal{H}^{4+\alpha}(l, k, H(0) \cap U)$ resp. $\mathcal{H}^{3+\alpha}(l, k+1, H(0) \cap U)$. Due to the first part for each $m \in \mathbb{N}$ there exists an analytic solution to equation (5.1) with the initial values $T_m = T_{0,m}$ and $\nabla_u T_m = T_{1,m}$. From corollary 5.2.9 we can follow that $\|T_m, U_+\|_{4+\alpha}$ is bounded as $m \to \infty$. By theorem 5.3.2 there exists a field $T \in \mathcal{H}^{4+\alpha}(l, k, U_+)$ and a subsequence $T_{m'}$ of T_m such that for all β, $0 \leq \beta \leq 4 + \alpha$, the weak limit of $\mathrm{D}^\beta T_{m'}$ exists and equals $\mathrm{D}^\beta T$.

Finally, since A_m, B_m, and C_m converge strongly to A, B, and C in appropriate Sobolev spaces we have

$$\sup_{U_+} |A - A_m| \to 0, \quad \sup_{U_+} |B - B_m| \to 0, \quad \text{and} \sup_{U_+} |C - C_m| \to 0. \tag{5.41}$$

By line (5.41) $L_{m'}(T_{m'}) \rightharpoonup L(T)$. On the other side $L_{m'}(T_{m'}) = F_{m'}$ and $F_{m'} \to F$ strongly, so $L(T) = F$. Furthermore, on $H(0) \cap U$ we obtain analogously $T_{m'} \rightharpoonup T$ and $\nabla_u T_{m'} \rightharpoonup \nabla_u T$, which have to be equal to T_0 and T_1 respectively. We can now conclude that T is a solution of the differential equation (5.1) with the given initial conditions. Uniqueness follows by theorem 5.2.7 and since each T_m satisfies the estimate (5.24) it will also be satisfied by T, thereby concluding the proof of the main theorem. *q. e. d.*

Proof of theorem 5.1.4: The proof for the n-dimensional case is very similar to that one above. In fact one calculates $\nabla_a S_\alpha^{ab}$ as before. It can be easily seen that all the terms are bounded due to the more restrictive differentiability conditions. Thus one obtains immediately the estimate

$$\|T, H(t) \cap U_+\|_{\alpha+1} \leq P_\alpha(\|T, H(0) \cap U_+\|_{\alpha+1} + \|F, U(t)\|_\alpha)$$

for all $\alpha \in \mathbb{N}_0$.

Everything else follows analogously from the Cauchy-Kovalevskaya theorem and the weak compactness theorem. Uniqueness is proven as in theorem 5.2.7. *q. e. d.*

A On Higher Order Energy Tensors

In proposition 5.2.8 we need energy tensors for derivatives of T. This chapter contains three lengthy calculations for these energy tensors.

A.1 Divergence of The Energy-Tensor

The calculation is similar to that one in the proof of proposition 5.2.1, equation (5.17). Let g and A be Lorentzian metrics on M and e be a Riemannian metric on M. ∇ denotes the covariant derivative with respect to g. We define the energy tensor for the m^{th}-order derivative of a tensor field T as follows

$$\begin{aligned} S_m^{ab} = \Bigg\{ & \left(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd}\right)\nabla_c\nabla_{e_m}\ldots\nabla_{e_1}T^{i_1\ldots i_l}{}_{j_1\ldots j_k} \\ & \nabla_d\nabla_{f_m}\ldots\nabla_{f_1}T^{p_1\ldots p_l}{}_{q_1\ldots q_k} \\ & -\frac{1}{2}A^{ab}\nabla_{e_m}\ldots\nabla_{e_1}T^{i_1\ldots i_l}{}_{j_1\ldots j_k}\nabla_{f_m}\ldots\nabla_{f_1}T^{p_1\ldots p_l}{}_{q_1\ldots q_k}\Bigg\} \\ & e^{e_1 f_1\ldots e_m f_m j_1 q_1\ldots j_k q_k}{}_{i_1 p_1\ldots i_l p_l}. \end{aligned}$$

Thus for the divergence of S_m we obtain

$$\begin{aligned} \nabla_a S_m^{ab} = \nabla_a \Bigg\{ & \left(A^{ac}A^{bd} - \frac{1}{2}A^{ab}A^{cd}\right)\nabla_c\nabla_{e_m}\ldots\nabla_{e_1}T^{i_1\ldots i_l}{}_{j_1\ldots j_k} \\ & \nabla_d\nabla_{f_m}\ldots\nabla_{f_1}T^{p_1\ldots p_l}{}_{q_1\ldots q_k} \\ & -\frac{1}{2}A^{ab}\nabla_{e_m}\ldots\nabla_{e_1}T^{i_1\ldots i_l}{}_{j_1\ldots j_k}\nabla_{f_m}\ldots\nabla_{f_1}T^{p_1\ldots p_l}{}_{q_1\ldots q_k}\Bigg\} \\ & e^{e_1 f_1\ldots e_m f_m j_1 q_1\ldots j_k q_k}{}_{i_1 p_1\ldots i_l p_l} \end{aligned}$$

$$
\begin{aligned}
&+\left\{\left(A^{ac}A^{bd}-\frac{1}{2}A^{ab}A^{cd}\right)\nabla_c\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\right.\\
&\nabla_d\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&\left.-\frac{1}{2}A^{ab}\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\right\}\\
&\nabla_a e^{e_1f_1\dots e_mf_mj_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}.
\end{aligned}
$$

As one can see this yields

$$
\nabla_a S^{ab}_m=\left\{A^{ac}A^{bd}\nabla_a\nabla_c\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\right. \tag{A.1a}
$$

$$
+A^{ac}A^{bd}\nabla_a\nabla_d\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\nabla_c\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k} \tag{A.1b}
$$

$$
-\frac{1}{2}A^{ab}A^{cd}\nabla_a\nabla_c\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k} \tag{A.1c}
$$

$$
\left.-\frac{1}{2}A^{ab}A^{cd}\nabla_a\nabla_d\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\nabla_c\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\right\} \tag{A.1d}
$$

$$
e^{e_1f_1\dots e_mf_mj_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}+\tilde{S}^b_m, \tag{A.1e}
$$

where

$$
\begin{aligned}
\tilde{S}^b_m=&\left\{\nabla_a\left(A^{ac}A^{bd}-\frac{1}{2}A^{ab}A^{cd}\right)\nabla_c\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\right.\\
&\nabla_d\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&\left.-\frac{1}{2}\nabla_a\left(A^{ab}\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\right)\right\}\\
&e^{e_1f_1\dots e_mf_mj_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}\\
&+\left\{\left(A^{ac}A^{bd}-\frac{1}{2}A^{ab}A^{cd}\right)\nabla_c\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\right.\\
&\nabla_d\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\
&\left.-\frac{1}{2}A^{ab}\nabla_{e_m}\dots\nabla_{e_1}T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_{f_m}\dots\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\right\}\\
&\nabla_a e^{e_1f_1\dots e_mf_mj_1q_1\dots j_kq_k}{}_{i_1p_1\dots i_lp_l}.
\end{aligned}
$$

By symmetry of A^{cd} in line (A.1c) and symmetry of e, we have

$$\begin{aligned}
\nabla_a S_m^{ab} =& (A^{ac} A^{bd} \nabla_a \nabla_c \nabla_{e_m} \dots \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_d \nabla_{f_m} \dots \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \quad \text{(A.2a)} \\
&+ A^{ac} A^{bd} \nabla_a \nabla_d \nabla_{f_m} \dots \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \nabla_c \nabla_{e_m} \dots \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \quad \text{(A.2b)} \\
&- A^{ab} A^{cd} \nabla_a \nabla_d \nabla_{f_m} \dots \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \nabla_c \nabla_{e_m} \dots \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k}) \quad \text{(A.2c)} \\
&e^{e_1 f_1 \dots e_m f_m j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l} + \tilde{S}_m^b. \quad \text{(A.2d)}
\end{aligned}$$

After renaming the indices $a \leftrightarrow d$, we interchange the derivatives ∇_a and ∇_d in line (A.2c), hence

$$\begin{aligned}
\nabla_a S_m^{ab} = \Bigg\{ & A^{ac} \nabla_a \nabla_c \nabla_{e_m} \dots \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} \nabla_d \nabla_{f_m} \dots \nabla_{f_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\
& A^{ac} \Bigg(- \sum_{r=1}^{k} \nabla_{f_m} \dots \nabla_{f_1} T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^u{}_{j_r da} \nabla_c \nabla_{e_m} \dots \nabla_{e_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\
& - A^{ac} \sum_{r=1}^{m} \nabla_{f_m} \dots \nabla_u \dots \nabla_{f_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^u{}_{f_r da} \nabla_c \nabla_{e_m} \dots \nabla_{e_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \\
& + A^{ac} \sum_{r=1}^{l} \nabla_{f_m} \dots \nabla_{f_1} T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_r}{}_{uda} \nabla_c \nabla_{e_m} \dots \nabla_{e_1} T^{p_1 \dots p_l}{}_{q_1 \dots q_k} \Bigg) \Bigg\} \\
& A^{bd} e^{e_1 f_1 \dots e_m f_m j_1 q_1 \dots j_k q_k}{}_{i_1 p_1 \dots i_l p_l} + \tilde{S}_m^b
\end{aligned}$$

holds.

A.2 Derivatives of The Differential Equation

In this section we give an expression for the m^{th} derivative of equation (5.1). We have

$$\begin{aligned}
\nabla_{e_m} \dots \nabla_{e_1} (A^{ac} \nabla_a \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k}) =& \nabla_{e_m} \dots \nabla_{e_1} F^{i_1 \dots i_l}{}_{j_1 \dots j_k} \\
&- \nabla_{e_m} \dots \nabla_{e_1} (B^{a q_1 \dots q_k i_1 \dots i_l}{}_{p_1 \dots p_l j_1 \dots j_k} \nabla_a T^{p_1 \dots p_l}{}_{q_1 \dots q_k}) \\
&- \nabla_{e_m} \dots \nabla_{e_1} (C^{q_1 \dots q_k i_1 \dots i_l}{}_{p_1 \dots p_l j_1 \dots j_k} T^{p_1 \dots p_l}{}_{q_1 \dots q_k}).
\end{aligned}$$

Thus

$$\begin{aligned}A^{ac}\nabla_{e_m}\dots\nabla_{e_1}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} = {}&\nabla_{e_m}\dots\nabla_{e_1}F^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &-\nabla_{e_m}\dots\nabla_{e_1}A^{ac}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &-\sum_{k=0}^{m-1}\sum_{r\in C_m^k}\prod_{i=1}^{m}(\nabla_{e_i}^{(1-r_i)})A^{ac}\prod_{j=1}^{m}(\nabla_{e_j}^{r_j})\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &-\sum_{k=0}^{m}\sum_{r\in C_m^k}\prod_{i=1}^{m}(\nabla_{e_i}^{(1-r_i)})B^{aq_1\dots q_k i_1\dots i_l}{}_{p_1\dots p_l j_1\dots j_k}\prod_{j=1}^{m}(\nabla_{e_j}^{r_j})\nabla_a T^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &-\sum_{k=0}^{m}\sum_{r\in C_m^k}\prod_{i=1}^{m}(\nabla_{e_i}^{(1-r_i)})C^{q_1\dots q_k i_1\dots i_l}{}_{p_1\dots p_l j_1\dots j_k}\prod_{j=1}^{m}(\nabla_{e_j}^{r_j})T^{i_1\dots i_l}{}_{j_1\dots j_k},\end{aligned}\tag{A.3}$$

where $C_m^k := \{r\in\{0,1\}^m \,|\, \sum_{i=1}^m r_i = k\}$, i.e. the set of combinations without repetition.

A.3 Ricci's Identity for Higher Order Derivatives

Let g be a symmetric Lorentzian metric with associated covariant derivative ∇, then the following generalization of the Ricci identity holds.

$$\begin{aligned}\nabla_{e_m}\dots\nabla_{e_1}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k} ={}&\nabla_{e_m}\dots\nabla_{e_2}\nabla_a\nabla_{e_1}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &+\sum_{r=1}^{k}\nabla_{e_m}\dots\nabla_{e_2}(\nabla_c T^{i_1\dots i_l}{}_{j_1\dots u\dots j_k}R^u{}_{j_r e_1 a})\\ &+\nabla_{e_m}\dots\nabla_{e_2}(\nabla_u T^{i_1\dots i_l}{}_{j_1\dots j_k}R^u{}_{c e_1 a})\\ &-\sum_{r=1}^{l}\nabla_{e_m}\dots\nabla_{e_2}(\nabla_c T^{i_1\dots u\dots i_l}{}_{j_1\dots j_k}R^{i_r}{}_{u e_1 a})\end{aligned}$$

$$\begin{aligned}={}&\nabla_a\nabla_{e_m}\dots\nabla_{e_1}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\\ &+\sum_{r=1}^{k}\nabla_{e_{m-1}}\dots\nabla_{e_1}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots u\dots j_k}R^u{}_{j_r e_m a}\\ &+\sum_{r=1}^{m-1}\nabla_{e_{m-1}}\dots\nabla_u\dots\nabla_{e_1}\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}R^u{}_{e_r e_m a}\\ &+\nabla_{e_{m-1}}\dots\nabla_{e_1}\nabla_u T^{i_1\dots i_l}{}_{j_1\dots j_k}R^u{}_{c e_m a}\\ &-\sum_{r=1}^{l}\nabla_{e_{m-1}}\dots\nabla_{e_1}\nabla_c T^{i_1\dots u\dots i_l}{}_{j_1\dots j_k}R^{i_r}{}_{u e_m a}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{r=1}^{k} \nabla_{e_m}(\nabla_{e_{m-2}} \dots \nabla_{e_1} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^{u}{}_{j_r e_{m-1} a}) \\
&+ \sum_{r=1}^{m-2} \nabla_{e_m}(\nabla_{e_{m-2}} \dots \nabla_u \dots \nabla_{e_1} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^{u}{}_{e_r e_{m-1} a}) \\
&+ \nabla_{e_m}(\nabla_{e_{m-2}} \dots \nabla_{e_1} \nabla_u T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^{u}{}_{c e_{m-1} a}) \\
&- \sum_{r=1}^{l} \nabla_{e_m}(\nabla_{e_{m-2}} \dots \nabla_{e_1} \nabla_c T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_s}{}_{u e_{m-1} a} \pm \dots \\
&+ \sum_{r=1}^{k} \nabla_{e_m} \dots \nabla_{e_2}(\nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^{u}{}_{j_r e_1 a}) + \nabla_{e_m} \dots \nabla_{e_2}(\nabla_u T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^{j}{}_{c e_1 a}) \\
&- \sum_{r=1}^{l} \nabla_{e_m} \dots \nabla_{e_2}(\nabla_c T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_r}{}_{u e_1 a})
\end{aligned}
$$

$$
\begin{aligned}
=& \nabla_a \nabla_c \nabla_{e_m} \dots \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} - \sum_{r=1}^{l} \nabla_a(\nabla_{e_{m-1}} \dots \nabla_{e_1} T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_r}{}_{u e_m c}) \\
&+ \nabla_a \left(\sum_{r=1}^{m-1} \nabla_{e_{m-1}} \dots \nabla_u \dots \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^{u}{}_{e_r e_m c} \right. \\
&\left. + \sum_{r=1}^{k} \nabla_{e_{m-1}} \dots \nabla_{e_1} T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^{u}{}_{j_r e_m c} \right) \\
&\pm \cdots + \sum_{r=1}^{k} \nabla_a \nabla_{e_m} \dots \nabla_{e_2}(T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^{u}{}_{j_r e_1 c}) \\
&- \sum_{r=1}^{l} \nabla_a \nabla_{e_m} \dots \nabla_{e_2}(T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_r}{}_{u e_1 c}) \\
&+ \sum_{r=1}^{k} \nabla_{e_{m-1}} \dots \nabla_{e_1} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^{u}{}_{j_r e_m a} \\
&+ \sum_{r=1}^{m-1} \nabla_{e_{m-1}} \dots \nabla_u \dots \nabla_{e_1} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^{u}{}_{e_r e_m a} \\
&+ \nabla_{e_{m-1}} \dots \nabla_{e_1} \nabla_u T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^{u}{}_{c e_m a} \\
&- \sum_{r=1}^{l} \nabla_{e_{m-1}} \dots \nabla_{e_1} \nabla_c T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_r}{}_{u e_m a} \\
&\pm \cdots + \nabla_{e_m} \dots \nabla_{e_2} \left(\sum_{r=1}^{k} \nabla_c T^{i_1 \dots i_l}{}_{j_1 \dots u \dots j_k} R^{u}{}_{j_r e_1 a} + \nabla_u T^{i_1 \dots i_l}{}_{j_1 \dots j_k} R^{u}{}_{c e_1 a} \right) \\
&- \sum_{r=1}^{l} \nabla_{e_m} \dots \nabla_{e_2}(\nabla_c T^{i_1 \dots u \dots i_l}{}_{j_1 \dots j_k} R^{i_r}{}_{u e_1 a})
\end{aligned}
\tag{A.4}
$$

B Estimates on The Coefficients

B.1 Inequalities for The Second Order Energy Tensor

In this section we provide some estimates on $\nabla_a S_2^{ab}$. We have by Appendix A.1 and equation (5.1)

$$\begin{aligned}\nabla_a S_2^{ab} =&[\nabla_{e_2}\nabla_{e_1}F^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d\nabla_{f_2}\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\ &-\nabla_{e_2}\nabla_{e_1}A^{ac}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d\nabla_{f_2}\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\ &-\nabla_{e_2}\nabla_{e_1}B^{a\bar{j}_1\dots\bar{j}_k i_1\dots i_l}{}_{\bar{i}_1\dots\bar{i}_l j_1\dots j_k}\nabla_a T^{\bar{i}_1\dots\bar{i}_l}{}_{\bar{j}_1\dots\bar{j}_k}\nabla_d\nabla_{f_2}\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\ &-\nabla_{e_2}\nabla_{e_1}C^{\bar{j}_1\dots\bar{j}_k i_1\dots i_l}{}_{\bar{i}_1\dots\bar{i}_l j_1\dots j_k}T^{\bar{i}_1\dots\bar{i}_l}{}_{\bar{j}_1\dots\bar{j}_k}\nabla_d\nabla_{f_2}\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\ &-\nabla_{e_2}\nabla_a\nabla_c T^{i_1\dots i_l}{}_{j_1\dots j_k}\nabla_d\nabla_{f_2}\nabla_{f_1}T^{p_1\dots p_l}{}_{q_1\dots q_k}\\ &-(\text{etc., etc.})]A^{bd}e^{e_1 f_1 e_2 f_2 j_1 q_1\dots j_k q_k}{}_{i_1 p_1\dots i_l p_l}\\ &+(\text{terms of lower order}).\end{aligned}$$

Thus (where $Q > 0$ denotes an arbitrary constant which is subject to change from line to line)

$$\begin{aligned}\int\limits_0^t\int\limits_{H(t')\cap\bar{U}_+}\nabla_a S_2^{ab}\,\mathrm{d}\sigma(e)\,\mathrm{d}t' \le Q\int\limits_0^t\int\limits_{H(t')\cap\bar{U}_+}&\{|\mathrm{D}^2F||\mathrm{D}^3T|+|\mathrm{D}^2A||\mathrm{D}^2T||\mathrm{D}^3T|\\ &+|\mathrm{D}^2B||\mathrm{D}T||\mathrm{D}^3T|+|\mathrm{D}^2C||T||\mathrm{D}^3T|+|\mathrm{D}A||\mathrm{D}^3T||\mathrm{D}^3T|\\ &+|\mathrm{D}B||\mathrm{D}^2T||\mathrm{D}^3T|+|\mathrm{D}C||\mathrm{D}T||\mathrm{D}^3T|\\ &+(\text{terms of lower order})\}\,\mathrm{d}\sigma(e)\,\mathrm{d}t'\\ \le Q\int\limits_0^t\int\limits_{H(t')\cap\bar{U}_+}&\{|\mathrm{D}^3T|^2+|\mathrm{D}^2F|^2+|\mathrm{D}^2A|^2|\mathrm{D}^2T|^2+|\mathrm{D}^2B|^2|\mathrm{D}T|^2\\ &+|\mathrm{D}^2C|^2|T|^2+|\mathrm{D}B|^2|\mathrm{D}^2T|^2+|\mathrm{D}C|^2|\mathrm{D}T|^2\\ &+(\text{lower order})\}\,\mathrm{d}\sigma(e)\,\mathrm{d}t'\end{aligned}$$

We now deal with every single term involving one of the coefficients A, B, or C. Hence we obtain:

(1)

$$\int_0^t \int_{H(t')\cap\bar{U}_+} |\mathrm{D}^2 A|^2 |\mathrm{D}^2 T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' = \int_0^t \left\|\mathrm{D}^2 B \otimes \mathrm{D}T, H(t')\cap U_+\right\|_0^2 \,\mathrm{d}t'$$

$$\begin{aligned}
\text{Lemma 3.2.9 } &\leq Q \int_0^t \left\|\mathrm{D}^2 A, H(t')\cap U_+\right\|_1^2 \left\|\mathrm{D}^2 T, H(t')\cap U_+\right\|_1^2 \,\mathrm{d}t' \\
&\leq Q \int_0^t \left\|A, H(t')\cap U_+\right\|_3^2 \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t' \\
\text{Lemma 3.2.11 } &\leq Q \int_0^t \left\|A, U_+\right\|_4^2 \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t' \\
&\leq Q \int_0^t \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t'
\end{aligned}$$

(2)

$$\begin{aligned}
\int_0^t \int_{H(t')\cap\bar{U}_+} |\mathrm{D}^2 B|^2 |\mathrm{D}T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' &= \int_0^t \left\|\mathrm{D}^2 B \otimes \mathrm{D}T, H(t')\cap U_+\right\|_0^2 \,\mathrm{d}t' \\
&\leq \int_0^t \left\|B \otimes \mathrm{D}T, H(t')\cap U_+\right\|_2^2 \,\mathrm{d}t' \\
\text{Lemma 3.2.9 } &\leq Q \int_0^t \left\|B, H(t')\cap U_+\right\|_2^2 \left\|\mathrm{D}T, H(t')\cap U_+\right\|_2^2 \,\mathrm{d}t' \\
&\leq Q \int_0^t \left\|B, H(t')\cap U_+\right\|_2^2 \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t' \\
\text{Lemma 3.2.11 } &\leq Q \int_0^t \left\|B, U_+\right\|_3^2 \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t' \\
&\leq Q \int_0^t \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t'
\end{aligned}$$

(3) Analogously to (2) we get

$$\int_0^t \int_{H(t')\cap\bar{U}_+} |\mathrm{D}^2 C||T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' \le Q \int_0^t \left\|T, H(t')\cap U_+\right\|_2^2 \,\mathrm{d}t'.$$

(4)

$$\int_0^t \int_{H(t')\cap\bar{U}_+} |\mathrm{D}B|^2|\mathrm{D}^2 T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' = \int_0^t \left\|\mathrm{D}B\otimes \mathrm{D}^2 T, H(t')\cap U_+\right\|_0^2 \,\mathrm{d}t'$$

$$\begin{aligned}
\text{Lemma 3.2.9 } &\le Q \int_0^t \left\|\mathrm{D}B, H(t')\cap U_+\right\|_1^2 \left\|\mathrm{D}^2 T, H(t')\cap U_+\right\|_1^2 \\
&\le Q \int_0^t \left\|B, H(t')\cap U_+\right\|_2^2 \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t' \\
\text{Lemma 3.2.11 } &\le Q \int_0^t \left\|B, U_+\right\|_3^2 \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t' \\
&\le Q \int_0^t \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t'
\end{aligned}$$

(5) As in (4) we have

$$\int_0^t \int_{H(t')\cap U_+} |\mathrm{D}C|^2|\mathrm{D}T^2| \,\mathrm{d}\sigma(e)\,\mathrm{d}t' \le Q \int_0^t \left\|T, H(t')\cap U_+\right\|_2^2 \,\mathrm{d}t'.$$

B.2 Inequalities for The Third Order Energy Tensor

By a similar calculation as in the previous subsection we obtain for $\nabla_a S_3^{ab}$ that

$$\begin{aligned}
\int_0^t \int_{H(t')\cap\bar{U}_+} \nabla_a S_3^{ab} \nabla_b t \,\mathrm{d}\sigma(e)\,\mathrm{d}t' \le Q \int_0^t \int_{H(t')\cap\bar{U}_+} &\{|\mathrm{D}^4 T|^2 + |\mathrm{D}^3 F|^2 \\
&+ |\mathrm{D}^3 A|^2|\mathrm{D}^2 T|^2 + |\mathrm{D}^3 B|^2|\mathrm{D}T|^2 + |\mathrm{D}^3 C|^2|T|^2 + |\mathrm{D}^2 A|^2|\mathrm{D}^3 T|^2 \\
&+ |\mathrm{D}^2 B|^2|\mathrm{D}^2 T|^2 + |\mathrm{D}^2 C|^2|\mathrm{D}T|^2 \\
&+ |\mathrm{D}B|^2|\mathrm{D}^3 T|^2 + |\mathrm{D}C|^2|\mathrm{D}^2 T|^2 + (\text{lower order})\} \,\mathrm{d}\sigma(e)\,\mathrm{d}t'.
\end{aligned}$$

In detail this yields:

(1)

$$
\begin{aligned}
\int_0^t \int_{H(t')\cap\bar{U}_+} |\mathrm{D}^3 A|^2 |\mathrm{D}^2 T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' &= \int_0^t \left\|\mathrm{D}^3 A \otimes \mathrm{D}^2 T, H(t')\cap U_+\right\|_0^2 \,\mathrm{d}t' \\
\text{Lemma 3.2.9 } &\leq Q \int_0^t \left\|\mathrm{D}^3 A, H(t')\cap U_+\right\|_1^2 \left\|\mathrm{D}^2 T, H(t')\cap U_+\right\|_1^2 \,\mathrm{d}t' \\
&\leq Q \int_0^t \left\|A, H(t')\cap U_+\right\|_4^2 \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t'
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_0^t \int_{H(t')\cap\bar{U}_+} |\mathrm{D}^3 B|^2 |\mathrm{D} T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' &= \int_0^t \left\|\mathrm{D}^3 B \otimes \mathrm{D} T, H(t')\cap U_+\right\|_0^2 \,\mathrm{d}t' \\
&\leq Q \int_0^t \left\|B \otimes \mathrm{D} T\right\|_3^2 \,\mathrm{d}t' \\
\text{Lemma 3.2.9 } &\leq Q \int_0^t \left\|B, H(t')\cap U_+\right\|_3^2 \left\|\mathrm{D} T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t' \\
&\leq Q \int_0^t \left\|B, H(t')\cap U_+\right\| \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t'
\end{aligned}
$$

(3) Following the calculation of (2) we obtain

$$
\begin{aligned}
&\int_0^t \int_{H(t')\cap\bar{U}_+} |\mathrm{D}^2 A|^2 |\mathrm{D}^3 T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' \\
&\qquad\qquad \leq \int_0^t \left\|C, H(t')\cap U_+\right\|_3^2 \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t'.
\end{aligned}
$$

(4)

$$\begin{aligned}
\int_0^t \int_{H(t')\cap\bar{U}_+} |\mathrm{D}^2A|^2|\mathrm{D}^3T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' &= \int_0^t \left\|\mathrm{D}^2A\otimes\mathrm{D}^3T, H(t')\cap U_+\right\|_0^2 \mathrm{d}t' \\
\text{Lemma 3.2.9} &\leq Q\int_0^t \left\|\mathrm{D}^2A, H(t')\cap U_+\right\|_1^2 \left\|\mathrm{D}^3T, H(t')\cap U_+\right\|_1^2 \mathrm{d}t' \\
&\leq Q\int_0^t \left\|A, H(t')\cap U_+\right\|_3^2 \left\|T, H(t')\cap U_+\right\|_4^2 \mathrm{d}t' \\
\text{Lemma 3.2.11} &\leq Q\int_0^t \left\|A, U_+\right\|_4^2 \left\|T, H(t')\cap U_+\right\|_4^2 \mathrm{d}t' \\
&\leq Q\int_0^t \left\|T, H(t')\cap U_+\right\|_4^2 \mathrm{d}t'
\end{aligned}$$

(5)

$$\begin{aligned}
\int_0^t \int_{H(t')\cap\bar{U}_+} |\mathrm{D}^2B|^2|\mathrm{D}^2T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' &= \int_0^t \left\|\mathrm{D}^2B\otimes\mathrm{D}^2T, H(t')\cap U_+\right\|_0^2 \mathrm{d}t' \\
\text{Lemma 3.2.9} &\leq Q\int_0^t \left\|B\otimes\mathrm{D}^2T, H(t')\cap U_+\right\|_2^2 \mathrm{d}t' \\
&\leq Q\int_0^t \left\|B, H(t')\cap U_+\right\|_2^2 \left\|\mathrm{D}^2T, H(t')\cap U_+\right\|_2^2 \mathrm{d}t' \\
&\leq Q\int_0^t \left\|B, H(t')\cap U_+\right\|_2^2 \left\|T, H(t')\cap U_+\right\|_4^2 \mathrm{d}t' \\
\text{Lemma 3.2.11} &\leq Q\int_0^t \left\|B, U_+\right\|_3^2 \left\|T, H(t')\cap U_+\right\|_4^2 \mathrm{d}t' \\
&\leq Q\int_0^t \left\|T, H(t')\cap U_+\right\|_4^2 \mathrm{d}t'
\end{aligned}$$

(6) Analogous to (5)

$$\int_0^t \int_{H(t')\cap U_+} |\mathrm{D}^2C|^2|\mathrm{D}T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' \leq Q\int_0^t \left\|T, H(t')\cap U_+\right\|_3^2 \mathrm{d}t'.$$

(7)

$$\begin{aligned}
\int_0^t \int_{H(t')\cap U_+} |\mathrm{D}B|^2|\mathrm{D}^3T|^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' &= \int_0^t \left\|\mathrm{D}B\otimes \mathrm{D}^3T, H(t')\cap U_+\right\|_0^2 \,\mathrm{d}t' \\
\text{Lemma 3.2.9 } &\leq Q\int_0^t \left\|\mathrm{D}B, H(t')\cap U_+\right\|_1^2 \left\|\mathrm{D}^3T, H(t')\cap U_+\right\|_1^2 \,\mathrm{d}t' \\
&\leq Q\int_0^t \left\|B, H(t')\cap U_+\right\|_2^2 \left\|T, H(t')\cap U_+\right\|_4^2 \,\mathrm{d}t' \\
\text{Lemma 3.2.11 } &\leq Q\int_0^t \left\|B, U_+\right\|_3^2 \left\|T, H(t')\cap U_+\right\|_4^2 \,\mathrm{d}t' \\
&\leq Q\int_0^t \left\|T, H(t')\cap U_+\right\|_4^2 \,\mathrm{d}t'
\end{aligned}$$

(8) In a similar way as in (7) we have

$$\int_0^t \int_{H(t')\cap \bar{U}_+} |\mathrm{DC}|^2|\mathrm{D}^2T^2 \,\mathrm{d}\sigma(e)\,\mathrm{d}t' \leq Q\int_0^t \left\|T, H(t')\cap U_+\right\|_3^2 \,\mathrm{d}t'.$$

All other expressions are of order ∇A, A, B, or C and therefore bounded by assumption (3) of proposition 5.2.1 (which follows from assumption (4) of proposition 5.2.8).

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Index

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