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On some Planar Hamiltonian Systems

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Contents

Preface

This thesis consists of two blocks: In the first block, comprising chapters one and two, we will have a glimpse at general mechanics, motivate some basic concepts and give fundamental definitions. We will prove some classical theorems, e.g. Noether's theorem, and we will have a look at symplectic vector spaces. On one hand, the theory is fairly abstract, on the other hand, it gives an impression why Hamiltonian systems are so important.
In the second block, comprising chapters three and four, we will have a somewhat closer look on existence and uniqueness questions related to Hamiltonian systems. Chapter three gives classical existence and uniqueness results. In Chapter four we present some recent results applicable to certain planar autonomous Hamiltonian systems with a discontinuous right-hand side.
The graphics were created with the Mac software "Grapher".

Chapter 1

Introduction

1.1 Hamiltonian Formulation in classical mechanics

"When I was in high school, my physics teacher- whose name was Mr. Bader - called me down one day after physics class and said, 'You look bored; I want to tell you something interesting.' Then he told me something which I found absolutely fascinating, and have, since then, always found fascinating. Every time the subject comes up, I work on it."

-Richard Feynman

This something Richard Feynman referred to was the principle of least action. One basic observation immediately leads to Lagrangian and Hamiltonian mechanics: It is that the path a particle takes according to the equations of motions is extremal. The equations of motion are given by *Newton's second law*:

$$\mathbf{F} = m\mathbf{a} \tag{1.1}$$

where $\mathbf{F}$ is the *force* and $\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2}$ is the *accelaration*.

We define the *Lagrangian* as the difference between kinetic and potential energy:

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{m}{2}\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} - U(\mathbf{x}) \tag{1.2}$$

The force is given by $f = -\nabla U$. To see that Newton's second law is extremal, we investigate a quantity defined as follows.
Let $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Then the *action* of $\mathcal{L}$ along a curve $\mathbf{x} : [t_0, t_1] \to \mathbb{R}^n$ is defined as

$$S := \int_{t_0}^{t_1} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt \tag{1.3}$$

If we take the Lagrangian as defined in equation (1.2), the units of the action are $[S] = [Js] = [Nms] = [kg\, m^2\, s^{-1}]$.

Theorem 1. *A curve* $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ *is an extremum of the action if and only if the equation*

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right) = \frac{\partial \mathcal{L}}{\partial \mathbf{x}} \tag{1.4}$$

holds along the curve $\mathbf{x}(t)$.

Proof. We compute the first variation of the action in the space $X = \{\gamma : [t_0, t_1] \to \mathbb{R}^n \text{ continuously differentiable } \mid \gamma(t_0) = \mathbf{x}(0), \gamma(t_1) = \mathbf{x}(t_1)\}$ along $\mathbf{x}$. Let h be C^1 with $h(t_0) = h(t_1) = 0$. We calculate using the chain rule and integration by parts:

$$\begin{aligned}
\delta S(\mathbf{x})h &= \lim_{\epsilon \to 0} \frac{S(\mathbf{x} + \epsilon h) - S(\mathbf{x})}{\epsilon} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{t_0}^{t_1} \mathcal{L}(\mathbf{x} + \epsilon h, \dot{\mathbf{x}} + \epsilon \dot{h})\, dt \\
&= \int_{t_0}^{t_1} \left\{ \frac{\partial \mathcal{L}}{\partial \mathbf{x}} h + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \dot{h} \right\} dt = \int_{t_0}^{t_1} \left\{ \frac{\partial \mathcal{L}}{\partial \mathbf{x}} h - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right) h \right\} dt + \left.\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} h\right|_{t_0}^{t_1} \\
&= \int_{t_0}^{t_1} \left\{ \frac{\partial \mathcal{L}}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right) \right\} h\, dt
\end{aligned} \tag{1.5}$$

So $\delta S(\mathbf{x})(h) = 0$ for all h defined as above is true, if and only if equation (1.4) holds along $\mathbf{x}$. □

Equation (1.4) is called *Euler-Lagrange equation*. If we compute it for the Lagrangian defined in (1.2), we obtain

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{d}{dt} m\dot{\mathbf{x}} + \nabla U(\mathbf{x}) = m\mathbf{a} - \mathbf{F} = 0 \tag{1.6}$$

and we are back at Newton's second law.

We can reformulate the equations in another way. We introduce the so called *canonical coordinates* via

$$\mathbf{q} = \mathbf{x} \quad , \quad \mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \tag{1.7}$$

Here, we have just used Cartesian coordinates, such that $\mathbf{q}$ describes the position of, say, n mass points and $\mathbf{p}$ describes their momenta. We define the *Hamiltonian* of our system by

$$H(\mathbf{p}, \mathbf{q}) := \mathbf{p} \cdot \dot{\mathbf{q}} - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) \tag{1.8}$$

For the Lagrangian in equation (1.2), the Hamiltonian is nothing but the total energy of the system, as a simple calculation shows. Now, we reformulate the Euler-Lagrange equation in terms of the Hamiltonian.

Theorem 2. *The Euler-Lagrange equation is equivalent to the system of* $2n$ *first-order ordinary differential equations given by*

$$\begin{cases} \dot{\mathbf{q}} = \dfrac{\partial H}{\partial \mathbf{p}} \\[2ex] \dot{\mathbf{p}} = -\dfrac{\partial H}{\partial \mathbf{q}} \end{cases} \tag{1.9}$$

Proof. First let $\mathbf{q}$ satisfy the Euler-Lagrange equation. The exterior derivative of H is given by

$$dH = \frac{\partial H}{\partial \mathbf{p}}\, d\mathbf{p} + \frac{\partial H}{\partial \mathbf{q}}\, d\mathbf{q} \tag{1.10}$$

but by definition of H it is also given by

$$dH = \dot{\mathbf{q}}\, d\mathbf{p} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}}\, d\mathbf{q} \tag{1.11}$$

because $\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}$ by assumption. Since both expressions must be the same, we have

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \quad , \quad \frac{\partial H}{\partial \mathbf{q}} = -\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \tag{1.12}$$

Invoking again equation (1.7), we obtain Hamilton's equations. The converse direction is prooved analogously. □

Let $(\mathbf{q}, \mathbf{p})$ be canonical coordinates in phase space. Then the *Poisson-bracket* of two functions f and g is defined as

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{1.13}$$

With this notation, we can write Hamilton's equations equivalently as

$$\begin{cases} \dot{\mathbf{q}} = \{\mathbf{p}, H\} \\ \dot{\mathbf{p}} = \{\mathbf{q}, H\} \end{cases} \tag{1.14}$$

Example 1. We consider a very simple mechanical system: a spring with a mass attached to it. The gravitational field of the earth will drag the mass down, on the other hand, the spring will pull it back. The Lagrangian for this system is given by $\mathcal{L}(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2$, where m is the mass, k is the stiffness of the spring and q is the position of the mass. Hamiltonian's equations read

$$\begin{cases} \dot{q} = \dfrac{p}{m} \\ \dot{p} = -kq \end{cases} \tag{1.15}$$

The solution to this linear system with initial data (q_0, p_0) is given by

$$\begin{aligned} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} &= \exp \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega t) & \frac{1}{\sqrt{km}} \sin(\omega t) \\ -\sqrt{km} \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \end{aligned}$$

where we have set $\omega := \sqrt{\frac{k}{m}}$.

1.2 Nonlinear Configuration spaces

We will only very briefly touch the case of nonlinear configuration spaces. They arise in the study of mechanical systems with constraints, that is, the mass points cannot shoot around in space freely. For example, the motion of a mass point depends on the motion of another mass point. We pass from the position space $\mathbb{R}^{3n}$ to some manifold.

We will start with a classical result: The flow of an Hamiltonian system preserves the volume in phase space - this is Liouville's theorem. In order to prove it, we need the following

Lemma 1. *Let A be a matrix. Then*

$$\det(I + tA) = 1 + t\,\mathrm{tr}(A) + O(t^2) \text{ as } t \to 0 \tag{1.16}$$

Proof. The function $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is smooth and so is the composition $\det(I + \,.\,A) : \mathbb{R} \to \mathbb{R}$. We compute the first derivative at zero using the Jacobi formula

$$\left.\frac{d}{dt}\right|_{t=0} \det(I + tA) = \left.\mathrm{tr}(\mathrm{adj}(I + tA)\frac{d}{dt}(I + tA))\right|_{t=0} = \mathrm{tr}(A) \tag{1.17}$$

where $\mathrm{adj}(M)$ denotes the adjugate of a matrix M. If we now expand the function $\det(I + \,.\,A)$ in a Taylor series around zero, we obtain formula (1.16). □

Now, we are ready to prove the following, very famous

Theorem 3. *(***Liouville's Theorem***)*
Let $\{g^t\}_{t\in\mathbb{R}}$ be the flow of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ and let $D \subset \mathbb{R}^{2n}$ be a bounded region in phase space. If $\nabla \cdot \boldsymbol{f} = 0$, then

$$\mathrm{vol}(g^t(D)) = \mathrm{vol}(D) \tag{1.18}$$

Proof. First observe that the flow can be expanded as

$$g^t(\mathbf{x}) = \mathbf{x} + \mathbf{f}(\mathbf{x})t + O(t^2) \text{ as } t \to 0 \tag{1.19}$$

We calculate by the change of variables formula

$$\mathrm{vol}(g^t(D)) = \int_{g^t(D)} 1\, d\mathbf{x} = \int_D \det \frac{\partial g^t(\mathbf{x})}{\partial \mathbf{x}}\, d\mathbf{x} \tag{1.20}$$

Equation (1.19) implies that

$$\frac{\partial g^t(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{I} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}} t + O(t^2) \text{ as } t \to 0 \tag{1.21}$$

Invoking Lemma 1 we conclude that

$$\mathrm{vol}(g^t(D)) = \int_D \left\{1 + t\nabla \cdot \mathbf{f} + O(t^2)\right\} d\mathbf{x} \text{ as } t \to 0 \tag{1.22}$$

and therefore

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(g^t(D)) = \int_D \nabla \cdot \mathbf{f}\, d\mathbf{x} = 0 \tag{1.23}$$

by the assumption made on $\mathbf{f}$. This proves our claim. □

Consider a baton twirling, that is, two point masses m_1, m_2 connected by a straight line. One of the two masses, say, m_1 could, at least theoretically, be at any point in space. However, if the position of m_1 is determined, the other mass can only be on a circle with center m_1. So the configuration space of our mechanical system is given by $\mathbb{R}^2 \times S^1$.

Let M be a smooth manifold and $\mathcal{L} : TM \to \mathbb{R}$ a smooth function on its tangent bundle. We define a functional on the vector space of all smooth curves $\gamma : [t_0, t_1] \to M$ via

$$\Phi(\gamma) = \int_{t_0}^{t_1} \mathcal{L}(\gamma(t), \dot{\gamma}(t))\, dt \tag{1.24}$$

where $\dot{\gamma}(t) \in T_{\gamma(t)}M$ is the velocity vector of γ. An extremal of the functional Φ is called *Lagrangian motion.* The pair $(M, \mathcal{L})$ is called *Lagrangian system.*

Totally analogously to the "flat" case, one obtains the following

Theorem 4. *The trajectory of a particle $\gamma(t)$ in local coordinates $\boldsymbol{q} = (q_1, ..., q_1)$ in a Lagrangian system $(M, \mathcal{L})$ is a solution to the Euler-Lagrange Equation*

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right) = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \tag{1.25}$$

If we are given a Riemannian manifold (M, g), we also have a natural notion of *kinetic energy*, namely $T = \frac{1}{2}g(v, v)$ for $v \in TM_x$, and a **potential energy** is simply a smooth function $U : M \to \mathbb{R}$. In this case, we can define a Lagrangian as the difference between kinetic and potential energy $\mathcal{L} := T - U \circ \pi$, where $\pi : TM \to M$ is the canonical projection.

Example 2. Consider a pendulum. The point mass m is forced to move on a circle of radius one, so the configuration space is no longer linear, but a manifold. We assume the potential energy to be homogenous in space $U(x, y) = -mgx$. Kinetic energy is given by $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$. In the chart $\phi : (\cos(q), \sin(q)) \to q$, the Lagrangian takes the form $\mathcal{L}(q, \dot{q}) = \frac{m}{2}\dot{q} + mg\cos(q)$. The Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = m\ddot{q} + mg\sin(q) = 0 \tag{1.26}$$

and Hamilton's equations are

$$\begin{cases} \dot{q} &= \dfrac{p}{m} \\ \dot{p} &= -mg\sin(q) \end{cases} \tag{1.27}$$

The phase plane of equation (1.27) and is plotted in Figure 1.1 .

We will now state and proof another classical, deep theorem: Noether's Theorem. Heuristically, it says that whenever the configuration space of a mechanical systems admits a symmetry, then there exists a conserved quantity corresponding to that symmetry. For instance, the conserved quantity that is related to the symmetry in time is the total energy of the system, the conserved quantity related to the symmetry in space is the total momentum and so forth.

Theorem 5. *(***Noether's Theorem***)*
Let $(M, \mathcal{L})$ be an Lagrangian system and let $\{h^s\}_{s\in\mathbb{R}}$ be a one-parameter group of diffeomorphisms $h^s : M \to M$ such that

$$\mathcal{L}(h^s(p), T_p h^s(\mathbf{v})) = \mathcal{L}(p, \mathbf{v}) \tag{1.28}$$

for all $\mathbf{v} \in T_pM$, for all $p \in M$ and for all $s \in \mathbb{R}$.
Then the system has a first integral $\mathcal{I} : TM \to \mathbb{R}$, which is given in local coordinates $\mathbf{q} = (q_1, ..., q_n)$ as

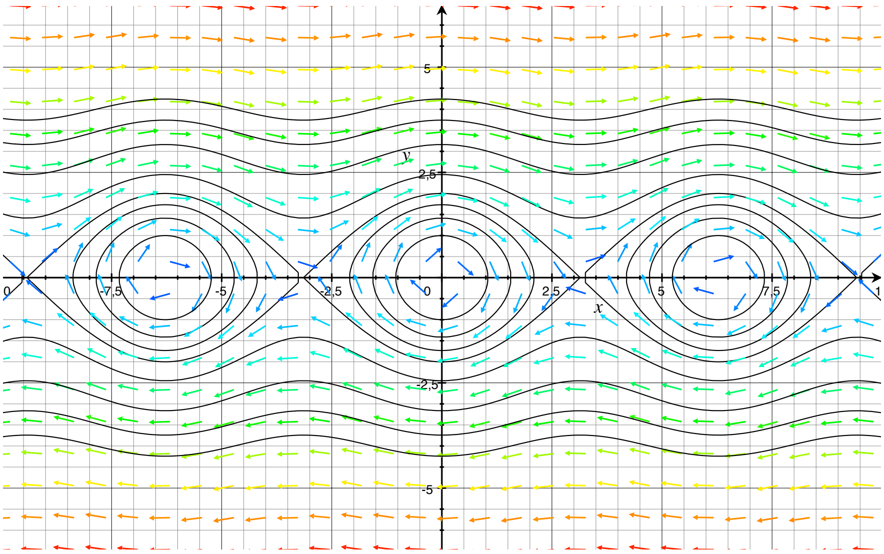

Figure 1.1: Vector field and some level sets of equation (1.27)

$$\mathcal{I}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \frac{dh^s(\mathbf{q})}{ds}\bigg|_{s=0} \tag{1.29}$$

Proof. Let us first consider $M = \mathbb{R}^n$ and let $\gamma : \mathbb{R} \to M, \mathbf{q} = \gamma(t)$ be a solution to Lagrange's equation. Then also $h^s \circ \gamma : \mathbb{R} \to M$ is a solution to Lagrange's equation by assumption.
Define $\Gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ by $\mathbf{q} = \Gamma(s,t) = h^s(\gamma(t))$. Differentiating Γ with respect to s we obtain:

$$0 = \frac{\partial \mathcal{L}}{\partial s}(\Gamma, \dot{\Gamma}) = \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot \frac{\partial \Gamma}{\partial s} + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \dot{\Gamma} \tag{1.30}$$

where a dot denotes a derivative with respect to t. The partial derivatives of $\mathcal{L}$ are taken at $\mathbf{q} = \Gamma(st)$ and $\dot{\mathbf{q}} = \dot{\Gamma}(s,t)$ respectively.
Because $\Gamma|_{s=const.} : \mathbb{R} \to \mathbb{R}^n$ satisfies Lageange's equation for any fixed s, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}(\Gamma(st), \dot{\Gamma}(s,t))\right) = \frac{\partial \mathcal{L}}{\partial \mathbf{q}}(\Gamma(s,t), \dot{\Gamma}(s,t)) \tag{1.31}$$

We now substitute the left-hand side of the above equation in equation (1.30) and obtain

$$0 = \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right)\frac{\partial \mathbf{q}}{\partial s} + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\left(\frac{d}{dt}\frac{\partial \mathbf{q}}{\partial s}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\frac{\partial \mathbf{q}}{\partial s}\right) = \frac{d}{dt}\mathcal{I} \tag{1.32}$$

Now, it is not difficult to show that the definition of $\mathcal{I}$ is chart-independent. Therefore, we have a conserved quantity on the whole of any manifold M. □

Example 3. Consider a spherical pendulum in the approximately homogenous gravitational force field of the earth. Its configuration manifold is S^2 and the associated Lagrangian is $\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$+mgz. We have a one-parameter group of transformations given by rotations about the z-axis $T^s : S^2 \to S^2$, that is

$$T^s(\mathbf{x}) = \begin{pmatrix} \cos(s) & -\sin(s) & 0 \\ \sin(s) & \cos(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.33}$$

It is easy to see that the Lagrangian is invariant under T^s for all $s \in \mathbb{R}$. How does T^s and $\mathcal{L}$ look like in a chart of S^2? We use the coordinates

$$\mathbf{q} : (\phi_1, \phi_2) \mapsto \begin{pmatrix} \sin(\phi_1)\cos(\phi_2) \\ \sin(\phi_1)\sin(\phi_2) \\ \cos(\phi_1) \end{pmatrix} \tag{1.34}$$

An annoying computation shows that $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}(\dot{\phi_1}^2 + \sin(\phi_1)^2\dot{\phi_2}^2) + mg\cos(\phi_1)$. Another annoying computation shows, that T^s in coordinates is just given by translation: $(\mathbf{q}^{-1} \circ T^s \circ \mathbf{q})(\phi_1, \phi_2) = (\phi_1, \phi_2 + s)$. Noether's theorem tells us now, that the quantity

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\frac{dT^s(\mathbf{q})}{ds}\right|_{s=0} = m\begin{pmatrix} \dot{\phi_1} \\ \sin(\phi_1)^2\dot{\phi_2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = m\sin{\phi_1}^2\dot{\phi_2} \tag{1.35}$$

is conserved. This is nothing but the angular momentum in direction of the z-axis.

Chapter 2

Symplectic Vector spaces

In this chapter, we take a step back and define Hamiltonian systems from an abstract point of view. This formulation is very useful, since many problems admit a Hamiltonian formulation, for instance, field theory. In this case, our underlying space is not finite dimensional, so we have to consider Hamilton's equations on a Banach space.
We first define *symplectic vector spaces*, which are the proper linear spaces to define Hamilton's equations. We define symplectic transformations and play a little with symplectic geometry. Finally, we formulate Hamilton's equations.

Definition 1. Let V be a real Banach space and let $\Omega : V \to \mathbb{R}$ be a continuous bilinear 2-form on V. The form is called *non-degenerate*, if $\Omega(v, w) = 0$ for all $w \in V$ implies that $v = 0$.

A *symplectic form* Ω is a non-degenerate, skew-symmetric and bilinear 2-form on V. The pair (V, Ω) is called *symplectic vector space*.

Example 4. Let V be a vector space and let V^* be its dual. We can define a 2-form Ω on their product space $W := V \times V^*$ via

$$\Omega((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2)$$

for $v_1, v_2 \in V$ and $\alpha_1, \alpha_2 \in V^*$. This form is called *canonical symplectic form* on W.

If we take a finite dimensional space V with basis $\{e_i\}_{i=1}^n$ and the corresponding dual basis $\{e^i\}_{i=1}^n$ of V^*, then the matrix representation of Ω in the basis $\{(e_1, 0), ..., (e_n, 0), (0, e^1), ..., (0, e^n)\}$ is given by

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{\text{-}I} & \mathbf{0} \end{pmatrix} \tag{2.1}$$

where $\mathbf{I}$ denotes the identity matrix of order n.

Definition 2. Let (V, Ω) and (W, Θ) be two symplectic vector spaces. A smooth map $S : V \to W$ is called *symplectic* or *symplectic transformation*, if

$$\Theta(\mathbf{D}S(v)v_1, \mathbf{D}S(v)v_2) = \Omega(v_1, v_2) \tag{2.2}$$

for all $v, v_1, v_2 \in V$.

In infinitely many dimensions, $\mathbf{D}S(v)$ is the Féchet Derivative of S at v. We recall that, given two Banach spaces $(V, \|.\|_V)$ and $(W, \|.\|_W)$, a function $F : U \to W$, $U \subseteq V$ open, is *Fréchet diffrerentiable* at a point $v \in V$ if there exists an linear operator $A_v \in L(V, W)$ such that

$$\lim_{h \to 0} \frac{\|F(v + h) - F(v) - A_v(h)\|_W}{\|h\|_V} = 0 \tag{2.3}$$

Every finite dimensional symplectic vector space has even dimension.

Symplectic transformations are the homomorphisms of symplectic vector spaces. If $\dim V = 2n < \infty$, the set of all symplectic maps on V forms a group, the *symplectic group* of order $2n$, which is denoted by $Sp(2n)$. One can easily show that for the matrix representation A of any symplectic transformation $\det(A) = 1$ holds. So $Sp(2n) \subseteq SL(2n, \mathbb{R})$.

If we are in the finite dimensional case, we call a basis $\mathcal{B} = \{e_1, ..., e_n\} \cup \{f_1, ..., f_n\}$ *smyplectic*, if

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0 \text{ and } \Omega(e_i, f_i) = 1 \tag{2.4}$$

for $1 \leq i, j \leq n, i \neq j$. The following theorem asserts that all finite dimensional vector spaces of the same dimension are isomorphic

Theorem 6. *Every finite dimensional symplectic vector space (V, Ω) has a symplectic basis. Given any non-zero vector $v \in V$, we can pick a symplectic basis that contains v.*

The proof is essentially the same as for an vector space together with an inner product and we will omit it. The symplectic group $Sp(2n)$ acts transitively on a symplectic vector space V: for any two vectors $v, w \in V$ there exists some $S \in Sp(2n)$ such that $S(v) = w$.

We can characterize symplectic transformations on finite dimensional spaces as follows. The proof can be found in [1].

Theorem 7. *Let (V, Ω) be a finite dimensional symplectic vector space. The following are equivalent:*

- *The map $S : V \to V$ is symplectic.*
- *Its matrix S in the standard symplectic coordinate representation satisfies $S^T JS = J$ with J being defined as in* (2.1)*.*
- *The transformation S takes some (and hence any) symplectic basis into a symplectic basis.*

The characteristic polynomial $p_S(\lambda)$ of a symplectic transformation S has a funny property: it is self-reciprocal, meaning that $p_S(\lambda) = \lambda^{\deg(p)} p_S(\frac{1}{\lambda})$. This implies, that if λ_0 is an eigenvalue of S, so is $\frac{1}{\lambda_0}$. The eigenvalues of a symplectic transformation come in quadruples and lie symmetric with respect the unit circle.

Definition 3. Let (V, Ω) be a symplectic vector space. A vector field $\mathbf{X} : V \to V$ is called *Hamiltonian*, if there exists a function $H \in C^1(V, \mathbb{R})$ such that

$$\Omega(\mathbf{X}(v), w) = \mathbf{D}H(v)(w) \tag{2.5}$$

for all $v, w \in V$. Such a function is called *Hamiltonian function*, *a Hamiltonian* or *energy function*. We write X_H to indicate, that X_H is a Hamiltonian vector field with Hamiltonian function H.

We can add a further condition to our list in Theorem 7, which also holds in the infinite dimensional case

Proposition 1. *Let (V, Ω) and (W, Θ) be two symplectic vector spaces and $F : V \to W$ be a diffeomorphism. Then F is a symplectic transformation if and only if the equation*

$$\mathbf{D}F(v)\mathbf{X}_{H\circ F}(v) = \mathbf{X}(F(v)) \tag{2.6}$$

for all Hamiltonian vector fields $\mathbf{X}_H$ *on* W *and all* $v \in V$ *holds.*

Proof. Let $v, w \in V$ be given, then

$$\begin{aligned}\Omega(\mathbf{X}_{H\circ F}(v), w) &= \mathbf{D}(H \circ F)(v)w = \mathbf{D}H(F(v)) \cdot \mathbf{F} = (v)w \\ &= \Theta(\mathbf{X}_H(F(v)), \mathbf{D}F(v)w)\end{aligned} \tag{2.7}$$

Assume first that F is symplectic. We have

$$\Theta(\mathbf{D}F(v)\mathbf{X}_{H\circ F}(v), \mathbf{D}F(v)w) = \Omega(\mathbf{X}_{H\circ F}(v), w) \tag{2.8}$$

and thus by nondegeneracy of Θ and because $\mathbf{D}F(v)$ is an isomorphism, equation (2.6) holds.

Conversely, assume that equation (2.6) holds. By equation (2.7) we have

$$\Theta(\mathbf{D}F(v)\mathbf{X}_{H\circ F}(v), \mathbf{D}F(v)w) = \Omega(\mathbf{X}_{H\circ F}(v), w) \tag{2.9}$$

for any $v \in V$ and any $H \in C^1(W, \mathbb{R})$. Let $v' \in V$ be arbitrary. If we pick $(H \circ F)(v) = \Omega(v', v)$, we have that $\mathbf{X}_{H\circ F}(v) = v'$ and hence, F is symplectic. □

The third name for a Hamiltonian function in Definition 3 is deliberately chosen, as the following theorem shows.

Theorem 8. *Let* $\gamma : \mathbb{R} \to V$ *be an integral curve of the Hamiltonian vector field* $\mathbf{X}_H$*. Then* $H(\gamma(t))$ *is constant.*

Proof. Differentiating according to the chain rule and the skew-symmetry of Ω yield

$$\begin{aligned}\frac{d}{dt}H(\gamma(t)) &= \mathbf{D}H(\gamma(t))\frac{d\gamma}{dt}(t) = \mathbf{D}H(\gamma(t))\mathbf{X}_H(\gamma(t)) \\ &= \Omega(\mathbf{X}_H(\gamma(t)), \mathbf{X}_H(\gamma(t))) = 0\end{aligned} \tag{2.10}$$

□

We are ready to formulate Hamilton's equations for linear spaces.

Definition 4. Let H be a Hamiltonian function with vector field $\mathbf{X}_H$, then *Hamilton's equations* for a curve $\gamma : \mathbb{R} \to V$ are defined as

$$\frac{d\gamma}{dt} = \mathbf{X}_H(\gamma) \tag{2.11}$$

Let us now look at two problems in an infinite dimensional vector space which admit a Hamiltonian formulation.

Example 5. Not only for mass points, but also for fluids the Hamiltonian approach proved to be useful. Suppose we have a bounded domain $\Omega \subseteq \mathbb{R}^3$ filled with an incompressible fluid. The motion of the fluid is given by a family of volume preserving diffeomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$. These are integral curves to a time dependent velocity field $\mathbf{v} : \Omega \times \mathbb{R} \to \Omega$, which is a solution to the Euler equations

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p \\ \nabla \cdot \mathbf{v} = 0 \\ \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \partial\Omega \end{cases} \tag{2.12}$$

where $\mathbf{n}$ is any outward pointing normal vector of the boundary of Ω.

Let now $\mathfrak{g}$ be the space of divergence-free vector fields tangent to the boundary $\partial\Omega$. For two vector fields $\mathbf{v} = (v_1, ..., v_n)$ and $\mathbf{w} = (w_1, ..., w_n)$ in $\mathfrak{g}$ we can define the negative Jacobi-Lie bracket as

$$([\mathbf{v}, \mathbf{w}]_L)_i = \sum_{j=1}^{n} \left(w_j \frac{\partial v_j}{\partial x_i} - v_j \frac{\partial w_i}{\partial x_j} \right) \tag{2.13}$$

Now, we introduce the so called *ideal fluid bracket* of functions F, G acting on the velocity field $\mathbf{v}$:

$$\{F, G\}(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot \left[\frac{\delta F}{\delta \mathbf{v}}, \frac{\delta G}{\delta \mathbf{v}} \right]_L dx \tag{2.14}$$

where $\frac{\delta F}{\delta \mathbf{v}}$ is the functional derivative defined by

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(\mathbf{v} + \epsilon\delta\mathbf{v}) = \int_\Omega \left\{ \delta\mathbf{v} \cdot \frac{\delta F}{\delta \mathbf{v}} \right\} dx^3 \tag{2.15}$$

If we now define our Hamiltonian to be the kinetic energy in the fluid, that is

$$H(\mathbf{v}) = \frac{1}{2}\int_\Omega \mathbf{v}\cdot\mathbf{v}\, dx^3 \tag{2.16}$$

one can show that Euler equations (2.12) are equivalent to Hamilton's equations for the ideal fluid bracket:

$$\dot{F} = \{F, H\} \tag{2.17}$$

for any function F on $\mathfrak{g}$. See [12].

Example 6. We want to model the unconstrained vibration of a homogenous membrane with mass density $\rho = 1$ in $\mathbb{R}^n$.
As our symplectic vector space, we take the cartesian product of, say, the space of smooth, rapidly decreasing real-valued functions on $\mathbb{R}^n \times \mathbb{R}$ times itself $V := C_0^\infty(\mathbb{R}^n \times \mathbb{R})^2$. Clearly, V is a vector space and we obtain a symplectic structure on V by setting

$$\Omega((\phi_1, \pi_1), (\phi_2, \pi_2)) = \int_{\mathbb{R}^n} \{\phi_1\pi_2 - \phi_2\pi_1\}\, d\mathbf{x} \tag{2.18}$$

One checks that (V, Ω) indeed is a symplectic vector space. An element $\phi \in C_0^\infty$ represents the displacement of the membrane from an n-dimensional hyperplane. We get a notion of kinetic energy by

$$T = \frac{1}{2}\int_{\mathbb{R}^n} \left(\frac{\partial\phi}{\partial t}(\mathbf{x}, t)\right)^2 d\mathbf{x} \tag{2.19}$$

and if we assume a linear restoring force, a notion of potential energy by

$$U = \frac{1}{2}\int_{\mathbb{R}^n} \|\nabla_{\mathbf{x}}\phi(\mathbf{x}, t)\|^2\, d\mathbf{x}. \tag{2.20}$$

The Hamiltonian is then the total energy of the system

$$H(\phi, \pi) = \frac{1}{2}\int_{\mathbb{R}^n} \left\{\pi(\mathbf{x}, t)^2 + \|\nabla_{\mathbf{x}}\phi(\mathbf{x}, t)\|^2\right\} d\mathbf{x} \tag{2.21}$$

If we differentiate our Hamiltonian according to the functional derivative defined in equation (2.15), Hamilton's equations take the form

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \dfrac{\delta H}{\delta \pi} = \pi \\ \dfrac{\partial \pi}{\partial t} = -\dfrac{\delta H}{\delta \phi} = \Delta \phi \end{cases} \tag{2.22}$$

This is nothing but the linear wave equation $\frac{\partial^2 \phi}{\partial t^2} = \Delta \phi$.

Chapter 3

General ODE-Theory

We return to ordinary differential equations in "flat" space to discuss the existence and uniqueness of solutions to Hamiltonian systems.

Theorem 9. *(***Peano's Theorem***)*
Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be open and $F : U \to \mathbb{R}$ be continuous. Then the system

$$\begin{cases} \dot{\mathbf{x}} = F(t, \mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{3.1}$$

possesses a local solution for any initial data $\mathbf{x}_0 \in U$.

A proof can, for instance, be found in [17].
This theorem is in general not true for Cauchy problems in Banach spaces. If the right-hand side has higher regularity, we can do better and obtain

Theorem 10. *(***Picard-Lindelöf Theorem***)*
Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be open and $F : U \to \mathbb{R}$ be continuous in the first variable and Lipschitz continuous in the second variable. Then the system (3.1) *posesses a unique local solution for any initial data $\mathbf{x}_0 \in U$.*

To emphasize the relevance of these two theorems, let us have a look at the following

Example 7. Consider the following one dimensional equation on $\mathbb{R}^+$:

$$\begin{cases} \dot{x} = \sqrt{|x|} \\ x(0) = x_0 \end{cases} \tag{3.2}$$

The right-hand side is not Lipschitz continuous around zero, since it is continuously differentiable on $\mathbb{R} \setminus \{0\}$, but its derivative at zero does not exist. If we take $x_0 = 0$, we get a one parameter set $\{x^\tau(t)\}_{\tau \in \mathbb{R}^+}$ of solutions, given by

$$x^\tau(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \tau \\ \frac{1}{4}(t-\tau)^2 & \text{if } t > \tau \end{cases} \tag{3.3}$$

A solution stays zero until some time τ and then evolves according to what we expect for an initial value not equal zero. Behavior like that is known as *Peano phenomenon*.

Is there something between Peano and Picard-Lindelöf?

We call a function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ and $\omega(r) > 0$ for $r > 0$ a *modulus of continuity*. A modulus of continuity satisfies the *Osgood condition*, if

$$\int_0^1 \frac{d\tau}{\omega(\tau)} = \infty \tag{3.4}$$

Theorem 11. (Osgood's criterion)
Suppose $\omega : [0, \infty) \to [0, \infty)$ is a modulus of continuity and satisfies the Osgood condition and suppose that the right hand side in equation (3.1) is defined on $[t_0, t_0 + T] \times B_R(\mathbf{x}_0)$ and satisfies

$$\|(F(t, \mathbf{x}) - F(t, \mathbf{y}))\| \leq \omega(\|\mathbf{x} - \mathbf{y}\|) \tag{3.5}$$

for all $t \in [t_0, t_0 + T]$ and all $\mathbf{x}, \mathbf{y} \in B_R(\mathbf{x}_0)$. Then equation (3.1) possesses a unique local solution.

For a discussion of Osgood's critierion, see for instance [4].
We say that a function F *admits* the modulus of continuity ω, if equation (3.5) holds for F and ω. For the modulus of continuity $\omega_1(t) = t$, we obtain the Lipschitz continuous functions, for $\omega_2(t) = t^\alpha$, $\alpha \in (0, 1)$ the Hölder continuous functions. Observe, that ω_1 satisfies the Osgood condition, whereas ω_2 does not.
A function, that admits the modulus of continuity $\omega(t) = t(1 + |\log(t)|)$ is called *almost Lipschitz continuous*. We compute

$$
\begin{aligned}
\int_0^1 \frac{dt}{\omega(t)} = \int_0^1 \frac{dt}{t(1+|\log(t)|)} &= \int_0^1 \frac{dt}{t(1-\log(t))} \\
&= -\log(1-\log(t))\Big|_0^1 = \infty
\end{aligned}
\tag{3.6}
$$

and infer that $\omega(t) = t(1+|\log(t)|)$ satisfies the Osgood condition.

Example 8. Consider the following initial value problem

$$
\begin{cases} \dot{x} = |x| \log |x| \\ x(0) = x_0 \end{cases}
\tag{3.7}
$$

The right hand side is not locally Lipschitz continuous around zero. Nevertheless, uniqueness of solutions is guaranteed, since the right-hand side is locally almost Lipschitz continuous around zero.
To see this, assume first that $0 < x < \frac{1}{e}$, $0 < \epsilon < 1$ and $x + \epsilon < \frac{1}{e}$, where $e = \exp(1)$. Since log is monotone, we deduce that

$$
\log(x+\epsilon) \geq \log\left(\frac{x^2+\epsilon^2}{x+\epsilon}\right) - \frac{\epsilon}{\epsilon + x}
$$

and from the concavity of log, it follows that

$$
\log(x+\epsilon) \geq \frac{\epsilon}{x+\epsilon}\log(\epsilon) + \frac{x}{x+\epsilon}\log(x) - \frac{\epsilon}{x+\epsilon}
$$

Rewriting the above equation leads to

$$
x\log(x) - (x+\epsilon)\log(x+\epsilon) \leq \epsilon(1-\log(\epsilon))
$$

which is precisely the almost Lipschitz condition, if we set $y = x + \epsilon$ and remember, that $|x|\log|x|$ is strictly decreasing on $(0, \frac{1}{e})$ and that $\epsilon, x+\epsilon < \frac{1}{e}$. Because $|x|\log|x|$ is even, the same holds true for $-\frac{1}{e} < x < 0$, $-1 < \epsilon < 0$, $x + \epsilon > -\frac{1}{e}$. The case $0 < x < \frac{1}{e}$, $-1 < \epsilon < 0$, $x + \epsilon > -\frac{1}{e}$ follows from the monotonicity of $\omega(t) = t(1+|\log(t)|)$. Putting this together, Osgood's criterion implies that the uniqueness of solutions is guaranteed.

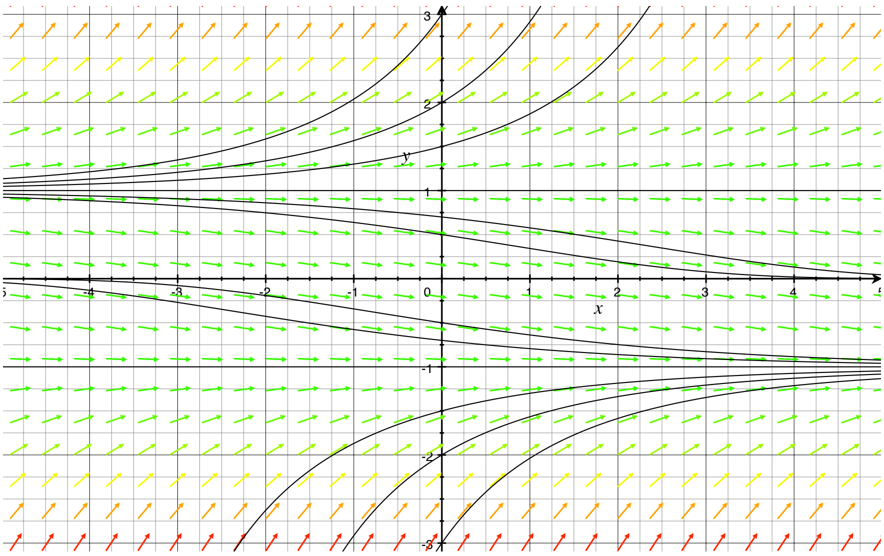

Figure 3.1: Vector field and some solutions of equation (3.7)

We will have a look at the planar Hamiltonian system

$$\begin{cases} \dot{x} = -\dfrac{\partial H}{\partial y}(x,y) \\ \dot{y} = \dfrac{\partial H}{\partial x}(x,y) \end{cases} \tag{3.8}$$

with a continuously differentiable Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ and initial data $(x_0, y_0) \in \mathbb{R}^2$. We present the proof of an existence and uniqueness theorem as it was obtained in [16], but first we need a lemma on one-dimensional equations.

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and consider the equation*

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \tag{3.9}$$

Existence and uniqueness of solutions is guaranteed as long as $f(x_0) \neq 0$.

We give the proof presented in [15].

Proof. Assume that the system (3.9) with $f(x_0) \neq 0$ has two different solutions x_1 and x_2 in a neighborhood U of $t = 0$. Then, because of the

continuity of f, it holds that $|f(x_1(t))| > 0$ and $|f(x_2(t))| > 0$ for all t in a (possibly smaller) neighborhood $V \subseteq U$ of $t = 0$.
Let us assume w.l.o.g. that $f(x_0) > 0$ so that there exists a time t_1 such that $x_1(t_1) \neq x_1(0)$, because otherwise x_1 would be constant on U and hence $f(x_0) = 0$. We can pick such a t_1, while at the same time $x_2(t_1) \neq x_1(t_1)$ and $f(x_1(t)), f(x_2(t)) > 0$ for all $t \in [0, t_1]$. Now, by the choice of t_1, it holds that $f(x) > 0$ for all $x \in [x_0, x_1(t_1)]$ and therefore there exists a time t_2 such that $x_2(t_2) = x_1(t_1)$. We integrate equation (3.9)

$$t_1 - t_0 = \int_{x_0}^{x_1(t_1)} \frac{dx_1}{f(x_1)} = \int_{x_0}^{x_1(t_1)} \frac{dx_2}{f(x_2)} = t_2 - t_0 \tag{3.10}$$

a contradiction. $\square$

In the case of Hamiltonian systems, we obtain the following result.

Theorem 12. *Given initial data* $(x_0, y_0) \in \mathbb{R}^2$ *such that* $\nabla H(x_0, y_0) \neq (0, 0)$. *Then the initial value problem associated to* (3.8) *has a unique, local solution.*

Proof. Let $(x_0, y_0) \in \mathbb{R}^2$ such that $\nabla H(x_0, y_0) \neq 0$. Solutions to (3.8) parameterize a level set of H, at least locally, that is

$$H(x(t), y(t)) = H(x_0, y_0) \tag{3.11}$$

By assumption, either $\frac{\partial H}{\partial x}$ or $\frac{\partial H}{\partial y}$ does not vanish at (x_0, y_0). Let us assume w.l.o.g. that $\frac{\partial H}{\partial x}(x_0, y_0) \neq 0$. Because of the implicit function theorem and in light of equation (3.11) there exists a neighborhood $U = (x_1, x_2) \times (y_1, y_2)$ of (x_0, y_0) and a C^1-function $f : (y_1, y_2) \to (x_1, x_2)$ with $f(y_0) = x_0$ and the property that: $(x, y) \in U$ satisfies (3.11) if and only if $f(y) = x$.
So if $(x(t), y(t))$ is a solution to (3.8), then $x(t) = f(y(t))$ has to hold. We have that $y(t)$ is a solution of the system

$$\begin{cases} \dot{y} = \dfrac{\partial H}{\partial x}(f(y), y) \\ y(0) = y_0 \end{cases} \tag{3.12}$$

Since H is continuously differentiable, the right-hand side $\frac{\partial H}{\partial x}(f(.), .)$ of the above equation is continuous. Remembering that $\frac{\partial H}{\partial x}(f(y_0), y_0) \neq 0$ by assumption, we deduce from Lemma 2 that equation (3.12) has a unique solution in an open neighborhood of y_0 where the right-hand side does not vanish. Now the uniqueness of a solution to (3.8) in an open neighborhood of (x_0, y_0) follows immediately. $\square$

Let us look at an example of a planar Hamiltonian system where uniqueness fails.

Example 9. Consider the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sqrt{|x|} \end{cases} \tag{3.13}$$

The associated Hamiltonian is given by $H(x, y) = \frac{y^2}{2} + \frac{2}{3}|x|^{\frac{1}{2}}x$, which has an equilibrium at (0,0). Peano phenomenon occurs at (0,0): we have a one-parameter family of solutions $\{(x^\tau(t), y^\tau(t))\}_{\tau\in\mathbb{R}}$ given by

$$x^\tau(t) = \begin{cases} 0 & \text{if } 0 \le t \le \tau \\ -\frac{1}{144}(t-\tau)^4 & \text{if } t > \tau \end{cases} \tag{3.14}$$

and

$$y^\tau(t) = \begin{cases} 0 & \text{if } 0 \le t \le \tau \\ -\frac{1}{36}(t-\tau)^3 & \text{if } t > \tau \end{cases} \tag{3.15}$$

similar to Example 7.

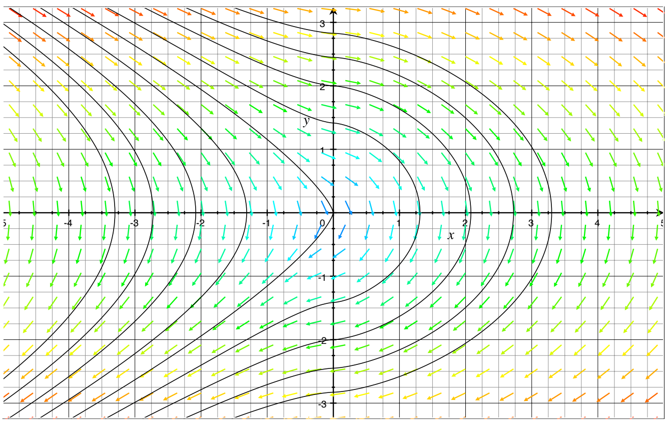

Figure 3.2: Vector field and some level sets of equation (3.13)

One can see a "kink" at (0,0).

Chapter 4

The case of a discontinuous right-hand side

In this chapter, we will investigate integral curves with a "kink", meaning they are not continuously differentiable.

4.1 One-dimensional equations

We will have a look at one-dimensional autonomous differential equations whose right-hand side has jump discontinuities. These considerations permit to gain some insight about differential equations with a discontinuous right-hand side.

Definition 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. A point x is called *jump discontinuity* of f, if the right-hand side and left-hand side limits of f at x exist and do not coincide:

$$f^+(x) := \lim_{y \to x+} f(y) \neq \lim_{y \to x-} f(y) =: f^-(x) \tag{4.1}$$

For the rest of this section, we will consider a special class of functions, which have no zeros, stay bounded and may be discontinuous. This requirements are important in order to make the variation of constants formula work.

Definition 6. Given an open interval $I \subset \mathbb{R}$ and $\varepsilon \in (0, 1)$, let $\mathcal{F}_I^\varepsilon$ be the class of functions $f : I \to \mathbb{R}$ having only jump discontinuities and such that either $\varepsilon < f(x) < \varepsilon^{-1}$ for all $x \in I$, or $-\varepsilon^{-1} < f(x) < -\varepsilon$ for all $x \in I$.

Note that the set of jump discontinuities of a function is at most countable and, given any countable subset $C \subset I$, there is a monotone function whose discontinuities occur exactly at the elements of C (see [18]). While a monotone function has only jump discontinuities (see [8]), the family $\mathcal{F}_I^\varepsilon$ clearly also comprises functions that are not monotone. For example, the difference of two increasing functions belongs to $\mathcal{F}_I^\varepsilon$ as soon as the range condition in Definition 1 is secured. For later use we point out that if $f \in \mathcal{F}_I^\varepsilon$, then $1/f \in \mathcal{F}_I^\varepsilon$, and both are Riemann integrable on compact subsets of I.

Let us now consider the autonomous scalar differential equation

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0, \end{cases} \tag{4.2}$$

for a function $f \in \mathcal{F}_I^\varepsilon$, where $I \subset \mathbb{R}$ is an open interval containing x_0.

Definition 7. A solution to (4.2) on $(-t_0, t_0) \subset I$ with $t_0 > 0$ is a continuous function $x : (-t_0, t_0) \to \mathbb{R}$ that is differentiable for all $t \in (-t_0, t_0)$, with exception of an at most countable set C, and such that $\dot{x}(t) = f(x(t))$ whenever $t \in I \setminus C$.

Proposition 2. *The notion of solution in Definition 7 coincides with that of a continuous solution to the integral equation*

$$x(t) = x_0 + \int_0^t f(x(s))\, ds, \qquad t \in (-t_0, t_0). \tag{4.3}$$

Proof. Indeed, for a solution according to Definition 2, the generalized form of the Fundamental Theorem of Calculus in [8] or [14] ensures that (4.3) holds. Note that the fact that the derivative fails to exist at an at most countable set of points is essential, and allowing failure on a set of measure zero could invalidate this statement (see [6, 8]).

Conversely, a continuous solution to (4.3) is strictly monotone on $(-t_0, t_0)$ and must be differentiable at a point of continuity of $f(x(t))$, where $\dot{x}(t) = f(x(t))$. The fact that $t \mapsto x(t)$ is strictly monotone, in combination with the defining property of $f \in \mathcal{F}_I^\varepsilon$, ensure that there is an at most countable subset of $(-t_0, t_0)$ where the map $t \mapsto f(x(t))$ is not continuous since a point of discontinuity of f is attained at most once by the function $t \mapsto x(t)$. Consequently we deal with a solution in the sense of Definition 7 and the claimed equivalence of the notion of solutions is proven. □

Lemma 3. *For $f \in \mathcal{F}_I^\varepsilon$ the problem (4.2) has a unique local solution.*

Proof. Since $f \in \mathcal{F}_I^\varepsilon$ ensures $1/f \in \mathcal{F}_I^\varepsilon$, the function

$$\Phi(\cdot, x_0) : I \to \mathbb{R}, \qquad \Phi(x;\, x_0) := \int_{x_0}^{x} \frac{ds}{f(s)}, \quad x \in I\,, \tag{4.4}$$

is well-defined, strictly monotone, and differentiable at every point $x \in I$ with exception of an at most countable set C. Since $\Phi(x_0;\, x_0) = 0$, there exists some $t_0 > 0$ such that the open interval $\Phi(I;\, x_0)$ contains $(-t_0, t_0)$.

One can easily check that

$$x(t) = \Phi^{-1}(t;\, x_0), \qquad t \in (-t_0, t_0)\,, \tag{4.5}$$

is a solution in the sense of Definition 2. Uniqueness follows immediately by noticing that if $x(t)$ is a solution, then $\Phi(x(t);\, x_0) = t$ for all $|t|$ small enough (so that the left-hand side is well-defined). □

Within our setting, the fact that $f \in \mathcal{F}_I^\varepsilon$ is bounded ensures that a local solution defined on some interval $(-t_0, t_0)$ can always be extended until it reaches the boundary of I, in accordance with the continuation theory [4, 5] for integral equations of type (4.3). We refrain from discussing growth conditions that would dispense of boundedness; for these matters see e.g. [3].

Example 10. We illustrate the above considerations for (4.2) with

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{for} \quad x < 0\,, \\ 2 & \text{for} \quad x \geq 0\,. \end{cases} \tag{4.6}$$

If $x_0 \geq 0$, then (4.4) takes the explicit form

$$\Phi(x;\, x_0) = \begin{cases} x - x_0/2 & \text{for} \quad x < 0\,, \\ (x - x_0)/2 & \text{for} \quad x \geq 0\,, \end{cases} \tag{4.7}$$

with the inverse $\Phi^{-1}(\cdot,\, x_0) : \mathbb{R} \to \mathbb{R}$ given by

$$\Phi^{-1}(y;\, x_0) = \begin{cases} y + x_0/2 & \text{for} \quad y < -\,x_0/2\,, \\ 2y + x_0 & \text{for} \quad y \geq -\,x_0/2\,. \end{cases} \tag{4.8}$$

In accordance to Lemma 1, note that the function $x(t) = \Phi^{-1}(t;\, x_0)$ is differentiable with $\dot{x}(t) = f(x(t))$ for all $t \in \mathbb{R} \setminus \{-x_0/2\}$. On the other hand, if $x_0 < 0$, then

$$\Phi(x;\,x_0) = \begin{cases} x - x_0 \text{ for } x < 0\,, \\ x/2 - x_0 \text{ for } x \geq 0\,, \end{cases} \quad \Phi^{-1}(y;\,x_0) = \begin{cases} y + x_0 \text{ for } y < -\,x_0\,, \\ 2(y + x_0) \text{ for } y \geq -\,x_0\,, \end{cases} \tag{4.9}$$

so that the unique solution

$$x(t) = \begin{cases} t + x_0 \quad \text{for} \quad t < -x_0\,, \\ 2(t + x_0) \quad \text{for} \quad t \geq -x_0\,, \end{cases} \tag{4.10}$$

fails to be differentiable only at $t = -x_0$.

The next examples show that the range requirement for $f \in \mathcal{F}_I^\varepsilon$ is crucial.

Example 11. Consider (4.2) with $x_0 = 0$ and

$$f(x) = \begin{cases} 1 \quad \text{if} \quad x < 0\,, \\ 0 \quad \text{for} \quad x = 0\,, \\ -1 \quad \text{if} \quad x > 0\,. \end{cases} \tag{4.11}$$

Clearly $x_0(t) = 0$ is a solution for $t \in \mathbb{R}$, but the functions

$$x_+(t) = \begin{cases} t \quad \text{if} \quad t < 0\,, \\ 0 \quad \text{if} \quad t \geq 0\,, \end{cases} \qquad x_-(t) = \begin{cases} -t \quad \text{if} \quad t < 0\,, \\ 0 \quad \text{if} \quad t \geq 0\,, \end{cases} \tag{4.12}$$

are also solutions on $\mathbb{R}$. Thus uniqueness fails.

Example 12. Consider (4.2) with $x_0 = 0$ and with a slight alteration of the right-hand side of Example 11, given by

$$f(x) = \begin{cases} 1 \quad \text{if} \quad x \leq 0\,, \\ -1 \quad \text{if} \quad x > 0\,. \end{cases} \tag{4.13}$$

Then there is no solution (in the sense of Definition 2) on an interval of the form $(-t_0,\, t_0)$ with $t_0 > 0$.

Assuming that $x : (-t_0,\, t_0) \to \mathbb{R}$ is a solution for some $t_0 > 0$, we first claim that we must have that $x(t) \leq 0$ for $t \in [0, t_0)$. Indeed, the existence of some $t_+ \in (0, t_0)$ with $x(t_+) > 0$ would permit us to infer by continuity that there is some $t_- \geq 0$ such that $x(s) > 0$ for all $s \in (t_-, t_+)$, while $x(t_-) = 0$. But this yields the contradiction

$$x(t_+) = x(t_+) - x(t_-) = \int_{t_-}^{t_+} f(x(s))\, ds = \int_{t_-}^{t_+} (-1)\, ds = t_- - t_+ < 0\,. \tag{4.14}$$

Even $x(t) \leq 0$ on $[0, t_0)$ is not possible since it leads to the contradiction

$$x(t) = x(t) - x(0) = \int_0^t f(x(s))\, ds = \int_0^t 1\, ds = t > 0, \qquad t \in (0, t_0)\,. \tag{4.15}$$

Consequently there are no solutions.

4.2 Planar autonomous Hamiltonian systems with a discontinuous right-hand side

We now investigate the problem of the local existence and uniqueness of solutions for the autonomous planar Hamiltonian system

$$\begin{cases} \dot{x} = -\dfrac{\partial H}{\partial y}(x, y) \\[2ex] \dot{y} = \dfrac{\partial H}{\partial x}(x, y) \end{cases} \tag{4.16}$$

with a discontinuous right-hand side. Specifically, we assume that for open intervals I_1, $I_2 \subset \mathbb{R}$, that $H : I_1 \times I_2 \to \mathbb{R}$ is a continuous function such that for some $\varepsilon \in (0, 1)$:

Definition 8. (Condition H1)
The function H is continuously differentiable on $I_1 \times I_2$, except for a closed set D such that at least one of the projections $D_1 \subset I_1$ and $D_2 \subset I_2$ on the axes is a countable set. We require further that at every point $(x, y) \in I_1 \times I_2$ the lateral limits

$$\left(\frac{\partial H}{\partial x}\right)^{\pm}(x,y) = \lim_{h\to 0^{\pm}} \frac{H(x+h,y)-H(x,y)}{h},$$
$$\left(\frac{\partial H}{\partial y}\right)^{\pm}(x,y) = \lim_{h\to 0^{\pm}} \frac{H(x,y+h)-H(x,y)}{h},$$

exist and the pairs of numbers $\left(\frac{\partial H}{\partial x}\right)^{\pm}(x,y)$ and $\left(\frac{\partial H}{\partial y}\right)^{\pm}(x,y)$ have the same sign and absolute values located in the interval $(\varepsilon, \varepsilon^{-1})$.

Since $D \subset D_1 \times D_2$, we deduce that D has Lebesgue measure zero, so that the gradient ∇H (in the sense of distributions) belongs to $L^\infty(I_1 \times I_2)$. Consequently, the function H is locally Lipschitz continuous on $I_1 \times I_2$ (see [7]).

Example 13. Examples of classes of continuous functions $H : (1,2) \times (1,2) \to \mathbb{R}$ with a discontinuous gradient satisfying (H1) are:

1. functions of the type $H(x,y) = H_0(x,y) + F(x)G(y)$;
2. functions of the type $H(x,y) = H_0(x,y) + F(x) + G(y)$;

where $H_0 : [1,2] \times [1,2] \to \mathbb{R}$ is continuously differentiable, while $F(x) = \int_0^x f(s)\,ds$ and $G(y) = \int_0^y g(s)\,ds$ for some monotone functions $f,\, g : [0,2] \to [1,2]$, with precisely one discontinuous on $(1,2)$. Both classes provide examples for which the set D can be uncountable.

Within the setting specified by (H1), a notion of solution is suggested by analogy with the considerations made in connection with Definition 7.

Definition 9. A solution $(x,y) : (-t_0,t_0) \to I_1 \times I_2$ to (4.16) with initial data (x_0,y_0) is a continuous function $t \mapsto (x(t),\, y(t))$ that is differentiable at all points $t \in (-t_0,\, t_0)$ with exception of an at most countable set C, with $(x(0),\, y(0)) = (x_0,\, y_0)$ and such that (4.16) holds for all $t \in (-t_0,\, t_0) \setminus C$.

Alternatively, we may define a solution to (4.16) with initial data (x_0,y_0) as a continuous function $t \mapsto (x(t),\, y(t))$ from $(-t_0,t_0)$ to $I_1 \times I_2$, such that

$$\begin{cases} x(t) = x_0 - \displaystyle\int_0^t \frac{\partial H}{\partial y}(x(s), y(s))\, ds\,, \\ \\ y(t) = y_0 + \displaystyle\int_0^t \frac{\partial H}{\partial x}(x(s), y(s))\, ds\,, \end{cases} \qquad t \in (-t_0, t_0)\,. \tag{4.17}$$

The fact that each function $t \mapsto x(t)$ and $t \mapsto y(t)$ is strictly monotone on $(-t_0, t_0)$ ensures that the curve $\{(x(t), y(t))\}_{t\in(-t_0,t_0)}$ intersects the set D of discontinuities of ∇H for t belonging to an at most countable set $C \subset (-t_0, t_0)$. For $t \in (-t_0, t_0) \setminus C$ we have that the curve is differentiable with a bounded tangent vector $(\dot{x}(t),\, \dot{y}(t)) = \Big(-\dfrac{\partial H}{\partial y}(x(t), y(t))\,,\, \dfrac{\partial H}{\partial x}(x(t), y(t))\Big)$, thus recovering Definition 9. Conversely, the fact that a solution in the sense of Definition 9 satisfies (4.17) follows at once from the generalized Fundamental Theorem of Calculus for functions that are differentiable except at countably many points [8].

Note that for a solution to (4.16) in the sense of Definition 9, we have that $\dfrac{d}{dt}\, H(x(t),\, y(t)) = 0$ at every $t \in (-t_0,\, t_0) \setminus C$. The map $H : I_1 \times I_2 \to \mathbb{R}$ being locally Lipschitz and $t \mapsto (x(t),\, y(t))$ being locally absolutely continuous, since its derivative exists and has absolute value in $(\varepsilon,\, \varepsilon^{-1})$ for all $t \in (-t_0, t_0) \setminus C$, ensure that the map $t \mapsto H(x(t),\, y(t))$ is locally absolutely continuous. Consequently, we have that

$$H(x(t),\, y(t)) = H(x_0, y_0) \qquad t \in (-t_0, t_0)\,. \tag{4.18}$$

Our approach towards local existence and uniqueness relies on exploiting the integral of motion (4.18) in combination with the following generalized version of the implicit function theorem, proved in [9, 11].

Lemma 4. (Implicit Function Theorem)
Suppose that $\mathcal{H} : O \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous mapping with $\mathcal{H}(x_0, y_0) = 0$ at some $(x_0, y_0) \in O$. Assume that there exist open neighborhoods $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ of x_0 and y_0, respectively, such that $A \times B \subset O$, and such that the map $y \mapsto \mathcal{H}(x, y) : B \to \mathbb{R}$ is one-to-one for each $x \in A$. Then there exist open neighborhoods $A_0 \subset A$ and $B_0 \subset B$ of x_0 and y_0, respectively, such that for all $x \in A_0$ the equation $\mathcal{H}(x, y) = 0$ has a unique solution $y = T(x)$ in B_0, and the mapping $T : A_0 \to B_0$ is continuous.

We can now proceed with our main result.

Theorem 13. *If H satisfies $(H1)$, then there exists a unique local solution of (4.16) with initial data $(x_0, y_0) \in I_1 \times I_2$.*

Proof. Say $\left(\frac{\partial H}{\partial x}\right)^{\pm}(x,y) > 0$ and $\left(\frac{\partial H}{\partial y}\right)^{\pm}(x,y) > 0$. Let us define $\mathcal{H}(x,y) = H(x,y) - H(x_0,y_0)$. Clearly $\mathcal{H}(x_0,y_0) = 0$. Moreover, for every fixed $x \in I_1$, the map $y \mapsto \mathcal{H}(x,y)$ is one-to-one on I_2. Indeed, we have

$$\mathcal{H}(x,y_2) - \mathcal{H}(x,y_1) = \int_{y_1}^{y_2} \frac{\partial H}{\partial y}(x,s)\,ds, \quad y_1, y_2 \in I_2, \tag{4.19}$$

since the map $s \mapsto \mathcal{H}(x,s)$ is locally Lipschitz continuous in I_2. Since $\frac{\partial H}{\partial y}$ is of one sign, failure of injectivity would imply that $\frac{\partial H}{\partial y}(x,s) = 0$ almost everywhere between y_1 and y_2, which is impossible for H satisfying (H1).

Applying Lemma 2, we deduce that there exist open intervals $A_0 \subseteq I_1$, $B_0 \subseteq I_2$, and a continuous function $T : A_0 \to B_0$ with $T(x_0) = y_0$, such that if $(x,y) \in A_0 \times B_0$ satisfy $\mathcal{H}(x,y) = 0$, then $y = T(x)$.

We claim that the map $x \mapsto T(x)$ is strictly monotone on A_0. For this, due to its continuity, it suffices to prove that $T : A_0 \to B_0$ is one-to-one. But the existence of $x_1 x_2$ in A_0 with $T(x_1) = T(x_2) = y \in B_0$ would yield $\mathcal{H}(x_1,y) = \mathcal{H}(x_2,y)$, so that $H(x_0,y_0) = H(x_1,y) = H(x_2,y)$. Due to the locally Lipschitz continuity of the function H and the fact that $\frac{\partial H}{\partial x}$ is of one sign, this is only possible if $\frac{\partial H}{\partial x}(s,y) = 0$ for almost every $s \in [x_1, x_2]$, a feature that is impossible for H satisfying (H1).

Since T is strictly monotone on A_0, it follows that $x \mapsto (x, T(x))$ is of bounded variation and therefore the zero level set of $\mathcal{H}$ near (x_0, y_0) is a curve of finite length. It therefore possesses a parametrization by arc length $(x(s), y(s))$, with $[x'(s)]^2 + [y'(s)]^2 = 1$ and $x'(s) < 0$ almost everywhere. Differentiating the relation $H(x(s), y(s)) = H(x_0, y_0)$ we obtain

$$\frac{\partial H}{\partial x}(x(s), y(s))x'(s) + \frac{\partial H}{\partial y}(x(s), y(s))y'(s) = 0 \quad \text{a.e.} \tag{4.20}$$

Rewriting the relation as

$$x'(s) = -\frac{\frac{\partial H}{\partial y}(x(s),y(s))}{\frac{\partial H}{\partial x}(x(s),y(s))}\,y'(s)\,, \qquad y'(s) = -\frac{\frac{\partial H}{\partial x}(x(s),y(s))}{\frac{\partial H}{\partial y}(x(s),y(s))}\,x'(s) \quad \text{a.e.} \tag{4.21}$$

Together with $[x'(s)]^2 + [y'(s)]^2 = 1$ a.e., this yields

$$\begin{aligned} y'(s) &= \frac{\frac{\partial H}{\partial x}(x(s),\, y(s))}{\sqrt{\left(\frac{\partial H}{\partial x}(x(s),\, y(s))\right)^2 + \left(\frac{\partial H}{\partial y}(x(s),\, y(s))\right)^2}} \quad \text{a.e.}\,, \\ x'(s) &= -\frac{\frac{\partial H}{\partial y}(x(s),\, y(s))}{\sqrt{\left(\frac{\partial H}{\partial x}(x(s),\, y(s))\right)^2 + \left(\frac{\partial H}{\partial y}(x(s),\, y(s))\right)^2}} \quad \text{a.e.} \end{aligned}$$

Since $(H1)$ holds, we deduce that

$$\frac{\epsilon^2}{\sqrt{2}} \le y'(s) \le \frac{1}{\epsilon^2\sqrt{2}}, \qquad -\frac{1}{\epsilon^2\sqrt{2}} \le x'(s) \le -\frac{\epsilon^2}{\sqrt{2}} \quad \text{a.e.} \tag{4.22}$$

Let $\gamma(s) := y'(s)/H_x(x(s),y(s)) = -x'(s)/H_y(x(s),y(s))$. The above estimates for $x'(s)$ and $y'(s)$ yield

$$\frac{\epsilon^3}{\sqrt{2}} \le \gamma(s) \le \frac{1}{\epsilon^2\sqrt{2}} \quad \text{a.e.} \tag{4.23}$$

Thus the function $\Gamma(r) := \int_0^r \gamma(s)\,ds$ is Lipschitz, hence differentiable a.e. with $\Gamma'(s) = \gamma(s)$ a.e. Moreover, its inverse is also Lipschitz and differentiable almost everywhere. Set $r = \Gamma(s)$ and $X(r) = x(s)$, $Y(r) = y(s)$. Then

$$X'(r) = x'(s)\frac{ds}{dr} = \frac{x'(s)}{\gamma(s)} = -\frac{\partial H}{\partial y}(X(r),Y(r)) \quad \text{a.e.} \tag{4.24}$$

and

$$Y'(r) = \frac{\partial H}{\partial x}(X(r),Y(r)) \quad \text{a.e.} \tag{4.25}$$

Integrating these two relations and using the fact that $(X(0), Y(0)) = (x_0, y_0)$ we obtain

$$\begin{cases} X(r) = x_0 - \displaystyle\int_0^r \frac{\partial H}{\partial y}(X(r), Y(r))\, ds\,, \\ \\ Y(r) = y_0 + \displaystyle\int_0^r \frac{\partial H}{\partial x}(X(r), Y(r))\, ds\,, \end{cases}$$

so that $(X(r), Y(r))$ is a continuous solution to the above integral equation and therefore, according to the remarks made after Definition 7 and Definition 9, is a solution to system (4.16) in the sense of Definition 9. Since it is a parametrization of a level set of H, uniqueness immediately follows. Indeed, there can be at most one parametrization which solves system (4.16) since the tangent vector only depends on the position. $\square$

We illustrate now the relevance of our main result.

Example 14. Let $H : (-1, 1) \times (-1, 1) \to \mathbb{R}$ be given by $H(x, y) = |x| + 2x + y$. One can easily check that (H1) holds for $\varepsilon = 1/4$, with $D = \{(0, y) : -1 < y < 1\}$ and $D_1 = \{0\}$. Consequently, our theorem ensures the local existence and uniqueness of solutions to the corresponding system (4.16) for any initial data $(x_0, y_0) \in (-1, 1) \times (-1, 1)$. The specific simple form of (4.16), namely

$$\begin{cases} \dot{x} = -1\,, \\ \dot{y} = 2 + \operatorname{sgn}(x)\,, \end{cases} \tag{4.26}$$

where $\operatorname{sgn}(x) = \dfrac{x}{|x|}$ for $x \neq 0$, actually permits us to solve the problem explicitly.

First of all, clearly $x(t) = x_0 - t$. Consequently, $\dot{y}(t) = 1$ for $t > x_0$, while $\dot{y}(t) = 3$ for $t < x_0$. Therefore

$$y(t) = \begin{cases} t + \alpha \quad \text{for} \quad t \geq x_0\,, \\ 3t + \alpha - 2x_0 \quad \text{for} \quad t < x_0\,, \end{cases} \tag{4.27}$$

with α chosen to accommodate the initial condition $y(0) = y_0$. A straightforward calculation yields $\alpha = y_0$ if $x_0 \leq 0$, while $\alpha = 2x_0 + y_0$ for $x_0 > 0$.

To find the maximal domain of definition of the solution one imposes the conditions $|x(t)| < 1$ and $|y(t)| < 1$. Depending on the values of x_0

and y_0, $y(t)$ might take on only one of the two possible expressions. For example, if $(x_0, y_0) = (-8/10, -9/10)$, then $x(t) = x_0 - t$ and $y(t) = t + \alpha$ throughout the domain of definition.

In general, the solution is defined for

$$\max\left\{\frac{2x_0 - y_0 - 1}{3}, -1 - y_0, x_0 - 1\right\} < t < \min\{x_0 + 1, 1 - y_0\} \tag{4.28}$$

if $x_0 \leq 0$, and for

$$\max\left\{-\frac{1 + y_0}{3}, x_0 - 1\right\} < t < \min\left\{x_0 + 1, 1 - 2x_0 - y_0, \frac{1 - y_0}{3}\right\} \tag{4.29}$$

if $x_0 > 0$.

Example 15. Let $H_0 : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. If $M = \sup\limits_{|x|,\,|y| \leq 1} \{|\nabla H(x, y)|\}$, let $H : (-1, 1) \times (-1, 1) \to \mathbb{R}$ be given by

$$H(x, y) = H_0(x, y) + M|x| + 3M(x + y). \tag{4.30}$$

One can easily check that (H1) holds for $\varepsilon = \min\left\{\frac{M}{2}, \frac{1}{6M}\right\}$, with $D = \{(0, y) : \ -1 < y < 1\}$ and $D_1 = \{0\}$. Consequently, our theorem ensures the local existence and uniqueness of solutions to the corresponding system (4.16) for any initial data $(x_0, y_0) \in (-1, 1) \times (-1, 1)$. In this general setting there can be no expectation of finding explicit solutions.

The next example illustrates the importance of the sign-condition in (H1).

Example 16. Let $H : (-1, 1) \times (-1, 1) \to \mathbb{R}$ be given by $H(x, y) = |x| - y^2/2$. We have $D = \{(0, y) : \ -1 < y < 1\}$ and $D_1 = \{0\}$, but each component of $\nabla H = (\operatorname{sgn}(x), -y)$ changes sign in $(-1, 1) \times (-1, 1)$, thus invalidating (H1). Note that both functions $t \mapsto (t^2/2,\, t)$ and $t \mapsto (-\,t^2/2,\, -\,t)$ solve (4.16) with initial data $(0, 0)$.

We will consider another, abstract condition on H under which the Cauchy problem has a unique solution. Let $H : I_1 \times I_2 \to \mathbb{R}$ be a continuous function such that for some $\varepsilon > 0$:

Definition 10. (Condition H2)
The function H is continuously differentiable on $I_1 \times I_2$ except for the set D, which is contained in the set

$$D' = \bigcup_{c \in C} \{(x, y) \in \mathbb{R}^2 : H(x, y) = c\} \tag{4.31}$$

where $C \subset \mathbb{R}$ is countable and each level set $\{(x, y) \in \mathbb{R}^2 : H(x, y) = c\}$ has Lebesgue-measure zero for $c \in C$. We require further that at every point $(x, y) \in I_1 \times I_2$ the lateral limits

$$\begin{aligned} \left(\frac{\partial H}{\partial x}\right)^{\pm}(x, y) &= \lim_{h \to 0^{\pm}} \frac{H(x + h, y) - H(x, y)}{h}, \\ \left(\frac{\partial H}{\partial y}\right)^{\pm}(x, y) &= \lim_{h \to 0^{\pm}} \frac{H(x, y + h) - H(x, y)}{h}, \end{aligned}$$

exist and the pairs of numbers $\left(\frac{\partial H}{\partial x}\right)^{\pm}(x, y)$ and $\left(\frac{\partial H}{\partial y}\right)^{\pm}(x, y)$ have the same sign and absolute values located in the interval $(\varepsilon, \varepsilon^{-1})$.

The 2-dimensional Lebesgue-measure of the set D is zero, since it is a subset of a countable union of sets of measure zero. Therefore, if H satisfies $(H2)$, then ∇H belongs to $L^{\infty}(I_1 \times I_2)$ and H is locally Lipschitz-continuous.

Theorem 14. *If H satisfies $(H2)$ and $H(x_0, y_0) \notin C$, then there exists a unique local solution of (4.16).*

Proof. Analogously to Theorem 1, we deduce from the Lipschitz continuity of H and the sign condition on $\frac{\partial H}{\partial y}$ that $y \mapsto \mathcal{H}(x, y)$ is one-to-one.

Applying Lemma 2, it follows that there exist open intervals $A_0 \subseteq I_1$, $B_0 \subseteq I_2$, and a continuous function $T : A_0 \to B_0$ with $T(x_0) = y_0$, such that if $(x, y) \in A_0 \times B_0$ satisfy $\mathcal{H}(x, y) = 0$, then $y = T(x)$.

Since the set D is a disjoint union of level sets and $H(x_0, y_0) \notin C$ it follows that the graph of T does not intersect with D. □

We illustrate Theorem 2 by means of the following example.

Example 17. Let $H : (-1, 1) \times (-1, 1) \to \mathbb{R}$ be given by $H(x, y) = |x + y| + 2(x + y)$. The set of discontinuities is given by $D = \{(x, y) \in \mathbb{R}^2 : x =$

$-y\} = \{(x, y) \in \mathbb{R}^2 : H(x, y) = 0\}$. Theorem 2 guarantees that for every initial value $(x_0, y_0) \in (-1, 1) \times (-1, 1)$ such that $x_0 \neq y_0$, there exists a unique solution. The Hamiltonian system is given by

$$\begin{cases} \dot{x} = -\operatorname{sgn}(x + y) - 2\,, \\ \dot{y} = \operatorname{sgn}(x + y) + 2\,, \end{cases} \tag{4.32}$$

By the Implicit Function Theorem, we know that locally x is a function of y. Setting $H_0 := H(x_0, y_0)$ we obtain

$$x(y) = \begin{cases} H_0/3 - y & \text{if } x_0 > -y_0\,, \\ H_0 - y & \text{if } x_0 < -y_0\,, \end{cases}$$

and the system reduces to

$$\begin{cases} \dot{y} = \operatorname{sgn}(H_0/3) + 2 & \text{if } x_0 > -y_0, \\ \dot{y} = \operatorname{sgn}(H_0) + 2 & \text{if } x_0 < -y_0\,. \end{cases}$$

Both equations have the solution $y(t) = y_0 + (\operatorname{sgn}(H_0) + 2)t$. Thus, the solution to equation (4.32) is given by $(x(t), y(t)) = (x_0 - (\operatorname{sgn}(H_0) + 2)t, y_0 + (\operatorname{sgn}(H_0) + 2)t)$.

Bibliography

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Second Edition, Springer, 1989.

[2] M. S. Berger, *Nonlinearity and functional analysis*, Academic Press, New York-London, 1977.

[3] A. Constantin, Global existence of solutions for perturbed differential equations, *Ann. Mat. Pura Appl.* **168** (1995), 237–299.

[4] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D. C. Heath and Co., Boston, 1965.

[5] C. Corduneanu, *Integral equations and applications*, Cambridge University Press, 1991.

[6] J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York-London, 1969.

[7] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992.

[8] E. Hewitt and K. Stromberg, *Real and abstract analysis*, New York: Springer, 1977.

[9] K. Jittorntrum, An implicit function theorem, *J. Optim. Th. Appl.* **25** (1978), 575–577.

[10] F. Kogelbauer, On the Cauchy Problem for planar autonomous Hamiltonian systems with a discontinuous right-hand side, *Monatshefte für Mathematik*, to appear (2013)

[11] S. Kumagai, An implicit function theorem: comment, *J. Optim. Th. Appl.* **31** (1980), 285–288.

[12] J. E. Marsden, T. S. Ratiu, *Introduction to Mechanics and Symmetrie*, Second Edition, Springer, 2002

[13] K. R. Meyer and G. R. Hall, *Introduction to Hamiltonian dynamical systems and the n-body problem*, Springer-Verlag, New York, 1992.

[14] I. P. Natanson, *Theory of functions of a real variable*, Frederick Ungar, New York, 1955.

[15] L. C. Piccinini, Remarks on a uniqueness problem for dynamical systems, *Boll. Un. Mat. Ital. A*, 18 (1981), 132-137.

[16] C. Rebelo, A note on the uniqueness of Cauchy problems associated to planar Hamiltonian systems, *Portugal. Math.* **57** (2000), 415– 419.

[17] G. Teschl, *Ordinary Differential Equations and Dynamical Systems*, Providence: American Mathematical Society, 2012.

[18] B. S. Thomson, J. B. Bruckner and A. M. Bruckner, *Elementary real analysis*, Prentice Hall, 2001.

[19] E. Wahlen, Uniqueness for autonomous planar differential equations and the Lagrangian formulation of water flows with vorticity, *J. Nonl. Math. Phys.* **11** (2004), 549–555.

Appendix

Deutsche Zusammenfassung

Diese Arbeit besteht aus zwei Teilen: Im ersten Teil, der Kapitel eins und zwei umfasst, werden wir uns kurz mit Mechanik befassen, werden einige grundlegende Konzepte motivieren und wichtige Begriffe definieren. Wir stellen einige klassische Sätze, wie das Theorem von Noether, vor und diskutieren symplektische Vektorräume. Einerseits ist die Theorie ziemlich abstrakt, andererseits wird dadurch deutlich, warum Hamiltonsche Systeme so wichtig sind.
Im zweiten Teil, der Kapitel drei und vier umfasst, werden wir uns eingehender mit der Existenz- und Eindeutigkeit von Lösungen Hamiltonscher Systeme auseinandersetzen. In Kapitel drei formulieren wir klassische Existenz- und Eindeutigkeitssätze. In Kapitel vier stellen wir einige neue Resultate über planare Hamiltonsche Systeme mit nicht stetiger rechter Seite vor.
Die Graphiken wurden mit der Mac Software "Grapher" erstellt.

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