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„The Existence of Well-Behaved Aggregate Production Functions in a Classical Model with Heterogeneous Capital and a Finite Technology: An Axiomatic Approach"

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# Contents

**1 Introduction** **1**

**2 On the classical model and some simplifying assumptions** **3**

**3 On technology and cost minimization** **6**

3.1 The Perron-Frobenius Theorem . . . . . 6

3.2 Definition of a finite technology . . . . . 8

3.3 Definition and existence of cost-minimizing techniques . . . . . 13

**4 On switch points, well-behaved aggregate production functions and regular economies** **21**

4.1 Definition and existence of switch points . . . . . 21

4.2 An important property of switch points . . . . . 25

4.3 Definition of well-behaved aggregate production functions and regular economies . . . . . 27

**5 On the existence of well-behaved aggregate production functions** **37**

5.1 Proof of the existence theorem for well-behaved aggregate production functions with finite technologies . . . . . 37

5.2 The "marginal productivity of capital" in regular economies . . 39

5.3 Consequences for non-regular economies . . . . . 41

**6 Conclusion** **43**

**Notes** **46**

**References** **55**

# 1 Introduction

Every student of economics makes at some point of his or her academic career the acquaintance of the so-called "well-behaved aggregate production function" (or "neoclassical macroeconomic production function"), that traditionally serves as description of the economy's production side in *aggregate* terms: There are two factors of production, "labor" and *aggregate* (physical) "capital" (optimally adjusted to each other in the "long-run"), which are *functionally* assigned to a quantity of *aggregate* "output". The conventional textbook version of the well-behaved aggregate production function is a *continuously differentiable* function, that exhibits the well known properties of "constant returns to scale" (or "linear homogeneity") and diminishing "marginal productivity" with respect to both factors. With this, well-behaved aggregate production functions are in a certain sense relevant to applied economics, especially in the field of *wage policy*, since they implicate that *competitive* and *cost-minimizing* producers apply *more* labor relative to capital in response to a *fall* in the (real) wage rate relative to the factor price of capital. That is, if labor becomes cheaper relative to capital, then producers have incentives to apply less "capital intensive" methods of production with smaller "labor productivity". Conversely, a *rise* in the wage rate relative to the factor price of capital (meaning that labor gets more expensive relative to the factor capital) implies *more* "capital intensive" production with *higher* "labor productivity". Accordingly, a relatively *lower* wage rate results in *higher* demand for labor and *smaller* demand for capital to produce the same quantity of aggregate output, while a relatively *higher* wage rate implicates *smaller* demand for labor and *higher* demand for capital to generate a *constant* output level. In such a (neoclassical) world (where this kind of "substitution" between the factors capital and labor prevails) every kind of (involuntary, non-frictional and non-structural) *unemployment* can be *reduced* by a wage rate, that *falls* relative to the factor price of capital.

The well-behaved aggregate production function was also a matter of controversies. The famous critique of Joan Robinson, who called the production function a "powerful instrument of miseducation" (Robinson, 1953, p. 81), initiated a series of debates among economists over the *logical consistency* of well-behaved aggregate production functions with the choice of techniques by *cost-minimizing* producers using *heterogeneous* capital (i.e. capital, that is composed of *non-commensurable* goods). These debates became known as "Cambridge Controversies in the Theory of Capital" (Harcourt, 1969).[1] With

reference to these controversies, Edwin Burmeister established a theorem, that gives an *equivalent* condition for the consistency of well-behaved aggregate production functions with cost-minimization. It reads: "[A well-behaved] aggregate steady-state production function $F(K)$ exists if and only if the economy is regular" (Burmeister, 1980, Theorem 4.5). The term "$F(K)$" refers to a production function in *intensive form* relating an *index of (aggregate) capital intensity*, indicated by the term "$K$", with an *index of labor productivity* (or "per capita consumption"). The theorem is formulated within a "classical" model which is characterized by *heterogeneous capital* and a *finite technology*, that is, the number of technical alternatives available for production is finite.[2] Burmeister called this kind of modeling a "Leontief-Sraffa model" with "non-differentiable technologies" (Burmeister, 1980, p. 102 and p. 133).

Burmeister set up the theorem without giving a *formal* proof or constructing a complete *axiomatic* framework, which might serve as basis for the formulation of such a proof. He only proved a version of the theorem with a *continuum* of technical alternatives, which he called "neoclassical" (or "differentiable, constant-returns-to-scale") technology (Burmeister, 1980, p. 119 and Theorem 4.4).[3] The object of the present work is to provide a *mathematical* proof for Burmeister's existence statement. This is also an appropriate way to make the reader clear that in models with heterogeneous capital an additional condition (the notion of the *regular economy*) is necessary to verify the existence of a well-behaved aggregate production and the conclusions drawn from it (like the negative relationship between the wage rate and the employment of labor at a constant output level). Aside from general production-theoretic and mathematical interests regarding the conditions for a logically consistent formulation of well-behaved aggregate production functions in models with heterogeneous capital, the reason why this topic should deserve closer attention is the importance of this kind of production functions, for it is the basis for many (growth and business cycle) models which are widely used in the fields of theoretic and applied macroeconomics. This is indicated by the fact that well-behaved aggregate production functions can be encountered in many macroeconomic textbooks (which present them in a partly unreflective way),[4] and are used for empirical research to support practical decisions in economic policy.[5]

This work is structured as follows. In Section 2 we briefly discuss how the classical approach is characterized and introduce some assumptions, which define the analytical framework for a (classical) general equilibrium model to describe a (closed) economy. Section 3 purports a specification of the for-

malism, which gives the model its mathematical structure. This especially concerns the definition of the *technology* and the economy's *price system*, on whose basis we formulate *equilibrium conditions*. As contrasted with Burmeister, we particularly put emphasis on building the *axiomatic structure* of the model. Based on this, we give in Section 4 the terms *regular economy* and *well-behaved aggregate (steady-state) production function* a definite mathematical meaning, so as to prove Burmeister's Theorem 4.5 in Section 5 by following a way, that is different from the one suggested by him. Thereby, any deviations from the model originally presented by Burmeister are explicitly mentioned. At the end, we give a brief discussion of the problems regarding the empirical dimension of the theorem.

# 2 On the classical model and some simplifying assumptions

We begin with a brief discussion of the structure of the "classical" (or "neoricardian") *long-run equilibrium* model. According to Kurz and Salvadori (2000a, pp. 154-5), its analytical core can be characterized by the following list of *exogenous* variables:

(a) The technology as set of available technical alternatives.

(b) The size and composition of the net output (as a bundle that contains a surplus of produced goods over the quantities used up in a process of *circular* production).[6]

(c) A distributive variable: The ruling rate of profit on advanced capital or the wage rate(s) for (different kinds of) labor.

(d) The available quantities of different kinds and qualities of natural (i.e. non-reproducible) resources such as land, mineral deposits and so on.

The only acting agents in this model are *profit-seeking* producers who are assumed to choose from the technology in order to *minimize costs*, while the exogenously given net output (dedicated to satisfy final demand) may be interpreted as "the needs and wants of the members of the different classes of society and the requirements of reproduction and capital accumulation" (Kurz and Salvadori, 2000a, p. 155). Under the condition of *free competition*, meaning free entry in and exit from all markets, the above list of exogenous variables

is sufficient to determine a *long-run equilibrium* with the following set of *endogenous* variables: The relative prices of produced commodities, the rate of profit, or alternatively the wage rate(s), the (scarcity) rents for natural resources and the quantities of goods necessary to produce the net output. In such an equilibrium all prices including the rate of profit and the wage rate(s) are supposed to be uniform throughout the entire economy, and commodity prices are governed by costs of production, that is, by the technology and the exogenous distributive variable. This analytically reflects the institution of free competition.[7] In the classical long-run equilibrium, the supply of natural resources is the only factor of production that is given in (absolute) quantities, whereas "the stock of equipment [or capital] and the amount of employment [of labor] have [optimally] been adjusted to each other" (Robinson, 1970, p. 310). With the rate of profit or the wage rate(s) as independent variable(s), the (functional) income distribution is not fully endogenously determined but partly exogenous.[8]

Now, we introduce some assumptions, many of which are (tacitly) supposed in models with well-behaved aggregate production functions.[9] Of course, these assumptions are restrictive because they abstract from certain aspects of empirical reality (as models do in general to provide a simplified picture of reality), and one may object that models based on them ignore facts, that are relevant for applied economics (and with respect to many economic problems, this is certainly true). But the main purpose of this work is not to offer a justification for every single assumption but to demonstrate that even in models, which are simplified by these assumptions, an additional *regularity condition* is required to prove the existence of well-behaved aggregate production functions, once capital goods are assumed to be heterogeneous.[10] In the following, the list, by which we introduce these (standard) assumptions:

(A1) Land and all kinds of *natural resources* are *abundant*.[11] This means that natural resources are "free goods" and thus *no (scarcity) rents* must be paid to their owners.[12]

(A2) The economy is characterized by *single production*, i.e. there are *no (desired or undesired) joint products* (that could cause "external effects").[13]

(A3) Production is a *circular* process in the sense of "Production of Commodities by Means of Commodities" (Sraffa, 1960), and the economy is *self-sustaining*: A *surplus* of all commodities can be produced, that is, more quantities of each commodity are generated than what is necessary

to use up in order to produce them (in other words, it is possible to generate a net output that contains positive amounts of each commodity).

(A4) The economy produces a *fixed number of commodities*, which *all* enter directly or indirectly as input goods the production of each commodity. [14] This means that, in the language of Piero Sraffa, all commodities are "basic" products.[15]

(A5) The "time"[16] required for production is *equal* along the entire spectrum of reproducible commodities, i.e. there is a *uniform production period* throughout the economy.[17] Additionally, for the production of *each commodity* the entire amounts of labor and capital are advanced as *input goods* at the *beginning* of a uniform production period and the whole quantity of the *single output good* makes its appearance at the *end* of the production period.

(A6) Capital goods are *fully* depreciated during the uniform production period, that is, only "circular capital" is used for production.[18]

(A7) The production of all commodities exhibits *constant returns to scale* in the sense of operating a production process at any intensity and without advantages or disadvantages in costs that result from the size of output.

(A8) Labor applied in production of each commodity is *homogeneous.*[19]

(A9) Wages are paid at the *end* of the production period.[20]

(A10) Only *one* commodity is used for *consumption.*

(A11) There are *no net investments* and all accumulated amounts of non-consumed commodities just serve to *replace* the amounts of capital used up in the production process. Consequently, the economic system remains in a *steady state* (or *stationary condition*) and the *only commodity*, which is represented by a *positive* amount in the net product, is the *consumption good.*[21]

(A12) The economy is in a state of *free competition* and *full information.* The rate of profit paid on advanced capital, the wage rate paid for homogeneous labor and the commodity prices are *known* and *uniform.*[22]

(A13) The (uniform) rate of profit is the *exogenous* distributive variable.

It is worth mentioning that with Assumption (A11) we tacitly postulated the given net output to be entirely used for consumption, irrespective of the ruling (functional or personal) income distribution: Neither profit-earners nor wage-earners have any inclination to save their income. Consequently, problems, arising from effective demand, are eliminated from our model by assumption. In other words, we assume "Say's law".[23] As a result, the absolute net output is independent of the rate of profit, but, as we shall see later, the net output in relation to the total quantity of labor, required for production, may vary with the rate of profit (and accordingly with the income distribution).

# 3 On technology and cost minimization

The content of this section involves a formalization of the model, that has been presented in Section 2. In particular, the task lies in a formal definition of a *finite technology* as well as in defining a *system of equilibrium prices* with a uniform rate of profit as the independent distributive variable. Based on this, we investigate the conditions regarding *cost-minimization*. Within the framework of the model, we are then able to define a *long-run equilibrium*. Throughout the rest of this section we suppose that there are $n \in \mathbb{N}$ commodities in the economy.

## 3.1 The Perron-Frobenius Theorem

In this subsection we formulate a version of the so-called "Perron-Frobenius Theorem", which is essential for the forthcoming analysis. This theorem asserts important facts on *semipositive* and *indecomposable* matrices.
First, we need to introduce the following relations:

**Definition 3.1.** *Let $A = (a_{ij})_{i=1,...m,j=1,...,n}$ and $B = (b_{ij})_{i=1,...,m,j=1,...,n}$ be two real $m \times n$ matrices, i.e. $A, B \in \mathbb{R}^{m\times n}$. If $a_{ij} \geq b_{ij}$ for any $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$, then $A \geq B$ is written. The expression $A \gneqq B$ means $A \geq B$ and $A \neq B$. If $a_{ij} > b_{ij}$ for any $i \in \{1..., m\}$ and $j \in \{1, ..., n\}$, then $A > B$ is written.*

Suppose that $\mathbf{0} \in \mathbb{R}^{m\times n}$ stands for the $m \times n$ zero matrix. Following Kurz and Salvadori (1995, p. 506), if for a real $m \times n$ matrix $A$ the statement $A \geq \mathbf{0}$ ($A \gneqq \mathbf{0}$, $A > \mathbf{0}$) is true, then we call $A$ *nonnegative* (*semipositive*, *positive*). We also use this definitions for vectors, which can be regarded as matrices with only one column or only one row.

For the next definition we need to consider a certain class of *elementary matrices*. Suppose that the elementary matrix $S \in \mathbb{R}^{n\times n}$ is a square matrix such that for any semipositive square matrix $A \in \mathbb{R}^{n\times n}$ the right-hand multiplication of $A$ with $S$ results in the transformation from the original matrix $A$ to the matrix product $AS$ by interchanging the position of two columns in $A$ (for example, the $i$-th column of $A$ changes its position with the $j$-th column of the same matrix). The subsequent left-hand multiplication of $AS$ with the transpose of $S$, denoted by $S^T$, results in a transformation from $AS$ to the matrix product $S^TAS$ by interchanging the positions of the corresponding rows in $AS$ (in our example, this means an interchange of the position of the $i$-th row of matrix $AS$ with its $j$-th row). Now, we are able to define *indecomposable* matrices; see also Kurz and Salvadori (1995, p. 516):

**Definition 3.2.** *Let $A \in \mathbb{R}^{n\times n}$ be semipositive. Matrix $A$ is called indecomposable, if there is no finite sequence of elementary matrices $S_1, ..., S_k \in \mathbb{R}^{n\times n}$ of the type described above such that*

$$S_1^T ... S_k^T A S_1 ... S_k = \begin{pmatrix} A_{11} & \boldsymbol{0} \\ A_{21} & A_{22} \end{pmatrix}, \tag{3.1}$$

*where the right-hand matrix in equation (3.1) is a block matrix composed of the submatrices $A_{11} \in \mathbb{R}^{m\times m}$, $A_{22} \in \mathbb{R}^{k\times k}$, $A_{21} \in \mathbb{R}^{k\times m}$ and the $m \times k$ zero matrix $\boldsymbol{0}$ with $m \in \mathbb{N}$ satisfying $m + k = n$.*

At last, we are ready to formulate the Perron-Frobenius Theorem:

**Theorem 3.1.** *Let matrix $A \in \mathbb{R}^{n\times n}$ be semipositive and indecomposable, and let the $n \times n$ unit matrix be associated with the expression $I_n$. Then the following statements are true:*

*(a) Matrix $A$ has a real eigenvalue. This implies that the largest real eigenvalue of $A$ exists.*

*(b) The largest real eigenvalue $\lambda_L$ of $A$ is positive.*

*(c) For any real number $\lambda > \lambda_L$ matrix $(\lambda I_n - A)$ is invertible and its inverse $(\lambda I_n - A)^{-1}$ is positive.*

*(d) Matrix $A$ has with respect to $\lambda_L$ a positive eigenvector, which is unique up to the multiplication by a scalar.*

*(e) Let $x \in \mathbb{R}^{n\times 1}$ be a nonnegative vector and $\lambda \in \mathbb{R}$. If $(\lambda I_n - A)x$ is semipositive, then $\lambda > \lambda_L$.*

*Proof.* A proof of these statements can be found in Kurz and Salvadori (1995, p. 513 and p. 517). □

As in Theorem 3.1, we use in the following sections expression $\lambda_L$ to denote the *largest real eigenvalue* for a semipositive and indecomposable $n \times n$ matrix $A$ and expression $I_n$ to denote the $n \times n$ *unit matrix.*

## 3.2 Definition of a finite technology

One of the exogenously given variables of the classical model is the technology; see item (a). In this subsection we provide a formal specification of it. In order to do so, we mathematically define the terms *process* and *technique* under the condition of single production, that is assumed in (A2).[24] We first define:

**Definition 3.3.** *Let a process to produce one unit of commodity $i \in \{1, ..., n\}$ be defined as a pair $(a, l)$ with the semipositive vector*

$$a = (a_1, ..., a_n) \in \mathbb{R}^{1 \times n}$$

*and the positive scalar*

$$l > 0.$$

*The scalar $a_j$ is called input coefficient for commodity $j \in \{1, ..., n\}$ as (physical) capital good and $l$ is called input coefficient for labor.*

Economically, for any $j \in \{1, ..., n\}$ input coefficient $a_j$ in Definition 3.3 stands for the amount of commodity $j$ needed to produce one unit of commodity $i$ and $l$ represents the amount of labor to produce one unit of commodity $i$. In order to produce one unit of each commodity simultaneously, a process $(a_i, l_i)$ must be activated for each commodity $i \in \{1, ..., n\}$. We define:

**Definition 3.4.** *Consider for each commodity $i \in \{1, ..., n\}$ a process $(a_i, l_i)$ in the sense of Definition 3.3 with $a_i = (a_{i1}, ..., a_{in})$. Then they constitute a unique technique represented by the pair $(A, \ell)$ with the semipositive matrix*

$$A := (a_{ij})_{i,j=1,...,n} \in \mathbb{R}^{n \times n}$$

*and the positive vector*

$$\ell := (l_1, ..., l_n)^T \in \mathbb{R}^{n \times 1}.$$

In other words, a technique is a combination of single processes to produce one unit of each of the $n$ commodities. Economically the matrix $A$ in Definition 3.4 contains the input coefficients for physical capital and the vector $\ell$ the input coefficients for labor, i.e. $a_{ij}$ and $l_i$ are for each $i, j \in \{1, ..., n\}$ input coefficients as explained above (note that the elements in the $i$-th row of $A$ and the $i$-th element of $\ell$ form together the process, which is used by technique $(A, \ell)$ to produce one unit of commodity $i$). With (A6), we have assumed that each capital good is fully depreciated during the uniform production period. This means that the input coefficient $a_{ij}$ can be interpreted as the amount of commodity $j$, that gets used up in the process of producing one unit of commodity $i$, and so matrix $A$ also contains the depreciated amounts of all capital goods. In (A7) we have postulated *constant returns to scale* or "alternative techniques of production which are all of the Leontief-Sraffa fixed-coefficient type" (Burmeister, 1980, p. 103): Any positive amount of any commodity can be produced by "scaling" the same process. At this point, recall that in classical analysis the size and composition of the *net output* is exogenous (item (b) in the list of exogenous variables). It is supposed to be produced by the application of a technique, which leads us to the following definition:

**Definition 3.5.** *Let the semipositive $n \times 1$ vector $C \in \mathbb{R}^{n\times 1}$ be the net output, and consider a technique $(A, \ell)$ in the sense of Definition 3.4. The net output $C$ is produced by technique $(A, \ell)$, if there is a semipositive vector $Q \in \mathbb{R}^{n\times 1}$ such that*

$$C = Q - A^T Q. \tag{3.2}$$

*Then $Q$ is called gross output with respect to net output $C$ and technique $(A, \ell)$.*

The system of equations (3.2) gives us a mathematical representation of a fundamental *input-output relation*, that must be satisfied in a system of circular and self-sustaining production. Expression $A^T Q$ stands for the vector containing the amounts of capital goods, that must be used up in order to produce gross output $Q$ by technique $(A, \ell)$ within a uniform production period (of course, the multiplication or "scaling" of matrix $A$ by gross output vector $Q$ is economically allowed by the assumption of constant returns to scale). The input vector $A^T Q$ must be deducted from $Q$ to extract the net output $C$ for the current production period and to command enough quantities of input goods to generate $C$ in the next production period. Next, we need to state the question under which circumstances it is possible to produce a given net output with a certain technique, or to put this into other words, under which conditions the system of linear equations (3.2) with a given vector $C$

and a given technique $(A, \ell)$ is solvable. In order to answer this question, the following theorem will be helpful:

**Theorem 3.2.** *Let $(A, \ell)$ be a technique and assume that $C \in \mathbb{R}^{n\times 1}$ is the net output, that is produced by this technique. Suppose that the semipositive matrix $A \in \mathbb{R}^{n\times n}$ is indecomposable with $\lambda_L$ as its largest real eigenvalue. If we assume that $\lambda_L < 1$, then there is a unique positive $n \times 1$ vector which is the (unique) gross output with respect to net output $C$ and technique $(A, \ell)$.*

*Proof.* The $n \times n$ matrix $A$ is semipositive and indecomposable with $\lambda_L < 1$. According to Theorem 3.1 (c), the inverse of matrix $I_n - A$ exists and is positive. The same is true for the inverse of matrix $I_n - A^T$. Thus, we can define the $n \times 1$ vector

$$\tilde{Q} := (I_n - A^T)^{-1} C. \tag{3.3}$$

It is clear that $\tilde{Q}$ is the unique solution of the system of linear equations (3.2), since

$$\tilde{Q} - A^T \tilde{Q} = (I_n - A^T)\tilde{Q} = C.$$

Thus, $\tilde{Q}$ is the unique gross output with respect to net output $C$ and technique $(A, \ell)$. As matrix $(I_n - A^T)^{-1}$ is positive and vector $C$ semipositive, vector $\tilde{Q}$ must be positive.

□

According to (A4), all of the $n$ commodities are basic products. With respect to a technique $(A, \ell)$, this is the case, if and only if matrix $A$ is indecomposable, i.e. if it is not possible to conduct operations as described in Definition 3.2 in such a way that we get a matrix of the same form like the one on the right-hand side of equation (3.1).[25] In addition, we have postulated with assumption (A3) that production can be organized in a way, so that the economy produces a surplus of each commodity over the depreciated amounts of the same commodity utilized as capital good. This can be done by technique $(A, \ell)$, if we further assume that $\lambda_L < 1$ making the definition of a unique positive gross output $\tilde{Q}$ with respect to each semipositive net output $C$ and technique $(A, \ell)$ by equation (3.3) possible. Since matrix $A^T$ is semipositive, the capital input vector $A^T\tilde{Q}$ is also positive, i.e. each commodity is needed directly or indirectly as input good to produce itself and every other commodity.

Taking into consideration what we have discussed above, we finally construct the technology as set of all techniques. We start by exogenously giving the set of all processes for each commodity $i \in \{1, ..., n\}$, which we denote by

$T_i$. As Definition 3.3 suggests, we can consider $T_i$ as subset of $\mathbb{R}_+^{1\times n} \times \mathbb{R}_{++}$, where $\mathbb{R}_+^{1\times n}$ stands for the set of all semipositive $1 \times n$ vectors and $\mathbb{R}_{++}$ for the set of all positive real numbers. According to Definition 3.4, it now makes sense to define the economy's *technology* as the Cartesian product

$$T := \times_{i=1}^{n} T_i,$$

which can be considered as subset of $\mathbb{R}_+^{n\times n} \times \mathbb{R}_{++}^{n\times 1}$ with $\mathbb{R}_+^{n\times n}$ as set of the semipositive $n \times n$ matrices and $\mathbb{R}_{++}^{n\times 1}$ as set of the positive $n \times 1$ vectors.[26] Of course, we are only interested in techniques that are able to produce any given net output in the sense of Definition 3.5. At the same time, all of the $n$ commodities are supposed to be basic products with respect to these techniques. Hence, in the light of Theorem 3.2, we postulate:

(P1) The technology $T$ satisfies the condition

$$T \subseteq \{(A, \ell) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_{++}^{n\times 1} \mid A \text{ is indecomposable with } \lambda_L < 1\}.$$

This postulate makes sure that each technique in $T$ is "feasible" with respect to (A3) and (A4). Moreover, we assume the total number of available techniques to be finite:

(P2) The technology $T$ is a finite set.[27]

With (P2) there exists a finite set $I$ with $|I| = |T|$ and a bijection from $I$ to $T$. The set $I$ can be considered as an *index set* for $T$, that is, any technique in $T$ can be uniquely identified by an index in $I$ and vice versa. This justifies to write the technique, which is identified by the index $d \in I$, as $({}_dA, {}_d\ell)$, where ${}_dA = ({}_da_{ij})_{i,j=1,\dots,n}$ stands for the matrix of input coefficients for capital and ${}_d\ell = ({}_dl_1, \dots, {}_dl_n)^T$ for the vector of input coefficients for labor.[28] The scalar ${}_d\lambda_L$ signifies the largest real eigenvalue associated with ${}_dA$. If we denote the exogenously given net output by $C = (C_1, \dots, C_n)^T \in \mathbb{R}^{n\times 1}$, we can define the positive unique gross output with respect to $C$ and technique $d$ by

$${}_dQ(C) := (I_n - ({}_dA)^T)^{-1}C, \tag{3.4}$$

as it is suggested by equation (3.3).[29] For a given net output $C$ the total (homogeneous) labor input ${}_dL(C)$ and the capital input vector ${}_dK(C) = ({}_dK_1(C), \dots, {}_dK_n(C))^T$ are determined by

$${}_dL(C) := ({}_d\ell)^T {}_dQ(C)$$

and

$$ {}_dK(C) := ({}_dA)^T {}_dQ(C), $$

respectively (note that, since ${}_d\ell$ is positive and ${}_dA$ is semipositive, ${}_dL(C)$ is a positive scalar and ${}_dK(C)$ is a positive $n \times 1$ vector). Hence, we can define a vector ${}_dk(C) = ({}_dk_1(C), ..., {}_dk_n(C))^T$ by

$$ {}_dk(C) := \frac{1}{{}_dL(C)} {}_dK(C), \tag{3.5} $$

and a vector ${}_dC(C) = ({}_dc_1(C), ..., {}_dc_n(C))^T$ by

$$ {}_dc(C) := \frac{1}{{}_dL(C)} C, \tag{3.6} $$

Through these definitions, we can prove the following Theorem:

**Theorem 3.3.** *Let $C \in \mathbb{R}^{n\times 1}$ be a semipositive vector and consider a technique $d \in I$. Then for any $\lambda > 0$ it follows that*

$$ {}_dk(\lambda C) = {}_dk(C) $$

*and*

$$ {}_dc(\lambda C) = {}_dc(C). $$

*Proof.* From identity (3.4) it follows that

$$ {}_dQ(\lambda C) = (I_n - ({}_dA)^T)^{-1} \lambda C = \lambda (I_n - ({}_dA)^T)^{-1} C = \lambda {}_dQ(C) $$

for any $\lambda > 0$. From this we obtain

$$ {}_dL(\lambda C) = ({}_d\ell)^T {}_dQ(\lambda C) = \lambda ({}_d\ell)^T {}_dQ(C) = \lambda {}_dL(C) $$

and

$$ {}_dK(\lambda C) = ({}_dA)^T {}_dQ(\lambda C) = \lambda ({}_dA)^T {}_dQ(C) = \lambda {}_dK(C). $$

With identities (3.5) and (3.6) we finally receive the conclusions of the theorem. □

Theorem 3.3 says that the vectors ${}_dk(C)$ and ${}_dc(C)$ remains constant, as long as *all* elements in $C$ are varied by a positive factor $\lambda$. With (A10) and (A11) we have assumed that there is only one consumption good and no capital accumulation. This means that the given net output $C = (C_1, ..., C_n)^T$ can be considered as pure consumption vector, that must necessarily be a semipositve vector consisting of only one strictly positive element. Without

loss of generality, we assume that commodity 1 is the consumption good, i.e. $C_1 > 0$ and $C_i = 0$ for each $i \in \{2, ..., n\}$, if $n > 1$. Under this condition, any variation of $C_1$ by a positive factor $\lambda$ means a variation of *all* elements in vector $C$ by the same factor. As a consequence of Theorem 3.3, the magnitudes ${}_dk(C)$ and ${}_dc(C)$ are constant for any positive $C_1$. For any technique $d \in I$, this justifies the definition of the constant vectors ${}_dk = ({}_dk_1, ..., {}_dk_n)^T$ and ${}_dc = ({}_dc_1, ..., {}_dc_n)^T$ by

$$ {}_dk := {}_dk(C) \tag{3.7} $$

with ${}_dk_i > 0$ for each $i \in \{1, ..., n\}$, and

$$ {}_dc := {}_dc(C) \tag{3.8} $$

with ${}_dc_1 > 0$ and ${}_dc_i = 0$ for any $i \in \{2, ..., n\}$, if $n > 1$. Therefore, each technique $d \in I$ can be uniquely characterized by the (constant) technical magnitudes ${}_dk_1, ..., {}_dk_n$ and ${}_dc_1$. Variable ${}_dk_i$ can be interpreted as capital intensity with respect to commodity $i$ (used as capital good by technique $d$) and ${}_dc_1$ represents the *per labor unit (net) output* of commodity 1,[30] that is, the *(net) labor productivity* by which technique $d$ produces a (net) output of commodity 1. It is important to stress that the capital intensities ${}_dk_1, ..., {}_dk_n$ are *heterogeneous* (since they are *physically* different), while the labor productivity ${}_dc_1$ is *homogeneous.*[31]

## 3.3 Definition and existence of cost-minimizing techniques

In this subsection we determine *which* techniques will be chosen from technology $T$ by profit seeking producers. In order to accomplish this, we first need to define a *system of prices* for each technique $d \in I$. Since according to assumption (A12) the economy is characterized by *free competition*,[32] the rate of profit on the value of advanced capital and the wage rate for homogeneous labor are assumed to be the *same* in every line of production. We define what Adam Smith called "natural prices" and Karl Marx "Produktionspreise", namely a system of prices which cover costs of production, but do not pay any extra profits over and above what is allowed by the uniform rate of profit; see also Kurz and Salvadori (1995, p. 1, p. 12, p. 29 and p. 98):

**Definition 3.6.** *Let $d \in I$ be a technique and consider the semipositive vector $y \in \mathbb{R}^{n\times 1}$. Then the triple $(p, w, r)$ with the semipositive vector $p \in \mathbb{R}^{n\times 1}$ and the two nonnegative scalars $w, r \in \mathbb{R}$ is called a system of production prices with respect to technique $d$, if the equations*

$$ p = (1 + r)_dAp + w_d\ell \tag{3.9} $$

*and*

$$y^T p = 1 \tag{3.10}$$

*are satisfied.*

Vector $p$ in Definition 3.6 stands for the $n$ *cost covering* commodity prices, the scalar $w$ signifies the uniform wage rate and the scalar $r \in \mathbb{R}$ represents the uniform rate of profit. The semipositive vector $y$ is a bundle of commodities serving as numeraire, in terms of which prices are measured. The right-hand side of equation (3.9) is to be considered as unit costs of production, that occur, when one unit is produced of each commodity by technique $d$. Vector ${}_dAp$ contains the values of advanced capital consisting of what producers have to pay for capital goods at (supply) prices $p$. Expression ${}_dAp$ also represents the values of depreciated capital. Vector $r_dAp$ contains the profits, that can be earned by capitalists, and, with free competition, the same vector also represents *opportunity costs* of (the value of) own capital. Altogether, the *unit capital costs* are summarized by vector $(1+r)_dAp$ (the rate of profit $r$ or the factor $1+r$ can be considered as some kind of "factor price of capital"). With assumption (A9), wages are paid at the end of the uniform production period. Hence, vector $w_d\ell$ represents the *unit labor costs.*[33] Unit Capital costs and unit labor costs form the total *unit costs of production* (note that, according to (A1), there are no costs caused by the use of land or any other kind of natural resources). The uniformity of the rate of profit ensures that under the condition of free competition capitalists cannot earn in different branches different amounts of profits in relation to the value of advanced capital: No extra profits can be made over the (opportunity) costs of (own) capital.[34] A uniform wage rate guarantees that workers cannot earn different wages in different lines of production. Thus, equations (3.9) and (3.10) form *equilibrium conditions* for an economy with free competition and full information. In accordance with assumption (A13), we take the rate of profit $r$ as *exogenous.* With a *given* numeraire bundle $y$,[35] this means that in equilibrium the commodity prices $p$ and the wage rate $w$ are *endogenously* determined at the rate of profit $r$.

The challenge is now to prove the *existence* of a price system with respect to technique $d$, i.e finding a pair of a commodity price vector and a wage rate satisfying the system of *linear equations* (3.9) and (3.10). For this purpose, we define

$${}_dR := \frac{1 - {}_d\lambda_L}{{}_d\lambda_L} \tag{3.11}$$

for any technique $d \in I$. Because of Theorem 3.1 (b) and postulate (P1), the inequalities $0 < {}_d\lambda_L < 1$ are true and hence ${}_dR > 0$. Therewith, we can

formulate the following theorem:

**Theorem 3.4.** *Let $d \in I$ be a technique. Then for any rate of profit $r \in [0, {}_dR]$ and any semipositive numeraire bundle $y \in \mathbb{R}^{n\times 1}$ there exists a unique system of production prices with respect to technique $d$.*

*Proof.* First we prove the theorem for the case $r = {}_dR$; see also Kurz and Salvadori (1995, pp. 97-8). Due to (P1), matrix ${}_dA$ is semipositive and indecomposable. According to theorem 3.1 (d), matrix ${}_dA$ has with respect to eigenvalue ${}_d\lambda_L$ a positive eigenvector $x \in \mathbb{R}^{n\times 1}$, that is unique except for the multiplication by a scalar. This implies that

$$x = \frac{1}{{}_d\lambda_L} {}_dAx.$$

From $r = {}_dR = \frac{1-{}_d\lambda_L}{{}_d\lambda_L}$ we receive ${}_d\lambda_L = \frac{1}{r+1}$ and hence

$$x = (1+r)_dAx.$$

Now, we define the vector

$$\overline{p} := \lambda x \tag{3.12}$$

for a positive real number $\lambda$. As a result, we obtain

$$\overline{p} = (1+r)_dA\overline{p} = (1+r)_dA\overline{p} + 0_d\ell$$

meaning that $\overline{p}$ satisfies the system of equations (3.9) for the case $w = 0$. If we define the positive scalar $\lambda$ by

$$\lambda := \frac{1}{y^Tx}, \tag{3.13}$$

then it is clear that $\overline{p}$ satisfies equation (3.10), that is,

$$y^T\overline{p} = 1$$

is satisfied. Hence, $(\overline{p}, 0, r)$ is the unique system of production prices with respect to technique $d$ at the rate of profit $r$.
Next, we consider the case $r < {}_dR$. Matrix ${}_dA$ is indecomposable and semipositive, and so is matrix $(1+r)_dA$. From $0 < {}_d\lambda_L$ and $r < {}_dR$ we can infer that $(1+r)_d\lambda_L < 1$. At the same time, $(1+r)_d\lambda_L$ is the largest real eigenvalue of $(1+r)_dA$. Thus according to Theorem 3.1 (c), the inverse matrix $(I_n - (1+r)_dA)^{-1}$ exists and is positive. If we define the positive scalar

$$w := \frac{1}{y^T(I_n - (1+r)_dA)^{-1}{}_d\ell}$$

and the positive vector

$$p := \frac{1}{y^T(I_n - (1+r)_d A)^{-1}{}_d\ell}(I_n - (1+r)_d A)^{-1}{}_d\ell,$$

then it can be easily shown that $(p, w, r)$ is the unique system of production prices with respect to technique $d$ at the rate $r$; see for example Kurz and Salvadori (1995, p. 98).

□

Hence, for any rate of profit in the closed interval $[0, {}_dR]$ and a given semipositive numeraire bundle the system of linear equations (3.9) and (3.10) has a unique solution with respect to technique $d \in I$.[36] For the rest of this section, we take the semipositive vector $y \in \mathbb{R}^{n\times 1}$ as numeraire bundle. If we define vector $\overline{p}$ like in the proof of Theorem 3.4 by (3.12) and (3.13), then the functions

$${}_dw(r) := \begin{cases} \frac{1}{y^T(I_n-(1+r)_d A)^{-1}{}_d\ell} & \text{if } r \in [0, {}_dR), \\ 0 & \text{if } r = {}_dR, \end{cases}$$

and

$${}_dp(r) = ({}_dp_1(r), ..., {}_dp_n(r))^T := \begin{cases} {}_dw(r)(I_n - (1+r)_d A)^{-1}{}_d\ell & \text{if } r \in [0, {}_dR), \\ \overline{p} & \text{if } r = {}_dR, \end{cases}$$

are well defined on $[0, {}_dR]$. As we have seen in the proof of Theorem 3.4, we have ${}_dw(r) \geq 0$ and ${}_dp(r) > \mathbf{0}$ for each $r \in [0, {}_dR]$. Therefore, the pair $({}_dp(r), {}_dw(r), r)$ forms the unique system of production prices with respect to technique $d$ for any rate of profit $r \in [0, {}_dR]$. Function ${}_dw$ is of particular interest with respect to functional income distribution. It namely relates the rate of profit to the (equilibrium) wage rate under the assumption that only technique $d$ is in use. Function ${}_dw$ has some important properties which are summarized by the following theorem:

**Theorem 3.5.** *Let $d \in I$ be a technique. Then function ${}_dw$ is continuous and strictly monotonically decreasing on $[0, {}_dR]$.*

*Proof.* Considering a rate of profit $r \in [0, {}_dR)$, we can write the inverse of matrix $I_n - (1+r)_d A$ as

$$(I_n - (1+r)_d A)^{-1} = \frac{1}{det(I_n - (1+r)_d A)}{}_dB(r), \tag{3.14}$$

where $det(I_n - (1+r)_d A)$ signifies the determinant of $I_n - (1+r)_d A$ and ${}_dB(r)$ represents the adjugate of the same matrix (i.e. the matrix of cofactors of

the elements of $I_n - (1+r)_d A$).[37] Expression $det(I_n - (1+r)_d A)$ represents a polynomial of degree $n$ in $r$, while the elements of ${}_dB(r)$ are polynomials of degree $n-1$ in $r$.[38] Next, we define the function

$$g(r) := \frac{y^T {}_dB(r)_d\ell}{det(I_n - (1+r)_d A)}$$

for any $r \in [0, {}_dR)$. Because of the fact that ${}_d\lambda$ is an eigenvalue of ${}_dA$ and identity (3.11), we get $det(I_n - (1+{}_dR)_d A) = 0$. Thus, $g$ is a rational function with the pole ${}_dR$. Since

$$y^T (I_n - (1+r)_d A)^{-1} {}_d\ell = g(r),$$

we can write

$${}_dw(r) = \frac{1}{g(r)}$$

for each $r \in [0, {}_dR)$. Therefore, we can conclude that ${}_dw$ is differentiable and hence continuous on $[0, {}_dR)$. In Kurz and Salvadori (1995, pp. 99) it is shown that the derivative of ${}_dw$ takes only negative values on $(0, {}_dR)$. This means that function ${}_dw$ is also strictly monotonically decreasing on $[0, {}_dR)$. From ${}_dw(r) > 0$ it follows $g(r) > 0$ for every $r \in [0, {}_dR)$. Function $g$ is therefore strictly monotonically increasing on the same interval. With ${}_dR$ as pole of function $g$, we can conclude that

$$\lim_{r \to {}_dR} g(r) = \infty$$

and thus

$$\lim_{r \to {}_dR} {}_dw(r) = 0 = {}_dw({}_dR),$$

meaning that function ${}_dw$ is also continuous at ${}_dR$. Therefore, it is continuous and strictly monotonically decreasing on the entire interval $[0, {}_dR]$. □

With the same technique in use, the rate of profit can only rise, if the wage rate falls.[39] As ${}_dw({}_dR) = 0$ for any technique $d \in I$, function ${}_dw(r)$ converges to 0 as $r$ approaches ${}_dR$. This means that at the rate of profit ${}_dR$ workers receive no remuneration, when technique $d$ is in use. In the context of our analysis, it makes sense to consider ${}_dR$ as the *maximum feasible rate of profit* with respect to technique $d$.[40]

So far, we have examined the determination of the price system under the condition that there is no switch in the applied technique. Now, we must deal with the determination of the production technique itself. This leads us directly to the choice of technique by cost-minimizing producers, whereby

we assume that $r \in [0, \max_{h \in I}\, {}_hR]$. This guarantees that there is always at least one technique, with respect to which the system of equations (3.9) and (3.10) is solvable with positive commodity prices and a nonnegative wage rate. We define *cost-minimizing techniques* as follows; see also Kurz and Salvadori (1995, pp. 128-9):

**Definition 3.7.** *Let $d \in I$ be a technique and consider $r \in [0, {}_dR]$. Technique $d$ minimizes costs at the rate of profit $r$, if for any technique $e \in I$ with $r \leq {}_eR$ the condition*

$$ {}_dp(r) \leq (1+r)_eA_dp(r) + {}_dw(r)_e\ell \tag{3.15} $$

*is satisfied.*

Why is any technique $d \in I$, that satisfies condition (3.15), to be considered as cost-minimizing? When we suppose that at a rate of profit $r \in [0, {}_dR]$ the ruling price system in the economy equals the system of production prices with respect to technique $d$, then any producer can sell or buy commodities at prices ${}_dp(r)$ and workers get paid at the wage rate ${}_dw(r)$. As a result, each producer, who uses another technique $e$ (with $r \leq {}_eR$) to produce one unit of each commodity, faces unit capital costs equal to $(1+r)_eA_dp(r)$ and unit labor costs equal to ${}_dw(r)_e\ell$. Recall that ${}_dp(r)$ are the incurred unit costs of production, when technique $d$ is used. Thus, any alternative technique generates in each line of production unit costs, that are not smaller than those generated by technique $d$: Profit seeking producers minimize (unit) costs by applying technique $d$. There are no possibilities to earn any extra profits by using an alternative technique, and accordingly there are no incentives to use a technique other than technique $d$. In addition, producers always yield the same rate of profit by technique $d$, no matter which branch of production they enter. Therefore, (3.15) is *another* equilibrium condition. When technique $d$ minimizes costs at the rate of profit $r \leq {}_dR$, then the system of prices $({}_dp(r), {}_dw(r), r)$ satisfies the conditions (3.9), (3.10) and (3.15), which define an *equilibrium* under free competition. The triple $({}_dp(r), {}_dw(r), r)$ can thus be considered as *long-run equilibrium prices* or "*long-period position*" (Kurz and Salvadori, 1995, p. 127-8).[41]

Before we proceed to show the existence of cost-minimizing techniques, we need to consider some more theorems. First we prove:

**Theorem 3.6.** *Let $d, e \in I$ be two techniques and consider $r \in [0, \min\{{}_dR, {}_eR\}]$. Then, the following two statements are true:*

*(a) If ${}_dw(r) < {}_ew(r)$, then technique $d$ does not minimize costs at $r$.*

(b) *If both technique $d$ and technique $e$ minimize costs at $r < \min\{{}_dR, {}_eR\}$, then ${}_dp(r) = {}_ep(r)$ and ${}_dw(r) = {}_ew(r)$.*

*Proof.* In Kurz and Salvadori (1995, p. 130), it is shown that statement (a) is true for the case $r < \min\{{}_dR, {}_eR\}$. It remains to provide a proof for the case $r = \min\{{}_dR, {}_eR\}$. Since ${}_ew(r) > {}_dw(r)$ is assumed, we have ${}_ew(r) > 0$. This means that $r < {}_eR$ and thus $r = {}_dR$. As a consequence, we have ${}_dw(r) = 0$ and ${}_dp(r) = \overline{p}$. Now, we assume that $d$ minimizes costs at $r$, that is,

$$(I_n - (1+r)_eA)_dp(r) \leq {}_dw(r)_e\ell.$$

Since $(I_n - (1+r)_eA)^{-1} > \mathbf{0}$ and ${}_dw(r) = 0$, we obtain ${}_dp(r) \leq \mathbf{0}$ and so the false statement $\overline{p} \leq \mathbf{0}$.
The proof of statement (b) can be found in Kurz and Salvadori (1995, pp. 130-1). □

An important consequence of Theorem 3.6 is that even with a technology containing more than one technique the negative relation between the rate of profit and the wage rate remains true, because the cost-minimizing technique yields the highest wage rate and the wage-profit relation with respect to a single technique is strictly monotonically decreasing; see Theorem 3.5. Unless there is any kind of technical progress (meaning an extension of $T$ by techniques with smaller input coefficients), the rate of profit (or the "factor price of capital") can only rise, when the wage rate falls.[42] Statement (b) in Theorem 3.6 tells us that long-run equilibrium prices are unique, whenever cost-minimizing techniques exist (no matter whether there is only one or more than one cost-minimizing technique).

The next theorem reads:

**Theorem 3.7.** *Let $d \in I$ be a technique and consider $r \in [0, {}_dR]$. In addition, let the numeraire bundle be positive. If there is a process $(a_1, ..., a_n, l) \in T_v$ with $v \in \{1, ..., n\}$ such that*

$${}_dp_v(r) > (1+r)\sum_{j=1}^{n} {}_dp_j(r)a_j + {}_dw(r)l,$$

*then there exits a technique $e \in I$ with ${}_ew(r) > {}_dw(r)$.*

*Proof.* We introduce a matrix $A = (a_{ij})_{i,j=1,...,n} \in \mathbb{R}^{n\times n}$ by defining $a_{vj} := a_j$ and $a_{ij} := {}_da_{ij}$ for any $i \in \{1, ..., n\}\backslash\{v\}$ and $j \in \{1, ..., n\}$. We further define a

vector $\ell = (l_1, ..., l_n)^T \in \mathbb{R}^{n\times 1}$ by $l_v := l$ and $l_i := {}_d l_i$ for any $i \in \{1, ..., n\}\backslash\{v\}$. Then, there is a technique $e \in I$ such that ${}_eA = A$ and ${}_e\ell = \ell$. We receive

$$ {}_dp(r) \gneqq (1+r)_eA_dp(r) + {}_dw(r)_e\ell. $$

From this we obtain

$$ (\frac{1}{1+r}I_n - {}_eA)_dp(r) \gneqq \frac{1}{1+r}{}_dw(r)_e\ell \geq \mathbf{0}. $$

Owing to postulate (P1), matrix ${}_eA$ is semipositive and indecomposable. With ${}_dp(r) > \mathbf{0}$ we thus get from Theorem 3.1 (e)

$$ \frac{1}{1+r} > {}_e\lambda_L = \frac{1}{1+{}_eR}, $$

and so $r < {}_eR$. In Kurz and Salvadori (1995, pp. 129-30) it is shown that ${}_ew(r) > {}_dw(r)$, if $r < {}_dR$. For the case $r = {}_dR$, we immediately receive

$$ {}_ew(r) > 0 = {}_dw(r). $$

□

The following theorem shows an *equivalent* characterization of cots-minimizing techniques:

**Theorem 3.8.** *Let $d \in I$ be a technique and consider $r \in [0, {}_dR]$. In addition, let the numeraire bundle be positive. Then technique $d$ minimizes costs at the rate of profit $r$, if and only if ${}_dw(r) \geq {}_ew(r)$ for any technique $e \in I$ with $r \leq {}_eR$.*

*Proof.* A complete proof can be found in Kurz and Salvadori (1995, p. 79) for the case $n = 2$, but it can be analogously applied to the general case $n \in \mathbb{N}$ (ibid., p. 131). To prove the "only if" part the contraposition of Theorem 3.6 (a) can be applied. Thus, the fact that technique $d$ minimizes costs at the rate $r$ implies ${}_dw(r) \geq {}_ew(r)$ for any technique $e \in I$ with $r \leq {}_eR$. The "if" part can be proved by using the contraposition of Theorem 3.7, and so if ${}_dw(r) \geq {}_ew(r)$ for any technique $e \in I$ with $r \leq {}_eR$, then it follows that

$$ {}_dp(r) \leq (1+r)_eA_dp(r) + {}_dw(r)_e\ell. $$

This means that technique $d$ is cost-minimizing at $r$. □

At last, we are finally able to formulate an *existence statement* for cost-minimizing techniques and thus for long-run equilibria. Up until now, we have not needed postulate (P2), but to prove the the existence of cost-minimizing techniques it is essential:[43]

**Theorem 3.9.** *For each rate of profit $r \in [0, \max_{h \in I}\, {}_hR]$ there exists a technique $d \in I$, which minimizes costs at $r$.*

*Proof.* A proof is provided by Kurz and Salvadori (1995, p. 79) for the case $n = 2$. It can be extent to the general case $n \in \mathbb{N}$ (ibid., p. 131). Because of postulate (P2), the technology $T$ is a finite set. As a consequence, there is a technique $d \in I$ such that ${}_dw(r) \geq {}_ew(r)$ for each technique $e \in I$ with $r \leq {}_eR$. This is true for any arbitrary semipositive and thus positive numeraire bundle. According to Theorem 3.8, technique $d$ is cost-minimizing at the rate of profit $r$.[44] □

Of course, this theorem does not tell us whether the cost-minimizing technique is unique at a given rate of profit in $[0, \max_{h \in I}\, {}_hR]$. This issue is discussed in the context of the existence of *switch points*, which is part of the content of Section 4.

# 4 On switch points, well-behaved aggregate production functions and regular economies

Based on the equilibrium concept, which we have introduced in Section 3, we give in this section a clear definition of both *well-behaved aggregate production functions* and *regular economies*. Thereby, the notion of *switch points* serves as an important auxiliary tool. Let again $n \in \mathbb{N}$ be the number of commodities that are produced in the economy. As before, we measure prices in terms of a semipositive numeraire bundle $y \in \mathbb{R}^{n \times 1}$.

## 4.1 Definition and existence of switch points

With Theorem 3.9 we know that at any rate of profit in the set of all feasible rates of profit $[0, \max_{h \in I}\, {}_hR]$ there always exist cost-minimizing techniques that are chosen by profit-seeking producers, and so long-run equilibrium prices are always uniquely determined; see Theorem 3.6 (b). We now make two additional postulates:

(AP1) For each technique $d \in I$ there is at least one rate of profit $r \in [0, \max_{h \in I}\, {}_hR]$ such that $d$ is the unique technique that minimizes costs at the rate $r$.[45]

(AP2) Two different techniques are characterized by different per labor unit outputs and different maximum rates of profit, i.e ${}_dc_1 \neq {}_ec_1$ and ${}_dR \neq {}_eR$ for any techniques $d, e \in I$ with $d \neq e$.

Of course we are only interested in technologies which contains at least two techniques. For this we further assume that $|I| \geq 2$.

We define *switch points* from one technique to another:

**Definition 4.1.** *Let $d, e \in I$ be two different techniques and consider $r \in (0, \min\{{}_dR, {}_eR\})$. The rate of profit $r$ is called a switch point from technique $d$ to technique $e$, if there exists $\epsilon > 0$ with $(r - \epsilon, r + \epsilon) \subseteq (0, \min\{{}_dR, {}_eR\})$ such that the following properties are satisfied:*

*(a) Technique $d$ is the unique cost-minimizing technique at any rate of profit in $(r - \epsilon, r)$.*

*(b) Technique $e$ is the unique cost-minimizing technique at any rate of profit in $(r, r + \epsilon)$.*

*(c) Technique $d$ and technique $e$ are both cost-minimizing at $r$.*

Definition 4.1 postulates that the set of all switch points, or any subset of it, must be disconnected. Now, we need to state the questions whether switch points exist, and whether their total number is finite. Before we answer them, we prove the following theorem:

**Theorem 4.1.** *Let $d \in I$ be a technique and consider $r \in [0, {}_dR]$. Then the following equation is satisfied:*

$$ {}_dp_1(r)_dc_1 = r({}_dp(r))^T{}_dk + {}_dw(r). \tag{4.16} $$

*Proof.* Vector ${}_dQ(C)$ is the gross output with respect to net output $C$ and technique $d$. To keep the notation in this proof as simple as possible, we define ${}_dQ := {}_dQ(C)$. It follows that

$$ C = {}_dQ - ({}_dA)^T{}_dQ. $$

If we multiply this equation with the price vector $({}_dp(r))^T$ from the left-hand side, we obtain

$$ ({}_dp(r))^TC = ({}_dp(r))^T({}_dQ - ({}_dA)^T{}_dQ) = ({}_dp(r))^T{}_dQ - ({}_dp(r))^T({}_dA)^T{}_dQ. $$

With $({}_dp(r), {}_dw(r), r)$ being the system of production prices with respect to technique $d$ at the rate of profit $r$, we then receive

$$ ({}_dp(r))^TC = ((1+r)_dA_dp(r) + {}_dw(r)_d\ell)^T{}_dQ - ({}_dp(r))^T({}_dA)^T{}_dQ = $$

$$ = (1+r)({}_dp(r))^T({}_dA)^T{}_dQ + {}_dw(r)({}_d\ell)^T{}_dQ - ({}_dp(r))^T({}_dA)^T{}_dQ = $$

$$= r({}_dp(r))^T({}_dA)^T{}_dQ + {}_dw(r)({}_d\ell)^T{}_dQ.$$

Because of the identities ${}_dL(C) = ({}_d\ell)^T{}_dQ$ and ${}_dK(C) = ({}_dA)^T{}_dQ$ as well as condition $({}_dp(r))^TC = {}_dp_1(r)C_1$, we get

$${}_dp_1(r)C_1 = r({}_dp(r))^T{}_dK(C) + {}_dw(r){}_dL(C).$$

If we divide this equation by ${}_dL(C)$, then with (3.5) and (3.6) we have

$${}_dp_1(r){}_dc_1(C) = r({}_dp(r))^T{}_dk(C) + {}_dw(r).$$

Through identities (3.7) and (3.8), we finally receive equation (4.16). □

The next theorem helps us to answer the questions regarding the existence and the number of switch points:

**Theorem 4.2.** *The set of all rates of profit in* $[0, \max_{h\in I}{}_hR]$*, at which more than one technique is cost-minimizing, is a finite and non-empty subset of the interval* $(0, \max_{h\in I}{}_hR)$*.*

*Proof.* Consider two different techniques $d, e \in I$. From Theorem 4.1 it follows that ${}_hp_1(0){}_hc_1 = {}_hw(0)$ for each technique $h \in I$, and so with $y = (1, 0, ..., 0)^T$, i.e ${}_dp_1(0) = {}_ep_1(0) = 1$, we have ${}_dw(0) = {}_dc_1$ and ${}_ew(0) = {}_ec_1$. At the same time, ${}_hw({}_hR) = 0$ for each technique $h \in I$. Because of Theorem 3.5, Theorem 3.6 (a) and (AP2), that postulates ${}_dc_1 \neq {}_ec_1$ and ${}_dR \neq {}_eR$, technique $d$ and technique $e$ cannot both be cost-minimizing at the rates of profit 0 and $min\{{}_dR, {}_eR\}$. Therefore, the set of all rates of profit, at which more than one technique is cost-minimizing, is a subset of $(0, \max_{h\in I}{}_hR)$. Now, we assume that technique $d$ and technique $e$ are both cost-minimizing at some $r \in (0, min_dR, {}_eR)$. Then, from Theorem 3.6 (b) it follows that ${}_dp(r) = {}_ep(r)$ and ${}_dw(r) = {}_ew(r)$. Hence, we can infer

$$\frac{1}{{}_dw(r)}{}_dp(r) = \frac{1}{{}_ew(r)}{}_ep(r). \tag{4.17}$$

Recall that ${}_ep(r)$ satisfies condition (3.9). Then from the statements ${}_dp(r) = {}_ep(r)$ and ${}_dw(r) = {}_ew(r)$ it follows that

$${}_ep(r) = (1 + r){}_eA{}_dp(r) + {}_dw(r){}_e\ell.$$

Because of this and the fact that ${}_dp(r)$ satisfies condition (3.9) as well, condition (4.17) is equivalent to

$$(1 + r)({}_dA - {}_eA)\frac{1}{{}_dw(r)}{}_dp(r) + ({}_d\ell - {}_e\ell) = 0.$$

If we again define ${}_dB(r)$ as adjugate of matrix $I_n - (1+r){}_dA$ (as in the proof of Theorem 3.5), then, through condition (3.14) and

$$\frac{1}{{}_dw(r)}{}_dp(r) = (I_n - (1+r){}_dA)^{-1}{}_d\ell,$$

it can be shown that (4.17) is equivalent to

$$(1+r)({}_dA - {}_eA){}_dB(r){}_d\ell + det(I_n - (1+r){}_dA)({}_d\ell - {}_e\ell) = 0. \tag{4.18}$$

The left-hand side of equation (4.18) is a vector of polynomials of degree $n$ in $r$. This means that $r$ is a root of several polynomials of degree $n$.[46] As any polynomial has finitely many different roots, there exit finitely many different rates of profit in $(0, min\{{}_dR, {}_eR\})$ at which both $d$ and $e$ are cost-minimizing. Since we have postulated that $T$ is finite, the set of all rates of profit in $[0, \max_{h\in I}{}_hR]$, at which more than one technique is cost-minimizing, must be finite.

Now, let the numeraire bundle be positive and recall that $|I| \geq 2$. Then, according to (AP1), there must be two different techniques $d, e \in I$ and two different rates of profit $r^{'}, r^{''} \in [0, max_{h\in I}{}_hR]$ such that $d$ is the unique cost-minimizing technique at $r^{'}$ and $e$ is the unique cost-minimizing technique at $r^{''}$. Without loss of generality, let $r^{'} < r^{''}$. According to Theorem 3.8, we have ${}_dw(r^{'}) > {}_hw(r^{'})$ for any technique $h \in I\backslash\{d\}$ with $r^{'} \leq {}_hR$ and ${}_ew(r^{''}) > {}_hw(r^{''})$ for any technique $h \in I \setminus \{e\}$ with $r^{''} \leq {}_hR$. Because of Theorem 3.5, function ${}_hw$ is continuous for any technique $h \in I$. Then, from $I$ being a finite set it follows that there must exist a technique $t \in I \setminus \{d\}$ and a rate of profit $\tilde{r}$ such that $r^{'} < \tilde{r} < r^{''}$ and ${}_dw(\tilde{r}) = {}_tw(\tilde{r}) \geq {}_hw(\tilde{r})$ for any technique $h \in I$ with $\tilde{r} \leq {}_hR$.[47] As a result of Theorem 3.8, both $d$ and $t$ are cost-minimizing at $\tilde{r}$. The set of all rates of profit in $[0, \max_{h\in I}{}_hR]$, at which more than one technique is cost-minimizing, is therefore not empty. □

Because of the last theorem, the total number of all rates of profit in $(0, \max_{h\in I}{}_hR)$, at which more than one technique is cost-minimizing, equals a natural number $k \in \mathbb{N}$ with $k \geq 1$. We denote the set of all such rates of profit by $\{r_1, ..., r_k\}$ assuming that $r_i < r_{i+1}$ for any $i \in \{1, ..., k-1\}$, if $k \geq 2$; note that the rate $r_i$ is positive for each $i \in \{1, ..., k\}$. This means that at any rate of profit in $[0, \max_{h\in I}{}_hR] \setminus \{r_1, ..., r_k\}$ the cost-minimizing technique must be unique, and so the set of all switch points is a subset of $\{r_1, ..., r_k\}$. The question is, whether there is a rate of profit in this set, which is a switch point in the sense of Definition 4.1. To answer this question, we define for convenience

$$r_0 := 0$$

and

$$r_{k+1} := \max_{h \in I} {}_h R.$$

Owing to the continuity of the wage-profit functions, for any $i \in \{0, ..., k\}$ there is a technique $d \in I$, which is the *unique* cost-minimizing technique at each rate of profit $r \in (r_i, r_{i+1})$. If this were not so, then there would be a rate of profit in the same interval, at which more than one technique is cost-minimizing. For the same reason, technique $d$ must as well be cost-minimizing at both the rates $r_i$ and $r_{i+1}$. Because of (AP1) and $|I| \geq 2$, there exists $i \in \{1, ..., k\}$ such that the technique, which is the unique cost-minimizing technique at each rate of profit $r \in (r_{i-1}, r_i)$, is *different* from the technique, which is the unique cost-minimizing technique at each rate of profit $r \in (r_i, r_{i+1})$. This means that $r_i$ is a switch point from one technique to another. Hence, there is at least one switch point in the sense of Definition 4.1 in the set $\{r_1, ..., r_k\}$. To keep things as simple as possible, we postulate:

(AP3) The set of all switch points is identical to the set $\{r_1, ..., r_k\}$.

(AP4) If a rate of profit $r \in \{r_1, ..., r_k\}$ is a switch point from technique $d$ to technique $e$, then $d$ and $e$ are the only cost-minimizing techniques at $r$.

Postulate (AP3) implies, that the total number of switch points equals $k$. It also excludes cases, where there exists $i \in \{1, ..., k\}$ such that the unique cost-minimizing technique is identical at all rates of profit in both $(r_{i-1}, r_i)$ and $(r_i, r_{i+1})$. For example, if $k \geq 3$, then at the rate of profit $r_1$ there is a switch from the technique, which is the unique cost-minimizing technique at all rates of profit in $[r_0, r_1)$, to a different technique, which is the unique cost-minimizing technique at all rates of profit in $(r_1, r_2)$. At the rate of profit $r_2$ a switch occurs to a different technique, which is the unique cost-minimizing technique at all rates of profit in $(r_2, r_3)$, and so on. Postulate (AP4) guarantees that for each $i \in \{1, ..., k\}$ the technique, which is cost-minimizing at each rate of profit in $(r_{i-1}, r_i)$, and the technique, which is cost-minimizing at each rate of profit in $(r_i, r_{i+1})$, are the only cost-minimizing techniques at the switch point $r_i$.

## 4.2 An important property of switch points

In this subsection, we discuss an important property of switch points. For this, we apply a notion, that can be used as characterization of what Burmeister considered to be a "regularity condition" (in regard to the quality of the reaction of per labor unit output to changes of the rate of profit) for the case of finite technologies (Burmeister, 1980, pp. 121-2). We define:

**Definition 4.2.** *Let $r \in \{r_1, ..., r_k\}$ be a switch point from technique $d$ to technique $e$. Then we define*

$$R(r) := \sum_{i=1}^{n} \frac{{}_e p_i(r)}{{}_e p_1(r)} ({}_e k_i - {}_d k_i). \tag{4.19}$$

The magnitude $R(r)$ in Definition 4.2 measures the aggregated changes in the heterogeneous capital intensities, that result from a switch from technique $d$ to technique $e$, through an evaluation with *relative* equilibrium commodity prices at switch point $r$ (because of Theorem 3.6 (b), in equilibrium the condition ${}_d p_i(r) = {}_e p_i(r)$ is satisfied for each $i \in \{1, ..., n\}$). In other words, $R(r)$ represents the so-called "real Wicksell effect", that occurs at the switch point $r$. Since in expression (4.19) the relative prices are measured in terms of commodity 1, the same is true for the magnitude $R(r)$ (we shall use Definition 4.2 to construct an *index of capital intensity* as well as an *index of labor productivity*, both necessary for the definition of *aggregate production functions*).

The following Theorem reads:

**Theorem 4.3.** *If $r \in \{r_1, ..., r_k\}$ is a switch point from technique $d$ to technique $e$, then*

$${}_e c_1 - {}_d c_1 = r R(r). \tag{4.20}$$

*Proof.* We apply Theorem 4.1 to obtain

$${}_d p_1(r) {}_d c_1 = r({}_d p(r))^T {}_d k + {}_d w(r)$$

and

$${}_e p_1(r) {}_e c_1 = r({}_e p(r))^T {}_e k + {}_e w(r).$$

From $r$ being a switch point from technique $d$ to technique $e$ and Theorem 3.6 (b), it follows that ${}_d p(r) = {}_e p(r)$ and ${}_d w(r) = {}_e w(r)$. This implies

$${}_e p_1(r)({}_e c_1 - {}_d c_1) = r({}_e p(r))^T({}_e k - {}_d k) = r \sum_{i=1}^{n} {}_e p_i(r)({}_e k_i - {}_d k_i)$$

and thus

$${}_e c_1 - {}_d c_1 = r \sum_{i=1}^{n} \frac{{}_e p_i(r)}{{}_e p_1(r)} ({}_e k_i - {}_d k_i).$$

With identity (4.19), we thus receive equation (4.20). □

Theorem 4.3 means that changes of per labor unit output due to switching of techniques, can be measured through changes in the capital intensities. Since in (AP2) we have postulated different per labor unit output levels for different techniques, we immediately conclude from Theorem 4.3 that $R(r) \neq 0$ for every switch point $r \in \{r_1, ..., r_k\}$.

## 4.3 Definition of well-behaved aggregate production functions and regular economies

We define at every switch point an *index of (aggregate) capital intensity* as well as an *index of labor productivity.* The main purpose of an index of capital intensity is to reflect changes in the *heterogeneous* capital intensities by a scalar number, while an index of labor productivity is supposed to do the same with respect to changes of per labor unit output of commodity 1. For defining an index of both capital intensity and labor productivity, we consider technique $d \in I$ as the unique cost-minimizing technique at every rate of profit $r \in [r_0, r_1)$. Next, we set

$$K_0 := \sum_{i=1}^{n} \frac{{}_d p_i(0)}{{}_d p_1(0)} {}_d k_i$$

and

$$c_0 := {}_d c_1$$

as "initial values" for an index of capital intensity and an index of labor productivity, respectively. With $K_0$ and $c_0$ an index of capital intensity and an index of labor productivity can be defined as follows:

**Definition 4.3.** *For each $i \in \{1, ..., k\}$ the index of capital intensity $K_i$ and the index of labor productivity $c_i$ is defined by*

$$K_i := K_0 + \sum_{j=1}^{i} R(r_j) \tag{4.21}$$

*and*

$$c_i := c_0 + \sum_{j=1}^{i} r_j R(r_j). \tag{4.22}$$

Since $R(r_i)$ is measured in terms of commodity 1 for each $i \in \{1, ..., k\}$, both the index of capital intensity and the index of labor productivity can be interpreted as magnitudes that are as well measured in terms of commodity 1. As a consequence of Theorem 4.3, we get $c_i = {}_e c_1$ for each $i \in \{1, ..., k\}$, if $r_i$ is a switch point from one technique to technique $e$, and so the index of labor productivity is an *exact* measure of the (homogeneous) per labor unit output.[48] For any $i \in \{1, ..., k\}$ the magnitude $K_i$ is defined as sum from $R(r_1)$ to $R(r_i)$, which is added to $K_0$. Since the "real Wicksell effect" $R(r_i)$ reflects changes in the physical (heterogeneous) capital intensities, occurring at switch point $r_i$, our index of capital intensity serves its purpose to describe

these changes in form of a scalar number.[49] Since $R(r) \neq 0$ for every switch point $r \in \{r_1, ..., r_k\}$, we have $K_{i-1} \neq K_i$ and $c_{i-1} \neq c_i$ for each $i \in \{1, ..., k\}$; see (AP2) and Theorem 4.3. Note that due to identities (4.21) and (4.22) if $K_{i-1} > K_i$ ($K_{i-1} < K_i$) for any $i \in \{1, ..., k\}$, then we have $c_{i-1} > c_i$ ($c_{i-1} < c_i$); see also the proof of Theorem 4.7.

One may wonder why we use an index of capital intensity which takes into account the "real Wicksell effect" but not the "price Wicksell effect", which means changes in the aggregated value of the heterogeneous capital intensities (i.e. the heterogeneous capital intensities as physical quantities times the respective equilibrium commodity prices at the various rates of profit), that result from mere changes in the equilibrium commodity prices. The reason is that an index of capital intensity is supposed to reflect switches of techniques (and thus the corresponding changes in the physical capital intensities), that occur at switch points, but not changes in commodity prices, that occur, when the rate of profit varies between two switch points (note that a "real Wicksell effect" can only be observed at switch points, while at all the other rates of profit we only face a "price Wicksell effect"). Of course, this does not mean that our index of capital intensity is entirely independent of the economy's price system, since it uses equilibrium prices at the switch points to aggregate (or evaluate) the corresponding changes in the heterogeneous capital intensities. Nevertheless, as we shall see in the course of this and the next section, the economy's condition of exhibiting "regular" (or "neoclassical") properties (with respect to the relationship between magnitudes like employment, the labor productivity and the wage rate) merely depends on the sign of the "real Wicksell effect" at the various switch points (but not on the pure "price Wicksell effect").[50] Besides, in contrast to the "price Wicksell effect", the sign of the "real Wicksell effect" is invariant to the choice of numeraire; see also Burmeister (1980, pp. 122-4).

Next, we discuss the relation of both the index of capital intensity and the index of labor productivity to equilibrium. For this purpose, we need to consider two different cases for the current rate of profit $r \in [0, \max_{h \in I\ h} R]$. First, we assume $r$ to be different from any switch point, that is, we assume $r \notin \{r_1, ..., r_k\}$. This case means that the cost-minimizing technique to produce the net output $C_1$ is unique at the rate $r$. If $r \in [0, r_1)$, this technique is characterized by the point $(K_0, c_0)$, meaning that at the rate $r$ the equilibrium is characterized by the index of capital intensity $K_0$ and the index of labor productivity $c_0$. If $k \geq 2$ and $r \in (r_i, r_{i+1})$ with $i \in \{1, ..., k-1\}$, the cost-minimizing technique is characterized by the pair $(K_i, c_i)$, meaning that at $r$

the equilibrium index of capital intensity is equal to $K_i$ and the equilibrium index of labor productivity is equal to $c_i$. If $r \in (r_k, \max_{h \in I\ h} R]$, the equilibrium is characterized by the pair $(K_k, c_k)$. In the second case, we assume that $r$ equals a switch point, i.e. there exists $i \in \{1, ..., k\}$ such that $r = r_i$ with $r_i$ representing a switch point from technique $d$ to technique $e$. According to (AP4) technique $d$ and technique $e$ are the only cost-minimizing techniques at the rate of profit $r$. At the same time, technique $d$ is characterized by the pair $(K_{i-1}, c_{i-1})$ and technique $e$ is characterized by the pair $(K_i, c_i)$. Of course, at switch point $r$ it does not matter to cost-minimizing producers if they apply technique $d$, technique $e$ or a combination of both techniques to produce the net output $C_1$. If we assume for an arbitrary number $\lambda \in [0, 1]$ that the amount $\lambda C_1$ is produced by technique $d$ and the amount $(1 - \lambda)C_1$ is produced by technique $e$, then in equilibrium the index of capital intensity is equal to $\lambda K_{i-1} + (1 - \lambda)K_i$ and the index of labor productivity is equal to $\lambda c_{i-1} + (1 - \lambda)c_i$. But each other convex linear combination of the pairs $(K_{i-1}, c_{i-1})$ and $(K_i, c_i)$ is also consistent with the equilibrium conditions at the rate of profit $r$. This means that the share of the net output $C_1$, which is generated by technique $d$, cannot be uniquely determined. In view of these considerations, it makes sense to define the interval of all possible equilibrium levels for the index of capital intensity by

$$V := [\min_{i \in \{0,...,k\}} K_i, \max_{i \in \{0,...,k\}} K_i]$$

and the interval of all possible equilibrium levels for the index of labor productivity by

$$H := [\min_{i \in \{0,...,k\}} c_i, \max_{i \in \{0,...,k\}} c_i],$$

so as to define the set $P \subseteq V \times H$ by

$$P := \{\lambda(K_{i-1}, c_{i-1}) + (1 - \lambda)(K_i, c_i) | i \in \{1, ..., k\}, \lambda \in [0, 1]\}.$$

Set $P$ can be conceived as *set of all equilibrium positions* with respect to the *index of capital intensity* and the *index of labor productivity.*

How can a situation with a switch point, at which a technique is characterized by a higher index of capital intensity than another technique, be interpreted? Consider $i \in \{1, ..., k\}$ and suppose that $r_i$ is a switch point from technique $d$ to technique $e$. This means that at $r_i$ technique $d \in I$ is characterized by the equilibrium position $(K_{i-1}, c_{i-1})$ and technique $e \in I$ is characterized by the equilibrium position $(K_i, c_i)$. Assume further that $K_{i-1} > K_i$, that is, technique $d$ exhibits a higher index of capital intensity than technique

$e$. By Definition 4.3 it can be shown that $c_{i-1} > c_i$, that is, technique $d$ is characterized by a higher index of labor productivity than technique $e$. Hence, to produce the net output $C_1$ technique $e$ requires a higher amount of labor than technique $d$, or to put this into other words, labor employed by technique $d$ is supported by higher quantities of at least one specific kind of capital good than labor employed by technique $e$ (otherwise technique $d$ would be strictly preferred to technique $e$ by cost-minimizing producers along the entire spectrum of feasible rates of profit; a contradiction to (AP1)). Following the terminology of Robinson (1953, pp. 91), technique $d$ can be called "more mechanised" than technique $e$ (or technique $e$ can be called "less mechanised" than technique $d$).

By means of the concept $P$, we are able to define *aggregate production functions* as follows:

**Definition 4.4.** *The set $P$ is represented by a function $F : V \to H$, if*

$$F(K) = c$$

*for all $(K, c) \in P$. Function $F$ is then called an aggregate production function with respect to $P$.*

If there exists an aggregate production function in the sense of Definition 4.4, the set of equilibrium positions $P$ will become the graph of this function. Every aggregate production function with respect to $P$ has the following important property:

**Theorem 4.4.** *Let $P$ be represented by function a $F$ and consider*

$$K := \lambda K_{i-1} + (1 - \lambda) K_i$$

*with $\lambda \in [0, 1]$ and $i \in \{1, ..., k\}$. Then it follows that*

$$F(K) = \lambda F(K_{i-1}) + (1 - \lambda) F(K_i).$$

*Proof.* Because of $(K_{i-1}, c_{i-1}) \in P$ and $(K_i, c_i) \in P$, we receive $F(K_{i-1}) = c_{i-1}$ and $F(K_i) = c_i$ from Definition 4.4. If we define

$$c := \lambda c_{i-1} + (1 - \lambda) c_i,$$

then $(K, c) \in P$ holds. Since $P$ is represented by function $F$, we have $F(K) = c$. From this, we finally conclude that

$$F(K) = \lambda F(K_{i-1}) + (1 - \lambda) F(K_i).$$

□

This theorem can be used to prove:

**Theorem 4.5.** *Let function $F$ be an aggregate production function with respect to $P$. Then:*

*(a) For each $i \in \{1, ..., k\}$,*

$$\frac{F(K_{i-1}) - F(K_i)}{K_{i-1} - K_i} = r_i.$$

*(b) For each $i \in \{1, ..., k\}$ and each $K \in [\min\{K_i, K_{i-1}\}, \max\{K_i, K_{i-1}\}]$,*

$$F(K) = r_i K + c_i - r_i K_i.$$

*Proof.* If set $P$ is represented by $F$, then from $(K_{i-1}, c_{i-1}), (K_i, c_i) \in P$ it follows that $F(K_{i-1}) = c_{i-1}$ and $F(K_i) = c_i$ for any $i \in \{1, ..., k\}$. By applying Definition 4.3, we thus get

$$\frac{F(K_{i-1}) - F(K_i)}{K_{i-1} - K_i} = \frac{c_{i-1} - c_i}{K_{i-1} - K_i} = \frac{\sum_{j=1}^{i-1} r_j R(r_j) - \sum_{j=1}^{i} r_j R(r_j)}{\sum_{j=1}^{i-1} R(r_j) - \sum_{j=1}^{i} R(r_j)} =$$

$$= \frac{-r_i R(r_i)}{-R(r_i)} = r_i.$$

This proves statement (a). Next, we prove statement (b). If we consider any $K \in [\min\{K_i, K_{i-1}\}, \max\{K_i, K_{i-1}\}]$, then there exists a (unique) number $\lambda \in [0, 1]$ such that $K = \lambda K_{i-1} + (1 - \lambda) K_i$. By Theorem 4.4 we obtain

$$F(K) = \lambda F(K_{i-1}) + (1 - \lambda) F(K_i).$$

With $F(K_i) = c_i$ and statement (a), we hence have

$$F(K) = \lambda F(K_{i-1}) + (1-\lambda) F(K_i) = \lambda \frac{F(K_{i-1}) - F(K_i)}{K_{i-1} - K_i} (K_{i-1} - K_i) + F(K_i) =$$

$$= \lambda r_i (K_{i-1} - K_i) + c_i - r_i K_i + r_i K_i = r_i (\lambda K_{i-1} + (1 - \lambda) K_i) + c_i - r_i K_i.$$

Finally, we receive $F(K) = r_i K + c_i - r_i K_i$.

□

According to Theorem 4.5 (a), for any $i \in \{1, ..., k\}$ switch point $r_i$ equals what can be considered as "marginal product of capital", that results from a switching of techniques occurring at $r_i$. If we assume that $r_i$ is a switch point from technique $d$ to technique $e$, then technique $d$ is represented by the pair $(K_{i-1}, F(K_{i-1}))$ and technique $e$ is represented by the pair $(K_i, F(K_i))$.

Switching from technique $d$ to technique $e$ thus results in a "marginal productivity of capital" as difference quotient determined through dividing the change of the index of labor productivity $F(K_{i-1}) - F(K_i)$ by the change of the index of capital intensity $K_{i-1} - K_i$.

At this point, we give the notion of *well-behaved aggregate (steady-state) production functions* for *finite* technologies an exact mathematical meaning; see also Burmeister (1980, p. 134):

**Definition 4.5.** *Let function $F$ be an aggregate production function with respect to $P$ in the sense of Definition 4.4. Then $F$ is called a well-behaved aggregate production function, if the following properties are fulfilled:*

*(a) Function $F$ is left and right differentiable at each $K$ in the interior of the interval $V$, and left or right differentiable at the boundary points of $V$.*

*(b) For each rate of profit $r \in [0, \max_{h \in I}\, {}_h R]$ there exists $\tilde{K} \in V$ such that*

$$\lim_{K \to \tilde{K}^-} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K} \geq r \geq \lim_{K \to \tilde{K}^+} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K}$$

*is satisfied.*[51]

*(c) $F$ is continuous.*

*(d) $F$ is strictly monotonically increasing.*

*(e) $F$ is concave.*

Definition 4.5 gives a discrete version of what is known from many textbooks as (neoclassical) macroeconomic production function (which is *continuously differentiable* on its entire domain). Properties (c) and (d) mean that the index of labor productivity continuously increases with higher levels of the index of capital intensity, while property (a) postulates that at all points in $V$ the one-sided limits of the difference quotient of $F$ exist. Property (b) can be interpreted as follows: If $P$ is represented by a well-behaved aggregate production function $F$, then for each rate of profit $r \in [0, \max_{h \in I}\, {}_h R]$ there is a technique (or a combination of techniques), which is characterized by the index of capital intensity $\tilde{K} \in V$ at the rate $r$. The value $\tilde{K}$ satisfies the inequality

$$\lim_{K \to \tilde{K}^-} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K} \geq r,$$

which means that the left-sided differential quotient (or the left-sided "marginal productivity of capital") of $F$ at $\tilde{K}$ is not smaller than the rate of profit $r$ (the "factor price of capital"). At same time, $\tilde{K}$ fulfills the inequality

$$r \geq \lim_{K \to \tilde{K}^+} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K},$$

meaning that the right-sided differential quotient (or the right-sided "marginal productivity of capital") of $F$ at $\tilde{K}$ is not larger than $r$ (note that $F$ need not be differentiable at $\tilde{K}$). Because of the concavity of $F$, represented by property (e), the left-sided differential quotient of $F$ at $\tilde{K}$ is not larger than the one-sided differential quotients (representing the one-sided limits of the respective difference quotient) of $F$ at any index of capital intensity smaller than $\tilde{K}$ and the right-sided differential quotient of $F$ at $\tilde{K}$ is not smaller than the one-sided differential quotients of $F$ at any index of capital intensity larger than $\tilde{K}$. As a result, $\tilde{K}$ represents an "optimal" index of capital intensity characterizing a technique (or a combination of techniques), which is cost-minimizing at the rate of profit $r$ (the same is true for $\tilde{K}$ being a boundary point of $V$).[52] We further discuss this matter in a more specific context in Subsection 5.2.

So far, we have not yet discussed whether $R(r_i)$ is positive or negative for given $i \in \{1, ..., k\}$. Now, we show that at switch point $r_1$ the sign of $R(r_1)$ is definite:

**Theorem 4.6.** *The inequality $R(r_1) < 0$ is always true.*

*Proof.* Let $r_1$ be a switch point from technique $d$ to technique $e$, meaning that technique $d$ is cost-minimizing at the rate of profit 0. Additionally, we assume $R(r_1) > 0$ (recall that the case $R(r_1) = 0$ has been excluded). From Theorem 4.3 it follows that ${}_ec_1 > {}_dc_1$. Because of Theorem 4.1, if we choose commodity 1 as numeraire (i.e $y = (1, 0, ..., 0)^T$), we receive ${}_dc_1 = {}_dw(0)$ and ${}_ec_1 = {}_ew(0)$; see also the proof of Theorem 4.2. This means that ${}_ew(0) > {}_dw(0)$. As a consequence Theorem 3.6 (a), technique $d$ does not minimize costs at the rate of profit 0; a contradiction arises.

□

Now, we can define what Burmeister (1980, pp. 121-2) considered as a "regular economy" for *finite* technologies:

**Definition 4.6.** *The economy is called regular, if*

$$R(r_i) < 0$$

*for any $i \in \{1, ..., k\}$.*

An important consequence of *regular economies* (i.e. economies, where the "real Wicksell effect" is always negative) is summarized by the following theorem:

**Theorem 4.7.** *If the economy is regular, then for each $i \in \{1, ..., k\}$ it follows that*

$$K_{i-1} > K_i$$

*and*

$$c_{i-1} > c_i.$$

*Proof.* From Definition 4.3 we immediately obtain $K_i - K_{i-1} = R(r_i)$ and $c_i - c_{i-1} = r_i R(r_i)$ for each $i \in \{1, ..., k\}$. The conclusions follows from $R(r_i) < 0$ and $r_i > 0$. □

In regular economies the index of capital intensity and the index of labor productivity are both smaller at higher switch points. As a consequence, if the current rate of profit exceeds a switch point (which means a smaller wage rate), then we have a switch to a technique, that exhibits smaller levels for the index of capital intensity and the index of labor productivity. Of course, such a technique requires higher (absolute) amounts of labor to produce the constant net output $C_1$. Therefore, a rise in the rate of profit (corresponding to a fall in the wage rate) never implies a fall in the demand for labor. If $r_i$ is a switch point from technique $d$ to technique $e$, then in regular economies it follows that ${}_dc_1 > {}_ec_1$ as a result of Theorem 4.3. With a constant net output $C_1$, we thus have ${}_dL < {}_eL$. This is the main essence of Burmeister's "regularity condition". A switch of this kind, namely a switch to a "more mechanised" technique to a "less mechanised" technique (that requires higher amounts of labor to produce $C_1$) as the result of a higher rate of profit (or higher "factor price of capital") can also be interpreted as a process in which the factor capital gets "substituted" by the factor labor as the result of a smaller wage rate (or smaller "factor price of labor"); see also Robinson (1953, pp. 92).

The following theorem provides a remarkable result for regular economies:

**Theorem 4.8.** *Let the economy be regular and let $F$ be an aggregate production function with respect to $P$. Then $F$ is a well-behaved aggregate production function in the sense of Definition 4.5.*

*Proof.* With the economy being regular, it follows from Theorem 4.7 that $K_i < K_{i-1}$ for each $i \in \{1, ..., k\}$. Consequently, we obtain

$$(\min\{K_i, K_{i-1}\}, \max\{K_i, K_{i-1}\}) = (K_i, K_{i-1}).$$

We need to show that $F$ satisfies the properties (a) to (e) in Definition 4.5.

First, we prove property (b) for any rate of profit $r \in [0, \max_{h \in I}{}_h R]$. Consider the case $r \in [r_1, r_k]$ and assume that $r$ is equal to a switch point, i.e. there exists $i \in \{1, ..., k\}$ such that $r = r_i$. Next, take any $\tilde{K} \in (K_i, K_{i-1})$. Due to Theorem 4.5 (b), we then have

$$F(\tilde{K}) = r_i \tilde{K} + c_i - r_i K_i.$$

Clearly, $F$ is differentiable at $\tilde{K}$ with the derivative $F'(\tilde{K}) = r_i$, and so

$$\lim_{K \to \tilde{K}^-} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K} = r = \lim_{K \to \tilde{K}^+} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K}.$$

Now, we assume that $r$ is not a switch point, that is, there is no $i \in \{1, ..., k\}$ such that $r = r_i$. Because of $r \in [r_1, r_k]$ this assumption makes only sense, if $k \geq 2$. It follows that there exists $i \in \{1, ..., k-1\}$ such that $r_i < r < r_{i+1}$. In addition, let $\tilde{K} = K_i$. We apply Theorem 4.5 (b) to obtain

$$F(K) = r_i K + c_i - r_i \tilde{K}$$

for any $K \in [\tilde{K}, K_{i-1}]$ and

$$F(K) = r_{i+1} K + c_{i+1} - r_{i+1} K_{i+1}$$

for any $K \in [K_{i+1}, \tilde{K}]$. Then,

$$\lim_{K \to \tilde{K}^+} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K} = \lim_{K \to \tilde{K}^+} \frac{r_i \tilde{K} + c_i - r_i \tilde{K} - (r_i K + c_i - r_i \tilde{K})}{\tilde{K} - K} = r_i$$

and

$$\lim_{K \to \tilde{K}^-} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K} =$$

$$= \lim_{K \to \tilde{K}^-} \frac{r_{i+1} \tilde{K} + c_{i+1} - r_{i+1} K_{i+1} - (r_{i+1} K + c_{i+1} - r_{i+1} K_{i+1})}{\tilde{K} - K} = r_{i+1}.$$

Thus, $F$ is left and right differentiable at $\tilde{K}$ with

$$\lim_{K \to \tilde{K}^-} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K} > r > \lim_{K \to \tilde{K}^+} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K}.$$

We have shown that $F$ fulfills property (b) for any rate of profit $r \in [r_1, r_k]$. Now, consider the case $r \in [0, r_1) \cup (r_k, \max_{h \in I}{}_h R]$. By application of Theorem 4.5 (b), we obtain

$$\lim_{K \to K_0^-} \frac{F(K_0) - F(K)}{K_0 - K} = \lim_{K \to K_0^-} \frac{r_1 K_0 + c_1 - r_1 K_1 - (r_1 K + c_1 - r_1 K_1)}{K_0 - K} = r_1$$

and

$$\lim_{K \to K_k^+} \frac{F(K_k) - F(K)}{K_k - K} = \lim_{K \to K_k^+} \frac{r_k K_k + c_k - r_k K_k - (r_k K + c_k - r_k K_k)}{K_k - K} = r_k.$$

As a consequence, function $F$ is left differentiable at the boundary point $K_0$ of the interval $V$ and right differentiable at the boundary point $K_k$ of the same interval. If we set $\tilde{K} = K_0$, then $F$ fulfills property (b) for any a rate of profit $r \in [0, r_1)$, and if we set $\tilde{K} = K_k$, then property (b) is satisfied for any rate of profit $r \in (r_k, \max_{h \in I\ h} R]$.

With property (b), we have also shown that function $F$ satisfies property (a).

Next, we prove statement (c). Because of Theorem 4.5 (b), function $F$ is clearly continuous on the open interval $(K_i, K_{i-1})$ for any $i \in \{1, ..., k\}$. Next, we consider $F$ at the point $K_i$ with $i \in \{1, ..., k-1\}$ for the case $k \geq 2$. By applying Theorem 4.5 (b) we get

$$\lim_{K \to {K_i}^+} F(K) = \lim_{K \to {K_i}^+} (r_i K + c_i - r_i K_i) = c_i = F(K_i).$$

With Theorem 4.5 (a) and Theorem 4.5 (b), we also have

$$\lim_{K \to {K_i}^-} F(K) = \lim_{K \to {K_i}^-} (r_{i+1} K + c_{i+1} - r_{i+1} K_{i+1}) =$$

$$= r_{i+1}(K_i - K_{i+1}) + c_{i+1} = \frac{F(K_i) - F(K_{i+1})}{K_i - K_{i+1}}(K_i - K_{i+1}) + c_{i+1} =$$

$$= \frac{c_i - c_{i+1}}{K_i - K_{i+1}}(K_i - K_{i+1}) + c_{i+1} = c_i = F(K_i).$$

Therefore, $\lim_{K \to K_i} F(K) = F(K_i)$ meaning that $F$ is continuous at $K_i$. By the same way, it can be shown that

$$\lim_{K \to {K_k}^+} F(K) = c_k = F(K_k)$$

and

$$\lim_{K \to {K_0}^-} F(K) = c_0 = F(K_0).$$

Function $F$ is also continuous at $K_0$ and $K_k$, and so $F$ is continuous on the entire interval $V$.

Now, we consider $K \in (K_i, K_{i-1})$ for any $i \in \{1, ..., k\}$. We apply Theorem 4.5 (b) to get

$$F(K) = r_i K + c_i - r_i K_i.$$

Then, $F$ is differentiable at $K$ with $F'(K) = r_i > 0$. Function $F$ is therefore strictly monotonically increasing on the interval $[K_i, K_{i-1}]$, and so on the entire

interval $V$. Property (d) is satisfied.

Finally, it remains to show that $F$ satisfies property (e). We know from Theorem 4.5 (b), that for each $i \in \{1, ..., k\}$ function $F$ is a polynomial of degree 1 on the interval $[K_i, K_{i-1}]$, whose first derivative is equal to $r_i$. If $k = 1$, then $V = [K_1, K_0]$ and $F$ is concave. If $k \geq 2$, then for any $i, j \in \{1, ..., k\}$ with $i < j$ we obtain

$$F^{'}(K^{''}) = r_j > r_i = F^{'}(K^{'})$$

for any $K^{''} \in (K_j, K_{j-1})$ and any $K^{'} \in (K_i, K_{i-1})$. Since

$$K^{''} < K_{j-1} \leq K_i < K^{'},$$

we can conclude that function $F$ is concave on $V$.

□

Theorem 4.8 says that in regular economies any aggregate production function is *well-behaved*, since it automatically satisfies all the properties of Definition 4.5.

# 5 On the existence of well-behaved aggregate production functions

In this section we finally provide a proof of Burmeister's Theorem 4.5, which is based on the results obtained in Section 4. In addition, we discuss the role of the "marginal productivity of capital" in determining cost-minimizing techniques in regular economies, and subsequently we depict the conditions prevailing in non-regular economies.

## 5.1 Proof of the existence theorem for well-behaved aggregate production functions with finite technologies

With Theorem 4.8 we have already achieved most of what is the aim of our work, namely to formulate and prove Burmeister's Theorem 4.5. By using the terms we have introduced in the course of subsection 4.3, we are now able to do so.[53] We give Theorem 4.5 the form:

**Theorem 5.1.** *The set $P$ is represented by a well-behaved aggregate production function in the sense of Definition 4.5, if and only if the economy is regular.*

*Proof.* To prove the "only if" part is relatively simple. Due to Theorem 4.7, we can define a function $F : V \to H$ by

$$F(K_i) := c_i \tag{5.23}$$

for each $i \in \{0, ..., k\}$ and

$$F(\lambda K_{i-1} + (1-\lambda)K_i) := \lambda c_{i-1} + (1-\lambda)c_i \tag{5.24}$$

for each $i \in \{1, ..., k\}$ and $\lambda \in [0, 1]$. Clearly, $F$ is an aggregate production function with respect to $P$. According to Theorem 4.8, function $F$ is hereby a well-behaved aggregate production function.

Now, we prove the "if" part for $k \geq 2$, because the economy is always regular in the case $k = 1$ (this is a consequence of Theorem 4.6). Moreover, we assume $P$ to be represented by a function $F$, but at the same time we suppose that the economy is not regular. Since $R(r_1) < 0$, there must exists $i \in \{2, ..., k\}$ such that $R(r_i) > 0$ (recall that $R(r_i) \neq 0$ for each $i \in \{1, ..., k\}$). Without loss of generality, we can assume that $R(r_2) > 0$. Then, we obtain $K_1 < K_2$, but we have $K_0 > K_1$ at the same time. From Theorem 4.5 (a) it follows that

$$\frac{F(K_1) - F(K_2)}{K_1 - K_2} = r_2.$$

With $F(K_1) = c_1$ and $F(K_2) = c_2$, we thus obtain

$$c_1 = r_2 K_1 + c_2 - r_2 K_2.$$

Next, consider $K \in \mathbb{R}$ with $K_1 < K \leq \min\{K_0, K_2\}$. Since $r_1 < r_2$, we get

$$r_1 K + c_1 - r_1 K_1 - c_1 = r_1(K - K_1) < r_2(K - K_1) =$$

$$= r_2 K + c_2 - r_2 K_2 - (r_2 K_1 + c_2 - r_2 K_2) = r_2 K + c_2 - r_2 K_2 - c_1.$$

From this we immediately receive

$$r_1 K + c_1 - r_1 K_1 < r_2 K + c_2 - r_2 K_2.$$

Because of Theorem 4.5 (b), for each $K \in [K_1, K_0] \cap [K_1, K_2]$ we have

$$F(K) = r_1 K + c_1 - r_1 K_1$$

and

$$F(K) = r_2 K + c_2 - r_2 K_2.$$

Consequently, there exist two different values for $F$ at any point in the non-empty set $(K_1, K_0] \cap (K_1, K_2]$, and so $P$ cannot be represented by a function in the sense of Definition 4.4.

□

## 5.2 The "marginal productivity of capital" in regular economies

If the economy is regular, then identities (5.23) and (5.24) define a *well-behaved aggregate production function* $F$, which *functionally* represents the *set of all equilibrium positions* $P$. Burmeister also used the term "'pseudo aggregate production function' across steady-state equilibria" to denote well-behaved aggregate production functions for "neoclassical" technologies (Burmeister, 1980, p. 133). Also, for finite technologies the well-behaved aggregate production function comes very close to what Robinson considered as a "pseudo-production function, showing the possible [steady-state] positions of [long-run] equilibrium, corresponding to various values of the rate of profit, in an imagined 'given state of technical knowledge' [i.e. with an exogenous technology]" (Robinson, 1970, p. 309).[54] Each equilibrium position represents a cost-minimizing technique (or a combination of cost-minimizing techniques) at the given rate of profit, that can be determined by means of the "marginal productivity of capital". In regular economies with $k$ switch points the production technology $T$ must necessarily contain $k+1$ techniques. Thus, it makes sense to specify the technology index set $I$ in the following way: $I := \{1, 2, ..., k+1\}$. Each index $i \in I$ represents the technique, which is the unique cost-minimizing technique at each rate of profit in $(r_{i-1}, r_i)$ and which is characterized by the equilibrium position $(K_{i-1}, c_{i-1})$. Rate $r_1$ is therefore the switch point from technique 1 to technique 2, rate $r_2$ is the switch point from technique 2 to technique 3, and so forth.

With the above specification, we can demonstrate the role of the "marginal productivity of capital" in the determination of cost-minimizing technique(s) and the corresponding "optimal" (range of the) index of capital intensity at any current rate of profit $r \in [0, \max_{h \in I\ h} R]$. We first assume $r$ to be different from any switch point, i.e $r \notin \{r_1, ..., r_k\}$. If in the case $k \geq 2$ the rate of profit $r$ is between two switch points, then there exists $i \in \{1, ..., k-1\}$ such that $r_i < r < r_{i+1}$. By Theorem 4.5 (a) we know that switch point $r_i$ corresponds to the "marginal productivity of capital" with respect to any switching between technique $i$ and technique $i+1$. Switch point $r_i$ also equals the right-sided differential quotient of $F$ at $K_i$; see also the proof of Theorem 4.8. This means that

$$r_i = \frac{F(K_{i-1}) - F(K_i)}{K_{i-1} - K_i} = \lim_{K \to K_i^+} \frac{F(K_i) - F(K)}{K_i - K}.$$

Switch point $r_{i+1}$ corresponds to the "marginal productivity of capital" with respect to any switching between technique $i+1$ and technique $i+2$; see

Theorem 4.5 (a). It is also equal to the left-sided differential quotient of $F$ at $K_i$; see also the proof of Theorem 4.8. This means that

$$r_{i+1} = \frac{F(K_i) - F(K_{i+1})}{K_i - K_{i+1}} = \lim_{K \to K_i^-} \frac{F(K_i) - F(K)}{K_i - K}.$$

Hence, the current rate of profit $r$ is larger than the "marginal productivity of capital" with respect to any switching between technique $i$ and technique $i+1$, but smaller than the "marginal productivity of capital" with respect to any switching between technique $i+1$ and technique $i+2$. At the same time, $r$ is larger than the right-sided differential quotient of $F$ at $K_i$, but smaller than the left-sided differential quotient of $F$ at the same value, that is,

$$\lim_{K \to K_i^-} \frac{F(K_i) - F(K)}{K_i - K} > r > \lim_{K \to K_i^+} \frac{F(K_i) - F(K)}{K_i - K}.$$

Because of the concavity of $F$, at any index of capital intensity lower (higher) than $K_i$ the one-sided differential quotients, or one-sided "marginal productivities of capital", are not smaller (larger) than the left-sided (right-sided) differential quotient at $K_i$. If we conceive $r$ as "factor price of capital", then it is rational for cost-minimizing producers to choose technique $i+1$ characterized by the index of capital intensity $K_i$ and the index of labor productivity $c_i (= {}_{i+1}c_1)$. The rate of profit $r$ thus determines the equilibrium position $(K_i, c_i)$ on the graph of $F$, that is, in the set $P$. For the same reason, if the current rate of profit is smaller than the first switch point, i.e. $0 \leq r < r_1$, then producers choose technique 1 characterized by the index of capital intensity $K_0$ and the index of labor productivity $c_0 (= {}_1c_1)$. The respective equilibrium position on the graph of $F$ is $(K_0, c_0)$. If the current rate of profit is larger than the $k$-th switch point, i.e. $r_k < r \leq \max_{h \in I} {}_hR$, then producers choose technique $k+1$ characterized by the index of capital intensity $K_k$ and the index of labor productivity $c_k (= {}_{k+1}c_1)$. The respective equilibrium position on the graph of $F$ is $(K_k, c_k)$. Now, we assume $r$ to be equal to a switch point, i.e. there exists $i \in \{1, ..., k\}$ such that $r = r_i$. Then the current rate of profit is equal to the "marginal productivity of capital" with respect to any switching between technique $i$ and technique $i+1$, and we have

$$F^{'}(K) = r_i = r$$

for any index of capital intensity $K \in (K_i, K_{i-1})$; see also the proof of Theorem 4.8. This means that the rate of profit $r$ is also equal to the differential quotient (or "marginal productivity of capital") of $F$ at $K$. Because of the concavity

of $F$, any convex linear combination of $K_i$ and $K_{i-1}$ represents an "optimal" index of capital intensity at the rate of profit $r$. At the same time, $K_i$ and $K_{i-1}$ are "optimal" themselves at this rate, since

$$\lim_{K \to K_i^+} \frac{F(K_i) - F(K)}{K_i - K} = \lim_{K \to K_{i-1}^-} \frac{F(K_{i-1}) - F(K)}{K_{i-1} - K} = r_i = r.$$

The "optimal" range of the index of capital intensity is therefore restricted to the interval $[K_i, K_{i-1}]$, meaning that at $r$ we have a continuum of equilibrium levels for the index of capital intensity as well as a continuum of equilibrium levels for the index of labor productivity, and in equilibrium the net output $C_1$ is produced by technique $i$, technique $i+1$ or a combination of both techniques. Of course, with well-behaved aggregate production functions we can only observe *regular* reactions of the index of labor productivity and thus of the (absolute) demand for labor to any variations in the rate of profit (and accordingly in the wage rate); see Subsection 4.3.[55]

It can be shown that in the special case of $n = 1$, meaning that only one commodity is produced, the economy is always regular, since under this condition function ${}_hw$ is a polynomial of degree 1 (or "linear") for each technique $h \in T$ (irrespective of the specific numeraire bundle). The same is true with any number of commodities, if there is for any technique $h \in T$ a real number $x > 0$ such that for any rate of profit $r \in [0, {}_hR]$ the condition

$${}_hp(r) = xv$$

with

$$v := (I_n - {}_hA)^{-1}{}_h\ell \in \mathbb{R}^{n \times 1}$$

is fulfilled. This means that the economy's system of (equilibrium) prices follows the "labor theory of value" (which is automatically satisfied by the one-commodity case).[56] It is equivalent to the case, where for any technique $h \in I$ there is a real number $z > 0$ such that

$${}_hA_h\ell = z_h\ell$$

is satisfied.[57] We do not prove these statements, but invite the reader to explore the arguments in Kurz and Salvadori (1995, pp. 51-3 and pp. 110-3).[58]

## 5.3 Consequences for non-regular economies

Any economy, that is not regular, always exhibits "paradoxical consumption behavior" (Burmeister, 1980, pp. 116-8); see also Samuelson (1966, pp. 576-7).

This simply means that the index for labor productivity becomes greater at some switch points. If we assume that $R(r_2) > 0$ (like in the proof of Theorem 5.1), then the inequalities $c_1 < c_2$ and $K_1 < K_2$ are true. Suppose that $r_2$ is a switch point from technique $d$ to technique $e$. Technique $d$ is then characterized by the index of capital intensity $K_1$ and the index of labor productivity $c_1$, while technique $e$ is characterized by the index of capital intensity $K_2$ and the index of labor productivity $c_2$. With constant net output $C_1$, we gain from $c_1 < c_2$ the inequality ${}_dL > {}_eL$. According to Definition 4.1, there exists $\epsilon > 0$ with $(r_2 - \epsilon, r_2 + \epsilon) \subseteq (0, \min\{{}_dR, {}_eR\})$ such that technique $d$ is the unique cost-minimizing technique at any rate of profit $r \in (r_2 - \epsilon, r_2)$ and technique $e$ is the unique cost-minimizing technique at any rate of profit $r \in (r_2, r_2 + \epsilon)$. Consequently, at any $r \in (r_2 - \epsilon, r_2)$ the index of capital intensity is $K_1$, the index of labor productivity is equal to $c_1$ and the absolute demand for labor is ${}_dL$, while at any $r \in (r_2, r_2 + \epsilon)$ the index of capital intensity is $K_2$, the index of labor productivity is $c_2$ and the absolute demand for labor is equal to ${}_eL$. This means that we face a situation, where we have production of net output $C_1$ with a higher index of capital intensity and a higher index of labor productivity at higher rates of profit.[59] On the one hand, this has the consequence that the demand for labor is smaller at lower wage rates, because for any $r^{'}, r^{''} \in (r_2 - \epsilon, r_2 + \epsilon)$ with $r^{'} < r_2 < r^{''}$ we have

$$ {}_dw(r^{'}) > {}_dw(r_2) = {}_ew(r_2) > {}_ew(r^{''}), $$

owing to Theorem 3.6 (b). On the other hand, the index of capital intensity is larger at lower wage rates, and "substitution" between the factors capital and labor in the conventional "neoclassical" sense ceases to exist as general law in non-regular economies. Such a "perverse relationship between wages and mechanisation" (Robinson, 1966, p. 110) occurs, when, for example, the technology exhibits the so called "reswitching of techniques" phenomenon. This means for $k \geq 2$ that there are $i, j \in \{1, ..., k\}$ with $i < j$ such that the unique cost-minimizing technique is the same at any rate of profit in both the intervals $(r_{i-1}, r_i)$ and $(r_j, r_{j+1})$.[60] In Garegnani (1966, pp. 566-7) we can find a numerical example with $n = |I| = k = 2$, that treats a case, where "reswitching of techniques" takes place. With Theorem 4.3, the reader can show that the economy, described there, is not regular: It is no longer guaranteed that the economy is regular, when capital is heterogeneous, i.e when $n > 1$ with all commodities being basic products.

# 6 Conclusion

Theorem 5.1 says that the *existence of well-behaved aggregate production functions* is *equivalent* to the condition, that defines *regular economies.* The regular economy is a property which is not automatically satisfied by models with heterogeneous capital; see Subsection 5.3. With regard to Theorem 5.1, this has the consequence that the existence of well-behaved aggregate production functions is not guaranteed in a more general production-theoretic framework.

On the one hand, the logical restriction of the validity of well-behaved aggregate production functions can be seen as crucial reason against their application. On the other hand, one might argue that cases, where the economy is not regular, are irrelevant for important empirical problems. In this regard, the interesting question is: If it is possible to empirically test whether real economies are regular and thus exhibit well-behaved aggregate production functions? Of course, to give a reasonable and definite answer to this question is very difficult and within the scope of this work a final conclusion on this matter cannot be drawn. But at least, we can briefly address the main difficulties regarding an empirical testing and a possible way to handle them. The model, which we have used in the previous sections, is based on assumptions, that are very restrictive and often hard to justify (like the assumptions on natural resources, the competitive situation, full information, "Say's law" and so on). With regard to the specific problem, this particularly concerns the assumption of a static technology (one of the exogenous variables, which are necessary for the determination of equilibrium with the respective endogenous variables): "[T]he argument concerns comparisons of equilibrium positions with different rates of profit and the same 'state of technical knowledge.' These are not found in nature and cannot be observed" (Naqvi and Robinson, 1967, p. 591). Even if we had good empirical reasons to accept all of the model's general assumptions (which would allow at least comparative statics with different technologies), the problem would remain to empirically determine the (current) technology, i.e. *all* of the relevant (old and new) techniques with their associated input coefficients.[61] Hence, the success of any attempt to test directly, whether a real economy is regular (in the sense of Definition 4.6) and thus characterized by a well-behaved aggregate production function, suffers from the difficulty in detecting all of the relevant empirical data. But despite of these problems, "ultimately theoretical models must be rejected or tentatively accepted on the basis of empirical evidence" (Burmeister, 1980, p. 101). Therefore, in order to decide, whether the postulate of the existence of well-behaved aggregate pro-

duction functions is of any value for applied economics (especially in designing a wage policy to reduce unemployment), empirical arguments must be involved as reasonably as possible. We think that one potential way to accomplish this is by considering the empirical quality in the relationships between observable magnitudes, that indicate variables like the wage rate, employment of labor, labor productivity or the rate of capital accumulation. On theoretic grounds, the quality of these relations depends on whether the economy is regular, that is, it depends on the existence of well-behaved production functions. We know from Section 4 that in regular economies a rising wage rate (a smaller rate of profit) never implies a decrease in both the index of capital intensity and the index of labor productivity, and in cases, where the current rate of profit falls below a switch point, a rising wage rate corresponds to an increase in both the index of capital intensity and the index of labor productivity. This means that with a constant net output a rising wage rate (that is, the factor labor becomes more expensive relative to the factor capital) never implicates an increasing (absolute) employment of labor and implies a decrease in the same variable, whenever the current rate of profit falls below a switch point.[62] On the other hand, if the economy is not regular, the index of capital intensity and the index of labor productivity are no longer monotonically increasing in the wage rate (monotonically decreasing in the rate of profit). As a result, for non-regular economies we always find a switch point, at which a rising wage rate corresponds to a decrease in both the index of capital intensity and the index of labor productivity. Consequently, a higher (absolute) amount of employment to produce a constant net output is associated with a higher wage rate (despite the fact that a higher wage rate means that the factor labor becomes more expensive relative to the factor capital).[63] This leads to the fact that in non-regular economies the "neoclassical nexus" between the respective variables is violated.[64] In consideration of this, if in an attempt to check, whether the actual relationships are consistent with the conclusions drawn from well behaved aggregate production functions, we find that empirical evidence contradicts what it is to be expected from regular economies (meaning that the assumption of the existence of well-behaved aggregate production functions leads to misleading results), then we must reject (or restrict in some way) the general hypothesis of an overall existence of well-behaved aggregate production functions.[65] [66] In such a case, we have to deal with the possibility of (the one or other kind of) "irregular" patterns in the relations between (real) wages and other relevant variables like labor productivity or employment.[67]

We conclude this work by pointing to the fact that the issue of the exis-

tence of well-behaved aggregate production functions can also be approached by a different way. While Burmeister (1980) used a (comparative-static) *classical long-run equilibrium* model with a given production technology as set of available techniques, Fisher (1969) discovered conditions for the existence of aggregate production functions on the basis of already existing microeconomic production functions without making any assumptions on the economy's price system. With respect to this analysis and on the motivation to figure out under which circumstances aggregate production functions in the form of Cobb-Douglas specifications provide good predictions on the wage rate, he performed some simulation experiments within the analytical framework of a dynamic model of a discrete sequence of *neoclassical short-run equilibria*, which are all characterized by their own *endowment* of each factor of production (Fisher, 1971). Fisher concluded that good predictions are obtained, as long as the wage share of the aggregate income remains roughly constant (even though the simulated economies did not satisfy the conditions for aggregating the underlying microeconomic production functions, which were specified as conventional Cobb-Douglas production functions).[68]

# Notes

1 A fine overview of the Cambridge Controversies can be found in Kurz and Salvadori (1995, pp. 443-55). We also recommend the reading of Felipe and Fisher (2003, pp. 214-21).

2 This model framework served as stage for the Cambridge Controversies; see among others Samuelson (1962), Levhari (1965), Garegnani (1966), Pasinetti (1966) as well as Naqvi and Robinson (1967).

3 This regards the existence of well-behaved aggregate production functions (in intensive form), that are continuously differentiable on their entire domain (i.e. the version of aggregate production functions as it is known from textbooks).

4 See among others Romer (2006), Acemoglu (2009), Blanchard (2011) and Mankiw (2013).

5 See, for example, Deutsche Bundesbank (2013, pp. 19-21). Of particular interest is also the report of the Sachverständigenrat zur Begutachtung der gesamtwirtschaftlichen Entwicklung (2003, pp. 364-6), where a model with a well-behaved aggregate (Cobb-Douglas) production function was used to recommend a wage policy, which involved a smaller increase of (real) wage rates in order to reduce unemployment in Germany.

6 This is a modification of the model used by Burmeister (1980, p. 103), who took the total available quantity of labor as given.

7 To get an overview of the idea of "free competition" see also Kurz and Salvadori (1995, pp. 1-41).

8 On the other hand, the list of exogenous variables of the "neoclassical" (mainstream) model is composed of the following elements (Kurz and Salvadori, 2000a, pp. 155-6): The production technology, the consumer's preferences, an initial endowment of goods (which can be either used as consumption goods or for productive purposes) and the distribution of property rights over this endowment among consumers. In this model not only producers, who are assumed to apply cost-minimizing methods of production, but also consumers, who are assumed to optimize their preferences, are among the acting agents. Both producers and consumers act as *price takers* (*perfect competition*). The resulting *general equilibrium*, which is determined through equating demand and supply functions (or correspondences) on all markets (including the factor markets), consists of the following endogenous variables: The quantities of goods, which are traded on the markets, and the relative prices (including the relative factor prices). In such a neoclassical *short-run equilibrium* not only natural resources but all goods (including the stock of physical capital and all kinds of labor) are given in quantities. Another important difference to the classical model is the fact that in the neoclassical model the entire income distribution is endogenously determined. All factor prices (as wage rates, rental rates, rates of interest and so on) are determined through supply and demand as scarcity prices, meaning that if an input good gets scarce (relative to other goods), it also gets more expensive (relative to other goods). Hence, input goods in excess supply are paid at reservation prices, which can be equal to zero. As a result, in neoclassical models the "Rule of Free Goods" (Kurz and Salvadori, 1993, pp. 140-2) can be applied to labor with the consequence that wages may fall to zero in case of (involuntary) unemployment.

9 Some of these assumptions were called "trick assumptions" by Robinson (1953, p. 85-6).

10 As a matter of fact, there are more complicated models, which incorporate a greater variety of empirical aspects. In the following notes, we briefly explain for some assumptions how they abstract from reality or simplify the model, that is used in this work.

11 Certainly, this is to be regarded as "bold assumption, not least with regard to advanced economies which are typically characterized by the scarcity of some of their lands and the depletion of some of their stocks of raw materials etc." (Kurz and Salvadori, 2000b, p. 15).

12 In case of totally appropriated natural resources, there must be competition among the owners of natural resources, which reduces any scarcity rent to a reservation price equal to zero. In other words, there are no monopolies in the possession of natural resources.

13 This is a special case of "free disposal", i.e. commodities not wanted ("discommodities" or "bads") can be disposed of at no costs, and so negative prices are not possible (the "owners" of bads do not pay anything in order to get rid of them). This assumption is very restrictive, since many disposal processes are in fact costly (Kurz and Salvadori, 2000b, p. 15-6).

14 An input good, which enters indirectly the production of a certain commodity as final product, means that it is used to (directly) produce other commodities, which are used to (directly or indirectly) produce the final product.

15 None of the commodities belongs to the class of "*non-basic* products" such as " 'luxury' products which are not used, whether as instruments of production or as articles of subsistence, in the production of others" (Sraffa, 1960, pp. 7-8). In the presence of non-basic products, the dependent distributive variable and the prices of basic products can be determined independently of the prices of non-basic products, which can be determined afterwards; see also Note 25. Thus, the exclusion of non-basic products just serves to simplify the analysis.

16 In this work we do not deal with "time" either as mathematical variable (which is explicitly incorporated into the model) or in the sense of Robinson's "historical time" with uncertain future events (Robinson, 1980).

17 This assumption also involves "the introduction of some fictitious products, occasionally referred to as 'semi-finished products' or 'work in progress'" (Kurz and Salvadori, 2000b, p. 16).

18 With this assumption we avoid the problem of "fixed capital" in the form of joint products (Kurz and Salvadori, 1995, pp. 186-208).

19 Or, "what amounts to the same thing, differences in quality are reduced to equivalent differences in quantity in terms of a given and constant structure of relative wage rates, the problem of the creation of skilled labour powers is set aside, and the work effort per hour is given and constant [...]" (Kurz and Salvadori, 2000b, p. 18).

20 This means that wages are paid "*post factum* as a share of the [...] [net] product" (Sraffa, 1960, p. 10). This assumption does not exclude the possibility that parts of wages can be paid as means of subsistence at the beginning of the production period. In such a case, capital, advanced at the beginning of the production period, contains wages (Kurz and Salvadori, 2000b, p. 16).

21 In other words, the analysis is based on *comparative statics*. This also means that the model is not able to describe dynamics in the sense of Joseph Schumpeter, since the technology is an independent and constant variable that cannot be changed through innovative behavior of entrepreneurs (or what Schumpeter called "Unternehmer").

22 From the viewpoint of an individual agent, the current uniform commodity prices (including the current uniform rate of profit and the current uniform wage rate) have ruled in the past and are also expected to rule in the future, when important decisions are made. This means that expectations concerning the future are alway fulfilled. This is what Robinson understood as "state of *tranquillity*" (Robinson, 1966, p. 59), which is a necessary condition of equilibrium. One might as well say that equilibrium is based on the concept of "logical

time", since there is no distinction between (irreversible) past, present and (uncertain) future events, that is, "no distinction between any one day and any other" (Robinson, 1980, pp. 219-20).

23 Every reader interested in the different versions of "Say's law" may study Eatwell and Robinson (1973, pp. 23-5, p. 27, pp. 43-4 and p. 50) as well as Blaug (1996, pp. 143-60).

24 We have borrowed these terms directly from Kurz and Salvadori (1995, pp. 127-8).

25 If it were possible to transform matrix $A$ into a matrix like the one on the right-hand side of equation (3.1), then we could separate $m$ commodities as potential basic products (represented by the input coefficient matrix $A_{11}$) from the remaining $n-m$ commodities, which would definitely not be basic products. This means that the $m$ potential basic products could be produced independently of the $n-m$ non-basic products by the technique constituted by matrix $A_{11}$ (which contains the respective input coefficients for capital) and by the respective input coefficients for labor taken from $\ell$. Thus, the endogenous distributive variable and the prices of these $m$ commodities would be determined independently of the prices of the non-basic products.

26 Set $T$ can be interpreted as what Robinson called a "book of blueprints representing a 'given state of technical knowledge,'" (Robinson, 1971, p. 255).

27 Of course, in order to be consistent with this postulate the set $T_i$ must be finite for each $i \in \{1, ..., n\}$.

28 Therefore, for an index set $I$ the technology can be rewritten as

$$T = \{({}_dA, {}_d\ell) \in \mathbb{R}^{n\times n}_{+} \times \mathbb{R}^{n\times 1}_{++} \mid d \in I\}.$$

29 This definition can be considered as general solution of the "open Leontief [input-output] model" (Kurz and Salvadori, 1995, pp. 395-6).

30 By the term "labor unit" we mean the (abstract) unit, in which the labor coefficients and therefore the magnitude ${}_dL(C)$ for any technique $d \in I$ are measured. The labor coefficients can be measured in working hours or in number of persons engaged. The latter case was implicitly assumed by Burmeister, who considered per capita quantities of the consumption good; see Burmeister (1980, p. 103).

31 Instead of the net output (of commodity 1) we can take the supply of available labor $\tilde{L} > 0$ as exogenous. Then, we can determine a net output $\tilde{C} = (\tilde{C}_1, 0, ..., 0)^T \in \mathbb{R}^{n\times 1}$ with $\tilde{C}_1 > 0$, so that $\tilde{L}$ is fully employed by a technique $d \in I$. If the net output $\tilde{C}$ is produced by technique $d \in I$, then the per labor unit (net) output of commodity 1 is equal to ${}_dc_1$. This means that $\tilde{C}_1 := {}_dc_1\tilde{L}$. Because of identity (3.6), we have

$${}_dc_1 = {}_dc_1(\tilde{C}) = \frac{1}{{}_dL(\tilde{C})}\tilde{C}_1.$$

Consequently, we obtain $\tilde{L} = {}_dL(\tilde{C})$.

32 With respect to the technology, this means that each producer has free access to each technique (i.e. no element in $T$ can be monopolized by any producer).

33 If wages were solely paid at the beginning of the production period, then they would become part of the advanced capital, that had to be entirely depreciated ("variables Kapital" in the Marxian sense). Condition (3.9) would appear as

$$p = (1 + r)({}_dAp + w_d\ell)$$

with $r({}_dAp + w_d\ell)$ as total profit vector and ${}_dAp + w_d\ell$ as vector of depreciated capital.

34 The system of equations (3.9) with respect to a technique $d \in T$ can be represented at the rate of profit $r$ for each $i \in \{1, ..., n\}$ in scalar form as follows:

$$p_i = (1 + r) \sum_{j=1}^{n} p_{j\,d} a_{ij} + w_d l_i.$$

Thus, with production prices as ruling price system the (uniform) rate of profit satisfies for each $i \in \{1, ..., n\}$ the equation

$$r = \frac{p_i - w_d l_i - \sum_{j=1}^{n} p_{j\,d} a_{ij}}{\sum_{j=1}^{n} p_{j\,d} a_{ij}}.$$

This means that the rate of profit $r$ equals the rate of return for the production of each commodity: For each line of production $i \in \{1, ..., n\}$ we have the same relation of the profits (the numerator of the above fraction) from selling one unit of commodity $i$ to the value of capital (the denominator) to produce one unit of that commodity. As a result, there are no incentives for capital owners to change from one line of production to another. If we assume that the rate of profit is equal to the ruling rate of interest on borrowed capital, then they are also no incentives to lend a value of financial capital for interest (instead of using it to buy physical capital goods in order to make profits). This means that no producer, who borrows financial capital, can make any extra profits over the interest, that must be paid to capital lenders, and no capital owner can earn any interest higher than the opportunity costs in form of profits, that can be earned with capital as means of production.

35 Without loss of generality, the numeraire can be changed to another semipositive vector $\tilde{y} \in \mathbb{R}^{n \times 1}$. The price system $(\frac{1}{\tilde{y}^T p} p, \frac{w}{\tilde{y}^T p}, r)$ is as well a system of production prices with respect to technique $d$, since it satisfies condition (3.9) and (with respect to $\tilde{y}$) equation (3.10). Taking a bundle of commodities as numeraire is consistent with the way followed by Burmeister in taking labor as numeraire (Burmeister, 1980, p. 105). If $w > 0$, then $(\frac{1}{w} p, 1, r)$ is a well defined price system which satisfies condition (3.9). In this case commodity prices are expressed in terms of "labor commanded" (Adam Smith's favored numeraire), or what John Maynard Keynes used to call "wage-unit"; see also Kurz and Salvadori (1995, p. 116).

36 Note that the determination of the solution is independent of the size of the given net output. This is the result of the "non-substitution theorem", which assumes a given rate of profit, constant returns to scale, homogeneous labor, no scarce land (or natural resources) and single production (Kurz and Salvadori, 1995, p. 26).

37 Statement (3.14) is verified in Havlicek (2008, p. 202) for any invertible matrix.

38 Taking account of the fact, that the elements of $I_n - (1 + r)_d A$ are polynomials of degree 1 in $r$, these statements can be proved by applying mathematical induction and the Laplace expansion (for computing determinants).

39 This is the famous inverse wage-profit relation that can be found among the arguments of David Ricardo and Karl Marx.

40 In Kurz and Salvadori (1995, p. 100) it is emphasized that any possible solution for the system of equations (3.9) and (3.10) at $r > {}_dR$ yields a negative wage rate or negative commodity prices (meaning that some of the commodities become "bads"). The former has no economic meaning with respect to free competition on the labor market, the latter clearly contradicts assumption (A2).

41 Such an equilibrium is consistent with the "law of one price" (as to wages and all commodity prices), which characterizes a state in which there are (neither to producers nor to workers)

any incentives to change from one line of production to another. In other words, no excess profits or excess wages can be accomplished through mere arbitrage operations.

42 Or to put this into other words, "in the absence of innovation both [the rate of profit and the wage rate] cannot go up" (Samuelson, 1962, pp. 196).

43 If we had, instead of a finite technology, a continuum of techniques (i.e. with $I$ as uncountable set), then, in order to prove the existence of cost-minimizing techniques at a rate of profit $r > 0$, condition $\max_{h \in I}\, {}_h w(r) < \infty$ would additionally be required; see also Kurz and Salvadori (1995, p. 131).

44 Note that the validity of Theorem 3.9 is independent of the choice of numeraire. If a system of production prices $(p, w, r)$ satisfies condition (3.15) with respect to a semipositive numeraire bundle, then so does the system of production prices $(\frac{1}{\tilde{y}^T p} p, \frac{w}{\tilde{y}^T p}, r)$ with respect to any other semipositive numeraire bundle $\tilde{y}$. Even if we take labor as numeraire, then the system of production prices $(\frac{1}{w} p, 1, r)$ with $w > 0$ satisfies condition (3.15) as well.

45 By this postulate, we exclude all techniques from $T$, that are never strictly preferred to other techniques by cost-minimizing producers. This exclusion regards either techniques, that do not minimize costs at any feasible rate of profit, or techniques, that are only cost-minimizing, when the same is true for some other techniques.

46 The reader may also explore the arguments provided by Bruno et al. (1966).

47 This result can be obtained from the intermediate value theorem for continuous functions.

48 If we relax assumption (A10) and take the net output $C$ as arbitrary semipositve vector, then it can be shown that for any technique $d \in I$ equation (4.16) will be generalized by

$$({}_d p(r))^T {}_d c(C) = r({}_d p(r))^T {}_d k(C) + {}_d w(r)$$

with ${}_d c(C)$ as vector of (net) labor productivities (or per labor unit outputs) for the various consumption goods with respect to technique $d$. If we define for any switch point $r$ from technique $d$ to technique $e$ the "real Wicksell effect" as

$$R(r) := \sum_{i=1}^{n} \frac{{}_e p_i(r)}{({}_e p(r))^T C} ({}_e k_i(C) - {}_d k_i(C)),$$

then it can be obtained

$$\frac{1}{({}_e p(r))^T C} ({}_e p(r))^T {}_e c(C) - \frac{1}{({}_e p(r))^T C} ({}_e p(r))^T {}_d c(C) = rR(r)$$

as generalization of equation (4.20). But the relaxation of assumption (A10) has a serious consequence. With respect to single techniques, the (heterogeneous) capital intensities and labor productivities are no longer independent of the absolute net output $C$, unless we only permit even variations of the elements of $C$ by a positive factor $\lambda > 0$; see Theorem 3.3. Under this condition, identities (4.21) and (4.22) define an index of capital intensity and an index of labor productivity (now measuring the change in the heterogeneous per labor unit outputs) respectively, which are both measured in terms of commodity bundle $C$ as (fixed) "composite commodity" (Robinson, 1953, p. 85), and the existence problem of well-behaved aggregate production functions can be treated in a similar way like in the case of only one consumption good.

49 In the case of the one-commodity economy (i.e. with $n = 1$), the index of capital intensity corresponds exactly to the (homogeneous) capital intensity (with respect to the single capital good). This means that $K_i = {}_e k_1$ at any switch point $r_i$ from one technique to technique $e$.

50 For the significance of these two types of "Wicksell-effect" see also Naqvi and Robinson (1967, p. 582).

51 If $\tilde{K}$ is a boundary point of $V$, at which $F$ is left (right) differentiable, then property (b) means

$$\lim_{K \to \tilde{K}^-} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K} \geq r \quad (r \geq \lim_{K \to \tilde{K}^+} \frac{F(\tilde{K}) - F(K)}{\tilde{K} - K}).$$

52 Property (b) in Definition 4.5 can be considered as the discrete version of the "marginal productivity of capital equals the factor price of capital" condition, while property (e) means that higher levels for the index of capital intensity are connected to "diminishing marginal productivity (return) of capital".

53 Burmeister suggested the methods found in Cass (1973, pp. 87-91) to construct a well-behaved aggregate production function for regular economies and to prove his Theorem 4.5 (Burmeister, 1980, p. 298). But instead, we have directly been inspired by his proof of Theorem 4.4 (Burmeister, 1980, pp. 131-3), which is the exact analogon of Theorem 4.5 for the case of "neoclassical" technologies, that is, with $I$ as uncountable set; see also Section 1. Again, any technique in $I$ is characterized by a unique per labor unit output and by unique capital intensities. Burmeister used a model, where for each $i \in \{1, ..., n\}$ and for any feasible rate of profit $r \geq 0$ the commodity price $p_i(r)$ and the capital intensity $k_i(r)$ as well as the per labor unit output of the (only) consumption good $c_1(r)$ (all determined by the cost-minimizing technique at $r$) are differentiable functions (Burmeister, 1980, pp. 119-21). In this case, the "real Wicksell effect" to characterize regular economies is defined by

$$R(r) := \sum_{i=1}^{n} \frac{p_i(r)}{p_1(r)} \frac{d}{dr} k_i(r).$$

This formula can be considered as analogon of equation (4.19). An economy is then regular, if $R(r) < 0$ for each feasible rate of profit $r$. As index for capital intensity Burmeister defined

$$K(r) := K_0 + \int_0^r R(\theta)\, d\theta,$$

that is, the analogon of equation (4.21). Any infinitesimal change in the per labor unit output with respect to the rate of profit is described by

$$\frac{d}{dr} c_1(r) = rR(r),$$

which is the analogon of equation (4.20). Hence, Burmeister was able to define the index for labor productivity as differentiable function with respect to the rate of profit by

$$c(r) = c_0 + \int_0^r \theta R(\theta)\, d\theta,$$

that is, the analogy of equation (4.22). In regular economies, both the index of capital intensity and the index of labor productivity are strictly monotonically decreasing in the rate of profit (i.e. each rise in the rate of profit implies a falling index of capital intensity as well as a falling index of labor productivity). It should be mentioned that in the context of "neoclassical" technologies there are no switch points in the sense of Definition 4.1. The reason is that the per labor unit output and the capital intensities are differentiable functions with respect to the rate of profit, and so the cost-minimizing technique can only be unique at each feasible rate of profit. This means that for two different techniques $d, e \in I$, that

determine (in the sense of Definition 3.6) at any feasible rate of profit the same commodity prices and the same wage rate, there must always be a third technique $t \in I \backslash \{d, e\}$ that yields a wage rate higher than those afforded by technique $d$ and technique $e$. Then, in the light of Theorem 3.6 (a), both technique $d$ and technique $e$ cannot be cost-minimizing at the respective rate of profit. For this matter, see also the intuitive arguments in Pasinetti (1969, pp. 516-9) and Garegnani (1970, pp. 410-4).

54 In many textbooks well-behaved aggregate production functions appear as preexisting *technical* relationship between *physical* quantities, whose existence is postulated, *before* any assumptions have been made on prices and equilibrium. In contrast, the equilibrium conditions of the economy's price system is *already* incorporated by the well-behaved aggregate production function, which is defined through (5.23) and (5.24), as we have used equilibrium prices at switch points to define the index of both capital intensity and labor productivity.

55 It must not be forgotten that we solely use comparative statics in this work, but no dynamics with time as explicit mathematical variable (and, of course, no dynamics at all in the Schumpeterian sense with $T$ being subject to changes by introduction of technical innovations). This also means that our model does not involve any mechanism that describes how the (steady-state) economy moves from one point on the graph of $F$ (or long-period position) to another through "historical time" (Robinson, 1980, p. 221).

56 Note that vector $v = (v_1, ..., v_n)^T$ satisfies condition $v = {}_hAv + {}_h\ell$. Thus, for each $i \in \{1, ..., n\}$ the i-th component $v_i$ of $v$ can be considered as total amount of labor ("labor embodied") to produce one unit of commodity $i$. It includes not only the input coefficient for labor ${}_hl_i$ (direct labor), but also the amount of labor to produce the quantities of the different capital goods necessary to produce one unit of commodity $i$ (indirect labor), which is represented by the sum $\sum_{j=1}^{n} {}_ha_{ij}v_j$ (with regard to this, the magnitude ${}_hL$ means the absolute amount of direct labor to produce the net output $C_1$ by technique $h$).

57 See also Garegnani (1970, pp. 408-16) for the significance of this condition in using Samuelson's "tools of the Surrogate Production Function [...] and Surrogate Capital to show how we can sometimes predict exactly [rather approximately] how certain quite complicated heterogeneous capital models [of two-commodity economies with finite technologies and one non-basic product satisfying the labor theory of value] will behave by treating them *as if* they had come from a simple generating [continuously differentiable well-behaved aggregate] production function [which is the surrogate production function with labor and *homogeneous* surrogate capital as factors of production to generate a *homogeneous* output] [...]" (Samuelson, 1962, p. 194).

58 Owing to Theorem 3.6 (b), in cases, where the economy's price system satisfies the "labor theory of value", the relative equilibrium commodity prices are constant and independent of the rate of profit (even if cost-minimizing producers switch from one technique to another). As a result, the "price Wicksell effect" between switch points is always zero, and we only have a negative "real Wicksell effect" at the various switch points. Therefore, changes in the index of capital intensity, defined by identity (4.21), corresponds exactly to changes in the aggregated value of the heterogeneous capital intensities.

59 A positive relation between the index of capital intensity and the rate of profit can be considered as what Samuelson called "reverse capital deepening" (Samuelson, 1966, pp. 581-2).

60 Note that this technique is characterized by two equilibrium points, namely by $(K_{i-1}, c_{i-1})$ and $(K_j, c_j)$, where $c_{i-1} = c_j$ due to Theorem 4.3 and identity (4.22).

61 According to Robinson, a given technology as "'book of blueprints [...], is not something which exists in nature. In real life techniques are blueprinted only when they are going to be used. The coexistence at a moment of time of a number of techniques that would be eligible at different rates of profit is conceived only as logical device to disentangle the ambiguity of 'substitution between labour and capital.' (Robinson, 1971, p. 255).

62 In case of regular economies with "neoclassical" technologies, the amount of applied labor to produce a constant final output is a continuous and strictly monotonically decreasing function with respect to the wage rate, i.e. a rising wage rate always implies a continuously falling level of employment.

63 On this occasion, it must be stressed again that the condition of a constant net output is the implication of "Say's law", meaning that variations in the rate of profit (i.e. in the functional income distribution) do not affect the amount of the net output. If we relax this assumption and allow for effects of the income distribution on the net output, then with a varied rate of profit the absolute amount of applied labor is not only influenced by switches of techniques, but also by changes in the net output. For a example, if we assume that the net output increases with a fall in the rate of profit (which corresponds to a rise in the wage rate), then, due to constant returns to scale, rising (absolute) employment of labor, which is associated with a higher wage rate, can be observed even in regular economies.

64 Zambelli (2004) attempted to estimate the probability that an economic system, which is simulated by a computer program (based on a classical long-run equilibrium model with a finite technology), does not exhibit "neoclassical properties". For this purpose, he defined for each feasible rate of profit the aggregated value of the heterogeneous capital intensities as (index for aggregate) capital intensity. He considered an economy as "neoclassical", if the so defined capital intensity is monotonically decreasing in the current rate of profit. To ensure that, the condition of regular economies (meaning a solely negative "real Wicksell effect" at switch points) must be supplemented by an additional regularity condition: The equilibrium commodity prices must react to an increase in the rate of profit, when it still remains between two switch points (i.e. when the heterogeneous physical capital intensities remain constant), in a way, so that the aggregated value of the heterogeneous capital intensities does not increase (i.e. the pure "prices Wicksell effect", occurring only between switch points, must not be positive). Consequently, economies, that are regular in the sense of Definition 4.6, are not necessarily neoclassical by this criterion. In contrast to the "real Wicksell effect", the sign of the "price Wicksell effect" depends on the choice of numeraire; see also Subsection 4.3. This means that whether an economy is neoclassical is also subject to the choice of numeraire, while the property of an economy being regular is independent of the numeraire. Zambelli reported a value of nearly 0.4 as estimated probability (calculated through the average of the observed relative frequencies of "neoclassical economies" as result of the simulations) that at a given numeraire bundle an economy (which is generated by the above mentioned computer program) is neoclassical (Zambelli, 2004, p. 112).

65 Such a hypothesis will remain in a certain sense valid, if we allow for technical progress (for example by comparative statics or any sort of dynamics), which changes an existent well-behaved aggregate production function. This is implied by the assumption that the economy remains regular after any extension of the technology by new techniques (with smaller input coefficients).

66 One might also speak of a hypothesis (or theory), that has been falsified in the sense of Karl Popper.

67 For example, in Flassbeck and Spiecker (1998, pp. 21-41), who considered realized empirical associations between the (real) rates of change in magnitudes like the average wage rate, the wage share, the average labor productivity, employment or macroeconomic investment (i.e. expenditure for capital goods), empirical arguments can be found, that appear to question the main conclusions drawn from well-behaved aggregate production functions (especially with respect to wage policy to handle unemployment).

68 This is a remarkable result, given the fact that in a neoclassical short-run equilibrium model, in which the technology is specified as aggregate Cobb-Douglas production function (and where factor prices are determined by the marginal product of the respective factors of production), the factor shares of the aggregate income are equal to the respective output elasticities (which are constant technological parameters). According to Fisher, it suggests that "the view that the constancy of labor's share is due to the presence of an aggregate Cobb-Douglas production function is mistaken. Causation runs the other way and the apparent success of aggregate Cobb-Douglas production functions is due to the relative constancy of labor's share" (Fisher, 1971, p. 306). Further comments on Fisher's finding can be found in Zambelli (2004, pp. 103-4).

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**Abstract:**

**Englisch**

The *well-behaved aggregate production function* (or *neoclassical macroeconomic production function*) is a concept, which is the basis of many (growth and business cycle) models. This kind of models is widely used in theoretic and applied macroeconomics. The well-behaved aggregate production function was also the main issue of the so-called *Cambridge Capital Controversies,* initiated by a famous paper of Robinson (1953). With reference to these controversies and within the framework of a *classical long-run equilibrium* with *heterogeneous capital* and a *finite technology*, Burmeister (1980) set up a mathematical theorem, which asserts the condition of *regular economies* to be *equivalent* to the *existence of well behaved aggregate production functions*. Neither a formal proof nor an exact axiomatic framework to develop a proof for this theorem was given by Burmeister. In this work we provide a *mathematical proof* within a (slightly modified) version of the model, under which Burmeister originally formulated the theorem. Thereby, we focus our attention on the construction of the axiomatic structure of the model, particularly with respect to the verification of the existence of *cost-minimizing techniques* and *switch points*, which are essential to define the term *regular economy*. At the end of this work, a brief discussion of the problems, regarding the empirical dimension of the theorem, is given.

**Deutsch**

Die *neoklassische aggregierte Produktionsfunktion* (oder *neoklassische makroökonomische Produktionsfunktion*) dient als Basis für viele Modelle (über Wachstum und Konjunktur), welche in der theoretischen und empirischen Makroökonomik Anwendung finden. Sie war Hauptstreitpunkt der sogenannten *Cambridge Kapitalkontroversen*, angestoßen durch den berühmten Aufsatz von Robinson (1953). Darauf bezugnehmend und im Rahmen eines Models des *klassischen Langfristgleichgewichts* mit *heterogenem Kapital* und *finiter Technologie*, formulierte Burmeister (1980) einen mathematischen Satz, welcher die *Äquivalenz* der Bedingung *regulärer Ökonomien* zur *Existenz neoklassischer aggregierter Produktionsfunktionen* behauptet. Dabei wurden von Burmeister weder ein formaler Beweis noch ein exakter axiomatischer Rahmen, der als Basis für die Entwicklung eines Beweises dienen könnte, geliefert. In der vorliegenden Arbeit stellen wir einen *mathematischen Beweis* dieses Satzes vor, und zwar innerhalb einer (leicht modifizierten) Version des Modellrahmens, unter dem Burmeister den Satz ursprünglich formulierte. Dabei legen wir ein besonderes Augenmerk auf den Aufbau der axiomatischen Strukturen des Models, insbesondere im Hinblick auf den Nachweis der Existenz *kostenminimierender Techniken* und sogenannter *Switch Points*, welche zur Definition des Begriffs der *regulären Ökonomie* notwendig sind. Am Ende dieser Arbeit wird eine kurze Diskussion über Probleme bezüglich der empirischen Dimension des Satzes geliefert.

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