MASTERARBEIT

Titel der Masterarbeit

On the dimension of Bergman spaces of square integrable holomorphic functions in one and several complex variables.

Verfasser

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Contents

1 Introduction

We fix notation and give a list of some important definitions and results from complex analysis, measure theory and the theory of partial differential operators before starting the discussion on the dimension of Bergman spaces. Especially in the revision of complex analysis and measure theory we skip the details, but we are more precise in the discussion of fundamental solutions to certain partial differential operators.

1.1 Basic definitions and notation

The open disc in $\mathbb{C}^n$ centered at a with radius h is denoted by $D(a,h)$. A set $\Omega \subseteq \mathbb{C}^n$ is called domain, if it's open and connected.
If $\Omega \subseteq \mathbb{C}^n$ is some open set with respect to (w.r.t.) Lebesgue measure (denoted by $d\lambda$), we denote the set of all holomorphic functions on Ω by $\mathcal{H}(\Omega)$, the space of compactly supported smooth (N-times continuously differentiable) functions by $C_0^\infty(\Omega)$ ($C_0^N(\Omega)$) and the space of N-times continuously differentiable functions by $C^N(\Omega)$. For simplicity we write $C(\Omega) := C^0(\Omega)$, denoting the space of continuous functions on Ω. Elements of $C_0^\infty(\Omega)$ are called test functions.
If $\Psi : \Omega \to \mathbb{R}$ is measurable w.r.t. Lebesgue measure, we frequently use weighted L^2-spaces. Put

$$\mathcal{N} := \{f : \Omega \to \mathbb{C} : f = 0 \text{ almost everywhere w.r.t. Lebesgue measure}\}$$

and for $1 \leq p < \infty$ we define the weighted L^p space by

$$L^p(\Omega,\Psi) := \{f : \Omega \to \mathbb{C} : \int_\Omega |f(z)|^p e^{-\Psi(z)} d\lambda(z) < \infty\}/\mathcal{N}.$$

By definition, the L^p spaces contains equivalence classes of functions. For simplicity we write for a function f satisfying $\int_\Omega |f|^p e^{-\Psi} d\lambda(z) < \infty$ that $f \in L^2(\Omega,\Psi)$, instead of $[f]_\sim \in L^p(\Omega,\Psi)$, which would be more precisely. We use the notation $[f]_\sim$ only at some point, where it is absolutely essential. In many situations we have a look at classical L^p-spaces as well and write $L^p(\Omega) := L^p(\Omega, 0)$. We equip these spaces with the following norms

$$\|f\|_{L^p(\Omega,\Psi)} := \left(\int_\Omega |f(z)|^p e^{-\Psi(z)} d\lambda(z)\right)^{1/p},$$

and we denote the classical norms by $\|f\|_{L^p(\Omega)} := \|f\|_{L^p(\Omega,0)}$. It is well known that the spaces $(L^p(\Omega,\Psi), \|.\|_{L^p(\Omega,\Psi)})$ are Banach spaces for $1 \leq p < \infty$ and even Hilbert spaces if $p = 2$ (with inner product denoted by $\langle .,. \rangle_{L^2(\Omega,\Psi)}$). We won't have a look at the case $p = \infty$. For more details see [E09, Kapitel VI, §2]. The space of p-integrable $(0,1)$-forms is denoted by $L^p_{0,1}(\Omega,\Psi)$.
Sometimes it is useful to have a look at the space of p-integrable functions

$$L^p(\Omega,\Psi,\text{loc}) := \{f \in L^p(K,\Psi) : \text{ for all compact subsets } K \subseteq \Omega\}.$$

and for simplicity we write $L^1(\Omega, \mathrm{loc})$ instead of $L^1(\Omega, 0, \mathrm{loc})$.
We define the supremum of a bounded function $f : \Omega \to \mathbb{C}$ by

$$\|f\|_{\infty,\Omega} := \sup_{z\in\Omega} |f(z)|.$$

The set of distributions on $\mathbb{R}^n$ is denoted by $\mathcal{D}'(\mathbb{R}^n)$, which contains the set of compactly supported distributions $\mathcal{E}'(\mathbb{R}^n)$. The pairing of a distribution λ and a function $\phi \in C_0^\infty(\mathbb{R}^n)$ is denoted by $\langle \lambda, \phi \rangle$. The convolution $\lambda * \xi$, where $\lambda \in \mathcal{D}'(\mathbb{R}^n)$ and $\xi \in \mathcal{E}'(\mathbb{R}^n)$, is defined by

$$\langle \lambda * \xi, f \rangle := \langle \lambda(x), \langle \xi(y), \phi(x+y) \rangle \rangle,$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$. In fact, this is a generalization of the convolution of some Borel measurable essentially bounded function f and $g \in L^1(\mathbb{R}^n)$, i.e.

$$g * f(z) := \int_{\mathbb{R}^n} f(y) g(z-y) d\lambda(z).$$

The support of a function (measure, distribution) f, is denoted by $\mathrm{supp} f$.
The σ-algebra of all Borel sets in $\mathbb{C}^n$ (with norm topology) is denoted by $\mathcal{B}(\mathbb{C}^n)$. If not otherwise specified the term 'measure' denotes a positive Borel measure.

1.2 A collection of useful theorems

The below results are used several times in the following chapters. Most of them require more work and therefore, we skip the details of the proofs.

1.2.1 Holomorphic functions

All of the stated theorems are classical results from complex analysis. If not otherwise specified their proofs can be found in Hörmander's book [Hö90/I, ch. 1 and 2].

Theorem 1.1 (Weierstrass)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and $\{f_n\}_{n=1}^\infty \subseteq \mathcal{H}(\Omega)$. If there exists a function $f : \Omega \to \mathbb{C}$ such that for every compact subset K of Ω*

$$\|f_n - f\|_{\infty,K} \to 0,$$

as $n \to \infty$, then $f \in \mathcal{H}(\Omega)$. Furthermore, for every compact subset K of Ω and $k \in \mathbb{N}$ we have

$$\|f_n^{(k)} - f^{(k)}\|_{\infty,K} \to 0,$$

as $n \to \infty$.

Proof. A proof can be found in [Re92, p.195]. □

Theorem 1.2 (inhomogeneous Cauchy formula)**.** *Let Ω be a bounded, open set in $\mathbb{C}$, such that $b\Omega$ is piecewise C^1 and positively oriented. If $u \in C^1(\overline{\Omega})$, then u can be written in the following form*

$$u(z) = \frac{1}{2\pi i} \left(\int_{\partial\Omega} \frac{u(\xi)}{\xi - z} d\xi + \int_\Omega \frac{\frac{\partial u}{\partial \overline{\xi}}(\xi)}{\xi - z} d\xi \wedge d\overline{\xi} \right),$$

for every $z \in \Omega$.

The previous theorem can be applied to prove the following statement.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{C}$ be open. For every compact set $K \subseteq \Omega$ and every open neighborhood $\omega \subseteq \Omega$ of K there are constants $C_j > 0$ for $j = 0, 1, 2, \dots$ such that*

$$\|\frac{\partial^j u}{\partial z^j}\|_{\infty,K} \leq C_j \|u\|_{L^1(\omega)},$$

for every $u \in \mathcal{H}(\Omega)$.

Proof. Let $u \in \mathcal{H}(\Omega)$ and $\phi \in C_0^\infty(\omega)$ such that $\phi = 1$ in a neighborhood V of K. Now apply the inhomogeneous Cauchy formula to the function $\phi u \in C_0^\infty(\omega)$ and note that $\frac{\partial}{\partial \bar{z}}\Big(\phi u\Big) = u\frac{\partial \phi}{\partial \bar{z}}$. Thus, we get

$$\begin{aligned}\phi(z)u(z) &= \frac{1}{2\pi i}\left(\int_{\partial\omega} \frac{u(\xi)\frac{\partial\phi}{\partial\bar{\xi}}(\xi)}{\xi - z} d\xi + \int_\omega u(\xi)\frac{\frac{\partial\phi}{\partial\bar{\xi}}(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}\right) \\ &= \frac{1}{2\pi i}\int_{\omega\setminus V} u(\xi)\frac{\frac{\partial\phi}{\partial\bar{\xi}}(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}.\end{aligned} \tag{1.1}$$

By construction we have that

$$\inf_{\xi\in\omega\setminus V, z\in K} |\xi - z| > 0$$

and the function $\frac{\partial\phi}{\partial\bar{z}}$ is bounded on $\omega\setminus V$. Thus, our claim holds true for $j = 0$. The general situation follows by differentiating equality (1.1). □

The previous proof can be adapted to gain the following result. We use the multi-index notation: For $\alpha \in \mathbb{N}^n$ we define

$$|\alpha| \coloneqq \alpha_1 + \cdots + \alpha_n,$$

$$\partial^\alpha u(z) \coloneqq \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}.$$

Theorem 1.4. *Let Ω be open in $\mathbb{C}^n$. For every compact set $K \subseteq \Omega$ and every open neighborhood ω of K there are constants C_α for all multi-indices $\alpha \in \mathbb{N}^n$ such that*

$$\|\partial^\alpha u\|_{\infty,K} \leq C_\alpha \|u\|_{L^1(\omega)},$$

for every $u \in \mathcal{H}(\Omega)$.

Idea of the Proof. Let $u \in \mathcal{H}(\Omega)$. Like in Theorem 1.3, you have to choose a function $\psi \in C_0^\infty(\omega)$. Now focus on the function ϕu and apply the inhomogeneous Cauchy formula. □

Another well known principle for holomorphic functions is the following maximum principle (see [Re92, p.191]).

Theorem 1.5 (Maximum Principle)**.** *Let Ω be a domain in $\mathbb{C}$ and $f \in \mathcal{H}(\Omega)$. If $z_0 \in \Omega$, take $r > 0$ such that $D \coloneqq D(z_0, r)$ relatively compact in Ω, then*

$$|f(z_0)| \leq \|f\|_{\infty,bD}.$$

Equality holds if and only if f is constant on Ω.

A very important theorem is Montel's theorem, which is a classical statement of complex analysis. Before we can go into more details we have to describe the involved topology.

Definition 1.6. *Let Ω be a domain in $\mathbb{C}$. A family $\mathcal{F} \subseteq \mathcal{H}(\Omega)$ is called bounded on $D \subseteq \Omega$ if there exists a positive number $M(D) > 0$, such that for every $f \in \mathcal{F}$ we have*

$$\|f\|_{\infty,D} < M(D).$$

The family $\mathcal{F}$ is called bounded if it is bounded on every compact subset of Ω.
We call the family $\mathcal{F} \subseteq \mathcal{H}(\Omega)$ normal, if every sequence $\{f_n\}_{n=0}^{\infty} \subseteq \mathcal{F}$ contains a subsequence $\{f_{n_k}\}_{k=0}^{\infty}$ converging uniformly on the compact subsets of Ω to a function $f \in \mathcal{H}(\Omega)$.

The above terms stem from the following metric defined on $\mathcal{H}(\Omega)$: We take a countable compact exhaustion $\{K_n\}_{n=1}^{\infty}$ of Ω and define a metric by

$$d_{\mathcal{H}(\Omega)}(f,g) := \sum_{n=1}^{\infty} \frac{\|f-g\|_{\infty,K_n}}{2^n(1+\|f-g\|_{\infty,K_n})}.$$

By Theorem 1.1 the space $(\mathcal{H}(\Omega), d_{\mathcal{H}(\Omega)})$ is complete. (With the seminorms $\|f\|_{\infty,K_n}$ the space $\mathcal{H}(\Omega)$ is a Fréchet space.)
This metric induces the topology of uniform convergence on compact subsets on $\mathcal{H}(\Omega)$. The topology does not depend on the particular choice of the exhaustion.

Theorem 1.7 (Montel)**.** *Let Ω be a domain in $\mathbb{C}$. Every bounded family $\mathcal{F} \subseteq \mathcal{H}(\Omega)$ is normal.*

Proof. A proof can be found in [Re95, ch. 7, §1.]. □

1.2.2 Compactly supported continuous functions

Each of the following results can be found in [Ra95].
Let X be a topological space. The set of all complex valued continuous compactly supported functions is denoted by $C_0(X)$. The norm $\|.\|_{\infty,X}$ defines a topology on $C_0(X)$.
We will frequently use the following approximation lemma.

Lemma 1.8 (Approximation)**.** *Let Ω be a domain in $\mathbb{C}$, $\phi \in C_0(\Omega)$ and U a relatively compact open subset of Ω such that supp$\phi \subseteq U$. Then there exists $\{\phi_n\}_{n\in\mathbb{N}} \in C_0^{\infty}(\Omega)$ such that supp$\phi_n \subseteq U$ for all $n \in \mathbb{N}$ and $\|\phi - \phi_n\|_{\infty,\Omega} \to 0$ as $n \to \infty$. Moreover, if $\phi \geq 0$, then the approximating sequence $\{\phi_n\}_{n\in\mathbb{N}}$ can be chosen such that $\phi_n \geq 0$ for each n.*

Proof. A proof can be found in [Ra95, ch. 3, p.72] and is based on a smoothing argument. □

A Riesz representation theorem gives a connection between the dual space of $C_0(X)$ and special Borel measures.

Definition 1.9 (Radon measure)**.** *A Borel measure μ on a topological space X is called a Radon measure, if $\mu(K) < \infty$ for every compact subset K of X.*

A functional F on $C_0(X)$ is called positive if

$$F(\phi) \geq 0$$

for all $\phi \in C_0(X)$ with $\phi \geq 0$ on X.

Theorem 1.10 (Riesz)**.** *Let X be a locally compact metric space. If F is a positive linear fuctional on $C_0(X)$, then there exists a unique Radon measure μ on X such that*

$$F(\phi) = \int_X \phi(z) d\mu(z),$$

for all $\phi \in C_0(X)$.

Proof. A proof can be found in [Ra95, appendix 3, p.211]. □

1.3 Fundamental solutions and regularity

In this section we collect some informative and worth knowing properties of fundamental solutions and special partial differential operators. The main result is that any distributional solution to $\Delta u = 0$ and $\frac{\partial u}{\partial \bar{z}} = 0$ are indeed smooth functions. This statement will be frequently used in the following chapters.
The represented definitions and theorems can be found in [Hö90/II, ch. 2–4].

Definition 1.11 (Linear partial differential operator)**.** *Let $n \geq 1$. An operator*

$$P := \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha,$$

where only finitely many (complex) constants a_α are non-zero, is called a linear partial differential operator.

Linear partial differential operators are defined on different spaces. We consider the situation, when $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$, where Ω denotes an open subset of $\mathbb{R}^n$.

Definition 1.12 (Fundamental solution)**.** *A distribution $\lambda \in \mathcal{D}'(\mathbb{R}^n)$ is called a fundamental solution to the linear differential operator P if*

$$P\lambda = \delta_0,$$

where δ_0 denotes the Dirac distribution at 0 (i.e. point evaluation at 0).

As we shall see later, fundamental solutions can be used to find 'arbitrary' distributional solutions of P. For now, we derive distributional solutions to the Laplace and the Cauchy-Riemann operator.
For this purpose, we recall some consequences of the Gauss-Green theorem: Let U be a bounded, open subset of $\mathbb{R}^n$ s.t. bU is C^1. If $u, v \in C^1(\overline{U})$, $j \in \{1, \ldots, n\}$ and

$\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ denotes the outward pointing unit normal vector at $x \in bU$ the following equalities hold true

$$\int_U u_{x_j} v d\lambda = -\int_U u v_{x_j} d\lambda + \int_{bU} u v \nu_j dS, \tag{1.2}$$

and

$$\int_U \Big(u\Delta v - v\Delta u\Big) d\lambda = \int_{bU} \Big(u\big(\mathrm{grad} v|\nu\big) - v\big(\mathrm{grad} u|\nu\big)\Big) dS. \tag{1.3}$$

Equality (1.2) is called *integration by parts formula.* A proof can be found in [Ev10, p.712].
We follow the lines of Hörmander [Hö90/II, p.80] to show the following result.

Theorem 1.13. *Let $n \in \mathbb{N}$ s.t. $n \geq 2$. The area of the unit sphere in $\mathbb{R}^n$ is denoted by $s(n)$. The function*

$$f(x) := \begin{cases} \frac{\log|x|}{s(2)} & \text{if } x \in \mathbb{R}^2\backslash\{0\}, \\ \frac{-|x|^{2-n}}{(n-2)s(n)} & \text{if } x \in \mathbb{R}^n\backslash\{0\}, \end{cases}$$

is a fundamental solution to the Laplace operator, i.e. $\Delta f = \delta_0$. Moreover, we have that $\partial_j f(x) := \frac{\partial f(x)}{\partial x_j} = \frac{x_j}{s(n)|x|^n}$ in the sense of distributions.

Proof. Take a test function $\phi \in C_0^\infty(\mathbb{R}^n)$. The integration by parts formula (1.2) gives us

$$\begin{aligned} \langle \partial_j f, \phi\rangle &= -\langle f, \partial_j \phi\rangle = -\lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} f(x)\partial_j\phi(x) d\lambda(x) \\ &= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \partial_j f(x)\phi(x)dx + \lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} f(x)\phi(x)\frac{x_j}{|x|}dS. \end{aligned}$$

A short computation shows that

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} f(x)\phi(x)\frac{x_j}{|x|}dS = \begin{cases} \mathcal{O}(\varepsilon\log\varepsilon) & \text{if } n=2, \\ \mathcal{O}(\varepsilon) & \text{if } n=2, \end{cases}$$

as $\varepsilon \to 0$. This proves that $\partial_j f(x) = \frac{x_j}{s(n)|x|^n}$ in the sense of distributions.
We note that $\Delta f(x) = 0$ for $x \in \mathbb{R}^n\backslash\{0\}$ and thus, equality (1.3) implies

$$\begin{aligned} \langle \Delta f, \phi\rangle = \langle f, \Delta\phi\rangle &= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\Big(f(x)\Delta\phi(x) - \phi(x)\Delta f(x)\Big)d\lambda(x) \\ &= \lim_{\varepsilon\to 0}\int_{|x|=\varepsilon}\big(\phi(x)\mathrm{grad} f(x) - f(x)\mathrm{grad}\phi(x)|\frac{x}{|x|}\big)dS. \end{aligned}$$

One can check that the latter integral tends to 0 as $\varepsilon \to 0$. By continuity of ϕ the other term tends to $\phi(0)$ as $\varepsilon \to 0$. Hence we have shown that $\langle\Delta f, \phi\rangle = \langle\delta_0, \phi\rangle$. □

As a consequence of the previous theorem we can find a fundamental solution to the Cauchy-Riemann operator (see [Hö90/II, p.81]).

Corollary 1.14. *The function $f(z) := \frac{1}{\pi z}$ for $z \in \mathbb{C}\backslash\{0\}$ is a fundamental solution to $\frac{\partial}{\partial \bar{z}}$.*

Proof. At first we note that $\Delta = 4\frac{\partial^2}{\partial \bar{z}\partial z}$. Thus by Theorem 1.13

$$\delta_0 = \Delta \frac{\log|z|}{2\pi} = 4\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}\frac{\log|z|}{2\pi}\right) = \frac{\partial}{\partial \bar{z}}\frac{1}{\pi z},$$

is satisfied in the sense of distributions. □

Another nice property of fundamental solutions is the following: Let λ be a fundamental solution to the linear partial differential operator P and $\eta \in \mathcal{E}'(\mathbb{R}^n)$. The convolution $\lambda * \eta = \eta * \lambda$ is well defined and one has

$$P(\lambda * \eta) = \lambda * P\eta = P\lambda * \eta = \delta_0 * \eta = \eta. \tag{1.4}$$

We recall the definition of the support of a distribution $\lambda \in \mathcal{D}'(\Omega)$,

$$\text{supp}\lambda = \{x : \nexists \text{ neighborhood } V \text{ of } x \text{ s.t. } \langle \lambda, \phi \rangle = 0 \text{ for all } \phi \in \mathcal{D}'(V)\}.$$

Another kind of support is the so-called singular support, which consists of all points x s.t. the distribution isn't equal to a smooth function in a neighborhood of x. More precisely: Let $\lambda \in \mathcal{D}'(\Omega)$, then

$$\text{sing supp}\lambda := \{x : \nexists \text{ neighborhood } V \text{ of } x \text{ s.t. } \lambda|_V \text{ is a smooth regular distribution}\}.$$

The fact that $\mathcal{D}'(\mathbb{R}^n) * C_0^\infty(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n)$, a well known property of convolution of distributions (see [Hö90/II, p.88]), allows us to prove the following lemma (see [Hö90/II, p.102–103]).

Lemma 1.15. *Let $\lambda_1 \in \mathcal{D}'(\mathbb{R}^n)$ and $\lambda_2 \in \mathcal{E}'(\mathbb{R}^n)$. Then*

$$\textit{sing supp}(\lambda_1 * \lambda_2) \subseteq \textit{sing supp}\lambda_1 + \textit{sing supp}\lambda_2.$$

Proof. Let χ be a smooth cut-off-function s.t. $\chi = 1$ in a neighborhood of sing suppλ_2. We define $\sigma_2 := \chi\lambda_2 \in \mathcal{D}'(\mathbb{R}^n)$ and $\eta_2 := (1-\chi)\lambda_2 \in C_0^\infty(\mathbb{R}^n)$ and note that

$$\lambda_2 = \sigma_2 + \eta_2.$$

As we have already stated, $\lambda_1 * \eta_2 \in C^\infty(\mathbb{R}^n)$ and $\lambda_1 * \sigma_2$ is a C^∞-function in

$$\{x : x - \text{supp}\sigma_2 \subseteq \mathbb{R}^n \backslash \text{sing supp}\lambda_1\}.$$

This means that

$$\begin{aligned}
\text{sing supp}\Big(\lambda_1 * \lambda_2\Big) &= \text{sing supp}\Big(\lambda_1 * \sigma_2 + \lambda_2 * \eta_2\Big) \\
&= \text{sing supp}\Big(\lambda_1 * \sigma_2\Big) \\
&\subseteq \text{supp}\sigma_2 + \text{sing supp}\lambda_1 \\
&\subseteq \text{supp}\chi + \text{sing supp}\lambda_1.
\end{aligned}$$

We can take χ s.t. $\chi = 1$ in a neighborhood arbitrarily close to sing suppλ_2. □

Now it's quite easy to prove the following theorem (see [Hö90/II, p.110]).

Theorem 1.16. *If the linear partial differential operator P has a fundamental solution λ with sing supp$\lambda = \{0\}$ and Ω is an open set in $\mathbb{R}^n$, then*

$$\textit{sing supp}\eta = \textit{sing supp}P\eta,$$

for every $\eta \in \mathcal{D}'(\Omega)$.

Proof. An immediate consequence of the definition of singular support is that

$$\text{sing supp}P\eta \subseteq \text{sing supp}\eta.$$

Let $\eta \in \mathcal{E}'(\Omega)$. Lemma 1.15 and equation (1.4) give us

$$\text{sing supp}\eta = \text{sing supp}\lambda * P\eta \subseteq \text{sing supp}P\eta.$$

This proves the claim for compactly supported distributions.
If $\eta \in \mathcal{D}'(\Omega)$, we take a compact exhaustion $\{K_m\}_{m=1}^{\infty}$ of Ω and a family $\{\chi_m\}_{m=1}^{\infty}$ of smooth cut-off functions s.t. $\chi_m|_{K_m} = 1$ and note that

$$\begin{aligned} K_m \cap \text{sing supp}P\eta &= K_m \cap \text{sing supp}P(\chi_m\eta) \\ &= K_m \cap \text{sing supp}\chi\eta \\ &= K_m \cap \text{sing supp}\eta, \end{aligned}$$

for every $m \in \mathbb{N}$. This proves the claim. □

As next we recall a localization theorem for distributions. We use this result to prove the claim on regularity of solutions to the Cauchy-Riemann operator and the Laplace operator at the beginning of this section.

Theorem 1.17 (Localization of distributions)**.** *Let $\eta_1, \eta_2 \in \mathcal{D}'(\Omega)$.*

(i) If every $x \in \Omega$ has a neighborhood V_x s.t. $\eta_1\Big|_{V_x} = 0$, then $\eta_1 = 0$.

(ii) If every $x \in \Omega$ has a neighborhood V_x s.t. $\eta_1\Big|_{V_x} = \eta_2\Big|_{V_x}$, then $\eta_1 = \eta_2$.

Proof. A proof of (i) can be found in [Hö90/II, p.41]. The second claim follows by applying the first result to the distribution $\eta_1 - \eta_2$. □

Now we are able to prove a regularity result for special distributions.

Lemma 1.18. *Let $\eta \in \mathcal{D}'(\Omega)$. If the singular support satisfies sing supp$\eta = \emptyset$, then there exists a function $g \in C^\infty(\Omega)$ s.t. $g = \eta$ in the sense of distributions.*

Proof. By the definition of the singular support, for every $x \in \Omega$ there exists an open neighborhood V_x and a smooth function $g_x \in C^\infty(V_x)$ s.t.

$$g_x = \eta\Big|_{V_x}. \tag{1.5}$$

Let $x \neq y$ such that $z \in V_x \cap V_y \neq \emptyset$. Thus, we can find an open neighborhood U_z of z s.t. $U_z \subseteq V_x \cap V_y$. For every $\psi \in C_0^\infty(U_z)$ we have

$$\langle g_x, \psi \rangle = \langle \eta, \psi \rangle = \langle g_y, \psi \rangle,$$

which proves that $\int_{U_z} \psi(w)(g_x(w) - g_y(w))d\lambda(w) = 0$ for every $\psi \in C_0^\infty(U_z)$. By continuity of g_x and g_y in U_z we have that

$$g_x(w) = g_y(w)$$

for every $w \in U_z$.
Therefore, the function g defined by

$$g(z) := g_x(z),$$

if $z \in V_x$, is well defined and lies in $C^\infty(\Omega)$. By Theorem 1.17 and (1.5) we even have that

$$\eta = g$$

in the sense of distributions. □

The following regularity result for special partial differential operators applies to the Cauchy-Riemann and Laplace operator.

Corollary 1.19. *Let $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ be a partial differential operator with fundamental solution $\lambda \in \mathcal{D}'(\Omega)$ satisfying sing supp$\lambda = \{0\}$ and $g \in C^\infty(\Omega)$. If a distribution $\eta \in \mathcal{D}'(\Omega)$ satisfies $P\eta = g$, then there exists a function $f \in C^\infty(\Omega)$ s.t. $\eta = f$ in the sense of distributions and $Pf = g$ is satisfied in the classical sense.*

Proof. By Theorem 1.16 we have

$$\text{sing supp}\eta = \text{sing supp}P\eta = \text{sing supp}g = \emptyset.$$

Thus we apply Lemma 1.18 to find a function $f \in C^\infty(\Omega)$ s.t. $\eta = f$ in the sense of distributions. Since all derivatives of f are continuous $Pf = g$ holds in the classical sense. □

We apply the previous corollary in the following way: Let P be a partial differential operator satisfying the assumptions of Corollary 1.19. If we find a solution to the inhomogeneous equation

$$Pf = g$$

(with smooth right hand side) in the sense of distributions, then we are able to find a version which is a classical smooth solution.

Example 1.A. The previous theorem applies to the Cauchy-Riemann operator and the Laplace operator. For example, if $\eta_1 \in \mathcal{D}'(\mathbb{C})$ and $\eta_2 \in \mathcal{D}'(\mathbb{R}^n)$ satisfy

$$\frac{\partial \eta_1}{\partial \overline{z}} = 0,$$

$$\Delta\eta_2 = 0,$$

then one can find smooth versions of η_1 and η_2 solving the homogeneous Cauchy-Riemann equation and Laplace equation.
In particular, if $\eta_1 \in L^1(\mathbb{C}, loc)$ and $\eta_1 \in L^1(\mathbb{R}^n, loc)$ $(n \geq 2)$, then by the previous corollary η_1 and η_2 are equal to a smooth function except on a set of Lebesgue measure zero. ♦

A version of Corollary 1.19 for the Cauchy-Riemann operator and the Laplace operator can be found in [BG91, p.252, Prop. 3.6.5.].

2 Subharmonic functions and related concepts

In this chapter we provide material on upper semicontinuous functions, subharmonic functions, plurisubharmonic functions and topics related to logarithmic potential theory. Among others, we prove the Riesz representation theorem for subharmonic functions, which is based on Theorem 1.10 and gives a connection between the theory of subharmonic functions and (logarithmic) potentials. Finally, we introduce the term of logarithmic capacity.
Most of the stated theorems can be found in [Co95, ch. 19], [Hö90/I, ch. 1 and 2], [R86, ch. 4] and [Ra95, ch. 2–5]. Nevertheless, these are well-known facts and can be found in several other books.

2.1 Upper semicontinuous functions

This section only provides basic material on upper semicontinuous functions. If not otherwise specified, Ω always denotes an open subset of $\mathbb{C}$.

Definition 2.1 (Upper semicontinuous)**.** *A function $u : \Omega \to [-\infty, \infty]$ is called upper semicontinuous if for every $s \in \mathbb{R}$ the set $\{z \in \Omega : u(z) < s\}$ is open.*

A function u is called lower semicontinuous if $-u$ is upper semicontinuous. It turns out that a function is continuous if and only if it is lower and upper semicontinuous.

Lemma 2.2. *A function $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous if and only if*

$$\limsup_{y \to x} u(y) \leq u(x), \tag{2.1}$$

for all $x \in \Omega$.

Proof. This claim is an easy consequence of the definition. $\square$

The following lemma (see [F99, p.40]) tells us that the infimum taken over an arbitrary family of upper semicontinuous functions is again upper semicontinuous. This is an interesting difference to families of continuous functions, where one can not expect that the infimum is again continuous.

Lemma 2.3. *Let I be an index-set and $u_j : \Omega \to [-\infty, \infty)$, for $j \in I$, be a family of upper semicontinuous functions. The function*

$$U(z) := \inf_{j \in I} u_j(z),$$

is upper semicontinuous.

Proof. Let $s \in \mathbb{R}$. Take $z_0 \in \Omega$ s.t. $U(z_0) < s$. By definition of U we can find an index $j \in I$ s.t. $u_j(z_0) < s$. Since u_j is upper semicontinuous we can find a neighborhood V_{z_0} of z_0 s.t.

$$U(z) \le u_j(z) < s,$$

for every $z \in V_{z_0}$. This implies that $\{z \in \Omega : U(z) < s\}$ is open. □

In some situations we will need an approximation lemma by continuous functions. For this purpose, we provide the following result, which can be found in [Co95, Thm. 3.6, p.219].

Lemma 2.4. *If $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous and $u \le M < \infty$ on Ω, then there exists a decreasing sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ on Ω s.t. $f_n \le M$ and for every $z \in \Omega$ we have $f_n(z) \searrow u(z)$ as $n \to \infty$.*

Proof. W.l.o.g. we may assume that u is not identically $-\infty$.
Define $f_n(z) \coloneqq \sup\{u(y) - n|z-y| : y \in \Omega\}$. Thus, by definition we have that

$$f_n(z) \ge u(y) - n|z-y| \ge u(y) - (n+1)|z-y|$$

for $z, y \in \Omega$ and $n \in \mathbb{N}$. This proves that $f_n(z) \ge f_{n+1}(z) \ge u(z)$.
Now we check, if f_n is uniformly continuous. For this purpose, let $\varepsilon > 0$, $n \ge 1$ and $z \in \Omega$. The definition of f_n tells us that there exists a point $y \in \Omega$ such that $f_n(z) < u(y) - n|z-y| + \varepsilon$.
If $|x-z| < \varepsilon$, we have that $|x-y| < \varepsilon + |z-y|$ and hence,

$$f_n(x) \ge u(y) - n|x-y| \ge u(y) - n|z-y| - n\varepsilon > f_n(z) - (n+1)\varepsilon,$$

for $|x-z| < \varepsilon$. A similar computation shows that $f_n(z) > f_n(x) - (n+1)\varepsilon$ for $|z-x| < \varepsilon$. Hence, $|f_n(x) - f_n(z)| < (n+1)\varepsilon$, whenever $|z-x| < \varepsilon$ and thus f_n is uniformly continuous.
Finally, we prove the claim of pointwise convergence. Fix $z \in \Omega$ and $\varepsilon > 0$. For this purpose, we assume that $u(z) > -\infty$. Since u is upper semicontinuous we can find a $\delta > 0$ s.t. $u(y) < u(z) + \varepsilon$ for $|z-y| < \delta$. Thus, for the same y we have

$$u(y) - n|z-y| \le u(z) + \varepsilon.$$

If $|y-z| \ge \delta$, we have

$$u(y) - n|z-y| \le u(y) - n\delta \le M - n\delta.$$

Choose n_0 s.t. for all $n \ge n_0$ we have $M - n\delta < u(z) + \varepsilon$. Thus by definition of f_n we have

$$u(z) \le f_n(z) \le u(z) + \varepsilon,$$

for $n \ge n_0$.
Otherwise, if $u(z) = -\infty$. We assume that $\limsup_{n\to\infty} f_n(z) = a > -\infty$. Since, $\{f_n\}_{n=1}^{\infty}$ is decreasing we have that $a = \lim_{n\to\infty} f_n(z)$. Fix $\varepsilon > 0$. Thus, we can find n_0 such that $|f_n(z) - a| < \varepsilon$ for all $n \ge n_0$. By definition of f_n we can choose y_n such that

$$-\varepsilon \le f_n(z) - a \le u(y_n) - n|z-y_n| + \varepsilon - a \le M - n|z-y_n| + \varepsilon - a.$$

Hence, we even have that $0 \leq n|z - y_n| \leq M - a + 2\varepsilon$ and so $y_n \to z$ as $n \to \infty$. Now we can apply Lemma 2.2 and gain

$$-\infty < a = \limsup_{n\to\infty} f_n(z) \leq \limsup_{n\to\infty} u(y_n) + \varepsilon \leq u(z) + \varepsilon = -\infty,$$

a contradiction. □

Another important question is if upper semicontinuous functions have upper bounds on compact sets (see for instance [Ra95, Thm. 2.1.2, p.25]).

Lemma 2.5. *Let u be an upper semicontinous function on Ω and let K be a compact subset of Ω. Then u is bounded above and attains its upper bound.*

Proof. The increasing sequence of sets $\Omega_n := \{z \in \Omega : u(z) < n\}$, for $n \in \mathbb{N}$, form an open cover of K. Therefore, there is $n_0 \in \mathbb{N}$ s.t. $K \subseteq \Omega_{n_0}$. This proves the first claim. Let $M := \sup_{z\in K} u(z)$. By the definition of upper semicontinuity, we have that for every $n \in \mathbb{N}$ the sets $\{z \in \Omega : u(z) < M - \frac{1}{n}\}$ are open and cannot cover K (since they don't have a finite subcover). Hence, $u(z) = M$ for at least one $z \in K$. □

2.2 Subharmonic functions

Unless stated otherwise, in this section Ω denotes a domain in $\mathbb{C}$ with the associated induced norm topology of $\mathbb{C}$.

Definition 2.6. *A function $\Psi : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called subharmonic if the following two properties are satisfied:*

(i) Ψ is upper semicontinuous,

(ii) Ψ satisfies the submean value property, i.e. for every $a \in \Omega$ there exists a number $r_a > 0$ such that $D(a, r_a) \subseteq \Omega$ and

$$\Psi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \Psi(a + re^{i\theta}) d\theta,$$

for every $0 < r < r_a$.

An upper semicontinuous function Ψ is called strictly subharmonic if the inequality in property *(ii)* is strict.
We call a function $h : \Omega \to [-\infty, \infty)$ harmonic if and only if h and $-h$ are subharmonic.
Harmonic functions satisfy that

$$h(a) = \frac{1}{2\pi} \int_0^{2\pi} h(a + re^{i\theta}) d\theta, \tag{2.2}$$

for $r \geq 0$ s.t. $D(a, r) \subseteq \Omega$. These are exactly the functions s.t. $\Delta h = 0$ (see [Ev10, Thm. 2&3, p.25/26]). Equality (2.2) is called mean value property.
By linearity of the integral, it's easy to see that the addition of two subharmonic functions is again subharmonic.

Let $g \in C(bD(0,1))$. Sometimes it is quite useful to have a solution of the following Dirichlet problem: Is there a function $u : \overline{D(0,1)} \to \mathbb{R}$ s.t.

$$\begin{aligned} u(z) &= g(z) \quad \text{for } z \in bD(0,1), \\ \Delta u &= 0 \qquad \text{on } D(0,1). \end{aligned} \tag{2.3}$$

Actually, we don't want to prove existence of a function satisfying (2.3), but we write down the most important theorem. To solve (2.3) we need the so-called Poisson kernel:

Definition 2.7. *The function $P_r(\theta) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ defined by*

$$P_r(\theta) := \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}$$

is called the Poisson kernel.

There are several other representations of the Poisson kernel (see for instance [Co78, p.256–257]). It's easy to see that $P_r(\theta)$ is 2π-periodic in θ. Another nice property is that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1.$$

A proof of this statement can be found in [Co78, p.257].

Theorem 2.8 (Poisson's formula for the disc)**.** *Let $g \in C(bD(0,1))$. The function $u : \overline{D(0,1))} \to \mathbb{R}$ defined by*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)g(re^{it})dt$$

for $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$, and $u(e^{i\theta}) = g(e^{i\theta})$ for $0 \leq \theta \leq 2\pi$, is continuous on $\overline{D(0,1)}$ and is a solution of the Dirichlet problem (2.3)*.*

Proof. A proof can be found in [Co78, p.257–258], or in [Ra95, p.8]. □

There are several equivalent definitions of subharmonic functions. The following can be found in [Co95, Thm. 4.5, p.221].

Theorem 2.9. *If $\Psi : \Omega \to [-\infty,\infty)$ is upper semicontinuous, then the following properties are equivalent:*

(i) Ψ is subharmonic.

(ii) Every compact subset K of Ω has the following property: If $h \in C(K)$, harmonic on K°, s.t. $u \leq h$ on bK, then $u \leq h$ on K.

(iii) Every closed disc $D \subseteq \Omega$ has the following property: If $h \in C(D)$, harmonic on D°, s.t. $u \leq h$ on bD, then $u \leq h$ on D.

Proof. $(i) \implies (ii)$. Let K be a compact subset of Ω and assume that $h \in C(K)$ harmonic on K° with $\Psi \leq h$ on bK. We put $u := \Psi - h$ and therefore, we have to check if $u \leq 0$ on K.
Suppose that there is $z_0 \in K$ with $u(z_0) > \varepsilon > 0$ and define $A := \{z \in K : u(z) \geq \varepsilon\}$. By definition A is nonempty and compact (u is upper semicontinuous). Since, $bK \cap A = \emptyset$, we have that $A \subseteq K^\circ$. Upper semicontinuity implies again, that there exists $a \in A$ s.t. $u(a) \geq u(z)$ for all $z \in A$ and thus for all $z \in K$.
O denotes the connected component of K° s.t. the point a lies in O. Now we put $B := \{z \in O : u(z) = u(a)\}$, which is a relatively closed nonempty subset of O. Take $w \in B$ and $r > 0$ s.t. $\overline{D(w,r)} \subseteq \Omega$ and thus, by connectedness, $\overline{D(w,r)} \subseteq O$. For $0 < s < r$ we have

$$2\pi u(w) \leq \int_0^{2\pi} u(w + se^{i\theta}) d\theta \leq 2\pi u(a) = 2\pi u(w),$$

and hence,

$$\int_0^{2\pi} \Big(u(a) - u(w + se^{i\theta}) \Big) \, d\theta = 0.$$

The integrand of the previous display is positive and thus we have that $\overline{D(w,r)} \subseteq B$. This proves that B is open and hence $B = O$.
If $x \in bO \subseteq bK$, then by Lemma 2.2 we have

$$0 \geq u(x) \geq \limsup_{z \in O, z \to x} u(z) = u(a) > 0,$$

a contradiction.
$(ii) \implies (iii)$. This implication is easy to see.
$(iii) \implies (i)$. Suppose that $\overline{D(a,r)} \subseteq \Omega$. Fix $0 < s < r$. By Lemma 2.4 we can find $\{f_n^s\}_{n=1}^\infty \subseteq C(bD(a,s))$ s.t. $f_n^s \searrow \Psi$ pointwise on $bD(a,s)$. By Theorem 2.8 we find a function $h_n^s \in C(\overline{D(a,s)})$ which is harmonic on $D(a,s)$ and satisfies $h_n^s|_{bD(a,s)} = f_n^s$. By property (iii) we have that $\Psi \leq h_n^s$ on $\overline{D(a,s)}$ and therefore, we use harmonicity of h_n^s to obtain

$$\Psi(a) \leq h_n^s(a) = \frac{1}{2\pi} \int_0^{2\pi} h_n^s(a + s_1 e^{i\theta}) d\theta,$$

for $s_1 < s$. By Fatou's lemma for $s_1 \nearrow s$ we have

$$2\pi \Psi(a) \leq \int_0^{2\pi} h_n^s(a + se^{i\theta}) d\theta = \int_0^{2\pi} f_n^s(a + se^{i\theta}) d\theta.$$

Finally, we apply the monotone convergence theorem to see that Ψ satisfies the submean value property. $\square$

Remark 2.A. Property (ii) of Theorem 2.9 is called maximum principle for subharmonic functions.

Before we can prove a smoothing lemma, we have to provide a technical lemma first (see [Co95, Thm. 4.9, p.223]).

Lemma 2.10. *If Ψ is subharmonic on Ω and $\overline{D(a,r)} \subseteq \Omega$, then $\int_0^{2\pi} \Psi(a + re^{i\theta})d\theta \searrow 2\pi\Psi(a)$ as $r \searrow 0$.*

Proof. We use the same method as in the proof of Theorem 2.9. For $0 \leq s \leq r$ put $\sigma(s) := \frac{1}{2\pi}\int_0^{2\pi} \Psi(a+se^{i\theta})d\theta$. We can find a sequence of continuous functions $\{f_n^s\}_{n=1}^{\infty}$ on $bD(a,s)$ (see Lemma 2.4) such that $f_n^s \searrow \Psi$ pointwise on $bD(a,s)$. Hence, we can find functions $h_n^s \in C(\overline{D(a,s)})$ s.t. h_n^s is harmonic on $D(a,s)$ and $h_n^s|_{bD(a,s)} = f_n^s$ (see Theorem 2.8). Thus by subharmonicity we have $h_n^s \geq \Psi$ on $\overline{D(a,s)}$. Let $0 \leq s_1 < s_2 < r$ and use the mean value property for harmonic functions and Fatou's lemma to see

$$\sigma(s_1) \leq \frac{1}{2\pi}\int_0^{2\pi} h_n^{s_2}(a + s_1e^{i\theta})d\theta = h_n^{s_2}(a) \leq \frac{1}{2\pi}\int_0^{2\pi} h_n^{s_2}(a + s_2e^{i\theta})d\theta.$$

Hence, apply monotone convergence theorem to get $\sigma(s_1) \leq \sigma(s_2)$.
By subharmonicity we have $\Psi(a) \leq \sigma(s)$ and thus by Fatou's lemma we have

$$\Psi(a) \leq \lim_{s\searrow 0}\sigma(s) \leq \frac{1}{2\pi}\int_0^{2\pi} \limsup_{s\searrow 0}\Psi(a + se^{i\theta})d\theta \leq \Psi(a).$$

Therefore, the claim holds true. □

As previously mentioned, we are now able to prove a smoothing lemma (see for instance [Co95, Prop. 4.19, p.226]). For this purpose we need mollifiers. Let $\chi \in C_0^{\infty}(\mathbb{C})$ radial symmetric, positive with $\int_{\mathbb{C}} \chi(z)d\lambda(z) = 1$ and $\mathrm{supp}\chi \subseteq D(0,1)$ (standard mollifier). Then we define

$$\chi^{\varepsilon}(z) := \frac{1}{\varepsilon^2}\chi(\frac{z}{\varepsilon}),$$

for $\varepsilon > 0$.

Lemma 2.11 (Smoothing)**.** *Let Ψ be a subharmonic function on Ω and $\varepsilon > 0$. The function $\Psi_{\varepsilon} := \Psi * \chi^{\varepsilon}$ (called a mollification of Ψ) has the following properties:*

(i) $\Psi_{\varepsilon} \in C^{\infty}(\mathbb{C})$.

(ii) If K is a compact subset of Ω, then $\|\Psi - \Psi_{\varepsilon}\|_{L^1(K)} \to 0$ as $\varepsilon \to 0$.

(iii) Ψ_{ε} is subharmonic on $\Omega_{\varepsilon} := \{z \in \Omega : d(z, b\Omega) > \varepsilon\}$.

(iv) For each $z \in \Omega$ we have that $\Psi_{\varepsilon}(z) \searrow \Psi(z)$ as $\varepsilon \searrow 0$.

Proof. (i) and (ii) are standard results and for a proof we refer to [Co95, ch. 18, Prop. 3.6, p.181] or [Ev10, appendix C, Thm. 7, p.714].
(iii). Let $z \in \Omega$. A change of variables argument gives us

$$\Psi_{\varepsilon}(z) = \int_{D(0,1)} \Psi(z - \varepsilon w)\chi(w)d\lambda(w). \tag{2.4}$$

If $r > 0$, such that $\overline{D(z,r)} \subseteq \Omega_{\varepsilon}$, we have by Fubini's theorem, that

$$\int_0^{2\pi} \Psi_{\varepsilon}(z + re^{i\theta})d\theta = \int_0^{2\pi}\int_{D(0,1)} \Psi(z + re^{i\theta} - \varepsilon w)\chi(w)d\lambda(z)d\theta$$

$$
\begin{aligned}
&= \int_{D(0,1)} \chi(w) \int_0^{2\pi} \Psi(z + re^{i\theta} - \varepsilon w) d\theta d\lambda(w) \\
&\geq 2\pi \int_{D(0,1)} \chi(w)\Psi(z - \varepsilon w) d\lambda(w) = 2\pi\Psi_\varepsilon(z).
\end{aligned}
$$

This proves the claim.
(iv). Let $0 < \varepsilon < \delta$, for a fixed $\delta > 0$. Inequality (2.4) and Lemma 2.10 gives

$$
\begin{aligned}
\Psi_\varepsilon(z) &= \int_{D(0,1)} \Psi(z - \varepsilon w)\chi(w) d\lambda(w) \\
&= \int_0^1 r\chi(r) \int_0^{2\pi} \Psi(z - \varepsilon e^{i\theta}) d\theta dr \\
&\leq \int_0^1 r\chi(r) \int_0^{2\pi} \Psi(z - \delta e^{i\theta}) d\theta dr = \Psi_\delta(z).
\end{aligned}
$$

A similar computation (with help of Lemma 2.10 and monotone convergence theorem) shows that $\Psi_\varepsilon(z) \searrow \Psi(z)$ as $\varepsilon \searrow 0$. □

An application of the previous smoothing lemma is the following localization theorem (see [Co95, Cor. 4.22, p.227]).

Lemma 2.12. *If $u \in L^1(\Omega, loc)$ such that, whenever K is a compact subset of Ω there is a subharmonic function Ψ_K on K with $\Psi_K = u$ a.e. w.r.t. Lebesgue measure on K. Then there is a subharmonic function Ψ on Ω s.t. $u = \Psi$ a.e. on Ω.*

Proof. Let u_ε be a mollification of u as in the previous lemma. Take a sufficiently small $\delta > 0$ and let K be a compact subset of $\{z \in \Omega : d(z, b\Omega) > 2\delta\}$. The set $K^\delta := \{z \in \Omega : d(z, K) \leq \delta\}$ is compact in Ω. By assumption we know that $u = \Phi$ a.e. on K^δ, for a subharmonic function Φ. By construction we have for $z \in K$ and $\varepsilon < \delta$ that

$$
\begin{aligned}
u_\varepsilon(z) &= \int_{\mathbb{C}} u(w)\chi^\varepsilon(z - w) d\lambda(w) = \int_{K^\varepsilon} u(w)\chi^\varepsilon(z - w) d\lambda(w) \\
&= \int_{K^\varepsilon} \Phi(w)\chi^\varepsilon(z - w) d\lambda(w) = \int_{\mathbb{C}} \Phi(w)\chi^\varepsilon(z - w) d\lambda(w) = \Phi_\varepsilon(z).
\end{aligned}
$$

Therefore, we have that $u_\varepsilon(z) \to \Phi(z)$ as $\varepsilon \to 0$ on K. Since K was arbitrary, $\Psi(z) := \lim_{\varepsilon\to 0} u_\varepsilon(z)$ exits for all $z \in \Omega$ (more precisely, if $K \subseteq K_1$ are compact subsets of Ω, we have that $\Psi_K(z) = \Psi_{K_1}(z)$ on K. Now define $\Psi(z) := \Psi_K(z)$ for $z \in K$, which is well defined by the previous argument and satisfies $\Psi(z) = \lim_{\varepsilon\to 0} u_\varepsilon(z)$ for all $z \in \Omega$). Locally Ψ satisfies the submean value property and Lemma 2.2, hence, Ψ is subharmonic. On the other hand $u_\varepsilon \to u$ in $L^1(K)$ and passing to a subsequence shows that $u = \Psi$ a.e. on Ω. □

One of the most famous equivalent definitions for a C^2 function being subharmonic is the following one:

Corollary 2.13. *Let $\Psi \in C^2(\Omega)$. Then Ψ is subharmonic if and only if $\Delta\Psi \geq 0$.*

Proof. A proof can be found in [R86, Prop. 48, p.87]. We prove a more general result later on. □

We will see that one can use Lemma 2.12 to prove Corollary 2.13.

Example 2.B. *If $f(z) = (z - a_1) \cdot \dots \cdot (z - a_n)g(z)$, where $g \in \mathcal{H}(\mathbb{C})$ and g has no zeros on $\mathbb{C}$, then $z \mapsto \log|f(z)|$ is harmonic on $\mathbb{C}\backslash\{a_1, \dots, a_n\}$ and subharmonic on $\mathbb{C}$.*
Since

$$\log|f(z)| = \sum_{k=1}^{n} \log|z - a_k| + \log|g(z)|,$$

it suffices to show the claim for each term.
The function $z \mapsto \log|g(z)|$ satisfies the assumptions of Corollary 2.13 and $\Delta \log|g(z)| = 0$. Hence, this function is harmonic on $\mathbb{C}$.
The functions $z \mapsto \log|z - a_k|$ are upper semicontinuous and satisfy $\Delta \log|z - a_k| = 0$ for $z \in \mathbb{C}\backslash\{a_k\}$. By Corollary 2.13 this proves the first claim. Since the submean value property is satisfied in a_k the second claim follows immediately. ♦

The following example is an application of Jensen's inequality.

Lemma 2.14 (Jensen's inequality)**.** *Let $X \subseteq \mathbb{R}^n$ Borel measurable and μ a probability Borel measure on X, $I \subseteq \mathbb{R}$ an interval, $f : X \to I$ μ-integrable and $\phi : I \to \mathbb{R}$ convex. Then $\int_X f(z)d\mu(z) \in I$ and*

$$\phi\left(\int_X f(z)d\mu(z)\right) \leq \int_X \phi(f(z))d\mu(z). \tag{2.5}$$

Proof. A proof can be found in [E09, p.221]. □

Example 2.C. *The function $\Psi : z \mapsto \max(0, \log|z|)$ is subharmonic.*
By definition Ψ is continuous and therefore, we only have to check if the second property of Definition 2.6 is satisfied. We use Jensen's inequality (2.5) and subharmonicity of $z \mapsto \log|z|$ (see Example 2.B) to see that

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi(a + re^{i\theta})d\theta \geq \max\left(0, \frac{1}{2\pi}\int_0^{2\pi} \log|a + re^{i\theta}|d\theta\right) \geq \Psi(a)$$

for appropriate $r > 0$ and therefore our claim holds true. ♦

As in the last example, it follows by Jensen's inequality (2.5) that if $\Psi : \Omega \to \mathbb{R}$ is subharmonic and $h : \mathbb{R} \to \mathbb{R}$ is convex, then $h \circ \Psi$ is again subharmonic, provided that $h \circ \Psi$ is upper semicontinuous.

Remark 2.D. By Corollary 2.13 a function $\Psi \in C^2(\Omega)$ is harmonic if and only if $\Delta\Psi = 0$. Furthermore, as we have seen in Theorem 1.13 the Laplace-operator has a fundamental solution with singular support $\{0\}$ and therefore, by Corollary 1.19, if a distribution $u \in \mathcal{D}'(\Omega)$ satisfies $\Delta u = 0$, then u can be considered to be a smooth function. Hence, every distributional solution of the homogeneous Laplace equation is a version of a classical solution and therefore a version of a harmonic function.

Sometimes it is quite useful to have an integrability statement (see [Ra95, Thm. 2.5.1, p.39]).

Theorem 2.15. *If $\Psi : \Omega \to [-\infty, \infty)$ is subharmonic and not identically $-\infty$, then Ψ is locally integrable w.r.t. Lebesgue measure.*

Proof. It suffices to show that for every $z \in \Omega$ there exits a number $r > 0$ s.t.

$$\int_{D(z,r)} |\Psi(w)| d\lambda(w) < \infty. \tag{2.6}$$

The claim will then follow by a compactness argument.
Let us denote the set of $z \in \Omega$ satisfying property (2.6) by A and its complement by B. We prove that A and B are open and $\Psi|_B = -\infty$. The claim will then follow by connectedness of Ω and the assumption that u is not identically $-\infty$.
If $z \in A$ and $r > 0$ such that (2.6) is satisfied, then it's easy to see that $D(z, r) \subseteq A$. This proves that A is open.
Let $z \in B$ and choose $r > 0$ s.t. $\overline{D(z, 2r)} \subseteq \Omega$. By definition of B we have that

$$\int_{D(z,r)} |\Psi(w)| d\lambda(w) = \infty.$$

Take $y \in D(z, r)$ and put $s(y) := r + |z - y|$. For $x \in D(z, r)$

$$|x - y| \leq |x - z| + |z - y| < r + |z - y| = s(y)$$

is satisfied and thus, we have $D(z, r) \subseteq D(y, s(y)) \subseteq D(z, 2r)$. On $D(y, s(y))$ the function Ψ is upper bounded and therefore,

$$\int_{D(y,s(y))} \Psi(w) d\lambda(w) = -\infty.$$

The submean value property of Ψ gives

$$\Psi(y) \leq \frac{1}{2\pi} \int_0^{2\pi} \Psi(y + he^{it}) dt \tag{2.7}$$

for $0 \leq h \leq s(y)$. Multiplying (2.7) by $2\pi h$ and integrating over $[0, s(y)]$ w.r.t. h gives

$$\pi r^2 \Psi(y) \leq \pi s(y)^2 \Psi(y) \leq \int_{D(y,s(y))} \Psi(w) d\lambda(w) = -\infty.$$

Therefore, $\Psi|_{D(z,r)} = -\infty$. This proves that B is open and $u|_B = -\infty$. □

A consequence of the above theorem is that the set

$$I_\Psi := \{z \in \Omega : \Psi(z) = -\infty\}$$

is of Lebesgue measure zero. We take a compact exhaustion $\{K_n\}_{n=1}^\infty$ of Ω and by Theorem 2.15 we note that

$$\int_{K_n} |\Psi(z)| d\lambda(z) < \infty,$$

and thus, for every $n \in \mathbb{N}$ the set $I_\Psi \cap K_n$ is of Lebesgue measure zero. By σ-subadditivity of the Lebesgue measure we even have that $I_\Psi = \bigcup_{n=1}^\infty (I_\Psi \cap K_n)$ is of Lebesgue measure zero.

Since we have defined the term subharmonicity for a larger number of functions, it is useful to have a similar statement to Corollary 2.13 for all subharmonic functions.

Definition 2.16. *A distribution $\lambda \in \mathcal{D}'(\Omega)$ is called positive if*

$$\langle \lambda, \phi \rangle \geq 0,$$

for every test function $\phi \geq 0$.

Theorem 2.17. *If Ψ is subharmonic on Ω and not identically $-\infty$ in any component of Ω, then $\Delta\Psi$ exists in the sense of distributions and we have*

$$\Delta\Psi \geq 0. \tag{2.8}$$

Proof. We follow the lines of [Co95, Thm. 5.6, p.231].
By Theorem 2.15 we know that Ψ is locally integrable and therefore can be interpreted as distribution in $\mathcal{D}'(\Omega)$ and hence $\Delta\Psi$ exists in the sense of distributions.
We have to show that $\int_{\mathbb{C}} \Psi(z)\Delta\phi(z)d\lambda(z) \geq 0$, whenever ϕ is a positive test function on Ω.
Let us fix a test function ϕ and choose a number $r_0 \in (0, d(\text{supp}\phi, b\Omega)/2)$. Put $\Omega_{r_0} := \{z \in \Omega : d(z, \text{supp}\phi) \leq r_0\}$. For arbitrary $0 < r < r_0$ we have that for $z \in \Omega_{r_0}$ the set $\overline{D(z,r)} \subseteq \Omega$ and the submean value property implies

$$\begin{aligned} 2\pi \int_{\Omega_{r_0}} \Psi(z)\phi(z)d\lambda(z) &\leq \int_{\Omega_{r_0}} \phi(z) \int_0^{2\pi} \Psi(z + re^{i\theta})d\theta d\lambda(z) \\ &= \int_{\Omega_{r_0}} \Psi(z) \int_0^{2\pi} \phi(z - re^{i\theta})d\theta d\lambda(z). \end{aligned}$$

This gives us that

$$0 \leq \int_{\Omega_{r_0}} \Psi(z) \left(\int_0^{2\pi} \phi(z - re^{i\theta})d\theta - 2\pi\phi(z) \right) d\lambda(z) \tag{2.9}$$

is satisfied. We can expand the function ϕ into a Taylor series and get with $z = x + iy$

$$\begin{aligned} \int_0^{2\pi} \phi(z - re^{i\theta})d\theta = \int_0^{2\pi} &\Bigg[\phi(z) - \left(\begin{pmatrix} \partial_x\phi(z) \\ \partial_y\phi(z) \end{pmatrix} \middle| \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \right) \\ &+ \frac{1}{2} \left(\begin{pmatrix} \partial_{xx}\phi(z) & \partial_{xy}\phi(z) \\ \partial_{xy}\phi(z) & \partial_{yy}\phi(z) \end{pmatrix} \cdot \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \middle| \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \right) + S(z - re^{i\theta}) \Bigg] d\theta \\ &= 2\pi\phi(z) + 2\pi r^2\Delta\phi(z) + S_1(z,r), \end{aligned}$$

where $S(z - r^{i\theta})$ is the remainder term of the Taylor series expansion and

$$S_1(z,r) = \int_0^{2\pi} S(z - re^{i\theta})d\theta.$$

By the definition of the term $S_1(z,r)$ there exists a constant $C > 0$ such that $|S_1(z,r)| \leq Cr^3$ for all $z \in \Omega_{r_0}$ and sufficiently small $r > 0$. Inserting this into (2.9) and dividing by r^2 gives

$$0 \leq \int_{\Omega_{r_0}} \Psi(z) \left(2\pi\Delta\phi(z) + \frac{S_1(z,r)}{r^2} \right) d\lambda(z)$$

$$= 2\pi \int_{\Omega_{r_0}} \Psi(z)\Delta\phi(z)d\lambda(z) + \int_{\Omega_{r_0}} \Psi(z)\frac{S_1(z,r)}{r^2}d\lambda(z).$$

For $r \to 0$ the second integral vanishes and therefore, by definition of r_0 we have

$$\int_{\Omega_{r_0}} \Psi(z)\Delta\phi(z)d\lambda(z) = \int_{\Omega} \Psi(z)\Delta\phi(z)d\lambda(z) \geq 0.$$

This proves the claim. □

In the next section we will see that if Ψ is a locally integrable function such that $\Delta\Psi \geq 0$, then it is subharmonic. This will give us a proof of Corollary 2.13.
The following result is based on [Ra95, Thm. 3.7.2, p.72].

Theorem 2.18. *Let $\Psi : \Omega \to [-\infty, \infty)$ be a subharmonic function not identically $-\infty$. Then $\Delta\Psi$ exists in the sense of distributions and can be identified with a unique Radon measure μ on Ω.*

Proof. By Theorem 2.17 $\Delta\Psi$ exists and satisfies $\Delta\Psi \geq 0$.
We show that for every relatively compact subset V of Ω there exists a positive constant C such that

$$|\langle \Delta\Psi, \phi\rangle| \leq C\|\phi\|_{\infty,\Omega} \tag{2.10}$$

if $\phi \in C_0^\infty(\Omega)$ such that $\mathrm{supp}\phi \subseteq V$.
Let $\Phi \in C_0^\infty(\Omega)$ with $0 \leq \Phi \leq 1$ and $\Phi|_V = 1$. By definition we have

$$-\|\phi\|_{\infty,\Omega}\Phi \leq \phi \leq \|\phi\|_{\infty,\Omega}\Phi.$$

Positivity of $\Delta\Psi$ gives

$$-\|\phi\|_{\infty,\Omega}\langle \Delta\Psi, \Phi\rangle \leq \langle \Delta\Psi, \phi\rangle \leq \|\phi\|_{\infty,\Omega}\langle \Delta\Psi, \Phi\rangle.$$

Put $C := \langle \Delta\Psi, \Phi\rangle$ and note that (2.10) is satisfied. Therefore, by Lemma 1.8 $\Delta\Psi$ can be extended to a positive linear functional F on $C_0(\Omega)$ and by Theorem 1.10 we can find a positive Radon measure μ such that

$$F(\phi) = \int_\Omega \phi(z)d\mu(z),$$

for $\phi \in C_0(\Omega)$. By construction for $\phi \in C_0^\infty(\Omega)$ we have

$$\langle \Delta\Psi, \phi\rangle = F(\phi) = \int_\Omega \phi(z)d\mu(z),$$

telling us that $\Delta\Psi = \mu$ in the sense of distributions. Uniqueness in Theorem 1.10 shows that the measure μ is unique. □

The importance of the following result ([Ra95, Thm. 2.4.8, p.38]) will become clear when we start studying basic properties of logarithmic potentials.

Theorem 2.19. *Let μ be a finite measure on Ω and U be an open subset of $\mathbb{C}$. If the function $v : U \times \Omega \to [-\infty, \infty)$ satisfies the following three properties:*

(i) v is measurable on $U \times \Omega$,

(ii) $z \mapsto v(z,w)$ is subharmonic on U for every $w \in \Omega$,

(iii) $z \mapsto \sup_{w\in\Omega} v(z,w)$ is locally bounded from above on U.

Then the function $\Psi(z) \coloneqq \int_\Omega v(z,w)d\mu(w)$ is subharmonic on U.

Proof. We show that Ψ is subharmonic on each relatively compact subset D of Ω. The claim will then follow from Lemma 2.12.
The third assumption tells us that $z \mapsto \sup_{w\in\Omega} v(z,w)$ is bounded above on D. Therefore, w.l.o.g. we can assume that $v \leq 0$ on $D \times \Omega$. Hence, if $z_n \to z$ in D, we have by Fatou's lemma and the second assumption that

$$\begin{aligned}\limsup_{n\to\infty} \Psi(z_n) &\leq \int_\Omega \limsup_{n\to\infty} v(z_n,w)d\mu(w)\\ &\leq \int_\Omega v(z,w)d\mu(w) = \Psi(z).\end{aligned}$$

Hence, by Lemma 2.2, Ψ is upper semicontinuous.
If $D(z,r) \subseteq D$, Fubini's theorem gives

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \Psi(z+re^{i\theta})d\theta &= \int_\Omega \frac{1}{2\pi}\int_0^{2\pi} v(z+re^{i\theta},w)d\theta d\mu(z)\\ &\geq \int_\Omega v(z,w)d\mu(z) = \Psi(z),\end{aligned}$$

telling us that Ψ is indeed subharmonic. □

A representation theorem by Riesz ([Ra95, Thm. 3.7.9, p.76]) tells us that every subharmonic function can be written locally as a sum of a harmonic function and a subharmonic function of the type occurring in the previous theorem. This theorem requires knowledge on the theory of logarithmic potentials.

Theorem 2.20 (Riesz)**.** *Let Ω be a domain in $\mathbb{C}$ and $\Psi : \Omega \to [-\infty,\infty)$ a subharmonic function, not identically $-\infty$. Then given a relatively compact open subset U of Ω, we can decompose Ψ as*

$$\Psi(z) = \frac{1}{2\pi}\int_U \log|z-w|d\mu(w) + h(z)$$

a.e. w.r.t. Lebesgue measure on U, where $\mu = \Delta\mu|_U$ and h is harmonic on U.

We proof this theorem in the following section. The above theorem shows another time the connection of a subharmonic functions and its Laplacian and leads us to the term of logarithmic potentials. This will be the task of the next section.
There is a slightly stronger version of the previous theorem, because one can get rid of the claim that equality holds true only a.e. w.r.t. Lebesgue measure. We won't need this more general result.

2.3 Logarithmic potentials

For a detailed introduction to the theory of (logarithmic) potentials in the plane we refer to [Ra95, ch. 3, p.53–p.77]. If not otherwise mentioned, all of the following results can be found in that book.
In what follows, μ always denotes a compactly supported finite Borel measure on $\mathbb{C}$.

Definition 2.21 (Potential)**.** *The function*

$$\mathcal{P}_\mu(z) := \frac{1}{2\pi}\int_{\mathbb{C}} \log|z-w|d\mu(w)$$

is called the (logarithmic) potential of μ.

As you may have noticed, we can interpret $\mathcal{P}_\mu$ as a convolution of distributions. Therefore, we consider $\frac{1}{2\pi}\log|z|$ to be in $\mathcal{D}'(\mathbb{C})$ and μ to be in $\mathcal{E}'(\mathbb{C})$. Hence, we can convolute these two distributions and gain

$$\mathcal{P}_\mu = \frac{1}{2\pi}\log|z| * \mu.$$

Since $\frac{1}{2\pi}\log|z|$ is the fundamental solution of the Laplacian in the plane (see Theorem 1.13), we see that

$$\Delta\mathcal{P}_\mu = \Delta(\log|z| * \mu) = (\Delta\frac{1}{2\pi}\log|z| * \mu) = \delta_0 * \mu = \mu.$$

Hence, we have proved the following corollary.

Corollary 2.22. *If μ is a compactly supported finite Borel measure on $\mathbb{C}$ then*

$$\Delta\mathcal{P}_\mu = \mu$$

is satisfied in the sense of distributions.

An immediate consequence of this corollary is the following result.

Corollary 2.23. *Let μ_1 and μ_2 be two compactly supported Borel measures. Furthermore let Ω be an open subset of $\mathbb{C}$ and $h : \Omega \to \mathbb{R}$ be harmonic, such that*

$$\mathcal{P}_{\mu_1}(z) = \mathcal{P}_{\mu_2}(z) + h(z),$$

for $z \in \Omega$. Then $\Delta\mathcal{P}_{\mu_1}|_\Omega = \Delta\mathcal{P}_{\mu_2}|_\Omega$.

Proof. The claim follows by linearity of the Laplace operator and the previous lemma. □

Corollary 2.22 tells us that the Laplacian of $\mathcal{P}_\mu$ is positive. Unfortunately, this doesn't give us some information about subharmonicity of logarithmic potentials. We capture this question in the following lemma.

Lemma 2.24. *The potential $\mathcal{P}_\mu$ is subharmonic on $\mathbb{C}$, even harmonic on $\mathbb{C}\backslash supp\mu$ and satisfies*

$$\mathcal{P}_\mu(z) = \mu(\mathbb{C}) \log(|z|) + \mathcal{O}(|z|^{-1}),$$

as $|z| \to \infty$.

Proof. Let $K \coloneqq \mathrm{supp}\mu$. Applying Theorem 2.19 with $v(z,w) \coloneqq \log|z-w|$ on $\mathbb{C} \times K$, we have that $\mathcal{P}_\mu$ is subharmonic on $\mathbb{C}$. The same theorem with $v(z,w) \coloneqq -\log(|z-w|)$ on $\mathbb{C}\backslash K \times K$ gives that $-\mathcal{P}_\mu$ is subharmonic on $\mathbb{C}\backslash K$. Therefore, the first and the second statement hold true.
If $z \neq 0$, we have

$$\mathcal{P}_\mu(z) = \mu(\mathbb{C}) \log|z| + \int_{\mathbb{C}} \log\left|1 - \frac{w}{z}\right| d\mu(w).$$

Since μ has compact support, we have by Fatou's lemma that

$$\begin{aligned}
\limsup_{|z|\to\infty} \left| |z| \int_{\mathbb{C}} \log\left|1 - \frac{w}{z}\right| d\mu(w) \right| &\leq \limsup_{|z|\to\infty} \int_{\mathbb{C}} \left| \log\left(1 + \left|\frac{w}{z}\right|\right)^{|z|} \right| d\mu(w) \\
&\leq \int_{\mathbb{C}} \limsup_{|z|\to\infty} \left| \log\left(1 + \left|\frac{w}{z}\right|\right)^{|z|} \right| d\mu(w) \\
&= \int_K |w| d\mu(w),
\end{aligned}$$

meaning that

$$\int_{\mathbb{C}} \log\left|1 - \frac{w}{z}\right| d\mu(w) = \mathcal{O}(|z|^{-1}),$$

as $|z| \to \infty$. □

As a consequence of the previous lemma, we have that logarithmic potentials are locally integrable and finite a.e. w.r.t. Lebesgue measure. This will be used later on in Theorem 5.1.

Lemma 2.25. *Let u_1 and u_2 be subharmonic functions on a domain Ω in $\mathbb{C}$, not identically to $-\infty$. If $\Delta u_1 = \Delta u_2$, then there exists a harmonic function h on Ω such that*

$$u_1(z) = u_2(z) + h(z)$$

for a.e. $z \in \Omega$.

Proof. Put $h_0 \coloneqq u_1 - u_2$. Since h_0 is locally integrable and satisfies $\Delta h_0 = 0$, by Remark 2.D and Example 1.A, there exists a harmonic function h s.t. $h_0 = h$ a.e. on Ω. The function h satisfies the required property. □

Now we can prove Theorem 2.20.

Proof of Theorem 2.20. Let $\mu := \Delta\Psi|_U$. By Theorem 2.18 μ is a compactly supported Borel measure. Since, $\Delta\mathcal{P}_\mu = \mu$ we can apply Lemma 2.25 to gain a harmonic function $h : U \to \mathbb{R}$ satisfying

$$\Psi(z) = \mathcal{P}_\mu(z) + h(z),$$

for a.e. $z \in U$. □

The following result can be found in [Co95, Thm. 5.6, p.231]. It is the converse of Theorem 2.17 and finally proves Corollary 2.13.

Theorem 2.26. *If Ψ is a locally integrable function on an open connected set Ω in $\mathbb{C}$, then the following conditions are equivalent.*

(i) The function Ψ is equal to a subharmonic function a.e. w.r.t. Lebesgue measure on Ω.

(ii) $\Delta\Psi \geq 0$ in the sense of distributions on Ω.

(iii) There is a positive measure μ on Ω such that if B is a bounded open set with $\overline{B} \subseteq \Omega$ and $\mu|_B =: \mu_B$, then there is a harmonic function h on B such that

$$\Psi|_B = \mathcal{P}_{\mu_B} + h$$

a.e. on B.

Proof. Most of the work has already been done.
$(i) \implies (ii)$. We can apply Theorem 2.17 and get the claim.
$(ii) \implies (iii)$. Like in the proof of Theorem 2.18 $\Delta\Psi$ is a positive Radon measure and therefore, a similar argument as in the proof of Theorem 2.25 gives the claim.
$(iii) \implies (i)$. This is an immediate consequence of Lemma 2.12. □

At the end of this section we derive the logarithmic potential of the Lebesgue measure restricted to an open disc. For this purpose, we will need the following formula.

Lemma 2.27 (Jensen's formula)**.** *Let $f \in \mathcal{H}(D_R(0))$, $f(0) \neq 0$, $0 < r < R$. We denote the zeros of f in $\overline{D_r(0)}$ by $a_1, \ldots, a_n$ (multiple zeros being repeated). Then we have*

$$f(0)\prod_{n=1}^{N}\frac{r}{|a_n|} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta\right).$$

Proof. A proof can be found in [A79, p.207]. □

The following lemma gives us a representation of a logarithmic potential w.r.t. Lebesgue measure restricted to a disc.

Lemma 2.28. *If $h > 0$ and $a \in \mathbb{C}$, then*

$$\frac{1}{2\pi}\int_{D(a,h)}\log|z-w|d\lambda(w) = \begin{cases}\frac{h^2}{2}\log h + \frac{1}{4}(|z-a|^2 - h^2), & \text{if } z \in D(a,h),\\ \frac{h^2}{2}\log|z-a|, & \text{if } z \notin D(a,h).\end{cases} \tag{2.11}$$

Proof. We first study the case $z \in D(a,h)$. Hence we can write $z = a + b$ and note that $|b| < h$. We use polar coordinates to obtain

$$\frac{1}{2\pi}\int_{D(a,h)} \log|z-w|d\lambda(w) = \int_0^h \frac{1}{2\pi}\int_0^{2\pi} \log|b - re^{i\phi}|d\phi\; r\; dr.$$

We consider three cases to derive the integral with respect to ϕ:

(i) $|b| > r$: Since $z \mapsto \log|z|$ is harmonic on $\mathbb{C}\backslash\{0\}$ we can use the mean value property for harmonic functions (2.2) to see that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|b - re^{i\phi}|d\phi = \log|b|.$$

(ii) $|b| < r$: This case is quite similar to the first one:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log|b - re^{i\phi}|d\phi &= \frac{1}{2\pi}\int_0^{2\pi} \log|be^{-i\phi} - r|d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|r - \bar{b}e^{i\phi}|d\phi = \log r,\end{aligned}$$

where we have used the mean value theorem for harmonic functions again.

(iii) $|b| = r$: Since $b = re^{i\theta}$ for some $\theta \in [0, 2\pi]$ the function $\phi \mapsto \log|b - re^{i\phi}|$ has a singularity at θ. But in this case we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log|re^{i\phi} - b|d\phi = \log|b|$$

by using Jensen's formula 2.27 for harmonic functions.

Using these results it is easy to see, that

$$\int_0^h \frac{1}{2\pi}\int_0^{2\pi} \log|b - re^{i\phi}|d\phi\; r\; dr = \frac{h^2}{2}\log h + \frac{1}{4}(|z-a|^2 - h^2),$$

which proves the first part of our claim.
If $z \notin D(a,h)$ we can directly apply the mean value property for harmonic functions and easily get the claim. □

2.4 Energy, equilibrium measure and capacity

The following results can be found in [Ra95, ch. 3 and 5]. In Chapter 3 we will use the term capacity to give a sufficient and necessary condition for a space of holomorphic functions being non-trivial. The goal of this section is to provide the term capacity and concepts, which are closely related to logarithmic potential theory.

Definition 2.29 (energy)**.** *For a compactly supported, finite Borel measure μ on $\mathbb{C}$ the number*

$$I(\mu) := 2\pi\int_{\mathbb{C}} \mathcal{P}_\mu(z)d\mu(z),$$

is called the energy of μ.

Example 2.E. *The (normalized) Lebesgue measure restricted to an open disc has finite energy.*
Let $h > 0$ and $a \in \mathbb{C}$. If μ denotes the Lebesgue measure restricted to $D(a,h)$, we have by Lemma 2.28

$$\begin{aligned}\frac{I(\mu)}{2\pi} = \int_{D(a,h)} \mathcal{P}_\mu(z)d\mu(z) &= \int_{D(a,h)} \left(\frac{h^2}{2}\log h + \frac{1}{4}(|z-a|^2 - h^2)\right)d\lambda(z)\\ &= \left(\frac{h^4\pi}{2}\log h + \frac{h^3}{12} - \frac{h^4\pi}{4}\right).\end{aligned}$$

Definition 2.30 (polar sets)**.** *A set $E \subseteq \mathbb{C}$ is called polar, if for every finite Borel measure $\mu \neq 0$, compactly supported in E, satisfies*

$$I(\mu) = -\infty.$$

An easy consequence of the definition is that finite sets are polar. The proof of the following theorem is more complicated (see [Ra95, p.56]).

Theorem 2.31. *Let μ be a finite, compactly supported Borel measure on $\mathbb{C}$ with finite energy. Then $\mu(E) = 0$ for every Borel polar set.*

Proof. Suppose that there exists a polar set E with $\mu(E) > 0$. Since every finite Borel measure on $\mathbb{C}$ is regular (see [Ra95, Thm. A.2.2]), we can find a compact subset K of E s.t. $\mu(K) > 0$. Therefore, we can define a compactly supported non-zero Borel measure by $\nu := \mu|_K$. Since μ has finite energy, the set suppμ contains more than two points and thus, $d := \sup\{|x-y| : x, y \in \text{supp}\mu\} > 0$. By definition of ν and d we have

$$\begin{aligned}I(\nu) &= \int_K \int_K \log\left|\frac{z-w}{d}\right| d\mu d\mu + \mu(K)^2 \log d\\ &\geq \int_{\mathbb{C}} \int_{\mathbb{C}} \log\left|\frac{z-w}{d}\right| d\mu d\mu + \mu(K)^2 \log d\\ &= I(\mu) - \mu(\mathbb{C})^2 \log d + \mu(K)^2 \log d > -\infty.\end{aligned}$$

Hence, E cannot be polar. □

The following lemma is an easy consequence of the previous theorem.

Lemma 2.32. *Every polar Borel set $E \subseteq \mathbb{C}$ has Lebesgue measure zero.*

Proof. By Example 2.E, for every $r > 0$ the Lebesgue measure restricted to $D(0,r)$ has finite energy. Thus the previous theorem gives that $E \cap D(0,r)$ has Lebesgue measure zero. The claim follows by the monotone convergence theorem. □

We don't need a deeper understanding of polar sets and therefore, we refer to [Ra95].

Based on the term of energy we can now define capacity of a subset of $\mathbb{C}$.

Definition 2.33 ((logarithmic) capacity)**.** *Let Ω be a subset of $\mathbb{C}$. The set of all compactly supported Borel probability measures on Ω is denoted by $\mathcal{M}_1^c(\Omega)$. The number*

$$Cap(\Omega) := \sup_{\mu\in\mathcal{M}_1^c(\Omega)} e^{I(\mu)} = \exp(\sup_{\mu\in\mathcal{M}_1^c(\Omega)} I(\mu))$$

is called (logarithmic) capacity of Ω

Example 2.F. (i) Finite sets have capacity zero.

(ii) Open sets have positive capacity. This can be seen by Example 2.E.

(iii) By definition polar sets have capacity zero.

It's an easy consequence of the definition that, if $\Omega_1 \subseteq \Omega_2$, then $\mathrm{Cap}(\Omega_1) \leq \mathrm{Cap}(\Omega_2)$. The capacity satisfies some regularity assumptions.

Lemma 2.34 (inner regularity)**.** *Let Ω be a subset of $\mathbb{C}$, then*

$$Cap(\Omega) = \sup\{Cap(K) : K \textit{ is a compact subset of } \Omega\}.$$

Proof. Let $M := \sup\{\mathrm{Cap}(K) : K$ is a compact subset of $\Omega\}$. Since every compact subset K of Ω satisfies $\mathrm{Cap}(K) \leq \mathrm{Cap}(\Omega)$ it suffices to prove that $M \geq \mathrm{Cap}(\Omega)$. Suppose $M < \mathrm{Cap}(\Omega)$. Thus there exists $\varepsilon > 0$ and a measure $\mu \in \mathcal{M}_1^c(\Omega)$, s.t. $M \leq \mathrm{Cap}(\Omega) - \varepsilon < e^{I(\mu)}$. Since $K := \mathrm{supp}\mu$ is a compact subset of Ω, we have that $\mathrm{Cap}(K) \geq e^{I(\mu)}$ and therefore,

$$M \leq \mathrm{Cap}(\Omega) - \varepsilon < M,$$

a contradiction. □

Driven by the definition of capacity we introduce the term of equilibrium measure.

Definition 2.35 (equilibrium measure)**.** *Let K denote a compact subset of $\mathbb{C}$. A measure $\nu \in \mathcal{M}_1^c(K)$ satisfying*

$$I(\nu) = \sup_{\mu \in \mathcal{M}_1^c(K)} I(\mu)$$

is called an equilibrium measure for K.

It turns out, that every compact subset has an equilibrium measure (and thus, the definition makes sense), but it may not be unique (consider finite sets). To prove existence we need a kind of convergence on $\mathcal{M}_1^c(K)$.

Definition 2.36 (weak*-convergence)**.** *Let K be a compact subset of $\mathbb{C}$. A sequence $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{M}_1^c(K)$ is weak*-convergent to $\mu \in \mathcal{M}_1^c(K)$ if for every $f \in C(K)$*

$$\int_K f(z) d\mu_n(z) \to \int_K f(z) d\mu(z),$$

as $n \to \infty$. We use the notation $\mu_n \xrightarrow{w^*} \mu$.

In our situation the following theorem will be quite useful.

Theorem 2.37. *Let K be a compact subset of $\mathbb{C}$. Every sequence in $\mathcal{M}_1^c(K)$ has a weak*-convergent subsequence.*

Proof. A proof can be found in [Ra95, p.216]. The idea is to use separability of $C(K)$ and the Riesz-representation theorem 1.10. □

We need another lemma to prove existence of equilibrium measures.

Lemma 2.38. *If* $\mu_n \xrightarrow{w^*} \mu$ *in* $\mathcal{M}_1^c(K)$*, then* $\limsup_{n\to\infty} I(\mu_n) \leq I(\mu)$*.*

Proof. For $f, g \in C(K)$ we have by definition of weak*-covergence, that

$$\int_K \int_K f(z)g(w)d\mu_n(z)d\mu_n(w) \to \int_K \int_K f(z)g(w)d\mu(z)d\mu(w),$$

as $n \to \infty$. Since the set $\mathrm{span}[f(z)g(w) : f, g \in C(K)]$ is dense in $C(K \times K)$ (by Stone-Weierstraß theorem; [He08, p.63]), for every $h(z, w) \in C(K \times K)$

$$\int_K \int_K h(z,w)d\mu_n(z)d\mu_n(w) \to \int_K \int_K h(z,w)d\mu(z)d\mu(w)$$

is satisfied. Define $h_m(z, w) \coloneqq \max(\log|z - w|, -m) \in C(K \times K)$ for every $m \in \mathbb{N}$ and note

$$\begin{aligned}
\limsup_{n\to\infty} I(\mu_n) &= \limsup_{n\to\infty} 2\pi \int_K \mathcal{P}_{\mu_n}(z)d\mu_n(z) \\
&\leq 2\pi \limsup_{n\to\infty} \int_K \int_K h_m(z,w)d\mu_n(w)d\mu_n(z) \\
&= 2\pi \int_K \int_K h_m(z,w)d\mu(w)d\mu(z).
\end{aligned}$$

The claim follows by the monotone convergence theorem. □

With help of the previous lemma we can prove existence of equilibrium measures.

Theorem 2.39. *Every compact set* K *has an equilibrium measure.*

Proof. We define $M \coloneqq \sup_{\mu\in\mathcal{M}_1^c(K)} I(\mu)$. Thus, we can find a sequence $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{M}_1^c(K)$ s.t. $I(\mu_n) \to M$ as $n \to \infty$. By Theorem 2.37 we can pick a subsequence μ_{n_k}, which has a weak*-limit $\nu \in \mathcal{M}_1^c(K)$. Lemma 2.38 tells us that

$$I(\nu) \geq \limsup_{n\to\infty} I(\mu_{n_k}) = M.$$

Hence, ν is an equilibrium measure of K. □

The previous theorem tells us that for a compact set K we can find a compactly supported measure ν such that

$$\mathrm{Cap}(K) = e^{I(\nu)}.$$

Another nice result is that if we have a compact set with positive capacity, then there exists a potential which is bounded from below.

Theorem 2.40. *Let* K *be a compact subset of* $\mathbb{C}$*, and let* ν *be an equilibrium measure for* K*. Then* $\mathcal{P}_\nu(z) \geq I(\nu)$ *for every* $z \in \mathbb{C}$*.*

Proof. A proof can be found in [Ra95, p.60/61]. □

We use Lemma 2.34 to prove the following regularity result.

Lemma 2.41. *Let* $\{K_n\}_{n=1}^{\infty}$ *be a decreasing sequence of compact subsets in* $\mathbb{C}$ *and* $K \coloneqq \bigcap_{n=1}^{\infty} K_n$, *then*

$$Cap(K) = \lim_{n\to\infty} Cap(K_n).$$

If $\{\Omega_n\}_{n=1}^{\infty}$ *is an increasing sequence of subsets in* $\mathbb{C}$ *and* $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, *then*

$$Cap(\Omega) = \lim_{n\to\infty} Cap(\Omega_n).$$

Proof. (i) By the definition of capacity we have that

$$\mathrm{Cap}(K_n) \geq \mathrm{Cap}(K_{n+1}) \geq \mathrm{Cap}(K), \tag{2.12}$$

for every $n \in \mathbb{N}$. On the other hand, by Theorem 2.39, there exist equilibrium measures μ_n for K_n s.t. $\mu_n \in \mathcal{M}_1^c(K_{n_0})$ for every $n \geq n_0$. Theorem 2.37 and a diagonal argument guarantee that we can pick a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ which has a weak*-limit $\mu \in \mathcal{M}_1^c(K_1)$ with supp$\mu \subseteq K$. Hence $e^{I(\mu)} \leq \mathrm{Cap}(K)$. Applying Lemma 2.38, gives

$$\limsup_{k\to\infty} I(\mu_{n_k}) \leq I(\mu).$$

Combining this with inequality (2.12) implies

$$\mathrm{Cap}(K) \leq \limsup_{k\to\infty} \mathrm{Cap}(K_{n_k}) \leq \mathrm{Cap}(K),$$

proving the first claim.
(ii) The same argument as above gives

$$\mathrm{Cap}(\Omega_n) \leq \mathrm{Cap}(\Omega_{n+1}) \leq \mathrm{Cap}(\Omega). \tag{2.13}$$

Let K be an arbitrary compact subset of Ω and ν an equilibrium measure of K. By monotone convergence theorem we have that $\nu(K \cap \Omega_n) \to \nu(K) = 1$ as $n \to \infty$. Since every finite Borel measure is regular (see for instance [Ra95, Thm. A.2.2]) we can find compact subsets $K_n \subseteq \Omega_n \cap K$ s.t. $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots$ and $\nu(K_n) \to 1$. Therefore, w.l.o.g. we may assume that $\nu(K_n) > 0$ and define

$$\mu_n \coloneqq \frac{\nu|_{K_n}}{\nu(K_n)} \in \mathcal{M}_1^c(K_n).$$

The energy of μ_n satisfies

$$I(\mu_n) = \frac{1}{\nu(K_n)^2} \int_K \int_K \log|z-w| \mathbb{1}_{K_n}(z) \mathbb{1}_{K_n}(w) d\nu(w) d\nu(z) \tag{2.14}$$

and by definition of capacity we have

$$\mathrm{Cap}(\Omega_n) \geq e^{I(\mu_n)}. \tag{2.15}$$

Thus, by equality (2.14), monotone convergence and $\nu(K_n) \to 1$ as $n \to \infty$ we have

$$\lim_{n\to\infty} I(\mu_n) = I(\nu).$$

The last equality and inequality (2.15) tell us

$$\liminf_{n\to\infty} \mathrm{Cap}(\Omega_n) \geq e^{I(\nu)} = \mathrm{Cap}(K).$$

Since K was arbitrary, Lemma 2.34 and inequality (2.13) give

$$\mathrm{Cap}(\Omega) \geq \liminf_{n\to\infty} \mathrm{Cap}(\Omega_n) \geq \mathrm{Cap}(\Omega).$$

This completes the proof. □

2.5 Related Topics: Green's function, harmonic measure and Robin constant

In this section we give an explicit definition of Green's function, Robin constant and harmonic measure, which are topics related to potential theory and subharmonic functions. We note down a few lemmas and theorems, but don't prove them. We focus on results, which will be used in Section 3.2, but don't try to make out a list of the 'most important' theorems. The following definitions, results and their proofs can be found in [Co95, ch. 19 and 21] and [Ra95, ch. 4].
The set $\mathbb{C}^*$ denotes the extended complex plane $\mathbb{C} \cup \{\infty\}$ and the boundary of a set $\Omega \subseteq \mathbb{C}^*$ is defined by

$$b_\infty \Omega := \begin{cases} b\Omega & \text{if } \Omega \text{ is bounded,} \\ b\Omega \cup \{\infty\} & \text{if } \Omega \text{ is unbounded,} \end{cases}$$

where $b\Omega$ denotes the classical boundary in $\mathbb{C}$. Let us start with a generalization of the term 'harmonicity'.

Definition 2.42 (harmonic at infinity)**.** *Let $\infty \in \Omega \subseteq \mathbb{C}^*$. A function $\Psi : \Omega \to \mathbb{R}$ is called harmonic at infinity if the function $\Psi(z^{-1})$ has an harmonic extension in a neighborhood of zero.*

Remark 2.G. More precisely, there exists a neighborhood V of zero and a harmonic function $h : V \to \mathbb{R}$ s.t. $h(z) = \Psi(z^{-1})$ for all $z \in V \backslash \{0\}$. Since the function h is continuous at zero, this implies that the following limit exists:

$$\lim_{z \to \infty} \Psi(z) = \lim_{z \to 0} \Psi(z^{-1}) = \lim_{z \to 0} h(z).$$

Definition 2.43 (Green's function)**.** *For an open subset $\Omega \subseteq \mathbb{C}^*$ Green's function $G_\Omega : \Omega \times \Omega \to (-\infty, \infty]$ is a function with the following properties:*

(i) For every $w \in \Omega$ the function $z \mapsto G_\Omega(z, w)$ is harmonic on $\Omega \backslash \{w\}$ and bounded outside each neighborhood of w.

(ii) $G_\Omega(w, w) = \infty$ and

$$G_\Omega(z, w) = \begin{cases} \log |z| + \mathcal{O}(1), & w = \infty, \\ -\log |z - w| + \mathcal{O}(1), & w \neq \infty, \end{cases}$$

as $z \to w$.

(iii) $G_\Omega(z, w) \to 0$ as $z \to z_0$, for every $z_0 \in b_\infty \Omega$ except on a polar set.

For a general subset of $\mathbb{C}^*$ a Green's function does not exist, but for many interesting subsets of $\mathbb{C}^*$ one can find a Green's function.

Example 2.H. *The function*

$$G_{D(0,1)} := \log \left| \frac{1 - z\overline{w}}{z - w} \right| \tag{2.16}$$

is a Green's function for the open disc $D(0,1)$.
Basic computations show that the properties (i) and (ii) of Definition 2.43 are satisfied. Property (iii) can be seen as follows: Let $z_0 \in bD(0,1)$ and note that

$$G_{D(0,1)}(z,w) = \log|1 - z\overline{w}| - \log|z-w| = \log|\overline{z_0} - z\overline{z_0}\overline{w}| - \log|z-w|.$$

The claim follows, because the logarithm is continuous and $|z_0 - w| \neq 0$. ♦

Remark 2.I. Similar computations show that the functions

$$G_{D(0,R)}(z,w) = \log\left|\frac{R^2 - z\overline{w}}{R(z-w)}\right|, \tag{2.17}$$

and

$$G_{\mathbb{C}^*\setminus\overline{D(0,R)}}(z,w) = \log\left|\frac{z|w|^2 - R^2 w}{R\overline{w}(w-z)}\right|. \tag{2.18}$$

are indeed Green's functions of the underlying open subset. Alternatively, we can use the following property of Green's function.

Lemma 2.44. *Let* Ω_1 *and* Ω_2 *be open and connected subsets of* $\mathbb{C}^*$ *such that* G_{Ω_1} *and* G_{Ω_2} *exist. If there exists a holomorphic function* $h : \Omega_2 \to \mathbb{C}^*$ *which is one-to-one and satisfies* $f(\Omega_2) = \Omega_1$*, then*

$$G_{\Omega_2}(z,w) = G_{\Omega_1}(h(z), h(w)),$$

for every $z, w \in \Omega_2$.

Proof. A proof can be found in [Co95, Prop. 9.2 (c), p.247]. □

Thus we can use the Green's function (2.16) and the maps $z \mapsto \frac{z}{R}$ and $z \mapsto \frac{R}{z}$ to show that (2.17) and (2.18) are Green's functions of the associated underlying set.
At this point it is not totally clear under which conditions a Green's function exists. For our applications the following two results will be crucial.

Lemma 2.45. *If the domain* $\Omega \subseteq \mathbb{C}^*$ *has a Green's function* G_Ω*, then* G_Ω *is uniquely determined.*

Proof. A proof can be found in [Co95, Prop. 9.2 (a), p.247]. □

Lemma 2.46. *If an open set* $\Omega \subseteq \mathbb{C}^*$ *doesn't have a Green's function, then* $\mathbb{C}^*\setminus\Omega$ *is totally disconnected (i.e. the only connected components of* $\mathbb{C}^*\setminus\Omega$ *are the single points).*

Proof. A proof can be found in [Co95, p.249]. □

This implies that if K is a compact set s.t. bK is a system of Jordan curves, then $\mathbb{C}^*\setminus K$ possesses a Green's function.
Occasionally, it is quite useful to have more information than property (iii) of Definition 2.43 about the limit behavior of Green's function at the boundary of the defining set. For this purpose we introduce the following term (for more information see [Ra95, ch. 4], or [Co78, ch. 10.4] and [Co95, ch. 19.10]).

Definition 2.47 (regular point). *Let Ω be a open subset of $\mathbb{C}^*$ and $z_0 \in b_\infty\Omega$. If there exists a neighborhood V of z_0 s.t. one can find a strictly negative subharmonic function $b : V \cap \Omega \to [-\infty, \infty)$ and $\lim_{z\to z_0} b(z) = 0$, then z_0 is called regular. Otherwise, the point z_0 is called irregular.*

Remark 2.J. In [Ra95, Def. 4.1.4], the function b appearing in 2.47 is called a barrier at z_0. As opposed to this, in the book [Co78] a barrier describes a family $\{b_r\}_{r>0}$ of functions with the following properties:

(i) $-b_r$ is subharmonic on $\Omega \cap D(z_0, r)$,

(ii) $\lim_{z\to z_0} b_r(z) = 0$ and $\lim_{z\to w} b_r(z) = 1$ for $w \in \Omega \cap b(z_0, r)$.

Also the term of regular points in [Co95] is a little bit different compared to our definition. The connection of these different approaches can be found in [Co95, ch. 19.10], where Theorem 10.1 and Proposition 10.4 are of particular interest.

For our applications we need the following theorem.

Theorem 2.48. *Let Ω be a domain in $\mathbb{C}^*$ and $z_0 \in b_\infty\Omega$. If z_0 lies in a connected component C of $b_\infty\Omega$ s.t. $\{z_0\} \subsetneq C$, then z_0 is a regular point.*

Proof. A proof can be found in [Ra95, p.92]. □

There are several characterizations of regular (and irregular points) which can be found in [Ra95, ch. 4.2], [Co95, ch. 19.10 and 21.6] and [HK76, ch. 2.6, p.61–63]. For our purpose, Theorem 2.48 will be good enough.
There is an interesting relationship between regular points and the limit behavior at the boundary of a Green's function.

Theorem 2.49. *Let Ω be a domain in $\mathbb{C}^*$ s.t. G_Ω exists. If $z_0 \in b_\infty\Omega$, then the following statements are equivalent:*

(i) z_0 is a regular point of Ω.

(ii) For every $w \in \Omega$ we have $\lim_{z\to z_0} G_\Omega(z, w) = 0$.

Proof. A proof can be found in [Co95, p.328] and in [Ra95, p.111]. □

An easy consequence of the previous theorems is the following lemma.

Corollary 2.50. *Let $K \subseteq \mathbb{C}^*$ be a compact set s.t. bK is a system of Jordan curves. Then every point $a \in bK$ satisfies*

$$\lim_{z\to a} G_{\mathbb{C}^*\setminus K}(z, w) = 0, \tag{2.19}$$

for every $w \in \mathbb{C}^*\setminus K$.

Proof. We apply Theorem 2.48 to see that each point lying in bK is a regular point and Theorem 2.49 to see that (2.19) is satisfied. □

Another related topic is the concept of harmonic measure.

Lemma 2.51. *Let $\Omega \subseteq \mathbb{C}^*$ open, s.t. G_Ω exists. For every point $a \in \Omega$ and $u \in C_{\mathbb{R}}(b_\infty\Omega) := \{f : b_\infty\Omega \to \mathbb{R},\ \text{continuous}\}$ we define*

$$\hat{u}(a) := \sup\Bigg\{\phi(a) : \phi \text{ is subharmonic on } \Omega, \text{ bounded from above and } \limsup_{z\to y}\phi(z) \le u(y) \text{ for every } y \in b_\infty\Omega\Bigg\}.$$

Then there exists a unique measure $\omega_a \in \mathcal{M}_1^c(b_\infty\Omega)$ such that

$$\int_{b_\infty\Omega} u(z)d\omega_a(z) = \hat{u}(a),$$

for every $u \in C_{\mathbb{R}}(b_\infty\Omega)$.

Proof. A proof can be found in [Co95, p.301]. □

Definition 2.52 (harmonic measure)**.** *Let Ω satisfy the assumptions of Lemma 2.51. The unique measure ω_a in Lemma 2.51 is called harmonic measure for Ω at a.*

In our applications we will need the following result, telling us that harmonic measures are not concentrated at any single point, which lies on the boundary of the domain.

Lemma 2.53. *Let Ω satisfy the assumptions of Lemma 2.51. For every $a \in b_\infty\Omega$ and $y \in \Omega$ we have $\omega_y(\{a\}) = 0$.*

Proof. See for instance, [Co95, p.307]. □

By the previous theorem, we can see that for an unbounded domain Ω containing the point ∞ we have that $\omega_\infty(\{\infty\}) = 0$. Hence, the equality $\omega_\infty(b\Omega) = \omega_\infty(b_\infty\Omega)$ holds true. If $b\Omega$ is compact, we have that $\text{supp}\,\omega_\infty \subseteq b\Omega$ (boundary in $\mathbb{C}$!) and therefore, we can even define the logarithmic potential of ω_∞.
There is an interesting connection between the logarithmic potential $\mathcal{P}_{\omega_\infty}$ and the following constant.

Definition 2.54 (Robin constant)**.** *Let K be a compact subset of $\mathbb{C}$. If the set $\mathbb{C}^*\backslash K$ has a Green's function $G_{\mathbb{C}^*\backslash K}$, then the constant*

$$\mathcal{R}(K) := \lim_{z\to\infty}\Big(G_{\mathbb{C}^*\backslash K}(z,\infty) - \log|z|\Big)$$

is called Robin constant of K.

By Definition 2.43 and Remark 2.G the Robin constant exists, and we note that by Lemma 2.45 the following equality holds true

$$\mathcal{R}(K) = \mathcal{R}(bK), \tag{2.20}$$

if both constants are well defined. This is a consequence of the fact that for large z Green's function of K and bK coincide.

Theorem 2.55. *Let K be a compact subset of $\mathbb{C}$ s.t. $G_{\mathbb{C}^*\backslash K}$ exists. Let W be the component of $\mathbb{C}^*\backslash K$ with $\infty \in W$. Then the logarithmic potential of the harmonic measure ω_∞ for W at infinity satisfies*

$$2\pi\mathcal{P}_{\omega_\infty}(z) = \begin{cases} G_{\mathbb{C}^*\backslash K}(z,\infty) - \mathcal{R}(K) & \text{if } z \in W, \\ -\mathcal{R}(K) & \text{if } z \notin \overline{W}, \end{cases}$$

and $2\pi\mathcal{P}_{\omega_\infty}(z) \geq -\mathcal{R}(K)$ for every $z \in \mathbb{C}$.

Proof. Can be found in [Co95, p.314]. □

There is another interesting connection to the (positive) capacity of a compact set.

Lemma 2.56. *If K has positive capacity and $\mathcal{R}(K)$ is well defined, then*

$$Cap(K) \geq e^{-\mathcal{R}(K)}.$$

Proof. Let W be the connected component of $\mathbb{C}^*\backslash K$ with $\infty \in W$. Thus there exists a harmonic measure ω_∞ for W and by Theorem 2.55 we have that $2\pi\mathcal{P}_{\omega_\infty}(z) \geq -\mathcal{R}(K)$ on $\mathbb{C}$. Thus,

$$\mathrm{Cap}(K) \geq e^{I(\omega_\infty)} = e^{2\pi\int_{bK}\mathcal{P}_{\omega_\infty}(z)d\omega_\infty(z)} \geq e^{-\mathcal{R}(K)}.$$

This proves the claim. □

Example 2.K. *The robin constant of $\overline{D(0,R)}$ for a positive number R is equal to $-\log R$.*
This is an easy consequence of Remark 2.I, where we have seen that

$$G_{\mathbb{C}^*\backslash\overline{D(0,R)}}(z,\infty) = \log\left|\frac{z}{R}\right|.$$

Thus, by the definition of $\mathcal{R}(\overline{D(0,R)})$ it's simple to verify the claim,

$$\mathcal{R}(\overline{D(0,R)}) = \lim_{z\to\infty}\left[\log\left|\frac{z}{R}\right| - \log|z|\right] = -\log R.$$

♦

For an application of the previous results see Section 3.2.

2.6 Plurisubharmonic functions

In this section we want to generalize the term of subharmonicity to subsets of $\mathbb{C}^n$ for $n > 1$. For our purpose, the following generalization is the right one.

Definition 2.57 (Plurisubharmonicity)**.** *Let Ω be open in $\mathbb{C}^n$.*
A function $\Psi : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called plurisubharmonic if

(i) Ψ is upper semicontinuous and

(ii) for every $a \in \Omega$ and every $w \in \mathbb{C}^n$ the function $\lambda \mapsto \Psi(a+\lambda w)$ is subharmonic on the set $\{\lambda \in \mathbb{C} : a + \lambda w \in \Omega\}$.

The following theorems can be found in [Hö90/I, ch. 2] and [R86, ch. 4].
In the first step we show, that there is a similar result for plurisubharmonic functions as Corollary 2.13 for subharmonic functions.

Corollary 2.58. *Let Ω be open in $\mathbb{C}^n$ and $\Psi \in C^2(\Omega)$. The function Ψ is plurisubharmonic if and only if the Levi matrix*

$$M_{\Psi(z)} := \left(\frac{\partial^2 \Psi}{\partial z_j \partial \overline{z_k}}(z) \right)_{j,k=1}^{n}$$

is positive semidefinite for every $z \in \Omega$.

Proof. Let $w \in \mathbb{C}^n$, $a \in \Omega$ and λ such that $a + \lambda w \in \Omega$. Then the following holds true

$$\begin{aligned} \frac{\partial^2}{\partial \lambda \partial \overline{\lambda}} \Psi(a + \lambda w) &= \frac{\partial}{\partial \lambda} \sum_{k=1}^{n} \frac{\partial \Psi}{\partial \overline{z_k}}(a + \lambda w)\overline{w_k} \\ &= w^* M_{\Psi(a+\lambda w)} w. \end{aligned}$$

Hence, by Corollary 2.13 the right hand side is nonnegative if and only if $\lambda \mapsto \Psi(a+\lambda w)$ is subharmonic. □

Example 2.L. *The function $\Psi : \mathbb{C}^2 \to \mathbb{R}$ defined by*

$$\Psi(z_1, z_2) := \log(1 + |z_1|^2 |z_2|^2)$$

is plurisubharmonic.
An easy computation shows that

$$M_{\Psi(z_1,z_2)} = \frac{1}{(1 + |z_1|^2|z_2|^2)^2} \begin{pmatrix} |z_2|^2 & \overline{z_1} z_2 \\ z_1 \overline{z_2} & |z_1|^2 \end{pmatrix}.$$

and therefore,

$$\begin{aligned} (\overline{w_1}, \overline{w_2}) M_{\Psi(z_1,z_2)} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \frac{|w_1|^2|z_2|^2 + z_1 \overline{z_2} \overline{w_2} w_1 + \overline{z_1} z_2 \overline{w_1} w_2 + |z_1|^2 |w_2|^2}{(1 + |z_1|^2|z_2|^2)^2} \\ &= \frac{|\overline{w_1} z_2 + \overline{w_2} z_1|^2}{(1 + |z_1|^2|z_2|^2)^2} \geq 0. \end{aligned}$$

Because $\Psi \in C^2(\mathbb{C}^2)$ we can apply Corollary 2.58 and get the claim. ♦

In higher dimensions Theorem 2.15 looks as follows:

Theorem 2.59. *Let $\Omega \subseteq \mathbb{C}^n$ be connected. If $\Psi : \Omega \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic and not identicall $-\infty$, then $\Psi \in L^1(\Omega, loc)$ and the set $\{z \in \Omega : \Psi(z) = -\infty\}$ is of Lebesgue measure zero.*

Proof. A proof can be found in [R86, Lemma 4.11, p.89] and is similar to the proof of Lemma 2.15. □

Example 2.M. *We show, that if* $\Psi : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ *is a plurisubharmonic function, then*

$$z \mapsto \Psi(z) + \max(0, \log|z|) =: \Psi_1(z)$$

is plurisubharmonic.
Actually, we only have to check if $\Psi_0 : z \mapsto \max(0, \log|z|)$ is plurisubharmonic. Since Ψ_0 is continuous, the function Ψ_1 is upper semicontinuous. Hence, we have to check if the second property of Definition 2.57 is satisfied. For this purpose let $a, w \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Since the function $\lambda \mapsto \log|a + \lambda w|$ satisfies,

$$\begin{aligned}
\Delta \log|a + \lambda w| &= 4\frac{\partial^2}{\partial\overline{\lambda}\partial\lambda} \log|a + \lambda w| \\
&= 2\frac{\partial}{\partial\overline{\lambda}} \frac{\sum_{j=1}^n w_j(\overline{a_j} + \overline{\lambda w_j})}{|a + \lambda w|^2} \\
&= 2\frac{|w|^2|a|^2 - |(w, a)|^2}{|a + \lambda w|^4} \geq 0,
\end{aligned}$$

it is subharmonic by Corollary 2.26. Hence, we note by Jensen's inequality (2.5) and the previous display, that for $z_0 \in \mathbb{C}$ and appropriate $r > 0$ we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \Psi_0(z_0 + re^{i\theta})d\theta &\geq \max(0, \frac{1}{2\pi}\int_0^{2\pi} \log|a + (z_0 + re^{i\theta})w|d\theta \\
&\geq \max(0, \log|a + z_0 w|) = \Psi_0(z_0).
\end{aligned}$$

This proves that the function Ψ_1 is plurisubharmonic. ♦

In many situations the following approximation lemma will be quite helpful.

Theorem 2.60. *Let* $\phi \geq 0$ *be a radial symmetric test function, supported in* $D(0, 1)$ *such that*

$$\int_{\mathbb{C}^n} \phi(z)d\lambda(z) = 1.$$

If Ψ *is plurisubharmonic in* Ω*, it follows that*

$$\Psi_\varepsilon(a) = \int_{\mathbb{C}^n} \Psi(a - \varepsilon z)\phi(z)d\lambda(z)$$

is again plurisubharmonic if $\varepsilon < \inf\{|a - z| : z \in \Omega^c\}$*. In addition* Ψ_ε *is smooth and* $\Psi_\varepsilon \searrow \Psi$ *if* $\varepsilon \searrow 0$.

Proof. A proof can be found in [Hö90/I, Thm. 2.6.3, p.45] and is similar to the proof of Lemma 2.11. □

In the following chapters we consider weighted Bergman spaces (see Section 3) with a subharmonic or plurisubharmonic weight function. For such spaces Theorem 2.26 and Corollary 2.58 are rather helpful.

3 Spaces of holomorphic functions

Weighted and unweighted Bergman spaces are of highest interest in the following theorems. This chapter provides the basic definitions, nice results and some examples of trivial and finite dimensional Bergman spaces. We prove a result due to Carleson, which provides a necessary and sufficient condition for the unweighted Bergman space under which it is non-trivial. For this purpose we use methods from Section 2.4 and 2.5.

3.1 Bergman space

We introduce the following space of holomorphic functions (see [H12, p.2] and [Hi00, p.325]):

Definition 3.1. *Let Ω be a domain in $\mathbb{C}^n$. For a measurable function $\Psi : \Omega \to [-\infty, \infty)$ we define the the weighted Bergman space*

$$A^2(\Omega, \Psi) := \{f \in \mathcal{H}(\Omega) : \int_\Omega |f|^2 e^{-\Psi} d\lambda(z) < \infty\}$$

with norm

$$\|.\|_{A^2(\Omega,\Psi)} := \left(\int_\Omega |f(z)|^2 e^{-\Psi} d\lambda(z)\right)^{1/2}.$$

As usual, we denote the space $A^2(\Omega, 0)$ by $A^2(\Omega)$ and call it (unweighted) Bergman space.

Remark 3.A. Although, this space does only contain classical functions and no equivalence classes of functions, we occasionally interpret $A^2(\Omega, \Psi)$ as a subset of $L^2(\Omega, \Psi)$. This is possible by using the following embedding:

$$\begin{aligned} \iota : A^2(\Omega, \Psi) &\hookrightarrow L^2(\Omega, \Psi), \\ f &\mapsto [f]_\sim, \end{aligned}$$

which is in fact an isometric homomorphism (therefore, injective).

With help of this remark one easily checks that the space $A^2(\Omega, \Psi)$ is a vector space. It turns out that it's even a Hilbert space:

Theorem 3.2. *The space $A^2(\Omega, \Psi)$ is a Hilbert space.*

Proof. The norm of $A^2(\Omega, \Psi)$ is induced by the following inner product

$$\langle f, g\rangle_\Psi := \int_\Omega f\overline{g} e^{-\Psi} d\lambda(z).$$

Hence we have to check if the space itself is complete. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $A^2(\Omega, \Psi)$. If K is a compact subset of Ω and ω an open and bounded neighborhood of K in Ω, then we see by using Cauchy-Schwarz inequality that

$$\begin{aligned}\int_\omega |f_k - f_l| d\lambda(z) &\leq \left(\int_\omega |f_k - f_l|^2 e^{-\Psi} d\lambda(z)\right)^{1/2} \left(\int_\omega e^{\Psi} d\lambda(z)\right)^{1/2} \\ &\leq \|f_k - f_l\|_{A^2(\Omega,\Psi)} \left(\int_{\overline{\omega}} e^{\Psi} d\lambda(z)\right)^{1/2}.\end{aligned}$$

Using Theorem 1.3, one sees that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}(\Omega)$ as well and therefore, we can find an entire function f such that $f_n \to f$ as $n \to \infty$.
Interpreting $\{f_n\}_{n=1}^{\infty}$ as a Cauchy sequence in $L^2(\Omega, \Psi)$ gives a function $f_* \in L^2(\Omega, \Psi)$ such that $f_n \to f_*$. Hence there exists a subsequence f_{n_k} such that $f_{n_k} \to f_*$ pointwise almost everywhere w.r.t. Lebesgue measure. This shows that $f = f_*$ and $\{f_n\}_{n=1}^{\infty}$ converges. □

Unfortunately the following lemma holds true.

Lemma 3.3. *The space $A^2(\mathbb{C}, h)$ is trivial for a harmonic weight $h : \mathbb{C} \to \mathbb{R}$.*

Proof. At first we show that $A^2(\mathbb{C}) = \{0\}$. Let $f \in A^2(\mathbb{C})$. Hence, by Theorem 1.3 we have that

$$|f(z)^2| \leq \int_{\mathbb{C}} |f(\zeta)|^2 d\lambda(\zeta),$$

for every $z \in \mathbb{C}$. Since $z \to f(z)^2$ is an entire function, we have by Liouville's Theorem that $f^2 = c$ for a constant $c \in \mathbb{C}$. Square integrability gives that $c = 0$ and therefore, $f = 0$.
If h is a harmonic weight function, then there exists a conjugate function k such that $g(z) := h(z) + ik(z)$ is entire. If $f \in A^2(\mathbb{C}, h)$, we have that the function $f(z)e^{\frac{-g(z)}{2}}$ is entire and satisfies

$$\begin{aligned}\int_{\mathbb{C}} \left|f(z)e^{\frac{-g(z)}{2}}\right|^2 d\lambda(z) &= \int_{\mathbb{C}} \left|f(z)e^{\frac{-h(z)}{2}} e^{\frac{-ik(z)}{2}}\right|^2 d\lambda(z) \\ &= \int_{\mathbb{C}} |f(z)|^2 e^{-h(z)} d\lambda(z) < \infty.\end{aligned}$$

By the previous step, we have that $fe^{\frac{-g}{2}} = 0$ and therefore, $f = 0$. □

Therefore, we will focus on subharmonic or plurisubharmonic weight functions Ψ and we are interested under which conditions the space $A^2(\Omega, \Psi)$ is nontrivial. This will be the task of the following chapters.
If $\Omega = \mathbb{C}^n$ we will write down at least sufficient conditions on Ψ for $A^2(\mathbb{C}^n, \Psi)$ being of infinite dimension.
Nevertheless, $A^2(\mathbb{C}, \Psi)$ can be very small, even for a subharmonic, but not harmonic, weight function Ψ.

Lemma 3.4. *The space $A^2(\mathbb{C}, \log(1 + |z|^2))$ is of dimension one.*

Proof. Let $\Psi(z) := \log(1+|z|^2)$. Since,

$$\int_{\mathbb{C}} e^{-\Psi(z)} d\lambda(z) = 2\pi \int_0^\infty \frac{dr}{1+r^2} = 2\pi \arctan(r)\Big|_0^\infty = \pi^2,$$

the space $A^2(\mathbb{C}, \Psi)$ contains constants.
If $f \in A^2(\mathbb{C}, \Psi)$ is not constant, then we can expand it into a power series,

$$f(z) = \sum_{k=0}^{\infty} a_n z^n,$$

with $a_n \in \mathbb{C}$ and there exists at least one $k > 0$ such that $a_k \neq 0$. For every $R > 1$ we have

$$\begin{aligned}
\infty >& \|f\|^2_{A^2(\mathbb{C},\Psi)} \geq \int_{\overline{D(0,R)}\setminus D(0,1)} f(z)\overline{f(z)} \frac{1}{1+|z|^2} d\lambda(z) \\
=& \sum_{n,m=0}^{\infty} a_n \overline{a_m} \int_1^R r \int_0^{2\pi} r^{n+m} e^{i(n-m)\theta} \frac{1}{1+r^2} d\theta dr \\
=& 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_1^R \frac{r^{2n+1}}{1+r^2} dr \\
\geq& \pi |a_k|^2 (\log(1+R^2) - \log(2)).
\end{aligned}$$

Hence, $R \to \infty$ gives a contradiction. □

3.2 Unweighted Bergman spaces on unbounded domains

In this section we discuss the space $A^2(\Omega)$ and show that under a special condition on Ω it's a non-trivial space. We can even give a necessary and sufficient condition for this type of Bergman space being non-trivial. For this purpose we follow the lines of [C67, ch. 6, p.73] and [Co95, p.347–351].

Theorem 3.5 (Carleson)**.** *Let Ω be a connected domain in $\mathbb{C}^*$, s.t. its complement E is compact in $\mathbb{C}$. The space $A^2(\Omega)$ contains non-trivial functions if and only if $Cap(E) > 0$.*

Remark 3.B. There is an interesting result by Wiegerinck. He proved that for a domain $\Omega \subseteq \mathbb{C}$, the Bergman space $A^2(\Omega)$ is either trivial or of infinite dimension (see [Wi84, Section 3]). This result closes the gap between Theorem 3.5 and our main purpose. In gerneral Wiegerinck's result does not hold true for weigthed Bergman spaces (Lemma 3.4 gives us a counter example).
In the same paper he showed that for every $k > 0$ there exists a domain (with certain properties) in $\mathbb{C}^2$ s.t. the associated Bergman space is k-dimensional.

Before we start proving Theorem 3.5 we have to provide some technical results first. Let us begin with the terms *Jordan curve* and *Jordan system*.

Definition 3.6 (Jordan curve and Jordan system)**.** *Let $a < b$ be numbers in $\mathbb{R}$. A closed curve $\gamma : [a, b] \to \mathbb{C}$ is called Jordan curve, if $\gamma(t) = \gamma(s)$ is equivalent to $s = t$ or $|t - s| = b - a$.*
A Jordan system is a collection of pairwise disjoint Jordan curves.

More information on Jordan curves can be found in [Co95, p.1–6]. A very important result is the Jordan Curve Theorem, which is rather difficult to prove.

Theorem 3.7. *Let γ be a Jordan curve, then $\mathbb{C}\backslash\gamma$ consists of two connected components, an unbounded connected component and a bounded connected component.*

Proof. A proof can be found in [W58, ch. 3]. □

Driven by the previous theorem, the bounded connected component of a Jordan curve γ is called the inside of γ and is denoted by insγ.
For a Jordan curve γ we define the index by

$$\mathrm{ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z},$$

for $z \in \mathbb{C}\backslash\gamma$. It can be proven that $z \mapsto \mathrm{ind}_\gamma(z)$ is a function with values in $\mathbb{Z}$. The index of a Jordan system $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ is defined by

$$\mathrm{ind}_\Gamma(z) := \sum_{k=1}^{n} \mathrm{ind}_{\gamma_k}(z),$$

for every $z \in \mathbb{C}\backslash\Gamma$. A Jordan system is called *positive* if the index ind_Γ is either 0 or 1. The inside of a positive Jordan system is defined by

$$\mathrm{ins}\Gamma := \{z \in \mathbb{C}\backslash\Gamma : \mathrm{ind}_\Gamma(z) = 1\}.$$

We use the following existence theorem for Jordan systems.

Theorem 3.8. *Let $\Omega \subseteq \mathbb{C}$ be open. For a compact subset E of Ω there exists a smooth positive Jordan system Γ such that*

$$E \subseteq ins\Gamma \subseteq \Omega.$$

Proof. A proof can be found in [Co95, p.4]. □

For the sufficiency of Theorem 3.5 we need another technical lemma.

Lemma 3.9. *Let E be a compact subset of $\mathbb{C}$ with strictly positive capacity, i.e. $Cap(E) > 0$. Then there exist compact subsets E_1, E_2 of E with strictly positive capacity and $E_1 \cap E_2 = \emptyset$.*

Proof. Since E is compact and its capacity is strictly positive, there exits (see Theorem 2.39) an equilibrium measure ν s.t.

$$-\infty < I(\nu) = \sup_{\mu \in \mathcal{M}_1^c(E)} I(\mu). \tag{3.1}$$

Thus, we define the compact set $K := \operatorname{supp}\nu$ and note that $0 < \operatorname{Cap}(K) = \operatorname{Cap}(E)$. By Lemma 2.41 there exists a compact set $E_1 \subsetneq K$, s.t. $\operatorname{Cap}(E_1) > 0$. Now we define the compactly supported measures

$$\nu_1 := \nu|_{E_1} \quad \text{and} \quad \nu_2 := \nu|_{K \setminus E_1} \neq 0.$$

By construction we have that $\nu = \nu_1 + \nu_2$ and thus

$$\begin{aligned} -\infty < I(\nu) = I(\nu_1 + \nu_2) &= 2\pi \int_E \Big(\mathcal{P}_{\nu_1}(z) + \mathcal{P}_{\nu_2}(z)\Big) d(\nu_1 + \nu_2)(z) \\ &\le I(\nu_1) + I(\nu_2) + 2\pi M_1 \nu_2(K \setminus E_1) + 2\pi M_2 \nu_1(E_1), \end{aligned}$$

where $M_1 := \max_{z \in E} \mathcal{P}_{\nu_1}(z)$ and $M_2 := \max_{z \in E} \mathcal{P}_{\nu_2}(z)$ (see Lemma 2.5). Thus

$$I\left(\frac{\nu_2}{\nu_2(K \setminus E_1)}\right) = \frac{I(\nu_2)}{(\nu_2(K \setminus E_1))^2} > -\infty,$$

proving that $\operatorname{Cap}(K \setminus E_1) > 0$. Again by Lemma 2.41 there exists a compact subset E_2 of $K \setminus E_1$ with positive capacity. $\square$

Remark 3.C. If E is a compact set of positive capacity, then there exists a compactly supported measure s.t. its logarithmic potential is bounded from below. This can be seen as follows: Again there exists an equilibrium measure satisfying inequality (3.1). Thus, the claim is an easy consequence of Theorem 2.40.

Theorem 3.10. *Let $\Omega \subseteq \mathbb{C}$ be open and $(X, \mathcal{S}, \mu)$ a measure space. If the function $f : \Omega \times X \to \mathbb{C}$ satisfies*

(i) $\int_X |f(z,x)| d\mu(x)$ is finite for every $z \in \Omega$,

(ii) for every $x \in X$ the $f(.,x) : \Omega \to \mathbb{C}$ is holomorphic and

(iii) For every compact disc $D \subseteq \Omega$ there exists an integrable function $g_D : X \to [0, \infty]$, s.t. for every $z \in D$ we have $|f(z,x)| \le g_D(x)$ for μ almost every $x \in X$,

then the function $F : \Omega \to \mathbb{C}$ defined by

$$F(z) := \int_X f(z,x) d\mu(x),$$

is holomorphic and satisfies

$$\frac{\partial^n F}{\partial z^n}(z) = \int_X \frac{\partial^n f}{\partial z^n}(z,x) d\mu(x),$$

for every $z \in \Omega$.

Proof. A proof can be found in [E09, §5, Satz 5.8]. $\square$

We need another inequality to prove Theorem 3.5.

Lemma 3.11. *There exists a constant $C \in \mathbb{R}$ such that for every $z_1, z_2 \in D(0,1)$ we have*

$$\int_{D(0,1)} \frac{d\lambda(z)}{|z_1 - z||z_2 - z|} \leq C + 2\pi \log\left(\frac{1}{|z_1 - z_2|}\right).$$

Proof. W.l.o.g. we may assume that $z_1 \neq z_2$. After applying some rotations we can w.l.o.g. assume that $z_1 = a, z_2 = b$, where $a, b \in \mathbb{R}$ and $a < b$.
Let $c := b - a > 0$ and $w = z - \frac{a+b}{2}$. A simple geometric argument shows that

$$\begin{aligned} D(0,1) &\subseteq D\left(a, \frac{7c}{8}\right) \cup D\left(b, \frac{7c}{8}\right) \cup \left(D\left(\frac{a+b}{2}, 3\right) \backslash D\left(\frac{a+b}{2}, \frac{5c}{2}\right)\right)^{\circ} \\ &=: D_1 \cup D_2 \cup D_3. \end{aligned}$$

Therefore, we have that

$$\int_{D(0,1)} \frac{d\lambda(z)}{|z-a||z-b|} \leq \sum_{j=1}^{3} \int_{D_j} \frac{d\lambda(z)}{|z-a||z-b|} =: \sum_{j=1}^{3} I_j.$$

We give appropriate upper bounds for the the integrals I_1, I_2 and I_3.
If $z \in D_1$ we have

$$|z-a||z-b| = |z-a||z-a-(b-a)| \geq |z-a|\,||z-a|-c| = |z-a|(c-|z-a|),$$

and hence,

$$\begin{aligned} I_1 &\leq \int_{D_1} \frac{d\lambda(z)}{|z-a|(c-|z-a|)} = 2\pi \int_0^{7c/8} \frac{rdr}{r(c-r)} \\ &= 2\pi\Big(-\log(c-r)\Big|_0^{7c/8}\Big) = 2\pi\Big(-\log\left(\frac{c}{8}\right) + \log(c)\Big) \\ &= 2\pi \log 8. \end{aligned}$$

A similar inequality shows that

$$I_2 \leq 2\pi \log(8).$$

Before we estimate I_3 from above we note that

$$\begin{aligned} (z-a)(z-b) &= \Big(z - \frac{a+b}{2} + \frac{c}{2}\Big)\Big(z - \frac{a+b}{2} - \frac{c}{2}\Big) \\ &= \Big(w + \frac{c}{2}\Big)\Big(w - \frac{c}{2}\Big). \end{aligned}$$

The reversed triangle inequality gives us

$$\begin{aligned} I_3 &= \int_{D_3} \frac{d\lambda(w)}{|w^2 - \frac{c^2}{4}|} \leq \int_{D_3} \frac{d\lambda(w)}{|w|^2 - \frac{c^2}{4}} = 2\pi \int_{5c/8}^{3} \frac{rdr}{r^2 - \frac{c^2}{4}} \\ &= \pi \log\left(9 - \frac{c^2}{4}\right) - \pi \log\left(\frac{9c^2}{64}\right) \leq \pi \log 9 - \pi \log\left(\frac{9c^2}{64}\right). \end{aligned}$$

Since

$$-\pi \log\left(\frac{9c^2}{64}\right) = 2\pi \log\left(\frac{8}{3c}\right) = 2\pi \log\left(\frac{8}{3}\right) + 2\pi \log\left(\frac{1}{|z_1 - z_2|}\right),$$

the constant $C := 2\pi \log\left(\frac{8}{3}\right) + 4\pi \log(8) + \pi \log(9)$ satisfies our claim. $\square$

At this point we have provided enough material to prove the sufficiency of Theorem 3.5. We follow the lines of [C67, ch. 6, p.73].

Sufficiency of Theorem 3.5. By Lemma 3.9, we can find compact subsets E_1 and E_2 of E such that $E_1 \cap E_2 = \emptyset$ and both sets have positive capacity. Therefore, by Remark 3.C, we can find probability measures μ_j for $j = 1, 2$, such that $\mu_j(E_j) = 1$ and $\mathcal{P}_{\mu_j} \geq M$, for a finite constant $M \in \mathbb{R}$. Put $\mu := \mu_1 - \mu_2$. We have a look at the following function

$$f(z) := \int_E \frac{d\mu(\zeta)}{\zeta - z}.$$

We apply Lemma 3.10 and note that $\overline{\partial} f(z) = 0$ for every $z \in \Omega$ and hence, $f \in \mathcal{H}(\Omega)$. Let us check whether f is non-trivial. For this purpose, by Theorem 3.8 we can choose a positively oriented piecewise C^1-Jordan curve γ s.t. E_1 is contained in insγ and $E_2 \subseteq E \backslash \overline{\text{ins}\gamma}$, and note that by Fubini's theorem

$$\begin{aligned}
\frac{1}{2\pi i} \int_\gamma f(z) dz &= \frac{1}{2\pi i} \int_\gamma \int_E \frac{d\mu(\zeta)}{\zeta - z} dz \\
&= \int_{E_1 \cup E_2} \mathbb{1}_{\text{ins}\gamma}(\zeta) d\mu(\zeta) \\
&= -\mu_1(E_1) \\
&= -1.
\end{aligned}$$

Hence, we have to check, if f is square integrable. For this purpose, take a number $R > 1$ such that $E \subseteq D(0, R/2)$. For $z \notin D(0, R)$ and $\zeta \in E$ we have

$$\frac{1}{\zeta - z} = \frac{-1}{z} \sum_{k=0}^{\infty} \left(\frac{\zeta}{z} \right)^k,$$

and for every $M \in \mathbb{N}$

$$\left| \frac{-1}{z} \sum_{k=0}^{M} \left(\frac{\zeta}{z} \right)^k \right| \leq \frac{1}{R} \sum_{k=0}^{\infty} \frac{1}{2^k}.$$

We apply dominated convergence theorem and the fact that $\mu(E) = 0$ to see that for $|z| > R$ the following representation holds true

$$f(z) = - \sum_{k=0}^{\infty} \frac{\int_E \zeta^k d\mu(\zeta)}{z^{k+1}} = \sum_{k=2}^{\infty} \frac{a_k}{z^k},$$

where $a_k := - \int_E \zeta^{k-1} d\mu(\zeta)$. We note that $|a_k| \leq 2(R/2)^{k-1}$ for every $k \geq 2$.
For $|z| > R^2$, we have for every $M \in \mathbb{N}$ that

$$\begin{aligned}
\left| \sum_{n,m=2}^{M} \frac{a_n \overline{a_m}}{z^n \overline{z}^m} \right| = \left| \sum_{k=2}^{M} \frac{a_k}{z^k} \right|^2 &\leq \sum_{n=2}^{M} \frac{|a_n|}{|z|^n} \sum_{m=2}^{M} \frac{|a_m|}{|z|^m} \\
&\leq \frac{4}{|z|^3} \sum_{n=2}^{\infty} \frac{(R/2)^{n-1}}{R^{2(n-2)}} \sum_{m=2}^{\infty} \frac{(R/2)^{m-1}}{R^{2(m-1)}}
\end{aligned}$$

$$\leq \frac{C_1(R)}{|z|^3},$$

where $C_1(R)$ is a positive constant depending on R and the left hand side tends pointwise to $|f(z)|^2$ as $M \to \infty$. Therefore, dominated convergence theorem gives

$$\begin{aligned}
\int_{|z|>R^2} |f(z)|^2 d\lambda(z) &= \sum_{n,m=2}^{\infty} a_k \overline{a_m} \int_{R^2}^{\infty} r \int_0^{2\pi} \frac{e^{-i(k-m)\theta}}{r^{k+m}} d\theta dr \\
&= \sum_{n=2}^{\infty} |a_k|^2 \int_{R^2}^{\infty} \frac{1}{r^{2k-1}} dr \\
&= \sum_{n=2}^{\infty} \frac{|a_k|^2}{(2n-2)R^{2(2n-2)}} \\
&\leq \sum_{n=2}^{\infty} \frac{1}{2n-2} \left(\frac{2(R/2)^{n-1}}{R^{2(n-1)}} \right)^2 \\
&\leq \sum_{n=2}^{\infty} \left(\frac{1}{R} \right)^{2(n-1)} < \infty.
\end{aligned}$$

Hence, we have to check integrability for $|z| \leq R^2$. By definition we have

$$|\mu| = \mu_1 + \mu_2. \tag{3.2}$$

We use Fubini's theorem and Lemma 3.11 to see that

$$\begin{aligned}
\int_{D(0,R^2)} |f(z)|^2 d\lambda(z) &= \int_{D(0,R^2)} \left| \int_E \frac{d\mu(\zeta)}{\zeta - z} \right| \left| \int_E \frac{d\mu(\zeta')}{\zeta' - z} \right| d\lambda(z) \\
&\leq \int_{D(0,R^2)} \int_E \frac{d|\mu|(\zeta)}{|\zeta - z|} \int_E \frac{d|\mu|(\zeta')}{|\zeta' - z|} d\lambda(z) \\
&= \int_E \int_E \int_{D(0,R^2)} \frac{1}{|z-\zeta'||z-\zeta'|} d\lambda(z) d|\mu|(\zeta) d|\mu|(\zeta') \\
&\leq C_2(R) + \int_E \int_E \log\left(\frac{1}{|\zeta - \zeta'|} \right) d|\mu|(\zeta') d|\mu|(\zeta) \\
&\overset{(3.2)}{=} C_2(R) - \int_E \left(\mathcal{P}_{\mu_1}(\zeta) + \mathcal{P}_{\mu_2}(\zeta) \right) d|\mu|(\zeta) \\
&\leq C_2(R) - 4M < \infty,
\end{aligned}$$

for a positive constant $C_2(R)$. Hence, $A^2(\Omega)$ contains at least one non-trivial function. $\square$

For the other direction of the proof we need some additional lemmas. The following statement can be found in [Co95, p.345].

Lemma 3.12. *Let $R > 0$. The function f is an element of the Bergman space $A^2(\mathbb{C} \setminus \overline{D(0,R)})$ if and only if there exist complex numbers a_n such that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^{-n}, \tag{3.3}$$

uniformly convergent on compact subsets of $\mathbb{C}\backslash\overline{D(0,R)}$ *with* $a_0 = a_1 = 0$ *and*

$$\sum_{n=2}^{\infty} |a_n|^2 \frac{R^{2-2n}}{2n-2} < \infty. \tag{3.4}$$

Proof. Let $f \in A^2(\mathbb{C}\backslash\overline{D(0,R)})$. On $D(0,\frac{1}{R})\backslash\{0\}$ we define the holomorphic function $g(z) := f(\frac{1}{z})$. We can extend it into a Laurent series of the form

$$g(z) := \sum_{n\in\mathbb{Z}} a_n z^n,$$

uniformly convergent on compact subsets of $D(0,\frac{1}{R})\backslash\{0\}$. Let $r > R$. Apply the transformation formula to the function $z \mapsto \frac{1}{z}$ and use uniform convergence to see

$$\begin{aligned} \infty > \int_{\mathbb{C}\backslash\overline{D(0,R)}} |f(z)|^2 d\lambda(z) &\geq \int_{D(0,\frac{1}{R})\backslash D(0,\frac{1}{r})} \frac{|g(z)|^2}{|z|^3} d\lambda(z) \\ &= 2\pi \sum_{n\in\mathbb{Z}} |a_n|^2 \int_{1/r}^{1/R} s^{2n-3} ds \\ &= 2\pi \sum_{n\neq 1} |a_n|^2 \frac{s^{2n-2}}{2n-2}\bigg|_{1/r}^{1/R} + 2\pi|a_1|^2 \log s\bigg|_{1/r}^{1/R}. \end{aligned} \tag{3.5}$$

Thus, for $n = 1$, we have

$$\infty > |a_1|^2\Big(\log(r) - \log(R)\Big).$$

Letting $r \to \infty$ shows that $a_1 = 0$. For $n \leq 0$, we have $2n - 2 < 0$ and thus, similar to the previous case, $a_n = 0$.
Therefore, g has a removable singularity at 0 and $g(z) = \sum_{n=2}^{\infty} a_n z^n$, uniformly convergent on compact subsets of $D(0,\frac{1}{R})$. By definition, this means that

$$f(z) = g(\frac{1}{z}) = \sum_{n=2}^{\infty} a_n z^{-n},$$

uniformly convergent on compact subsets of $\mathbb{C}\backslash\overline{D(0,R)}$.
Similar to (3.5) one sees that by uniform convergence and the monotone convergence theorem

$$\begin{aligned} \infty &> \int_{\mathbb{C}\backslash\overline{D(0,R)}} |f(z)|^2 d\lambda(z) \\ &= \int_{\mathbb{C}\backslash\overline{D(0,R)}} \sum_{n,m=2}^{\infty} a_n\overline{a_m} z^{-n-m} d\lambda(z) \\ &= \int_R^{\infty}\int_0^{2\pi} \sum_{n,m=2}^{\infty} a_n\overline{a_m} s^{-n-m+1} e^{-i(n-m)\theta} d\theta ds \\ &= \int_R^{\infty} 2\pi \sum_{n=2}^{\infty} |a_n|^2 \frac{1}{s^{2n-1}} ds \\ &= 2\pi \sum_{n=2}^{\infty} |a_n|^2 \int_R^{\infty} \frac{1}{s^{2n-1}} ds = 2\pi \sum_{n=2}^{\infty} |a_n|^2 \frac{R^{2-2n}}{2n-2} \end{aligned} \tag{3.6}$$

is satisfied. This proves the first direction.
Otherwise, if f is represented by (3.3) it is a holomorphic function on $\mathbb{C}\backslash\overline{D(0,R)}$. Similar to (3.6) we use uniform convergence to interchange summation w.r.t. n and integration w.r.t. θ and note that f is square integrable. This proves the claim. $\square$

For the other direction of Theorem 3.5 we use a version of the (complex) Stokes' theorem. A proof can be found in [R86, p.117]. One can also use the real version of Green's formula (see [He06, p.557]) to prove the required version. In this situation one only has to use that $dz = dx + idy$.

Theorem 3.13 (Green's formula)**.** *Let Γ be a system of smooth positive Jordan curves and $\Omega := ins\Gamma$. If $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ and $\frac{\partial u}{\partial \overline{z}}$ is integrable over Ω, then*

$$\int_\Gamma u(z)dz = 2i \int_\Omega \frac{\partial u(z)}{\partial \overline{z}} d\lambda(z).$$

In addition we want to use the following remark on Section 2.5.

Remark 3.D. In the following theorem we apply the concepts presented in Section 2.5 as follows: Let K be a compact set in $\mathbb{C}$ s.t. bK is bounded by a system of (smooth) Jordan curves. Thus the set $\mathbb{C}^*\backslash\left(\mathbb{C}^*\backslash K\right) = K$ and $\mathbb{C}^*\backslash\left(\mathbb{C}^*\backslash bK\right) = bK$ are not totally disconnected and thus by Lemma 2.46 $\mathbb{C}^*\backslash K$ and $\mathbb{C}^*\backslash bK$ have a Green's function. By Lemma 2.45 the functions $G_{\mathbb{C}^*\backslash K}$ and $G_{\mathbb{C}^*\backslash bK}$ coincide on the unbounded domain $\mathbb{C}^*\backslash K$ and therefore, we even have that $\mathcal{R}(K) = \mathcal{R}(bK)$ (see equality (2.20)).
By Lemma 2.51 the set $\mathbb{C}^*\backslash K$ has a harmonic measure ω_∞ and by Lemma 2.53 $\mathrm{supp}\omega_\infty \subseteq bK$. Therefore, the logarithmic potential $\mathcal{P}_{\omega_\infty}$ is well defined.

We follow the lines of [Co95, p.350–352], to prove the necessity of Theorem 3.5.

Necessity of Theorem 3.5. We assume that $\mathrm{Cap}(E) = 0$ and show that $A^2(\Omega) = \{0\}$. For this purpose, let $f \in A^2(\Omega)$. Since E is compact, we can find a positive number $R > 0$, s.t. $E \subseteq D(0,R)$. By Lemma 3.12, we can find complex numbers a_n for $n \geq 2$ s.t.

$$f(z) = \sum_{n=2}^{\infty} a_n z^{-n}, \text{ for } |z| > R,$$

uniformly convergent on compact subsets of $\mathbb{C}\backslash\overline{D(0,R)}$. By assumption Ω is connected and hence, it suffices to show that $a_n = 0$ for every n. This will imply that $f = 0$ on Ω.
Let γ be a smooth Jordan curve in $\mathbb{C}\backslash\overline{D(0,R)}$ and $r > R$ s.t. γ lies in $D(0,r)$. If γ and $bD(0,r)$ have the same orientation, we have

$$\int_\gamma f(z)z^k dz = \int_{bD(0,r)} f(z)z^k dz = \sum_{n=2}^{\infty} \int_{bD(0,r)} a_n z^{k-n} dz = 2\pi r a_{k+1}.$$

Thus, it suffices to show that for $k \geq 1$ we have $\int_\Gamma f(z)z^k dz = 0$, where Γ denotes a system of smooth Jordan curves with $E \subseteq \mathrm{ins}\Gamma$.

For this purpose, let Γ be a system of positively oriented smooth Jordan curves in $D(0,R)\backslash E$ (see Theorem 3.8) s.t. $E \subseteq \mathrm{ins}\Gamma$ and $\mathbb{C}^*\backslash\Gamma$ has finitely many connected components. We define

$$K := \Gamma \cup \mathrm{ins}\Gamma \quad \text{and} \quad W := \mathbb{C}^*\backslash K.$$

Since K is bounded and bK is a system of smooth Jordan curves, Remark 3.D shows that the Robin constant $\mathcal{R}(K)$ exists and it's equal to $\mathcal{R}(bK)$. By the same remark there exists a Green's function G_W and a harmonic measure ω_∞ for W at ∞ supported in Γ. We define the function

$$u(z) := -2\pi\mathcal{R}(K)^{-1}\mathcal{P}_{\omega_\infty}(z) = -\mathcal{R}(K)^{-1}\int_{bK}\log|z-w|d\omega_\infty(w).$$

By Theorem 2.55 we can write

$$u(z) = \begin{cases} 1 - \mathcal{R}(K)^{-1}G_W(z,\infty) & \text{if } z \in W \\ 1 & \text{if } z \notin \overline{W}. \end{cases}$$

By Corollary 2.50 we even have that $u(z) = 1$ for $z \notin W$ and therefore, it is continuous on $\mathbb{C}^*$ and harmonic on W.
Define $U := \{z : u(z) > 0\} \supseteq K$ and note that U is open. We even have that U is bounded: Let $|z| > R+1$. Note that $d(\Gamma, z) > 1$ and thus, $|z-w|^{-1} < 1$ for all $w \in \Gamma$. This implies that

$$u(z) = -2\pi\mathcal{R}(K)^{-1}\mathcal{P}_{\omega_\infty}(z) < 0.$$

Hence, $U \subseteq \overline{D(0,R+1)}$. For a small $\varepsilon > 0$ we choose for each connected component of U exactly one smooth Jordan curve γ from the level set $\{z : u(z) = 1-\varepsilon\} \subseteq U \cap W$ s.t. the system Γ_ε of the chosen Jordan curves satisfies $\Gamma \subseteq \mathrm{ins}\Gamma_\varepsilon$ and $\Gamma_\varepsilon \cup bU$ is a positive Jordan system. We note that the curves in bU must be positively oriented and the curves in Γ_ε must be negatively oriented.
Let

$$V_\varepsilon := \mathrm{ins}\Big(\Gamma_\varepsilon \cup bU\Big)$$

Because $u|_{bU} = 0$ and $bV_\varepsilon = \Gamma_\varepsilon \cup bU$, we have for a holomorphic function h in a neighborhood of $\overline{W} = \mathbb{C}^*\backslash\mathrm{ins}\Gamma$ that

$$\int_\Gamma h(z)dz = -\int_{\Gamma_\varepsilon} h(z)dz = -\frac{1}{1-\varepsilon}\int_{\Gamma_\varepsilon} h(z)u(z)dz = -\frac{1}{1-\varepsilon}\int_{bV_\varepsilon} h(z)u(z)dz \tag{3.7}$$

and by Green's theorem 3.13 we get

$$\int_\Gamma h(z)dz = -\frac{2i}{1-\varepsilon}\int_{V_\varepsilon}\frac{\partial\Big(h(z)u(z)\Big)}{\partial\overline{z}}d\lambda(z) = -\frac{2i}{1-\varepsilon}\int_{V_\varepsilon} h(z)\frac{\partial u(z)}{\partial\overline{z}}d\lambda(z). \tag{3.8}$$

Replacing h by $\frac{\partial u}{\partial z}$ (and thus only holomorphic on a neighborhood of $\mathbb{C}^*\backslash\mathrm{ins}\Gamma_\varepsilon$) in (3.7) and use the same computation as in (3.8) to obtain

$$\int_{\Gamma_\varepsilon}\frac{\partial u}{\partial z}(z)dz = \frac{2i}{1-\varepsilon}\int_{V_\varepsilon}\left|\frac{\partial u}{\partial z}(z)\right|^2 d\lambda(z). \tag{3.9}$$

Let us fix $k \in \mathbb{C}$ and $f \in A^2(\Omega)$. We apply equality (3.8), Cauchy-Schwarz inequality and $V_\varepsilon \subseteq U \subseteq \overline{D(0, R+1)}$, and use equality (3.9) to see that

$$\begin{aligned}\left|\int_\Gamma z^k f(z)dz\right|^2 &= \frac{4}{(1-\varepsilon)^2}\left|\int_{V_\varepsilon} z^k f(z)\frac{\partial u}{\partial \overline{z}}(z)d\lambda(z)\right|^2 \\ &\leq \frac{4}{(1-\varepsilon)^2}\int_{V_\varepsilon}\left|z^k f(z)\right|^2 d\lambda(z)\int_{V_\varepsilon}\left|\frac{\partial u}{\partial z}(z)\right|^2 d\lambda(z) \qquad (3.10) \\ &\leq \frac{2(R+1)^k}{i(1-\varepsilon)}\|f\|^2_{L^2(\Omega)}\int_{\Gamma_\varepsilon}\frac{\partial u}{\partial z}(z)dz.\end{aligned}$$

In the sense of distributions we have

$$\begin{aligned}\frac{\partial u}{\partial z}(z) &= -\frac{1}{\mathcal{R}(K)}\frac{\partial(\log|z| * \omega_\infty)}{\partial z} = -\frac{1}{2\mathcal{R}(K)}\left(\frac{1}{z} * \omega_\infty\right) \\ &= -\frac{1}{2\mathcal{R}(K)}\int_\Gamma \frac{1}{z-\zeta}d\omega_\infty(\zeta).\end{aligned}$$

Since u is harmonic on W (hence $C^\infty(W)$), we even have that the previous equality holds true in the classical sense on W. Therefore, by Fubini's theorem and $\Gamma_\varepsilon \subseteq W$

$$\begin{aligned}\int_{\Gamma_\varepsilon}\frac{\partial u}{\partial z}(z)dz &= \frac{-1}{2\mathcal{R}(K)}\int_{\Gamma_\varepsilon}\int_\Gamma \frac{1}{z-\zeta}d\omega_\infty(\zeta)dz \\ &= \frac{-1}{2\mathcal{R}(K)}\int_\Gamma\int_{\Gamma_\varepsilon}\frac{1}{z-\zeta}dz d\omega_\infty(\zeta) \\ &= \frac{-\pi i}{\mathcal{R}(K)}\int_\Gamma\Big(-\mathbb{1}_{\mathrm{ins}\Gamma_\varepsilon}(\zeta)\Big)d\omega_\infty(\zeta) \\ &= \frac{\pi i}{\mathcal{R}(K)},\end{aligned}$$

where we have used that each Jordan curve of Γ_ε is negatively oriented and $\mathrm{supp}\omega_\infty \subseteq \Gamma \subseteq \mathrm{ins}\Gamma_\varepsilon$. Since ε can be chosen arbitrarily small, the previous inequality and (3.10) give

$$\left|\int_\Gamma z^k f(z)d\lambda(z)\right|^2 \leq \frac{2\pi(R+1)^k}{\mathcal{R}(K)}\|f\|^2_{L^2(\Omega)}. \qquad (3.11)$$

We can now find compact sets K_m satisfying $K_m \supseteq K_{m+1}$ and $E = \bigcap_{m=1}^\infty K_m$ s.t. bK_m are smooth Jordan systems and $\mathbb{C}^*\backslash bK_m$ has finitely many connected components. By Lemma 2.56, Lemma 2.41 and the assumption that $\mathrm{Cap}(E) = 0$ we have that

$$0 = \lim_{m\to\infty}\mathrm{Cap}(K_m) \geq \lim_{m\to\infty} e^{-\mathcal{R}(K_m)}.$$

This implies that $\lim_{m\to\infty}\mathcal{R}(K_m) = \infty$. Since the right hand side of (3.11) remains constant for the different Jordan curves bK_m we have that

$$\left|\int_\Gamma z^k f(z)d\lambda(z)\right|^2 = \left|\int_{bK_1} z^k f(z)d\lambda(z)\right|^2 = \lim_{m\to\infty}\left|\int_{bK_m} z^k f(z)d\lambda(z)\right|^2 = 0.$$

This proves the claim. □

4 Hörmander's theorem and some consequences

In this chapter we state Hörmander's theorem in one variable and in several variables. The key idea of proving this theorem is to show that the so-called basic estimates hold true and then apply the Hahn-Banach's theorem. Unfortunately in several variables the basic estimates are very difficult to prove and therefore we won't give a detailed argument. In one dimension the situation is much easier.

4.1 Hörmander's theorem in one variable

The idea of the proof can be found in Hörmander's book [Hö07, p.248–251]. As usual Ω denotes an open subset of $\mathbb{C}$.
We have to provide a very important theorem of functional analysis first.

Theorem 4.1 (Hahn-Banach)**.** *Let E be a vector space over a field $\mathbb{K}$, p a seminorm on E and F a linear subspace of E. If a linear functional λ on F satisfies*

$$|\lambda(x)| \leq p(x) \quad \textit{for all } x \in F,$$

then there exists an extension Λ of λ to the whole space E, such that

$$|\Lambda(x)| \leq p(x) \quad \textit{for all } x \in E.$$

Proof. A proof can be found in [He06, p.229]. □

Let $f \in C_0^\infty(\Omega)$, $g := \frac{\partial f}{\partial \bar{z}}$ and $\Psi : \Omega \to [-\infty, \infty)$ a smooth subharmonic weight function. In this situation integration by parts (1.2) gives

$$\langle g, \phi \rangle_{L^2(\Omega,\Psi)} = \int_\Omega g(z)\overline{\phi(z)}e^{-\Psi(z)}d\lambda(z) = -\int_\Omega f(z)\frac{\partial\left(\overline{\phi(z)}e^{-\Psi(z)}\right)}{\partial \bar{z}}d\lambda(z), \tag{4.1}$$

for every $\phi \in C_0^\infty(\Omega)$. For that reason we define the operator

$$\delta\phi(z) := e^{\Psi(z)}\frac{\partial\left(\phi(z)e^{-\Psi(z)}\right)}{\partial z} = \frac{\partial\phi(z)}{\partial z} - \phi(z)\frac{\partial\Psi(z)}{\partial z} \tag{4.2}$$

with domain $C_0^\infty(\Omega)$. Inserting this into equality (4.1) implies

$$\langle \frac{\partial f}{\partial \bar{z}}, \phi \rangle_{L^2(\Omega,\Psi)} = \langle g, \phi \rangle_{L^2(\Omega,\Psi)} = \langle f, \delta\phi \rangle_{L^2(\Omega,\Psi)}.$$

The following lemma will be the key idea to prove Hörmander's theorem.

Lemma 4.2. *Let Ψ be a strictly subharmonic smooth function on Ω. If $f \in L^2(\Omega, loc)$, then there exists a function $u \in L^2(\Omega, \Psi)$ such that $\frac{\partial u}{\partial \bar{z}} = f$ and*

$$\|u\|^2_{L^2(\Omega,\Psi)} \leq 4 \int_\Omega |f(z)|^2 \frac{e^{-\Psi(z)}}{\Delta\Psi(z)} d\lambda(z),$$

provided that the right hand side is finite.

Proof. Let $\delta : C_0^\infty(\Omega) \subseteq L^2(\Omega, \Psi) \to C_0^\infty(\Omega)$ defined by (4.2). We note that the image of δ is a linear subspace of $L^2(\Omega, \Psi)$. For every $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}
\|\delta\phi\|^2_{L^2(\Omega,\Psi)} &= \int_\Omega \overline{\frac{\partial\left(\phi(z)e^{-\Psi(z)}\right)}{\partial z}} \delta\phi(z) d\lambda(z) \\
&= -\int_\Omega \overline{\phi(z)} e^{-\Psi(z)} \frac{\partial(\delta\phi(z))}{\partial \bar{z}} d\lambda(z) \\
&= -\int_\Omega \delta\left(\frac{\partial \phi(z)}{\partial \bar{z}}\right) \overline{\phi(z)} e^{-\Psi(z)} d\lambda(z) + \int_\Omega \frac{\partial^2 \Psi(z)}{\partial z \partial \bar{z}} |\phi(z)|^2 e^{-\Psi(z)} d\lambda(z) \qquad (4.3) \\
&= \int_\Omega \left|\frac{\partial \phi(z)}{\partial \bar{z}}\right|^2 e^{-\Psi(z)} d\lambda(z) + \frac{1}{4}\int_\Omega |\phi(z)|^2 \Delta\Psi(z) e^{-\Psi(z)} d\lambda(z) \\
&\geq \frac{1}{4}\int_\Omega |\phi(z)|^2 \Delta\Psi(z) e^{-\Psi(z)} d\lambda(z).
\end{aligned}$$

Now we define a linear functional $D : \overline{\delta(C_0^\infty(\Omega))} \subseteq L^2(\Omega, \Psi) \to \mathbb{C}$ by

$$D(\overline{\delta\phi}) := -\int_\Omega f(z)\overline{\phi(z)} e^{-\Psi(z)} d\lambda(z),$$

for every $\phi \in C_0^\infty(\Omega)$. By Cauchy-Schwarz inequality and inequality (4.3) we have

$$\begin{aligned}
|D(\overline{\delta\phi})|^2 &= \left|\int_\Omega f(z)\overline{\phi(z)} e^{-\Psi(z)} d\lambda(z)\right|^2 \\
&\leq \int_\Omega |f(z)|^2 \frac{e^{-\Psi(z)}}{\Delta\Psi(z)} d\lambda(z) \int_\Omega |\phi(z)|^2 \Delta\Psi(z) e^{-\Psi(z)} d\lambda(z) \\
&\leq 4\|\delta\phi\|^2_{L^2(\Omega,\Psi)} \int_\Omega |f(z)|^2 \frac{e^{-\Psi(z)}}{\Delta\Psi(z)} d\lambda(z).
\end{aligned}$$

Therefore, by the Hahn-Banach Theorem 4.1 we can extend D to a linear functional D' on $L^2(\Omega, \Psi)$ with norm lower equal than $2(\int_\Omega |f|^2 e^{-\Psi}/\Delta\Psi d\lambda)^{1/2}$ and by the Riesz theorem (for linear functionals on Hilbert spaces) we can find a function $\overline{u} \in L^2(\Omega, \Psi)$ such that

$$D'(h) = \int_\Omega h(z) u(z) e^{-\Psi(z)} d\lambda(z),$$

for every $h \in L^2(\Omega, \Psi)$ and

$$\|u\|^2_{L^2(\Omega,\Psi)} \leq 4 \int_\Omega |f(z)|^2 \frac{e^{-\Psi(z)}}{\Delta\Psi(z)} d\lambda(z).$$

Therefore, for every $\phi \in C_0^\infty(\Omega)$ we even have

$$\begin{aligned}\langle f, \overline{\phi} e^{-\Psi}\rangle &= \int_\Omega f(z)\overline{\phi(z)} e^{-\Psi(z)} d\lambda(z) = -D(\overline{\delta\phi}) = -D'(\overline{\delta\phi}) \\ &= -\int_\Omega \overline{\delta\phi(z)} u(z) e^{-\Psi(z)} d\lambda(z) = -\int_\Omega \frac{\partial\left(\overline{\phi(z)} e^{-\Psi(z)}\right)}{\partial \overline{z}} u(z) d\lambda(z) \\ &= -\langle u, \frac{\partial\left(\overline{\phi} e^{-\Psi}\right)}{\partial \overline{z}}\rangle,\end{aligned}$$

(where $\langle ., .\rangle$ denotes the pairing of distributions) meaning that $\frac{\partial u}{\partial \overline{z}} = f$ in the sense of distributions. □

Before we can prove the general Hörmander L^2-estimates we have to provide some material about weak convergence. We need this type of convergence only for weighted L^2-spaces and hence, we only consider this special case. For more information see [E09, Kapitel VI, §5.4] or [He06, Kapitel IV, p.186–189].

Definition 4.3 (weak L^2-convergence)**.** *Let Ω be an open subset of $\mathbb{C}$ and $\Psi : \Omega \to [-\infty, \infty)$ a subharmonic weight. A sequence $\{f_n\}_{n=1}^\infty \subseteq L^2(\Omega, \Psi)$ converges weakly to $f \in L^2(\Omega, \Psi)$ if*

$$\int_\Omega f_n(z)\overline{g(z)} e^{-\Psi(z)} d\lambda(z) \to \int_\Omega f(z)\overline{g(z)} e^{-\Psi(z)} d\lambda(z),$$

as $n \to \infty$, for every $g \in L^2(\Omega, \Psi)$.

It's easy to see that every sequence converging in $(L^2, \|.\|_{L^2})$ is weakly convergent and the limits coincide, but a weakly convergent sequence may not be convergent in $(L^2, \|.\|_{L^2})$.

Remark 4.A. The proof of Hörmander's theorem requires the following results:

(i) Weak limits are unique.

Proof. This can be seen by contradiction. □

(ii) Every bounded sequence in $L^2(\Omega, \Psi)$ has a weak convergent subsequence.

Proof. See [He06, p.187]. The claim can be shown by using a diagonal argument, the Bolzano-Weierstraß theorem in one dimension and the Hilbert space structure of L^2. □

(iii) If the sequence $\{f_n\}_{n=1}^\infty \subseteq L^2(\Omega, \Psi)$ converges weakly to $f \in L^2(\Omega, \Psi)$, then

$$\|f\|_{L^2(\Omega,\Psi)} \leq \liminf_{n\to\infty} \|f_n\|_{L^2(\Omega,\Psi)}.$$

Proof. By Cauchy-Schwarz inequality we have

$$\left| \int_\Omega f_n(z)\overline{f(z)}e^{-\Psi(z)}d\lambda(z) \right| \leq \|f_n\|_{L^2(\Omega,\Psi)}\|f\|_{L^2(\Omega,\Psi)}$$

and by definition

$$\int_\Omega f_n(z)\overline{f(z)}e^{-\Psi(z)}d\lambda(z) \to \|f\|^2_{L^2(\Omega,\Psi)}.$$

Thus, the claim follows immediately. □

(iv) If the weight function Ψ is continuous, then weak convergence implies convergence in the sense of distributions.

Proof. By assumption $e^{-\Psi}$ is locally integrable. Thus, $C_0^\infty(\Omega) \subseteq L^2(\Omega, \Psi)$. □

The general Hörmander L^2-estimates follow by the smoothing Lemma 2.11.

Theorem 4.4 (Hörmander)**.** *Let Ψ be a subharmonic function on Ω. For $f \in L^2(\Omega, \Psi)$ there exists a function $u \in L^2(\Omega, loc)$ such that $\frac{\partial u}{\partial \bar{z}} = f$ and*

$$2\int_\Omega |u(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^2} d\lambda(z) \leq \|f\|^2_{L^2(\Omega,\Psi)}.$$

Proof. At first we assume that $\Psi \in C_0^\infty(\Omega)$. We put $\Phi(z) := \Psi(z) + 2\log(1+|z|^2)$ and note that

$$\Delta\Phi(z) \geq 2\Delta \log(1+|z|^2) = \frac{8}{(1+|z|^2)^2} > 0.$$

Hence, we have

$$\frac{e^{-\Phi(z)}}{\Delta\Phi(z)} \leq \frac{e^{-\Phi(z)}(1+|z|^2)^2}{8} = \frac{e^{-\Psi(z)}}{8}.$$

We can apply Lemma 4.2 to the weight Φ and get the claim.
Now we consider the case $\Psi \notin C_0^\infty(\Omega)$. Take a sequence of relatively compact sets Ω_n, increasing to Ω. By Lemma 2.11, we can find for every $n \in \mathbb{N}$ a smooth subharmonic function Ψ_n on Ω_n with $\Psi_n \searrow \Psi$ on Ω. By the previous step we can find a solution u_n such that

$$2\int_{\Omega_n} |u_n(z)|^2 \frac{e^{-\Psi_n(z)}}{(1+|z|^2)^2} d\lambda(z) \leq \|f\|_{L^2(\Omega,\Psi)}, \tag{4.4}$$

for every $n \in \mathbb{N}$. Put $\Phi_n(z) := \Psi_n(z) + 2\log(1+|z|^2)$. By Remark 4.A (*ii*) we can find a subsequence $u^{(n)}_{n_k}$ converging weakly to $u^{(n)}$ in $L^2(\Omega_n, \Phi_n)$ for every n. Actually we know by smoothness of Φ_n that $u^{(n)} \in L^2(\Omega, \mathrm{loc})$. Since weak limits are unique and $\Phi_{n+1} \leq \Phi_n$, as well as $\Omega_{1/n} \nearrow \Omega$ we can define a function $u \in L^2(\Omega, \mathrm{loc})$, s.t. $u^{(n)}_{n_k}$ converges weakly to u in $L^2(\Omega_n, \Phi_n)$ (define $u(z) := u^{(n)}$ on Ω_n and note that by the uniqueness of weak limits this function is well defined in $L^2(\Omega, \mathrm{loc})$). Remark 4.A (*iv*) tells us that $\frac{\partial u}{\partial \bar{z}} = f$ on Ω. Finally inequality (4.4) and Remark 4.A (*iii*) imply

$$2\|u\|^2_{L^2(\Omega_{1/n},\Phi_n)} \leq \|f\|_{L^2(\Omega,\Psi)}.$$

Letting $n \to \infty$, the claim follows by the monotone convergence theorem. □

Remark 4.B. The previous theorem gives only a solution to the inhomogeneous Cauchy-Riemann-equation in the sense of distribution. Nevertheless, it can be used to find holomorphic functions. Since the Cauchy-Riemann operator has a fundamental solution with singular support equal to $\{0\}$ (see Corollary 1.14) we can apply Corollary 1.19 to see that every distributional solution to $\frac{\partial}{\partial \bar{z}}$ with smooth right hand side is in fact a classical solution.
In example, if we have found a distribution satisfying $\frac{\partial f}{\partial \bar{z}} = \frac{\partial \phi}{\partial \bar{z}}$ on $\mathbb{C}$, where ϕ is some known C^∞ function, then there exists a holomorphic version of $f - \phi$. Hence, for this operator every weak solution can be considered to be a classical solution.

4.2 Hörmander's theorem in several variables

The following theorems and proofs can be found in [Hö90/I, ch. 4] and in [H12, ch. 7]. We want to write them down only in the case of $(0,1)$-forms, but they hold true for (p,q)-forms as well. More information can be found in [H12].
In several variables the following theorem replaces Lemma 4.2.

Theorem 4.5. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and $\Psi : \Omega \to \mathbb{R}$ be a real valued function in $C^2(\Omega)$ such that there exists a positive continuous function c on Ω satisfying*

$$c(z)|w|^2 = c(z)\sum_{k=1}^{n} |w_k|^2 \leq \sum_{k,l=1}^{n} \frac{\partial^2 \Psi(z)}{\partial z_l \partial \overline{z_k}} w_l \overline{w_k},$$

for $z \in \Omega$ and $w \in \mathbb{C}^n$. If $g \in L^2_{0,1}(\Omega, \Psi)$ and $\overline{\partial} g = 0$, it follows that one can find $f \in L^2(\Omega, \Psi)$ with $\overline{\partial} f = g$ and

$$\int_\Omega |f|^2 e^{-\Psi} d\lambda \leq 2 \int_\Omega |g|^2 \frac{e^{-\Psi}}{c} d\lambda, \tag{4.5}$$

provided that the right hand side is finite.

Proof. A proof can be found in [H12, p.57–58]. Similar to the proof of Lemma 4.2 one requires the Hahn-Banach theorem 4.1 and the inequality of Lemma 4.2 is replaced by the so-called basic estimates. □

Our next aim is to show a very important theorem, which guarantees existence of a solution to $\overline{\partial}$-equation satisfying some special L^2-inequality. The proof is quite similar to the one-dimensional case. Weak convergence is defined as in Definition 4.3, but now we allow Ω to be a subset of $\mathbb{C}^n$. Remark 4.A still holds true for this generalization of weak convergence.

Theorem 4.6 (Hörmander)**.** *Let Ω be a smoothly bounded pseudoconvex set in $\mathbb{C}^n$ and Ψ any plurisubharmonic function in Ω. For every $g \in L^2_{0,1}(\Omega, \Psi)$ with $\overline{\partial} g = 0$ there is a solution $u \in L^2(\Omega, loc)$ of the equation $\overline{\partial} u = g$ such that*

$$\int_\Omega |u(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^2} d\lambda(z) \leq \int_\Omega |g(z)|^2 e^{-\Psi(z)} d\lambda(z). \tag{4.6}$$

Proof. We split the proof into two steps. In the first step we assume that Ψ satisfies a special regularity condition. In the second step we use a approximation argument to prove the general result.
$\underline{\Psi \in C^2(\Omega)}$: We can apply the previous Theorem 4.5 to the function $\Psi(z) + 2\log(1+|z|^2)$, because

$$\sum_{j,k=1}^{n} w_j \overline{w_k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} 2\log(1+|z|^2) = \frac{|w|^2(1+|z|^2 - |\sum_{j=1}^{n} w_j \overline{z_j}|^2)}{(1+|z|^2)^2} \geq \frac{2|w|^2}{(1+|z|^2)^2}.$$

Put $c(z) := \frac{2}{(1+|z|^2)^2}$ in Theorem 4.5 and therefore, gain a function $u \in L^2(\Omega, \mathrm{loc})$ such that $\overline{\partial} u = g$ and (4.6) is satisfied.
$\underline{\Psi \notin C^2(\Omega)}$: By assumption there exists some plurisubharmonic function $s \in C^\infty(\Omega)$ such that

$$\Omega_r := \{z \in \Omega : s(z) < r\}$$

relatively compact in Ω, for any $r \in \mathbb{R}$. By definition the open sets Ω_r are smoothly bounded and pseudoconvex. We apply Theorem 2.60 to obtain plurisubharmonic functions Ψ_ε defined in $\Omega_{r(\varepsilon)}$ such that $\Psi_\varepsilon \searrow \Psi$ and $r(\varepsilon) \to \infty$ if $\varepsilon \to 0$. The previous step gives us a function $u_\varepsilon \in L^2(\Omega_{r(\varepsilon)}, \Psi_\varepsilon)$ such that $\overline{\partial} u_\varepsilon = g$ in $\Omega_{r(\varepsilon)}$ and

$$\int_{\Omega_{r(\varepsilon)}} |u_\varepsilon(z)|^2 \frac{e^{-\Psi_\varepsilon(z)}}{(1+|z|^2)^2} d\lambda(z) \leq \int_{\Omega_{r(\varepsilon)}} |g(z)|^2 e^{-\Psi_\varepsilon(z)} d\lambda(z) \leq \int_\Omega |g(z)|^2 e^{-\Psi(z)} d\lambda(z).$$

Since $\Psi_\varepsilon \searrow \Psi$ as $\varepsilon \searrow 0$ we can find a sufficiently small $\varepsilon_0 > 0$ such that for arbitrary $r \in \mathbb{R}$ the following holds true

$$\int_{\Omega_r} |u_\varepsilon(z)|^2 \frac{e^{-\Psi_{\varepsilon_0}(z)}}{(1+|z|^2)^2} d\lambda(z) \leq \int_\Omega |g(z)|^2 e^{-\Psi(z)} d\lambda(z).$$

Using Remark 4.A we obtain some subsequence u_{ε_j} converging weakly to some $u \in L^2(\Omega, \mathrm{loc})$. Therefore, as $\varepsilon_0 \searrow 0$ we have by monotone convergence theorem that

$$\int_\Omega |u(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^2} d\lambda(z) \leq \int_\Omega |g(z)|^2 e^{-\Psi(z)} d\lambda(z).$$

Since $\overline{\partial}$ is a continuous operator on the space of distributions, we even have $\overline{\partial} u = g$. $\square$

4.3 Applications of Hörmander's theorem

The next theorem will teach us how to apply Hörmander's theorem. All of the following theorems and lemmas can be found in [Hö90/I, ch. 4].

Theorem 4.7. *Let Ψ be a plurisubharmonic function in the pseudoconvex open set $\Omega \subseteq \mathbb{C}^n$. If $z^0 \in \Omega$ and $e^{-\Psi}$ is integrable in a neighborhood of z^0 then one can find a holomorphic function u in Ω such that $u(z^0) = 1$ and*

$$\int_\Omega |u(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^{3n}} d\lambda(z) < \infty.$$

Proof. We follow Hörmander's lines.
W.l.o.g. we can assume that $z^0 = 0$ and by assumption we can find a polydisc

$$D := \{z \in \Omega : |z_j| < r, j = 1, \ldots, n\},$$

where $e^{-\Psi}$ is integrable. For $k = 0, 1, \ldots, n$ we define the sets

$$\Omega_k := \{z \in \Omega : |z_j| < r \text{ for } j > k\}.$$

At this point we note that $\Omega_0 = D, \Omega_n = \Omega$ and $\Omega_k \subseteq \Omega_{k+1}$.
Claim: *For every $k = 0, 1, \ldots, n$ there exists a holomorphic function u_k in Ω_k such that $u_k(0) = 1$ and*

$$\int_{\Omega_k} |u_k(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^{3k}} d\lambda(z) < \infty. \tag{4.7}$$

We prove this claim by induction.
If $k = 0$, the function $u_0 \equiv 1$ has the desired properties.
Assume that $0 < k \leq n$ and u_{k-1} has already been constructed. Choose a function $\phi \in C_0^\infty(\mathbb{C})$ such that supp$\phi \subseteq D(0, \frac{r}{2})$, the restriction of ϕ to $D(z_k, \frac{r}{3})$ is equal to 1 and $0 \leq \phi \leq 1$. Define

$$u_k(z) := \phi(z_k) u_{k-1}(z) - z_k v(z),$$

where v is to be chosen in such a way that u_k is holomorphic. Thus, we must prove existence of v and therefore, have to solve

$$\overline{\partial} v = \frac{u_{k-1} \overline{\partial} \phi}{z_k} =: f.$$

Note that $u_{k-1}(z) \overline{\partial} \phi(z_k) = 0$ on $\Omega_k \backslash \Omega_{k-1}$ and hence, by induction hypothesis

$$\int_{\Omega_k} |f(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^{3(k-1)}} d\lambda(z) < \infty$$

is satisfied. We can apply Theorem 4.6 to the plurisubharmonic function $\Psi(z) + 3(k-1)\log(1+|z|^2)$ and gain a function v satisfying

$$\int_{\Omega_k} |v(z)|^2 e^{-\Psi(z)} (1+|z|^2)^{3k-1} d\lambda(z) < \infty.$$

By definition of u_k and induction hypothesis this shows that (4.7) holds true.
The construction of u_n even shows that: $u_n(0) = u_{n-1}(0) = \cdots = u_0(0) = 1$. This proves the theorem. □

We can use Theorem 4.6 once more to prove the following theorem (see [Hö90/I, p.99]).

Theorem 4.8. *Let Ψ be a plurisubharmonic function in $\mathbb{C}^n$ and define*

$$A_\Psi \coloneqq \{u \in \mathcal{H}(\mathbb{C}) : \exists N \in \mathbb{N} \ s.t. \ \int_{\mathbb{C}} |u(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^N} d\lambda(z) < \infty\}.$$

Then the closure - w.r.t. to the topology of uniform convergence on compact subsets - of A_Ψ in $\mathcal{H}(\mathbb{C})$ contains all $u \in \mathcal{H}(\mathbb{C})$ such that $|u|^2 e^{-\Psi}$ is locally integrable and it is equal to $\mathcal{H}(\mathbb{C})$ iff $e^{-\psi}$ is locally integrable.

Proof. Let h be an entire function such that $|h|^2 e^{-\Psi}$ is locally integrable. Hence, we must approximate h in $D(0, R)$ for arbitrary $R > 0$ by functions in A_Ψ.
For this purpose, choose a $\phi \in C_0^\infty(\mathbb{C}^n)$ such that $\phi|_{D(0,R+1)} = 1$ and define $k \coloneqq \phi h$. For every $|z| < R + 1$ we have that

$$f(z) \coloneqq \overline{\partial} k(z) = h(z) \overline{\partial} \phi(z) = 0,$$

where we have used, that h is an entire function. By construction we gain that f has compact support. Let

$$\Psi_t(z) \coloneqq \Psi(z) + \max\left(0, t \log\left(\frac{|z|}{(R+1)}\right)\right)$$

and note that these functions are plurisubharmonic (see Example 2.M) and $e^{-\Psi_t} \to e^{-\Psi}$ pointwise as $t \to \infty$. By monotone convergence theorem we have

$$\int_{\mathbb{C}} |f(z)|^2 e^{-\Psi_t(z)} d\lambda(z) \to 0,$$

as $t \to \infty$. We apply Theorem 4.6 to get functions g_t satisfying $\overline{\partial} g_t = f$ and

$$\int_{\mathbb{C}} |g_t(z)|^2 \frac{e^{-\Psi_t(z)}}{(1+|z|^2)^2} d\lambda(z) \to 0 \tag{4.8}$$

as $t \to \infty$. By construction we even have that $\overline{\partial} g_t(z) = 0$ if $|z| < R + 1$ and

$$\int_{D(0,R+1)} |g_t(z)|^2 d\lambda(z) \le C \int_{\mathbb{C}} |g_t(z)|^2 \frac{e^{-\Psi_t(z)}}{(1+|z|^2)^2} d\lambda(z) \tag{4.9}$$

for a positive constant C, depending on R. Combining (4.8) and (4.9) gives

$$\int_{D(0,R+1)} |g_t(z)|^2 d\lambda(z) \to 0$$

as $t \to \infty$. Theorem 1.4 gives that $g_t(z) \to 0$ uniformly on compact subsets in $D(0, R)$ as $t \to \infty$. By construction we even have

$$\int_{\mathbb{C}} |g_t(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^{2+t/2}} d\lambda(z) =$$
$$= \int_{D(0,R+1)} |g_t(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^{2+t/2}} d\lambda(z) + \int_{\mathbb{C}\setminus D(0,R+1)} |g_t(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^{2+t/2}} d\lambda(z)$$

$$
\begin{aligned}
&\leq \int_{D(0,R+1)} |g_t(z)|^2 \frac{e^{-\Psi(z)}}{(1+|z|^2)^2} d\lambda(z) + \int_{\mathbb{C}\setminus D(0,R+1)} |g_t(z)|^2 \frac{(R+1)^t e^{-\Psi(z)}}{|z|^t (1+|z|^2)^2} d\lambda(z) \\
&= \int_{D(0,R+1)} |g_t(z)|^2 \frac{e^{-\Psi_t(z)}}{(1+|z|^2)^2} d\lambda(z) + \int_{\mathbb{C}\setminus D(0,R+1)} |g_t(z)|^2 \frac{e^{-\Psi_t(z)}}{(1+|z|^2)^2} d\lambda(z) \\
&= \int_{\mathbb{C}} |g_t(z)|^2 \frac{e^{-\Psi_t(z)}}{(1+|z|^2)^2} d\lambda(z) < \infty,
\end{aligned}
$$

thus, setting $N \coloneqq 2 + t/2$ we see that the functions $k - g_t$ lie in A_Ψ. Hence the first claim holds true.
The second claim can be easily seen from the first one: If $e^{-\Psi}$ is locally integrable, then by continuity also $|u|^2 e^{-\Psi}$ is locally integrable for any $u \in \mathcal{H}(\mathbb{C})$. The converse holds true since $1 \in \mathcal{H}(\mathbb{C})$. $\square$

In the previous theorems we have seen how to prove Hörmander's L^2-estimate. We have applied it in several theorems to make some 'arbitrary' function holomorphic in such a way, that special properties are satisfied. Exactly the same argument will be seen in the following chapter (especially in the proof of Theorem 5.1) another time.

5 Under which conditions is the weighted Bergman space of infinite dimension?

In this chapter we derive sufficient conditions under which the weighted Bergman space $A^2(\mathbb{C}^n, \Psi)$ is of infinite dimension. In Section 5.1 we focus on the case $n = 1$ and prove a theorem due to Rozenblum and Shirokov, and establish a condition on the weight function under which the Bergman space is finite dimensional. In Section 5.2 we have a look at the higher dimensional case $n \geq 1$ and discuss a condition given by Shigekawa. In the higher dimensional case, the discussed condition is not necessary for $A^2(\mathbb{C}^n, \Psi)$ to be infinite dimensional. We compare the previous conditions in the last Section 5.3 and give some examples, where the weighted Bergman space is of infinite dimension, but Shigekawa's condition is not satisfied.

5.1 Weighted Bergman space in one variable

In this section we follow the lines of Rozenblum and Shirokov (see [RS06, Thm. 3.2, p.144]) to prove the following theorem:

Theorem 5.1 (Rozenblum, Shirokov)**.** *If $\Psi \in C^2(\mathbb{C}, \mathbb{R})$ is a subharmonic function, such that*

$$\int_{\mathbb{C}} \Delta\Psi(z) d\lambda(z) = \infty, \tag{RS}$$

then $A^2(\mathbb{C}, \Psi)$ is infinite dimensional.

At the end of this section we give a sufficient condition for $A^2(\mathbb{C}, \Psi)$ to be finite dimensional.
The proof of Theorem 5.1 is quite long and therefore, we prepare some technical tools in Section 5.1.1 first.

5.1.1 First preparations

We provide some useful inequalities. These results are used later on in the proof of Theorem 5.1.

Lemma 5.2. *Let $0 < R_1 < R_2$. There exists a negative number $M < 0$ such that for every $w \in D(0, R_1)$ and every $z \in \mathbb{C} \backslash D(0, R_2)$ the following inequality holds true*

$$\log |z - w| \geq M + \log |z|. \tag{5.1}$$

Proof. Let $w \in D(0, R_1)$ and $z \in \mathbb{C} \backslash D(0, R_2)$ be arbitrary. Since $z \neq 0$, we have

$$\log |z - w| \geq \log \left(\frac{|z| - |w|}{|z|} |z| \right)$$

$$\geq \log\left(1 - \frac{R_1}{R_2}\right) + \log|z|.$$

Therefore, the claim holds true for $M \coloneqq \log\left(1 - \frac{R_1}{R_2}\right)$. ☐

We will need another inequality at the end of Section 5.1.2.

Lemma 5.3. *If $|z| \geq \sqrt{\sqrt{2}+1}$ we have*

$$\frac{1}{|z|^4} \leq \frac{2}{(1+|z|^2)^2}. \tag{5.2}$$

Proof. The inequality follows directly by using some basic algebraic manipulation. ☐

After providing all these technical lemmas, we can start to prove Theorem 5.1.

5.1.2 Proof of Theorem 5.1

The main idea of the proof is to construct an arbitrary number of functions, $f_1, \ldots, f_n$, say, contained in $A^2(\mathbb{C}, \Psi)$, in such a way, that there exist n points $z_1, \ldots, z_n$ satisfying $f_j(z_k) = 0$ for $k < j$ and $f_j(z_j) = 1$. In fact, if they weren't linearly independent, there would be numbers $a_1, \ldots, a_{n-1}$ in $\mathbb{C}$ such that

$$a_1 f_1 + a_2 f_2 + \cdots + a_{n-1} f_{n-1} = f_n.$$

Inserting z_1 gives $a_1 = 0$. Repeating this argument for $z_2, \ldots, z_{n-1}$ shows that

$$a_1 = a_2 = \cdots = a_{n-1} = 0.$$

Therefore, the constructed functions will be linearly independent.

Remark 5.A. As we will see at the end of the proof, it suffices to show existence of one of the functions, f_n say. Existence for the other functions will follow in a similar way. The strategy to find f_n is as follows: At first we have to make a clever choice of the points $z_1, \ldots, z_n$. We consider a test function ϕ such that $\phi(z_k) = 0$ for $k = 1, \ldots, n-1$ and $\phi = 1$ in a small neighborhood of z_n. Hörmander's theorem 4.6 applied to the function $\frac{\partial \phi}{\partial \bar{z}}$ and slightly different weights U^h (compared to Ψ), and Montel's theorem 1.7 will tell us how to change the function ϕ in such a way that we get the desired function f_n.

The following proof was given by Rozenblum and Shirokov in 2006. We exactly follow their lines.

Proof of Theorem 5.1. . We split the proof into five steps.
Step 1: *In the first step we only provide some notation and definitions of sets and measures. The complex numbers $z_1, \ldots, z_n$, positive numbers $R_1, \ldots, R_n, \delta$ and subharmonic functions $\psi_1, U_k, U'_{k,h}$ for $k = 1, \ldots, n$ and $0 < h \leq \delta/2$ will be introduced as well.*

Let $n > 0$ be a number in $\mathbb{N}$. The locally finite Borel measure induced by $\Delta\Psi$ is denoted by μ. Thus, by (RS) we can find a number $R_1 > \frac{1}{2}\sqrt{\sqrt{2}+1}$ such that

$$\infty > M_0 \coloneqq \mu(D(0, R_1)) \geq 46,$$

and numbers $R_k > R_{k-1}$ for $k = 2, \ldots, n$ in such a way that

$$\mu(\Omega_k) \geq 20,$$

where $\Omega_k := D(0, R_k) \backslash D(0, R_{k-1})$ for $k = 2, \ldots, n$ and $\Omega_1 := D(0, R_1)$.
Now we define some compactly supported Borel measures on $\mathbb{C}$ and have a look at the associated logarithmic potentials. This approach will give us subharmonic functions, to which we apply Hörmander's theorem.
Define the first measure by

$$\mu_0(.) := \frac{26}{M_0}\mu(. \cap \Omega_1),$$

and note that $\mu_0 \leq \mu$ since $\frac{26}{M_0} < 1$. The logarithmic potential of μ_0 is defined as

$$\Psi_0(z) := \mathcal{P}_{\mu_0}(z) = \frac{1}{2\pi}\int_{\Omega_1} \log|z - w| d\mu_0(w)$$

and therefore $\Delta\Psi_0 = \mu_0$ is satisfied, since μ_0 is compactly supported (see Corollary 2.22). We also have some information about the asymptotic behavior of Ψ_0 at infinity. In particular by dominated convergence theorem we get

$$\Psi_0(z) \sim \frac{13}{\pi}\log|z|$$

for $|z|$ large.
Let $\Psi_1(z) := \Psi(z) - \Psi_0(z)$ and define $\mu_1 := \Delta\Psi_1 = \Delta\Psi - \Delta\Psi_0 = \mu - \mu_0 \geq 0$, showing that Ψ_1 is again subharmonic and

$$\infty > M_k := \mu_1(\Omega_k) = \mu(\Omega_k) - \mu_0(\Omega_k) \geq 20,$$

for $k = 1, \ldots, n$.
Similarly to the construction of μ_0 we define

$$\nu_k(.) := \frac{20}{M_k}\mu_1(. \cap \Omega_k), \text{ for } k = 1, \ldots, n.$$

These measures satisfy

$$\nu_k \leq \mu_1$$

and arguing as above, we obtain that the logarithmic potentials $U_k := \mathcal{P}_{\nu_k}$ of ν_k have asymptotic growth rate $U_k \sim \frac{10}{\pi}\log|z|$ as $|z| \to \infty$. Since U_k is finite almost everywhere w.r.t. Lebesgue measure, we can take numbers $z_k \in \Omega_k^\circ$ for $k = 1, \ldots, n$ such that $|U_k(z_k)| < \infty$. Therefore, by construction we have $z_k \neq z_l$ if $k \neq l$. We can also find some small $\delta > 0$ such that

$$D(z_k, \delta) \subseteq \Omega_k^\circ,$$

for $k = 1, \ldots, n$.

To prove the stated theorem we need another family of measures depending on δ. Let $0 < h \leq \frac{\delta}{2}$ and define

$$\nu'_{k,h}(.) := \frac{20}{\pi h^2} \int_{\mathbb{C}} \mathbb{1}_{\,.\,\cap D(z_k,h)}(w) d\lambda(w).$$

By definition we have

$$\nu_k(\Omega_k) = \nu'_{k,h}(\Omega_k) = 20. \tag{5.3}$$

We denote the logarithmic potentials of $\nu'_{k,h}$ by $U'_{k,h} := \mathcal{P}_{\nu'_{k,h}}$ and note that

$$U'_{k,h}(z) \sim \frac{10}{\pi} \log |z| \sim U_k(z) \tag{5.4}$$

for $|z|$ large.

Step 2: *For every* $0 < h \leq \delta/2$ *the function*

$$U^h(z) := \Psi_1(z) + \sum_{k=1}^{n} (U'_{k,h}(z) - U_k(z))$$

is subharmonic and satisfies the following inequality

$$|U^h(z) - \Psi_1(z)| \leq \frac{20 \log 2}{\pi} n, \ \text{if } |z| \geq 2R_n, \tag{5.5}$$

and there is a constant $C_1(\delta, R_n)$*, independent of* h*, such that,*

$$U^h(z) - \Psi_1(z) \geq -2nC_1(\delta, R_n), \ \text{for } |z| \leq 2R_n \text{ and } z \notin \bigcup_{k=1}^{n} D(z_k, \delta). \tag{5.6}$$

Remark 5.B. As suggested in Remark 5.A, this step gives us a family of weight functions U^h. In our situation these functions differ from the original weight function Ψ only in a harmless way. For large z this can be expected by statement (5.4). Actually, U^h is comparable to the weight function $\Psi_1 = \Psi - \Psi_0$, but (since we want to apply Hörmander's theorem only to compactly supported functions) by Lemma 2.5 this doesn't matter. The lack of knowledge on the behavior of the functions $U^h - \Psi_1$ in the discs $D(z_k, \delta)$ is concealed by a special choice of test functions.
An important part of this step is, that all of the appearing bounds for the difference $U^h - \Psi_1$ do not depend on the small parameter h. This allows us to apply Montel's theorem 1.7 in *Step 4*.

A straight-forward computation shows that U^h is again subharmonic. The following argument illustrates that the function U^h satisfies the inequality (5.5) as well. In fact, for $z \neq 0$ we have

$$\begin{aligned} U'_{k,h}(z) - U_k(z) &= \frac{1}{2\pi} \int_{\Omega_k} \log |z - w| d(\nu'_{k,h} - \nu_k)(w) \\ &= \frac{1}{2\pi} \int_{\Omega_k} \log |z| d(\nu'_{k,h} - \nu_k)(w) + \frac{1}{2\pi} \int_{\Omega_k} \log |1 - \frac{w}{z}| d(\nu'_{k,h} - \nu_k)(w) \end{aligned}$$

$$\underset{(5.3)}{=} \frac{1}{2\pi} \int_{\Omega_k} \log|1 - \frac{w}{z}| d(\nu'_{k,h} - \nu_k)(w).$$

Since for $|w| \leq R_n$ and $|z| \geq 2R_n$ the integrand satisfies

$$\begin{aligned} -\log 2 = \log(1 - \frac{1}{2}) &\leq \log|1 - |\frac{w}{z}|| \\ &\leq \log|1 - \frac{w}{z}| \\ &\leq \log(1 + |\frac{w}{z}|) \leq \log(1 + \frac{R_n}{2R_n}) \leq \log 2, \end{aligned}$$

which means that $|\log|1 - \frac{w}{z}|| \leq \log 2$ and therefore,

$$\begin{aligned} |U'_{k,h}(z) - U_k(z)| &\leq \frac{1}{2\pi} \int_{\Omega_k} \left|\log|1 - \frac{w}{z}|\right| |d(\nu'_{k,h} - \nu_k)(w)| \\ &\leq \frac{\log 2}{2\pi} \int_{\Omega_k} d(\nu'_{k,h} + \nu_k)(w) = \frac{20 \log 2}{\pi}. \end{aligned}$$

Hence, for $|z| \geq 2R_n$ we have

$$|U^h(z) - \Psi_1(z)| \leq \frac{20 \log 2}{\pi} n.$$

The next argument will give us a lower bound for $U^h(z) - \Psi_1(z)$ such that (5.6) is satisfied. Let $z \in \mathbb{C}$ such that $|z| \leq 2R_n$ and $z \notin D(z_k, \delta)$. Hence, for $w \in D(z_k, h)$ we get

$$\begin{aligned} |z - w| = |z - z_k + (z_k - w)| &\geq ||z - z_k| - |w - z_k|| \\ &\geq |z - z_k| - |w - z_k| > \delta - h \geq \frac{\delta}{2} > 0, \end{aligned} \tag{5.7}$$

and

$$|z - w| \leq |z| + |w| \leq 2R_n + R_n = 3R_n. \tag{5.8}$$

After combining the inequalities (5.7) and (5.8) and using monotonicity of log we get

$$\log \frac{\delta}{2} \leq \log|z - w| \leq \log(3R_n),$$

which in fact gives

$$|\log|z - w|| \leq \max(|\log(\frac{\delta}{2})|, |\log(3R_n)|) =: C_0(\delta, R_n).$$

One can apply the previous inequality to get

$$\begin{aligned} |U'_{k,h}(z)| &\leq \frac{1}{2\pi} \int_{D(z_k,h)} \frac{20}{h^2\pi} |\log|z - w|| \, d\lambda(w) \\ &\leq C_0(\delta, R_n) \frac{10}{\pi} =: C_1(\delta, R_n), \end{aligned} \tag{5.9}$$

if $|z| \leq 2R_n$ and $z \notin D(z_k, h)$. Another consequence of inequality (5.8), where in fact we have only used that $w \in \Omega_k$, is the inequality

$$U_k(z) = \frac{1}{2\pi} \int_{\Omega_k} \log|z - w| d\nu_k(w) \leq \frac{20}{2\pi} \log(3R_n) \leq C_1(\delta, R_n), \tag{5.10}$$

if $|z| \leq 2R_n$. Combining the two inequalities (5.9) and (5.10), shows that for $|z| \leq 2R_n$ and $z \notin \bigcup_{k=1}^n D(z_k, \delta)$ one can establish the lower bound

$$U^h(z) - \Psi_1(z) = \sum_{k=1}^n (U'_{k,h}(z) - U_k(z)) \geq -nC_1(\delta, R_n) - nC_1(\delta, R_n) = -2nC_1(\delta, R_n),$$

where the constant $C_1(\delta, R_n)$ depends on δ and R_n, but not on h.

Step 3: *Let $\phi \in C_0^\infty(\mathbb{C})$ such that $\phi|_{D(z_k,\delta)} = 0$ for $k < n$ and $\phi|_{D(z_n,\delta)} = 1$, as well as supp$\phi \subseteq D(0, 2R_n)$ is satisfied. For every $0 < h \leq \delta/2$ there exists a function $g_h : \mathbb{C} \to \mathbb{C}$ such that $\frac{\partial g_h}{\partial \bar{z}} = \frac{\partial \phi}{\partial \bar{z}}$ and a finite constant $K > 0$, independent of h, such that the inequalities*

$$\int_{\mathbb{C}} |g_h(z)|^2 \frac{e^{-U^h(z)}}{(1 + |z|^2)^2} d\lambda(z) \leq K, \tag{5.11}$$

$$\int_{R \leq |z| \leq 2R} |g_h(z)|^2 e^{-\Psi_1(z)} d\lambda(z) \leq 2^{20n/\pi}(1 + (2R)^2)^2 K, \ \textit{for } R \geq 2R_n, \tag{5.12}$$

and

$$\int_{|z| \geq 2R_n} |g_h(z)|^2 \frac{e^{-\Psi_1(z)}}{(1 + |z|^2)^2} d\lambda(z) \leq 2^{20n/\pi} K \tag{5.13}$$

are satisfied. Thus the integrals can be estimated from above by a constant, independent of h.

As we have already seen, U^h is a subharmonic function and hence we want to apply Hörmander's theorem (i.e. Theorem 4.6) to solve the Cauchy-Riemann equation with right hand side $\frac{\partial \phi}{\partial \bar{z}}$. We must guarantee that the function $\frac{\partial \phi}{\partial \bar{z}}$ is an element of $L^2(\mathbb{C}, U^h)$. Let $S := \text{supp}\phi \setminus \bigcup_{k=1}^n D(z_k, \delta)$, then we easily see by (5.6) that

$$\begin{aligned}
\int_{\mathbb{C}} |\frac{\partial \phi(z)}{\partial \bar{z}}|^2 e^{-U_h(z)} d\lambda(z) &= \int_S |\frac{\partial \phi(z)}{\partial \bar{z}}|^2 e^{-U_h(z)} d\lambda(z) \\
&\leq e^{2nC_1(\delta,R_n)} \int_S |\frac{\partial \phi(z)}{\partial \bar{z}}|^2 e^{-\Psi_1(z)} d\lambda(z) \\
&\leq e^{2nC_1(\delta,R_n)} \max_{z \in S} |\frac{\partial \phi(z)}{\partial \bar{z}}|^2 \int_S e^{-\Psi(z)} e^{\Psi_0(z)} d\lambda(z) =: K(\delta, n, R_n),
\end{aligned} \tag{5.14}$$

is satisfied. Because we are integrating over a compact set, we note that by subharmonicity of Ψ_0 and continuity of Ψ the last integral must be bounded and therefore,

$\frac{\partial\phi}{\partial\bar{z}} \in L^2(\mathbb{C}, U^h)$. For simplicity we write $K = K(\delta, n, R_n)$, which is a finite number independent of h. Hence, by Theorem 4.6 we gain a function g_h in $L^2(\mathbb{C}, \mathrm{loc})$ such that $\frac{\partial g_h}{\partial\bar{z}} = \frac{\partial\phi}{\partial\bar{z}}$ and

$$\int_{\mathbb{C}} |g_h(z)|^2 \frac{e^{-U^h(z)}}{(1+|z|^2)^2} d\lambda(z) \leq \int_{\mathbb{C}} |\frac{\partial\phi(z)}{\partial\bar{z}}|^2 e^{-U^h(z)} d\lambda(z) = \int_S |\frac{\partial\phi(z)}{\partial\bar{z}}|^2 e^{-U^h(z)} d\lambda(z),$$

where the last equality holds true by definition of ϕ and S. Combining the last line and inequality (5.14) gives

$$\int_{\mathbb{C}} |g_h(z)|^2 \frac{e^{-U^h(z)}}{(1+|z|^2)^2} d\lambda(z) \leq K(\delta, n, R_n),$$

a uniform upper bound in h, which is in fact inequality (5.11).
Inequality (5.5) shows that for every $R \geq 2R_n$ we have

$$\frac{1}{(1+(2R)^2)^2} \int_{R\leq|z|\leq 2R} |g_h(z)|^2 e^{-\Psi_1(z)} d\lambda(z) \leq e^{\frac{20\log 2}{\pi} n} \int_{\mathbb{C}} |g_h(z)|^2 \frac{e^{-U^h(z)}}{(1+|z|^2)^2} d\lambda(z),$$

which implies

$$\int_{R\leq|z|\leq 2R} |g_h(z)|^2 e^{-\Psi_1(z)} d\lambda(z) \leq 2^{20n/\pi} (1+(2R)^2)^2 K.$$

By using inequality (5.5) another time the following inequality holds true

$$\begin{aligned}\int_{|z|\geq 2R_n} |g_h(z)|^2 \frac{e^{-\Psi_1(z)}}{(1+|z|^2)^2} d\lambda(z) &\leq e^{\frac{20\log 2}{\pi} n} \int_{|z|\geq 2R_n} |g_h(z)|^2 \frac{e^{-U^h(z)}}{(1+|z|^2)^2} d\lambda(z) \\ &\leq 2^{20n/\pi} \int_{\mathbb{C}} |g_h(z)|^2 \frac{e^{-U^h(z)}}{(1+|z|^2)^2} d\lambda(z) \leq 2^{20n/\pi} K.\end{aligned}$$

Step 4: *There exists a function $f_n \in A^2(\mathbb{C}, \Psi)$ such that*

$$f_n(z_k) = \begin{cases} 0, & \text{if } k < n, \\ 1, & \text{if } k = n. \end{cases}$$

The previous arguments hold for every $0 < h \leq \frac{\delta}{2}$. Therefore, we can define a family of functions $f_h := \phi - g_h$. By definition of g_h we know that $\frac{\partial f_h}{\partial\bar{z}} = 0$ in the sense of distributions and hence, by Remark 4.B, w.l.o.g. we may assume that f_h is an entire function. If $R \geq 2R_n$ we even conclude, that by Cauchy-Schwarz inequality and our choice of suppϕ we have

$$\begin{aligned}\int_{R\leq|z|\leq 2R} |f_h| d\lambda &\leq \left(\int_{R\leq|z|\leq 2R} |f_h|^2 e^{-\Psi_1} d\lambda\right)^{1/2} \left(\int_{R\leq|z|\leq 2R} e^{\Psi_1} d\lambda\right)^{1/2} \\ &= \left(\int_{R\leq|z|\leq 2R} |g_h|^2 e^{-\Psi_1} d\lambda\right)^{1/2} \left(\int_{R\leq|z|\leq 2R} e^{\Psi_1} d\lambda\right)^{1/2},\end{aligned} \tag{5.15}$$

where the latter integral is finite since Ψ_1 is subharmonic and therefore has a maximum on compact sets. The first integral on the right hand side is finite as a consequence of (5.12). All occurring bounds do not depend on h.
It follows from (5.15), that the family $\{f_h\}_{0<h\leq\delta/2}$ is uniformly bounded in $L^1(D(0,2R)\backslash\overline{D(0,R)})$ on every annulus contained in $R<|z|<2R$. Thus, by Theorem 1.3 and Definition 1.6, the family is uniformly bounded in $\mathcal{H}(D(0,2R)\backslash\overline{D(0,R)})$ and by the maximum principle, Theorem 1.5, we even have that $\{f_h\}_{0<h\leq d/2}$ is uniformly bounded in $\mathcal{H}(D(0,2R))$. Since R can be chosen arbitrarily large, $\{f_h\}_{0<h\leq\delta/2}$ is uniformly bounded in $\mathcal{H}(\mathbb{C})$. Let us pick a sequence h_l s.t. $h_l\to 0$ as $l\to\infty$ and note that by Montel's theorem 1.7 we can extract a subsequence of $\{f_{h_l}\}_{l=1}^{\infty}$ (w.l.o.g. the sequence itself) s.t. f_{h_l} converges to a function f in $\mathcal{H}(\mathbb{C})$ w.r.t. the topology of uniform convergence on compact subsets of $\mathbb{C}$.
Since ϕ doesn't depend on h_l, we have

$$g_{h_l}=\phi-f_{h_l}\to\phi-f=:g$$

uniformly on compact sets for $l\to\infty$.
We are now going to show, that $g(z_k)=0$ for every $k=1,\dots,n$. As we have proved in Lemma 2.28, the potentials $U'_{k,h}(z)$ can be written in the form

$$U'_{k,h}(z)=\begin{cases}\frac{10}{\pi}\log h+\frac{5}{\pi h^2}(|z-z_k|^2-h^2), & z\in D(z_k,h),\\ \frac{10}{\pi}\log|z-z_k|, & z\notin D(z_k,h).\end{cases} \tag{5.16}$$

If we assume that $g(z_k)\neq 0$ there is some l_0 such that $|g_{h_l}(z)|>\varepsilon>0$ for every $l\geq l_0$ and $z\in D(z_k,h_{l_0})$, by uniform convergence. Using subharmonicity of Ψ_1 and the explicit formula of $U'_{j,h}$ (if $j\neq k$) we find a constant

$$\begin{aligned}C_2&:=\min_{z\in D(z_k,h_{l_0})}(e^{-\Psi_1}e^{-\sum_{j\neq k}U'_{j,h}})\\&\overset{(2.11)}{=}\min_{z\in D(z_k,h_{l_0})}\left(e^{-\Psi_1}\prod_{j\neq k}\frac{1}{|z-z_j|^{10/\pi}}\right)>0,\end{aligned} \tag{5.17}$$

independent of h. Using formula (5.16) another time, we easily see that for $z\in D(z_k,h)$

$$U'_{k,h}(z)\leq\frac{10}{\pi}\log h, \tag{5.18}$$

is satisfied and thus,

$$e^{-U'_{k,h}(z)}\geq h^{-10/\pi}, \tag{5.19}$$

if $z\in D(z_k,h)$. By definition of U_j, if $j\neq k$, we have for $z\in D(z_k,h)$ that

$$\begin{aligned}e^{\sum_{j\neq k}U_j(z)}&\geq e^{\sum_{j\neq k}\frac{10\log(\delta)}{\pi}}\\&=\delta^{10(n-1)/\pi}.\end{aligned} \tag{5.20}$$

We combine the inequalities (5.11), (5.17), (5.20) and (5.19) to obtain for $l \geq l_0$ that

$$\begin{aligned}
K(\delta, n, R_n) &\geq \int_{D(z_k,h_l)} |g_{h_l}(z)|^2 \frac{e^{-U^{h_l}(z)}}{(1+|z|^2)^2} d\lambda(z) \\
&= \int_{D(z_k,h_l)} |g_{h_l}(z)|^2 \frac{e^{-\Psi_1(z)} e^{-U'_{k,h_l}(z)} e^{U_k(z)} e^{-\sum_{j\neq k} U'_{j,h_l}(z)} e^{\sum_{j\neq k} U_j(z)}}{(1+|z|^2)^2} d\lambda(z) \\
&\geq C_2 \delta^{10(n-1)/\pi} \int_{D(z_k,h_l)} |g_{h_l}(z)|^2 \frac{e^{U_k(z)} e^{-U'_{k,h_l}(z)}}{(1+|z|^2)^2} d\lambda(z) \\
&\geq \frac{C_2 \delta^{10(n-1)/\pi} h_l^{-10/\pi}}{(1+R_n^2)^2} \int_{D(z_k,h_l)} |g_{h_l}(z)|^2 e^{U_k(z)} d\lambda(z) \\
&\geq \frac{C_2 \delta^{10(n-1)/\pi} h_l^{-10/\pi} \varepsilon^2}{(1+R_n^2)^2} \int_{D(z_k,h_l)} e^{U_k(z)} d\lambda(z) \\
&\geq \frac{C_2 \delta^{10(n-1)/\pi} h_l^{-10/\pi} \varepsilon^2 h_l^2 \pi}{(1+R_n^2)^2} e^{U_k(z_k)} = C_2' \frac{h_l^2}{h_l^{10/\pi}},
\end{aligned} \tag{5.21}$$

where we have used the submean value property for the subharmonic function e^{U_k} in the last estimate. Since $10/\pi > 2$ and because $C_2' > 0$ is independent of h_l the last term is unbounded as $h_l \to 0$. This is a contradiction to $K < \infty$ in equation (5.14).

Finally, since $g(z_k) = 0$, the function $f(z) = \phi(z) - g(z)$ satisfies

$$f(z_k) = \begin{cases} 0, & \text{if } k < n, \\ 1, & \text{if } k = n. \end{cases}$$

It remains to show that f lies in the space $A^2(\mathbb{C}, \Psi)$. By definition we have that $\Psi = \Psi_1 + \Psi_0$ and hence we can write

$$\int_{\mathbb{C}} |f(z)|^2 e^{-\Psi(z)} d\lambda(z) = \int_{\mathbb{C}} |f(z)|^2 e^{-\Psi_1(z) - \Psi_0(z)} d\lambda(z).$$

We can apply inequality (5.1) to get by the definition of Ψ_0

$$\begin{aligned}
e^{-\Psi_0(z)} &= \exp\left(-\frac{1}{2\pi}\int_{\Omega_1} \log|z-w| d\mu_0(w)\right) \\
&\overset{(5.1)}{\leq} C_3 \exp\left(-\frac{26}{2\pi M_0}\int_{\Omega_1} \Delta\Psi_0(w) dw \log|z|\right) \\
&= C_3 \exp(-13/\pi \log|z|) < \frac{C_3}{|z|^4}
\end{aligned} \tag{5.22}$$

for a constant $C_3 > 0$ and $|z| \geq 2R_n$. We split the $A^2(\mathbb{C}, \Psi)$-norm of f into two parts

$$\begin{aligned}
\int_{\mathbb{C}} |f(z)|^2 e^{-\Psi(z)} d\lambda(z) &= \int_{D(0,2R_n)} |f(z)|^2 e^{-\Psi(z)} d\lambda(z) + \int_{\mathbb{C}\setminus D(0,2R_n)} |f(z)|^2 e^{-\Psi(z)} d\lambda(z) \\
&=: I + J.
\end{aligned} \tag{5.23}$$

The first integral I is bounded since we are integrating over a compact set and the integrand itself is continuous. The second integral can be handled as follows

$$\begin{aligned}
J &\overset{(5.22)}{\leq} C_3 \int_{\mathbb{C}\setminus D(0,2R_n)} |f(z)|^2 e^{-\Psi_1(z)} \frac{1}{|z|^4} d\lambda(z) \\
&\overset{(5.2)}{\leq} 2C_3 \int_{\mathbb{C}\setminus D(0,2R_n)} |f(z)|^2 \frac{e^{-\Psi_1(z)}}{(1+|z|^2)^2} d\lambda(z) \\
&= 2C_3 \int_{\mathbb{C}\setminus D(0,2R_n)} |g(z)|^2 \frac{e^{-\Psi_1(z)}}{(1+|z|^2)^2} d\lambda(z) \\
&\overset{(5.13)}{\leq} 4C_3 2^{20n/\pi} K \\
&\overset{(5.14)}{<} \infty,
\end{aligned}$$

where we have used the particular choice of suppϕ again.
Therefore, the constructed function f lies in $A^2(\mathbb{C}, \Psi)$. Set $f = f_n$.

Step 5: *For arbitrary n in $\mathbb{N}$ there exist linearly independent functions $f_1, \ldots, f_n$ in $A^2(\mathbb{C}, \Psi)$. Therefore, Theorem 5.1 holds true.*

Since n was arbitrary in Step 1, we have already seen, how to construct the function f_n. The other functions f_k, for $k < n$, are constructed in the same way, except of a slight modification of Step 3. There one has to choose $\phi \in C_0^\infty(\mathbb{C})$ such that $\phi|_{D(z_j,\delta)} = 0$ for $j < k$ and $\phi|_{D(z_k,\delta)} = 1$. One can proceed as above to obtain the functions $f_1, \ldots, f_n$ such that $f_j(z_k) = 0$ if $j < k$ and $f_k(z_k) = 1$. As mentioned at the beginning of Section 5.1.2 they are linearly independent. □

Attentive readers may have noticed that one can change the assumption $\Psi \in C^2(\mathbb{C})$ and the claim still holds true. We have used the C^2 assumption only in the inequalities (5.14) and (5.23). Hence, it would be enough to assume that $e^{-\Psi}$ is locally integrable. Otherwise we would not be able to guarantee that $\frac{\partial\phi}{\partial\bar{z}}$ and f lie in $L^2(\Omega, \Psi)$. Unfortunately this assumption is not true for all subharmonic functions. For example the function $z \mapsto \log(|z|^2)$ is subharmonic on $\mathbb{C}$, but $e^{-\log|z|^2} = \frac{1}{|z|^2}$ is not locally integrable at zero.

Remark 5.C. One could obtain similar inequalities and subharmonic functions as in *Step 1* and *Step 2* for a finite measure which is not compactly supported in the plane, provided that the weight function is C^2. For this purpose one has to choose some sufficiently small constants (depending on n) instead of the constants $20, 26$ and 46 appearing in *Step 1*. Nevertheless, the theorem does not hold true for this more general situation. A counterexample would be the space $A^2(\mathbb{C}, \log(1+|z|^2)) = \mathbb{C}$ (see Example 3.4). The main problem is, that one also has to change the functions $U'_{k,h}$ in such a way that inequality (5.18) would only be satisfied for a constant smaller than 2 (instead of $10/\pi$) and therefore, we would not be able to derive at a contradiction with help of inequality (5.21).

5.1.3 Some notes

As we have seen in Lemma 3.4 there exist weighted Bergman spaces of finite dimension as well. We can apply this lemma to find a larger number of finite dimensional weighted Bergman spaces. Therefore, we adapt a method seen in Lemma 3.3.

For $x \in \mathbb{R}$ we denote the unique $m \in \mathbb{Z}$ s.t. $m-1 < x \leq m$ by $\lceil x \rceil$ and

$$\lceil x \rceil^+ := \max(0, \lceil x \rceil).$$

Lemma 5.4. *Let $R > 0$ and $c \geq 0$. The space*

$$\mathcal{T}_c := \mathcal{H}(\mathbb{C}) \cap \{f : \int_{\mathbb{C}\setminus D(0,R)} \frac{|f(z)|^2}{|z|^c} d\lambda(z) < \infty\}$$

has dimension $\lceil \frac{c}{2} - 1 \rceil^+$ and

$$\mathcal{T}_c = \begin{cases} span[1, z, \ldots, z^{\lceil \frac{1}{c} - 1 \rceil - 1}] & c > 2, \\ \{0\} & c \leq 2. \end{cases}$$

Proof. W.l.o.g. we assume that $R \geq 1$. The general result follows by continuity of the integrand.
Let $m \in \mathbb{N}$, then

$$\int_{\mathbb{C}\setminus D(0,R)} \frac{|z^m|^2}{|z|^c} d\lambda(z) < \infty$$

if and only if $m < \frac{c}{2} - 1$. Therefore, we have $\dim \mathcal{T}_c \geq \lceil \frac{c}{2} - 1 \rceil^+$.
Let $c > 2$. Hence, $\lceil \frac{c}{2} - 1 \rceil - 1 \geq 0$. If $f \in \mathcal{T}_c$, we can expand it into the following power series

$$f(z) = \sum_{k=0}^{\lceil \frac{c}{2} - 1 \rceil - 1} a_k z^k + \sum_{k=\lceil \frac{c}{2} - 1 \rceil}^{\infty} a_k z^k =: f_1(z) + f_2(z),$$

with $a_k \in \mathbb{C}$. The first step shows that $f_1 \in \mathcal{T}_c$ and thus, $f_2 \in \mathcal{T}_c$. We can write,

$$f_2(z) = z^{\lceil \frac{c}{2} - 1 \rceil} g(z),$$

for a $g \in \mathcal{H}(\mathbb{C})$. Since $2\lceil \frac{c}{2} - 1 \rceil - c \geq c - 2 - c = -2$, we have for $|z| \geq R$

$$|z|^{2\lceil \frac{c}{2} - 1 \rceil - c} \geq |z|^{-2} \geq \frac{1}{1 + |z|^2}.$$

Therefore, $g \in A^2(\mathbb{C}, \log(1 + |z|^2))$. By Lemma 3.4, we know that $g(z) = a_{\lceil \frac{c}{2} - 1 \rceil}$ and by the first step of the proof we even have that $f_2(z) = 0$.
Assume that $c \leq 2$ (equivalent to $\lceil \frac{c}{2} - 1 \rceil - 1 < 0$). Hence, we have for $|z| \geq R$, that $|z|^c \leq |z|^2$ and

$$\frac{1}{|z|^c} \geq \frac{1}{|z|^2} \geq \frac{1}{1 + |z|^2}.$$

Therefore, if $f \in \mathcal{T}_c$, we again have that $f \in A^2(\mathbb{C}, \log(1 + |z|^2))$ and so by Lemma 3.4 $f(z) = a$, for a $a \in \mathbb{C}$. By the first step of the proof we have that $f(z) = a = 0$. □

Theorem 5.5. *If Ψ is a subharmonic function such that*

$$\Psi(z) = 2\pi\mathcal{P}_\mu + h(z),$$

for a compactly supported positive measure μ with $\mu(\mathbb{C}) < \infty$ and a harmonic function $h : \mathbb{C} \to \mathbb{R}$. Then the space $A^2(\mathbb{C}, \Psi)$ is of finite dimension. Moreover,

$$dim\Big(A^2(\mathbb{C}, \Psi)\Big) \leq \lceil \frac{\mu(\mathbb{C})}{2} - 1\rceil^+.$$

The assumption of Theorem 5.5 implies

$$\int_{\mathbb{C}} \Delta\Psi(z) d\lambda(z) = \mu(\mathbb{C}) < \infty,$$

and hence, it is comparable to the assumption (RS) of Theorem 5.1.

Proof. Let $R > 0$ such that $\text{supp}\mu \subseteq D(0, R)$. For $|z| > 2R$ we have

$$\begin{aligned} 2\pi\mathcal{P}_\mu(z) &= \int_{\mathbb{C}} \log|z - w| d\mu(w) \\ &= \log|z|\mu(\mathbb{C}) + \int_{D(0,R)} \log|1 - \frac{w}{z}| d\mu(w) \\ &\leq \log|z|\mu(\mathbb{C}) + \log\frac{3}{2}\mu(\mathbb{C}). \end{aligned} \tag{5.24}$$

Similarly to the proof of Lemma 3.3 we can find a conjugate function k to h such that $g(z) := \frac{h(z)+ik(z)}{2}$ is entire.
If $f \in A^2(\mathbb{C}, \Psi)$, then inequality (5.24) gives

$$\begin{aligned} \infty > \|f\|_{A^2(\mathbb{C},\Psi)} &\geq \int_{\mathbb{C}\setminus D(0,2R)} |f(z)|^2 e^{-\Psi} d\lambda(z) \\ &\geq e^{\log\frac{2}{3}\mu(\mathbb{C})} \int_{\mathbb{C}\setminus D(0,2R)} |f(z)|^2 \frac{e^{-h(z)}}{|z|^{\mu(\mathbb{C})}} d\lambda(z) \\ &= e^{\log\frac{2}{3}\mu(\mathbb{C})} \int_{\mathbb{C}\setminus D(0,2R)} |f(z)e^{-g(z)}|^2 \frac{1}{|z|^{\mu(\mathbb{C})}} d\lambda(z). \end{aligned}$$

Since $f(z)e^{-g(z)} \in \mathcal{H}(\mathbb{C})$ we have by Lemma 5.4 that $fe^{-g} \in \mathcal{T}_{\mu(\mathbb{C})}$ and therefore,

$$A^2(\mathbb{C}, \Psi) \subseteq \mathcal{T}_{\mu(\mathbb{C})} e^g.$$

This proves the claim. □

By the Riesz's representation theorem, the previous theorem applies exactly to the subharmonic functions Ψ such that $\Delta\Psi$ has compact support.

5.2 Weighted Bergman space in several variables

In this section we prove a statement given by Shigekawa in 1990 (see [S91, Lemma 3.4, p.278]). Nevertheless his proof is quite difficult and very long and therefore, we follow the lines of Mehrabian (see [M08]) and Haslinger (see [H12]).

Theorem 5.6 (Shigekawa)**.** *Let $\Psi : \mathbb{C}^n \to \mathbb{R}$ be a C^∞ function. If η denotes the lowest eigenvalue of the Levi matrix*

$$i\partial\overline{\partial}\Psi(z) := \left(\frac{\partial^2 \Psi}{\partial z_j \partial \overline{z_k}}\right)_{j,k=1}^n$$

and satisfies

$$\lim_{|z|\to\infty} |z|^2 \eta(z) = \infty, \tag{S}$$

then the space $A^2(\mathbb{C}^n, \Psi)$ is of infinite dimension.

Before going into more details of the proof we have to provide a technical lemma first. This lemma is used in the proof given by Shigekawa as well as in the proof of Mehrabian, which is stated below.

Lemma 5.7. *Let $\xi \in \mathbb{C}^n$ and $K > 0$ and define $g(z) := \log(1 + K|z-\xi|^2)$. Then for each $w \in \mathbb{C}^n$ we have*

$$\frac{K}{(1+K|z-\xi|^2)^2}|w|^2 \leq i\partial\overline{\partial}g(w,w)(z) \leq \frac{K}{1+K|z-\xi|^2}|w|^2.$$

Proof. We derive each component of the associated Levi matrix,

$$\begin{aligned}
\frac{\partial^2 g}{\partial z_j \partial \overline{z_k}}(z) &= \frac{\partial}{\partial z_j} \frac{K(z_k - \xi_k)}{1+K|z-\xi|^2} \\
&= \frac{\delta_{k,j} K(1+K|z-\xi|^2) - K^2 (z_k - \xi_k)\overline{(z_j - \xi_j)}}{(1+K|z-\xi|^2)^2} \\
&= \frac{K}{(1+K|z-\xi|^2)^2} \left(\delta_{k,j}(1+K|z-\xi|^2) - K^2(z_k - \xi_k)\overline{(z_j-\xi_j)}\right),
\end{aligned}$$

where $\delta_{k,j}$ denotes the Kronecker-Delta. Hence, the Levi matrix is the sum of a multiple of the identity and a hermitian matrix and thus, we have

$$i\partial\overline{\partial}g(w,w)(z) = \frac{K}{(1+K|z-\xi|^2)^2}\left((1+K|z-\xi|^2)|w|^2 - K|(w, z-\xi)|^2\right),$$

which implies by using Cauchy-Schwarz inequality that

$$\begin{aligned}
\frac{K}{(1+K|z-\xi|^2)^2}|w|^2 &= \frac{K}{(1+K|z-\xi|^2)^2}\left((1+K|z-\xi|^2)|w|^2 - K|w|^2|z-\xi|^2\right) \\
&\leq i\partial\overline{\partial}g(w,w)(z) \leq \frac{K}{1+K|z-\xi|^2}|w|^2,
\end{aligned}$$

therefore, the claim holds true. □

After providing the above technical lemma, we now prove Shigekawa's theorem. The idea of the proof is to change the function Ψ in such a way, that it becomes strictly plurisubharmonic. Then we apply Theorem 4.7 to this new function and gain a non-trivial entire function f such that $fp \in A^2(\mathbb{C}^n, e^{-\Psi})$ for infinitely many polynomials p.

Proof of Theorem 5.6. Assumption (S) allows us to find a strictly positive constant K such that

$$i\partial\overline{\partial}\Psi(w,w)(z) \geq -K|w|^2,$$

for every z, w in $\mathbb{C}^n$, and that $i\partial\overline{\partial}\Psi(z)$ is strictly positive for large $|z|$. Let

$$g_\xi(z) \coloneqq \log(1 + 4K|z-\xi|^2),$$

and note that by Lemma 5.7 we have that

$$i\partial\overline{\partial}g_\xi(w,w)(z) \geq \frac{4K}{(1+4K|z-\xi|^2)^2}|w|^2.$$

Therefore, for $|z-\xi| \leq \frac{1}{\sqrt{4K}}$, we have

$$i\partial\overline{\partial}g_\xi(w,w)(z) \geq K|w|^2.$$

Since $i\partial\overline{\partial}\Psi(w,w)(z)$ is negative at most on a compact set in $\mathbb{C}^n$, there exist finitely many points $\xi_1, \dots, \xi_M$ in $\mathbb{C}^n$ such that this compact set is covered by the balls $D(\xi_l, \frac{1}{\sqrt{4K}})$. W.l.o.g. we may assume that $M \geq 2$ and therefore the function

$$\tilde{\Psi}(z) \coloneqq \Psi(z) + \sum_{k=1}^{M} g_k(z)$$

is strictly plurisubharmonic, where $g_k(z) \coloneqq g_{\xi_k}(z)$ for $k = 1, \dots, M$. Let us denote the lowest eigenvalue of $i\partial\overline{\partial}\tilde{\Psi}$ by $\tilde{\eta}$. Therefore, by assumption (S) we again have that

$$\lim_{|z|\to\infty} |z|^2\tilde{\eta}(z) = \infty.$$

Hence, for each $N \in \mathbb{N}$ we find a number $R > 0$ such that

$$\tilde{\eta}(z) \geq \frac{N+M+1}{|z|^2},$$

for every $|z| > R$. By construction we even have that $\tilde{\eta}_0 \coloneqq \inf\{\tilde{\eta}(z) : |z| \leq R\} > 0$ and

$$\kappa \coloneqq \frac{\tilde{\eta}_0}{2(N+M)} > 0.$$

We define the function

$$\Phi(z) \coloneqq \tilde{\Psi}(z) - (N+M)\log(1+\kappa|z|^2).$$

By definition we see that $e^{-\Phi}$ is locally integrable and we will see that it is even strictly plurisubharmonic. Notice that by Lemma 5.7 we have

$$i\partial\overline{\partial}\Phi(w,w)(z) \geq |w|^2\left(\tilde{\eta}(z) - \frac{(N+M)\kappa}{1+\kappa|z|^2}\right).$$

By definition of $\tilde{\eta}_0$ we have for $|z| \leq R$ that

$$\tilde{\eta}(z) - \frac{(N+M)\kappa}{1+\kappa|z|^2} \geq \tilde{\eta}_0 - (N+M)\kappa = \tilde{\eta}_0 - \frac{(N+M)\tilde{\eta}_0}{2(N+M)} = \frac{\tilde{\eta}_0}{2} > 0$$

and for $|z| > R$ we obtain

$$\tilde{\eta}(z) - \frac{(N+M)\kappa}{1+\kappa|z|^2} \geq \frac{N+M+1}{|z|^2} - \frac{N+M}{|z|^2} = \frac{1}{|z|^2},$$

which implies that Φ is strictly plurisubharmonic.
Thus, Theorem 4.7 guarantees that there exists an entire function f with $f(0) = 1$ and

$$\int_{\mathbb{C}^n} |f(z)|^2 \frac{e^{-\Phi(z)}}{(1+|z|^2)^{3n}} d\lambda(z) < \infty.$$

Now let $\tilde{N} := N - 3n$ and get

$$\begin{aligned}
&\int_{\mathbb{C}^n} |f(z)|^2 (1+|z|^2)^{\tilde{N}} e^{-\Psi(z)} d\lambda(z) = \\
&= \int_{\mathbb{C}^n} \underbrace{\frac{\prod_{k=1}^M (1+4K|z-\xi_k|^2)}{(1+\kappa|z|^2)^{N+M}}}_{=:h(z)} |f(z)|^2 (1+|z|^2)^{\tilde{N}} e^{-\Phi(z)} d\lambda(z) \\
&\leq \sup_{z\in\mathbb{C}^n} (h(z) \int_{\mathbb{C}^n} |f(z)|^2 \frac{e^{-\Phi(z)}}{(1+|z|^2)^{3n}} d\lambda(z) < \infty,
\end{aligned}$$

since $\sup_{z\in\mathbb{C}^n}\{h(z)\} < \infty$, because the nominator and denominator are of the same degree. Hence, $fp \in A^2(\mathbb{C}^n, e^{\Psi})$ for any polynomial p of degree lower than $\tilde{N}$. Because $N = \tilde{N} + 3n$ was arbitrary, we have shown the claim. □

In the next section we will give examples, where it becomes clear that Shigekawa's condition is not necessary.

5.3 Examples

We begin with a comparison of the two conditions under which the weighted Bergman space $A^2(\mathbb{C}, \Psi)$ is of infinite dimension. Finally, we give examples to illustrate that Shigekawa's condition fails to be necessary, in both the one dimensional and the higher dimensional case.

5.3.1 Comparison between Rozenblum-Shirokov and Shigekawa

We assume that Ψ is a subharmonic $C^\infty(\mathbb{C})$ function. We have seen two sufficient conditions for $A^2(\mathbb{C}, \Psi)$ being of infinite dimension:

$$\int_{\mathbb{C}} \Delta\Psi(z) d\lambda(z) = \infty \tag{RS}$$

and

$$\lim_{|z|\to\infty} |z|^2 \Delta\Psi(z) = \infty. \tag{S'}$$

Although, condition (S') can be formulated in a more general setting for application in higher dimensions (see Theorem 5.6).

Remark 5.D. In one dimension it is easy to see that if condition (S') is satisfied then also (RS) holds true:
By assumption (S') we can find a number $R > 0$, such that

$$|z|^2 \Delta\Psi(z) > 1, \quad \text{for all} \quad |z| \geq R.$$

Therefore, combining subharmonicity of Ψ with the inequality (RS) gives

$$\int_{\mathbb{C}} \Delta\Psi(z) d\lambda(z) \geq \int_{\mathbb{C}\setminus D(0,R)} \frac{|z|^2}{|z|^2} \Delta\Psi(z) d\lambda(z) \geq \int_{\mathbb{C}\setminus D(0,R)} \frac{1}{|z|^2} d\lambda(z).$$

The latter integral is unbounded and thus, condition (RS) is satisfied. ♦

We should note that even the condition

$$\lim_{|z|\to\infty} |z|^2 \Delta\Psi(z) > 0,$$

implies that (RS) is satisfied.

At this point one may assume that the two conditions are in fact equivalent, but this is not true as the following example teaches us:

Example 5.E. *We check if* $\Psi(z) := (\log(1+|z|^2))^2$ *satisfies condition* (S') *or* (RS). Indeed, Ψ is subharmonic on $\mathbb{C}$: Clearly, Ψ is a $C^2(\mathbb{C})$-function and satisfies

$$\begin{aligned} \Delta\Psi(z) &= 4 \frac{\partial^2}{\partial z \partial \overline{z}} \Big(\log(1+|z|^2)\Big)^2 \\ &= 8 \left(\frac{\partial}{\partial z} \frac{\log(1+|z|^2) z}{1+|z|^2} \right) \\ &= 8 \left(\frac{|z|^2 + \log(1+|z|^2)}{(1+|z|^2)^2} \right) \geq 0. \end{aligned}$$

Therefore, by Corollary 2.13 the function Ψ is subharmonic.
We can insert this in (S') and gain

$$\lim_{|z|\to\infty} 8 \left(\frac{|z|^4 + |z|^2 \log(1+|z|^2)}{(1+|z|^2)^2} \right) = 8. \tag{5.25}$$

Thus the condition given by Shigekawa is not satisfied.
We prove that the condition by Rozenblum and Shirokov is satisfied. By inspecting the proof of Remark 5.D we have already seen, that (5.25) implies (RS). Anyway, we do it again: For every $R > 0$ the following inequality holds true

$$\begin{aligned}\int_{\mathbb{C}} \Delta\Psi(z)d\lambda(z) &= \int_0^{2\pi}\int_0^{\infty} \frac{r^2+\log(1+r^2)}{(1+r^2)^2} r dr d\theta \\ &= \pi \int_0^{\infty} \frac{s+\log(1+s)}{(1+s)^2} ds \\ &\geq \pi \int_1^{R} \frac{s}{(1+s)^2} ds \\ &\geq \frac{\pi}{4}\int_1^R \frac{1}{s} ds = \frac{\pi\log(R)}{4}.\end{aligned}$$

Hence as $R \to \infty$ we have seen that (RS) is satisfied. ♦

This example proves that condition (RS) is weaker and even in the plane Shigekawa's condition (S) fails to be a necessary condition.

5.3.2 Shigekawas condition is not sharp

We have already seen, that the condition (S) fails to be a necessary condition under which $A^2(\mathbb{C}, \Psi)$ is infinite dimensional (for a special subharmonic function Ψ). In this section we give an example to show that even in the higher dimensional case Shigekawa's condition is not necessary. The below-mentioned example can be found in [H12, p.62].

Example 5.F. *We give an example in $\mathbb{C}^2$ to see that condition* (S) *in Theorem 5.6 is not necessary for $A^2(\mathbb{C}^2, \Psi)$ being of infinite dimension for special Ψ. For this purpose we have a look at the weight function $\Psi(z_1, z_2) := |z_1|^2|z_2|^2 + |z_2|^4$.*
An easy computation shows that the Levi matrix has the following form

$$i\partial\overline{\partial}\Psi(z_1, z_2) = \begin{pmatrix} |z_2|^2 & \overline{z_1}z_2 \\ \overline{z_2}z_1 & |z_1|^2 + 4|z_2|^2 \end{pmatrix}$$

and therefore it has the following two eigenvalues

$$\begin{aligned}\eta_1(z_1, z_2) &= \frac{1}{2}\left(5|z_2|^2 + |z_1|^2 - \sqrt{9|z_2|^4 + 10|z_1|^2|z_2|^2 + |z_1|^4}\right), \\ \eta_2(z_1, z_2) &= \frac{1}{2}\left(5|z_2|^2 + |z_1|^2 + \sqrt{9|z_2|^4 + 10|z_1|^2|z_2|^2 + |z_1|^4}\right).\end{aligned}$$

We check if the assumption (S) is satisfied along the z_1-axis and see that for the lowest eigenvalue η_1 we have

$$\lim_{|z_1|\to\infty} |z_1|^2\eta_1(z_1, 0) = 0$$

and we have seen that condition (S) is not satisfied.
Nevertheless, the space $A^2(\mathbb{C}^2, \Psi)$ is of infinite dimension. We recognize that the functions $f(z_1, z_2) := z_2^k$ lie in the space $A^2(\mathbb{C}^2, \Psi)$ for every $k \in \mathbb{N}$, because

$$\int_{\mathbb{C}^2} |f(z)|^2 e^{-\Psi(z)} d\lambda(z) = 4\pi \int_0^{\infty}\int_0^{\infty} r_2^{2k} e^{-(r_1^2 r_2^2 + r_2^4)} r_1 dr_1 r_2 dr_2$$

$$
\begin{aligned}
&= 4\pi \int_0^\infty \left(\int_0^\infty e^{-r_1^2 r_2^2} r_1 r_2^2 dr_1 \right) r_2^{2k-1} e^{-r_2^4} dr_2 \\
&= 2\pi \int_0^\infty \int_0^\infty e^{-s} ds r_2^{2k-1} e^{-r_2^4} dr_2 \\
&= 2\pi \int_0^\infty r_2^{2k-1} e^{-r_2^4} dr_2 < \infty
\end{aligned}
$$

is satisfied. The functions f_k are linearly independent and therefore, the space $A^2(\mathbb{C}^2, \Psi)$ is of infinite dimension. ♦

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Appendix

Abstract

The aim of this thesis is to establish, discuss and compare different sufficient conditions under which unweighted and weighted Bergman spaces are non-trivial or even infinite dimensional. We focus on Bergman spaces of square integrable holomorphic functions over unbounded domains in $\mathbb{C}$ and $\mathbb{C}^n$. In the unweighted case we give a sufficient and necessary condition on the underlying set in $\mathbb{C}$, under which the associated Bergman space is non-trivial. Concerning weighted Bergman spaces we consider the defining sets $\mathbb{C}$ and $\mathbb{C}^n$ and impose different conditions on the weight function.

For convenience we first provide a list of technical tools from topics related to potential theory, which are later applied to both the unweighted and weighted case.

In the first chapter we introduce a few definitions, give a short revision of important theorems from complex analysis, measure theory and discuss some results on fundamental solutions of certain partial differential operators.

In the second chapter we establish (following [Ra95],[Co95],[Hö90/I] and [R86]) some important properties of subharmonic functions, derive several characterizations of subharmonicity and give some examples of subharmonic functions. Furthermore, we derive some nice results from logarithmic potential theory and, among others, prove the Riesz representation theorem for subharmonic functions, a connection between the theory of subharmonic functions and (logarithmic) potential theory. Finally, we define some useful terms related to potential theory, such as the logarithmic capacity of subsets in the complex plane.

In the third chapter we state the definition of the most important objects of this work: Bergman spaces. We derive a list of basic properties and give examples of trivial and finite dimensional Bergman spaces. In the end we prove a theorem due to Carleson (see [C67] and [Co95]), which provides a necessary and sufficient condition under which the unweighted Bergman space in one complex variable over an unbounded domain is infinite dimensional. We will see that this is possible without using Hörmander's L^2-estimates or other existence theorems for solutions to the inhomogeneous Cauchy-Riemann equation.

In the fourth chapter we derive (following [Hö90/I], [Hö07] and [H12]) Hörmander's L^2-estimates in one dimension and give a sketch of the proof in higher dimensions. We also give some applications.

In the last chapter we follow the lines of Rozenblum and Shirokov (see [RS06]) to give a sufficient condition under which weighted Bergman spaces in the complex plane are infinite dimensional. We also give a condition on the weight function, which causes the associated Bergman space to be finite dimensional. Moreover, we establish a theorem by Shigekawa (see [S91]), which even applies in higher dimensions. We compare the two theorems and give an example of a weight function which satisfies the requirements of Rozenblum and Shirokov, but does not fulfill the conditions of Shigekawa's theorem. To conclude we give another example which shows that Shigekawa's condition fails to

be necessary in higher dimensions as well.

Zusammenfassung

Das Ziel dieser Arbeit ist verschiedene hinreichende Bedingungen für ungewichtete und gewichtete Bergmanräume herzuleiten, zu erklären und zu vergleichen, sodass der betrachtete Bergmanraum von unendlicher Dimension oder zumindest nicht trivial ist. Dazu beschränken wir uns auf Bergmanräume quadratintegrierbarer holomorpher Funktionen über unbeschränkten Gebieten in $\mathbb{C}$ oder $\mathbb{C}^n$. Im ungewichteten Fall nennen wir eine hinreichende und notwendige Bedingung an das zugrundeliegende Gebiet in $\mathbb{C}$, sodass der Bergmanraum nicht trivial ist. Im Vergleich dazu betrachten wir im gewichteten Fall nur Bergmanräume mit Grundmenge $\mathbb{C}$ oder $\mathbb{C}^n$ und stellen in diesem Fall verschiedene Bedingungen an das Gewicht.

Zunächst geben wir grundlegende Ergebnisse aus dem Bereich der Potentialtheorie an, welche wir sowohl im ungewichteten als auch im gewichteten Fall anwenden.

Wir beginnen im ersten Kapitel mit grundsätzlichen Definitionen und wiederholen grundlegende Ergebnisse aus dem Bereich der komplexen Analysis, Maßtheorie und über Fundamentallösungen spezieller partieller Differentialoperatoren.

Im zweiten Kapitel gehen wir ([Ra95],[Co95],[Hö90/I] und [R86] folgend) näher auf wichtige Eigenschaften von subharmonischen Funktionen ein, leiten einige Charakterisierungen des Begriffes der Subharmonizität her und geben Beispiele für subharmonische Funktionen an. Des Weiteren begründen wir einige interessante Resultate aus dem Bereich der logarithmischen Potentialtheorie und beweisen unter anderem den Darstellungssatz von Riesz für subharmonische Funktionen, einer Verbindung der Theorie über subharmonische Funktionen und der (logarithmischen) Potentialtheorie. Letztendlich definieren wir wichtige Begriffe aus dem Bereich der logarithmischen Potentialtheorie, wie beispielsweise die logarithmische Kapazität von Teilmengen der komplexen Ebene.

Im dritten Kapitel beschäftigen wir uns mit dem wichtigsten Begriff dieser Arbeit: Bergmanräumen. Wir leiten grundlegende Eigenschaften her und führen einige Beispiele von trivialen und endlich dimensionalen Bergmanräumen an. Letztendlich beweisen wir einen Satz von Carleson (siehe [C67] und [Co95]), welcher eine hinreichende und notwendige Bedingung liefert, sodass der ungewichtete Bergmanraum in einer komplexen Veränderlichen nicht trivial ist. Dazu werden weder die L^2-Abschätzungen von Hörmander noch jegliche andere Lösungen für die inhomogene Cauchy-Riemann Gleichung benötigt.

Im vierten Kapitel beweisen wir ([Hö90/I], [Hö07] und [H12] folgend) die L^2-Abschätzungen von Hörmander in einer Dimension und erklären wie man diese auch in höheren Dimensionen herleiten kann. Außerdem führen wir einige Anwendungen an.

Im letzten Kapitel behandeln wir einen Satz von Rozenblum und Shirokov (siehe [RS06]) und geben eine hinreichende Bedingung für den gewichteten Bergmanraum in der komplexen Ebene an, sodass dieser von unendlicher Dimension ist. Außerdem wird eine hinreichende Bedingung für das Gewicht beschrieben, sodass der zugehörige Bergmanraum nur von endlicher Dimension ist. Des Weiteren beweisen wir einen Satz von Shigekawa (siehe [S91]), welcher auch in höheren Dimensionen anwendbar ist. Wir vergleichen die Sätze und geben ein Beispiel einer Gewichtsfunktion an, welche die

Voraussetzungen im Satz von Rozenblum und Shirokov erfüllt, jedoch den Bedingungen im Satz von Shigekawa widerspricht. Abschließend führen wir ein weiteres Beispiel an, welches verdeutlicht, dass die Bedingung von Shigekawa auch im höher dimensionalen Fall nicht notwendig ist.

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