MASTERARBEIT

Titel der Masterarbeit

"A coordinate independent treatment of the Standard Model Lagrangian"

Verfasser

Harald Ringbauer, BSc BSc

angestrebter akademischer Grad

Master of Science (Msc)

Wien im August 2013

Studienkennzahl lt. Studienblatt:	A 066 821
Studienrichtung lt. Studienblatt:	Mathematik
Betreuer:	Ass.-Prof. Mag. Dr. Stefan Haller, Privatdoz.

Contents

Abstract

The aim of this master thesis is to discuss the mathematical setting for the Standard Model Lagrangian within the framework of principal bundles, principal connections and associated bundles. The formalism developed here allows to state the Lagrangian density for classical fields over a curved spacetime equipped with a spin structure. Applied to standard Minkowski space it yields the usual Standard Model Lagrangian. The main part of this thesis focuses on the construction of kinetic terms for fermionic fields and how to model them with spinor bundles. The latter are irreducible complex Clifford bundle modules. We construct a Dirac operator on them and discuss the existence of invariant forms. Therefore we study $\mathrm{Spin}_{r,s}$ invariant sesquilinear forms on a complex spinor space of dimension $r + s = 2k$.

Zusammenfassung

Das Ziel dieser Masterarbeit ist eine Ausarbeitung des mathematischen Rahmens für die Standard Modell Lagrange-Dichte mithilfe von Prinzipalbündeln, Prinzipalbündelzusammenhängen und assoziierten Vektorbündeln. Der entwickelte Formalismus erlaubt es, die Langrangedichte für klassische Felder über gekrümmten Raumzeiten ausgestattet mit einer Spinstruktur zu formulieren. Angewandt auf den Minkowski-Raum ergibt er die übliche Standard Modell Lagrange-Funktion. Ein großer Teil der Arbeit beschäftigt sich mit der Konstruktion von kinetischen Termen für fermionische Felder und wie man diese mit Spinor-Bündel modelliert. Letztere sind irreduzible komplexe Clifford-Bündel-Moduln und auf ihnen wird ein Dirac-Operator konstruiert. Zu diesem Zweck werden $\mathrm{Spin}_{r,s}$ invariante Sesquilinearformen auf einem komplexen Spinorraum der Dimension $r + s = 2k$ untersucht.

Acknowledgements

I wish to express my gratitude to my supervisor Stefan Haller for the useful comments, remarks and engagement through the learning process of this master thesis.
In addition, I would like to thank Johannes Horak for his detailed and high quality proof reading.
Above all, I am indebted to my parents for their support throughout the years.

Introduction

0.1 Motivation

The Standard Model of particle physics is an integral part of the present-day physical understanding of the world we live in. Developed throughout the mid till late 20th century it is without doubt the most successful of quantum field theories. It describes three of the four known fundamental forces with high accuracy. Together with Einstein's general theory of relativity, which describes gravity, they constitute the foundations of our current world view. Although lacking proven mathematical consistency and leaving some unexplained phenomena, it is believed by many physicists that it will be the basis of further development of physics.

The core of a Quantum Field Theory like the Standard Model is the so called **Lagrangian**, a simple function from which the dynamics of the entire system can be derived. Usually it is stated in a coordinate dependent form best suited for subsequent calculations and physical understanding. The mathematical frame work for the Standard Model Lagrangian is to formulate it as a "**Yang–Mills theory**" which is a general variant of **gauge theory**. The aim of this work is to discuss the mathematical framework. We discuss a more general formalism than neccessary for the Standard Model: We allow curved space times and generalize many things to arbitrary (or at last even) dimensions.

In the first chapter we introduce the general concepts for constructing such a mathematical framework. In the second chapter we discuss the construction of terms that appear in the Standard Model Lagrangian in a general context. In chapter three we verify that applied to the Standard Model and expressed in coordinates these terms match the Standard Model Lagrangian of particle physics, which can be found in the appendix.

0.2 The Standard Model Lagrangian

To motivate the definitions in the upcoming chapters we give a short and informal review of the Standard Model Lagrangian in the to be developed formalism.

Let X be a spacetime, i.e. a smooth semi Riemannian manifold of signature $(1,3)$. Let it be equipped with a spin structure $\mathrm{Spin}_{1,3} \hookrightarrow S(X) \to X$. The Standard Model has $U := U(1) \times SU(2) \times SU(3)$ as internal symmetry group, which is modeled by a (trivial) principal bundle $U(X)$ over X. This bundle is spliced with $S(X)$ to a $\mathrm{Spin}_{1,3} \times U$ principal bundle.

Central to the Standard Model is a unitary representation $\rho : U \to GL(V)$ which decomposes into irreducible parts $\rho_1 \oplus \rho_2 \cdots \oplus \rho_n$. Each irreducible representation appears exactly three times. This corresponds to the experimental fact that there are three generation of fermions. Furthermore there is a complex representation $\Delta_{1,3} : \mathrm{Spin}_{1,3} \to GL(S)$ which is the restriction of an irreducible complex Clifford algebra representation $\mathrm{Spin}_{1,3} \subset Cl_{1,3} \to \mathrm{End}(S)$ on a four dimensional space called Spinor space. This representation will split into two inequivalent irreducible parts $\Delta_{1,3} = \Delta^+_{1,3} \oplus \Delta^-_{1,3}$. We obtain a representation of the full symmetry group by $\Delta^+_{1,3} \times \rho$.

Classical fermionic fields are described as sections $\psi = \psi_1 \oplus \cdots \oplus \psi_n$ of the corresponding associated bundle $E^+ = E^+_1 \oplus \cdots \oplus E^+_n$. In the Standard Model there is a scalar field called Higgs field. It is described as a section ϕ of the associated bundle $\tilde{E}$ to a trivial representation of $\mathrm{Spin}_{1,3}$ and a unitary representation $\tilde{\rho} : U \to GL(\tilde{V})$. The so called gauge fields are described as principal connection ω on the principal bundle $U(X)$.

The classical Standard Model Lagrangian is then a functional:

$$L(\omega, \psi, \phi) = \int_X \mathcal{L}(\omega, \psi, \phi)(x) \ \mathbf{vol}(x)$$

where the Lagrange density $\mathcal{L}$ is an element of $C^\infty(X)$. We can decompose it into a sum of typical terms:

First there are the Yang Mills terms which depend solely on the principal connection ω. Each ω has a corresponding curvature form Ω which can be interpreted as section of the bundle $\Lambda^2(T^*X) \otimes (U(X) \times_{\mathrm{Ad}} \mathfrak{g})$. With multiples of the Killing forms of the $\mathfrak{u}(1)$, $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ parts in the Lie algebra and the metric on X we can construct a bilinear form $b = b_1 + b_2 + b_3$ on that bundle. We are free to choose any multiples of the Killing forms on each part of the Lie algebra and thus obtains three coupling constants g_1, g_2 and $g_3 \in \mathbb{R}$. This leads to a term

$$\omega \mapsto b(\Omega, \Omega) = g_1 b_1(\Omega_1, \Omega_1) + g_2 b_2(\Omega_2, \Omega_2) + g_3 b_3(\Omega_3, \Omega_3)$$

in the Standard Model Lagrangian density.

Next consider the kinetic terms. These depend on both the sections of the associated bundles and on the connection forms. The associated bundle E to the representation $\Delta \times \rho$ splits into $E = E^+ \oplus E^-$, therefore the fermionic fields can be embedded into the sections of E. Since Δ is the restriction of a Clifford algebra representation we have an action of the Clifford bundle $Cl(X)$ on $E = E^+ \oplus E^-$. The principal connection ω and the Levi-Civita connection on $S(X)$ induce a vector bundle connection ∇^ω on the associated bundle E. The last two facts will allow us to define a Dirac operator D on E, which restricts to a map $D^+ : E^+ \to E^-$. Furthermore we will construct a $\mathrm{Spin}_{1,3}$ invariant Hermitian product on the spinor space S which with respect to the splitting $S = S^+ + S^-$ can be written as a sum of invariant sesquilinear pairings $h_1 : S^+ \times S^- \to \mathbb{C}$ and $h_2 : S^- \times S^+ \to \mathbb{C}$. Together with the invariant Hermitian forms of the unitary representation ρ this pairings will lift to sesquilinear pairings $h_1 : E^+ \times E^- \to \mathbb{C}$ and $h_2 : E^- \times E^+ \to \mathbb{C}$. With this we obtain a term:

$$(\omega, \psi) \mapsto h_1(\psi, D^+\psi)$$

To make it real we add the complex conjugate to it. This yields the kinetic term of the fermionic fields in the Lagrangian density. The Higgs field ϕ has a different type of kinetic term which is conceptionally similar to the Yang Mills terms. With the metric of X and the invariant Hermitian form of the unitary representation on $\tilde{V}$ we obtain a Hermitian form $\tilde{h}$ on the complex vector bundle $\tilde{E} \otimes T^*X$. We define:

$$(\phi, \omega) \mapsto \nabla^\omega \phi \mapsto \tilde{h}(\nabla^\omega \phi, \nabla^\omega \phi)$$

and obtain the kinetic term of the Higgs field in the Lagrangian density.

The last classes of terms are the polynomial terms. They solely depend on the sections of the associated vector bundles. We will construct $\mathrm{Spin}_{1,3} \times U$ invariant multilinear forms $q_{ij} : V_i \times \tilde{V} \times \overline{V_j} \to \mathbb{C}$ for some i, j. These give rise to multilinear maps q_{ij} on the sections of the corresponding associated bundles. We define:

$$(\psi_i, \phi, \psi_j) \mapsto (\psi_i, \phi, \overline{\psi_j}) \mapsto m_{ij} q_{ij}(\psi_i, \phi, \overline{\psi_j})$$

where $m_{ij} \in \mathbb{C}$ are elements of a mass matrix which are parameters of the Standard Model. Adding the complex conjugates to these terms results in the Yukawa terms in the Standard Model. They are the basis of the Higgs mechanism.

There is another polynomial term in the Standard model, depending solely on the Higgs field. The invariant Hermitian form of the unitary representation on $\tilde{V}$ yields a Hermitian form s on the corresponding associated bundle. With this we arrive at:

$$\phi \mapsto -m_h^2(q(\phi) - \frac{c^2}{2})^2/(2c^2)$$

as term in the Lagrangian density. Here m_h and c are real parameters of the Standard Model named Higgs mass and Higgs vacuum expectation value. The whole term is named Higgs mass term.

Chapter 1

Important concepts

In this chapter the central concepts needed to formulate the Standard Model Lagrangian are discussed.

1.1 Lie groups

We begin with basic facts about Lie groups and algebras. However, before we give their definition we try to informally motivate the use of symmetry groups in physics: In physical models often a **reference frame** or basis has to be chosen, with respect to which a physical process is described. Each frame is a priori on equal footing, but often certain frames are better suited for calculations than others. In more involved theories it is often necessary to use frames where expressions are not too complicated.

Let V be a finite dimensional vector space over a field K. If we want to do calculations and not only abstractly add vectors we need to choose an ordered basis $\{e_i\}$, called a frame. With respect to this basis each vector can be described with coordinates: $v = v^i e_i$

We can think of a frame as an isomorphism of K-vector spaces from K^n, called coordinate vector space, to V. This correspondence is one-to-one. Choosing another basis yields a different isomorphism. When switching between two different bases, this induces an automorphism of the coordinate vector space. On the other hand each automorphism of the coordinate vector space induces a new coordinate isomorphism. We obtain:

Proposition 1.1.1 *Let V be a vector space. The set of frames can be identified with the set of isomorphisms from a fixed coordinate vector space V_0 to V. The automorphism group of V_0 then acts freely and transitively from the right on the set of all frames.*

This construction can be extended to an arbitrary category, for instance real vector spaces with an inner product of a certain signature. The process of picking a basis to describe a physical problem corresponds to choosing a coordinate isomorphism. Two different frames are linked by an element of the automorphism group. If we would choose a specific coordinate isomorphism, all of them can be identified with an element of the automorphism group. This identification, however, depends on an arbitrary choice.

1.1.1 Definition

For many objects of interest the automorphism group, often called structure group, is a **Lie group**. In this work the following standard definition is used [FH]:

Definition A Lie group is a smooth real manifold which is endowed with a compatible structure of a group:

- The group multiplication $G \times G \to G$ is smooth.
- The group inversion $G \to G$ is smooth.

Definition A **Lie algebra** is a vector space with a skew-symmetric bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

The tangent space at the identity TG_e of a Lie group G can be canonically endowed with the structure of a **Lie Algebra** $\mathfrak{g}$: For $g \in G$ let L_g denote the left multiplication $G \to G$. A vector field V is said to be left invariant if $L_g^* V = V$ for all g in G. It can be shown that left invariant vector fields are closed under the Lie bracket $[\ ,\ ]$ and that the evaluation at e provides an isomorphism to the tangent space at e. More explicitly, if v, w in TG_e there exist unique left invariant vector fields V, W such that $V(e) = v$ and $W(e) = w$. Then define $[v, w] := [V, W](e)$.

The theory of Lie groups and their connection to Lie algebras is very profound. A comprehensive modern book on that subject is [FH]. It can be shown with the so called Baker-Campbell-Hausdorff formula that it is possible to express the group multiplication locally in terms of the Lie algebra and bracket. For this reason Lie algebras, which are easier to investigate due to their linear structure, play a prominent role in the representation theory of Lie groups.

1.1.2 Important examples of Lie groups

A basic example of a Lie group is the general linear group $GL_n(\mathbb{R})$ of invertible $n \times n$ matrices. This is an open subset of the vector space of all $n \times n$ matrices $M_n(\mathbb{R})$ and its manifold structure is obtained in the obvious way where we take the matrix entries as coordinates. It is clear that the multiplication map is differentiable. It follows from Cramer's formula that the inversion is as well. The automorphism group of any real n-dimensional vector space $GL(V)$ is a Lie group: By choosing a basis we obtain a manifold structure(which is the same for all bases) and a Lie group isomorphism to $GL_n(\mathbb{R})$.

The tangent space at the identity of $GL_n(\mathbb{R})$ can be identified with $M_n(\mathbb{R})$, the space of all $n \times n$-matrices. It can be shown that upon this identification the Lie algebra bracket is given by the commutator of the matrices. Hence for a matrix Lie group the Lie algebra can be identified with a subspace of $M_n(\mathbb{R})$ with the commutator as bracket.

Many important Lie groups are isomorphic to subgroups of $GL_n(\mathbb{R})$. This implies that they can be interpreted as matrix Lie groups. They can often be described as subgroups preserving some structure on $\mathbb{R}^n$. This can be for example the volume element. In this case we obtain the Lie group $SL_n\mathbb{R}$, the group of $n \times n$-matrices with determinant 1. We of course have to verify that this is a submanifold, which can be done by applying the implicit function theorem to the defining function $\det(A) - 1 = 0$.

In the Standard Model a certain kind of Lie groups play a central role, the **unitary groups** $U(n)$. First recall some definitions to avoid inconsistence due to notation:

Definition Let V and W be a finite dimensional complex vector spaces. A map

$$\rho : V \times W \to \mathbb{C}$$

is called **sesquilinear** if it is conjugate linear in the first argument and linear in the second. A sesquilinear form

$$S : V \times V \to \mathbb{C}$$

is called **Hermitian** if

$$S(v,w) = \overline{S(w,v)} \quad \forall v,w \in V$$

and **skew Hermitian** if

$$S(v,w) = -\overline{S(w,v)} \quad \forall v,w \in V$$

A Hermitian form h on V is called **inner product** if it is positive definite, i.e.:

$$h(v,v) \geq 0$$

for all $v \in V$ with equality only for $v = 0$.

Then we can define:

Definition Let V be a finite dimensional complex vector space endowed with an inner product h. The unitary group $U(V)$ is the group of all automorphisms of V which preserve h, i.e.

$$U(V) := \{g \in GL(V) : g^*h = h\}$$

If we choose $\mathbb{C}^n$ as vector space with the standard Hermitian inner product $H(v, w) = \bar{v}^t \cdot w$, $U(\mathbb{C}^n)$ is isomorphic to the group of $n \times n$ complex matrices A satisfying $\bar{A}^T A = 1$. From this it can be deduced from the implicit function theorem (see [FH] that this group can be embedded as a compact real submanifold of dimension n^2 into the real manifold $GL_n(\mathbb{C}) \subset GL_{2n}(\mathbb{R})$. Since it is also a subgroup this group is a Lie group. By the usual convention $U(\mathbb{C}^n)$ is denoted by $U(n)$. Each finite dimensional vector space V with inner product is isomorphic to $\mathbb{C}^n$ endowed with the standard Hermitian inner product. Thus all unitary groups are isomorphic to a $U(n)$. The standard complex representation $U(n) \to GL_n(\mathbb{C})$ where each matrix is represented by itself is called the **standard representation**.

Definition The **special unitary group** $SU(n)$ is the subgroup of $U(n)$ formed by matrices of determinant 1.

With the implicit function theorem and the defining property $\det(A) - 1 = 0$ we obtain that $SU(n)$ is a real submanifold of $U(n)$ of dimension $n^2 - 1$ and hence also a Lie group. Again we have a standard representation where each matrix in $SU(n)$ is simply represented by itself.

Via differentiating the defining property it can be shown that the Lie algebra of $U(n)$ only contains **skew Hermitian** matrices. For reasons of dimensions this has to be the whole Lie algebra $\mathfrak{u}(n)$. Differentiating the defining property of $SU(n)$ implies that the matrices in its Lie algebra are traceless. Again for reasons of dimension the Lie algebra $\mathfrak{su}(n)$ is exactly the space of traceless skew Hermitian matrices.

Another kind of Lie groups that appears in this work are the indefinite orthogonal groups:

Definition Let V be a finite dimensional vector space endowed with a non-degenerate, symmetric bilinear form q. The **orthogonal group** $\mathrm{O}(V, q)$ is defined to be the group of all automorphisms of V that preserve the quadratic form, i.e.:

$$\mathrm{O}(V, q) := \{g \in GL(V) : g^*q = q\}$$

Let $\mathbb{R}^{r,s}$ denote the standard $r+s$ dimensional vector space with the standard bilinear form of signature (r,s):

$$q_{r,s}(\mathbf{x},\mathbf{y}) = x_1y_1 + \ldots + x_ry_r - x_{r+1}y_{r+1} - \ldots - x_{r+s}y_{r+s}$$

We define $\mathrm{O}(r,s) := \mathrm{O}(\mathbb{R}^{r,s})$. Via the standard basis of $\mathbb{R}^{r,s}$ we obtain a isomorphism of $\mathrm{O}(r,s)$ to the group of all matrices M that satisfy $M^T\eta M = \eta$, where

$$\eta = \mathrm{diag}(\underbrace{1,\cdots,1}_{r},\underbrace{-1,\cdots,-1}_{s}).$$

By using the implicit function theorem it can be shown that $\mathrm{O}(r,s)$ is a $n(n-1)/2$ dimensional Lie group. Since by Sylvester's law of inertia symmetric bilinear forms can always be diagonalized, all indefinite orthogonal groups are isomorphic to a $\mathrm{O}(\mathbb{R}^{r,s})$. Hence they are a Lie group.

We again have the subgroup consisting of elements with determinant 1:

$$\mathrm{SO}(V,q) := \{g \in \mathrm{O}(V,q) : \det(g) = 1\}$$

which is called **indefinite special orthogonal group**. Accordingly we define $\mathrm{SO}(r,s) := \mathrm{SO}(\mathbb{R}^{r,s})$. It can be shown by using the implicit function theorem that this is a Lie group(see [FH]). In the next section we will discuss some topological properties of $\mathrm{O}(r,s)$ and $\mathrm{SO}(r,s)$.

1.1.3 Lie group representations

Definition A representation of a Lie group G is a Lie group homomorphism from G into $GL(V)$, the automorphism group of a vector space; i.e. a smooth group homomorphism.

Definition A representation of a Lie Algebra $\mathfrak{g}$ is a Lie algebra homomorphism from $\mathfrak{g}$ into $\mathfrak{gl}(V)$, the endomorphism group of a vector space; i.e. a linear map that preserves Lie brackets.

Differentiating Lie group homomorphisms at the identity and the so called exponential map yield the following theorem: (see [FH], page 119 for a proof)

Theorem 1.1.2 ([FH], page 109) *If G and H are Lie groups with G connected and simply connected, the Lie group homomorphism $G \to H$ are in one-to-one correspondence with Lie algebra homomorphisms of the associated Lie algebras.*

Corollary 1.1.3 ([FH], page 109) *This implies in particular that representations of a connected and simply connected Lie Group are in one-to-one correspondence with representations of its Lie algebra.*

A Lie group has a canonical representation of its Lie algebra, the so called adjoint action: First let G act on itself via conjugation:

$$\mathrm{conj} : G \to \mathrm{Aut}(G), \quad \mathrm{conj}_g(h) = ghg^{-1} \tag{1.1}$$

This action is smooth and $\mathrm{conj}_g(e) = e$, so differentiating conj_g at the neutral element maps $\mathfrak{g}$ isomorphically to $\mathfrak{g}$. This map is denoted:

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g} \tag{1.2}$$

Because $\mathrm{conj}_{gh} = \mathrm{conj}_g \circ \mathrm{conj}_h$, differentiating yields $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \circ \mathrm{Ad}_h$. So the mapping $g \to \mathrm{Ad}_g$ is a representation, the so called **adjoint representation** of G. It can be shown that Ad_g is an automorphism of the Lie algebra $\mathfrak{g}$, i.e. $[x, y] = [\mathrm{Ad}_g(x), \mathrm{Ad}_g(y)]$ for all $x, y \in \mathfrak{g}$. Notice that for matrix Lie groups and the canonical identification of their Lie algebras with matrices

$$\mathrm{Ad}_g(h) = ghg^{-1} \quad \forall g \in G, h \in \mathfrak{g}$$

where the right-hand side connotation is matrix multiplication.

We can differentiate $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ at the identity to obtain a map:

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{der}(\mathfrak{g}) \tag{1.3}$$

It can be shown that $\mathrm{ad}(x)(y) = [x, y]$ for all $x, y \in \mathfrak{g}$(see [FH]), and this is another common way to define the Lie bracket on the tangent space at the identity of a Lie group.

1.2 Spacetime

In this section we introduce the concept of spacetime manifolds. We begin with dimension four, which is most important in physics, and then generalize to arbitrary dimension. First introduce the concept of a Minkowski vector space:

Definition A Minkowski vector space is a four-dimensional real vector space with a non-degenerate, symmetric bilinear form with signature $(1, 3) = (+, -, -, -)$.[1]

This definition was introduced by Minkowski in order to describe the mathematical structure of special relativity. Elements of a Minkowski space are often called **events** or **fourvectors**. Choosing a basis, or equivalently (1.1.1) an isomorphism to coordinate vector space $\mathbb{R}^{1,3}$ yields 4 coordinates for each event: (x_0, x_1, x_2, x_3) Beginning with the index 0 is a widespread and generally accepted notation in physics. There x_0 is regarded as time coordinate and the other three components (x_1, x_2, x_3) as space coordinates. The

[1] In mathematics and general relativity often also the signature $(3, 1)$ is used

automorphism group of the Minkowski spacetime, $\mathrm{O}(1,3)$, is the so called **Lorentz group**. This is perhaps the most important and prominent group in 20th century physics and plays a central role in this work. There are a lot of books about Minkowski space and its prominent role in Mathematical physics, [SU] or [N2] offer a good introduction.

1.2.1 Spacetime manifolds

Within the framework of Minkowski space we can not describe gravitational effects. Gravitation seems to curve the worldlines of particles independent of their inner constitution like their charge or mass. Minkowski space alone does not allow such natural "grooves". Einsteins made the observation that at least locally every point has a neighborhood that looks like Minkowski space. The natural question is now "How small?". The answer expressed in turn of the century mathematical terminology is we need an "**infinitesimal neighborhood**" to make this statement exact. This allows to give a modern definition: [N2]

Definition A **spacetime** is a 4-dimensional smooth manifold X with a semi-Riemannian metric $\mathbf{g}$ of index $(1,3)$, called a **Lorentz mectric**. Thus, for each $x \in X$ the tangent space in x is isomorphic to $\mathbb{R}^{1,3}$.

Smooth 4-manifolds are a rich subject, and many of them admit a Lorentzian metric.[2] There are physical restrictions of which spacetime manifolds are of real physical significance. For example, compact spacetime manifolds always admit closed timelike curves(curves with always timelike tangent vectors, i.e. vectors v with $g(v,v) > 0$)[N1]. This would have some bizarre physical interpretations, so compact spacetimes are usually disregarded. Another important restriction follows shortly, when we have the right tools to state it. The Minkoswki space $\mathbb{R}^{1,3}$ itself can be regarded as spacetime manifold. The underlying manifold is of course $\mathbb{R}^4$ and all its tangent spaces can be canonically identified with $\mathbb{R}^{1,3}$ and inherit the Minkowski product. In standard coordinates the metric tensor is obviously constant and hence the curvature tensor of $\mathbb{R}^{1,3}$ vanishes.

Minkowski space plays a central role in the concept of spacetime manifolds. For each point on a spacetime manifold the tangent space is isomorphic to $\mathbb{R}^{1,3}$. This isomorphism is not unique. A coordinate isomorphism into $\mathbb{R}^{1,3}$ has to be picked to locally describe things such as the momentum of a particle. Choosing another basis corresponds to applying an element of the automorphism group of $\mathbb{R}^{1,3}$.

[2] Many here means all non compact ones and all compact ones with Euler characteristic 0, see [O]

1.2.2 Lorentz group

As already mentioned in (1.2), the Lorentz group is defined in the following way:

Definition The Lorentz group is the indefinite orthogonal group $\mathrm{O}(1,3)$, i.e. the group of all invertible linear maps $\mathbb{R}^{1,3} \to \mathbb{R}^{1,3}$ that preserve the bilinear form. An element of the Lorentz group is called a **Lorentz transformation**.

Next we discuss some basic topological properties of the indefinite orthogonal groups $\mathrm{O}(r,s)$(1.1.2). A central observation is that the polar decomposition of $\mathrm{O}(r,s) \subset GL_n(\mathbb{C})$ gives rise to a homotopy equivalence

$$\mathrm{O}(r,s) \to \mathrm{O}(r) \times \mathrm{O}(s).$$

See ([K], proposition 1.143) for details. It is a classical result that $\mathrm{SO}(n)$ is connected and for $n \geq 3$ the fundamental group is $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$(see [K], proposition 1.136). We can deduce the following facts:

- For $r, s \neq 0$ the group $\mathrm{SO}(r,s)$ has exactly two connected components.
- For $r, s \neq 0$ the group $\mathrm{O}(r,s)$ has four connected components and $\pi_0(\mathrm{O}(r,s)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Klein four-group.
- For $s \geq 3$ the fundamental group of $\mathrm{SO}^0(1,s) = \mathrm{O}^0(1,s)$ is $\pi_1(\mathrm{SO}^0(1,s)) \cong \mathbb{Z}_2$

Thus the Lorentz group $\mathrm{O}(1,3)$ has four connected components and the fundamental group of its connected component $\mathrm{O}^0(1,3)$ is $\mathbb{Z}_2$.

In physics usually a time ordering is needed: If there are two events in Minkowski space which are separated by a non space-like vector(i.e. a vector such that $b(v,v) > 0$) then all observers have to agree upon which event happened first, i.e. has a smaller time coordinate. It is also a physical fact that admissible bases are always linked by an orientation preserving Lorentz transformation, that is a transformation with determinant one. With the above homotopy equivalence it can be shown that all time and orientation preserving Lorentz transformations are exactly the connected component of the Lorentz group, $\mathrm{SO}^0(1,3)$.

This means that the underlying space is still Minkowski space, but we have to restrict the automorphism group. We can take care of this by adding more structure to Minkowski space, so that the automorphism group will shrink to $\mathrm{SO}^0(1,3)$. To describe this structure a closer look at timelike vectors(vectors v with $b(v,v) > 0$) is needed. The set containing all of them decomposes into two connected components, which is best shown in coordinates where this sets correspond to $x_0 > 0$ and $x_0 < 0$.

Definition Time oriented and **oriented** Minkowski space is a four dimensional real vector space with a non-degenerate, symmetric bilinear form with signature $(+, -, -, -)$ and additionally a fixed orientation as well as a fixed time orientation. Time orientation corresponds to assigning to one of the connected components of timelike vectors the label "**future directed**" and to the other the label "**past directed**"

It can be easily verified that the automorphism group of time oriented and oriented Minkowski space is the connected component of the Lorentz group $\mathrm{SO}^0(1,3)$.

Now we are ready to introduce the concept of a **timeoriented spacetime manifold**. We want to have a spacetime manifold where each tangent space additionally has the structure of a time oriented Minkowski space. This time orientation should of course be smooth. As with orientation there are various equivalent ways to define this. We will describe the smooth structure with smooth timelike vector fields:

Definition A **time orientatable** spacetime is a spacetime X where the two connected components of timelike vectors in each tangent space can be labeled future and past directed in a way such that the following holds:
Let V be an arbitrary smooth local timelike vector field defined over a coordinate neighborhood $U \subseteq X$, i.e. $g_p(V_p, V_p) > 0$ for each $p \in U$. Then V is either future or past directed, that means V_p is future directed for each $p \in U$ or V_p is past directed for each $p \in U$.
A spacetime X equipped with such a labelling is called **time oriented**.

It is easily verified that on each connected component of a time orientable manifold exactly two time orientations exist, and one is just the reverse of the other. In the next section after introducing some useful concepts a more convenient definition of time orientation is given.

Thus we have two non trivial conditions for a spacetime allowing us to rule out "unphysical" behavior, namely the well known orientation and the time orientation. A third one will follow shortly from the fact that not $\mathrm{SO}^0(1,3)$ is the right automorphism group, but its double cover. To formulate and understand this third restriction the right framework is discussed in the next section.

We can easily generalize the concept of a spacetime to each dimension. It is then defined as a smooth n-dimensional manifold with a semi-Riemannian metric of signature $(1, s)$. Everything generalizes straightforward. Here the group of orientation and time orientation preserving automorphisms of $\mathbb{R}^{1,s}$ is the connected component $\mathrm{SO}(1,s)^0$ of $\mathrm{O}(1,s)$. By the topological facts discussed above the fundamental group of it is $\pi_1(\mathrm{SO}(1,s)^0) \cong \mathbb{Z}_2$.

1.3 Principal bundles

In this section another important concept to formulate the Standard Model Lagrangian in a geometrical way is introduced, the formalism of principal bundles. The definitions and notation will follow [N2].

1.3.1 Definition

Definition Let X be a differentiable manifold and G a Lie group. A **smooth principal bundle** over X with structure group G consists of a differentiable manifold P, a smooth map $\mathcal{P} : P \to X$ onto X and a smooth right action

$$\sigma : P \times G \to P, \quad \sigma(p, g) = p \cdot g$$

such that the following two conditions are satisfied:

- σ preserves the fibers of $\mathcal{P}$, i.e. $\mathcal{P}(p \cdot g) = \mathcal{P}(p)$ for all $p \in P$ and all $g \in G$
- **Local triviality**: For each $x \in X$ there exists an open set V in X containing x_0 and a diffeomorphism $\Psi : \mathcal{P}^{-1}(V) \to V \times G$ of the form $\Psi(p) = (\mathcal{P}(p), \psi(p))$, where $\psi : \mathcal{P}^{-1}(V) \to G$ satisfies $\psi(p \cdot g) = \psi(p) \cdot g$ for all $p \in \mathcal{P}^{-1}(V)$ and all $g \in G$.

The pair (V, Ψ) is called a **local trivialization** of the G-bundle and a family of local trivializations $\{V_j, \Psi_j\}_{j \in J}$ such that the V_j s cover X is called a **trivializing cover** or principal bundle atlas. A principal bundle will be denoted by $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$.

The Cartesian product $U \times G$ gives rise to a principle bundle structure called a **trivial** bundle. Every principal bundle is isomorphic to a trivial bundle via a trivialization. But this trivialization is not unique because there is no distinct element in a fiber of a principal bundle. Every element is treated equally.

1.3.2 Connections

Connections on principal bundles will turn out to be an essential object in physical gauge field theories such as the Standard Model and will describe the interaction of “forces” and “matter” in their Lagrangian. The fibers over a point x, $\mathcal{P}^{-1}(x)$ give rise to the concept of ’vertical’ in a bundle:

Definition Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle and p be a point of P. Then by applying the right action $\sigma : G \to P$ we obtain a diffeomorphism from G to the fiber containing p. By differentiating the right action we obtain a linear isomorphism from the Lie algebra $\mathfrak{g}$ of G to a subspace of the tangent space $T_p(P)$. This subspace is called **vertical subspace** $\mathrm{Vert}_p(P) \subset T_p(P)$.

The vertical subspace is exactly the kernel of the derivative of the projection map $\mathcal{P}$:

$$\mathrm{Vert}_p(P) = \ker(T\mathcal{P})_p$$

All vertical subspaces together $\mathrm{Vert}(P) := \ker(T\mathcal{P}) \subset TP$ span a subbundle called vertical bundle of the tangent bundle.

Each $A \in \mathfrak{g}$ gives rise to a vector field $A^{\#}$ on P via the differential of the map $\sigma_p(g) = p \cdot g$ at the neutral element e:

$$A^{\#}(p) = (T\sigma_p)_e(A) = \frac{d}{dt}(p \cdot \exp(tA))|_{t=0}$$

These **fundamental vector fields** span the vertical bundle.

Then we introduce the following concept:

Definition A (principal) **connection form** on a principal bundle $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ with action σ is a smooth $\mathfrak{g}$-valued 1-form ω on P which satisfies

1. $(\sigma_g)^*\omega = Ad_{g^{-1}} \circ \omega$ for all $g \in G$
2. $\omega(A^{\#}) = A$ for all A in $\mathfrak{g}$

Given a connection form we can immediately define the **horizontal subspace** $\mathrm{Hor}_p(P)$ of $T_p(p)$ by

$$\mathrm{Hor}_p(P) = \{v \in T_p(P) : \omega_p(v) = 0\}$$

All horizontal subspaces together span a subbundle of the tangent bundle, the **horizontal bundle**. We can prove that the tangent bundle is the direct sum of a horizontal and the vertical bundle, i.e.:

$$T_p(P) = \mathrm{Hor}_p(P) \oplus \mathrm{Vert}_p(P) \quad \forall p \in P$$

Proof Suppose v is in $\mathrm{Vert}_p(P) \cap \mathrm{Hor}_p(P)$. By identifying the vertical subspace with the Lie algebra of G the tangent vector v would be the $A^{\#}(p)$ for a $A \in \mathfrak{g}$. But then $\omega_p(v) = \omega_p(A^{\#}(p)) = A$ by (1.3.2)(2). Therefore by definition of the horizontal subspace $v = 0$, thus the two subspaces have a trivial intersection. The dimension of $\mathrm{Hor}_p(P)$ by its definition as Kernel of a linear map is the dimension of $T_p(P)$ minus the dimension of $\mathrm{Vert}_p(P)$, so dim $T_p(P)$ =dim $\mathrm{Vert}_p(P)$ + dim $\mathrm{Hor}_p(P)$. ∎

The horizontal subspace is invariant under the action of G on P in the following sense:

$$(T\sigma_g)_p \,\mathrm{Hor}_p(P) = \mathrm{Hor}_{p \cdot g}(P)$$

which follows from (1.3.2)(1).

Remark The horizontal subbundle is a distribution on the manifold P because it is defined as the kernel of a smooth form. This process can be reversed, it can be shown that every distribution $\mathcal{D}$ on P that satisfies $T_p(P) = \mathcal{D}(p) \oplus \operatorname{Vert}_p P$ and $(T\sigma_g)_p \mathcal{D}_p(p) = \mathcal{D}_{p \cdot g}(P)$ gives rise to a connection form ω on P where $\operatorname{Hor}_p(P) = \mathcal{D}(p)$ by using the connection to project onto the vertical subspace. We could equally well use the distribution picture to define connections and this gives a more intuitively accessible method of relating to a connection form. See [N1] for details. But for further use in this work the given definition is better suited.

1.3.3 Curvature

Next we define the curvature of a connection form. Let d denote the usual exterior derivative.

Definition Let ω be a connection form on a smooth principal bundle $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$. Then the curvature $\Omega \in \Lambda^2(P, \mathfrak{g})$ is defined to be the horizontal part of the exterior derivative of ω:

$$\Omega_p(v, w) := (d\omega)_p(v^H, w^H) \tag{1.4}$$

The curvature form is bilinear, skew symmetric and smooth. How to fill this definition with life and give intuitive interpretations(obstruction to flatness, parallel transport along closed curves...) can be found in [N1].

Our definition of curvature(1.3.3) is, although short, not very suited for calculations. To this end we first need the following definition:

$$[\omega, \omega]_p(v, w) := [\omega_p(v), \omega_p(w)] \ \forall v, w \in T_p(P) \tag{1.5}$$

Because of the skew symmetry of the Lie bracket $[\omega, \omega]$ is a $\mathfrak{g}$ valued exterior two form. Now the way to the famous **Cartan Structure Equation** is paved:

Proposition 1.3.1 *Let Ω be the curvature form of ω as in (1.3.3). Then:*

$$\Omega = d\omega + [\omega, \omega] \tag{1.6}$$

We can use the obvious skew symmetry of both sides, fix a $p \in P$ and $v, w \in T_p(P)$ and decompose them in vertical and horizontal parts to reduce the proof to the following cases:

1. v and w both horizontal
2. v and w both vertical
3. v vertical and w horizontal

For the full proof with all details see [N2].

1.4 Associated bundles

In this section another concept central to the formulation of the Standard Model Lagrangian is be introduced, the so called associated bundle. Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a smooth principal G-bundle with right action σ. Let V be a finite dimensional vector space with a smooth representation of G. This defines a smooth left action of G on $P \times V$:

$$((p, v), g) \to (p \cdot g^{-1}, g \cdot v)$$

We use the same dot to indicate all actions of G. It will be clear from the context which one to use. The orbits of this action induce an equivalence relation on $P \times V$. We denote the equivalence class of (p, g) as $[p, g]$. Let $P \times_G V$ be the set of equivalence classes and give it the quotient topology determined by the quotient map $\mathcal{Q}$.
Now we can define a projection mapping

$$\mathcal{P}_G : P \times_G V \to X, \quad \mathcal{P}_G([p, v]) = \mathcal{P}(p)$$

This is well defined, because the right action on the principal bundle is fiber preserving. For $x \in X$ and $p \in P$ such that $\mathcal{P}(p) = x$ the fiber above $x \in X$, $\mathcal{P}_G^{-1}(x)$, is given by the set $\{[p, v] : v \in V\}$ because G acts transitively on the fibers of P.

Now we fix a $x_o \in X$ and let (U, Ψ) be a **trivialization** of P containing x_0. There is a canonical **associated cross section** s given by:

$$s(x) := \Psi^{-1}(x, e)$$

With this we define a map:

$$\tilde{\Phi} : U \times V \to \mathcal{P}_G^{-1}(U), \quad \tilde{\Phi}(x, v) = [s(x), v]$$

which is continuous since it is the composition of continuous maps. By using that the right action acts freely and transitively we conclude that $\tilde{\Phi}$ has an inverse:

$$\tilde{\Psi} : \mathcal{P}_G^{-1}(U) \to U \times V, \quad \tilde{\Psi}([s(x), v] = (x, v)$$

To show that $\tilde{\Psi}$ is continuous, we observe that $\tilde{\Psi} \circ \mathcal{Q}$ is given by $(p, v) \to [p, v] \to (\mathcal{P}(p), g \cdot v)$ for a certain g. This shows that $\tilde{\Phi} = \tilde{\Psi}^{-1}$ and $\tilde{\Psi}$ are **homeomorphisms**.

This shows that the structure $(P \times_G V, X, \mathcal{P}_G, V)$ is a locally trivial vector bundle with the same trivializing neighborhoods as the original principal bundle. To verify that it can be equipped with a unique differentiable structure such that the projection $\mathcal{P}_G$ is smooth and each $\tilde{\Psi}$ is a diffeomorphism it is enough to show that the 'coordinate change' maps

$$\tilde{\Psi}_i \circ \tilde{\Psi}_j^{-1} : (U_j \cap U_i) \times V \to (U_j \cap U_i) \times V$$

are diffeomorphisms. A direct calculation shows that in fact

$$\tilde{\Psi}_i \circ \tilde{\Psi}_j^{-1}(x, v) = (x, g_{ji}(x) \cdot v)$$

where g_{ji} is the unique smooth transition function of the original pincipal bundle: $U_j \cap U_i \to G$ such that $s_j(x) = s_i(x) \cdot g_{ij(x)}$.
Because the action on the vector space is smooth we are finished. This leads to the following definition:

Definition $(P \times_G V, X, \mathcal{P}_G, V)$ is a **smooth vector bundle**, the so called **associated bundle**. The trivializing neighborhoods of X of the initial principal bundle are again trivializing neighborhoods so we can take these as trivializing cover. The transition functions are given by the action on V of the corresponding original transition functions. The fibers are now isomorphic to V.

Remark The same construction of surgically replacing fibers can be done with any manifold F on which G acts smoothly on the left. The process is exactly the same, just literally replace V with F. In this work we only need the concept of associated vector bundles.

If the representation of G on V is denoted by ρ and the original principal bundle by P, we denote the associated vector bundle by:

$$\mathcal{P}_\rho : P \times_\rho V \to X$$

An important example of an associated bundle is the vector bundle associated to the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ of a principal bundle $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$. This bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ is called the **adjoint bundle** and we denoted it by $\mathrm{Ad}\ P$.

1.4.1 Frame bundles

Given a finite dimensional vector bundle $\pi : E \to X$ we can use all frames for all fibers to construct a principal bundle. Recall from (1.1.1) that a frame is an ordered basis or, equivalently an isomorphism from coordinate vector space K^n. We give an outline of this process. Let $L(E)_x$ denote the set of all frames for a fiber over $x \in X$(interpreted as vector space) and define: $L(E) = \bigcup_{x \in X} L(E)_x$. We have a projection map:

$$\mathcal{P}_L : L(E) \to X, \quad \mathcal{P}_L(p) = x \text{ for } p \in L(E)_x$$

The automorphism group of the coordinate vector space K^n is the general linear group $GL_n(K)$. By (1.1.1) we obtain a transitive and free right action on $L(E)_x$. This induces an action on $L(E)$ which is fiber preserving. The objective now is to provide $L(E)$ with a topology and manifold structure such that

$$GL_n(K) \to L(E) \xrightarrow{\mathcal{P}_L} M$$

is a smooth principal $GL_n(K)$ bundle over M, called the **(linear) frame bundle** of X. To do so, we fix a x_0 in M. Let U be a trivializing neighborhood of x_0 in E with trivializing map $\tilde{\psi} : \pi^{-1}(U) \to U \times K^n$, which gives for each point x in U an isomorphism $\tilde{\psi_x}$ from K^n to the fiber over x.

Each frame in $L(E)_x$ is an isomorphism p from K^n to the fiber as well, so there is a unique automorphism $g_\psi(p)$ of K^n(an element of $GL_n(K)$ such that $p = \tilde{\psi_x} \circ g_\psi(p)$ This gives rise to a map

$$\psi : \mathcal{P}_L^{-1}(U) \to U \times GL_n(K) : \psi(p) = (\mathcal{P}_L(p), g_\psi(p))$$

These ψ will become the trivializing maps of $L(E)$. We see that ψ is bijective. We can now use the topology on U and on $GL_n(K)$ to induce a topology on $L(E)$: A subset $\mathcal{U}$ of $L(E)$ is declared to be open if and only if for each trivialization (U, ψ) $\psi(\mathcal{U} \cap \mathcal{P}_L^{-1}(U))$ is open in $U \times GL_n(K)$. It is easily verified that this really is a **topology** on $L(E)$ and makes $\mathcal{P}_L$ a continuous map.

Next we show how $L(E)$ can be made a topological manifold: To obtain charts, we need to show that the trivializations ψ are **homeomorphisms** or, equivalently, continuous and open. For the latter, let W be an open set in $\mathcal{P}_L^{-1}(U)$. Because $\mathcal{P}_L^{-1}(U)$ is open in $L(E)$, W is also open in $L(E)$. By the definition of the topology on $L(E)$, $\psi(W \cap \mathcal{P}_L^{-1}(U)) = \psi(W)$ is open in $U \times GL_n(K)$ which shows that each trivialization is **open**.

To show continuity of ψ, let (U, ψ) and (V, ρ) be two trivializations with $U \cap V \neq \emptyset$. Then take a closer look at:

$$\psi \circ \rho^{-1} : (U \cap V) \times GL_n(K) \to (U \cap V) \times GL_n(K)$$

By definition of ψ and ρ each p in $\mathcal{P}_L^{-1}(U \cap V)$ can be expressed as $p = \tilde{\psi_x} \circ g_\psi(p)$ and $p = \tilde{\rho_x} \circ g_\rho(p)$. From this it can be deduced that $g_\psi(p) = \tilde{\psi_x}^{-1} \circ \tilde{\rho_x} \circ g_\rho(p)$. It follows that $\psi \circ \rho^{-1}(x, g)$ is $(x, \tilde{\psi_x}^{-1} \circ \tilde{\rho_x} \circ g)$ which is composition of smooth maps and hence a diffeomorphism. [3].

[3] Observe that the map $\tilde{\psi_x}^{-1} \circ \tilde{\rho_x}$ is an element of $GL_n(K)$ and corresponds to the transition map of the original trivializations of the bundle E

Now the continuity of ψ can be proven. Let Z be an open set in $U \times GL_n(K)$. We have to show that $\psi^{-1}(Z)$ is open in $L(E)$. By definition of the topology on $L(E)$ this is exactly the case if for each trivialization (V, ρ) the set $\rho(\psi^{-1}(Z) \cap \mathcal{P}_L^{-1}(V))$ is open in $U \times GL_n(K)$. A brief calculation shows that:

$$\rho(\psi^{-1}(Z) \cap \mathcal{P}_L^{-1}(V)) = (\rho \circ \psi^{-1})(Z \cap (U \times GL_n(K))$$

which is open because as just shown $\rho \circ \psi^{-1}$ is a diffeomorphism. This shows that ψ is continuous. By the above it is also open and hence a **homeomorphism**.

It is not hard to verify that $L(E)$ is Hausdorff. Together with the fact that the trivializations are homeomorphisms this makes $L(E)$ a **topological manifold**. Its dimension is $m + n^2$, where m denotes the rank of the vector bundle E. How to define a smooth structure on it is clear by the above observation that the trivialization changes are diffeomorphisms. Within trivializations it can be shown that $\mathcal{P}_L$ as well as the right action are smooth. It follows from the definition of the trivializations that they respect the right action. Eventually we arrive at the following result:

Proposition 1.4.1 *Given a finite dimensional vector bundle E of rank n, the* ***(linear) frame bundle*** *as constructed above is a principal bundle $GL_n(K) \hookrightarrow L(E) \xrightarrow{\mathcal{P}_L} X$. A trivializing cover of E is again a trivializing cover.*

An important example of a frame bundle is the associated frame bundle to the tangent bundle of a smooth manifold X, we denote it by $L(X)$.

Given a smooth vector bundle E we can construct its linear frame bundle $L(E)$ and subsequently the associated bundle to $L(E)$ via the natural representation of $GL_n(K)$ on K^n. Due to the construction of associated bundles they are a quotient of $L(E) \times K^n$. More specifically, an element in the associated bundle is the orbit of the action of $GL_n(K)$ on $L(X) \times K^n$:

$$[p, v] = \{(p \cdot g^{-1}, g \cdot v) : g \in GL_n(K)\} \tag{1.7}$$

With the interpretation of p as an isomorphism from K^n to the fiber in E over $\mathcal{P}_L(p)$ and observing that the map

$$\omega : L(E) \times K^n \to E, \omega(p, v) = p(v)$$

is constant on the orbits (1.7) we see that ω factors to a map

$$\Omega : L(E) \times_{GL_n(K)} K^n \to E$$

This map is fiber preserving and linear. By trivializations it can be shown that it is also a diffeomorphism. Thus we obtain an isomorphism of smooth vector bundles over X.

This shows how closely related these two constructions are. All vector bundles of rank n are isomorphic to a vector bundle associated to a principal $GL_n(K)$ bundle.

1.4.2 Orthonormal frame bundles

Let $\pi : E^{'} \to M$ be a vector bundle carrying additional structure on its fibers in such a way that the automorphism group of a typical fiber $V^{'}$ can be interpreted as a Lie subgroup of $GL(V^{'})$. Then we can carry out a construction similar to the frame bundle construction (1.4.1). Let $F(E^{'})_x$ be the set of all frames for a fiber over $x \in M$, i.e. all isomorphisms of $V^{'}$ to the fiber over x. Then define $F(E^{'}) = \bigcup_{x \in X} F(E^{'})_x$, and the natural projection map $\mathcal{P}_F : F(E^{'}) \to X, \mathcal{P}_F(p) = x$ for $p \in F(E^{'})_x$. Then we can exactly repeat the linear frame bundle construction to obtain a principal bundle called **associated frame bundle**. An exact formulation of this concept will not be given. We rather look at some explicit examples:

- Let X be a semi-Riemannian manifold of signature (p, q). For each $x \in M$ exists a neighborhood such that the tangent bundle TM can be trivialized via an orthonormal frame. Thus it can be interpreted as vector bundle with typical fiber $\mathbb{R}^{p,q}$. The **orthonormal frame bundle** is the $\mathrm{O}(p, q)$ bundle we obtain by repeating the construction of the linear frame bundle for orthonormal frames of the tangent bundle. This principal bundle will be denoted as:

$$\mathrm{O}(p, q) \hookrightarrow \mathrm{O}(p, q)(X) \xrightarrow{\mathcal{P}_L} X \tag{1.8}$$

- Let X be an oriented and time oriented spacetime as defined in (1.2.1).As above its tangent bundle can be interpreted as a vector bundle where the fibers are isomorphic to oriented and time-oriented Minkowski space. Its automorphism group is the connected component of the Lorentz group $\mathrm{SO}^0(1, 3)$. Repeating the construction of the linear frame bundle for orthonormal, oriented and time oriented frames yields the principal bundle:

$$\mathrm{SO}^0(1, 3) \hookrightarrow \mathcal{L}(X) \xrightarrow{\mathcal{P}_L} X \tag{1.9}$$

The latter bundle called **oriented, time oriented orthonormal frame bundle** will turn out to be a core part for the formulation of the Standard model.

Another application of the associated bundle and frame bundle concept is a description of **time orientation** and **space orientation** of spacetimes.

The pseudo orthogonal group $\mathrm{O}(p,q)$ allows two important homomorphisms into $\mathbb{Z}_2$: The so called space orientation character σ_+ and the time orientation character σ_-. These characters map the respective orientation preserving elements in $\mathrm{O}(p,q)$ to 1 and the respective orientation flipping elements to -1. Given a semi Riemannian manifold X and its frame bundle $\mathrm{O}_{p,q}(X)$, we can construct the following associated bundles:

$$\mathrm{O}(p,q) \times_{\sigma_+} \mathbb{Z}_2 \tag{1.10}$$

$$\mathrm{O}(p,q) \times_{\sigma_-} \mathbb{Z}_2 \tag{1.11}$$

These are called **space orientation bundle** and **time orientation bundle**. A global section of these bundles is called space orientation or time orientation of the semi Riemannian manifold X. If such a space or time orientation exists, which means that the bundle has a global trivialization, X is called space respectively time orientable. For a Lorentzian manifold $(p = 1)$ the previously defined time orientation(1.2.2) of a spacetime is seen to coincide with this more general concept: We could equally well construct the time orientation bundle by letting the frame bundle act on the set of connected components of timelike vectors. Then a continuous choice of such a connected component corresponds to a section of the time orientation bundle.

Remark Let us reflect upon the physical interpretations of the concepts of this section: The associated frame bundle construction takes all possible frames in each tangent space of spacetime and groups them into one object, the associated frame bundle. If a physicist wants to describe a process happening in a neighborhood of a point, he has to pick a frame in each point, which is a smooth section of the associated bundle or equivalently a local trivialization. Of course he also needs now a formalism to describe objects. This will turn out to be sections of associated bundles to the frame bundle, so basically vectors which transform in a certain way under a basis change.

1.4.3 Tensorial forms

In this section a certain kind of vector valued forms on principal bundles, the so called tensorial forms are introduced. The notation and proofs follow [N2].

Definition Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle with right action σ. Let V be a vector space and $\rho : G \to GL(V)$ be a representation of G on V, denoted by $(\rho(g))(v) = g \cdot v$.
A V-valued k-form ϕ on P is

- **pseudotensorial** of type ρ if and only if the following equation holds
$$\sigma_g^* \phi = g^{-1} \cdot \phi \quad \forall g \in G \tag{1.12}$$

- **tensorial** of type ρ if and only if it is pseudotensorial of type ρ and **horizontal** in the sense that $\phi_p(v_1, \cdots, v_k) = 0$ if one of the $v_1, \cdots, v_k \in T_p(P)$ is vertical.[4] A 0-form is taken to be vacuously horizontal. The set of all V-valued k-forms on P that are tensorial of type ρ provided with the obvious real vector space structure will be denoted by $\Lambda^k_\rho(P, V)$

That pseudotensorial forms are a larger class than tensorial ones can be seen by a connection form ω: By its definition (1.3.2) it is a pseudotensorial form of type Ad, but not tensorial(indeed the kernel of ω_p is $\mathrm{Hor}_p(P)$ and not $\mathrm{Vert}_p(P)$).

Lemma 1.4.2 *Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle and ϕ a pseudotensorial form of type ρ with the same notations as above. Then the exterior derivative $d\phi$ is also pseudotensorial of type ρ*

Proof With the facts that d is a natural transformation and that the action of g on V is linear we obtain:

$$\sigma_g^*(d\phi) = d(\sigma_g^*\phi) = d(g^{-1} \cdot \phi) = g^{-1} \cdot d\phi$$

∎

So d is a differential operator on pseudotensorial forms. To get a similar differential operator on tensorial forms, we need a principal connection on $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$:

Lemma 1.4.3 *Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle with a principal connection form ω and τ a pseudotensorial form of type ρ with the same notation as above. Then the V-valued k-form τ^H on P defined by*

$$\tau_p^H(v_1, \cdots, v_k) = \tau_p(v_1^H, \cdots, v_k^H) \tag{1.13}$$

is tensorial of type ρ. Here v^H is the horizontal part of $v \in T_p(P)$ determined by the connection form ω.

Proof The defined τ^H is a smooth V-valued k-form: It is skew-symmetric, C^∞-linear and as composition of smooth projections and a smooth form smooth. We see directly from the definition that τ^H is horizontal. To show that τ^H is pseudotensioral we use the fact that the action of G on tangent vectors via the differential of σ respects the splitting in horizontal and vertical parts:

$$((T\sigma_g)_p(v))^H = (T\sigma_g)_p(v^H) \ \forall g \in G, p \in P, v \in T_p(P) \tag{1.14}$$

[4]Note that we do not need a connection to define horizontal.

This follows from observing that $(T\sigma_g)_p$ maps vertical vectors to vertical ones and $(T\sigma_g)_p$ maps horizontal vectors to horizontal ones. The latter follows directly from the invariance of the horizontal space. Because the splitting of a tangent vector in its horizontal and vertical part is unique we obtain (1.14).

Let $\mathbf{v}$ denote the k arguments $v_1, \cdots, v_k$ of τ. With this we get the following:

$$
\begin{aligned}
(\sigma_g^*\tau^H)_p(\mathbf{v}) = \tau^H_{p\cdot g}((T\sigma_g)_p\mathbf{v}) = \tau_{p\cdot g}(((T\sigma_g)_p\mathbf{v})^H) \\
= (\sigma_g^*\tau)_p(\mathbf{v}^H) = g^{-1}\cdot\tau_p(\mathbf{v}^H) = g^{-1}\cdot\tau_p^H(\mathbf{v})
\end{aligned}
$$

This shows that τ^H is also pseudotensorial, hence it is tensorial. ∎

We obtain a covariant exterior derivative on pseudotensorial forms:

Proposition 1.4.4 *Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle with a connection form ω and ϕ a pseudotensorial k-form of type ρ in the same notation as above. Then the* ***covariant exterior derivative*** *$d^\omega\phi$ of ϕ, defined by*

$$(d^\omega\phi)_p(v_1, \cdots, v_{k+1}) = (d\phi)^H_p(v_1, \cdots, v_{k+1}) = (d\phi)_p(v_1^H, \cdots, v_{k+1}^H) \quad (1.15)$$

is a tensorial $(k+1)$-form of type ρ. In particular,

$$d^\omega : \Lambda^k_\rho(P,V) \to \Lambda^{k+1}_\rho(P,V) \quad (1.16)$$

A principal connection form on a principal bundle gives rise to a canonical covariant derivative on tensorial forms. We have already seen an example of an action of a covariant derivative: The curvature form Ω (1.3.3) is the covariant exterior derivative of the connection form ($\Omega = d^\omega\omega$). As with the curvature in its original definition, the definition above is inconvenient form for calculations. But if viewed as operator on tensorial forms $\Lambda^k_\rho(P,V) \to \Lambda^{k+1}_\rho(P,V)$ there is again a useful equation similar to the Cartan structure equation, which is given below. First w define the following product of forms:

Definition Let ρ be a smooth representation of a Lie group G on a vector space V. Then consider the Lie algebra homomorphism $\rho' : \mathfrak{g} \to \mathfrak{gl}(V)$ obtained by differentiating the Lie group homomorphism $\rho : G \to GL(V)$. The result is a bilinear map:

$$\mathfrak{gl}(V) \times V \to V, \quad (A, v) \mapsto A \cdot v := \rho'(A)(v) \quad (1.17)$$

We then define a product $\dot\wedge$ for $\mathfrak{g}$-valued k-forms α and V-valued l-forms β:

$$\alpha \,\dot\wedge\, \beta(v_1, \cdots v_{k+l}) := \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}} (-1)^{\mathrm{sgn}(\sigma)}\alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)})\cdot\beta(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

where we sum over all permutations in S_{k+l}

With this we can write down the following form of the covariant derivative:

Proposition 1.4.5 *Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle with a principal connection form ω and ϕ a tensorial k-form of type ρ in the same notation as above. Then the following equation holds:*

$$d^\omega \phi = d\phi + \omega \dot{\wedge} \phi$$

The proof is analogous to the proof of the Cartan structure equation(1.3.1). We fix a $p \in P$ and $v_1 \dots v_{k+1} \in T_p(P)$ and decompose them in vertical and horizontal parts. By using the multilinearity the proof reduces to the following cases:

1. Each $v_1, \cdots, v_{k+1}$ is horizontal.
2. Two or more of $v_1, \dots, v_{k+1}$ are vertical.
3. Precisely one of $v_1, \dots, v_{k+1}$ is vertical and the rest horizontal.

The first two cases are trivial, the third requires some work. The full proof can be found in [N2].

Next we state and prove a fact that is central for this thesis: Tensorial forms correspond to forms on the base manifold with values in an associated bundle.

Theorem 1.4.6 *Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle with a connection form ω and let $\rho : G \to GL(V)$ be a representation of the structure group G on a finite dimensional vector space. Then there exists a canonical linear isomorphism from the space of tensorial k-forms of type ρ, $\Lambda^k_\rho(P, V)$ to the space of k-forms on X with values in the associated bundle $P \times_\rho V$, $\Lambda^k(X, P \times_\rho V) = \Gamma(\Lambda^q T^* X \otimes P \times_\rho V)$.*

Proof First recall that each p in P induces a linear isomorphism of V with the fiber over $x = \mathcal{P}(p)$ in the principal bundle by:

$$p : V \to (P \times_\rho V)_x, \quad p(v) = [p, v]$$

Now define a map

$$\iota : \Lambda^k_\rho(P, V) \to \Lambda^k(X, P \times_\rho V)$$

by pointwise defining:

$$\iota(\tau)_x(v_1, \cdots v_k) := [\tilde{x}, \tau_{\tilde{x}}(\tilde{v_1}, \cdots, \tilde{v_k})] \tag{1.18}$$

where $\tilde{x} \in P$ is an arbitrary element in the fiber over x: $\mathcal{P}^{-1}(x)$ and $\tilde{v_i}$ are tangent vectors in $\tilde{x}$ such that $(T\mathcal{P})_{\tilde{x}}(\tilde{v_i}) = v_i$

We have to show that ι is well defined. This is seen by noting that:

- $(T\mathcal{P})_{\tilde{x}}(\tilde{v_i}) = v_i$ and $(T\mathcal{P})_{\tilde{x}}(\tilde{\tilde{v_i}}) = v_i$ implies that $\tilde{v_i} = \tilde{\tilde{v_i}} + h$, where h is vertical.
- $\tau_{\tilde{x}\cdot g}(T\sigma_g(\tilde{v_1}), \cdots, T\sigma_g(\tilde{v_k})) = (g^*\sigma)_{\tilde{x}}(\tilde{v_1}, \cdots, \tilde{v_k})) = g^{-1}\tau_{\tilde{x}}(\tilde{v_1}, \cdots, \tilde{v_k})$ by using that τ is tensorial.

Thus $[\tilde{\tilde{x}}, \tau_{\tilde{\tilde{x}}}(\tilde{\tilde{v_1}}, \cdots, \tilde{\tilde{v_k}})] = [\tilde{x}, \tau_{\tilde{x}}(\tilde{v_1}, \cdots, \tilde{v_k})]$ and ι_x is well defined. By using charts we can see that it depends smoothly on x, so we obtain an element $\iota(\tau)$ in $\Lambda^k(X, P \times_\rho V)$. The map ι is linear.

Now define a map:

$$\kappa : \Lambda^k(X, P \times_\rho V) \to \Lambda^k_\rho(P, V)$$

by defining pointwise:

$$\kappa(\phi)_p(\tilde{v_1}, \cdots \tilde{v_k}) := p^{-1} \circ \phi_x(v_1, \cdots v_k) \tag{1.19}$$

where as above $T\mathcal{P}(\tilde{v_i}) = v_i$ and $\mathcal{P}(p) = x$. The map κ_x is well defined, linear and depends smoothly on x. Therefore we obtain a linear map κ into the V-valued k-forms on P. It is left to prove that $\kappa(\phi)$ is tensorial, which follows from:

$$\begin{aligned} g^*\kappa(\phi)_p(v_1, \cdots v_k) =& \kappa(\phi)_{p\cdot g}(T\sigma_g\tilde{v_1}, \cdots T\sigma_g\tilde{v_k}) = \\ =& (p \cdot g)^{-1} \circ \phi_x(v_1, \cdots v_k) = g^{-1} \cdot \kappa(\phi)_p(v_1, \cdots v_k) \end{aligned}$$

It is horizontal since for vertical $\tilde{v_i}$ the projection $v_i = T\mathcal{P}(\tilde{v_i})$ is 0.
The two constructions are inverse to each other, which can directly be verified from their pointwise definitions.((1.18) and (1.19)) ∎

Given a section of the principal bundle P, the above map $\iota : \Lambda^k_\rho(P, V) \to \Lambda^k(X, P \times_\rho V)$ can be written in the following form:

Proposition 1.4.7 *Let $s : X \to P$ be a section of the principal bundle P and τ be a tensorial k-form on P. Then the above map $\iota : \Lambda^k_\rho(P, V) \to \Lambda^k(X, P \times_\rho V)$ is given by:*

$$\iota(\tau)_x(v_1, \dots v_k) = [s(x), (s^*\tau)_x(v_1, \dots v_k)] \tag{1.20}$$

Proof By the definition of the pullback:

$$(s^*\tau)_x(v_1, \dots v_k) = \tau_{s(x)}((Ts)_x v_1, \dots, (Ts)_x v_n)$$

But $\mathcal{P} \circ s = \mathrm{id}$, thus

$$(T\mathcal{P})_{s(x)}((Ts)_x v_i) = v_i$$

This shows that the right-hand side of (1.20) can be chosen in the definition of ι in (1.18). ∎

A special case of identification (1.4.6) is the case $k = 0$. The theorem then allows us to identify smooth sections of an associated bundle $P \times_\rho V$ with equivariant functions $P \to V$. Here a function $f : P \to V$ is called equivariant if $f(p \cdot g) = \rho(g^{-1})f(p)$ for all g in G.

One also has now a means of defining a covariant exterior derivative for associated bundle valued forms:

Proposition 1.4.8 *Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle with a connection form ω and let $\rho : G \to GL(V)$ be a representation of the structure group G on a finite dimensional vector space. Then ω induces a canonical exterior derivative on the associated bundle $P \times_\rho V$ valued forms:*

$$\nabla^\omega : \Lambda^k(X, P \times_\rho V) \to \Lambda^{k+1}(X, P \times_\rho V)$$

Proof Use (1.4.6) and the covariant exterior derivative d^ω: (1.4.4) to obtain a map:

$$\nabla^\omega : \Lambda^k(X, P \times_\rho V) \overset{\iota^{-1}}{\cong} \Lambda^k_\rho(P, V) \xrightarrow{d^\omega} \Lambda^{k+1}_\rho(P, V) \overset{\iota}{\cong} \Lambda^{k+1}(X, P \times_\rho V)$$

∎

It turns out that this canonical exterior derivative in fact is a vector bundle connection, as already indicated by the notation:

For $k = 0$ we obtain a map

$$\nabla^\omega : \ \Gamma(P \times_\rho V) \to \Gamma(T^*X \otimes P \times_\rho V) \tag{1.21}$$

We can verify by direct calculation using (1.4.5) and the explicit isomorphism (1.4.6) that this operator also fulfills the Leibnitz rule:

$$\nabla^\omega(f\sigma) = \nabla^\omega f + \sigma \otimes df$$

for all smooth functions f on X and sections σ of $P \times_\rho V$. Therefore ∇^ω is indeed a **vector bundle connection**.

Remark If $G = Gl(V)$ and the representation ρ is infinitesimaly effective, i.e. $T\rho_e : \mathfrak{g} \to \mathfrak{gl}(V)$ is injective, each vector bundle connection ∇ on $P \times_\rho V$ gives rise to a principal connection on P and the two constructions are inverse to each other(See [M], theorem 19.9).

The induced vector bundle connection ∇^ω has a property which is useful later on:

Proposition 1.4.9 *Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle equipped with a connection form ω and let $\rho_1 : G \to GL(V)$ and $\rho_2 : G \to GL(W)$ be representations of the structure group G on finite-dimensional vector spaces V and W. Furthermore let $\phi : V \to W$ be an equivariant map, i.e.:*

$$\phi \circ \rho_1(g) = \rho_2(g) \circ \phi \quad \forall g \in G$$

Then:

- *The equivariant map $\phi : V \to W$ induces a vector bundle homomorphism on the associated bundles:*

$$\tilde{\phi} : \Gamma(P \times_{\rho_1} V) \to \Gamma(P \times_{\rho_2} W)$$

- *For such a $\tilde{\phi}$:*

$$\nabla^{\omega}_{X}(\tilde{\phi} s) = \tilde{\phi}(\nabla^{\omega}_{X} s) \tag{1.22}$$

for all vector fields X in $\Gamma(TM)$ and all sections s of $P \times_{\rho_1} V$

Proof Let s be a section of $P \times_{\rho_1} V$. By the above theorem (1.4.6) this corresponds to a tensorial 0-form on P with values in V. Let us denote this map $P \to V$ again by s. We then define a map:

$$\tilde{s} : P \xrightarrow{s} V \xrightarrow{\phi} W, \quad \tilde{s} = \phi \circ s$$

Since ϕ is equivariant, $\tilde{s}$ is also tensorial and therefore corresponds to a section of $P \times_{\rho_2} V$ which we also denote by $\tilde{s}$. This map is $C^{\infty}(X)$ linear and thus yields the desired vector bundle homomorphism:

$$\tilde{\phi} : \Gamma(P \times_{\rho_1} V) \to \Gamma(P \times_{\rho_2} W), \quad s \mapsto \tilde{s}$$

To prove the second point it is sufficient to show that the corresponding tensorial forms of both sides in (1.22) agree. Let $\tilde{X}$ be a vector field on P such that $T\mathcal{P}(\tilde{X}) = X$. Then the left-hand side is given in terms of tensorial forms by:

$$(d^{\omega}(\phi \circ s))(\tilde{X}) = d(\phi \circ s)(\tilde{X}^{H}) = \phi \circ (ds)(\tilde{X}^{H})$$

In the last step we used that $d(\phi \circ s) = \phi \circ ds$, which holds because ϕ is a linear map. The right-hand side in terms of tensorial forms becomes:

$$\phi \circ (d^{\omega} s)(\tilde{X}) = \phi \circ (ds)(\tilde{X}^{H})$$

Hence both side of (1.22) agree. ∎

Coordinate formula

Here an explicit coordinate formula for the induced covariant derivative ∇^ω is calculated. Let $G \hookrightarrow P \stackrel{\mathcal{P}}{\to} X$ be a principal bundle equipped with a connection form ω and an associated bundle $P \times_\rho V$. Then, as above, there exists a vector bundle connection ∇^ω on $P \times_\rho V$ which we express in local coordinates.

Let U be a coordinate neighborhood of a point $x \in X$ with coordinate functions $(x_1, \dots x_n) : U \to W \subset \mathbb{R}^n$ and $s : \mathbb{R}^n \to U \to P$ be a local section(with respect to the coordinates) of the principal bundle. Then we can interpret a section $\sigma : W \to P \times_\rho V$ as a map:

$$\sigma : W \to P \times_\rho V, \quad \sigma(\mathbf{x}) = [s(\mathbf{x}), v(\mathbf{x})]$$

where v is a map from W to the vector space V. Given a fixed section of the principal bundle $s(\mathbf{x})$, we can describe a section of the associated bundle via $v : W \to V$. The aim is to express ∇^ω in terms of the section s, the map v and the coordinate one forms dx^i.

To this end we advance in the following way: Let $\tilde{\sigma}$ denote the tensorial 0-form that corresponds to σ. It is a equivariant map $\mathcal{P}^{-1}(W) \to V$. From equation (1.4.5) we now that:

$$\widetilde{\nabla^\omega \sigma} = d\tilde{\sigma} + \omega \dot{\wedge} \tilde{\sigma}$$

Now express this in coordinates, where we use that the section s gives rise to a local trivialization $W \times G$ of P. Let $(y_1, \dots y_l)$ be coordinate functions of a neighborhood of the identity in G with $g(0) = e$. Then we obtain coordinate functions $(x_1, \dots x_n, y_1 \dots y_l)$ of a neighborhood N of $s(\mathbf{x})$. Next we want to state $\tilde{\sigma} : \mathcal{P}^{-1}(W) \to V$ with respect to these coordinates. Let $p \in W$ be of the form $s(x_1, \dots x_n) \cdot g(y_1, \dots y_l) = s(\mathbf{x}) \cdot g(\mathbf{y})$ It follows from the explicit isomorphism that:

$$\tilde{\sigma}_p = p^{-1} \circ \sigma_x = g(\mathbf{y}) \cdot v(\mathbf{x})$$

For the exterior derivative term we obtain:

$$d\tilde{\sigma}_p = \sum_{i=1}^{l} \frac{\partial}{\partial y_i} g(\mathbf{y}) \cdot v(\mathbf{x})\ dy^i + \sum_{j=1}^{n} g(\mathbf{y}) \cdot \frac{\partial}{\partial x_j} v(\mathbf{x})\ dx^j$$

Subsequently we translate this expression back into an associated bundle valued one form via the explicit isomorphism ι. We are interested in the coefficients with respect to the coordinate one forms dx^i:

$$\iota(d\tilde{\sigma})^i_x = \iota(d\tilde{\sigma}_p)_{\mathbf{x}}(\partial x_i) = [s(\mathbf{x}), d\tilde{\sigma}_{s(\mathbf{x})}(\bar{\partial}_{x_i})] = [s(\mathbf{x}), \frac{\partial}{\partial x_i} v(\mathbf{x})]$$

where we have used that in our coordinates ∂x_i on P is a vertical lift of ∂x_i on X. This yields the desired expression for the first term. The second is calculated analogously. First we obtain:

$$(\omega \dot{\wedge} \tilde{\sigma})^i_p = \omega^i(\mathbf{x}, \mathbf{y}) \cdot g(\mathbf{y}) \cdot v(\mathbf{x})$$

where $\sum_{i=1}^{n+l} \omega_i dx^i$ is the coordinate expression of ω. Therefore:

$$\iota(\omega \dot{\wedge} \tilde{\sigma})^i_x = [s(x), (\omega \dot{\wedge} \tilde{\sigma})_{s(x)}(\bar{\partial_{x_i}})] = [s(x), \omega^i(\mathbf{x}) \cdot v(\mathbf{x})]$$

where $\omega_i(\mathbf{x}) := \omega_i(\mathbf{x}, \mathbf{0}) = \omega_i(s(\mathbf{x}))$. Note that $\omega_i(\mathbf{x})$ depends on the local cross section s. By its definition $\omega_i(\mathbf{x}) = \omega_{s(\mathbf{x})}((Ts)_x(\partial x_i))$, and by the definition of the pullback this is $(s^*\omega)_x(\partial x_i) = (s^*\omega)_{\mathbf{x}i}$

We arrive at the following result:

Proposition 1.4.10 *With the notation as above the following identity holds:*

$$(\nabla^\omega(\sigma))_{\mathbf{x}} = \sum_{i=1}^{n} [s(\mathbf{x}), (\frac{\partial}{\partial x_i} + (s^*\omega)_i(\mathbf{x})) \cdot v(\mathbf{x})]\ dx^i$$

1.4.4 Spliced bundles

When construction the Standard Model Lagrangian we will be in the following situation: Given two principal bundles over the same space X we want to "splice" them together into a single bundle. Consider the two bundles over a given space X:

$$G_1 \hookrightarrow P_1 \xrightarrow{\mathcal{P}_1} X$$
$$G_2 \hookrightarrow P_2 \xrightarrow{\mathcal{P}_2} X$$

The total space of the spliced bundle is given by:

$$P_1 \circ P_2 = \{(p_1, p_2\} \in P_1 \times P_2 : \mathcal{P}_1(p_1) = \mathcal{P}_2(p_2)\}$$

which is a smooth submanifold of $\mathcal{P}_1 \times \mathcal{P}_2$. One can define the projection:

$$\mathcal{P}_{12}(p_1, p_2) : \mathcal{P}_1 \circ \mathcal{P}_2 \to X, \quad \mathcal{P}_{12}(p_1, p_2) = \mathcal{P}_1(p_1) = \mathcal{P}_2(p_2)$$

which is a smooth map. We obtain a smooth right action of $G_1 \times G_2$ on $\mathcal{P}_1 \circ \mathcal{P}_2$ by:

$$(p_1, p_2) \cdot (g_1, g_2) = (p_1 \cdot g_1, p_2 \cdot g_2)$$

In total this gives a smooth principal $G_1 \times G_2$ bundle over X, the **spliced bundle**:

$$G_1 \times G_2 \hookrightarrow P_1 \circ P_2 \xrightarrow{\mathcal{P}_{12}} X$$

We have projection maps $\pi_1 : P_1 \circ P_2 \to P_1$ and $\pi_2 : P_1 \circ P_2 \to P_2$ given by $\pi_i(p_1, p_2) := p_i$ for $i = 1, 2$. We can use them to lift connections from P_1 and P_2. Observe that the Lie algebra of $G_1 \times G_2$ is a direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Given a connection ω_1 on P_1 we can therefore pull it back to a $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ valued form $\pi_1^*\omega_1$ by identifying $\mathfrak{g}_1$ with $\mathfrak{g}_1 \times 0$ in $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Given a connection ω_2 on P_2 we analogously obtain a $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ valued form $\pi_2^*\omega_2$. This leads to the following:

Proposition 1.4.11 *The form $\pi_1^*\omega_1 + \pi_2^*\omega_2$ defined above is a principal connection on $P_{12} : P_1 \circ P_2 \to X$*

Proof First we show that $\pi_1^*\omega_1$ is **pseudotensorial**(1.4.3) of type Ad. Here g_1 denotes both the action of g_1 on the spliced bundle $P_1 \circ P_2$ and on P_1, the dot denotes the adjoint action on the Lie algebra valued forms.

$$\begin{aligned}(g_1, g_2)^*\pi_1^*\omega_1 = (\pi_1 \circ (g_1, g_2))^*\omega_1 = (g_1 \circ \pi_1)^*\omega_1 \\ = \pi_1^* g_1^* \omega_1 = \pi_1^*(g_1^{-1} \cdot \omega_1) = (g_1, g_2)^{-1} \cdot (\pi_1^*\omega)\end{aligned}$$

The same argument yields that $\pi_2^*\omega_2$ is pseudotensorial, and therefore $\pi_1^*\omega_1 + \pi_2^*\omega_2$ as well.

Next we observe what $\pi_1^*\omega$ gives on **vertical** vectors $T\sigma^{(p_1,p_2)}(A_1, A_2)$, where (A_1, A_2) denotes an element in $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\sigma^{(p_1,p_2)}(g_1, g_2)$ is given by $(p_1, p_2) \cdot (g_1, g_2)$.

$$\begin{aligned}(\pi_1^*\omega_1)_{(p_1,p_2)}((T\sigma_{(p_1,p_2)})_e(A_1, A_2)) = (\omega_1)_{p_1}((T\pi_1)_{(p_1,p_2)})(T\sigma_{(p_1,p_2)})_e(A_1, A_2)) = \\ (\omega_1)_{p_1}((T\sigma_{p_1})_e A_1) = A_1\end{aligned}$$

where the last line is interpreted as an embedding of $\mathfrak{g}_1$ in $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ as above. In combination with the same calculation for $\pi_2^*\omega$ this yields:

$$(\pi_1^*\omega_1 + \pi_2^*\omega_2)_{(p_1,p_2)}(T\sigma_e^{(p_1,p_2)})(A_1, A_2)) = (A^1, A^2)$$

Hence $\pi_1^*\omega_1 + \pi_2^*\omega_2$ also fulfills the second condition of being a principal connection. ∎

This construction allows us to splice principal bundles with given connections into one principal bundle with an attached connection. The above constructions generalize to any finite number of bundles.

Let $\rho_1 : G_1 \to GL(V_1)$ and $\rho_2 : G_2 \to GL(V_2)$ be two finite dimensional representations of G_1 and G_2 on two vector spaces V_1 and V_2 over the same field. We want to associate a bundle to the spliced bundle $P_1 \circ P_2$ via these two representations. This is achieved by linearly extending the action of $G_1 \times G_2$ given by $\rho_1 \times \rho_2$ on $V_1 \times V_2$ to a representation:

$$\rho_1 \times \rho_2 : G_1 \times G_2 \to GL(V_1 \otimes V_2)$$

Thus there is an associated bundle:

$$P_1 \circ P_2 \times_{(\rho_1 \times \rho_2)} (V_1 \otimes V_2)$$

This bundle can be understood in the following way:

Proposition 1.4.12 *There is a canonical isomorphism:*

$$P_1 \circ P_2 \times_{(\rho_1 \times \rho_2)} (V_1 \otimes V_2) \cong (P_1 \times_{\rho_1} V_1) \otimes (P_2 \times_{\rho_2} V_2)$$

Proof This isomorphism is constructed fiberwise. Let x be in X and (p_1, p_2) be an element in the fiber of $P_1 \circ P_2$. Let σ_x^1 and σ_x^2 be elements of $(P_1 \times_{\rho_1} V_1)$ resp. $(P_2 \times_{\rho_2} V_2)$ in the fiber over x. Then there exist unique $v \in V_1$ and $w \in V_2$ such that $\sigma_x^1 = [p_1, v]$ and $\sigma_x^2 = [p_2, w]$. Then we can define a map:

$$\iota_x : (P_1 \times_{\rho_1} V_1)_x \times (P_2 \times_{\rho_2} V_2)_x \to (P_1 \circ P_2 \times_{(\rho_1 \times \rho_2)} (V_1 \otimes V_2))_x : \\ [p_1, v] \times [p_2, w] \to [(p_1, p_2), v \otimes w] \quad (1.23)$$

Changing (p_1, p_2) to another point in the fiber over x does not affect this map and it is linear in both components. Because of the universal property of the tensor product ι_x extends to the tensor product $((P_1 \times_{\rho_1} V_1) \otimes (P_2 \times_{\rho_2} V_2))_x$. By observing how it behaves on a basis it can be deduced that this is a fiberwise isomorphism. By choosing a local section $s(x) = (p_1(x), p_2(x))$ of $P_1 \circ P_2$ we can verify that ι_x depends smoothly on x. ∎

Now let ω_1 be a principal connection on P_1 and ω_2 be a prinicpal connection on P_2. As above (1.4.11) we obtain a principal connection $\pi_1^* \omega_1 + \pi_2^* \omega_2$ on $P_1 \circ P_2$. With (1.21) we get vector bundle connections ∇^{ω_1} on $E_1 := (P_1 \times_{\rho_1} V_1)$, ∇^{ω_2} on $E_2 := (P_2 \times_{\rho_2} V_2)$ and $\nabla^{\omega_1 + \omega_2}$ on $E_1 \otimes E_2 \cong (P_1 \circ P_2 \times_{(\rho_1 \times \rho_2)} (V_1 \otimes V_2))$
Via the coordinate form (1.4.10) of $\nabla^{\omega_1 + \omega_2}$ the following result can be shown:

Proposition 1.4.13 *With the notation as above the following equation holds for all sections σ of $(P_1 \times_{\rho_1} V_1)$ and e of $(P_2 \times_{\rho_2} V_2)$:*

$$\nabla^{\omega_1 + \omega_2}(\sigma \otimes e) = (\nabla^{\omega_1} \sigma) \otimes e + \sigma \otimes (\nabla^{\omega_2} e)$$

Therefore $\nabla^{\omega_1 + \omega_2}$ is the canonical ***tensor product connection*** *of ∇^{ω_1} and ∇^{ω_2}.*

1.5 Spin structure

1.5.1 Clifford algebras

In section (1.2.2) the connected component of the Lorentz group $\mathrm{SO}(1,3)^0$ is established as the structure group which links admissible frames. With an underlying space time manifold X modelling gravity we obtain as described above (1.9) a principal bundle $\mathrm{SO}(1,3)^0 \hookrightarrow \mathcal{L}(X) \xrightarrow{\mathcal{P}_L} X$ as central object. But as already mentioned we need more structure to describe the Standard Model Lagrangian. The fermionic fields do not transform via a representation of $\mathrm{SO}(1,3)^0$, but via a representation of its universal cover. In this section we show how this universal cover is embedded in the so called Clifford algebra of $\mathbb{R}^{1,3}$. The notation and presentation follows [LM].

Definition Let V be a finite dimensional vector space over a field k with characteristic $\neq 2$ and q be a quadratic form on V. Let

$$\mathcal{T}(V) = \sum_{i=0}^{\infty} \bigotimes^{i} V \tag{1.24}$$

denote the tensor algebra of V. Define $\mathcal{I}_q$ to be the ideal in $\mathcal{V}$ generated by all elements of the form $v \otimes v - q(v)\mathbb{1}$[5] for $v \in V$. Then the **Clifford algebra** $Cl(V,q)$ associated to V and q is an associative algebra with unit defined as:

$$Cl(V,q) := \mathcal{T}(V)/\mathcal{I}_q \tag{1.25}$$

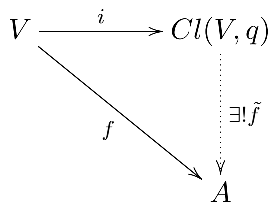
It can be shown([LM]) that the map from V in $Cl(V,q)$ which is determined by the embedding of V in $\mathcal{T}(V)$ is injective. This gives a natural embedding $V \subset Cl(V,q)$. We see that the Clifford algebra is generated by the vector space V subject to the relation

$$v \cdot v = q(v). \tag{1.26}$$

This results in the following universal characterization of a Clifford algebra:

Proposition 1.5.1 ([LM], Proposition 1.1) *Let $f : V \to \mathcal{A}$ be a linear map from a k-vector space V with quadratic form q into an associative k-algebra with unit, such that $f(v) \cdot f(v) = q(v)1$ for all $v \in V$. Then f extends uniquely to a k-algebra homomorphism $\tilde{f} : Cl(V,q) \to \mathcal{A}$. The algebra $Cl(V,q)$ is the up to isomorphism unique associative k-algebra with this property.*

[5] In mathematical literature, for instance [LM], often $v \otimes v + q(v)$ is used, but both definitions are common.



The discussion above shows that $Cl(V,q)$ has this universal property: Any linear map $f : V \to \mathcal{A}$ extends to a unique algebra homomorphism $\mathcal{T}(V) \to \mathcal{A}$ which is 0 on the ideal and therefore descends to $Cl(V,q)$. Objects defined by an universal property are always unique up to canonical isomorphism.

A Clifford algebra has an important natural splitting. In order to define it we observe that by the universal property of the Clifford algebra the map $\alpha(v) = -v$ on V uniquely extends to a automorphism $\tilde{f} : Cl(V,q) \to Cl(V,q)$. It can readily be verified that $\tilde{f}$ is an involution because f is. This follows from the following fact: Let g and h be two automorphisms of V which preserve the quadratic form q(i.e. in $O(V)$). Then by the universal property we obtain $\widetilde{g \circ h} = \tilde{g} \circ \tilde{h}$. The eigenspaces of $\tilde{f}$ to $+1$ and -1 now give a decomposition:

$$Cl(V,q) = Cl^0(V,q) \oplus Cl^1(V,q)$$

where $Cl^i(V,q)$ denotes the eigenspace to $(-1)^i$ of α. Furthermore $\alpha(v \cdot w) = \alpha(v) \cdot \alpha(w)$ holds for all $v, w \in Cl(V,q)$. Thus $\alpha : Cl(V,q) \to Cl(V,q)$ is an automorphism. By applying the above argument we additionally obtain a group homomorphism: $O(V,q) \to \mathrm{Aut}(Cl)$.

A Clifford algebra has a natural filtration. This structure comes from the filtration $\mathcal{T}^0 \subset \mathcal{T}^2 \subset \ldots \subset \mathcal{T}(V)$ of the tensor algebra, defined by $\mathcal{T}^r := \sum_{s \le r} \bigotimes^s V$. We have $\mathcal{T}^r \otimes \mathcal{T}^s \subseteq \mathcal{T}^{r+s}$ for all r, s. We set $\mathcal{F}^i = \pi_q(\mathcal{T}^i)$ to obtain a filtration $\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \ldots \subset Cl(V,q)$, which has the property that $\mathcal{F}^r \cdot \mathcal{F}^s \subseteq \mathcal{F}^{r+s}$ for all r, s. With this structure $Cl(V,q)$ is a **filtered algebra**. It follows that the multiplication map descends to a map: $(\mathcal{F}^r/\mathcal{F}^{r-1}) \times (\mathcal{F}^s/\mathcal{F}^{s-1}) \to (\mathcal{F}^{r+s}/\mathcal{F}^{r+s-1})$ for all r, s. With this approach we arrive at the **associated graded algebra** $\bigoplus_{r \ge 0}(\mathcal{F}^r/\mathcal{F}^{r-1})$
It can be shown that the associated graded algebra of $Cl(V,q)$ is naturally isomorphic to the exterior algebra $\Lambda^* V$. This follows from the fact that by (1.26) multiplication in $Cl(V,q)$ is skew in the highest order terms with respect to the filtration. This leads to an important isomorphism:

Proposition 1.5.2 ([LM], Proposition 1.3) *There is a canonical vector space isomorphism:*

$$\Lambda^* \xrightarrow{\cong} Cl(V,q)$$

compatible with the filtrations.

Proof Define a map

$$f : V \times \cdots \times V \to Cl(V,q), \quad f(v_1,\ldots,v_r) = \frac{1}{r!}\sum_\sigma \mathrm{sign}(\sigma)v_{\sigma(1)}\ldots v_{\sigma(r)}$$

where we take the sum over all elements σ of the symmetric group. Then f descends to a linear map $\tilde{f} : \Lambda^r V \to Cl(V,q)$. By applying the associated graded algebra structure of $Cl(V,q)$ it can be shown that for $r = 0,\ldots n$ $\tilde{f}$ is injective and the direct sum of these maps is an isomorphism. See [LM]) for details.

With this isomorphism we obtain that for a n-dimensional vector space V with quadratic form q the vector space dimension of the $Cl(V,q)$ is $\mathbf{2^n}$. If we have a q-orthogonal basis $e_1, e_2, \cdots, e_n$ of V, then the isomorphism shows that the products of the form $e_{i_1} \cdot e_{i_2} \cdots e_{i_k}$ with $i_1 < i_2 < \cdots i_k$ are a basis of $Cl(V,q)$.

1.5.2 The groups Pin and Spin

Next we define the subset of units in the Clifford algebra:

$$Cl^\times(V,q) := \{\phi \in Cl(V,q) : \exists \phi^{-1} \in Cl(V,q) : \phi^{-1}\phi = \phi\phi^{-1} = 1\}$$

With the multiplicative structure of the Clifford algebra this is a group. It contains all elements $v \in V$ with $q(v) \neq 0$, since the inverse of v is given by $\frac{v}{q(v)}$. We define the following two subgroups of $Cl^\times(V,q)$:

Definition

- $\mathrm{Pin}(V,q) := \{v_1 \ldots v_r | v_j \in V, q(v_j) = \pm 1 \ \forall j\}$
- $\mathrm{Spin}(V,q) := \mathrm{Pin}(V,q) \cap Cl^0(V,q)$

There is a representation of $\mathrm{Pin}(V,q)$ on $GL(V)$ where unit vectors $v \in V$ act via a reflection on a hyperplane normal to v:

Definition

$$\widetilde{Ad} : \mathrm{Pin} \to GL(V), \quad \widetilde{Ad}_\phi(v) := \alpha(\phi)v\phi^{-1} \tag{1.27}$$

That $\widetilde{Ad}_\phi$ really is an element in $GL(V)$ will be explained in the next proposition. Since α is an automorphism of the Clifford algebra, this modified adjoint representation is a representation.

By using that $v^{-1} = \frac{v}{q(v)}$ we can verify the following equation:

Proposition 1.5.3 *Let $v \in V \subset Cl(V,q)$ be an element with $q(v) \neq 0$. Then*

$$\widetilde{Ad}_v(w) = \alpha(v)wv^{-1} = w - 2\frac{q(v,w)}{q(v)}v \quad \forall w \in V$$

where the action of $\widetilde{Ad}$ is extended to all $v \in V$ with $q(v) \neq 0$

An immediate consequence of this is that $\widetilde{Ad}_v$ is an element in $GL(V)$ which preserves the quadratic form, i.e. an element of the orthogonal group $\mathrm{O}(V)$.

From now on let the underlying field be $\mathbb{R}$ or $\mathbb{C}$ and q be non degenerate. By restricting $\widetilde{Ad}$ we obtain a representation $\mathrm{Pin}(V,q) \to \mathrm{O}(V,q)$, since $\widetilde{Ad}_v$ for $v \in V$ with $q(v) \neq 0$ is in $\mathrm{O}(V,q)$ and Pin is by definition generated by such elements. In the following we discuss the kernel and image of this representation.

Theorem 1.5.4 (Cartan-Dieudonné) *Let q be a non-degenerate quadratic form on a finite dimensional vector space V. Then every element $g \in \mathrm{O}(V,q)$ can be written as a product of r reflections where $r \leq dim(V)$*

$$g = \rho_{v_1} \circ \cdots \circ \rho_{v_r} \tag{1.28}$$

A proof by induction for the dimension can be found in [G].

Since over $\mathbb{R}$ and $\mathbb{C}$ all reflection vectors can be normalized to length ± 1 the representation $\widetilde{Ad}$ of Pin contains all reflections. Hence the representation is surjective. Another important property is that reflections have determinant -1. This follows because the orthogonal hyperplane to the reflection vector is an eigenspace to 1, while the reflection vector is an eigenvector to -1.

Since Spin is generated by an even number of reflections representation $\widetilde{Ad}$ of Pin restricts to a representation: $\widetilde{Ad} : \mathrm{Spin}(V,q) \to \mathrm{SO}(V,q)$. The above theorem of Cartan-Dieudonné ensures that this representation is surjective.

The question regarding the kernel of our representations $\widetilde{Ad}$ requires a more elaborate discussion. Because of equation(1.27) it contains $\pm 1 \in \mathrm{Spin}(V,q) \subset \mathrm{Pin}(V,q)$. This is actually the full kernel. See [LM] for details.

In combination the staements above yield the following:

Theorem 1.5.5 ([LM], Theorem 2.9.) *Let V be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ and q be a non-degenerate quadratic form on V. Then there are short exact sequences:*

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(V,q) \xrightarrow{\widetilde{Ad}} \mathrm{O}(V,q) \to 1$$

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(V,q) \xrightarrow{\widetilde{Ad}} \mathrm{SO}(V,q) \to 1$$

For this work the real case is of special interest. We introduce some notation for brevity. We preveously defined: $\mathrm{O}(r,s) = \mathrm{O}(\mathbb{R}^{r,s})$ and $\mathrm{SO}(r,s) = \mathrm{SO}(\mathbb{R}^{r,s})$. For simplicity we introduce the notation $\mathrm{Pin}_{r,s} := \mathrm{Pin}(\mathbb{R}^{r,s})$ and $\mathrm{Spin}_{r,s} := \mathrm{Spin}(\mathbb{R}^{r,s})$. We suppress the second index if it is zero, so for example $\mathrm{Spin}_n = \mathrm{Spin}_{n,0}$

With the topological facts about the indefinite orthogonal groups we discussed earlier (1.2.2) we arrive at the following corollary of the above theorem (1.5.5):

Corollary 1.5.6 ([LM], Theorem 2.10) *There are short exact sequences*

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_r \xrightarrow{\mathrm{Spin}} \mathrm{SO}(r) \to 1$$
$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}^0_{1,s} \xrightarrow{\mathrm{Spin}} \mathrm{SO}^0(1,s) \to 1$$

These sequences represent the universal covering homomorphisms of the groups on the right-hand side for all $r, s \geq 3$. In this context the map $\widetilde{Ad}$ is from now on denoted by Spin.

Proof We have to show that the coverings are non trivial. Since the kernel in each case is $\{1, -1\}$ it is enough to join -1 to 1 by a path. To this end we choose orthogonal vectors e_1, e_2 with $q(e_1) = q(e_2) = \pm 1$. Then $\gamma(t) = \pm\cos(2t) + e_1 e_2 \sin(2t) = (e_1\cos(t) + e_2\sin(t))(e_2\sin(t) - e_1\cos(t))$ is such a path. ∎

1.5.3 Spin manifolds

From now on $\mathrm{Spin}_{r,s}$ denotes the **connected component** $\mathrm{Spin}^0_{r,s}$. Recall from (1.9) that each oriented and time oriented spacetime manifold X has a corresponding bundle $\mathrm{SO}(1,s)^0 \hookrightarrow \mathcal{L}(X) \xrightarrow{\mathcal{P}_L} X$. Our aim is to lift this bundle for $s \geq 3$ to one with structure group $\mathrm{Spin}_{1,s}$, the double cover of $\mathrm{SO}(1,s)^0$. Consequently we require a bundle

$$\mathrm{Spin}_{1,s} \hookrightarrow S(X) \xrightarrow{\mathcal{P}_S} X \tag{1.29}$$

and a map from $S(X)$ to $\mathcal{L}(X)$ that, restricted to fibers, is the covering map Spin:

Definition A **spin structure** for X consists of a principal $\mathrm{Spin}_{1,s}$-bundle over X (1.29) and a map

$$\lambda : S(X) \to \mathcal{L}(X) \tag{1.30}$$

such that

$$\mathcal{P}_L(\lambda(p)) = \mathcal{P}_S(p)$$

and

$$\lambda(p \cdot g) = \lambda(p) \cdot \mathrm{Spin}(g)$$

Therefore the following diagram commutes:

$$\begin{array}{ccccc} S(X) \times \mathrm{Spin}_{1,s} & \xrightarrow{\ \cdot\ } & S(X) & \xrightarrow{\mathcal{P}_S} & X \\ \downarrow{\scriptstyle \lambda \times \mathrm{Spin}} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle id_X} \\ \mathcal{L}_X \times \mathrm{SO}(1,s)^0 & \xrightarrow[\ \cdot\ ]{} & \mathcal{L}(X) & \xrightarrow[\mathcal{P}_{\mathcal{L}}]{} & X \end{array}$$

After orientation and time orientation we encounter the third condition for a spacetime that rules out "unphysical" behavior: Not every oriented and time oriented spacetime X admits a spin structure. However, it is required to describe fundamental fermions. Therefore spacetime manifolds without spin structure are dismissed. The question whether X allows a spin structure is a purely topological one. In [N2] it is shown that the vanishing of the "second Stiefel-Whitney class" of X, which is a certain Čech cohomology class $\omega_2(X) \in \check{H}^2(X;\mathbb{Z}_2)$, is a necessary and sufficient condition for the existence of a spin structure on X.

1.5.4 Levi-Civita connection and its lift

It is a basic result in Riemannian geometry that on the tangent bundle of a semi-Riemannian manifold exists a unique torsion free metric connection. In our context this translates to(see [M]):

Proposition 1.5.7 *Let (X, g) be a semi-Riemannian manifold and let $\mathrm{SO}(r, s)(X)$ denote its oriented tangent frame bundle. Then there exists an unique principal connection on $\mathrm{SO}(r, s)(X)$ with the property that its torsion tensor vanishes identically. This connection is called the* ***Levi-Civita connection****.*

The Levi-Civita connection is a important tool in general relativity and plays a role in the formulation of the standard model over curved spacetimes. On spacetimes with spin structure it can be lifted in the obvious way. Each connection on the oriented tangent frame bundle can be lifted:

Proposition 1.5.8 *Let X be spacetime with spin structure $\lambda : S(X) \to \mathcal{L}(X)$, see (1.5.3). Let ω be a principal connection on $\mathcal{L}(X)$. Then $\lambda^*\omega$ is a principal connection on $S(X)$.*

Proof First we show that $\lambda^*\omega$ is pseudotensorial. Let g and $\mathrm{Spin}(g)$ denote the right action of g on $S(X)$ resp. $\mathrm{Spin}(g) \in \mathrm{SO}(1, s)$ on $\mathcal{L}(X)$.

$$\begin{aligned} g^*\lambda^*(\omega) &= (\lambda \circ g)^*\omega = (\mathrm{Spin}(g) \circ \lambda)^*\omega \\ &= \lambda^* \mathrm{Spin}(g)^*\omega = \lambda^*(\mathrm{Spin}(g)^{-1} \cdot \omega) = g^{-1} \cdot (\lambda^*\omega) \end{aligned}$$

Then we observe how $\lambda^*\omega$ acts on vertical vectors. Let $\sigma^p(g) = p \cdot g$ denote the action of the groups in a point p of the bundles. By differentiating the covering map Spin : $\mathrm{Spin}_{1,s} \to \mathrm{SO}(1,s)^0$ we otain a canonical isomorphism of the Lie algebras of these groups. Let A denote an arbitrary element in this Lie algebra. Then:

$$(\lambda^*\omega)_p((T\sigma_p)_e A) = \omega_{\lambda(p)}((T\lambda)_p(T\sigma_p)_e A) = \omega_{\lambda(p)}((T(\lambda \circ \sigma_p))_e A) = \omega_{\lambda(p)}((T\sigma_{\lambda(p)})_e A) = A$$

This shows that $\lambda^*\omega$ fulfills both required properties of a principal connection and the proof is complete.

So every principal connection on $\mathcal{L}(X)$ can be lifted to a connection on $S(X)$. In particular we have:

Corollary 1.5.9 *The Levi-Civita connection can be uniquely lifted to a connection on $S(X)$.*

1.5.5 Representations of Clifford algebras

This subsection discusses the construction of certain representations of the Spin and Pin groups central to this work. As seen above they can be identified with subgroups of the Clifford algebra. Therefore we begin to construct representations of Clifford algebras which later give rise to representations of the Spin group.

Definition ([LM], definition 5.1) Let V be a vector space over a field k and q be a quadratic form on V. Let $K \supseteq k$ be a field containing k. Then a K-representation of the Clifford algebra $Cl(V,q)$ is a k-algebra homomorphism

$$\rho : Cl(V,q) \to \mathrm{Hom}_K(W)$$

into the algebra of linear transformations of a finite dimensional vector space W over K. The space W is called a $Cl(V,q)$ module over K, or a **Clifford module**.

For us the cases K=$\mathbb{R}$ or $\mathbb{C}$ are interesting. A complex vector vector space W can be regarded as real vector space V with a real linear map $J : W \to W$ such that $J^2 = -\mathbf{1}$. Consequently a complex representation of a real Clifford algebra is a real representation ρ that commutes with J.

We mainly consider complex representations of real Clifford algebras. Note that any such representation automatically extends to a representation of $Cl_{r,s} \otimes_{\mathbb{R}} \mathbb{C}$, the complexification of the Clifford algebra. By using the universal property this algebra is easily seen to be isomorphic to the Clifford algebra $Cl(\mathbb{C}^{r+s}, q \otimes \mathbb{C})$ of the complexified quadratic form. But on $\mathbb{C}^n$ all non-degenerate quadratic forms are equivalent, thus this algebra is isomorphic to $\mathbb{C}l_n := Cl(\mathbb{C}^n, q_\mathbb{C})$ where $q_\mathbb{C}$ is the standard quadratic form on $\mathbb{C}^n$. On the other hand any representation of $\mathbb{C}l_n$ restricts to a complex representation of $Cl_{r,s}$ This shows that it is sufficent to study complex representations of $\mathbb{C}l_n$ to investigate which complex representations of $Cl_{r,s}$ can occur. We start with the former and it is useful to describe $\mathbb{C}l_n$ as matrix algebras.

One efficient way of doing so(see [LM] for details) is to show that $\mathbb{C}l_1 \cong \mathbb{C} \oplus \mathbb{C}$ and $\mathbb{C}l_2 \cong \mathbb{C}(2)$ where $\mathbb{C}(2)$ denotes the algebra of 2×2 complex matrices. Then by directly defining a map we can produce an isomorphism:

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_\mathbb{C} \mathbb{C}l_2$$

By induction this proves the following important result:

Proposition 1.5.10 *There are isomorphisms of algebras*

$$\mathbb{C}l_{2k} \cong \mathbb{C}(2^k) \tag{1.31}$$

$$\mathbb{C}l_{2k+1} \cong \mathbb{C}(2^k) \oplus \mathbb{C}(2^k) \tag{1.32}$$

Therfore all Clifford algebras of complex vector spaces with non degenerate forms are seen to be isomorphic to complex matrix algebras or the direct sum of two. A similar statement holds for the real indefinite Clifford algebras. However, in that case real, complex and quaternionic matrix algebras can appear(see [LM]). In this work we focus on $\mathbb{C}l_{2k}$. A direct construction of such an isomorphism is given. This approach follows [BW] At first define the following complex 2×2 matrices:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{1}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

With this define:

$$P_i = \mathbf{1}' \otimes \cdots \otimes \mathbf{1}' \otimes P \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \tag{1.33}$$

$$Q_i = \mathbf{1}' \otimes \cdots \otimes \mathbf{1}' \otimes Q \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \tag{1.34}$$

where the operation $\otimes$ denotes the usual tensor product of matrices. It is no longer important to distinguish between Ps and Qs. We will refer to each of them with the symbol P_i, where the index ranges from $i = 1$ to $i = 2k$. A simple calculation shows that the following relations hold:

$$P_i^2 = 1 \quad \text{for } i = 1, 2, \ldots, 2k \tag{1.35}$$

$$P_i P_j = -P_j P_i \quad \text{for all pairs } i \neq j \tag{1.36}$$

By counting dimensions we see that the P_i act on $\mathbb{C}^{2^k}$ and subsequently the complex algebra A generated by the P_i is a subalgebra of the $2^k \times 2^k$ complex matrix algebra. By observing how the P_i acts on the standard basis vectors we can verify that A is the full matrix algebra. Applying the universal property we obtain a $\mathbb{C}$-algebra homomorphism from $\mathbb{C}l(2k) \to A$. Since they have the same dimension as complex vector spaces this is a **isomorphism**. This constitutes a direct proof of the first part of (1.31) and we have direct access to such an isomorphism.

Next we state some further basic facts concerning representations of Clifford algebras.

[LM], page 31 Let $V, q, k \subseteq K$ be as in definition 1.5.5.

- A K-representation $\rho : Cl(V,q) \to \mathrm{Hom}_K(W,W)$ is said to be **reducible** if the vector space W can be written as a non-trivial direct sum $W = W_1 \oplus W_2$ such that each W_i for $i = 1,2$ is invariant under $\rho(\phi)$ for all $\phi \in Cl(V,q)$.
- A representation is called **irreducible** if it is not reducible.
- Two representations $\rho_j : Cl(V,q) \to \mathrm{Hom}_K(W_j;W_j)$ for j=1,2 are said to be **equivalent** if there exists a K-linear isomorphism $E : W_1 \to W_2$ such that $E \circ \rho_1(\phi) \circ E^{-1} = \rho_2(\phi)$ for all $\phi \in Cl(V,q)$.

Remark Usually representations are called "irreducible" if there are no proper invariant subspaces. Since $\mathrm{Cl}(V,q)$ is the algebra of a finite group(the reader is referred to [LM], Proposition 5.4) the two concepts are equivalent.

It follows directly from the definition above that every representation of a Clifford algebra can be decomposed into a **direct sum** of irreducible ones: If a representation is not irreducible we can decompose it further. Because of the finite dimension of the representation this process must stop at some point. Thus we are interested in irreducible representations. In our case we work with matrix algebras where the representation theory is particularly simple:

Theorem 1.5.11 ([LM], 5.6.) *Let $K = \mathbb{R}$ or $\mathbb{C}$ and denote by $K(n)$ the ring of $n \times n$ K-matrices. Then the natural representation ρ of $K(n)$ on the vector space K^n is up to equivalence the only irreducible K-representation of $K(n)$.*

Proof This follows from the classical fact that the algebras $K(n)$ are simple and simple algebras only have one irreducible representation up to equivalence. ([L])

Applied to the even dimensional complex Clifford algebras this yields corollary:

Corollary 1.5.12

- *All irreducible complex representations of* $\mathbb{C}l_{2k}$ *are equivalent.*
- *All irreducible complex representations of the real Clifford algebra* $\mathbb{C}l_{r,s}$ *with* $r+s$ *even are equivalent.*

Thus we obtain a complete classification of all representations of $\mathbb{C}l_{2k}$ up to equivalence and we also have explicitly constructed such a representation. Next we restrict it to a Spin-representation.

1.5.6 Spin representations

Recall the following canonical embeddings:

$$\mathrm{Spin}_{r,s} \subset Cl^0_{r,s} \subset \mathbb{C}l^0_{r+s} \subset \mathbb{C}l_{r+s}$$

Thus each complex representation of $\mathbb{C}l_{r+s}$ gives rise to a complex representation of $\mathrm{Spin}_{r,s}$. Reducible representations restrict to reducible ones, thus we start with irreducible representations of $\mathbb{C}l_{r+s}$. For $r+s=2k$ even we know that up to equivalence there is only one complex representation of $\mathbb{C}l_{r+s}$. This representation acts on a 2^k dimensional complex vector space S. So we can define:

Definition Let $r+s=2k$. The **complex spinor representation** of $\mathrm{Spin}_{r,s}$ is the equivalence class of the representation

$$\Delta_{r,s} : \mathrm{Spin}_{r,s} \to GL_{\mathbb{C}}(S)$$

given by restricting an irreducible complex representation $\mathbb{C}l_{2k} \to Hom_{\mathbb{C}}(S,S)$. The 2^k dimensional complex vector space S is called **spinor space**.

If we had restricted to representations of $\mathrm{Pin}_{1,3}$ then this representation would be irreducible since $\mathrm{Pin}_{1,3}$ contains an additive basis of $Cl_{r,s}$. For λ_n it is not possible to apply such an argument and we will show that in fact λ decomposes into two inequivalent irreducible representations.

To do so we first introduce the so called complex **(oriented) volume element** in $\mathbb{C}l_{2k}$. Choose an orientation on $\mathbb{C}^{2k}$ and let $e_1,\ldots,e_{2k}$ be any positively oriented orthonormal basis. Then the associated volume element is defined to be the product:

$$\omega = i^k e_1 \cdots e_{2k} \tag{1.37}$$

This definition is independent of the choice of the orthonormal basis since for any other positively oriented orthonormal basis $\tilde{e} = g \cdot e$ with $g \in O_{2k}(\mathbb{C})$ the identity $\tilde{e}_1 \ldots \tilde{e}_{2k} = \det(g) e_1 \cdots e_{2k} = e_1 \cdots e_{2k}$ holds.

A direct calculation shows that ω anticommutes with every element $v \in \mathbb{C}^{2k} \subset \mathbb{C}l_{2k}$ and that its square is $\mathbf{1}$:

$$v\omega = -\omega v \quad \forall v \in V \tag{1.38}$$

$$\omega^2 = \mathbf{1} \tag{1.39}$$

Next let $\phi : \mathbb{C}l_{2k} \to \mathrm{Hom}_{\mathbb{C}}(S, S)$ be an irreducible complex representation on a vector space S. Then the two above statements also hold for $\phi(\omega)$. Thus we define two projection operators:

$$\pi^{\pm} \in \mathrm{Hom}_{\mathbb{C}}(S, S) : \frac{1}{2}(\mathbf{1} \pm \phi(\omega)) \tag{1.40}$$

This yields a direct sum decomposition $S = S^+ \oplus S^-$. Note that S^+ and S^- are the eigenspaces to 1 respectively -1 of $\phi(\omega)$. Since ω commutes with each element in the even subalgebra $\mathbb{C}l^0_{2k}$, $\phi(\omega)$ and hence $\pi^{\pm}$ commute with the action of $\mathbb{C}l^0_{2k}$. This shows that the spaces S^+ and S^- are $\mathbb{C}l^0_n$ invariant. Since the spin group sits inside $\mathbb{C}l^0_n$, we have shown the following:

Proposition 1.5.13 *With the above projections a complex spinor representation $\Delta_{2k} : \mathrm{Spin}_{r,s} \to GL_{\mathbb{C}}(S)$ into a spinor space S decomposes into two representations:*

$$\Delta_{2k} = \Delta^+_{2k} \oplus \Delta^-_{2k}$$

These representations are called ***chiral representations*** *or half spin representations.*

Note that the chiral representations are defined via spinor representations, so the left and right representations are inherently linked by this.

Next we take a look at the realization of these representations via the Weyl-Brauer matrices.(1.33). First observe by a short calculation that the volume element in this representation of $\mathbb{C}l_{2k}$ is given by:

$$\phi(\omega) = i^k\phi(e_1)\cdots\phi(e_{2k}) = i^k\Pi^{2k}_{i=1}P_i = (-1)^s\mathbf{1'} \otimes \mathbf{1'} \otimes \ldots \otimes \mathbf{1'} = (-1)^s \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.41}$$

where $s \in \mathbb{N}$ depends on k, but the factor $(-1)^s$ is not of relevance, as the sign of ω can be changed by choosing a different orientation. We directly observe that the eigenspaces of $\phi(\omega)$ to 1 and -1 are each of dimension $2^{(k-1)}$. Thus $\Delta^{\pm}_{2k}$ are each $2^{(k-1)}$ dimensional representations. But they can not be equivalent, which the following short argument shows:

The real vector space $\mathbb{R}^{r,s}$ inside $\mathbb{C}l_{2k}$ contains elements of the form $i^{r(j)}e_j$, where $i^{r(j)}$ with $r(j) \in \mathbb{N}$ is a phase factor that depends on the signature of the bilinear form and j runs from 1 to $2k$. Now $\mathrm{Spin}_{r,s}$, by definition, contains even products of such elements. In particular a scalar multiple $c\omega$ of ω. But $\phi(c\omega) = c\phi(\omega)$ and that shows that $\Delta^+_{2k}(c\omega) = c\mathbf{1}$, while $\Delta^-_{2k}(c\omega) = -c\mathbf{1}$. So these representations are **inequivalent**, as scalar multiples of the identity are invariant under equivalence transformations.

We can also show that the chiral representations are **irreducible**. To this end we need the following fact: The map $f : \mathbb{C}^n \to \mathbb{C}l^0_{n+1}$ given by $f(e_i) = ie_{n+1}e_i$ where the e_i denote the standard basis has the property that $f(v)^2 = q(v)$. Therefore by the universal property (1.5.1) f extends to an algebra homomorphism $\tilde{f}$. Checking on a linear basis shows that this is in fact an isomorphism:

$$\tilde{f} : \mathbb{C}l_n \xrightarrow{\cong} \mathbb{C}l^0_{n+1}$$

The chiral representations $\Delta^\pm_{r,s}$ are restrictions of representations of $\mathbb{C}l^0_{2k} \cong \mathbb{C}l_{2k-1}$. But as mentioned above (1.31) it can be shown that $\mathbb{C}l_{2k-1} \cong \mathbb{C}(2^{k-1}) \oplus \mathbb{C}(2^{2k-1})$, so $\mathbb{C}l_{2k-1}$ has two equivalence classes of irreducible 2^{k-1}-dimensional representations. By reasons of dimension $\mathbb{C}l^0_{2k}$ is irreducibly represented on S^+ and S^-. Since $\mathrm{Spin}_{r,s}$ contains an additive basis of $\mathbb{C}l^0_{2k}$ the chiral representations $\Delta^\pm_{r,s}$ are also **irreducible**.

For the construction of kinetic terms we will need $\mathrm{Spin}_{r,s}$ invariant Hermitian forms on the spinor space. To understand if there are any and how to construct them, some basic representation theory is needed.

Definition Let $\rho : G \to GL(V)$ be a representation of a group. Then there exists an unique representation $\rho^* : G \to GL(V^*)$ on the dual $V^* = \hom(V, \mathbb{C})$ of V which respects the natural pairing(denoted by $\langle\ ,\ \rangle$):

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$$

for all g in G, v in V and v^* in V^*. In other words the **dual representation** is defined by:

$$\rho^*(g) = \rho(g^{-1})^t : V^* \to V^*$$

Given a basis of V and thereby of V^* by choosing the dual basis we can verify that the matrix representation of $\rho^*(g)$ with respect to the latter basis is the transposed inverse of the matrix representation of $\rho(g)$ with respect to the first basis.

Definition Let $\rho : G \to GL(V)$ be a complex representation of a group. Let $\bar{V}$ be the complex conjugate vector space of V; this is the vector space consisting of all elements $\{\bar{v} : v \in V\}$ with addition and scalar multiplication in such a way that the map $C : V \to \bar{V}, C(v) = \bar{v}$ is antilinear. Then there exists a unique representation $\bar{\rho} : G \to GL(\bar{V})$ that makes the following diagram commutative:

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ {\scriptstyle C}\downarrow & & \downarrow{\scriptstyle C} \\ \bar{V} & \xrightarrow[\bar{\rho}(g)]{} & \bar{V} \end{array}$$

This **complex conjugate representation** is given by:

$$\bar{\rho}(g) \cdot \bar{v} = \overline{\rho(g) \cdot v}$$

Note that since C is antilinear a representation need not be isomorphic to its complex conjugate representation. Given a basis of V and thereby of $\bar{V}$(via C) we can verify that the matrix representation of $\bar{\rho}(g)$ with respect to the latter basis is the complex conjugate of the matrix representation of $\rho(g)$ with respect to the first basis. Notice that the complex conjugates of two equivalent representations are still equivalent, and the same statement holds for the dual representations.

1.5.7 Spin invariant forms

Here we discuss sesquilinear forms on a spinor space. Let S be a spinor space, i.e. we have an irreducible complex representation $\mathbb{C}l_{2k} \to \mathrm{Hom}_{\mathbb{C}}(S, S)$. Recall that we have an embedding of $\mathbb{R}^{r,s} \subset \mathbb{C}^{2k} \subset \mathbb{C}l_n$ and thus by the universal property of the Clifford algebra an embedding $Cl(\mathbb{R}^{r,s})$ in $\mathbb{C}l_n$ which gives rise to the spinor representation $\Delta_{r,s} : \mathrm{Spin}_{r,s} \to GL_{\mathbb{C}}(S)$. Next we investigate $\mathrm{Spin}_{r,s}$ invariant forms on S.

Begin by defining the **Clifford group**. This is the finite group $F_n \subset \mathbb{C}l_n$ generated by the standard orthonormal basis $(e_1, e_2 \ldots, e_n)$ of $\mathbb{C}^n$. Choose an inner product on S and average it over the finite group F_n. Thus we obtain a F_n invariant inner product $h_0 = \langle\ ,\ \rangle$ on S. In particular:

$$\langle e_i x, e_i y \rangle = \langle x, y \rangle \quad \forall x, y \in S$$

It follows that all e_i are Hermitian, i.e.:

$$\langle e_i x, y \rangle = \langle x, e_i y \rangle \quad \forall x, y \in S$$

Next we construct a $\mathrm{Spin}_{r,s}$ invariant Hermitian form based on h_0. Let $(\tilde{e}_1, \tilde{e}_2 \cdots, \tilde{e}_n)$ denote the standard orthonormal basis of $\mathbb{R}^{r,s}$. The above embedding $\mathbb{R}^{r,s} \to \mathbb{C}^n$ restricts to:

$$\tilde{e}_j \to \begin{cases} e_j & \text{for } 1 \le j \le r \\ ie_j & \text{for } r+1 \le j \le n \end{cases}$$

Note that the factor i makes $\tilde{e}_j = ie_j$ skew Hermitian. The inner product h_0 is in general not $\mathrm{Spin}_{r,s}$ invariant. Therefore we modify it:

$$h : S \times S \to \mathbb{C}, \quad h(x,y) := h_0(x, By) \tag{1.42}$$

where

$$B := i^{(\frac{r(r-1)}{2})} e_1 \cdot e_2 \cdot \dots \cdot e_r.$$

A short calculation shows that B is Hermitian and invertible. Thus h is a non degenerate Hermitian form. Furthermore for r even B anticommutes with e_j for $1 \le j \le r$ and commutes with e_j for $r+1 \le j \le n$, and vice versa for r odd. With this we are able to prove:

Proposition 1.5.14 *The Hermitian form h is invariant under the connected component of the spin group* $\mathrm{Spin}_{r,s}$*, i.e. $g^*h = h$ for all $g \in \mathrm{Spin}_{r,s}$.*

Proof We verifiy this by direct calculation. Let $v = v^j \tilde{e}_j$ be in $\mathbb{R}^{r,s}$ and x, y in S. We obtain:

$$\begin{aligned} h(vx,y) &= \sum_{j=1}^{n} h_0(v^j \tilde{e}_j x, By) \\ &= \sum_{j=1}^{r} h_0(x, v^j \tilde{e}_j By) + \sum_{j=r+1}^{n} h_0(x, -v^j \tilde{e}_j By) \\ &= (-1)^{r+1} \sum_{j=1}^{n} h_0(x, Bv^j \tilde{e}_j y) = (-1)^{r+1} h_0(x, Bvy) = (-1)^{r+1} h(x, vy) \end{aligned}$$

So Clifford multiplication is skew up to a possible sign depending on r. It follows that:

$$h(vx, vy) = (-1)^{r+1} h(x, vvy) = (-1)^{r+1} q(v) h(x,y)$$

The entire spin group is by definition (1.5.2) generated by elements of the form vw where $q(v) = \pm 1$ and $q(w) = \pm 1$. We obtain:

$$h(vwx, vwy) = q(v) q(w) h(x,y)$$

We can verify that for the connected component $\mathrm{Spin}_{r,s}$ of the spin group always $q(v) = q(w)$: The tensor algebra $\mathcal{T}(V)$ has an involution, given on pure tensors by the reversal of order: $v_1 \otimes \dots \otimes v_j \mapsto v_j \otimes \dots \otimes v_1$ This map preserves the ideal and so descends to an antiautomorphism:

$$(\ )^t : Cl(V,q) \to Cl(V,q)$$

With this we define a smooth map:

$$\mathrm{Spin}_{r,s} \subset Cl_{r,s} \to \{\pm 1\} \subset \mathbb{R}, \quad v \mapsto v^t v$$

On the connected component this map has to be constant, which shows that for $vw \in \mathrm{Spin}_{r,s}$ $q(v) = q(w)$. Thus h is in fact invariant under $\mathrm{Spin}_{r,s}$. ∎

Note that since h is Hermitian $h(x,x)$ is **real** for all x in S. But h does not need to be an inner product(positive definite), only in the case $r = 0$ when B is the identity. This means that the spinor representation of Spin_n is an unitary representation. The existence of such an invariant Hermitian product can also be deduced from the compactness of Spin_n. For $r, s > 1$ on the other hand $\mathrm{Spin}_{r,s}$ is not compact which implies that there can be no faithful finite dimensional unitary representations.

Next investigate how the Hermitian form looks like with regard to the splitting of $S = S^+ \oplus S^-$ for dimension $r + s = 2k = n$. By definition (1.5.7) B is in $\mathbb{C}l^0$ if r is even and in $\mathbb{C}l^1$ if r is odd. So in the first case B maps from S^+ to S^+ and S^- to S^- whereas in the second case its restriction to S^+ has its image in S^- and vice versa. Furthermore a short calculation shows that the volume element (1.37) $\omega = i^k e_1 \cdots e_{2k}$ is Hermitian with respect to h_0, i.e. $h_0(x, \omega y) = h_0(\omega x, y)$, so the eigenspaces $S^\pm$ of ω to ± 1 are h_0-orthogonal. Thus for r even they are additionaly h orthogonal:

$$\begin{aligned} h(x^+ + x^-, y^+ + y^-) &= h_0(x^+ + x^-, By^+ + By^-) = \\ &h_0(x^+, By^+) + h_0(x^-, By^-) = h(x^+, y^+) + h(x^-, y^-) \end{aligned}$$

where $x = x^+ + x^-$ denotes the canonical splitting of $x \in S$ into chiral parts. Note that h restricts to two invariant Hermitian forms $h = h_+ + h_-$ on the chiral spinor spaces $S^\pm$.

If r is odd, which is our main interest, we have:

$$h(v^+ + v^-, w^+ + w^-) = h(v^+, w^-) + h(v^-, w^+) \tag{1.43}$$

Notice that the sesquilinear forms $h_1(v,w) = h(v^+, w^-)$ and $h_2(v,w) = h(v^-, w^+)$ are also $\mathrm{Spin}_{r,s}$ invariant. However, they are not Hermitian since

$$\begin{aligned} \overline{h_1(v,w)} &= \overline{h(v^+, w^-)} = h(w^-, v^+) = h_2(w,v) \\ \overline{h_2(v,w)} &= h_1(w,v) \end{aligned}$$

In fact linear combinations of the form $\alpha h_1 + \alpha h_2$ and $i\beta h_1 - i\beta h_2$ for $\alpha, \beta \in \mathbb{R}$ are Hermitian.

Next we discuss 'how many' invariant Hermitian forms exist on the spinor space. First we note that the invariant sesquilinear forms on a vector space V span a complex vector space, and the invariant Hermitian forms a real vector space. A sesquilinear form can be regarded as a complex bilinear map: $\bar{V} \times V \to \mathbb{C}$. And by the universal property such maps are in one to one correspondence with complex linear maps $s : \bar{V} \otimes V \to \mathbb{C}$. Equivalently each sesquilinear form can be regarded as a linear map:

$$h : V \to \bar{V}^*$$

To be invariant under a representation ρ of a group G on V means that the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\;h\;} & \bar{V}^* \\ \downarrow{\scriptstyle \rho(g)} & & \downarrow{\scriptstyle \bar{\rho}^*(g)} \\ V & \xrightarrow[\;h\;]{} & \bar{V}^* \end{array}$$

Maps that commute with an action of a group are called **equivariant**. For a spinor space of even dimension $S = S^+ \oplus S^-$ and the above introduced invariant $\mathrm{Spin}_{r,s}$ invariant Hermitian form $h = h_1 + h_2$ we obtain for r odd $\mathrm{Spin}_{r,s}$ equivariant linear maps:

$$\begin{aligned} h_1 :& S^+ \to \overline{S^-}^* \\ h_2 :& S^- \to \overline{S^+}^* \end{aligned}$$

Since h is non degenerate these maps h_1 and h_2 are invertible. Assume that $\tilde{h}_1$ is another $\mathrm{Spin}_{r,s}$ invariant map: $S^+ \to \overline{S^-}^*$. This gives rise to the following commuting diagram:

$$\begin{array}{ccccc} S^+ & \xrightarrow{\;\tilde{h}_1\;} & \overline{S^-}^* & \xrightarrow{\;h_1^{-1}\;} & S^+ \\ \downarrow{\scriptstyle \Delta_{r,s}(g)} & & \downarrow{\scriptstyle \bar{\Delta}^*_{r,s}(g)} & & \downarrow{\scriptstyle \Delta_{r,s}(g)} \\ S^+ & \xrightarrow[\;\tilde{h}_1\;]{} & \overline{S^-}^* & \xrightarrow[\;h_1^{-1}\;]{} & S^+ \end{array}$$

In general the composition of equivariant maps is again equivariant. So $h_1^{-1} \circ \tilde{h}_1$ is a $\mathrm{Spin}_{r,s}$ equivariant map $S^+ \to S^-$. Since S^+ is irreducible we can apply a part of **Schur's lemma** ([FH]) which states that such an equivariant automorphism has to be a scalar multiple of the identity $\lambda\mathbf{1}$ for $\lambda \in \mathbb{C}$. It follows that $\tilde{h}_1 = \lambda h_1$. A analogous argument shows that all $\mathrm{Spin}_{r,s}$

equivariant maps $S^- \to \overline{S^+}^*$ have to be scalar multiples of h_2.

Assume that we have an equivariant map:

$$h_3 : S^+ \to \overline{S^+}^*$$

Composing with $h_2^{-1} : \overline{S^+}^* \to S^-$ we obtain an equivariant map $S^+ \to S^-$. Such a map yields a equivalence of the representations $\Delta^+_{r,s}$ and $\Delta^-_{r,s}$, but above we proved that this representations are inequivalent. Thus h_3 has to be trivial. In an analogous way we can verify that each equivariant map $h_4 : S^- \to \overline{S^-}^*$ has to be trivial.

By putting the previous results together we obtain the following result:

Proposition 1.5.15 *Let $n = 2k = r + s$ be even and r odd. Then each* $\mathrm{Spin}_{r,s}$ *invariant sesquilinear form $h : S \times S \to \mathbb{C}$ on the complex spinor space S can be written as complex linear combination of h_1 and h_2 (1.43). Thus the complex dimension of* $\mathrm{Spin}_{r,s}$ *invariant sesquilinear forms on the spinor space is 2.*

Proof Let h be such an invariant sesquilinear form, i.e. an equivariant map:

$$s : S^+ \oplus S^- \to \overline{S^+ \oplus S^-}^* \cong \overline{S^+}^* \oplus \overline{S^-}^*$$

We can decompose s into $s = s_1 + s_2 + s_3 + s_4$, where

$$s_1 : S^+ \to \overline{S^-}^* \quad s_2 : S^- \to \overline{S^+}^*$$
$$s_3 : S^+ \to \overline{S^+}^* \quad s_4 : S^- \to \overline{S^-}^*$$

With the above results, s_3 and s_4 have to be 0 whereas s_1 and s_2 have to be scalar multiples of h_1 and h_2. ∎

Corollary 1.5.16 *Let $n = 2k = r + s$ be even and r odd. Then the dimension of the real vector space of* $\mathrm{Spin}_{r,s}$ *invariant Hermitian forms on the complex spinor space S is 2. Each Hermitian form can be written as linear combination of $h_1 + h_2$ and $ih_1 - ih_2$.*

Proof Note that each invariant sesquilinear form can uniquely be written as direct sum of an invariant Hermitian and an invariant skew Hermitian form by

$$h(v, w) = \frac{1}{2}(h(v, w) + \overline{h(w, v)}) + \frac{1}{2}(h(v, w) - \overline{h(w, v)})$$

Furthermore we obtain a real vector space isomorphism from the invariant Hermitian forms to the invariant skew Hermitian forms by $\alpha(h)(v, w) = ih(v, w)$. Thus the dimension of the real vector space of $\mathrm{Spin}_{r,s}$ invariant Hermitian forms on the complex spinor space is 2. We arrive at the conclusion that the above linear independent Hermitian forms span this space. ∎

An analogous argument shows that in case of r odd the space of invariant Hermitian forms is again 2. All invariant sesquilinear forms and Hermitian forms can be written as complex or real linear combinations of h_+ and h_-. In this work we are mainly interested in r odd because for spacetimes we have $r = 1$.

1.5.8 Clifford and spinor bundles

In this section the concepts of Clifford and spinor bundles will be introduced where we will use the notation and approach of [LM]. Recall from (1.4) the construction of an associated vector bundle. Now use that orthogonal transformations of $\mathbb{R}^{r,s}$ with the universal property of the Clifford algebra (1.5.1) uniquely extend to automorphisms of the Clifford algebra. This leads to a representation:

$$cl(\rho_{r,s}) : \mathrm{SO}(r,s) \to \mathrm{Aut}(Cl(\mathbb{R}^{r,s})). \tag{1.44}$$

Let X be a spacetime with a spin structure with corresponding principal $\mathrm{Spin}_{r,s}$-bundle $S(X)$. Via the spin homomorphism $cl(\rho_{r,s})$ extends to a representation of $\mathrm{Spin}_{r,s}$. This allows us to define the following associated bundle:

Definition

$$Cl(X) = S(X) \times_{cl(\rho_{r,s})} Cl(\mathbb{R}^{r,s}) \tag{1.45}$$

It is evident that $Cl(X)$ is in fact a bundle of Clifford algebras over X since $cl(\rho_{r,s})$ is an algebra homomorphism. Thus each fiber is isomorphic to a Clifford algebra. Note that the Clifford bundle can also be defined when there is no spin structure on X: The representation of $\mathrm{Spin}_{r,s}$ results from a representation of $\mathrm{SO}(r,s)$ and there the transition functions of $Cl(X)$ can be interpreted to actually take values in $\mathrm{SO}(r,s)$, which leads to the following canonical isomorphism:

$$Cl(X) \cong SO(r,s)(X) \times_{cl(\rho_{r,s})} Cl(\mathbb{R}^{r,s}) \tag{1.46}$$

Thus the Clifford bundle can be interpreted as an associated bundle to the frame bundle of positively oriented orthonormal frames. This exists for every oriented semi-Riemannian manifold.

This leads to an alternative description of the Clifford bundle: The tangent bundle of a semi-Riemannian manifold is a bundle where the fibers are equipped with a bilinear form g or equivalently, a quadratic form q. We can show(what is not done here, but consult [LM]) that in fact the Clifford bundle can be defined as the following quotient bundle:

$$Cl(X) = (\sum_{i=0}^{\infty} \bigotimes^{i} TX)/I(TX) \tag{1.47}$$

where $I(TX)$ is the bundle whichs fiber over $x \in X$ is the two-sided ideal in $\sum_{i=0}^{\infty} \bigotimes^i TX_x$ generated by elements $v \otimes v - q(v)^2$ for $v \in TX_x$.
Via the fiberwise embedding of TX_x in $Cl(X)$ we obtain an injective map from TX into $Cl(X)$.

Definition Let X be a spacetime with a spin structure $S(X)$ as above. Let M be a Clifford module for $Cl(\mathbb{R}^{r,s})$. By restricting to the $\mathrm{Spin}_{r,s} \subset Cl(\mathbb{R}^{r,s})$, we obtain a representation μ of $\mathrm{Spin}_{r,s}$. This allows us to define the following bundle:

$$\mathrm{Spin}(X) = S(X) \times_{\mu} M \tag{1.48}$$

A bundle of this form is a **spinor bundle**

Finally we arive at the following central proposition:

Proposition 1.5.17 *Let* $\mathrm{Spin}(X)$ *be a spinor bundle. Then* $\mathrm{Spin}(X)$ *is a bundle of modules over the bundle of algebras* $Cl(X)$*. In particular the sections of the spinor bundle are a module over the sections of the Clifford bundle.*

Proof Recall from (1.44) that the representation of $\mathrm{Spin}_{r,s}$, with respect to which the Clifford bundle was constructed, arises from an action of $\mathrm{SO}(r,s)$ on the Clifford algebra $Cl(\mathbb{R}^{r,s})$. But such algebra automorphisms can be described directly. Recall from (1.27) the following representation of $\mathrm{Spin}_{r,s}$:

$$\mathrm{Ad} : \mathrm{Spin}_{r,s} \to \mathrm{Aut}(Cl(\mathbb{R}^{r,s})), \quad \mathrm{Ad}_g(\phi) = g\phi g^{-1}$$

By equation (1.5.3) it can be seen that Ad_g and $cl(\rho_{r,s})(g)$ induce the same action on $V \subset \mathrm{Cl}(\mathbb{R}^{r,s})$ and since both are algebra automorphisms they coincide on the entire Clifford algebra. It follows that:

$$Cl(X) = S(X) \times_{Ad} Cl(\mathbb{R}^{r,s})$$

Now observe that the diagram

$$\begin{array}{ccc} Cl(\mathbb{R}^{r,s}) \times M & \xrightarrow{\mu} & M \\ \downarrow{\scriptstyle \rho_g} & & \downarrow{\scriptstyle \rho'_g} \\ Cl(\mathbb{R}^{r,s}) \times M & \xrightarrow[\mu]{} & M \end{array}$$

given by

$$\begin{array}{ccc} (\phi, m) & \longrightarrow & (\phi m) \\ \downarrow & & \downarrow \\ (g\phi g^{-1}, gm) & \longrightarrow & (g\phi m) \end{array}$$

commutes. It follows that the induced map $Cl(\mathbb{R}^{r,s}) \otimes M \to M$ is $\mathrm{Spin}_{r,s}$ equivariant. By proposition (1.4.9) such an equivariant map gives rise to a vector bundle homomorphism:

$$\mu : Cl(X) \otimes \mathrm{Spin}(X) \to \mathrm{Spin}(X) \tag{1.49}$$

which is seen to have the desired properties.

Two spinor bundles $S_1(X)$ and $S_2(X)$ are called **equivalent** if they are equivalent as bundles of $Cl(X)$ modules. A spinor bundle is called **irreducible** if it does not split up in a direct sum of $Cl(X)$ modules. From the earlier derived results of dimension and number of equivalence classes of Clifford modules we can deduce that on connected X the same results hold for Spinor bundles.

A Clifford bundle $Cl(X)$ is defined as an associated bundle to the $\mathrm{Spin}_{r,q}$ principal bundle $S(X)$. Recall from (1.4.8) that a given principal connection on $S(X)$ gives rise to a covariant derivative on associated bundles. So given a principal connection ω on $S(X)$ we obtain a vector bundle connection ∇ on the Clifford bundle. This connection has the following property:

Proposition 1.5.18 ([LM], proposition 4.8) *The covariant derivative ∇ on $Cl(X)$ acts as a derivation on the algebra of sections:*

$$\nabla(\phi \cdot \psi) = (\nabla\phi) \cdot \psi + \phi \cdot (\nabla\psi) \tag{1.50}$$

for any two sections ϕ and ψ of $Cl(X)$. Furthermore on the subbundle $TX \subset Cl(X)$ the covariant derivative agrees with the usual covariant derivative, i.e. the derivative induced by the representation $\rho_{r,s}$ on $\mathbb{R}^{r,s}$.

Proof Clifford multiplication yields a map

$$Cl(\mathbb{R}^{r,s}) \otimes Cl(\mathbb{R}^{r,s}) \to Cl(\mathbb{R}^{r,s})$$

which is $\mathrm{Spin}_{r,s}$ equivariant since it acts on $\mathrm{Cl}(\mathbb{R}^{r,s})$ via conjugation. So by proposition (1.4.9) we obtain a corresponding vector bundle homomorphism:

$$m : Cl(X) \otimes Cl(X) \to Cl(X)$$

which is the Clifford multiplication on the Clifford bundle. The second part of (1.4.9) tells us that $m \circ \nabla_Y(\phi \otimes \psi) = \nabla_Y(\phi \cdot \psi)$ for all vector fields Y and sections ϕ, ψ of $Cl(X)$. Via proposition (1.4.13) this directly yields equation (1.50).
$\mathrm{Spin}_{r,s}$ acts on $\mathbb{R}^{r,s} \subset Cl(\mathbb{R}^{r,s})$ via $\rho_{r,s}$ and the tangent bundle TX can be regarded as associated bundle $S(X) \times_{\rho_{r,s}} \mathbb{R}^{r,s}$. This shows the second assertion.

Next let $\mathrm{Spin}(X)$ be a spinor bundle defined as above. Again this is an associated bundle to $S(X)$. Given a connection ω on $S(X)$ we obtain a connection ∇ on $\mathrm{Spin}(X)$. With respect to the above introduced action of sections of the Clifford bundle on a spinor bundle we arrive at the following:

Proposition 1.5.19 ([LM], proposition 4.11) *The covariant derivative* ∇ *on* $\mathrm{Spin}(X)$ *acts as a derivation with respect to the module structure over* $Cl(X)$*, i.e.:*

$$\nabla(\phi \cdot \sigma) = (\nabla\phi) \cdot \sigma + \phi \cdot (\nabla\sigma) \tag{1.51}$$

for any section ϕ *of* $Cl(X)$ *and any section* σ *of* $\mathrm{Spin}(X)$.

Proof The proof is essentially analogous to (1.5.9). Recall the Clifford multiplication map (1.49):

$$\mu : Cl(X) \otimes \mathrm{Spin}(X) \to \mathrm{Spin}(X)$$

which is induced by an equivariant map $Cl(\mathbb{R}^{r,s}) \times M \to M$. Then we again use proposition (1.4.9). ∎

We are now in a position to define an important class of complex spinor bundles:

Definition Let X be a connected oriented and time oriented semi Riemannian manifold of signature $(1, s)$ where $1 + s = 2k$ is even. Assume that X is equipped with a spin structure. In this dimension there exists, as shown above, an unique up to equivalence irreducible complex Clifford module of complex dimension 2^k. The complex spinor bundle which originates from this module is the **Dirac spinor bundle** $\mathbf{D}(X)$.

Here we show that as above we can split $\mathbf{D}$ into a direct sum of $Cl^0(X)$ modules. Since X is oriented, we can choose an orientation and define a global section of $Cl(X, \mathbb{C})$ by setting at each point $x \in X$:

$$\omega = i^m e_1 \cdots e_{2m} \tag{1.52}$$

for any positively oriented orthonormal basis $\{e_1, \dots, e_{2m}\}$. In each fiber this is exactly the volume element we defined above. We can now use the $+1$ and -1 eigenbundles to obtain the desired direct sum splitting:

$$\mathbf{D}(X) = \mathbf{D}(X)^+ \oplus \mathbf{D}(X)^- \tag{1.53}$$

These bundles can be written as an associated bundle: Let $\Delta^{\pm}_{1,s}$ denote the two complex chiral representations of $\mathrm{Spin}_{1,s}$. Then there is a canonical vector bundle isomorphism:

$$\mathbf{D}(X)^{\pm} \cong S(X) \times_{\Delta^{\pm}_{1,s}} S^{\pm}$$

Now assume that there is a given unit section e of the tangent bundle, i.e. a map: $e : X \to TX$ such that $||e(x)|| = 1$ for each x. In our setting such a section always exists, the time orientability of X ensures that. Observe that ω anticommutes with e: $\omega e = -e\omega$. Thus multiplication by e swaps the $+1$ and -1 eigenbundles of ω. We obtain bundle maps:

$$\mu_e : \mathbf{D}(X)^+ \to \mathbf{D}(X)^-, \quad \mu_e(\sigma) = e \cdot \sigma$$
$$\mu_e : \mathbf{D}(X)^- \to \mathbf{D}(X)^+, \quad \mu_e(\sigma) = e \cdot \sigma$$

Since $e \cdot e = 1$ it is clear that these maps are in fact **isomorphisms**.

1.5.9 The Dirac operator

Next we define an important first-order differential operator.

Definition Let X be a semi Riemannian manifold with Clifford bundle $Cl(X) \supset TX$ and let S be any bundle of left modules over $Cl(X)$. Assume that S is furnished with a connection ∇. Then we can define the following operator:

$$D : \Gamma(X, S) \xrightarrow{\nabla} \Gamma(X, T^*X \otimes S) \xrightarrow{\cong} \Gamma(X, TX \otimes S) \xrightarrow{c} \Gamma(X, S)$$

Here the isomorphism $TX \cong T^*X$ induced by the metric is used and c denotes the canonical map $TX \otimes S \to S$ induced by the Clifford module multiplication. This operator $D : \Gamma(S) \to \Gamma(S)$ is the **Dirac operator**.

Let us express the Dirac operator in orthonormal coordinates. For $x \in X$ let $(e_1, \dots, e_n)$ denote a local orthonormal frame of TX. Let σ be a section of S. The covariant derivative of σ is given in coordinates by: $\sum_{i=1}^n e^i \otimes \nabla_{e_i}\sigma$, where e^i denotes the dual coframe to e_i. By the definition of the Dirac operator we have to apply the canonical sharp isomorphism $\sharp : T^*X \to TX$ to e^i, and we obtain that $\sharp(e^i) = q(e_i)e_i$. Then, according to the above definition, we have to apply the Clifford multiplication. Thus we obtain:

$$D\sigma = \sum_{i=1}^{n} q(e_i)e_i \cdot \nabla_{e_i}\sigma \tag{1.54}$$

which is the expression of the Dirac operator with respect to a local orthonormal frame.

Now apply the Dirac operator to spinor bundles. In this setting we have a space time X with spin structure $S(X)$, an associated spinor bundle $\mathrm{Spin}(X)$ and the canonical Clifford bundle $Cl(X)$. By (1.5.9) we have a Riemannian connection form ω on $S(X)$, and by (1.5.9) and (1.5.19) we obtain an algebra connection ∇ on $Cl(X)$ and a connection ∇ on $\mathrm{Spin}(X)$ which respects the Clifford module structure. We can now define the Dirac operator $\Gamma(\mathrm{Spin}(X)) \to \Gamma(\mathrm{Spin}(X))$. The same construction for irreducible spinor bundles of Riemannian manifolds leads to the so called **Atiyah-Singer operator**, a fundamental operator in spin geometry. However, we are interested in the semi Riemannian case.

In particular we can carry out this construction for an irreducible complex spinor bundle, the Dirac spinor bundle $\mathbf{D}(X)$. We obtain a Dirac operator:

$$D : \Gamma(\mathbf{D}(X)) \to \Gamma(\mathbf{D}(X)) \tag{1.55}$$

Next we investigate how the Dirac operator behaves with respect to the splitting: $\mathbf{D}(X) = \mathbf{D}(X)^+ \oplus \mathbf{D}(X)^-$. First observe that the volume element ω is parallel with respect to the Clifford algebra connection. To prove this, choose a local orthonormal tangent frame field $(e_1, \ldots, e_n)$ so that $(\nabla e_j)_x = 0$ for all j. Since the associated derivation on the Clifford bundle is an algebra derivation and coincides with the usual covariant derivative on TX we obtain

$$\nabla\omega = \nabla(e_1 \cdots e_n) = (\nabla e_1)e_2 \cdots e_n + \ldots + e_1 \cdots e_{n-1}\nabla e_n = 0 \tag{1.56}$$

so ω is indeed parallel. We can prove that ∇ preserves $\mathbf{D}(X)^{\pm}$. This follows from these bundles being characterized as ± 1 eigenbundles of the module action of ω on $\mathbf{D}$. Let σ be a section in $\mathbf{D}^+$:

$$\nabla\sigma = \nabla(\omega \cdot \sigma) = \omega \cdot (\nabla\sigma)$$

Thus $\nabla\sigma$ is also in the $+1$ eigenbundle of ω, and an analogous argument holds for the -1 eigenbundle. The full Dirac operator D is a composition of the covariant derivative and the Clifford multiplication of TX. The latter operation anticommutes with ω, therefore it swaps the $+1$ and -1 eigenbundle of ω. We arrive at the following result:

Proposition 1.5.20 *A Dirac operator D on a Dirac spinor bundle $\mathbf{D} = \mathbf{D}^+ \oplus \mathbf{D}^-$ has the form:*

$$D = D^+ + D^-$$

where

$$D^+ : \Gamma(\mathbf{D}^+) \to \Gamma(\mathbf{D}^-)$$
$$D^- : \Gamma(\mathbf{D}^-) \to \Gamma(\mathbf{D}^+)$$

The next goal is to lift the $\mathrm{Spin}_{r,s}$ invariant sesquilinear products introduced in (1.43) to products on the Dirac spinor bundle. Recall from (1.5.8) that this bundle was defined as associated bundle to the principal $\mathrm{Spin}_{1,s}$ bundle $S(X)$ via the complex spinor representation:

$$D(X) = S(X) \times_{\Delta_{1,s}} S$$

Recall from (1.5.14) that there exists a $\Delta_{1,s}$ invariant Hermitian form h on S. It can be interpreted as equivariant linear map $S \to \bar{S}^*$. By proposition (1.4.9 this gives rise to a vector bundle homomorphism:

$$\mathbf{D}(X) = S(X) \times_{\Delta_{1,s}} S \to S(X) \times_{\overline{\Delta_{1,s}}^*} \bar{S}^* = \overline{\mathbf{D}(X)}^*$$

This corresponds to a map:

$$h : \Gamma(\mathbf{D}(X)) \times \Gamma(\mathbf{D}(X)) \to C^\infty(X, \mathbb{C})$$

which is $C^\infty(X, \mathbb{C})$ antilinear in the first component and $C^\infty(X, \mathbb{C})$ linear in the second. An analogous argument holds in a more general situation, we arrive at the following:

Proposition 1.5.21 *Let $G \hookrightarrow P \xrightarrow{\mathcal{P}} X$ be a principal bundle, let $\rho : G \to GL(V)$ be a complex representation of the structure group, $E = P \times_\rho V$ the associated bundle and h a ρ-invariant sesquilinear form on V. Via the above construction we then can lift h to a bilinear map:*

$$h : \Gamma(E) \times \Gamma(E) \to C^\infty(X, \mathbb{C})$$

which is $C^\infty(X, \mathbb{C})$ antilinear in the first component and $C^\infty(X, \mathbb{C})$ linear in the second. This is called a ***sesquilinear form*** *on E.*
If h is Hermitian the lifted map is Hermitian as well, i.e. $h(\sigma_1, \sigma_2) = \overline{h(\sigma_2, \sigma_1)}$ for all sections σ_1, σ_2 of E.

In particular we can lift h to a Hermitian form on $\mathbf{D}(X)$. Recall the splitting of $h(v, w) = h_1(v^+, w^-) + h_2(v^-, w^+)$ for all $v, w \in S$. This yields a decomposition of the form on the Dirac spinor bundle $h = h_1 + h_2$ where h_1 and h_2 can be interpreted as maps:

$$h_1 : \Gamma(\mathbf{D}^+) \times \Gamma(\mathbf{D}^-) \to C^\infty(X, \mathbb{C}) \tag{1.57}$$
$$h_2 : \Gamma(\mathbf{D}^-) \times \Gamma(\mathbf{D}^+) \to C^\infty(X, \mathbb{C}) \tag{1.58}$$

Both are antilinear in the first component and linear in the second. By (1.5.15) each sesquilinear form on $\mathbf{D}$ can be written as a $C^\infty(X, \mathbb{C})$ linear combination of them.

Lifted sesquilinear forms on associated bundles which come from an invariant sesquilinear form on a vector space are parallel:

Proposition 1.5.22 *With the notation and setting from (1.5.21) the followings holds:*

$$Yh(\sigma_1, \sigma_2) = h(\nabla_Y \sigma_1, \sigma_2) + h(\sigma_1, \nabla_Y \sigma_2)$$

for all $Y \in TX$ *and* $\sigma_1, \sigma_2 \in \Gamma(E)$

Proof The invariant sesquilinear form h on V corresponds to a equivariant linear map:

$$\tilde{h} : V \otimes \bar{V} \to \mathbb{C}$$

By proposition (1.4.9) this corresponds to a vector bundle homomorphism:

$$\tilde{h} : \Gamma(E \otimes \bar{E}) \to C^\infty(X, \mathbb{C})$$

For sections of σ_1, σ_2 of E the equation $\tilde{h}(\sigma_1 \otimes \overline{\sigma_2}) = h(\sigma_1, \sigma_2)$ holds. Using the second part of (1.4.9) we arrive at:

$$\tilde{h}(\nabla_Y(\sigma_1 \otimes \overline{\sigma_2})) = \nabla_Y \tilde{h}(\sigma_1 \otimes \overline{\sigma_2})$$

from which the proposition follows. ∎

Chapter 2

Terms of the Lagrangian

We are ready to write down suitable terms for a Lagrangian. Let X denote a spacetime. We assume that there is a principal bundle $G \hookrightarrow P(X) \to X$, where G is the symmetry group of the system. Furthermore we have a finite number of fields ψ_i which are described as sections of associated vector bundles to $P(X)$. Because of the semi Riemannian structure on this oriented manifold there exists a canonical volume form **vol**, with respect to which we can integrate smooth functions in $C^\infty(X)$. This volume form is defined as the unique element λ in $\Lambda^n(X)$ such that, for any x in X and any oriented orthonormal basis $(e_1, \dots e_n)$ for $T_x(X)$, $\lambda_x(e_1, \dots e_n) = 1$.

A typical Lagrangian is of the form:

$$L = \int_X \mathcal{L}(x)\ \mathbf{vol}(x)$$

Here $\mathcal{L}(x)$ denotes the **Lagrangian density**, which is a smooth function on X (a element in $C^\infty(X)$). It typically depends on a principal connection ω on $P(X)$ and on the fields ψ_i. The Lagrangian is a functional:

$$L(\omega, \psi_1, \dots \psi_n) = \int_X (\mathcal{L}(\omega, \psi_1, \dots \psi_n))(x)\ \mathbf{vol}(x)$$

Quantum field theories in particle physics such as quantum electrodynamics or the Standard Model are usually formulated in terms of the Lagrangian density $\mathcal{L}$, which is often simply called Lagrangian by physicists. In the next sections we will write down typical parts of it:

2.1 Kinetic terms

With the developed framework we are ready to write down kinetic terms in a quantum field theory describing spin $\frac{1}{2}$ fermions. We begin with a toy example without any inner symmetries. Let X be of even dimension and be equipped with a spin structure, i.e. a principal $\mathrm{Spin}_{1,s}$ bundle $S(X)$ with the properties (1.5.3). Then a fermionic field is described as section ψ of the associated **Dirac spinor bundle D**. (1.5.8) Let h, h_1, and h_2 be the Hermitian forms on $\mathbf{D}$ introduced in (1.57). Let ∇ be the associated connection on $\mathbf{D}$ to the Levi-Civita connection on $S(X)$. This gives rise to the Dirac operator D on $\mathbf{D}$ (1.55). We will use maps of the form $\Gamma(\mathbf{D}) \to C^\infty(X)$ to obtain terms in the Lagrangian density. A kinetic term has to involve the associated connection. With the tools at our hand we can construct such a map. First define:

$$K : \Gamma(\mathbf{D}) \to C^\infty(X, \mathbb{C}), \quad \psi \to h(\psi, D\psi). \tag{2.1}$$

Notice how this map behaves with respect to the splitting of $D = D^+ + D^-$:

$$\psi = \psi^+ + \psi^- \xmapsto{D^+ + D^-} D^+\psi^+ + D^-\psi^- \mapsto h_1(\psi^+, D^+\psi^+) + h_2(\psi^-, D^-\psi^-)$$

In particular we have a decomposition $K = K^1 + K^2$, where K^1 and K^2 can be interpreted as maps

$$K_1 : \Gamma(\mathbf{D}^+) \to C^\infty(X, \mathbb{C}) \tag{2.2}$$

$$K_2 : \Gamma(\mathbf{D}^-) \to C^\infty(X, \mathbb{C}) \tag{2.3}$$

Note that if we started with any sesquilinear form $\tilde{h} = \alpha h_1 + \beta h_2$ on $\mathbf{D}$ where $\alpha, \beta \in \mathbb{C}$ we would have a corresponding splitting $\tilde{K} = \alpha K_1 + \beta K_2$.

An admissible term in the Lagrangian density should be real valued. This can be achieved by adding the complex conjugate: $\tilde{K} = K + \bar{K}$. In physics K also gains an additional factor i since eventually this gives the 'right' equations. But that is beyond the scope of this work, the aim here is to build a fitting mathematical framework.

After explaining this toy model we write down kinetic terms in a framework which is applied in physics. Additional inner symmetries come into play and in many relevant gauge theories they are modeled by associated bundles to principal bundles of unitary groups(1.1.2). The usual way to describe both the inner symmetries and the spacetime symmetry $\mathrm{Spin}_{1,s}$ is to splice their principle bundles to obtain a new principal bundle which has the product group as structure group.

How does this fit into the framework developed so far? Let as above $S(X)$ be the principal $\mathrm{Spin}_{1,s}$ bundle of a spacetime with spin structure and let $U(X)$ denote a principal bundle of an unitary group. Furthermore assume that there exists a complex representation $\rho : U \to U(V) \subset GL(V)$, i.e. V is equipped with a positive definite Hermitian form that is invariant under the representation of U. According to proposition (1.5.21) we can lift this Hermitian form to a Hermitian form on the associated bundle $E := U(X) \times_\rho V$.

We arrive at an associated bundle $(S(X) \circ U(X)) \times_{(\Delta^{2k} \times \rho)} (S \otimes V)$. By proposition (1.4.12) it is canonically isomorphic to $\mathbf{D} \otimes E$. Both bundles carry Hermitian forms, and there exists a distinguished Hermitian form on their tensor product due to the following standard result:

Proposition 2.1.1 *Let V and W be complex vector bundles with Hermitian forms h_1 and h_2. There then exists a Hermitian form h on their tensor product such that:*

$$h(v_1 \otimes w_1, v_2 \otimes w_2) = h_1(v_1, v_2)h_2(w_1, w_2)$$

for all sections v_1 and v_2 of V and w_1 and w_2 of W

Thus we obtain a Hermitian form $\tilde{h}$ on $\mathbf{D} \otimes E$. Recall that the Spin invariant form h on $\mathbf{D}$ splits into two sesquilinear forms h_1 and h_2. This yields a splitting $\tilde{h} = \tilde{h_1} + \tilde{h_2}$.

Furthermore $\mathbf{D} \otimes E$ can be made a bundle of left modules over the Clifford bundle $Cl(X)$. This is carried out in the obvious way: For $\phi \in \Gamma(Cl(X)), \sigma \in \Gamma(\mathbf{D})$ and $e \in \Gamma(E)$ the module multiplication is defined by:

$$\phi \cdot (\sigma \otimes e) = (\phi \cdot \sigma) \otimes e$$

By linearly extending we obtain a Clifford action on the whole bundle.

Next let us assume that we have a given principal connection ω^E on the principal bundle $U(X)$. Furthermore let ω^R denote the Levi-Civita connection on $S(X)$. With (1.21) we obtain vector bundle connections ∇^R and ∇^E on the associated bundles $\mathbf{D}$ and E. Combined they yield a principal connection on the spliced bundle $S(X) \circ U(X)$. This in turn gives rise to a vector bundle connection ∇^{R+E} on the associated bundle $\mathbf{D} \otimes E$. By proposition (1.4.13) this is the canonical tensor product connection. Thus for all sections of the form $\sigma \otimes e$:

$$\nabla^{R+E}(\sigma \otimes e) = (\nabla^R \sigma) \otimes e + \sigma \otimes (\nabla^E e)$$

We can show that this derivation is a **module derivation**, i.e. compatible with the Clifford multiplication:

$$\nabla^{R+E}(\phi \cdot (\sigma \otimes e)) = \nabla^{R+E}((\phi \cdot \sigma) \otimes e) = (\nabla^R(\phi \cdot \sigma)) \otimes e + (\phi \cdot \sigma) \otimes (\nabla^E e) =$$
$$= (\nabla\phi) \cdot (\sigma \otimes e) + \phi \cdot ((\nabla^R \sigma) \otimes e) + \phi \cdot (\sigma \otimes (\nabla^E e)) = (\nabla\phi)(\sigma \otimes e) + \phi \cdot (\nabla^{R+E}(\sigma \otimes e))$$

for all sections of $\mathbf{D} \otimes E$ of the form $\sigma \otimes e$ and $\phi \in \Gamma(Cl(X))$. By linearly extending it then holds for all sections of $\mathbf{D} \otimes E$.

The splitting $\mathbf{D} = \mathbf{D}^+ \oplus \mathbf{D}^-$ carries over to a splitting:

$$\mathbf{D} \otimes E = (\mathbf{D}^+ \otimes E) \oplus (\mathbf{D}^- \otimes E)$$

The bundle $\mathbf{D}^\pm \otimes E$ is canonically isomorphic to the associated bundle $(S(X) \circ U(X)) \times_{(\Delta^\pm_{1,s} \times \rho)} (S^\pm \otimes V)$. So the chiral representations take the place of the spinor representation, which motivates the following naming:

In physics, fermionic fields are described as sections of $\mathbf{D} \otimes E$. From now on we refer to it as **fermionic bundle** and to the bundles $\mathbf{D}^\pm \otimes E$ as **chiral fermionic bundles**.

Via the action of the Clifford bundle on the fermionic bundle we obtain an action of the complexified Clifford bundle $Cl(X) \otimes \mathbb{C}$. Due to the definition of this action the chiral fermionic bundles can be characterized in the following way:

Proposition 2.1.2 *The chiral fermionic bundles $\mathbf{D}^\pm \otimes E$ are the ± 1 eigenbundles of the action of the volume element $\omega \in Cl(X) \otimes \mathbb{C}$*

The bundle $\mathbf{D} \otimes E$ is a Clifford module and carries a connection. Thus by (1.5.9) there exists a Dirac operator D on it. By using that the vector bundle connection is a Clifford module connection and by repeating the proof that lead to (1.5.20) we obtain that $\tilde{D}$ splits into two parts $\tilde{D}^+ + \tilde{D}^-$. These can be understood as operators:

$$\tilde{D}^+ : \Gamma(\mathbf{D}^+ \otimes E) \to \Gamma(\mathbf{D}^- \otimes E)$$
$$\tilde{D}^- : \Gamma(\mathbf{D}^- \otimes E) \to \Gamma(\mathbf{D}^+ \otimes E)$$

Starting from this we construct suitable terms for the Lagrangian. Recall that the Hermitian form $\tilde{h}$ on the fermionic bundle $\mathbf{D} \otimes E$ splits into the sum of sesqilinear forms $\tilde{h}_1 + \tilde{h}_2$. With the splitting of the fermionic bundle into the chiral fermionic bundles these can be interpreted as maps:

$$\tilde{h}_1 : \Gamma(\mathbf{D}^+ \otimes E) \times \Gamma(\mathbf{D}^- \otimes E) \to C^\infty(X, \mathbb{C})$$
$$\tilde{h}_2 : \Gamma(\mathbf{D}^- \otimes E) \times \Gamma(\mathbf{D}^+ \otimes E) \to C^\infty(X, \mathbb{C})$$

which are $C^\infty(X, \mathbb{C})$ antilinear in the first component and $C^\infty(X, \mathbb{C})$ linear in the second.

With all structures lifted to the spliced bundle and the associated fermionic bundle we can now write down kinetic terms analogous to before(2.1):

$$\tilde{K} : \Gamma(\mathbf{D} \otimes E) \to C^\infty(X, \mathbb{C}),\ \psi \to \tilde{h}(\psi, D\psi)$$

Again we have a splitting $\tilde{K} = \tilde{K}_1 + \tilde{K}_2$ where $\tilde{K}^1$ and $\tilde{K}^2$ can be interpreted as maps

$$\tilde{K}_1 : \Gamma(\mathbf{D}^+ \otimes E) \to C^\infty(X, \mathbb{C})$$
$$\tilde{K}_2 : \Gamma(\mathbf{D}^- \otimes E) \to C^\infty(X, \mathbb{C})$$

By adding their complex conjugates to make the Lagrangian density real we arrive at kinetic terms for the chiral fermionic bundles.

All kinetic terms in the Standard Model are of this form with one exception: The kinetic term for the Higgs field. This field is not fermionic, it transforms trivially under spacetime transformations. This is the defining property of a scalar. It can be described as section of an associated bundle $(S(X) \circ U(X)) \times_{1\times\rho} (\mathbb{C} \otimes V)$ where 1 denotes the trivial representation and ρ is, as before, an unitary representation, i.e. a complex representation that preserves a Hermitian product on V. We refer to a bundle of this form as **scalar bundle**. This associated bundle is by proposition (1.4.12) canonically isomorphic to $(S(X) \times_1 \mathbb{C}) \otimes (U(X) \times_\rho V)$. But the first bundle is just the trivial complex line bundle, thus we obtain an isomorphism:

$$(S(X) \times_1 \mathbb{C}) \otimes (U(X) \times_\rho V) \cong U(X) \times_\rho V =: E$$

To write down a kinetic term lift the Hermitian form h on V as before to a Hermitian form $h : \Gamma(E) \times \Gamma(E) \to C^\infty(X, \mathbb{C})$. Then note that for a semi Riemannian manifold X the metric gives rise to a symmetric bilinear form $g : \Gamma(T^*X) \times \Gamma(T^*X) \to C^\infty(X)$.

These two forms can be lifted to a unique Hermitian form on the real tensor product bundle with the canonical complex structure $T^*X \otimes E$ by the defining property(2.1.1) above:

$$\tilde{h}(v_1 \otimes \sigma_1, v_2 \otimes \sigma_2) = g(v_1, v_2)\ h(\sigma_1, \sigma_2)$$

for all sections v_1, v_2 of T^*X and σ_1, σ_2 of E. Given a connection on $U(X)$ we have an associated connection ∇ on E. We can write down the following kinetic term:

$$K : \Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes T^*X) \xrightarrow{\tilde{h}} C^\infty(X) \tag{2.4}$$

Since $\tilde{h}$ is Hermitian, terms of the form $\tilde{h}(\sigma, \sigma)$ are real valued for all sections $\sigma \in \Gamma(E \otimes T^*X)$. Thus we do not need to add the complex conjugate.

2.2 Polynomial terms

In this section another class of terms in a typical Lagrangian is discussed. Contrary to kinetic terms no derivatives are involved in polynomial terms. There is no dependence on the principal connection at all.

The general setting is the following: Let G be a Lie group, X be a space-time and $P(X)$ a principal bundle over X with structure group G. Furthermore let $\rho_i : G \to GL(V_i)$ be a complex representation of G on a vector space V_i, where $i = 1, \dots, n$. Then, as in (1.5.6), there are unique complex conjugate representations $\bar{\rho}_i$ of G on $\bar{V}_i$. Let E_i denote the associated bundle $P(X) \times_{\rho_i} V_i$ and $\bar{E}_i$ the complex conjugate associated bundle $P(X) \times_{\overline{\rho_i}} \bar{V}_i$. The canonical antilinear map $C : V_i \to \bar{V}_i$ is equivariant, so analogous to proposition (1.4.9) we obtain a $C^\infty(X, \mathbb{C})$ antilinear map

$$C : \Gamma(E_i) \to \Gamma(\bar{E}_i).$$

Assume now that we have given a G invariant multilinear form

$$q : V_{i_1} \times V_{i_2} \times \cdots \times V_{i_k} \times \overline{V_{j_1}} \times \cdots \times \overline{V_{j_l}} \longrightarrow \mathbb{C} \tag{2.5}$$

where the indices are in $\{1, \cdots n\}$. By the universal property of the tensor product such maps are in one-to-one correspondence with G invariant linear maps:

$$q : V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_k} \otimes \overline{V_{j_1}} \otimes \cdots \otimes \overline{V_{j_l}} \longrightarrow \mathbb{C}$$

Analogous to the lift of invariant Hermitian forms (1.5.21) it can be shown that q lifts to a $C^\infty(X, \mathbb{C})$ multilinear map:

$$q : \Gamma(E_{i_1}) \times \Gamma(E_{i_2}) \times \cdots \times \Gamma(E_{i_k}) \times \Gamma(\overline{E_{j_1}}) \cdots \times \Gamma(\overline{E_{j_l}}) \longrightarrow C^\infty(X, \mathbb{C})$$

Via composition with $C : E_j \to \bar{E}_j$ we obtain a map:

$$p : \Gamma(E_{i_1}) \times \Gamma(E_{i_2}) \times \cdots \times \Gamma(E_{i_k}) \times \Gamma(E_{j_1}) \times \cdots \times \Gamma(E_{j_l}) \longrightarrow C^\infty(X, \mathbb{C})$$

By defining $\tilde{p} := p + \overline{p}$ we arrive at a suitable term $\tilde{p}$ for the Lagrangian density.

Next we discuss how to construct invariant multilinear forms as in (2.5). The group G acts on $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_k} \otimes \overline{V_{j_1}} \cdots \otimes \overline{V_{j_l}}$ via the tensor product representation. Most of the groups important in physics have the property that all their representations are completely reducible, i.e. they are isomorphic to a direct sum of irreducible representations. The following Lemma is useful:

Lemma 2.2.1 *Let $\rho : G \to GL(V)$ be a complete reducible representation of group G on a complex vector space V. Then the space of G invariant linear maps $V \to \mathbb{C}$ is isomorphic to the space of G invariant vectors in V.*

Proof Let $V \cong A_1 \oplus A_2 \oplus \cdots \oplus A_m$ be a decomposition of V into irreducible parts. Let $\rho_i : G \to A_i$ denote the corresponding irreducible representations. For each G invariant map $q : V \to \mathbb{C}$ we have the following commuting diagram:

$$\begin{array}{ccccc} A_1 \oplus A_2 \oplus \cdots \oplus A_m & \xrightarrow{\cong} & V & & \\ \Big\downarrow{\scriptstyle (\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_m)(g)} & & \Big\downarrow{\scriptstyle \rho} & \searrow^{q} & \\ A_1 \oplus A_2 \oplus \cdots \oplus A_m & \xrightarrow[\cong]{} & V & \xrightarrow[q]{} & \mathbb{C} \end{array}$$

for all $g \in G$. We can decompose $q = q_1 + q_2 + \ldots + q_m$ where q_j is the restriction of q to A_j. The kernel of each q_j is an invariant subspace. If A_j is not the trivial representation, so of dimension ≥ 2, the kernel is not empty and due to the irreducibility of A_j it has to be A_j. Thus q_j is the 0 map. Therefore only the parts of q on the trivial representations can be non trivial. This shows that the space of all invariant maps $V \to \mathbb{C}$ is isomorphic to the dual of the space of invariant vectors in V and hence to the space of invariant vectors in V. ∎

Note that the isomorphism is not unique. An immediate consequence of this lemma is:

Proposition 2.2.2 *With the notation as above and for completely reducible representations: The complex vector space of G invariant multilinear maps $V_{i_1} \times V_{i_2} \times \cdots \times V_{i_k} \times \overline{V_{j_1}} \cdots \times \overline{V_{j_l}} \longrightarrow \mathbb{C}$ is isomorphic to the complex vector space of G invariant vectors in $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_k} \otimes \overline{V_{j_1}} \cdots \otimes \overline{V_{j_l}}$*

2.3 Yang Mills terms

The third class of terms in the Lagrangian density only depends on principal connections. The general context here is: As before let X be a spacetime. Let G be a Lie group and assume we have a principal bundle $P(X)$ over X with structure group G. Let ω be a principal connection on $P(X)$. Let Ω denote the Lie algebra $\mathfrak{g}$ valued corresponding curvature form on $P(X)$. By (1.4.4) this is a tensorial two form of type Ad, so with (1.4.6) we can interprete Ω as a two form on X with values in the associated bundle $P(X) \times_{\mathrm{Ad}} \mathfrak{g}$. This bundle is usually referred to as the **adjoint bundle** $\mathrm{Ad}(P)$. Then Ω can be interpreted as a section of the real vector bundle $\Lambda^2 T^*M \otimes \mathrm{Ad}(P)$

Next observe that the semi Riemannian metric g on X gives rise to a symmetric bilinear form on the two forms $\Lambda^2 T^*X$: Extend the metric analogously to (2.1.1) to a symmetric bilinear form on $g : T^*X \otimes T^*X \to C^\infty(X)$ and then restrict g to $\Lambda^2 T^*X \subset (T^*X \otimes T^*X)$.

Given an Ad invariant bilinear form on the Lie algebra $\mathfrak{g}$ we can lift this as above to a $C^\infty(X)$ bilinear form on the sections of $\mathrm{Ad}(P)$. Then we can lift the two symmetric bilinear forms on the sections of $\Lambda^2 T^*X$ and Ad P analogous to (2.1.1) to a symmetric form:

$$b : \Gamma(\Lambda^2 T^*X \otimes \mathrm{Ad}(P)) \times \Gamma(\Lambda^2 T^*X \otimes \mathrm{Ad}(P)) \to C^\infty(X)$$

which is $C^\infty(X)$ linear in each component. So interpreting Ω as a section of $\Lambda^2 T^*X \otimes (\mathrm{Ad}\ P)$, by $b(\Omega, \Omega)$ we obtain an element in $C^\infty(X)$. This is a suitable Lagrangian density. Note that it only depends on the connection form ω.

In the construction above we need an Ad invariant bilinear form on the Lie algebra $\mathfrak{g}$. For many relevant Lie groups such a form can readily be constructed. Recall from (1.3) the adjoint representation:

$$ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad}(x)(y) = [x, y]$$

With this we can define the following map for any finite dimensional real Lie algebra:

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad B(x, y) = \mathrm{trace}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$$

which is seen to be symmetric and bilinear. This form is commonly known as **Killing form** and plays a prominent role in the classification of semi simple Lie algebras. It has many useful properties, but here we require only that it is invariant under automorphisms s of the Lie algebra. In our case, with the Lie algebra derived from a Lie group G, this yields:

$$B(\mathrm{Ad}_g(x), \mathrm{Ad}_g(y)) = B(x, y)$$

for all $x, y \in \mathfrak{g}$ and for all $g \in G$. Thus the Killing form is a form with the desired property of Ad invariance.

In physics the special unitary groups $SU(n)$ as defined in (1.1.2) often appear as symmetry groups. By the identification of the Lie algebra $\mathfrak{su}(n)$ with Hermitian skew symmetric trace free matrices and the commutator as the Lie bracket it can be shown that the following formula for the Killing form B holds(see ([FH]), Exercise 14.36):

$$B(X, Y) = 2n\, \mathrm{trace}(XY) \tag{2.6}$$

for all $X, Y \in \mathfrak{su}(n)$. With this equation the Killing form can be calculated in a straightforward way. We also have that $B(X,X) = -2n\,\mathrm{trace}(X^*X)$, where X^* denotes the Hermitian conjugate of X. From this it follows that the Killing form is negative definite on $\mathfrak{su}(n)$. More generally, there is a theorem that the Killing form on Lie algebras of semisimple Lie groups is always non-degenerate. ([FH]).

Chapter 3

The Lagrangian of the Standard Model

With the framework developed in the previous chapters we are ready to write down the Standard Model Lagrangian in a coordinate independent way (0.2). We then show that expressed in coordinates the introduced terms match the one used in physics(see appendix A). The aim here is to reproduce the Standard Model Lagrangian in a mathematical exact framework, not to understand the (physical) meanings of the terms.

3.1 General setting

We begin our discussion with the Minkowski spacetime (1.2) $X = \mathbb{R}^{1,3}$ endowed with the canonical structure of a spacetime manifold and a chosen orientation and time orientation. The tangent bundle TX can be trivialized globally by using the coordinate vector fields as basis. Thus we have a isomorphism $TX \cong (X \times \mathbb{R}^{1,3})$. Recall from (1.9) that X has a corresponding oriented, time oriented orthonormal frame bundle $\mathrm{SO}^0(1,3) \hookrightarrow \mathcal{L}(X) \xrightarrow{\mathcal{P}_L} X$. The tangent space in each point is canonically isomorphic to $\mathbb{R}^{1,3}$ and thus we have a canonical trivialization of the oriented, timeoriented orthonormal frame bundle, i.e. there is a isomorphism: $\mathcal{L}(X) \cong X \times SO^0(1,3)$.

The tangent space at a point (x, g) in $\mathcal{L}(X)$ can be written as a direct sum $TX_x \oplus (T\mathcal{L}_+^\uparrow)_g$. This gives rise to a Lie algebra valued one form ω_0 on $\mathcal{L}(X)$ by defining:

$$\omega_{0(x,g)}(v, w) := (T\sigma_{g^{-1}})_g w \tag{3.1}$$

where $\sigma_g(g_2) = g_2 \cdot g$ denotes the right action of $\mathcal{L}_+^\uparrow$ on itself. It can directly be verified that ω is a connection one form. Its kernel in (x, g) is TX_x. In fact it can be shown that the torsion tensor of ω vanishes. Thus this connection is the Levi Civita connection.

Furthermore we can construct a Spin structure on X (1.5.3). We define $S(X) = X \times \mathrm{Spin}_{1,3}$ and

$$\lambda : S(X) = (X \times \mathrm{Spin}_{1,3}) \xrightarrow{\mathrm{id}\times\mathrm{Spin}} (X \times \mathcal{L}_+^\uparrow) \cong \mathcal{L}(X)$$

It is obvious that $S(X)$ and λ together are a Spin structure.

The inner symmetry group of the Standard Model is the product group $U(1) \times SU(2) \times SU(3)$. Let $U(X) = X \times (U(1) \times SU(2) \times SU(3))$ denote the trivial principal bundle over X. Note that this is the spliced bundle of $X \times U(1)$, $X \times SU(2)$ and $X \times SU(3)$. We then splice $U(X)$ with the spin structure bundle $S(X)$ to obtain the full symmetry bundle:

$$\mathrm{Spin}_{1,3} \times U(1) \times SU(2) \times SU(3) \hookrightarrow (P(X) = S(X) \circ U(X)) \to X$$

Now we are exactly in the situation explained in section (2.1). Fermionic fields are described as sections of associated bundles to $P(X)$, the chiral fermionic bundles. For their construction we need to describe representations of the inner symmetry group of the Standard Model.

Moreover the Lagrangian also depends on principal connections, in physics called gauge fields. Let ω_1, ω_2 and ω_3 denote principal connections on trivial $U(1)$, $SU(2)$ and $SU(3)$-bundles. As in (1.4.11) we obtain a principal connection $\omega_1 + \omega_2 + \omega_3$ on $U(X)$. Splicing again with the Levi-Civita connection yields a principal connection on the full symmetry bundle $P(X)$.

3.2 The representations of the Standard Model

As mentioned in the previous section fermionic fields in the Standard Model are described as sections of chiral fermionic bundles, i.e. associated bundles to $(S(X) \circ U(X)) \times_{(\Delta^{\pm}_{1,3} \times \rho)} (\mathbb{C}^2 \otimes V)$ where $\rho : (U(1) \times SU(2) \times SU(3)) \to GL(V)$ is a representation of the inner symmetry group. First, we discuss the representations appearing in the Standard Model:

Begin with the unitary group $\mathbf{U(1)}$. By definition this group consists of all complex numbers c satisfying $\bar{c}c = 1$, i.e. the unit circle. Each element in $U(1)$ can be written as $e^{i\theta}$. In the Standard Model all occuring representations of $U(1)$ are complex of dimension one and are of the following form:

$$e^{i\theta} \cdot z = e^{3iY\theta} z \tag{3.2}$$

The parameter Y has to be a multiple of $\frac{1}{3}$ otherwise this is not a valid representation. The factor three can be attributed to historical development and is of no mathematical significance. Such a representation is denoted by the value of Y and the one dimensional vector space on which it acts by

$\mathbb{C}_Y$. It can be verified that all these representations preserve the standard Hermitian product on $\mathbb{C}$, thus they are unitary representations.

Next we discuss the representations of $\mathbf{SU(2)}$. As mentioned in (1.1.2) we have the standard representation where each matrix in $SU(2)$ is simply represented by itself. We denote this representation for reasons of dimension by 2, and similarly the one dimensional trivial representation by 1. It follows directly from the definition of $SU(2)$ that 2 preserves the standard Hermitian product on $\mathbb{C}^2$. Furthermore let $\overline{2}$ denote the **conjugation representation** of the standard representation given on $\mathbb{C}^2$ by $g \to \bar{g}$. These two representations are equivalent. We can verify this with the following argument: We define $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then for any complex 2×2 matrix g with determinant 1:

$$BgB^{-1} = (g^T)^{-1} \tag{3.3}$$

However, for all g in $SU(2) : \bar{g}^T = g^{-1}$, and therefore $BgB^{-1} = (g^T)^{-1} = \bar{g}$. Thus the standard representation of $SU(2)$ and its conjugate are equivalent.

At last we discuss representations of $\mathbf{SU(3)}$. Again we have the standard representation which preserves the standard Hermitian product on $\mathbb{C}^3$. For reasons of dimension we will denote it by 3. Analogously to $SU(2)$ we have the conjugate representation of 3: $g \to \bar{g}$ and we denote it by $\bar{3}$. However, this time 3 and $\bar{3}$ are not equivalent. Let 1 denote the trivial representation on $\mathbb{C}$, again for reasons of dimension.

With the notation defined, we are ready to explicitly write down the representations of fermionic fields in the Standard Model, and additionally the Higgs field representation:

The Standard Model representations			
Name	Symbol	$\text{Spin}_{1,3}$	$SU(3) \times SU(2) \times U(1)$
Left-handed quark	Q_L	Δ^+	$(3, 2, \frac{1}{3})$
Right-handed quark(up)	u_R	Δ^-	$(3, 1, \frac{4}{3})$
Right-handed quark(down)	d_R	Δ^-	$(3, 1, -\frac{2}{3})$
Left-handed lepton	L_L	Δ^+	$(1, 2, -1)$
Right-handed electron	e_R	Δ^-	$(1, 1, -2)$
Higgs field	ϕ	trivial	$(1, 2, 1)$

It is an experimental fact that each fermion appears in three generations. Thus each fermionic field carries a generation index, which we suppress for the moment. The reason for this is that most of the terms in the Lagrangian are identical for all three generations. The Lagrangian density can equivalently be described in terms of the conjugate representations, which corresponds to antiparticles. We will see that conjugation changes left to right and vice versa. Thus, by conjugating the right-handed fields everything can be described in terms of left handed fields, which is common in particle physics.

Note that all the above representations are irreducible. This follows from the fact that, given a product of groups $G_1 \times G_2$, irreducible representations of G_1 and G_2 tensor to a irreducible representation of $G_1 \times G_2$. The representations of $U(1), SU(2)$ and $SU(3)$ mentioned above are all irreducible. Thus their tensor products are as well.

3.3 Gauge terms

The first terms in the Standard Model we explicitly write down are the Yang Mills terms from the previous chapter(2.3). These only depend on the connections ω_1, ω_2 and ω_3 on the trivial $U(1)$, $SU(2)$ and $SU(3)$ bundles. In order to write down a Yang Mills term for each of these bundles we need Ad invariant forms on the Lie algebra of these groups. For $SU(2)$ and $SU(3)$ these will be scalar multiples of the Killing form.

The Lie algebra $\mathfrak{u}(1)$ is one dimensional and hence the Lie bracket vanishes. Thus the Killing form is the 0 form. It is easily verified that $U(1)$ is abelian. Therefore the representation Ad is trivial and any symmetric bilinear form on the Lie algebra $\mathfrak{u}(1)$ is suitable. Applying the matrix interpretation $\mathfrak{u}(1)$ can be seen to be the space of pure imaginary numbers. By using the element i as basis this is canoncially isomorphic to $\mathbb{R}$, and the only symmetric bilinear forms there are multiples of the standard one.

Next we discuss the problem of stating the Yang Mills terms in coordinates. To this end we need the curvature forms Ω_1, Ω_2 and Ω_3 of the principal connections ω_1, ω_2 and ω_3. Therefore we first show how to generally calculate a curvature form Ω in terms of a principal bundle connection ω on a principal bundle $G \hookrightarrow P \to X$. The curvature form is identified with an associated bundle valued 2-form $\tilde{\Omega}$. To express it in coordinates we need a local cross-section $s : U \to P$, where U is a open subset of X. Then by (1.4.7) this curvature is of the following form:

$$\tilde{\Omega}_x = [s(x), (s^*\Omega)_x] \tag{3.4}$$

Our aim is to express $\mathcal{F} := (s^*\Omega)_x$, in physics called **local field strength** ([N2]), in terms of $\mathcal{A} := s^*\omega$, commonly called the **local gauge potential**. We can pull back both sides of the Cartan structure equation(1.3.1) by s

and immediately obtain:

$$s^*\Omega = ds^*\omega + [s^*\omega, s^*\omega] \tag{3.5}$$

Assume now that the domain $U \subset X$ of the cross-section s is also a coordinate neighborhood for a chart V with coordinate functions $x_1, \dots x_n$. Then the local field strength and the local gauge potential can be written in coordinate form $\mathcal{A} = \mathcal{A}_\alpha dx^\alpha$ and $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\alpha\beta}dx^\alpha dx^\beta$, where the $\mathcal{A}_\alpha$ and $\mathcal{F}_{\alpha\beta}$ denote $\mathcal{G}$-valued functions on U. Here we have used the **Einstein notation**, where an index which appears twice implies summation over all values of that index. Plugging this into (3.4) yields:

$$\tilde{\Omega}_x = \sum_{\alpha<\beta}[s(x), (\mathcal{F}_{\alpha\beta})_x]dx^\alpha dx^\beta \tag{3.6}$$

A calculation using (3.5) shows that:

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha + [\mathcal{A}_\alpha, \mathcal{A}_\beta] \tag{3.7}$$

After choosing a local section $\tilde{\Omega}$ can be described in terms of the components of $\mathcal{A} = s^*\omega$.

3.3.1 U(1) part

Next we begin to write down the Yang Mills term with respect to coordinates. We start with the $U(1)$ part. Let b denote a form on its Lie algebra $\mathfrak{u}(1)$, and g the symmetric bilinear form on T^*X induced by the metric. Recall from (2.3) that we have a symmetric form:

$$p : \Gamma(\Lambda^2 T^*X \otimes (\mathrm{Ad}\ P)) \times \Gamma(\Lambda^2 T^*X \otimes (\mathrm{Ad}\ P)) \longrightarrow C^\infty(X)$$

The Lagrangian density is then given by $p(\tilde{\Omega}, \tilde{\Omega})$. Substituting with (3.6) yields:

$$p(\frac{1}{2}\sum_{\alpha\beta}[s, (\mathcal{F}_{\alpha\beta})]dx^\alpha dx^\beta, \sum_{\gamma\delta}\frac{1}{2}[s, (\mathcal{F}_{\gamma\delta})]dx^\gamma dx^\delta) =$$
$$\frac{1}{4}\sum_{\alpha\beta}\sum_{\gamma\delta} b(\mathcal{F}_{\alpha\beta}, \mathcal{F}_{\gamma\delta})g(dx^\alpha, dx^\gamma)g(dx^\beta, dx^\delta) =$$
$$\frac{1}{4}\sum_{\alpha\beta}\sum_{\gamma\delta} b(\mathcal{F}_{\alpha\beta}, \mathcal{F}_{\gamma\delta})g^{\alpha\gamma}g^{\beta\delta} = \frac{1}{4}\sum_{\alpha\beta}\sum_{\gamma\delta} b(\mathcal{F}_{\alpha\beta}, \mathcal{F}_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta}) = \frac{1}{4}\sum_{\alpha\beta} b(\mathcal{F}_{\alpha\beta}, \mathcal{F}^{\alpha\beta})$$

The notation implicitly introduced in the last line is common in physics. It is referred to as 'raising an index with the metric'. Next we identify the

Lie algebra in the canonical way with matrices and think of the Lie algebra valued function $\mathcal{F}_{\alpha\beta}$ as matrix valued. In case of $\mathfrak{u}(1)$ these are purely imaginary numbers. Due to the above b is a multiple of the symmetric product $\tilde{b}(v,w) = 4 \cdot 9 \cdot v \cdot w$ for all purely imaginary numbers. The factor 9 is introduced by taking the vector $\frac{1}{3}i$ as basis vector while the first factor is conventional. So $b = (g_1)^{-2}\tilde{b}$, where g_1 is a real number. Inserting this in we have:

$$p(\widetilde{\Omega_1}, \widetilde{\Omega_1}) = \frac{1}{4}(\frac{6}{g_1}\mathcal{F}_{\alpha\beta} \cdot \frac{6}{g_1}\mathcal{F}^{\alpha\beta})$$

Here the Einstein summation convention is used. The constant g_1 can not be explained by the Standard Model and is called a coupling constant. It determines how strong the gauge terms of the Lagrangian density are in comparison to the interaction terms. In physics it usually absorbed it in the Lie algebra by defining:

$$B_\mu := \frac{6}{ig_1}\mathcal{A}_\mu; \quad B_{\mu\nu} := \frac{6}{ig_1}\mathcal{F}_{\mu\nu} \tag{3.8}$$

The factor i enters to make the matrices Hermitian. With this notation we arrive at:

$$p(\widetilde{\Omega_1}, \widetilde{\Omega_1}) = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

This is exactly the Standard Model Lagrangian density, see appendix A. Note that equation (3.7) becomes:

$$\mathcal{B}_{\mu\nu} = \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu$$

because the Lie bracket vanishes for $\mathfrak{u}(1)$ and the multiplicative factors cancel out.

3.3.2 SU(2) part

Repeating the calculation for the $U(1)$ part for the $SU(2)$ part we obtain:

$$p(\widetilde{\Omega_2}, \widetilde{\Omega_2}) = \frac{1}{4}\sum_{\alpha\beta} b(\mathcal{F}_{\alpha\beta}, \mathcal{F}^{\alpha\beta})$$

Interpreting $\mathcal{F}^{\alpha\beta}$ as traceless 2×2 matrix valued functions, we can define on them a form(2.6) proportional to the Killing form:

$$\tilde{b}(X,Y) = 2\,\mathrm{trace}(X \circ Y)$$

Again we use a scalar multiple: $b = \frac{1}{(g_2)^2}\tilde{b}$, where g_2 is a nonnegative real number. Interpreting again $\mathcal{F}^{\alpha\beta}$ as traceless skew Hermitian valued functions we obtain:

$$p(\tilde{\Omega}, \tilde{\Omega}) = \frac{1}{2}(\mathrm{trace}((g_2)^{-1}\mathcal{F}_{\alpha\beta} \cdot (g_2)^{-1}\mathcal{F}^{\alpha\beta})) = -\frac{1}{8}(\mathrm{trace}(\frac{2}{ig_2}\mathcal{F}_{\alpha\beta} \cdot \frac{2}{ig_2}\mathcal{F}^{\alpha\beta}))$$

As above we can absorb this factors in the Lie algebra and define:

$$W_\mu := \frac{2}{ig_2}\mathcal{A}_\mu; \quad W_{\mu\nu} := \frac{2}{ig_2}\mathcal{F}_{\mu\nu} \tag{3.9}$$

Note that these can be interpreted as 2×2 tracefree Hermitian matrix valued functions. We arrive at:

$$p(\widetilde{\Omega_2}, \widetilde{\Omega_2}) = -\frac{1}{8}\,\mathrm{trace}(W_{\mu\nu}W^{\mu\nu})$$

This is a Standard Model Lagrangian term(A). A calculation shows that equation (3.7) becomes:

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + \frac{ig_2}{2}(W_\mu W_\nu - W_\nu W_\mu).$$

3.3.3 SU(3) part

The $SU(3)$ part is treated in an analogous way to $SU(2)$. Again we choose:

$$\tilde{b}(X, Y) = 2\,\mathrm{trace}(XY)$$

as reference bilinear form on $\mathfrak{su}(2)$ and $b = \frac{1}{g^2}\tilde{b}$ we obtain:

$$p(\widetilde{\Omega_3}, \widetilde{\Omega_3}) = \frac{1}{2}(\mathrm{trace}(g^{-1}\mathcal{F}_{\alpha\beta} \cdot g^{-1}\mathcal{F}^{\alpha\beta}))$$

We can once again absorb the factors in the Lie algebra by defining:

$$G_\mu := \frac{1}{ig}\mathcal{A}_\mu; \quad G_{\mu\nu} := \frac{1}{ig}\mathcal{F}_{\mu\nu} \tag{3.10}$$

which are tracefree Hermitian 3×3 matrix valued functions. We arrive at

$$p(\widetilde{\Omega_3}, \widetilde{\Omega_3}) = -\frac{1}{2}\,\mathrm{trace}(G_{\mu\nu}G^{\mu\nu}) \tag{3.11}$$

as $\mathfrak{su}(3)$ term in the Lagrangian density. A calculation shows that equation (3.7) becomes:

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + ig(G_\mu G_\nu - G_\nu G_\mu)$$

This concludes the treatment of the Yang Mills part of the Standard Model Lagrangian density.

3.4 Fermionic kinetic terms

We now write down the kinetic part for a fermionic field in the Standard Model. Recall that they can be described as sections of chiral fermionic bundles $\mathbf{D}^{\pm} \otimes E$. In kinetic terms the Dirac operator appears. In order to write it down in coordinates we need an explicit complex representation of the Clifford algebra $Cl_{1,3}$ in the 4×4-complex matrices. We construct such a representation via the so called **Dirac matrices**. First define the following set of contravariant Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

With the definition of the Pauli matrices:

$$\sigma^\mu = [\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}]; \quad \tilde{\sigma}^\mu = [\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3]$$

the Dirac matrices can be compactly written in the form: $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}$.

The covariant matrices are defined as: $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu = \{\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3\}$. For the covariant Dirac matrices the following equation holds:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}1$$

We obtain an embedding of the Minkowski space in the set of 4×4 matrices:

$$\iota : \mathbb{R}^{1,3} \to \mathrm{End}(\mathbb{C}^4) : \quad v^\mu e_\mu \to v^\mu \gamma_\mu$$

where e_μ denotes the standard basis of $\mathbb{R}^{1,3}$. We see that $\iota(v) \cdot \iota(v) = q(v)$, so by the universal property of the Clifford algebra (1.5.1) the embedding uniquely extends to a map $\iota : Cl_{1,3} \to \mathrm{End}(\mathbb{C}^4)$. We then extend ι further in the natural way to a algebra homomorphism:
$\iota : \mathbb{C}l_4 \to \mathrm{End}(\mathbb{C}^4)$. For dimensional reasons this has to be an isomorphism, and we have explicitly constructed an irreducible complex Clifford module for $Cl_{1,3}$. Restricting to $\mathrm{Spin}_{1,3} \subset Cl_{1,3} \subset \mathbb{C}l_4$ we obtain a complex spinor representation: $\Delta_{1,3} : \mathrm{Spin}_{1,3} \to GL_4(\mathbb{C})$ and $\mathbb{C}^4$ becomes a spinor space.

With respect to the canonical isomorphism: $SL(2,\mathbb{C}) \cong \mathrm{Spin}_{1,3}$ it can be shown that $\Delta_{1,3}$ takes the following form([N2]):

$$\Delta_{1,3} : SL(2,\mathbb{C}) \to GL_4(\mathbb{C}), \quad \Delta_{1,3}(g) = \begin{pmatrix} g & 0 \\ 0 & (\bar{g}^T)^{-1} \end{pmatrix}$$

We can immediately extract the splitting into the chiral representations $\Delta^{\pm}_{1,3}$: On the first two components of $\mathbb{C}^4$ we have a representation $\Delta^{+}_{1,3}(g) = g$, and on the last two components $\Delta^{-}_{1,3}(g) = (\bar{g}^T)^{-1}$. Thus we expect the volume element to be diagonal. By definition (1.37) it is given by:

$$\omega = i^2\gamma_0 i\gamma_1 i\gamma_2 i\gamma_3 = i\gamma_0\gamma_1\gamma_2\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The chiral projections $\frac{1}{2}(\mathbf{1} \pm \omega)$ are projections on the first respectively last two components. We can also write down a Hermitian, $\mathrm{Spin}_{1,3}$ invariant form:

$$h : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}, \quad h(v,w) = \bar{v}^T\gamma^0 w = \bar{v_1}w_3 + \bar{v_2}w_4 + \bar{v_3}w_1 + \bar{v_4}w_2$$

Furthermore, as in (1.43), we have a splitting $h = h_1 + h_2$ into two $\mathrm{Spin}_{1,3}$ invariant sesquilinear forms:

$$h(v^+ + v^-, w^+ + w^-) = h_1(v^+, w^-) + h_2(v^-, w^+)$$

Note that in coordinates $\mathbb{C}^2 \cong S^+$ and $\mathbb{C}^2 \cong S^-$ the form $h_1 : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ is just the **standard Hermitian form** on $\mathbb{C}^2$.

With this explicit description of the spinor representations we are ready to write down the kintetic terms in coordinates.

3.4.1 The covariant derivative

Let ψ denote a section of a chiral fermionic bundle $\mathbf{D}^{\pm} \otimes E$ of the the Standard Model. Then we choose a section $s : \mathbb{R}^{1,3} \subset U \to (S(X) \circ U(X))$ of which the first component is the element (x, e) with respect to the canonical identification $S(X) \cong (X \times \mathcal{L}^{\uparrow}_{+})$. It follows from (3.1) that the pullback $s^*\omega_0$ of the Levi civita connection ω_0 on $S(X)$ is 0. With respect to such a section we obtain:

$$\psi(\mathbf{x}) = [s(\mathbf{x}), v(\mathbf{x})] \tag{3.12}$$

for a unique $v : U \to (\mathbb{C}^2 \otimes V)$. Note that V is the tensor product of standard complex vector spaces $\mathbb{C}^i$s, so via the standard isomorphism we can interprete v as map from U to $\mathbb{C}^n$ for some $n \in \mathbb{N}$.

The first step in constructing a kinetic term for such a section is applying the Dirac operator. By equation (1.54) we first have to apply the covariant derivative $\nabla_{\partial x_\mu}$ where ∇ denotes the covariant derivative that is associated to a principal connection ω. We calculate this in coordinates according to (1.4.10):

$$(\nabla_{\partial x_\mu}\psi)(\mathbf{x}) = [s(\mathbf{x}), (\frac{\partial}{\partial x_\mu} + (s^*\omega)_\mu(\mathbf{x}))\cdot v(\mathbf{x})] \tag{3.13}$$

Let

$$D_\mu = (\frac{\partial}{\partial x_\mu} + (s^*\omega)_\mu(\mathbf{x}))$$

The principal connection ω is the sum of the connections of the $X \times U(1)$, $X \times SU(2)$, $X \times SU(3)$ and $S(X)$ connections ω_1, ω_2, ω_3 and ω_0. With definitions (3.8), (3.9) and (3.10) we arrive at:

$$(s^*\omega)^\mu = \mathcal{A}^1_\mu + \mathcal{A}^2_\mu + \mathcal{A}^3_\mu = \frac{ig_1}{6}B_\mu + \frac{ig_2}{2}W_\mu + igG_\mu$$

where we have used the fact that $s^*\omega_0$ vanishes.

For a trivial representation the action of the Lie algebra on a vector defined via (1.17) is given by $A \cdot v = 0$. In the $SU(2)$ and $SU(3)$ parts we only have standard representations for which the action of the Lie Algebra is given by matrix multiplication: $\mathcal{A}_\mu \cdot v = \mathcal{A}_\mu v$. For the $U(1)$ representations (3.2) we directly calculate:

$$\mathcal{A}_\mu \cdot v = \frac{d}{d\theta}(e^{\theta\mathcal{A}_\mu}\cdot v) = \frac{d}{d\theta}(e^{3Y\theta\mathcal{A}_\mu})v = 3Y\mathcal{A}_\mu v \tag{3.14}$$

Expressing everything in terms of B_μ, W_μ and G_μ we can write down D_μ for all the gauge terms:

The covariant derivatives		
Name	Symbol	D_μ
Left-handed quark	Q_L	$\partial_\mu + \frac{ig_1}{6}B_\mu + \frac{ig_2}{2}W_\mu + igG_\mu$
Right-handed quark(up)	u_R	$\partial_\mu + \frac{i2g_1}{3}B_\mu + igG_\mu$
Right-handed quark(down)	d_R	$\partial_\mu - \frac{ig_1}{3}B_\mu + igG_\mu$
Left-handed lepton	L_L	$\partial_\mu - \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}W_\mu$
Right-handed electron	e_R	$\partial_\mu - ig_1B_\mu$

3.4.2 Clifford multiplication

The next step in order to obtain the Dirac operator is to carry out the Clifford multiplication with $q(e_\mu)e_\mu$. For our chosen cross section $s : \mathbb{R}^{1,3} \to S(X) \circ U(X)$ which is the trivial section on $S(X)$ Clifford multiplication by the tangent vector $\partial x_\mu \subset TX_\mathbf{x} \subset Cl_{1,3}$ on a section of a fermionic bundle is given by:

$$e_\mu \cdot [s(\mathbf{x}), v(\mathbf{x})] = [s(\mathbf{x}), \gamma_\mu v(\mathbf{x})]$$

Note that $q(\partial x_\mu)\gamma_\mu = \gamma^\mu$, thus the Dirac operator on a fermionic bundle is given by:

$$\tilde{D} : \Gamma(\mathbf{D} \otimes E) \to \Gamma(\mathbf{D} \otimes E), \quad \tilde{D}[s(\mathbf{x}), v(\mathbf{x})] = [s(\mathbf{x}), \gamma^\mu D_\mu v(\mathbf{x})]$$

With respect to the splitting of the fermionic bundle in the chiral fermionic bundles we arrive at:

$$\begin{aligned} \tilde{D^+} :&\Gamma(\mathbf{D}^+ \otimes E) \to \Gamma(\mathbf{D}^- \otimes E), \quad \tilde{D}^+[s(\mathbf{x}), v(\mathbf{x})] = [s(\mathbf{x}), \bar{\sigma}^\mu D_\mu v(\mathbf{x})] \\ \tilde{D^-} :&\Gamma(\mathbf{D}^- \otimes E) \to \Gamma(\mathbf{D}^+ \otimes E), \quad \tilde{D}^-[s(\mathbf{x}), v(\mathbf{x})] = [s(\mathbf{x}), \sigma^\mu D_\mu v(\mathbf{x}] \end{aligned}$$

The kinetic term for a left handed chiral fermionic field is given by (2.2):

$$\tilde{h_{1x}}([s(\mathbf{x}), v(\mathbf{x})], [s(\mathbf{x}), \bar{\sigma}^\mu D_\mu v(\mathbf{x})] = k(v(\mathbf{x}), \bar{\sigma}^\mu D_\mu v(\mathbf{x}))$$

where k denotes the invariant Hermitian form on $\mathbb{C}^2 \otimes V$. Note that V is a tensor product of $\mathbb{C}^i$s. In every component we have the standard Hermitian form on $\mathbb{C}^j$ and on $\mathbb{C}^2$ we also have the standard Hermitian form. With the interpretation of v as map $\mathbb{R}^{1,3} \to \mathbb{C}^n$ we see that k is the standard Hermitian form on $\mathbb{C}^n$ and hence:

$$k(v(\mathbf{x}), \bar{\sigma}^\mu D_\mu v(\mathbf{x})) = \overline{v(\mathbf{x})}^T \bar{\sigma}^\mu D_\mu v(\mathbf{x}) \tag{3.15}$$

Thus the kinetic term for a left handed fermionic field is given by:

$$K : \Gamma(\mathbf{D}^+ \otimes E) \to C^\infty(X), \quad ([s, v_L]) = \overline{v_L}^T \bar{\sigma}^\mu i D_\mu v_L + (c.c.)$$

where c.c. denotes the complex conjugate. As earlier explained the factor i is introduced for physical reasons. Completely analogous we obtain for a right handed fermionic field:

$$K : \Gamma(\mathbf{D}^- \otimes E) \to C^\infty(X), \quad ([s, v_R]) = \overline{v_R}^T \sigma^\mu i D_\mu v_R + (c.c.)$$

In physics sections of fermionic bundles are regarded as sections of vectors of operator valued distributions called quantum fields. However, this is beyond the scope of this work. All the equations for the Standard Model so far are for 'classical fields', but in physics we would need it for 'quantum fields'. Lagrangian densities then are not real functions on the spacetime, but rather operator valued distributions. The complex conjugate is replaced by the Hermitian conjugate, and in the above terms the (c.c) is replaced by (h.c.) and $\bar{v}$ by v^*.

3.5 Higgs field terms

In the Standard Model there is one field which is different from the others, the **Higgs field**. This field is central to the whole Standard Model, as via the so called Higgs mechanism it gives rise to mass terms for the gauge and fermionic fields. By (3.2) the Higgs field is a scalar bundle and described as a section of the associated bundles to the trivial representation of $\mathrm{Spin}_{1,3}$ and the $(1,2,1)$ representation of $U(1)$, $SU(2)$ and $SU(3)$.

In (2.4) the kinetic term for such a scalar bundle is described. We now write it down in coordinates. With respect to a section $s : \mathbb{R}^{1,3} \to S(X) \circ U(X)$, which is trivial on the $S(X)$ component as in (3.4.1) we can write a Higgs field ϕ as:

$$\phi(\mathbf{x}) = [s(\mathbf{x}), v(\mathbf{x})]$$

First we have to calculate its covariant derivative. As in (3.13) we arrive at:

$$(\nabla_{\partial x_\mu}\phi)(\mathbf{x}) = [s(\mathbf{x}), D_\mu v(\mathbf{x})]$$

where

$$D_\mu v = (\frac{\partial}{\partial x_\mu} + (s^*\omega)_\mu)v = (\frac{\partial}{\partial x_\mu} + \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}W_\mu)v.$$

Next we write down the full kinetic term in coordinates:

$$K_x(\phi,\phi) = \tilde{h}_x(\sum_\mu [s(\mathbf{x}), D_\mu v(\mathbf{x})]dx^\mu, \sum_\nu [s(\mathbf{x}), D_\nu v(\mathbf{x})]dx^\nu) =$$
$$\sum_{\mu\nu} g(dx^\mu, dx^\nu) h(D_\mu v(\mathbf{x}), D_\nu v(\mathbf{x})) = \sum_{\mu\nu} h(D_\mu v(\mathbf{x}), D_\nu v(\mathbf{x}) g^{\mu\nu}) =$$
$$\sum_\mu h(D_\mu v(\mathbf{x}), D^\mu v(\mathbf{x}))$$

The representation $(1,2,1)$ acts on $(\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}) \cong \mathbb{C}^2$, and h is the standard Hermitian product on the latter space. Using the Einstein convention we have:

$$K(\phi,\phi) = (\overline{D_\mu v})^T D^\mu v$$

This is the coordinate form of the Lagrangian density for the Higgs field.

There is another term in the Lagrangian density which solely depends on the Higgs field, the **Higgs mass term**. This is a simple polynomial term.

Notice that the $(1,2,1)$ representations is a unitary representation, and it is the standard Hermitian form h on $\mathbb{C}^2 \cong \mathbb{C}\otimes\mathbb{C}^2\otimes\mathbb{C}$ which is invariant under this representation. We can immediately lift this to an invariant form P on the Higgs bundle. With notations as above we arrive at a polynomial term:

$$P_x(\phi,\phi) = P([s(\mathbf{x}),v(\mathbf{x})],[s(\mathbf{x}),v(\mathbf{x})]) = h(v(\mathbf{x}),v(\mathbf{x})) = \overline{v(\mathbf{x})}^T v(\mathbf{x})$$

The Higgs mass term is given by $-m_h^2[P(\phi,\phi)-\frac{c^2}{2}]^2/(2c^2)$, where m_h and c are real parameters of the Standard Model called Higgs mass and Higgs vacuum expectation value. This is a valid term in the Lagrangian density. Written down in coordinates we obtain:

$$-m_h^2(\overline{v}^T v - \frac{c^2}{2})^2/(2c^2)$$

3.6 Yukawa coupling terms

In this section we treat polynomial terms which depend on the Higgs field and fermionic fields but are independent of the connection. They all are of similar form: The complex conjugate of a left handed fermionic field, a scalar Higgs field and a right handed fermionic field appears. In physics such terms are called Yukawa terms. To construct this polynomial terms we need to find invariant multilinear forms on the vector spaces of the representations and their conjugate representations.

To this end we first revisit the representations of $\mathrm{Spin}_{1,3} \cong SL(2,\mathbb{C})$. For the chiral fermionic fields we have:

$$\Delta^+_{1,3} : SL(2,\mathbb{C}) \to GL_2(\mathbb{C}) : \quad \Delta^+_{1,3}(g) = g \tag{3.16}$$
$$\Delta^-_{1,3} : SL(2,\mathbb{C}) \to GL_2(\mathbb{C}) : \quad \Delta^-_{1,3}(g) = (\bar{g}^T)^{-1} \tag{3.17}$$

Now assume we have given sections of the left-handed lepton bundle $(1,2,-1)$, the scalar Higgs bundle $(1,2,1)$ and the right-handed electron bundle $(1,1,-2)$. With respect to a section s of the principle bundle we can write these as:

$$[s(\mathbf{x}),L_L(\mathbf{x})] \quad [s(\mathbf{x}),\phi(\mathbf{x})] \quad [s(\mathbf{x}),e_R(\mathbf{x})]$$

where

$$L_L : \mathbb{R}^{1,3} \to \mathbb{C}^2\otimes\mathbb{C}\otimes\mathbb{C}^2\otimes\mathbb{C}_{-1} =: V_L$$
$$\phi : \mathbb{R}^{1,3} \to \mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}^2\otimes\mathbb{C}_{1} =: V_\phi$$
$$e_R : \mathbb{R}^{1,3} \to \mathbb{C}^2\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}_{-2} =: V_R$$

There exists a multilinear form:

$$b : (\overline{V_L}) \times V_\phi \times V_R \to \mathbb{C}, \quad b(\overline{L_L}, \phi, e_R) = \overline{L_L}^T \phi e_R \tag{3.18}$$

A calculation using (3.16) shows that this is invariant under $SL(2,\mathbb{C}) \times U(1) \times SU(2) \times SU(3)$. The SU(2) group action on ϕ and L_L cancels out, as well as the $SL(2,\mathbb{C})$ action on L_L and e_R. By adding the complex conjugate this gives rise to a admissable term in the Lagrangian. An additional factor $-\frac{\sqrt{2}}{v}$ where v is the Higgs vacuum expectation value occurs in the Standard model.

Recall that there are three generations of Leptons. We could write down such a term for each generation as implicitly done before. However the matter it is more complicated. We could use right and left handed leptons from different generations and couple them as in (3.19). This "mixing" happens in the Standard Model and is described by a complex 3×3 matrix M^e called **mass matrix**. We arrive at the following term in the Lagrangian density:

$$-\frac{\sqrt{2}}{v}\overline{L_L}^T \phi M^e e_R + (c.c.)$$

where the generation indices are suppressed.

A analogous term exists for the quark fields. Here we have sections of the left handed quark bundle $(3, 2, \frac{1}{3})$, the scalar Higgs bundle $(1, 2, 1)$, the right handed up quark bundle $(3, 1, \frac{4}{3})$ and the right handed down quark bundle $(3, 1, -\frac{2}{3})$. With respect to a section s of the principle bundle we can write these as:

$$[s(\mathbf{x}), Q_L(\mathbf{x})] \quad [s(\mathbf{x}), \phi(\mathbf{x})] \quad [s(\mathbf{x}), u_R(\mathbf{x})] \quad [s(\mathbf{x}), d_R(\mathbf{x})]$$

where

$$\begin{aligned} Q_L :& \mathbb{R}^{1,3} \to \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{\frac{1}{3}} =: V_L \\ \phi :& \mathbb{R}^{1,3} \to \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1 =: V_\phi \\ u_R :& \mathbb{R}^{1,3} \to \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{\frac{4}{3}} =: V_{u_R} \\ d_R :& \mathbb{R}^{1,3} \to \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}_{-\frac{2}{3}} =: V_{d_R} \end{aligned}$$

Again we have a multilinear form:

$$b : (\overline{V_L}) \times V_\phi \times V_{d_R} \to \mathbb{C}, \quad b(\overline{Q_L}, \phi, d_R) = \overline{Q_L}^T \phi d_R \tag{3.19}$$

As above this is a invariant under $SL(2,\mathbb{C}) \times U(1) \times SU(2) \times SU(3)$ as a calculation shows: The $SU(3)$ and $SL(2,\mathbb{C})$ parts in $\overline{Q_L}^T$ and d_R cancel out, the $SU(2)$ parts in $\overline{Q_L}^T$, ϕ and the $U(1)$ terms in all parts as well.

Analogously to the term for leptons we obtain a term in the Lagrangian density:

$$-\frac{\sqrt{2}}{v}\overline{Q_L}^T\phi M^d d_R + (c.c.)$$

As above here M^d is a complex 3×3 mass matrix that describes how different generations mix.

When writing down a similar term for up quarks we encounter a problem. The first candidate $\overline{Q_L}^T\phi u_R$ is not $U(1)$ invariant. A solution for this would be to look a the term $\overline{Q_L}^T\overline{\phi}u_R$, which is $U(1)$ invariant. But now a new problem occurs, we have two conjugate representations of $SU(2)$ and these do not cancel out. However, recall from (3.3) that the conjugate and standard representation of $SU(2)$ are equivalent: $BgB^{-1} = \bar{g}$. With this we can construct the following invariant multilinear form:

$$b : \overline{V_L} \times \overline{V_\phi} \times V_{u_R} \to \mathbb{C}, \quad b(\overline{Q_L}, \overline{\phi}, u_R) = (B^{-1}\overline{Q_L})^T\overline{\phi}d_R$$

where B^{-1} acts on the $SU(2)$ part. The full term in the Lagrangian density is then described by:

$$-\frac{\sqrt{2}}{v}(B^{-1}\overline{Q_L})^T\overline{\phi}M^u d_R + (c.c.)$$

where M^u is a complex 3×3 matrix that describes the mixing of the different generations.

This concludes the treatment of the terms in the Standard Model Lagrangian. We were able to write them down with respect to a section s of the principal bundle which is the form that appears in particle physics. Recall that in physcis quantum fields have to be used. Consequently all complex conjugations have to be replaced by Hermitian conjugates. Furthermore the order is important in all terms as operator valued distributions do not have to commute.

Appendix A

The full Standard Model Lagrangian in coordinates

Standard Model Lagrangian (including neutrino mass terms)
From *An Introduction to the Standard Model of Particle Physics, 2nd Edition*,
W. N. Cottingham and D. A. Greenwood, Cambridge University Press, Cambridge, 2007,
Extracted by J.A. Shifflett, updated from Particle Data Group tables at pdg.lbl.gov, 28 Mar 2013.

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}tr(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}) - \frac{1}{2}tr(\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}) && \text{(U(1), SU(2) and SU(3) gauge terms)}\\
& +(\bar\nu_L, \bar e_L)\,\tilde\sigma^\mu iD_\mu \begin{pmatrix}\nu_L\\ e_L\end{pmatrix} + \bar e_R\sigma^\mu iD_\mu e_R + \bar\nu_R\sigma^\mu iD_\mu\nu_R + \text{(h.c.)} && \text{(lepton dynamical term)}\\
& -\frac{\sqrt2}{v}\left[(\bar\nu_L,\bar e_L)\,\phi M^e e_R + \bar e_R\bar M^e\bar\phi\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\right] && \text{(electron, muon, tauon mass term)}\\
& -\frac{\sqrt2}{v}\left[(-\bar e_L,\bar\nu_L)\,\phi^* M^\nu \nu_R + \bar\nu_R\bar M^\nu\phi^T\begin{pmatrix}-e_L\\ \nu_L\end{pmatrix}\right] && \text{(neutrino mass term)}\\
& +(\bar u_L, \bar d_L)\,\tilde\sigma^\mu iD_\mu \begin{pmatrix}u_L\\ d_L\end{pmatrix} + \bar u_R\sigma^\mu iD_\mu u_R + \bar d_R\sigma^\mu iD_\mu d_R + \text{(h.c.)} && \text{(quark dynamical term)}\\
& -\frac{\sqrt2}{v}\left[(\bar u_L,\bar d_L)\,\phi M^d d_R + \bar d_R\bar M^d\bar\phi\begin{pmatrix}u_L\\ d_L\end{pmatrix}\right] && \text{(down, strange, bottom mass term)}\\
& -\frac{\sqrt2}{v}\left[(-\bar d_L,\bar u_L)\,\phi^* M^u u_R + \bar u_R\bar M^u\phi^T\begin{pmatrix}-d_L\\ u_L\end{pmatrix}\right] && \text{(up, charmed, top mass term)}\\
& +\overline{(D_\mu\phi)}D^\mu\phi - m_h^2[\bar\phi\phi - v^2/2]^2/2v^2. && \text{(Higgs dynamical and mass term)} \quad (1)
\end{aligned}$$

where (h.c.) means Hermitian conjugate of preceeding terms, $\bar\psi = \text{(h.c.)}\psi = \psi^\dagger = \psi^{*T}$, and the derivative operators are

$$D_\mu\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} = \left[\partial_\mu - \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu\right]\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}, \quad D_\mu\begin{pmatrix}u_L\\ d_L\end{pmatrix} = \left[\partial_\mu + \frac{ig_1}{6}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu + ig\mathbf{G}_\mu\right]\begin{pmatrix}u_L\\ d_L\end{pmatrix}, \tag{2}$$

$$D_\mu\nu_R = \partial_\mu\nu_R, \quad D_\mu e_R = [\partial_\mu - ig_1B_\mu]\,e_R, \quad D_\mu u_R = \left[\partial_\mu + \frac{i2g_1}{3}B_\mu + ig\mathbf{G}_\mu\right]u_R, \quad D_\mu d_R = \left[\partial_\mu - \frac{ig_1}{3}B_\mu + ig\mathbf{G}_\mu\right]d_R, \tag{3}$$

$$D_\mu\phi = \left[\partial_\mu + \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu\right]\phi. \tag{4}$$

ϕ is a 2-component complex Higgs field. Since $\mathcal{L}$ is $SU(2)$ gauge invariant, a gauge can be chosen so ϕ has the form

$$\phi^T = (0, v+h)/\sqrt2\,, \qquad <\phi>_0^T = \text{(expectation value of } \phi) = (0, v)/\sqrt2\,, \tag{5}$$

where v is a real constant such that $\mathcal{L}_\phi = \overline{(\partial_\mu\phi)}\partial^\mu\phi - m_h^2[\bar\phi\phi - v^2/2]^2/2v^2$ is minimized, and h is a residual Higgs field. B_μ, $\mathbf{W}_\mu$ and $\mathbf{G}_\mu$ are the gauge boson vector potentials, and $\mathbf{W}_\mu$ and $\mathbf{G}_\mu$ are composed of 2×2 and 3×3 traceless Hermitian matrices. Their associated field tensors are

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad \mathbf{W}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu + ig_2(\mathbf{W}_\mu\mathbf{W}_\nu - \mathbf{W}_\nu\mathbf{W}_\mu)/2, \quad \mathbf{G}_{\mu\nu} = \partial_\mu\mathbf{G}_\nu - \partial_\nu\mathbf{G}_\mu + ig(\mathbf{G}_\mu\mathbf{G}_\nu - \mathbf{G}_\nu\mathbf{G}_\mu). \tag{6}$$

The non-matrix $A_\mu, Z_\mu, W_\mu^\pm$ bosons are mixtures of $\mathbf{W}_\mu$ and B_μ components, according to the weak mixing angle θ_w,

$$A_\mu = W_{11\mu}sin\theta_w + B_\mu cos\theta_w, \qquad Z_\mu = W_{11\mu}cos\theta_w - B_\mu sin\theta_w, \qquad W_\mu^+ = W_\mu^{-*} = W_{12\mu}/\sqrt2, \tag{7}$$

$$B_\mu = A_\mu cos\theta_w - Z_\mu sin\theta_w, \quad W_{11\mu} = -W_{22\mu} = A_\mu sin\theta_w + Z_\mu cos\theta_w, \quad W_{12\mu} = W_{21\mu}^* = \sqrt2\,W_\mu^+, \quad sin^2\theta_w = .2315(4). \tag{8}$$

The fermions include the leptons e_R, e_L, ν_R, ν_L and quarks u_R, u_L, d_R, d_L. They all have implicit 3-component generation indices, $e_i=(e,\mu,\tau)$, $\nu_i=(\nu_e,\nu_\mu,\nu_\tau)$, $u_i=(u,c,t)$, $d_i=(d,s,b)$, which contract into the fermion mass matrices $M^e_{ij}, M^\nu_{ij}, M^u_{ij}, M^d_{ij}$, and implicit 2-component indices which contract into the Pauli matrices,

$$\sigma^\mu = \left[\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right], \quad \tilde\sigma^\mu = [\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3], \quad tr(\sigma^i) = 0, \quad \sigma^{\mu\dagger} = \sigma^\mu, \quad tr(\sigma^\mu\sigma^\nu) = 2\delta^{\mu\nu}. \tag{9}$$

The quarks also have implicit 3-component color indices which contract into $\mathbf{G}_\mu$. So $\mathcal{L}$ really has implicit sums over 3-component generation indices, 2-component Pauli indices, 3-component color indices in the quark terms, and 2-component $SU(2)$ indices in $(\bar\nu_L, \bar e_L), (\bar u_L, \bar d_L), (-\bar e_L, \bar\nu_L), (-\bar d_L, \bar u_L), \bar\phi$, $\mathbf{W}_\mu$, $\binom{\nu_L}{e_L}, \binom{u_L}{d_L}, \binom{-e_L}{\nu_L}, \binom{-d_L}{u_L}, \phi$.

Appendix B

Curriculum Vitae

Personal Information

Name	Harald Ringbauer
Date of Birth	1st of April 1989
Place of Birth	Mistelbach
Citizenship	Austrian

Education

10/2011-02/2012	Erasmus studies at Ruperto Carola University Heidelberg, Germany
since 2011	M.Sc. in Mathematics, University of Vienna Specialization "Geometry and Topology" Adviser: Stefan Haller
2008-2011	B.Sc. in Mathematics, University of Vienna Bachelor thesis: Spin groups as Double Covers of SO(n) Adviser: Stefan Haller
2008-2011	B.Sc. in Physics, University of Vienna Bachelor thesis: Representation theory of the Lorentz Group Adviser: Jakob Yngvason
2003-2007	Grammar school: Bundesoberstufenrealgymnasium Mistelbach

Work experience

10/2012-02/2013	Teaching Assistant University of Natural Resources and Life Sciences (BOKU)
07-09/2011	IAESTE placement at the Quantum Optics group, University of Southampton, United Kingdom
07-08/2009	Work placement at Allianz Insurance Group, Programming in Vienna, Austria
08/2007	GEN-AU Summer School, work placement at Veterinarian University Vienna Genetics research and production of Knock-Out Mice

Vienna, 18th August, 2013

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