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DISSERTATION

p-adic Automorphic L-functions

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Introduction

Beginning with the work of Kubota and Leopoldt, it has become apparent that there should also be a p-adic theory of L-functions. This theory parallels the theory of complex L-functions, for instance there are p-adic analogues of the famous class number formula, but there are also important results genuine to the p-adic theory, for example the Main Conjecture of Iwasawa theory. This conjecture, which has been proven by Mazur-Wiles and by Wiles, cf. [MW84] and [Wil90], establishes a strong tie between p-adic L-functions and properties of algebraic number fields. One expects that properties of special values of p-adic L-functions contain deep arithmetic information similar to the special values of complex L-functions.

p-adic Interpolation We fix a prime $p \in \mathbb{N}$ with $p > 3$. To construct the p-adic analogue of a complex L-function means to p-adically interpolate the values of the complex L-function at certain points. We illustrate this by taking the example of the Dirichlet L-function: Kubota and Leopoldt showed in [KL64] that for any Dirichlet character χ there exists a p-adic meromorphic function $L_p(\chi, \ ) : B \to \mathbb{C}_p$ with $\mathbb{C}_p$ the Tate-field and B a certain subset of $\mathbb{C}_p$, such that

$$L_p(\chi, 1-k) = (1 - \chi\omega^{-k}(p)p^{k-1})L(\chi\omega^{-k}, 1-k) \tag{1}$$

for all $k \in \mathbb{N}_{>0}$, where ω denotes the Teichmuller character at p. Roughly spoken, the p-adic Dirichlet L-function $L_p(\chi, \ )$ coincides at all negative integers with the complex Dirichlet L-function, i.e. it interpolates the values of the complex Dirichlet L-function at negative integers. We should mention, that the existence of a continuous function L_p satisfying equation (1) is equivalent to the fact that the values $L(\chi\omega^{-k}, 1-k)$, $k \geq 1$, satisfy certain congruence relations. Note, that in equation (1) the values $(1 - \chi\omega^{-k}(p)p^{k-1})L(\chi\omega^{-k}, 1-k)$ are algebraic and hence, by fixing an embedding $i_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, can be regarded as lying in $\mathbb{C}_p$. For arbitrary automorphic L-functions, it is the Conjecture of Deligne which proposes points where the values of the complex L-function are algebraic up to a constant and thus can be viewed as lying in $\mathbb{C}_p$. Let us make this precise. In the following, let π be a cohomological cuspidal representation of GL_n. Since we are actually interested in the case $n = 3$ and the cases where n is even resp. odd require slightly different treatment, we now assume n to be odd. We call a pair (χ, l) consisting of an idèle class character χ of finite order, i.e. $\chi_\infty = \mathbb{1}$ or $\chi_\infty = \mathrm{sgn}$, and an integer l critical for the representation π if l is critical for $\pi \otimes \chi$. Note, that if (χ', l) is critical for π then all the pairs (χ, l) with $\chi_\infty = \chi'_\infty$ are critical for π. The Conjecture of Deligne, cf. [Del79], predicts in particular that there exist periods $\Omega_l(\pi) \in \mathbb{C}^*$ such that for all critical pairs (χ, l)

$$\frac{L(\pi \otimes \chi, l)}{\Omega_l(\pi)} \in \bar{\mathbb{Q}}. \tag{2}$$

The p-adic automorphic L-function interpolating the values $L(\pi \otimes \chi, l)/\Omega_l(\pi)$ with (χ, l) critical will be a $\mathbb{C}_p$-valued function on the set X_p of continuous p-adic characters $\mathbb{Z}_p^* \to \mathbb{C}_p^*$. One can naturally identify X_p with a finite disjoint union of open balls in $\mathbb{C}_p$ of radius 1, (cf. section 5.2), hence it makes sense to speak about p-adic analytic functions on X_p with values in $\mathbb{C}_p$. Furthermore, the idèle class characters $\chi = \prod_\ell \chi_\ell$ of finite order and of conductor a power of p embed into X_p via $\chi \mapsto \chi_p \in X_p$. We are now ready to define the notion of a p-adic automorphic L-function: A p-adic analytic function $L_p : X_p \to \mathbb{C}_p$ is called a p-adic L-function attached to

π and to the critical integers on the left hand side of the functional equation if it satisfies the following interpolation property

$$L_p(x^l\chi_p) = C(\chi,l)\frac{L(\pi\otimes\chi,l)}{\Omega_l(\pi)} \tag{3}$$

for all critical pairs (χ,l) of π such that $l<1/2$ and the conductor of χ equals a power of p. Here, $C(\chi,l)$ is a "simple" factor in $\mathbb{C}^*$ and x^l denotes the $\mathbb{C}_p$-valued p-adic character $x\mapsto x^l$ on $\mathbb{Z}_p^*$, in particular $x^l\chi_p$ lies in X_p. The investigation of the existence of p-adic automorphic L-functions has led to a new branch of p-adic integration theory: Amice, Manin, Mazur, Velu, Višik and others found that the p-adic analytic functions f of logarithmic growth on X_p are exactly the Mellin transforms of p-adic h-admissible measures μ on $\mathbb{Z}_p^*$, i.e. they are of the form

$$f(\xi) = \int\limits_{\mathbb{Z}_p^*} \xi\, d\mu.$$

Thus, to prove the existence of a p-adic automorphic L-function L_p it suffices to construct an h-admissible measure μ whose Mellin transform at the p-adic characters $x^l\chi_p$ evaluates to $C(\chi,l)L(\pi\otimes\chi,l)/\Omega_l(\pi)$.

p-adic L-functions for $\mathbf{GL}_3$ Let now π be a cohomological cuspidal representation of GL_3 over $\mathbb{Q}$. The set of critical pairs (χ,l) of π can be described as follows. There exists a natural number l_π such that (χ,l) is critical for π if and only if l is lying in the interval $[1-l_\pi,l_\pi]$, (which is centered at $1/2$), and if l is smaller resp. greater than $1/2$, then χ_∞ and l are of different parities resp. of the same parity. For technical reasons, we now fix a certain pair of idèle class characters (η_0,η_1) with $\eta_{0,\infty}=\mathbb{1}$ and $\eta_{1,\infty}=\mathrm{sgn}$. In the following, η_l with $l\in\mathbb{Z}$ denotes the character η_0 if l is even and the character η_1 if l is odd. Hence, the set of critical pairs consists of all $(\chi\eta_l,l)$ with $\chi_\infty=\mathbb{1}$ and $1\le l\le l_\pi$ and of all $(\chi\eta_{l+1},l)$ with $\chi_\infty=\mathbb{1}$ and $1-l_\pi\le l\le 0$. In our situation, the Conjecture of Deligne is verified in [Mah00], Corollary 2, and can be reformulated as follows: For any integer $l\in[1-l_\pi,0]$ there exists a period $\Omega_l(\pi)$ such that

$$\frac{L(\pi\otimes\eta_{l+1}\chi,l)}{\Omega_l(\pi)}$$

is algebraic for all characters χ with infinity component $\chi_\infty=\mathbb{1}$. Thus, to construct a p-adic L-function L_p we have to p-adically interpolate the values $L(\pi\otimes\eta_{l+1}\chi,l)/\Omega_l(\pi)$ with $1-l_\pi\le l\le 0$ and $\chi_\infty=\mathbb{1}$, which basically amounts to interpolating the values $L(\pi\otimes\chi,l)/\Omega_l(\pi)$ for all critical pairs (χ,l) with $l<1/2$, cf. equation (3).

Theorem (cf. Corollary 5.8)**.** *There exists a p-adic analytic function $L_p=L_p(\ ,\pi):X_p\to\mathbb{C}_p$ satisfying the following properties:*

i) *For all idèle class characters $\chi:\mathbb{Q}^*\backslash\mathbb{A}^*\to\mathbb{C}^*$ of conductor p^e, $e\ge 2$, and with infinity component $\chi_\infty=\mathbb{1}$ and for all integers l with $1-l_\pi\le l\le 0$ we have*

$$L_p(x^l\chi_p) = C(\chi,l)\,\frac{L(\pi\otimes\eta_{l+1}\chi,l)}{\Omega(\pi)}, \tag{4}$$

where $C(\chi,l)\in\mathbb{C}^*$ is an explicit factor. If χ is a character with conductor a p-power and infinity component $\chi_\infty=\mathrm{sgn}$, then $L_p(x^l\chi_p)$ vanishes.

ii) *The function L_p is of logarithmic growth. For instance, if π is p-ordinary, cf. Appendix C, then L_p is equal to $o(\log_p^{6l_\pi-3}(\cdot))$.*

We note that the function L_p is not uniquely determined by these properties.

Let us say a few words about the proof of the Theorem. As indicated above, the strategy is to find an h-admissible measure whose Mellin transform satisfies interpolation property (4). To this end, in [Mah00] there is constructed a p-adic distribution $\mu_{\pi,l}$ on $\mathbb{Z}_p^*$ for every integer l in $[1-l_\pi, 0]$ such that $\int \chi_p \, d\mu_{\pi,l}$ can be related to $L(\pi \otimes \eta_{1+l}\chi, l)$ for all idèle class characters χ of level p^e and with infinity component $\chi_\infty = \mathbb{1}$. If π is cohomological with respect to the trivial representation, then $l_\pi = 1$, i.e. the pairs $(\chi, 0)$ with $\chi_\infty = \mathrm{sgn}$ are the only critical pairs on the left hand side of the functional equation, and it is shown in [Mah00] that the distribution $\mu_{\pi,0}$ extends (non-uniquely) to an h-admissible measure.

In this thesis, we generalize the result in [Mah00] to cuspidal representations π of GL_3 which are cohomological with respect to a non-trivial coefficient system, i.e. l_π is greater than 1. This means, we have to show that the distributions $\mu_{\pi,l}$ with $1 - l_\pi \leq l \leq 0$ are h-admissible and thus extend (non-uniquely) to a h-admissible measure $\widetilde{\mu}_\pi$ on $\mathbb{Z}_p^*$ whose Mellin transform at the p-adic characters $x^l\chi_p$ evaluates, up to a certain factor, to $L(\pi \otimes \eta_{l+1}\chi, l)$ for all integers $l \in [1-l_\pi, 0]$ and all idèle class characters χ of level p^e and with $\chi_\infty = \mathbb{1}$. It relies on cohomological methods to show that the distributions $\mu_{\pi,l}$ are h-admissible, meaning that the p-adic absolute values of $\mu_{\pi,l}(a + p^e\mathbb{Z}_p)$ do not grow unreasonably fast for e to infinity. The crucial point is that the distributions $\mu_{\pi,l}$ can be related to a certain pairing in cohomology. Let us explain this in more detail. Due to our assumptions on π there exists a rational representation (ρ, V) of GL_3 such that for a sufficiently small compact open subgroup $K_3^f \subset \mathrm{GL}_3(\mathbb{A}_f)$ the K_3^f-invariant vectors $\pi_f^{K_3^f}$ in the finite part π_f of π can be embedded into the cohomology group $H_c^2(S_3(K_3^f), \mathcal{V})$, where $S_n(K_n^f)$ is the manifold defined by

$$S_n(K_n^f) = \mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A}) / \mathrm{SO}_n(\mathbb{R}) Z_n^0(\mathbb{R}) K_n^f$$

for any compact open subgroup K_n^f in $\mathrm{GL}_n(\mathbb{A}_f)$ and $\mathcal{V}$ is the locally constant sheaf defined on $S_3(K_3^f)$ by the representation (ρ, V). It follows from the work of Harder, cf. [Har87b], that the finite part $\Pi_f(\chi)$ of the induced representation

$$\Pi(\chi) = \mathrm{Ind}_{B_2(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}(\eta' |\cdot|^{l-1/2}, \eta\chi |\cdot|^{-(l-1/2)}),$$

where η and η' are certain idèle class characters of finite order, occurs in the first cohomology of the boundary $\partial\bar{S}_2(K_2^f)$ of the Borel-Serre compactification with respect to a uniquely determined coefficient system (κ, W). Via the Eisenstein operator Eis, which defines a section to the restriction map $\mathrm{res} : H^1(S_2(K_2^f), \mathcal{W}) \to H^1(\partial\bar{S}_2(K_2^f), \mathcal{W})$, ($\mathcal{W}$ denotes the sheaf induced by the representation (κ, W)), the representation $\Pi_f(\chi)$ injects into the first cohomology group $H^1(S_2(K_2^f), \mathcal{W})$. In [Mah00], there is constructed a pairing in cohomology

$$\langle \, , \, \rangle : H_c^2(\widetilde{S}_3, \mathcal{V}) \times H^1(\widetilde{S}_2, \mathcal{W}) \longrightarrow \mathbb{C}, \tag{5}$$

where the tilde over the symbols S_3 and S_2 signals that we have changed over to the direct limit over all compact open subgroups $K_n^f \subset \mathrm{GL}_n(\mathbb{A}_f)$. Moreover, there are chosen an element ω in $H_c^2(\widetilde{S}_3, \mathcal{V})$ lying in the image of π_f and an element ψ_{p^e} in $\bigoplus_\chi \Pi(\chi)$ with χ running over all even Dirichlet characters of conductor p^e, such that

$$\mu_{\pi,l}(\epsilon + p^e\mathbb{Z}_p) = \langle r_{u(\epsilon,e)}\omega, \mathrm{Eis}(\psi_{p^e}) \rangle,$$

where $r_{u(\epsilon,e)}$ denotes right translation in cohomology by a certain unipotent matrix $u(\epsilon,e) \in \mathrm{GL}_3(\mathbb{A}_f)$. Since on the integral cohomology the pairing $\langle\ ,\ \rangle$ takes integral values, we will obtain an upper bound for the p-adic growth of the values $\mu_{\pi,l}(\epsilon + p^e\mathbb{Z})$, if we can compute (or at least estimate) the denominators of the elements $\mathrm{Eis}(\psi_{p^e})$ and of the elements $r_{u(\epsilon,e)}\omega$ for e to infinity. Most of chapter 4 is dedicated to the computation of bounds for the denominators of the Eisenstein cohomology classes $\mathrm{Eis}(\psi_{p^e})$ via Harder's method. We proceed along the lines of [Mah00], where bounds for the denominators of $\mathrm{Eis}(\psi_{p^e})$ are determined under the assumption that π is cohomological with respect to the trivial coefficient system, i.e. (ρ,V) and (κ,W) are the trivial representations and $l_\pi = 1$. If l_π is greater than 1, then the representations (ρ,V) and, in general, (κ,W) are non-trivial, thus the sheaf $\mathcal{W}$ is not necessarily constant and the "Dirichlet series" $S_{x,y,k}$ occurring in the computation of the "inner integral" in the case $l_\pi = 1$ in [Mah00] have to be replaced by the more general "Dirichlet series" $S_{x,y,k,\mu}$ depending on an additional parameter μ ranging between 0 and $\dim(W) - 1$. Moreover, in contrast to the case $l_\pi = 1$, the restrictions of $\mathrm{Eis}(\psi_{p^e})$ to the boundary of the Borel-Serre compactification do not lie in the integral cohomology and thus contribute to the denominators of $\mathrm{Eis}(\psi_{p^e})$ for $l_\pi > 1$. Finally, in the case $l_\pi = 1$, i.e. (ρ,V) is trivial, the denominators of $r_{u(\epsilon,e)}\omega \in H^2_c(\widetilde{S}_3,\mathcal{V})$ with $e \in \mathbb{N}$ are bounded, whereas, in the case $l_\pi > 1$, the denominators of $r_{u(\epsilon,e)}\omega$ are growing for e to infinity, cf. Proposition 4.13. Thus, in contrast to the case $l_\pi = 1$, not only the denominators of the Eisenstein classes $\mathrm{Eis}(\psi_{p^e})$ but also the denominators of the elements $r_{u(\epsilon,e)}\omega$ contribute to the denominators of the distributions $\mu_{\pi,l}$.

Functional Equation Having established the existence of the p-adic L-function $L_p(\ ,\pi)$ attached to π and to the critical integers on the left hand side of the functional equation, we will head for the p-adic functional equation. This includes constructing the p-adic L-function $\boldsymbol{L}_p(\ ,\pi)$ on X_p attached to the representation π and to the critical integers on the right hand side of the functional equation. Our strategy is to p-adically interpolate the complex ε-factor with the Euler factor at the place p removed in order to obtain the p-adic ε-factor ε_p, which is a bounded p-adic analytic function, and then to define $\boldsymbol{L}_p$ by

$$\boldsymbol{L}_p(\xi,\pi) = \varepsilon_p(\xi^{-1})L_p(\xi^{-1}x,\check{\pi}), \quad \xi \in X_p, \tag{6}$$

where $\check{\pi}$ denotes the dual representation of π. This yields the following extension of the Theorem stated above, (cf. Theorem 5.12):

There exists a p-adic analytic L-function $\boldsymbol{L}_p : X_p \to \mathbb{C}_p$ satisfying the following properties:

iii) *For the integers l with $1 \le l \le l_\pi$ and for all characters $\chi : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of conductor a p-power p^e, $e \ge 2$, and with infinity component $\chi_\infty = \mathbb{1}$ we have*

$$\boldsymbol{L}_p(x^l\chi_p) = C'(\chi,l)L(\pi\otimes\chi\eta_l,l), \tag{7}$$

where $C'(\chi,l) \in \mathbb{C}^*$ is an explicit factor. This means, $\boldsymbol{L}_p$ interpolates the critical pairs on the right hand side of the functional equation.

iv) *The function $\boldsymbol{L}_p$ is of logarithmic growth. For instance, if π is p-ordinary, then $\boldsymbol{L}_p$ is equal to $o(\log_p^{6l_\pi-3}(\cdot))$.*

v) *By definition, $\boldsymbol{L}_p$ and L_p satisfy the p-adic functional equation (6).*

We note that the proof of the interpolation property (7) relies on the behavior of the ε-factor under induction of representations.

From the Riemann Zeta-Function to Automorphic L-functions We list some of the complex L-functions for which p-adic analogues have been constructed. As mentioned above, the first example, dating back to 1964, is the Kubota-Leopoldt L-function, which is the p-adic Dirichlet L-function, i.e. it is the L-function attached to automorphic representations of GL_1 over $\mathbb{Q}$, cf. [KL64], [Lan90] and [Was97]. Katz has treated the case of GL_1 over CM-fields, cf. [Kat78]. Cassou-Noguès as well as Deligne and Ribet have treated the case of representations of GL_1 over totally real fields, cf. [CN79] and [DR80]. Colmez and Schneps have constructed p-adic L-functions attached to representations of GL_1 over imaginary extensions of $\mathbb{Q}$, cf. [CS92]. For automorphic representations of the group GL_2, p-adic L-functions have been constructed by a number of people: Manin as well as Mazur and Swinnerton-Dyer have constructed p-adic L-functions attached to automorphic representations of GL_2 over $\mathbb{Q}$, cf. [Man73] and [MSD74]. Manin has generalized his constructions to GL_2 over totally real number fields, cf. [Man76]. Kurčanov has constructed p-adic L-functions attached to automorphic representations of GL_2 over CM fields, cf. [Kur79], and Haran has given a construction of p-adic L-functions for GL_2 over arbitrary number fields, cf. [Har87a]. Moreover, in [Sch88] and [Hid90] we find a construction of p-adic L-functions which are attached to a special kind of automorphic representations of GL_3, namely representations which are Jacquet-Gelbart lifts of automorphic representations of GL_2. Finally, in [Mah00] the existence of p-adic L-functions attached to cuspidal representations of GL_3 which have non-trivial cohomology with respect to the trivial coefficient system is proven.

Content of this Thesis Chapters 1 and 2 present background material stated without proofs. In **chapter 1** we start with an introduction to automorphic forms concentrating on Whittaker models and Eisenstein series. The last section of chapter 1 outlines the construction of (complex) automorphic L-functions via Rankin-Selberg integrals. In **chapter 2** we gather the basic results on the cohomology of arithmetic subgroups starting from the Borel-Serre compactification to the relation between sheaf, de Rham and relative Lie algebra cohomology. Moreover, the cuspidal and the Eisenstein cohomology are introduced. These results are all stated in adèlic language. The last section of chapter 2 describes how a pairing of sheaves induces a pairing in cohomology, which will later be applied to construct the pairing (5). **Chapter 3** recalls the construction and cohomological interpretation of the distributions $\mu_{\pi,l}$ given in [Mah00], this is based on the definition of the complex automorphic L-function via Rankin-Selberg integrals explained in chapter 1.4. In the central **chapter 4** we compute bounds for the denominators of the Eisenstein cohomology classes $\mathrm{Eis}(\psi_{p^e})$ and of the cuspidal cohomological class $r_{u(\epsilon,e)}\omega$. **Chapter 5** contains the main result on the existence of p-adic L-functions and establishes a p-adic functional equation. The **Appendix** is concerned with some technical details; Appendix A shows that a certain map between manifolds is proper which allows one to transport cohomology with compact supports via this map, Appendix B confirms that the integral cohomology of arithmetic subgroups indeed form lattices in ordinary cohomology, as intuition suggests, and Appendix C determines the Hodge polygons of unitary cuspidal automorphic representations of GL_3 over $\mathbb{Q}$ and discusses p-ordinariness.

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Notations

Throughout, we fix the following notations.

We denote by $K_{n,\infty}$ the compact subgroup $\mathrm{SO}_n(\mathbb{R}) < \mathrm{GL}_n(\mathbb{R})$ and by $Z_n^0(\mathbb{R})$ the connected component of 1 of the center of $\mathrm{GL}_n(\mathbb{R})$. As usual, the group of upper triangular matrices in GL_n is denoted by B_n.
The Lie algebras of $\mathrm{GL}_n(\mathbb{R})$ resp. $\mathrm{SO}_n(\mathbb{R})$ are denoted by gl_n resp. so_n. We write $\mathfrak{z}(\mathrm{gl}_n)$ for the center of the universal enveloping algebra $\mathcal{U}(\mathrm{gl}_n)$ and $\mathfrak{z}(\mathrm{gl}_{n,\mathbb{C}})$ for the center of the universal enveloping algebra $\mathcal{U}(\mathrm{gl}_{n,\mathbb{C}})$ of the complexified Lie algebra $\mathrm{gl}_{n,\mathbb{C}}$ of $\mathrm{GL}_n(\mathbb{R})$.
We make use of the following level groups:

- $K(n,p^e) \subset \mathrm{GL}_n(\mathbb{Z}_p)$ is the group of matrices in $\mathrm{GL}_n(\mathbb{Z}_p)$ congruent to the identity matrix modulo p^e,
- $K_0(n,p^e) \subset \mathrm{GL}_n(\mathbb{Z}_p)$ is the group of matrices in $\mathrm{GL}_n(\mathbb{Z}_p)$ whose elements in the last row apart from the element on the diagonal are congruent to zero modulo p^e,
- $K_1(n,p^e) \subset \mathrm{GL}_n(\mathbb{Z}_p)$ is the group of matrices in $\mathrm{GL}_n(\mathbb{Z}_p)$ whose last row is congruent to the last row of the identity matrix modulo p^e,
- $\mathcal{I} \subset \mathrm{GL}_n(\mathbb{Z}_p)$ denotes the Iwahori subgroup consisting of the matrices in $\mathrm{GL}_n(\mathbb{Z}_p)$ which are congruent to upper triangular matrices modulo p.

The ring of adèles of $\mathbb{Q}$ is denoted by $\mathbb{A}$, the subgroup of idèles is denoted by $\mathbb{A}^*$.
We fix an additive character $\tau = \otimes_\ell \tau_\ell : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^*$. From chapter 3 on we will assume the conductor of τ to be $\mathbb{Z}$, i.e. $\mathbb{Z}_\ell$ is the largest ideal contained in the kernel of τ_ℓ for ℓ finite. In this case we denote by $G(\eta_p)$ for any character η_p on $\mathbb{Z}_p^*$ of level p^e the Gauss sum attached to η_p and to the additive character τ_p, i.e.

$$G(\eta_p) := \sum_{j \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^*} \eta_p(j)\tau_p\left(\frac{j}{p^e}\right).$$

For a Dirichlet character χ we denote by $L(\chi,s)$ the L-function associated to χ.
We denote by $\mathbb{C}_\ell$ the completion of an algebraic closure of $\mathbb{Q}_\ell$ and by $|\cdot|_\ell$ the absolute value on $\mathbb{C}_\ell$ normalized by $|\ell|_\ell = \ell^{-1}$. We fix an embedding $i_\ell : \bar{\mathbb{Q}} \to \mathbb{C}_\ell$ and also denote by $|\cdot|_\ell$ the absolute value on $\bar{\mathbb{Q}}$ induced by i_ℓ.
We denote by "Ind" unitary induction and by "ind" non-unitary induction.
For any group G and an abelian group M with a G-action, i.e. M is a G-module, we call $M_G := M/\langle (1-g)m, g \in G, m \in M\rangle$ the module of G-coinvariants.
If X is a topological space and A is an abelian group, we denote by $\underline{A}$ the sheaf of locally constant A-valued functions on X.

1. Automorphic Forms and Their L-functions

In this chapter we summarize some basic results on automorphic forms. This can be found at various places, see for instance [Bum97], chapter 3 or [Kud03a], chapter 7.
We want to emphasize that our aim in this chapter is not to give an overview but to prepare for later applications.

1.1. Basic Notations

Here we give the definition of a (cuspidal) automorphic representation, explain its decomposition as a restricted tensor product and introduce the contragredient representation.

We set $K = O_n(\mathbb{R})\prod_\ell \mathrm{GL}_n(\mathbb{Z}_\ell)$. Let ω be an idèle class character, $\omega : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$. A function f on $\mathrm{GL}_n(\mathbb{A})$ is called smooth if for every $g \in \mathrm{GL}_n(\mathbb{A})$ there exists a neighborhood N of g and a smooth function f_g on $\mathrm{GL}_n(\mathbb{R})$ such that $f(h_\infty h_f) = f_g(h_\infty)$ for all $h_\infty h_f \in N \subset \mathrm{GL}_n(\mathbb{R})\mathrm{GL}_n(\mathbb{A}_f)$.

Definition 1.1 (Automorphic Forms)**.** *A smooth function $\varphi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ is called a (K-finite) automorphic form on $\mathrm{GL}_n(\mathbb{A})$ with central character ω if it satisfies*

- *$\varphi(zg) = \omega(z)\varphi(g)$ for all $g \in \mathrm{GL}_n(\mathbb{A})$, $z \in Z(\mathbb{A})$,*
- *$\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \mathrm{GL}_n(\mathbb{Q})$,*
- *the space $\langle g \mapsto \varphi(gk) \,|\, k \in K\rangle$ is finite-dimensional,*
- *the space $\left\langle g \mapsto X\varphi(g) \,|\, X \in \mathfrak{z}(\mathrm{gl}_{n,\mathbb{C}})\right\rangle$ is finite-dimensional,*
- *φ is of moderate growth, i.e. for any norm $\|\ \|$ on $\mathrm{GL}_n(\mathbb{A})$ there exists a positive integer r and a constant $C \in \mathbb{R}$ such that*

$$|\varphi(g)| \leq C\|g\|^r.$$

We will denote the space of K-finite automorphic forms with central character ω by

$$\mathcal{A}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega).$$

An automorphic form φ on $\mathrm{GL}_n(\mathbb{A})$ is called cuspidal or a cusp form if

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)dn = 0$$

for every unipotent radical N of a proper parabolic subgroup of GL_n. The space of cuspidal forms with central character ω will be denoted by $\mathcal{A}_0(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$.

It is a fundamental fact, generally attributed to Gelfand and Piatetski-Shapiro, that cusp forms are not only of moderate growth but are rapidly decreasing. The following version of this statement can be found in [Cog03], Theorem 2.2.

Theorem 1.2. *If φ is a cusp form, then it is rapidly decreasing modulo the center $Z(\mathbb{A})$ on a fundamental domain $\mathcal{F}$ for $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})$, i.e. there exists an integer r such that for all $N \in \mathbb{N}$ there exists a constant $C \in \mathbb{R}$ with*

$$|\varphi(zg)| \leq C|z|^r \|g\|^{-N} \qquad \text{for all } g \in \mathcal{F} \cap SL_n(\mathbb{A}) \text{ and } z \in Z(\mathbb{A}).$$

The group $\mathrm{GL}_n(\mathbb{A}_f)$ acts on the space $\mathcal{A}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$ by right translation. This action cannot be extended to the group $\mathrm{GL}_n(\mathbb{A})$ since the right translation by elements in $\mathrm{GL}_n(\mathbb{R})$ does not preserve the K-finiteness. Though, $\mathcal{A}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$ is a $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module.

Definition 1.3 (Automorphic Representations)**.** *A complex vector space V which is an irreducible $(\mathrm{gl}_n, O_n(\mathbb{R}))$- and $\mathrm{GL}_n(\mathbb{A}_f)$-module is called an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ if it can be realized as a quotient of a subspace of $\mathcal{A}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$. It is called a cuspidal (automorphic) representation if it can be realized as a subspace of $\mathcal{A}_0(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$.*

It can be deduced from Theorem 1.2 that cusp forms are square integrable modulo the center, i.e. we have for all cusp forms φ

$$\int\limits_{Z(\mathbb{A})\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})} |\varphi(g)|^2 |\omega(\det(g))|^{-1} dg \; < \infty.$$

In particular, the space $\mathcal{A}_0(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$ is unitary if ω is unitary. On the other hand, it follows directly from the definitions that the central character of a unitary representation is unitary, so a cuspidal representation is unitary if and only if its central character is unitary. One knows that the space of cusp forms lies in the discrete part of the square integrable functions, see for instance [Bum97], Theorem 3.3.2. Thus, in the definition of cuspidal automorphic representations above there is no need to speak about a quotient of a subspace as in the case of automorphic forms since the space of cusp forms $\mathcal{A}_0(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$ is a Hilbert space direct sum of irreducible modules.

Definition 1.4 (Admissible Representations)**.** *A complex vector space V which is a $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module and a $\mathrm{GL}_n(\mathbb{A}_f)$-module is called admissible (or an admissible representation of $\mathrm{GL}_n(\mathbb{A})$) if every vector in V is K-finite and the isotypical part $V(\rho)$ is finite dimensional for every irreducible finite-dimensional representation ρ of K.*
A $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module V is called admissible if for any irreducible representation ρ of $O_n(\mathbb{R})$ the ρ-isotypical part in V is finite-dimensional.
For any prime ℓ a representation (π, V) of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ is called smooth if the stabilizer of each vector $v \in V$ is open. It is called admissible if it is smooth and for any compact open subgroup $K' \subset \mathrm{GL}_n(\mathbb{Q}_\ell)$ the space $V^{K'}$ is finite-dimensional or equivalently if it is smooth and for any irreducible representation ρ of $\mathrm{GL}_n(\mathbb{Z}_\ell)$ the ρ-isotypical part in V is finite-dimensional.

The reason for our interest in admissible representations is that automorphic representations can be shown to be admissible, cf. Theorem 3.1 in [Cog03].
The next result was proven by Flath, cf. [Fla79], by reinterpreting the irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$ as irreducible "admissible" representations of a certain algebra $\mathcal{H}$, called the Hecke algebra.

Theorem 1.5. *Let (π, V) be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$. Then there exist an irreducible admissible $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module (π_∞, V_∞) and for each prime ℓ an irreducible admissible representation (π_ℓ, V_ℓ) of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ such that for almost all primes ℓ the representations (π_ℓ, V_ℓ) are spherical and such that π is the restricted tensor product of the local representations, we write*

$$\pi = \pi_\infty \otimes \otimes'_{\ell<\infty} \pi_\ell.$$

Let (π, V) be an admissible representation of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ for some prime ℓ. There is a natural representation $(\check{\pi}, V^*)$ of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ on the dual space V^*: For $g \in \mathrm{GL}_n(\mathbb{Q}_\ell)$, $\Lambda \in V^*$ and $v \in V$ we set

$$(\check{\pi}(g)\Lambda)(v) := \Lambda(\pi(g^{-1})v). \tag{1.1}$$

Let us denote by $\check{V}$ the subspace of linear functionals $\Lambda : V \to \mathbb{C}$ which are K'-invariant for some open subgroup K' of $\mathrm{GL}_n(\mathbb{Q}_\ell)$. Then $(\check{\pi}, \check{V})$, where $\check{\pi}$ denotes the restriction of π to $\check{V}$, is a smooth representation of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ and it is not hard to show that it is admissible, cf. [Bum97], page 428. We call $(\check{\pi}, \check{V})$ the contragredient representation of (π, V).
Similarly, we define the contragredient representation $(\check{\pi}, \check{V})$ of an admissible $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module (π, V). Let $\check{V}$ be the space of all linear functionals Λ on V that are zero on the ρ-isotypical parts of V for all but finitely many irreducible representations ρ of $O_n(\mathbb{R})$. The action of $O_n(\mathbb{R})$ on $\check{V}$ is defined analogously to equation (1.1) and the action of gl_n is given by

$$(\check{\pi}(X)\Lambda)(v) := -\Lambda(\pi(X)v)$$

for $X \in \mathrm{gl}_n$, $\Lambda \in \check{V}$ and $v \in V$. Then, $(\check{\pi}, \check{V})$ is an admissible $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module.
Theorem 1.5 enables us to convey these local definitions to the global situation, i.e. to admissible representations (π, V) of $\mathrm{GL}_n(\mathbb{A})$. Let $\pi = \otimes'_\ell \pi_\ell$ and $V = \otimes'_\ell V_\ell$. The contragredient representation $(\check{\pi}, \check{V})$ is given by the admissible representation $(\otimes'_\ell \check{\pi}, \otimes'_\ell \check{V}_\ell)$.

Theorem 1.6. *Let (π, V) be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ for some prime ℓ. Then its contragredient $(\check{\pi}, \check{V})$ can be realized as the representation $(\widetilde{\pi}, V)$ with the group action*

$$\widetilde{\pi}(g)(v) = \pi({}^t g^{-1})(v)$$

for $g \in \mathrm{GL}_n(\mathbb{Q}_\ell)$ and $v \in V$. A similar statement is true, if (π, V) is a $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module.

In the non-Archimedean case this Theorem is due to Gelfand and Kajdan, cf. [GK75] Theorem A. The Archimedean statement for $n = 2$ is treated in exercise 2.5.8 in [Bum97].

Corollary 1.7. *Let (π, V) be an automorphic cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ with central character ω and assume that $V \subset \mathcal{A}_0(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A}), \omega)$. Then the contragredient representation $(\check{\pi}, \check{V})$ can be realized on the space $V' \subset \mathcal{A}_0(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A}), \omega^{-1})$ given by the functions of the form $g \mapsto \phi({}^t g^{-1})$ for $\phi \in V$. In particular, the contragredient representation of a cuspidal representation is again cuspidal.*

This follows from Theorem 1.6. One only has to note that the representation (π, V'), where π denotes right translation on the space of cuspidal forms V' constructed in the Corollary, and the representation $(\widetilde{\pi}, V)$, where the action of $\mathrm{GL}_n(\mathbb{A}_f)$ is given by $\widetilde{\pi}(g)(f) = \pi({}^t g^{-1})(f)$ for $f \in V$ and the action of $(\mathrm{gl}_n, O_n(\mathbb{R}))$ is defined analogously, are equivalent.

1.2. Whittaker Models

The construction of automorphic L-functions via Rankin-Selberg integrals, which will be outlined in section 1.4, heavily depends on the notation of Whittaker models. As the name suggests, a Whittaker model for a representation (π, V) is a certain manageable representative in the class of representations equivalent to (π, V). This does not exist for all representations (π, V).

We fix an additive character $\tau : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^*$. It induces a character $\tau : N_n(\mathbb{Q})\backslash N_n(\mathbb{A}) \to \mathbb{C}^*$ as follows

$$\tau \begin{pmatrix} 1 & x_{12} & x_{13} & \dots & x_{1n} \\ & 1 & x_{23} & \dots & x_{2n} \\ & & \ddots & & \\ & & & & 1 \end{pmatrix} = \tau(x_{12} + x_{23} + \dots x_{n-1,n}).$$

The character τ decomposes into a product of local characters $\tau = \prod_\ell \tau_\ell$ where τ_ℓ is an additive character on $\mathbb{Q}_\ell$ if $\ell < \infty$ and on $\mathbb{R}$ if $\ell = \infty$. Similarly as in the global setting, the characters τ_ℓ induce characters on $N_n(\mathbb{Q}_\ell)$ and $N_n(\mathbb{R})$. We denote by $\mathcal{W}(\tau)$ the space of smooth functions $\varphi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ such that

- $\varphi(ng) = \tau(n)\varphi(g)$ for $n \in N_n(\mathbb{A})$ and $g \in \mathrm{GL}_n(\mathbb{A})$,
- the space $\langle \varphi(gk) \,|\, k \in K \rangle$ is finite-dimensional and
- φ is of moderate growth.

With the action induced by right translation, the vector space $\mathcal{W}(\tau)$ is a $\mathrm{GL}_n(\mathbb{A}_f)$- and a $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module.

Definition 1.8 (Global Whittaker Models)**.** *Let (π, V) be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$. An invariant subspace in $\mathcal{W}(\tau)$ which is isomorphic to (π, V) is called a Whittaker model for (π, V). It is denoted by $\mathcal{W}(\pi, \tau)$ and we call (π, V) generic if it admits a Whittaker model.*

Theorem 1.5 asserts that any irreducible admissible representation (π, V) is the restricted tensor product of local representations. Of course, this does not imply that for instance the cusp forms in a cuspidal representations (π, V) can be written as products of functions on the local groups $\mathrm{GL}_n(\mathbb{Q}_\ell)$. Though, a statement of this kind is true for Whittaker models: If (π, V) is generic then the functions in $\mathcal{W}(\pi, \tau)$ decompose into products of functions on the local groups $\mathrm{GL}_n(\mathbb{Q}_\ell)$. This will be made precise in Theorem 1.11.
First we need to extend the notation of a Whittaker model to the case where (π, V) is a $(\mathrm{gl}_n, O_n(\mathbb{R}))$- or a $\mathrm{GL}_n(\mathbb{Q}_\ell)$-module. Analogously to the global case the local Whittaker model $\mathcal{W}(\pi, \tau_\ell)$ will consist of functions on $\mathrm{GL}_n(\mathbb{R})$ resp. on $\mathrm{GL}_n(\mathbb{Q}_\ell)$ if $\ell = \infty$ resp. $\ell < \infty$.

Definition 1.9 (Local Whittaker Models)**.** *Let (π, V) be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{Q}_\ell)$. We call (π, V) generic if there exists a space $\mathcal{W}(\pi, \tau_\ell)$ of functions $W : \mathrm{GL}_n(\mathbb{Q}_\ell) \to \mathbb{C}$ satisfying $W(ng) = \tau_\ell(n)W(g)$ such that $\mathcal{W}(\pi, \tau_\ell)$ is a $\mathrm{GL}_n(\mathbb{Q}_\ell)$-module under right translation which is isomorphic to π.*
Let (π, V) be an irreducible admissible $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module. We call (π, V) generic if there exists a space $\mathcal{W}(\pi, \tau_\infty)$ of smooth, $O_n(\mathbb{R})$-finite functions $W : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}$ of moderate growth and satisfying $W(ng) = \tau_\infty(n)W(g)$, such that $W(\pi, \tau_\infty)$ is a $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module (with respect to the action induced by right translation) which is isomorphic to (π, V).
In both cases the space $\mathcal{W}(\pi, \tau_\ell)$ is called the Whittaker model of (π, V).

Remark 1.10. *If one considers the bigger space of "smooth" automorphic forms instead of the K-finite automorphic forms, the notion of Whittaker models becomes easier to handle. In particular, a "smooth" automorphic representation is generic if and only if it admits a non-trivial continuous Whittaker functional, see the Remark in [Cog03], lecture 4, section 2.*

The next result is central to the theory of Whittaker models for $\mathrm{GL}_n(\mathbb{A})$. In particular, it gives the desired decomposition of global Whittaker functions into products of local Whittaker functions.

Theorem 1.11. *Let (π, V) be an irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$ with decomposition $\pi = \otimes'_\ell \pi_\ell$. Then (π, V) has a Whittaker model $\mathcal{W}(\pi, \tau)$ if and only if each of the local factors (π_ℓ, V_ℓ) admits a Whittaker model $\mathcal{W}(\pi_\ell, \tau_\ell)$.*
If (π, V) is generic then $\mathcal{W}(\pi, \tau)$ is unique and consists of all finite linear combinations of functions of the form $W(g) = \prod_\ell W_\ell(g_\ell)$ where $W_\ell \in \mathcal{W}(\pi_\ell, \tau_\ell)$ and for almost all ℓ the function W_ℓ is the normalized spherical element W_ℓ° in $\mathcal{W}(\pi_\ell, \tau_\ell)$, i.e. $W_\ell^\circ(gk) = W_\ell^\circ(g)$ for $g \in \mathrm{GL}_n(\mathbb{Q}_\ell)$ and $k \in \mathrm{GL}_n(\mathbb{Z}_\ell)$ and $W_\ell^\circ(1) = 1$. Every equimorphism $(\pi, V) \xrightarrow{\sim} \mathcal{W}(\pi, V)$ maps pure tensors to pure tensors.

For a proof, at least in the case $n = 2$, see [Bum97], Theorem 3.5.4.
The next result gives an explicit description of the Whittaker model of a cuspidal representation (π, V) and of the morphism from (π, V) to $\mathcal{W}(\pi, \tau)$.

Theorem 1.12. *Let (π, V) be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and assume that $V \subset \mathcal{A}_0(\mathrm{GL}_n(\mathbb{Q}), \mathrm{GL}_n(\mathbb{A}), \omega)$, where ω is a character of $\mathbb{Q}^* \backslash \mathbb{A}^*$. For $\varphi \in V$ and $g \in \mathrm{GL}_n(\mathbb{A})$ we set*

$$W_\varphi(g) = \int\limits_{N_n(\mathbb{Q}) \backslash N_n(\mathbb{A})} \varphi(ng)\tau(-n)dn.$$

Then the space $\mathcal{W}$ of functions W_φ is a Whittaker model for (π, V).
We have the following Fourier expansion

$$\varphi(g) = \sum_{\gamma \in N_{n-1}(\mathbb{Q}) \backslash \mathrm{GL}_{n-1}(\mathbb{Q})} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right) \quad \text{for } g \in \mathrm{GL}_n(\mathbb{A}).$$

Consult [Bum97], Theorem 3.5.5 for a proof in the case $n = 2$ and [Cog00], Theorem 1.1 for a proof of the Fourier expansion for arbitrary n. (In fact, proving the Fourier expansion is the only difficulty in the step from $n = 2$ to arbitrary n, with the Fourier expansion in hand the proof of [Bum97], Theorem 3.5.5 generalizes directly to arbitrary n.)

Remark 1.13. *Let $\pi = \otimes'_\ell \pi_\ell$ be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$. If ℓ is a prime with π_ℓ spherical, then there exist unramified characters $\chi_1, \dots \chi_n$ of $\mathbb{Q}_\ell^*$ such that*

$$\pi_\ell = \mathrm{Ind}_{B_n(\mathbb{Q}_\ell)}^{\mathrm{GL}_n(\mathbb{Q}_\ell)}(\chi_1, \dots \chi_n).$$

This follows from some general results in the representation theory of GL_n over non-Archimedean fields: Firstly, it is a fundamental fact that every irreducible spherical representation π of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ can be realized as a subquotient Q of $\mathrm{Ind}_{B_n(\mathbb{Q}_\ell)}^{\mathrm{GL}_n(\mathbb{Q}_\ell)}(\chi_1, \dots \chi_n)$ for some unramified characters $\chi_1, \dots \chi_n$ with $\chi_i^{-1}\chi_j \neq |\cdot|$ for $i < j$, cf. section 2.1 in [Kud94]. On the other hand, according to Theorem 2.3.1 in [Kud94], if the subquotient Q is generic then it coincides with $\mathrm{Ind}(\chi_1, \dots \chi_n)$. Consequently, every irreducible generic spherical representation of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ is of the form $\mathrm{Ind}(\chi_1, \dots \chi_n)$ with unramified characters $\chi_1, \dots \chi_n$ and since every cuspidal representation is generic, the claim follows.

The next Corollary computes the Whittaker model of the contragredient representation, similarly as Corollary 1.7 it follows from Theorem 1.6.

Corollary 1.14. *Let (π, V) be an irreducible admissible $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module or an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ or of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ for a prime ℓ. If (π, V) is generic with Whittaker model $\mathcal{W}(\pi, \tau)$ then its contragredient $(\check{\pi}, \check{V})$ is also generic and its Whittaker model is given by*

$$\mathcal{W}(\check{\pi}, \tau^{-1}) = \{g \mapsto W(w_n\, {}^t g^{-1}) : \ W \in \mathcal{W}(\pi, \tau)\},$$

where w_n denotes the Weyl element $\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$.

1.3. Eisenstein Series

In this section, we construct a certain intertwining operator from an admissible representation of $\mathrm{GL}_n(\mathbb{A})$ to the space of automorphic forms $\mathcal{A}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega)$. This intertwining operator and the elements in its image will be called Eisenstein series. We restrict ourselves to $n = 2$ since this is sufficient for later applications.

Let χ_1 and χ_2 be idèle class characters with $\chi_1 = \xi_1|\cdot|^{t_1}$ and $\chi_2 = \xi_2|\cdot|^{t_2}$ for unitary idèle class characters ξ_1 and ξ_2 and real numbers t_1 and t_2. (Note that the existence of this decomposition is not a restriction on the characters χ_1 and χ_2: The identification $\mathbb{Q}^*\backslash\mathbb{A}^* = \hat{\mathbb{Z}}^* \cdot \mathbb{R}^*_{>0}$ shows that any character $\chi : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ is given by a product of a unitary character ξ and a power of the norm function $|\cdot|^r$ with $r \in \mathbb{R}$.)
We define an admissible induced representation $\mathrm{Ind}(\chi_1, \chi_2) = (\pi_{\chi_1,\chi_2}, V_{\chi_1,\chi_2})$ of $\mathrm{GL}_2(\mathbb{A})$: The representation space V_{χ_1,χ_2} consists of all smooth K-finite functions f on $\mathrm{GL}_2(\mathbb{A})$ satisfying

$$f\left(\begin{pmatrix} b_1 & x \\ & b_2 \end{pmatrix} g\right) = \chi_1(b_1)\chi_2(b_2)\left|\frac{b_1}{b_2}\right|^{\frac{1}{2}} f(g)$$

for $g \in \mathrm{GL}_2(\mathbb{A}), b_1, b_2 \in \mathbb{A}^*$ and $x \in \mathbb{A}$. As usual, $\mathrm{GL}_2(\mathbb{A}_f)$ acts on V_{χ_1,χ_2} by right translation and the $(\mathrm{gl}_2, O_2(\mathbb{R}))$-module structure is given by right translation resp. differentiation of right translation. The Eisenstein series Eis is an intertwining operator

$$\mathrm{Eis} : \mathrm{Ind}(\chi_1, \chi_2) \longrightarrow \mathcal{A}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}), \omega), \tag{1.2}$$

where the central character of the automorphic forms on the right hand side is given by $\omega = \chi_1\chi_2$. For $f \in V_{\chi_1,\chi_2}$ and $g \in \mathrm{GL}_2(\mathbb{A})$ we set

$$\mathrm{Eis}(f)(g) = \sum_{\gamma \in B_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{Q})} f(\gamma g), \tag{1.3}$$

provided this sum converges. One checks that if this sum converges, then $\mathrm{Eis}(f)$ indeed lies in the space of automorphic forms with central character ω. To discuss the matter of convergence and analytic continuation we have to find a way to vary the pair of characters (χ_1, χ_2) continuously. For $s_i \in \mathbb{C}$ we set $\chi_i(s_i) = \xi_i|\cdot|^{s_i}$ for $i = 1, 2$. We introduce the auxiliary variable $s = \frac{1}{2}(s_1 - s_2 + 1)$.

Proposition 1.15. *For $f \in V_{\chi_1,\chi_2}$ the sum (1.3) is absolutely convergent if* $\mathrm{re}(s) > 1$.

For any pair of complex numbers (s_1, s_2) we choose an element f_{s_1,s_2} in the representation space $V_{\chi_1(s_1),\chi_2(s_2)}$. The assignment $(s_1, s_2) \mapsto f_{s_1,s_2}$ is called a flat section if the restriction of f_{s_1,s_2} to K is independent of the pair (s_1, s_2). Note, that a function $f \in V_{\chi_1,\chi_2}$ is already uniquely determined by its restriction to K since we have $\mathrm{GL}_2(\mathbb{A}) = B_2(\mathbb{A}) \cdot K$ by the global Iwasawa decomposition.

Theorem 1.16. *Let $\{f_{s_1,s_2}\}_{(s_1,s_2)}$ be a flat section. Then for fixed $c \in \mathbb{C}$ and $g \in \mathrm{GL}_2(\mathbb{A})$ the mapping $s \mapsto \mathrm{Eis}(f_{s+c-\frac{1}{2},-s+c+\frac{1}{2}})(g)$ defined by equation (1.3) for $\mathrm{re}(s) > 1$ has a meromorphic continuation to $\mathbb{C}$, which we also denote by* Eis.
More precisely, the function $s \mapsto L_S(\xi_1\xi_2^{-1}, 2s)\mathrm{Eis}(f_{s+c-\frac{1}{2},-s+c+\frac{1}{2}})(g)$ is entire unless there exists $\nu \in \mathbb{C}$ with $\xi_1\xi_2^{-1} = |\cdot|^\nu$ in which case the only possible pole is at $s = 1 - \frac{\nu}{2}$. Here, $L_S(\xi_1\xi_2^{-1}, s)$ denotes the partial L-function with respect to a certain finite set S of places ℓ.

A proof can be found in [Bum97], Theorem 3.7.1.
Let χ_1 and χ_2 be arbitrary idèle class characters. Since any element $f \in V_{\chi_1,\chi_2}$ can be embedded into a flat section (cf. [Bum97] Proposition 3.7.1), Theorem 1.16 provides us with a mapping Eis from V_{χ_1,χ_2} to the set of complex functions on $\mathrm{GL}_2(\mathbb{A})$. It can be shown that $\mathrm{Eis}(f)$ is an automorphic form with central character $\chi_1\chi_2$ and that Eis is an intertwining operator (cf. [Bum97], Proposition 3.7.3). This completes the construction of the operator Eis : $\mathrm{Ind}(\chi_1, \chi_2) \to \mathcal{A}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}), \omega)$.

Although the Eisenstein series are not square integrable modulo the center, the rapid decay of cusp forms implies that an appropriate product of a cusp form and an Eisenstein series is square integrable modulo the center. The next result (cf. [Bum97], Proposition 3.7.4) states that the integral over such a product is zero, one says that Eisenstein series are orthogonal to cusp forms.

Theorem 1.17. *Let (π, V) be a cuspidal automorphic representation with unitary central character ω and let χ_1 and χ_2 be two idèle class characters with $\omega = \chi_1\chi_2$. Then for $\phi \in V$ and $f \in V_{\chi_1,\chi_2}$ we find*

$$\int\limits_{Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})} \overline{\phi(g)}\,\mathrm{Eis}(f)(g)\,dg = 0.$$

1.4. Automorphic L-functions

Throughout this section, let (π, V) resp. (π', V') be a unitary irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ resp. of $\mathrm{GL}_m(\mathbb{A})$ with $n \geq m$. Let $(\pi, V) = \otimes'_\ell(\pi_\ell, V_\ell)$ and $(\pi', V') = \otimes'_\ell(\pi'_\ell, V'_\ell)$ be their decompositions into restricted tensor products of local representations. We also fix an additive character $\tau : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^*$. Via Rankin-Selberg integrals we will associate to each pair (π_ℓ, π'_ℓ) of local representations an L-factor $L(\pi_\ell \times \pi'_\ell, s)$ and an ε-factor $\varepsilon(\pi_\ell \times \pi'_\ell, \tau_\ell, s)$. We will see that the product $\prod_\ell L(\pi_\ell \times \pi'_\ell, s)$ is absolutely convergent for s with $\mathrm{re}(s) >> 0$ and has a meromorphic continuation $L(\pi \times \pi', s)$ to all of $\mathbb{C}$. The local ε-factor $\varepsilon(\pi_\ell \times \pi'_\ell, \tau_\ell, s)$ equals one for almost all places ℓ and hence the product $\varepsilon(\pi \times \pi', s) := \prod_\ell \varepsilon(\pi_\ell \times \pi'_\ell, \tau_\ell, s)$ is absolutely convergent for all $s \in \mathbb{C}$. We obtain the following functional equation

$$L(\pi \times \pi', s) = \varepsilon(\pi \times \pi', s)L(\check{\pi} \times \check{\pi}', 1 - s),$$

where $\check{\pi}$ resp. $\check{\pi}'$ denotes the contragredient representation of π resp. π'.
The cases $n > m$ and $n = m$ require individual (but similar) treatment. In view of later applications we will restrict ourselves to the case $n > m$ which we consider sufficient to convey the basic ideas behind the construction of L-functions via Rankin-Selberg integrals.
In this section we closely follow the exposition of Cogdell, [Cog00], where one can find more details and a treatment of the case $n = m$.

1.4.1. Eulerian Integrals

Our starting point are the following Rankin-Selberg integrals: For $\varphi \in V$ and $\varphi' \in V'$ we set

$$I(s,\varphi,\varphi') = \int\limits_{\mathrm{GL}_m(\mathbb{Q})\backslash\mathrm{GL}_m(\mathbb{A})} \int\limits_{Y_{n,m}(\mathbb{Q})\backslash Y_{n,m}(\mathbb{A})} \varphi\left(y\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix}\right)\tau^{-1}(y)\,dy\;\varphi'(h)|\det(h)|^{s-\frac{n-m}{2}}\,dh,$$

where $Y_{n,m}$ denotes the unipotent radical of the standard parabolic subgroup attached to the partition $(m+1,1,\dots 1)$. Note that the subgroup $Y_{n,n-1}$ is trivial and thus in the case $m=n-1$ the integral $I(s,\varphi,\varphi')$ simplifies as follows

$$I(s,\varphi,\varphi') = \int\limits_{\mathrm{GL}_{n-1}(\mathbb{Q})\backslash\mathrm{GL}_{n-1}(\mathbb{A})} \varphi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}\varphi'(h)|\det(h)|^{s-\frac{1}{2}}\,dh.$$

Let us denote by $W_\varphi \in \mathcal{W}(\pi,\tau)$ resp. $W_{\varphi'} \in \mathcal{W}(\pi',\tau^{-1})$ the Whittaker functions associated to φ resp. φ' as introduced in Theorem 1.12. According to Theorem 1.11 the Whittaker function W_φ of a pure tensor $\varphi = \otimes'_\ell\varphi_\ell \in \otimes'_\ell\pi_\ell$ splits into the product of local Whittaker functions $W_\varphi = \prod_\ell W_{\varphi_\ell}$. This is the crucial ingredient in the proof of the following decomposition (cf. [Cog00], section 2.2.2), for pure vectors $\varphi = \otimes'_\ell\varphi_\ell$ and $\varphi' = \otimes'_\ell\varphi'_\ell$ we have

$$I(s,\varphi,\varphi') = \prod_\ell \int\limits_{N_m(\mathbb{Q}_\ell)\backslash\mathrm{GL}_m(\mathbb{Q}_\ell)} W_{\varphi_\ell}\begin{pmatrix} h_\ell & 0 \\ 0 & I_{n-m}\end{pmatrix}W_{\varphi'_\ell}(h_\ell)|\det(h_\ell)|_\ell^{s-\frac{n-m}{2}}\,dh_\ell.$$

Thus, we have found that the integral $I(s,\varphi,\varphi')$ is Eulerian, i.e. decomposes into the product of local integrals. The local integrals on the right hand side, which we want to denote by $\Psi_\ell(s,W_{\varphi_\ell},W_{\varphi'_\ell})$ in the following, are absolutely convergent for $\mathrm{re}(s) >> 0$. In the next section we will explain how the set of local integrals $\Psi_\ell(s,W_{\varphi_\ell},W_{\varphi'_\ell})$ with $\varphi \in \pi$ and $\varphi' \in \pi'$ encodes the local L-factor $L(\pi_\ell,\pi'_\ell,s)$.

Now we want to take first steps towards the functional equation. Let $g \mapsto g^\iota := {}^tg^{-1}$ be the standard involution on GL_n. Corollary 1.7 states that the automorphic form $\widetilde{\varphi}(g) := \varphi(g^\iota)$ lies in the contragredient of (π,V). The Whittaker function $W_{\widetilde{\varphi}}$ corresponding to $\widetilde{\varphi}$ is given by $W_{\widetilde{\varphi}}(g) = W_\varphi(w_n g^\iota)$, where w_n denotes the Weyl element $\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$.

Applying the variable transformation $h \mapsto h^\iota$ to the integral $I(s,\varphi,\varphi')$ we get the following functional equation for the global integrals

$$I(s,\varphi,\varphi') = \widetilde{I}(1-s,\widetilde{\varphi},\widetilde{\varphi}'),$$

where the right hand side is defined by

$$\widetilde{I}(s,\varphi,\varphi') = \int\int \varphi\left(y^\iota\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix}\right)\tau^{-1}(y)\,dy\;\varphi'(h)|\det(h)|^{s-1+\frac{n-m}{2}}\,dh$$

and as before the integral attached to the variable h is over $\mathrm{GL}_m(\mathbb{Q})\backslash\mathrm{GL}_m(\mathbb{A})$ and the integral attached to the variable y is over $Y_{n,m}(\mathbb{Q})\backslash Y_{n,m}(\mathbb{A})$.

Again it can be shown that the integral $\widetilde{I}(s,\varphi,\varphi')$ is Eulerian, cf. [Cog00], Theorem 2.1. We have

$$\widetilde{I}(s,\varphi,\varphi') = \prod_\ell \widetilde{\Psi}_\ell(s,\rho(w_{n,m})W_{\varphi_\ell},W_{\varphi'_\ell}),$$

where $\rho(w_{n,m})$ denotes right translation by the Weyl element $w_{n,m} := \left(\begin{smallmatrix} I_m & \\ & w_{n-m} \end{smallmatrix}\right)$ and the local integrals are given by

$$\widetilde{\Psi}_\ell(s, W_{\varphi_\ell}, W_{\varphi'_\ell}) = \int\int W_{\varphi_\ell} \begin{pmatrix} h_\ell & & \\ x_\ell & I_{n-m-1} & \\ & & 1 \end{pmatrix} dx_\ell \, W_{\varphi'_\ell}(h_\ell) |\det(h_\ell)|^{s-\frac{n-m}{2}} \, dh_\ell,$$

where the integral attached to h_ℓ is over $N_m(\mathbb{Q}_\ell)\backslash \mathrm{GL}_m(\mathbb{Q}_\ell)$ and the integral attached to x_ℓ is over $M_{n-m-1,m}(\mathbb{Q}_\ell)$.

1.4.2. Local L-functions

Next, we have to analyse the families of local integrals $\Psi_\ell(s, W_{\varphi_\ell}, W_{\varphi'_\ell})$ and $\widetilde{\Psi}_\ell(s, W_{\varphi_\ell}, W_{\varphi'_\ell})$.
Let first ℓ be a non-Archimedean place. For any pair of Whittaker functions $W \in \mathcal{W}(\pi_\ell, \tau_\ell)$ and $W' \in \mathcal{W}(\pi'_\ell, \tau_\ell^{-1})$ the integral $\Psi_\ell(s, W, W')$ is absolutely convergent for $\mathrm{re}(s) >> 0$ and may be extended meromorphically to a rational function in ℓ^{-s}. Moreover, there exists a rational function $f(\ell^{-s})$ such that every function $\Psi_\ell(s, W, W')$ with $W \in \mathcal{W}(\pi_\ell, \tau_\ell)$ and $W' \in \mathcal{W}(\pi'_\ell, \tau_\ell^{-1})$ is of the form $P(\ell^{-s})f(\ell^{-s})$ where $P(\ell^{-s})$ is a polynomial function in ℓ^{-s}; in other words the functions $\Psi_\ell(s, W, W')$ can be written with a common denominator. These results on the local integrals may be deduced from a fundamental result on the asymptotics of Whittaker functions, cf. [Cog00], Proposition 3.1 and Proposition 3.2.
Let us denote by $\mathcal{I}(\pi_\ell, \pi'_\ell)$ the complex linear span of the local integrals $\Psi_\ell(s, W, W')$ in $\mathbb{C}(\ell^{-s})$. An integral manipulation shows that $\mathcal{I}(\pi_\ell, \pi'_\ell)$ is a $\mathbb{C}[\ell^s, \ell^{-s}]$-module, thus we have found that $\mathcal{I}(\pi_\ell, \pi'_\ell)$ is a fractional $\mathbb{C}[\ell^s, \ell^{-s}]$-module in $\mathbb{C}(\ell^{-s})$. Since $\mathbb{C}[\ell^s, \ell^{-s}]$ is a principal ideal domain, $\mathcal{I}(\pi_\ell, \pi'_\ell)$ is generated by a single element $Q_{\pi_\ell, \pi'_\ell}(\ell^{-s}) \in \mathbb{C}(\ell^{-s})$. One can show that $\mathcal{I}(\pi_\ell, \pi'_\ell)$ contains the constant function 1 and thus $Q_{\pi_\ell, \pi'_\ell}(\ell^{-s})$ may be chosen as the inverse of a polynomial function, $Q_{\pi_\ell, \pi'_\ell}(\ell^{-s}) = P_{\pi_\ell, \pi'_\ell}(\ell^s, \ell^{-s})^{-1}$. Since ℓ^{-s} and ℓ^s are units in $\mathbb{C}[\ell^s, \ell^{-s}]$ we may normalize the generator to be of the form $P_{\pi_\ell, \pi'_\ell}(\ell^{-s})^{-1}$ with $P_{\pi_\ell, \pi'_\ell}(0) = 1$.

Definition 1.18 (Local L-function)**.** *We define the local L-function $L(\pi_\ell \times \pi'_\ell, s)$ attached to the pair of representations (π_ℓ, π'_ℓ) as the normalized generator $P_{\pi_\ell, \pi'_\ell}(\ell^{-s})^{-1}$ of the fractional ideal $\mathcal{I}(\pi_\ell, \pi'_\ell)$.*

One can show that the integrals $\widetilde{\Psi}_\ell(s, W, W')$ for $W \in \mathcal{W}(\pi_\ell, \tau_\ell)$ $W' \in \mathcal{W}(\pi'_\ell, \tau_\ell^{-1})$ are also absolutely convergent for $\mathrm{re}(s) >> 0$ and may be continued meromorphically to a rational function in ℓ^{-s}. Moreover, the complex linear span of the local integrals $\widetilde{\Psi}_\ell(s, W, W')$ coincides with the fractional ideal $\mathcal{I}(\pi_\ell, \pi'_\ell)$ spanned by the local integrals $\Psi_\ell(s, W, W')$.

The functional equation for the global integral $I(s, \varphi, \varphi')$ and its Eulerian decomposition,

$$\begin{array}{ccc} I(s, \varphi, \varphi') & = & I(1-s, \widetilde{\varphi}, \widetilde{\varphi}) \\ \| & & \| \\ \prod_\ell \Psi_\ell(s, W_{\varphi_\ell}, W_{\varphi'_\ell}) & & \prod_\ell \widetilde{\Psi}_\ell(1-s, \rho(w_{n,m}) W_{\widetilde{\varphi_\ell}}, W_{\widetilde{\varphi}'_\ell}), \end{array}$$

suggests that the local functional equation should relate the integrals $\Psi_\ell(s, W_{\varphi_\ell}, W_{\varphi'_\ell})$ and $\widetilde{\Psi}_\ell(1-s, \rho(w_{n,m}) W_{\widetilde{\varphi_\ell}}, W_{\widetilde{\varphi}'_\ell})$. Whereas we have obtained the functional equation for the global integrals from a simple variable transformation, there is a clever idea and some work behind the proof of the local functional equation, cf. [JPSS83], section (2.7). Let us explain this shortly. For every s we can regard $\Psi_\ell(s, W, W')$ and $\widetilde{\Psi}_\ell(1-s, \rho(w_{n,m})\widetilde{W}, \widetilde{W'})$ as a bilinear form on $\mathcal{W}(\pi_\ell, \tau_\ell) \times$

$\mathcal{W}(\pi'_\ell, \tau_\ell^{-1})$, where $\widetilde{W}(g) = W(w_n\, {}^t g^{-1})$ and analogously for $\widetilde{W}'$. A manipulation of the integrals shows that for every s these two bilinear forms belong to the space $\mathcal{B}_s$ consisting of the bilinear forms B_s on $\mathcal{W}(\pi_\ell, \tau_\ell) \times \mathcal{W}(\pi'_\ell, \tau_\ell^{-1})$ which satisfy

$$B_s\left(\pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix} W, \pi'(h)W'\right) = |\det(h)|^{-s+\frac{n-m}{2}} B_s(s, W, W')$$

and

$$B_s(\pi(y)W, W') = \tau(y)B_s(W, W')$$

for $h \in \mathrm{GL}_m(\mathbb{Q}_\ell)$ and for $y \in Y_{n,m}(\mathbb{Q}_\ell)$.
On the other hand, one can prove that except for a finite number of values ℓ^{-s} the space $\mathcal{B}_s$ is at most one-dimensional. As a consequence we obtain the following local functional equation.

Theorem 1.19. *There is a rational function $\gamma(s, \pi_\ell \times \pi'_\ell, \tau_\ell) \in \mathbb{C}(\ell^{-s})$ such that*

$$\widetilde{\Psi}_\ell(1-s, \rho(w_{n,m})\widetilde{W}, \widetilde{W}') = \omega'(-1)^{n-1}\, \gamma(s, \pi_\ell \times \pi'_\ell, \tau_\ell)\, \Psi_\ell(s, W, W')$$

for $W \in \mathcal{W}(\pi_\ell, \tau_\ell)$ and $W' \in \mathcal{W}(\pi'_\ell, \tau_\ell^{-1})$. (Here ω' denotes the central character of π'_ℓ.)

The rationality of $\gamma(s, \pi_\ell \times \pi'_\ell, \tau_\ell)$ follows from the fact that it is a ratio of rational functions. We are now able to define the local ε-factor.

Definition 1.20 (Local ε-factor)**.** *We call*

$$\varepsilon(\pi_\ell \times \pi'_\ell, \tau_\ell, s) := \frac{\gamma(s, \pi_\ell \times \pi'_\ell, \tau_\ell) L(\pi_\ell \times \pi'_\ell, s)}{L(\check{\pi}_\ell \times \check{\pi}'_\ell, 1-s)}$$

the local ε-factor attached to the representations π_ℓ and π'_ℓ.

Let us fix some notation, instead of $L(\pi_\ell \times 1, s)$ we write $L(\pi_\ell, s)$ and similarly for all other cases where this convention applies, for instance for $\varepsilon(\pi_\ell, \tau_\ell, s)$.
Proposition 3.5 in [Cog00] states that the ε-factor $\varepsilon(\pi_\ell \times \pi'_\ell, \tau_\ell, s)$ is a monomial function of the form $c\,\ell^{-fs}$. If we assume τ_ℓ to be unramified, then the constant f in $\varepsilon(\pi_\ell, \tau_\ell, s) = c\,\ell^{-fs}$ is equal to the conductor of the representation π.
Let now π_ℓ, π'_ℓ and τ_ℓ be unramified. In Remark 1.13 we have seen that there exist unramified characters $\mu_1, \dots \mu_n$ and $\mu'_1, \dots \mu'_m$ of $\mathbb{Q}_\ell^*$ such that $\pi_\ell \simeq \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(\mu_1 \otimes \dots \otimes \mu_n)$ and $\pi'_\ell \simeq \mathrm{Ind}_{B_m}^{\mathrm{GL}_m}(\mu'_1 \otimes \dots \otimes \mu'_m)$. One can show that in this case the local L-function is given by the following formula

$$L(\pi_\ell \times \pi'_\ell, s) = \prod_{i,j}(1 - \mu_i(\ell)\mu'_j(\ell)\ell^{-s})^{-1} \tag{1.4}$$

and that the ε-factor is trivial, i.e. $\varepsilon(\pi_\ell \times \pi'_\ell, \tau_\ell, s) \equiv 1$, cf. [Cog00], section 3.1.3.
One should mention that this construction of local L-functions applies to any pair of generic irreducible representations π_ℓ and π'_ℓ not necessarily occurring in the decomposition of cuspidal representations.

We still have to give a definition of the local L-factor in the Archimedean case, so let $\ell = \infty$, i.e. $\mathbb{Q}_\ell = \mathbb{R}$. Although the results are essentially the same as in the non-Archimedean case, the approach is different. To define the local L-factor $L(\pi_\ell \times \pi'_\ell, s)$ one makes use of the local Langlands correspondence, which establishes a bijection between the set of infinitesimal equivalence

classes of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{R})$ and the set of equivalence classes of n-dimensional semi-simple complex representations of the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$, cf. Theorem 2 in [Kna94]. Taking the K-invariant vectors in an admissible irreducible representation of $\mathrm{GL}_n(\mathbb{R})$ yields an admissible irreducible $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module and Theorem 7 in [Bal97] states, that every irreducible admissible $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module can be obtained in this way. So the infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{R})$ correspond to the equivalence classes of irreducible admissible $(\mathrm{gl}_n, O_n(\mathbb{R}))$-modules. Thus, to the $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module π_∞ resp. π'_∞ is associated an n- resp. m-dimensional semi-simple representation $\sigma(\pi_\infty)$ resp. $\sigma(\pi'_\infty)$ of $W_{\mathbb{R}}$.
One can show that every finite-dimensional semi-simple representation σ of $W_{\mathbb{R}}$ is fully reducible and that each irreducible constituent has dimension one or two. In [Kna94] one can find a complete list of the irreducible representations σ of $W_{\mathbb{R}}$ and to each of these representation is associated an L-function $L(\sigma, s)$ and an ε-factor $\varepsilon(\sigma, \tau_\infty, s)$. If σ is reducible one defines $L(\sigma, s)$ as the product of the L-functions of the irreducible constituents of σ and analogously for the ε-factor.
The L-function attached to π_∞ and π'_∞ is defined as the L-function of the nm-dimensional representation $\sigma(\pi_\infty) \otimes \sigma(\pi'_\infty)$ of $W_{\mathbb{R}}$. Jacquet and Shalika show in [JS90] that this definition fits into the framework of Rankin-Selberg integrals, i.e. roughly spoken they prove that $L(\pi_\infty \times \pi'_\infty, s)$ is the "minimal common denominator" of the integrals $\Psi_\infty(s, W, W')$ with $W \in \mathcal{W}(\pi_\infty, \tau_\infty)$ and $W' \in \mathcal{W}(\pi'_\infty, \tau_\infty^{-1})$. In particular, they prove that the local integrals $\Psi_\infty(s, W, W')$ converge absolutely for $\mathrm{re}(s) >> 0$, that these integrals can be extended to meromorphic functions on $\mathbb{C}$ and that the quotients $\frac{\Psi_\infty(s,W,W')}{L(\pi_\infty \times \pi'_\infty, s)}$ are entire functions. Furthermore, they set $\varepsilon(\pi_\infty \times \pi'_\infty, \tau_\infty, s) := \varepsilon(\sigma(\pi_\infty) \otimes \sigma(\pi'_\infty), \tau_\infty, s)$ and verify the following functional equation

$$\frac{\widetilde{\Psi}_\infty(1-s, \rho(w_{n,m})\widetilde{W}, \widetilde{W'})}{L(\check{\pi}_\infty \times \check{\pi}'_\infty, 1-s)} = \varepsilon(\pi_\infty \times \pi'_\infty, \tau_\infty, s)\omega'(-1)^{n-1} \frac{\Psi_\infty(s, W, W')}{L(\pi_\infty \times \pi'_\infty, s)},$$

where ω' is the central character of π'_∞.
For instance if $\pi_\infty \simeq \mathrm{Ind}(\mu_1 \otimes \ldots \otimes \mu_n)$ is irreducible with unramified characters $\mu_i(x) = |x|^{r_i}$ of $\mathbb{R}^*$ for complex numbers r_i then the formulas (3.6) and (3.7) in [Kna94] imply

$$L(\pi_\infty, s) = \prod_{i=1}^{n} \pi^{\frac{-s-r_i}{2}} \, \Gamma\left(\frac{s+r_i}{2}\right)$$

and $\varepsilon(\pi_\infty, \tau_\infty, s) \equiv 1$ if $\tau_\infty(x) = e^{2\pi i x}$.
As in the non-Archimedean case this construction enables us to associate an L-factor to an arbitrary pair of a generic irreducible $(\mathrm{gl}_n, O_n(\mathbb{R}))$-module π_∞ and a generic irreducible $(\mathrm{gl}_m, O_m(\mathbb{R}))$-module π'_∞.

1.4.3. Global L-functions

Let S be a finite set of places ℓ containing $\ell = \infty$ such that for all $\ell \notin S$ the representations π_ℓ and π'_ℓ and the character τ_ℓ are unramified. For $\ell \notin S$ we know that $\pi_\ell \simeq \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(\mu_{\ell,1} \otimes \ldots \otimes \mu_{\ell,n})$ and that $\pi'_\ell \simeq \mathrm{Ind}_{B_m}^{\mathrm{GL}_m}(\mu'_{\ell,1} \otimes \ldots \otimes \mu'_{\ell,m})$ with $\mu_{\ell,i}$ and $\mu'_{\ell,j}$ unramified characters of $\mathbb{Q}_\ell^*$, cf. Remark 1.13. According to the Corollary of Theorem 3.3 in [Cog00] the absolute values of $\mu_i(\ell)$ and of $\mu'_j(\ell)$ are smaller than $\ell^{1/2}$. Thus, the product

$$\prod_{\ell \notin S} L(\pi_\ell \times \pi'_\ell, s) = \prod_{\ell \notin S} \prod_{\substack{1 \le i \le n \\ 1 \le j \le m}} (1 - \ell^{-s}\mu_{\ell,i}(\ell)\mu'_{\ell,j}(\ell))^{-1}$$

converges absolutely for $\mathrm{re}(s) >> 0$. Since S is a finite set, the product $\prod_\ell L(\pi_\ell \times \pi'_\ell, s)$ also converges absolutely for $\mathrm{re}(s) >> 0$ and we set

$$L(\pi \times \pi', s) := \prod_\ell L(\pi_\ell \times \pi'_\ell, s)$$

for $\mathrm{re}(s)$ sufficiently large. We want to make use of the analytic properties of the global and local integrals to establish a meromorphic continuation of the function $L(\pi \times \pi', s)$ to all of $\mathbb{C}$. Let $\varphi = \otimes' \varphi_\ell \in V$ and $\varphi' = \otimes' \varphi'_\ell \in V'$ be cuspidal forms such that their components φ_ℓ and φ'_ℓ are spherical vectors for all $\ell \notin S$. Then the local integral $\Psi_\ell(s, W_{\varphi_\ell}, W_{\varphi'_\ell})$ coincides with the local L-factor $L(\pi_\ell \times \pi'_\ell, s)$ for every $\ell \notin S$. Thus, we obtain for s with $\mathrm{re}(s) >> 0$ that

$$I(s, \varphi, \varphi') = \left(\prod_{\ell \in S} \frac{\Psi_\ell(s, W_{\varphi_\ell}, W_{\varphi'_\ell})}{L(\pi_\ell \times \pi'_\ell, s)} \right) L(\pi \times \pi', s). \tag{1.5}$$

Since the quotients occurring on the right hand side are analytic by the definition of the local L-factor and since the integral $I(s, \varphi, \varphi')$ is absolutely convergent for $s \in \mathbb{C}$ and in particular defines an analytic function we conclude that $L(\pi \times \pi', s)$ can be meromorphically continued to all of $\mathbb{C}$. This function $L(\pi \times \pi', s)$ is called the automorphic L-function attached to π and π'. A more sophisticated investigation of equation (1.5) shows that $L(\pi \times \pi', s)$ is even an analytic function, cf. [Cog00], section 4.2.
Analogously to the L-function we define the global ε-factor as the product of the local ε-factors, $\varepsilon(\pi \times \pi', s) := \prod_\ell \varepsilon(\pi_\ell \times \pi'_\ell, \tau_\ell, s)$. Since we have seen above that $\varepsilon(\pi_\ell \times \pi'_\ell, \tau_\ell, s)$ is trivial for all $\ell \notin S$ this product is actually a finite product and we do not have to worry about convergence. In contrast to the local ε-factors the global ε-factor is indeed independent of the additive character τ as the notation suggests. This follows for instance from the functional equation

$$L(\pi \times \pi', s) = \varepsilon(\pi \times \pi', s) L(\check{\pi} \times \check{\pi}', 1 - s),$$

which may be deduced from the functional equations for the global and local integrals, cf. [Cog00], Theorem 4.1.

The following notation is crucial for the construction of p-adic automorphic L-functions. A number $s_0 \in \frac{n+1}{2} + \mathbb{Z}$ is called a critical integer for π if $L(\pi_\infty, s)$ and $L(\check{\pi}_\infty, 1 - s)$ do not have poles at $s = s_0$. We emphasize that if π is a unitary irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ with n even, then the set of critical integers for π is a subset of $\frac{1}{2} + \mathbb{Z}$. Of course, if π_∞ is selfdual then the set of critical integers is symmetric with respect to $1/2$, i.e. if s is a critical integer then so is $1 - s$. A conjecture of Deligne in particular states that for any cuspidal automorphic representation π and a critical integer s_0 of π there exists a complex number $\Omega(\pi, s_0)$ such that

$$\frac{L(\pi \otimes \chi, s_0)}{\Omega(\pi, s_0)} \in \bar{\mathbb{Q}}$$

for all characters $\chi : \mathbb{Q}^* \backslash \mathbb{A}^* \to \mathbb{C}$ of finite order and with infinity component $\chi_\infty = \mathbb{1}$.

2. Cohomology

In the first section we introduce certain locally constant coefficient systems $\mathcal{V}$ on manifolds $S_n(K^f)$ stemming from rational representations (ρ, V) of GL_n. The whole chapter is dedicated to the study of their sheaf cohomology. In section 2.2 the manifold $S_n(K^f)$ is embedded into a compact manifold $\bar{S}_n(K^f)$, its Borel-Serre compactification. Section 2.3 explains how the sheaf cohomology of $S_n(K^f)$ with respect to $\mathcal{V}$ can be interpreted as de Rham cohomology and relative Lie algebra cohomology. The two following sections, section 2.4 and section 2.5 construct certain subspaces in the cohomology of $S_n(K^f)$, the cuspidal cohomology and the Eisenstein cohomology. In the last section we will see that under suitable conditions a pairing of sheaves gives rise to a pairing of cohomology groups.

2.1. Basic Notations

Let K^f be a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$. We introduce the differentiable manifolds

$$S_n(K^f) := \mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f K_{n,\infty} Z_n^0(\mathbb{R})$$

and

$$F_n(K^f) := \mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f K_{n,\infty} \cong \mathbb{R}^*_{>0} \times S_n(K^f).$$

It is a consequence of the strong approximation for SL_n that there exist finitely many elements $g_1, \dots g_m \in \mathrm{GL}_n(\mathbb{A}_f)$ such that

$$S_n(K^f) = \bigsqcup_{i=1}^{m} \Gamma^{g_i}\backslash \mathrm{GL}_n(\mathbb{R})/K_{n,\infty} Z_n^0(\mathbb{R}), \tag{2.1}$$

where $\Gamma^{g_i} := \mathrm{GL}_n(\mathbb{Q}) \cap g_i K^f g_i^{-1}$, with $\mathrm{GL}_n(\mathbb{Q})$ diagonally embedded into $\mathrm{GL}_n(\mathbb{A}_f)$, is an arithmetic subgroup of $\mathrm{GL}_n(\mathbb{Q})$, cf. [Har82], section 1.1. We call K^f neat, if the subgroups Γ^{g_i}, for $1 \leq i \leq m$, are torsion-free. In this case Γ^{g_i} acts fixed-point-free and properly discontinuously on the quotient $X := \mathrm{GL}_n(\mathbb{R})/K_{n,\infty} Z_n^0(\mathbb{R})$ and thus $\Gamma^{g_i}\backslash X$ is locally homeomorphic to X. So if K^f is neat, then $S_n(K^f)$ is a finite disjoint union of locally symmetric spaces. We will later make use of the fact that for any compact open subgroup K^f one can find a normal compact open subgroup L^f in K^f, which is neat.
Let $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$ be a finite-dimensional representation of algebraic groups over $\mathbb{Q}$, in particular V is a $\mathbb{Q}$-vector space. We want to define a sheaf $\mathcal{V}_{K^f}$ or shorter $\mathcal{V}$ of $\mathbb{Q}$-modules on the space $S_n(K_f)$ by describing its sections $\mathcal{V}_{K^f}(U)$ over an open subset $U \subset S_n(K^f)$. Let $\pi : \mathrm{GL}_n(\mathbb{A})/K^f K_{n,\infty} Z_n^0(\mathbb{R}) \to S_n(K^f)$ be the projection. We set

$$\mathcal{V}(U) = \left\{ s : \pi^{-1}(U) \longrightarrow V \;\middle|\; \begin{array}{l} s \text{ is locally constant and} \\ s(\gamma u) = \rho(\gamma) s(u) \text{ for } \gamma \in \mathrm{GL}_n(\mathbb{Q}), u \in \pi^{-1}(U) \end{array} \right\}. \tag{2.2}$$

If F is an arbitrary field containing $\mathbb{Q}$ we define the sheaf $\mathcal{V}_F$ of F-modules by replacing V with $V_F = V \otimes_{\mathbb{Q}} F$ in the definition (2.2), equivalently we could set $\mathcal{V}_F = \mathcal{V} \otimes_{\underline{Q}} \underline{F}$.

Let L^f be a compact open subgroup of K^f and denote by p the canonical mapping $p : S_n(L^f) \to S_n(K^f)$. This induces a map in cohomology

$$H^i(S_n(K^f), \mathcal{V}_{K^f}) \longrightarrow H^i(S_n(L^f), p^*\mathcal{V}_{K^f}), \tag{2.3}$$

where $p^*\mathcal{V}_{K^f}$ is the inverse image of $\mathcal{V}_{K^f}$, cf. [Har08b], section 4.4.3 or [Bre67], example II.6.3. Noting that $p^*\mathcal{V}_{K^f}$ coincides with the sheaf $\mathcal{V}_{L^f}$ on the manifold $S_n(L^f)$, we may define

$$H^i(\widetilde{S}_n, \mathcal{V}) := \varinjlim_{K^f} H^i(S_n(K^f), \mathcal{V}_{K^f}).$$

On this limit we have an action of the group $\mathrm{GL}_n(\mathbb{A}_f)$ as follows, cf. [Har87b], section (1.2.1). An element $g = (1, g_f)$ in $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{A}_f)$ defines by multiplication from the right an isomorphism of differentiable manifolds

$$r_g : S_n(K^f) \longrightarrow S_n(g_f^{-1} K_f g_f).$$

Consequently, as in (2.3), we obtain a map in cohomology

$$r_g^* : H^i(S_n(g_f^{-1} K^f g_f), \mathcal{V}) \longrightarrow H^i(S_n(K^f), r_g^*\mathcal{V}).$$

One can show that $r_g^*\mathcal{V}$ can be identified with the sheaf $\mathcal{V}$ on $S_n(K^f)$ and that r_g^* is compatible with the map (2.3). Thus, we finally obtain an action of $\mathrm{GL}_n(\mathbb{A}_f)$ on the direct limit $H^i(\widetilde{S}_n, \mathcal{V})$ which we also denote by r_g^*.
Since p and r_g are proper maps, the same considerations apply to the cohomology with compact supports and we can define cohomology spaces $H_c^i(\widetilde{S}_n, \mathcal{V})$ and again obtain an action r_g^* of $\mathrm{GL}_n(\mathbb{A}_f)$ on them.

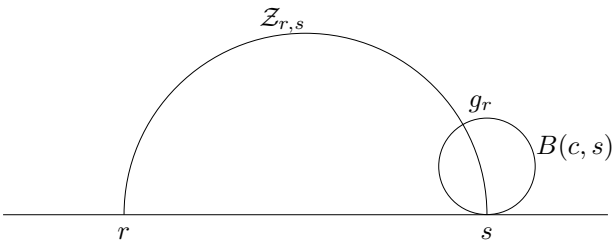
2.2. Some Facts about the Borel-Serre Compactification

In this section we want to summarize the main properties of the Borel-Serre compactification and illustrate its construction in the easiest case, namely for the upper half plane. This special case is important for later applications, cf. section 4.4. More details on the Borel-Serre compactification can be found in [BS73] or in [BJ06], section III.5.

We abbreviate $\mathrm{GL}_n(\mathbb{R})/K_{n,\infty}Z_n^0(\mathbb{R})$ to $X_n = X$. The Borel-Serre compactification $\bar{X}$ of X is a real analytic manifold with corners whose interior is equal to X. The action of $\mathrm{GL}_n(\mathbb{Q})$ on X extends to $\bar{X}$, cf. Proposition III.5.13 in [BJ06], clearly it preserves the boundary $\partial\bar{X} := \bar{X} - X$. For an arithmetic subgroup Γ of $\mathrm{GL}_n(\mathbb{Q})$ the space $\Gamma\backslash\bar{X}$ is a compact Hausdorff space, cf. Proposition III.5.14, and, what sets the Borel-Serre compactification apart from other compactifications as for instance the Baily-Borel compactification, is that the inclusion $\Gamma\backslash X \subset \Gamma\backslash\bar{X}$ is a homotopy equivalence.
In the case $n = 2$ the Borel-Serre compactification is comparatively easy to describe. We write $\mathbb{H} = \mathrm{GL}_2^+(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})Z_2^0(\mathbb{R})$ for the upper half plane. The space X_2 equals the disjoint union of two copies of $\mathbb{H}$ and we have $\bar{X} = \bar{\mathbb{H}} \sqcup \bar{\mathbb{H}}$, i.e. we are reduced to describe the Borel-Serre compactification $\bar{\mathbb{H}}$ of the upper half plane. On the level of sets, $\bar{\mathbb{H}}$ is the disjoint union

$$\bar{\mathbb{H}} = \mathbb{H} \sqcup \bigsqcup_{s \in \mathbb{P}^1(\mathbb{Q})} B(c_s, s),$$

where c_s is an arbitrary real number for every $s \in \mathbb{P}^1(\mathbb{Q})$ and the boundary component $B(c_s, s)$ consists of all points $g \in \mathbb{H}$ whose distance to s equals c_s (in the Poincaré metric). For any $r \in \mathbb{P}^1(\mathbb{R})\backslash\{s\}$ the intersection of $B(c, s), c \in \mathbb{R}$ with the geodesic $\mathcal{Z}_{r,s}$ joining r and s consists of a single point $g_r \in \mathbb{H}$, see figure.



The assignment $r \mapsto g_r$ yields a natural identification $\mathbb{P}^1(\mathbb{R})\backslash\{s\} \leftrightarrow B(c,s)$. If we do not want to specify a realization $B(c,s)$ or $\mathbb{P}^1(\mathbb{R})\backslash\{s\}$ of the boundary component at the cusp s, we will also denote this boundary component by $\mathbb{H}_{s,\infty}$. We write $\{r\}_s$ for the point in $\mathbb{H}_{s,\infty}$ corresponding to $r \in \mathbb{P}^1(\mathbb{R})\backslash\{s\}$. The group $\mathrm{GL}_2(\mathbb{Q})$ operates on the boundary by $\gamma\{r\}_s := \{\gamma r\}_{\gamma s}$ for $\gamma \in \mathrm{GL}_2(\mathbb{Q})$. For instance the setwise stabilizer of $\mathbb{H}_{\infty,\infty}$ in $\mathrm{GL}_2(\mathbb{Q})$ is the standard Borel subgroup $B_2(\mathbb{Q})$.

For a compact open subgroup K^f of $\mathrm{GL}_n(\mathbb{A}_f)$ we obtain the Borel-Serre compactification $\bar{S}_n(K^f)$ of $S_n(K^f)$ via the interpretation

$$S_n(K^f) = \bigsqcup_{i=1}^{m} \Gamma^{g_i}\backslash X_n, \tag{2.4}$$

where $g_1, \dots g_m$ are appropriate elements in $\mathrm{GL}_n(\mathbb{A}_f)$ and Γ^{g_i} is the arithmetic subgroup $\mathrm{GL}_n(\mathbb{Q}) \cap g_i K^f g_i^{-1}$ of $\mathrm{GL}_n(\mathbb{Q})$: we simply set $\bar{S}_n(K^f) = \bigsqcup \Gamma^{g_i}\backslash \bar{X}_n$.
Let $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$ be a finite-dimensional, rational representation and let $\mathcal{V}$ be the corresponding sheaf on $S_n(K^f)$. We have already mentioned that the embedding $i : S_n(K^f) \hookrightarrow \bar{S}_n(K^f)$ is a homotopy equivalence, thus we obtain an isomorphism

$$H^\bullet(S_n(K^f), \mathcal{V}) \cong H^\bullet(\bar{S}_n(K^f), i_*\mathcal{V})$$

and we usually write $\mathcal{V}$ instead of $i_*\mathcal{V}$.
Similarly, one constructs the boundary $\partial\bar{S}_n(K^f)$ of the Borel-Serre compactification $\bar{S}_n(K^f)$ by making use of equation (2.4) and we write $\mathcal{V}$ for the sheaf $j^*i_*\mathcal{V}$ on $\partial\bar{S}_n(K^f)$, where j denotes the embedding $j : \partial\bar{S}_n(K^f) \hookrightarrow \bar{S}_n(K^f)$. Fortunately, in the case $n = 2$ we are not reliant on this definition for cohomological matters since in [Har87b], section 2.1 and in [Har82], Proposition 3.1 it is shown that the spaces $\partial\bar{S}_2(K^f)$ and $B_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K^f K_{2,\infty} Z_2^0(\mathbb{R})$ are homotopy equivalent. If p denotes the quotient map

$$p : B_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K^f K_{2,\infty} Z_2^0(\mathbb{R}) \longrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K^f K_{2,\infty} Z_2^0(\mathbb{R}),$$

then we have

$$H^\bullet(\partial\bar{S}_2(K^f), j^*i_*\mathcal{V}) \cong H^\bullet(B_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K^f K_{2,\infty} Z_2^0(\mathbb{R}), p^*\mathcal{V}).$$

Moreover, the quotient map p induces the canonical restriction map from the cohomology of $S_2(K^f)$ to the cohomology of the boundary $\partial\bar{S}_2(K^f)$. Again, we simply write $\mathcal{V}$ for the sheaf $p^*\mathcal{V}$.
As before, we define $H^\bullet(\partial\widetilde{\bar{S}}_n, \mathcal{W})$ or shortened $H^\bullet(\partial\widetilde{S}_n, \mathcal{W})$ to be the direct limit of the cohomology spaces $H^\bullet(\partial\bar{S}_n(K^f), \mathcal{W})$ with regard to the directed set of compact open subgroups $\{K^f\}$.

We want to close this section with a fundamental sequence in cohomology which establishes a relation between ordinary cohomology, the cohomology of the boundary and cohomology with compact supports. According to Proposition 4.7.1 in [Har08b], we have

$$H_c^i(S_n(K^f), \mathcal{V}_\mathbb{C}) = H^i(\bar{S}_n(K^f), i_!(\mathcal{V}_\mathbb{C})),$$

where $i_!(\mathcal{V}_\mathbb{C})$ denotes the sheaf on $\bar{S}_n(K^f)$ obtained from $\mathcal{V}_\mathbb{C}$ by extension by zero, i.e.

$$i_!(\mathcal{V}_\mathbb{C})(U) = \left\{ s \in \mathcal{V}_\mathbb{C}(U \cap S_n(K^f)) \,\middle|\, \text{the closure of supp}(s) \text{ in } \bar{S}_n(K^f) \text{ lies in } S_n(K^f) \right\}$$

for any open set U in $\bar{S}_n(K^f)$. One can show that the short exact sequence of sheaves on $\bar{S}_n(K^f)$

$$0 \to i_!(\mathcal{V}_\mathbb{C}) \to i_*(\mathcal{V}_\mathbb{C}) \to i_*(\mathcal{V}_\mathbb{C})/i_!(\mathcal{V}_\mathbb{C}) \to 0$$

gives rise to the long exact sequence in cohomology

$$\ldots \to H^{i-1}(\partial\bar{S}_n(K^f), \mathcal{V}_\mathbb{C}) \to H^i_c(S_n(K^f), \mathcal{V}_\mathbb{C}) \to H^i(S_n(K^f), \mathcal{V}_\mathbb{C}) \to H^i(\partial\bar{S}_n(K^f), \mathcal{V}_\mathbb{C}) \to \ldots, \tag{2.5}$$

cf. [Har08a], III, section 2.1. The image of $H^i_c(S_n(K^f), \mathcal{V}_\mathbb{C}) \to H^i(S_n(K^f), \mathcal{V}_\mathbb{C})$ is called the inner cohomology.

2.3. De Rham Cohomology and Relative Lie Algebra Cohomology

If we are interested in the cohomology groups $H^i(S_n(K^f), \mathcal{V}_\mathbb{C})$ we can make use of the de Rham cohomology. In the whole section let K^f be a neat compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$, then for small open subsets $U \subset S_n(K^f)$ we know that $\mathcal{V}_\mathbb{C}(U)$ is isomorpic to $V_\mathbb{C} \cong \mathbb{C}^m$. We define the sheaf $\mathcal{V}_{\mathbb{C},\infty}$ of smooth sections in $\mathcal{V}_\mathbb{C}$ by

$$\mathcal{V}_{\mathbb{C},\infty}(U) = \Big\{ \sum_{i=1}^m f_i s_i \,\Big|\, \{s_i\}_i \text{ basis of } \mathcal{V}(U), f_i \in \mathcal{C}^\infty(U) \Big\}.$$

As usual, let us denote by Ω^p the sheaf of smooth p-forms on $S_n(K^f)$ and by $\mathcal{C}^\infty$ the sheaf of smooth $\mathbb{C}$-valued functions on $S_n(K^f)$. In [Har08b], section 4.10, the de Rham complex is constructed

$$0 \to \mathcal{V}_\mathbb{C} \to \mathcal{V}_{\mathbb{C},\infty} \to \mathcal{V}_{\mathbb{C},\infty} \otimes_{\mathcal{C}^\infty} \Omega^1 \to \mathcal{V}_{\mathbb{C},\infty} \otimes_{\mathcal{C}^\infty} \Omega^2 \to \ldots$$

and it is shown that this is an acyclic resolution of the sheaf $\mathcal{V}_\mathbb{C}$. Let us abbreviate the sheaves $\mathcal{V}_{\mathbb{C},\infty} \otimes_{\mathcal{C}^\infty} \Omega^p$ to $\Omega^p_\infty(\mathcal{V})$ and for $p = 0$ we set $\Omega^0_\infty(\mathcal{V}) = \mathcal{V}_{\mathbb{C},\infty}$. We can now formulate the following Theorem.

Theorem 2.1 (de Rham)**.** *We have an isomorphism*

$$H^i(S_n(K^f), \mathcal{V}_\mathbb{C}) = H^i(\Omega^\bullet_\infty(\mathcal{V})(S_n(K^f))),$$

where $\Omega^\bullet_\infty(\mathcal{V})(S_n(K^f))$ as usual denotes the set of global sections of the sheaf $\Omega^\bullet_\infty(\mathcal{V})$. There is an analogous statement for the cohomology with compact supports

$$H^i_c(S_n(K^f), \mathcal{V}_\mathbb{C}) = H^i(\Omega^\bullet_\infty(\mathcal{V})_c(S_n(K^f))),$$

where $\Omega^\bullet_\infty(\mathcal{V})_c(S_n(K^f))$ denotes the subset of sections in $\Omega^\bullet_\infty(\mathcal{V})(S_n(K^f))$ which have compact support.

Actually, one can compute the cohomology groups $H^i(M,\mathcal{F})$ via the de Rham cohomology for any manifold M and any local system $\mathcal{F}$ of finite-dimensional $\mathbb{C}$-vector spaces. In particular, we have not made use of the fact that the sheaf $\mathcal{V}_{\mathbb{C}}$ comes from a rational representation (ρ, V). Though, this will be crucial for the next considerations, we want to explain that, in our special situation, the de Rham cohomology groups are canonically isomorphic to the relative Lie algebra cohomology groups still to be defined.
Inside the Lie algebra gl_n we have the Lie algebra $\mathrm{k} := \mathrm{so}_n\mathrm{Lie}(\mathrm{Z}_n^0(\mathbb{R}))$ of the group $K^\infty := K_{n,\infty}Z_n^0(\mathbb{R})$. As usual, we denote by Ad the adjoint action of $\mathrm{GL}_n(\mathbb{R})$ on the Lie algebra gl_n. Following Harder, cf. section 3.1.2 in [Har08a], we will call a $\mathbb{C}$-vector space W which is acted on by the Lie algebra gl_n and the group K^∞ a generalized $(\mathrm{gl}_n, K^\infty)$-module, if the action of K^∞ is differentiable (but not necessarily locally finite) and the actions of gl_n and K^∞ satisfy the usual compatibility conditions, i.e. the action of gl_n restricted to k is the derivative of the action of K^∞ and we have

$$(\mathrm{Ad}(k)X)w = k(X(k^{-1}w)$$

for $k \in K^\infty$, $X \in \mathrm{gl}_n$ and $w \in W$. The adjoint action Ad of $\mathrm{GL}_n(\mathbb{R})$ on gl_n induces an action of K^∞ on the space $\Lambda^i(\mathrm{gl}_n/\mathrm{k})$. Thus, for a generalized $(\mathrm{gl}_n, K^\infty)$-module W we can consider the space of K^∞-invariant homomorphisms from $\Lambda^i(\mathrm{gl}_n/\mathrm{k})$ to W denoted by $\mathrm{Hom}_{K^\infty}(\Lambda^i(\mathrm{gl}_n/\mathrm{k}), W)$. We note that the images of these homomorphisms are pointwise k-invariant subsets of W. We have the following complex

$$0 \to W \to \mathrm{Hom}_{K^\infty}(\Lambda^1(\mathrm{gl}_n/\mathrm{k}), W) \to \mathrm{Hom}_{K^\infty}(\Lambda^2(\mathrm{gl}_n/\mathrm{k}), W) \to \dots, \tag{2.6}$$

where the differential is given by

$$d\omega(X_0, \dots X_q) = \sum_{i=0}^{q}(-1)^i X_i\,(\omega(X_0, \dots, \hat{X}_i, \dots X_q)) + \sum_{0\le i<j\le q}(-1)^{i+j}\omega([X_i, X_j], X_0, \dots \hat{X}_i, \dots \hat{X}_j \dots X_q)$$

for $\omega \in \mathrm{Hom}_{K^\infty}(\Lambda^q(\mathrm{gl}_n/\mathrm{k}), W)$ and $X_0, \dots X_q \in \mathrm{gl}_n/\mathrm{k}$. As usual, the "hat" over the i-th entry means that we skip the i-th entry.
We abbreviate the complex defined in (2.6) to $\mathrm{Hom}_{K^\infty}(\Lambda^\bullet(\mathrm{gl}_n/\mathrm{k}), W)$ and define the $(\mathrm{gl}_n, K^\infty)$-cohomology or relative Lie algebra cohomology as the cohomology of this complex

$$H^i(\mathrm{gl}_n, K^\infty, W) = H^i(\mathrm{Hom}_{K^\infty}(\Lambda^\bullet(\mathrm{gl}_n/\mathrm{k}), W)).$$

Next, we establish the connection to the de Rham cohomology. Let us consider the space $\mathcal{C}_\infty(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f)$ of smooth complex-valued functions f on $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f$. Here f is called smooth if $f(\ , g_f) : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}$ is smooth for all $g_f \in \mathrm{GL}_n(\mathbb{A}_f)$. On this space of smooth functions the group $\mathrm{GL}_n(\mathbb{A})$ acts by right translation, in particular it is a generalized $(\mathrm{gl}_n, K_{n,\infty}Z_n^0(\mathbb{R}))$-module. There is a canonical isomorphism of complexes

$$\begin{aligned}\mathrm{Hom}_{K^\infty}(\Lambda^\bullet(\mathrm{gl}_n/\mathrm{k}), \mathcal{C}_\infty(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f) \otimes V_{\mathbb{C}}) &\longrightarrow \Omega^\bullet_\infty(\mathcal{V})(S_n(K^f)),\\ \omega &\longmapsto \widetilde{\omega},\end{aligned}$$

cf. [Har08a], chapter III, section 3.1.2.
Let us describe shortly how a given element ω on the left hand side gives rise to a differential form $\widetilde{\omega}$. Denote by x_0 the image of the identity $e \in \mathrm{GL}_n(\mathbb{R})$ under the projection $\mathrm{GL}_n(\mathbb{R}) \to$

$X = \mathrm{GL}_n(\mathbb{R})/K^\infty$ and analogously we define $e_f \in \mathrm{GL}_n(\mathbb{A}_f)/K^f$ as the image of the identity in $\mathrm{GL}_n(\mathbb{A}_f)$. We identify the tangent space of $X \times \mathrm{GL}_n(\mathbb{A}_f)/K^f$ in $x_0 \times e_f$ with gl_n/k.
As before we denote by π the projection $\pi : \mathrm{GL}_n(\mathbb{A})/K^\infty K^f \to S_n(K^f)$ and we write D_π for its derivative. For any element $x \times h_f \in X \times \mathrm{GL}_n(\mathbb{A}_f)/K^f$ and any p-tuple $(Y_0, \dots Y_{p-1})$ of tangent vectors at $x \times h_f$, we have to define $\widetilde{\omega}(D_\pi(Y_0), \dots D_\pi(Y_{p-1})) \in \mathcal{V}_{\mathbb{C},x\times h_f}$. Here $\mathcal{V}_{\mathbb{C},x\times h_f}$ denotes the stalk of the sheaf $\mathcal{V}_\mathbb{C}$ in $\pi(x \times h_f)$, explicitly

$$\mathcal{V}_{\mathbb{C},x\times h_f} = \{f : \pi^{-1}(\pi(x \times h_f)) \to V_\mathbb{C} \mid f(\gamma(x \times h_f)) = \gamma f(x \times h_f),\ \gamma \in \mathrm{GL}_n(\mathbb{Q})\}.$$

We can find an element $(g_\infty, g_f) \in \mathrm{GL}_n(\mathbb{A})$ such that $(g_\infty, g_f)(x_0 \times e_f) = x \times h_f$. The derivative $D_{(g_\infty^{-1}, g_f^{-1})}$ of the left translation by $(g_\infty^{-1}, g_f^{-1})$ maps $(Y_0, \dots Y_{p-1})$ to a p-tuple of tangent vectors $(X_0, \dots X_{p-1})$ at $x_0 \times e_f$. Now we can define the element $\widetilde{\omega}$ as follows

$$\widetilde{\omega}(D_\pi(Y_0), \dots D_\pi(Y_{p-1}))(x \times h_f) = g_\infty^{-1}\omega(X_0, \dots X_{p-1})(g_\infty, g_f).$$

We obtain the following result.

Theorem 2.2. *We have an isomorphism*

$$H^i(S_n(K^f), \mathcal{V}_\mathbb{C}) = H^i(\mathrm{gl}_n, K^\infty, \mathcal{C}_\infty(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f) \otimes V_\mathbb{C})$$

and similarly for the cohomology with compact supports

$$H^i_c(S_n(K^f), \mathcal{V}_\mathbb{C}) = H^i(\mathrm{gl}_n, K^\infty, \mathcal{C}_{\infty,c}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f) \otimes V_\mathbb{C}),$$

where $\mathcal{C}_{\infty,c}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f)$ denotes the set of functions in $\mathcal{C}_\infty(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f)$ with compact supports.

Passing to the limit we get an isomorphism of $\mathrm{GL}_n(\mathbb{A}_f)$-modules

$$H^i(\widetilde{S}_n, \mathcal{V}_\mathbb{C}) = H^i(\mathrm{gl}_n, K^\infty, \mathcal{C}_\infty(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})) \otimes V_\mathbb{C}),$$

where $\mathcal{C}_\infty(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}))$ consists of all functions f on $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})$ which are smooth as functions $f(\ , g_f) : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}$ for every $g_f \in \mathrm{GL}_n(\mathbb{A}_f)$ and are right invariant under the transformation of a suitably small open compact subgroup $K' \subset \mathrm{GL}_n(\mathbb{A}_f)$. Again an analogous statement is true for the cohomology with compact supports.

Remark 2.3. *For $n = 2$ the results in this section can be directly adapted to the cohomology of the boundary. The de Rham theorem for the cohomology of the boundary reads as follows*

$$H^i(\partial\bar{S}_2(K^f), \mathcal{V}_\mathbb{C}) = H^i(\Omega^\bullet_\infty(\mathcal{V})(\mathrm{B}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K^\infty K^f)),$$

where now $\Omega^j_\infty(\mathcal{V})$ is a sheaf on $\mathrm{B}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K^\infty K^f$. Again, the de Rham complex can be interpreted as a complex attached to relative Lie algebra cohomology and we obtain

$$H^i(\partial\bar{S}_2(K^f), \mathcal{V}_\mathbb{C}) = H^i(\mathrm{gl}_2, K^\infty, \mathcal{C}_\infty(\mathrm{B}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K^f) \otimes V_\mathbb{C}).$$

Similar statements hold for the inductive limit of cohomology groups.

2.4. Cuspidal Cohomology

As in the last section we write K^∞ for the subgroup $K_{n,\infty}Z_n^0(\mathbb{R})$ of $\mathrm{GL}_n(\mathbb{R})$ and k for its Lie algebra.
Let W be an irreducible admissible $(\mathrm{gl}_n, K^\infty)$-module. By a standard argument, the center $\mathfrak{z}(\mathrm{gl}_n)$ of the universal enveloping algebra $\mathfrak{U}(\mathrm{gl}_n)$ acts on W by scalars, i.e. there exists a homomorphism $\chi_W : \mathfrak{z}(\mathrm{gl}_n) \to \mathbb{C}^*$, such that $zw = \chi_W(z)w$ for all $z \in \mathfrak{z}(\mathrm{gl}_n)$ and $w \in W$, cf. [Har87c] chapter III, section 3.7.11.2. This homomorphism is called the infinitesimal character of the $(\mathrm{gl}_n, K^\infty)$-module W. Of course, if (ρ, V) is a rational representation we can consider $V_\mathbb{C}$ as a $(\mathrm{gl}_n, K^\infty)$-module. The rational representation (ρ, V) is called absolutely irreducible if it is irreducible over $\mathbb{C}$. For a proof of the following Lemma we refer to [BW00], chapter I, Theorem 5.3 or [Har08a], chapter III, page 24.

Lemma 2.4 (Wigner's Lemma)**.** *Let W be an irreducible admissible $(\mathrm{gl}_n, K^\infty)$-module and let (ρ, V) be an absolutely irreducible rational representation. Then the non-vanishing of the relative Lie algebra cohomology $H^\bullet(\mathrm{gl}_n, K^\infty, W \otimes V_\mathbb{C})$ implies that the infinitesimal character χ_W of W and the infinitesimal character $\chi_{\check{V}_\mathbb{C}}$ of the contragredient representation $\check{V}_\mathbb{C}$ of $V_\mathbb{C}$ coincide, i.e. $\chi_W(z) = \chi_{\check{V}_\mathbb{C}}(z)$ for all $z \in \mathfrak{z}(\mathrm{gl}_n)$.*

Let now (ρ, V) be an absolutely irreducible rational representation with central character $\omega : Z(\mathrm{GL}_n(\mathbb{C})) \cong \mathbb{C}^* \to \mathbb{C}^*$. Via the unique decomposition $\mathbb{A}^* = \mathbb{R}^*_{>0}\mathbb{Q}^*\hat{\mathbb{Z}}^*$ we may extend ω to an idèle class character $\bar{\omega}$ by requiring

$$\bar{\omega}(trk) = \omega(t)$$

for $t \in \mathbb{R}^*_{>0}$, $r \in \mathbb{Q}^*$ and $k \in \hat{\mathbb{Z}}^*$. (More explicitly, one can construct the character $\bar{\omega}$ as follows: The character $\omega|_{\mathbb{R}^*_{>0}}$ is of the form $|\cdot|^z$ with $z \in \mathbb{C}$. The idéle class character $\bar{\omega}$ is then equal to $|\cdot|^z$ where now $|\cdot|$ means the absolute value on $\mathbb{A}^*$.) We define the Hilbert space

$$L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})$$

as the set of all functions f on $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f$ which transform under the connected component $Z^0(\mathbb{R})$ of the center of $\mathrm{GL}_n(\mathbb{R})$ by ω^{-1}, i.e.

$$f(zg) = \omega^{-1}(z)f(g), \qquad z \in \mathbb{R}^*,\ g \in \mathrm{GL}_n(\mathbb{A}),$$

and satisfy

$$\int\limits_{Z^0(\mathbb{R})\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})} |f(g)|^2\ |\bar{\omega}(\det g)|^{\frac{2}{n}} dg\ < \infty.$$

(Here, $|z|^{1/n}$ with $z \in \mathbb{C}$ denotes the positive real n-th root of $|z|$.) Right translation induces a natural $\mathrm{GL}_n(\mathbb{R}) \times \mathcal{H}_{K^f}$-action on the space $L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})$, where $\mathcal{H}_{K^f}$ denotes the Hecke algebra consisting of the compactly supported, K^f-invariant, complex-valued functions on $\mathrm{GL}_n(\mathbb{A}_f)$. With this action, $L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})$ is a quasi-unitary $\mathrm{GL}_n(\mathbb{R})$-module, i.e. there exists a character ν on $\mathbb{R}^*$, such that the action of $\mathrm{GL}_n(\mathbb{R})$ twisted by the character ν is unitary, (in our situation: take the character $|\omega|^{\frac{1}{n}}$). The discrete part $L^2_{\mathrm{disc}}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})$ is defined as the closure of the sum of all irreducible closed $\mathrm{GL}_n(\mathbb{R}) \times \mathcal{H}_{K^f}$-invariant subspaces in $L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})$. We have

$$L^2_{\mathrm{disc}}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1}) = \overline{\bigoplus_\pi L^2_{\mathrm{disc}}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})(\pi)},$$

where $\pi = \pi_\infty \otimes \pi_f$ runs over the irreducible quasi-unitary $\mathrm{GL}_n(\mathbb{R}) \times \mathcal{H}_{K^f}$-modules and as usual $L^2_{\mathrm{disc}}(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})(\pi)$ denotes the isotypical component corresponding to π.
In the following, we denote by $\widehat{\mathrm{GL}_n(\mathbb{R})}$ the set of all quasi-unitary irreducible representations of $\mathrm{GL}_n(\mathbb{R})$. It is explained in [Har08a], chapter III, section 3.2, that every quasi-unitary $\mathrm{GL}_n(\mathbb{R})$-module $(\pi_\infty, H_{\pi_\infty}) \in \widehat{\mathrm{GL}_n(\mathbb{R})}$ has finite multiplicity in $L^2_{\mathrm{disc}}(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})$, i.e. the space

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{R})}(H_{\pi_\infty}, L^2_{\mathrm{disc}}(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1}))$$

is finite-dimensional. We set

$$\mathrm{Coh}(V) = \left\{ \pi_\infty \in \widehat{\mathrm{GL}_n(\mathbb{R})} \;\middle|\; H^\bullet(\mathrm{gl}_n, K^\infty, \pi_\infty^{K^\infty} \otimes V_{\mathbb{C}}) \neq 0 \right\},$$

where $\pi_\infty^{K^\infty}$ denotes the $(\mathrm{gl}_n, K^\infty)$-module of K^∞-finite vectors in π_∞. It is well known that since the representation $(\pi_\infty, H_{\pi_\infty})$ is quasi-unitary, $\pi_\infty^{K^\infty}$ is an admissible $(\mathrm{gl}_n, K^\infty)$-module. Thus, the Lemma of Wigner implies that for all $\pi_\infty \in \mathrm{Coh}(V)$ the infinitesimal character $\chi_{\pi_\infty^{K^\infty}}$ of the $(\mathrm{gl}_n, K^\infty)$-module $\pi_\infty^{K^\infty}$ coincides with the infinitesimal character $\chi_{\check{V}_{\mathbb{C}}}$ of the representation $\check{V}_{\mathbb{C}}$. On the other hand, we know by a fundamental result of Harish-Chandra that for a given character $\chi : \mathfrak{z}(\mathrm{gl}_n) \to \mathbb{C}^*$ there are up to isomorphy only finitely many irreducible admissible $(\mathrm{gl}_n, K^\infty)$-modules with infinitesimal character χ. Consequently, $\mathrm{Coh}(V)$ is a finite set. Putting all this together, we find that

$$H_{\mathrm{Coh}(V)} := \bigoplus_{\pi : \pi_\infty \in \mathrm{Coh}(V)} L^2_{\mathrm{disc}}(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})(\pi), \tag{2.7}$$

where π runs over all unitary irreducible $\mathrm{GL}_n(\mathbb{R}) \times \mathcal{H}_{K^f}$-modules, is a finite sum of irreducible modules. A function $f \in L^2_{\mathrm{disc}}(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})$ is called a cusp form if for all proper parabolic subgroups $P \subset \mathrm{GL}_n$ with unipotent radical N the integral

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ug)du$$

is defined for almost all $g \in \mathrm{GL}_n(\mathbb{A})$ and equals zero for almost all g. The space of cusp forms, denoted by $L^2_{\mathrm{cusp}}(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})$, is a $\mathrm{GL}_n(\mathbb{R}) \times \mathcal{H}_{K^f}$-submodule. Analogously to definition (2.7), we set

$$H_{\mathrm{Coh}(V),\mathrm{cusp}} := \bigoplus_{\pi : \pi_\infty \in \mathrm{Coh}(V)} L^2_{\mathrm{cusp}}(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f, \omega^{-1})(\pi).$$

Of course, we have $H_{\mathrm{Coh}(V),\mathrm{cusp}} \subset H_{\mathrm{Coh}(V)}$ and in particular $H_{\mathrm{Coh}(V),\mathrm{cusp}}$ is a finite sum of irreducible modules. We have an inclusion $H^{K^\infty}_{\mathrm{Coh}(V)} \hookrightarrow \mathcal{C}_\infty(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K^f)$ and this induces a map in relative Lie algebra cohomology

$$I_d : H^\bullet(\mathrm{gl}_n, K^\infty, H^{K^\infty}_{\mathrm{Coh}(V)} \otimes V_{\mathbb{C}}) \longrightarrow H^\bullet(S_n(K^f), \mathcal{V}_{\mathbb{C}}),$$

cf. Theorem 2.2. By a result of Borel, cf. [Har08a] chapter III, page 30, the restriction of the map I_d to

$$I_{d,0} : H^\bullet(\mathrm{gl}_n, K^\infty, H^{K^\infty}_{\mathrm{Coh}(V),\mathrm{cusp}} \otimes V_{\mathbb{C}}) \longrightarrow H^\bullet(S_n(K^f), \mathcal{V}_{\mathbb{C}}) \tag{2.8}$$

is injective. Its image is denoted by $H^\bullet_{\mathrm{cusp}}(S_n(K^f), \mathcal{V}_{\mathbb{C}})$ and is called cuspidal cohomology. We also obtain a subspace $H^\bullet_{\mathrm{cusp}}(\widetilde{S}_n, \mathcal{V}_{\mathbb{C}})$ in the direct limit $H^\bullet(\widetilde{S}_n, \mathcal{V}_{\mathbb{C}})$. According to [Clo90], page 123, the cuspidal cohomology also defines a subspace in the cohomology with compact supports $H^2_{\mathrm{cusp}}(\widetilde{S}_n, \mathcal{V}_{\mathbb{C}}) \leq H^2_c(\widetilde{S}_n, \mathcal{V}_{\mathbb{C}})$.

2.5. Eisenstein Cohomology

Using Eisenstein series Harder has constructed a section to the restriction map

$$\mathrm{res} : H^1(\widetilde{S}_2, \mathcal{V}_{\mathbb{C}}) \longrightarrow H^1(\partial\widetilde{S}_2, \mathcal{V}_{\mathbb{C}}).$$

To describe this section we first have to give a description of the cohomology of the boundary, cf. Theorem 1 in [Har87b]. Let $M = M(d,\nu)$ be the rational representation of GL_2 of dimension $d+1$ and with central character $x \mapsto x^{d+2\nu}$. For an explicit realization of this representation as a space of homogeneous polynomials we refer to the beginning of chapter 4 below.

Theorem 2.5. *As a $\mathrm{GL}_2(\mathbb{A}_f)$- module the cohomology of the boundary is given by*

$$H^i(\partial\widetilde{S}_2, \mathcal{M}_{\mathbb{C}}) = \begin{cases} \bigoplus\limits_{\phi\in S_0} \mathrm{ind}_{B_2(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)} \phi_f & \text{for } i=0, \\ \bigoplus\limits_{\phi\in S_1} \mathrm{ind}_{B_2(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)} \phi_f & \text{for } i=1, \end{cases}$$

where S_0 resp. S_1 denotes the set of continuous homomorphisms $\phi = \phi_\infty\phi_f : T_2(\mathbb{Q})\backslash T_2(\mathbb{A}) \to \mathbb{C}^*$ whose infinity components ϕ_∞ satisfy

$$\phi_\infty \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = t_1^{-d}(t_1t_2)^{-\nu} \qquad \text{for } t_1, t_2 \in \mathbb{R}^* \text{ with } t_1t_2 > 0 \tag{2.9}$$

resp.

$$\phi_\infty \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = t_1^{1-\nu}t_2^{-d-\nu-1} \qquad \text{for } t_1, t_2 \in \mathbb{R}^* \text{ with } t_1t_2 > 0. \tag{2.10}$$

For $i > 1$ the cohomology of the boundary $H^i(\partial\widetilde{S}_2, \mathcal{M}_{\mathbb{C}})$ is trivial.

Remark 2.6. *The reason why Theorem 1 in [Har87b] looks more complicated than our formulation is that it treats the general case, i.e. GL_2 over an arbitrary number field. For instance, the vector space $\mathscr{H}(T/Z)$ occurring in Theorem 1 in [Har87b] is trivial in our situation, i.e. for GL_2 over $\mathbb{Q}$. At first glance also the ranges of summation look different: The sum in [Har87b] runs over all characters $\phi = \phi_\infty\phi_f : T_2(\mathbb{Q})\backslash T_2(\mathbb{A}) \to \mathbb{C}^*$ such that their infinity components ϕ_∞ fulfill equation (2.9) or equation (2.10) for $t_1 > 0$ and $t_2 > 0$ and such that $\widetilde{\phi}_f$ is trivial on the subset $\bar{Z} := \{\pm 1\}$ of $T_2(\mathbb{R})$. Looking into the definition of $\widetilde{\phi}_f$ in [Har87b], section 2.5.4, we find that $\widetilde{\phi}_f$ is trivial on $\bar{Z}$ if and only if ϕ_∞ restricted to $\bar{Z}$ also fulfills equation (2.9) resp. equation (2.10). This explains why the range of summation in the Theorem above coincides with the range of summation in Theorem 1 in [Har87b].*

In the Theorem above, "ind" denotes non-unitary induction, i.e. $\mathrm{ind}_{B_2(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)} \phi_f$ can be realized as the set of functions f on $\mathrm{GL}_2(\mathbb{A}_f)$ which satisfy $f(bg) = \phi_f(b)f(g)$ for $b \in B_2(\mathbb{A}_f)$ and $g \in \mathrm{GL}_2(\mathbb{A}_f)$ and which are right invariant under some compact open subgroup $K' \subset \mathrm{GL}_2(\mathbb{A}_f)$. Similarly, we denote by $\mathrm{ind}_{B_2(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\phi_\infty$ the $K_{2,\infty}Z_2^0(\mathbb{R})$-finite non-unitarily induced representation. We will abbreviate $\mathrm{ind}_{B_2(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}\phi_\infty$ to V_{ϕ_∞}, $\mathrm{ind}_{B_2(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)}\phi_f$ to V_{ϕ_f} and $V_{\phi_f} \otimes V_{\phi_\infty}$ to V_ϕ.
Let now be $i \in \{0,1\}$ and fix a character ϕ in S_i. Realizing the cohomology of the boundary as relative Lie algebra cohomology, cf. Remark 2.3, we can describe the embedding $\mathrm{ind}_{B_2(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)}\phi_f \hookrightarrow H^i(\partial\widetilde{S}_2, \mathcal{M}_{\mathbb{C}})$ explicitly. On page 69 in [Har87b] it is shown that

$$H^q(\mathrm{gl}_2, K_{2,\infty}Z_2^0(\mathbb{R}), V_{\phi_\infty} \otimes M_{\mathbb{C}}) = \begin{cases} \mathbb{C} & \text{if } q = i \text{ or } q = i+1, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. the $(\mathrm{gl}_2, K_{2,\infty}Z_2^0(\mathbb{R}))$-module $V_{\phi_\infty} \otimes M_{\mathbb{C}}$ has nontrivial cohomology in the degrees 0 and 1 if ϕ lies in S_0 and if ϕ lies in S_1 its cohomology is nontrivial for the degrees 1 and 2. Let e be a generator of $H^i(\mathrm{gl}_2, K_{2,\infty}, V_{\phi_\infty} \otimes M_{\mathbb{C}})$. An embedding of V_{ϕ_f} into the cohomology of the boundary is then given by

$$\begin{aligned} V_{\phi_f} &\longhookrightarrow H^i(\partial\widetilde{S}_2, \mathcal{M}_{\mathbb{C}}) = H^i(\mathrm{gl}_2, K_{2,\infty}Z_2^0(\mathbb{R}), \mathcal{C}_\infty(\mathrm{B}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})) \otimes M_{\mathbb{C}}), \\ \varphi_f &\longmapsto e \otimes \varphi_f, \end{aligned} \tag{2.11}$$

cf. pages 79 and 80 in [Har87b].

Before we can cite Theorem 2 in [Har87b] which describes the image of the restriction map res in the cohomology of the boundary we have to introduce some more notation.
For any character ϕ on $T(\mathbb{Q})\backslash T(\mathbb{A})$ we denote by ϕ^1 its restriction to the subtorus $T^1(\mathbb{A}) \cong \mathbb{A}^*$ consisting of the elements in $T(\mathbb{A})$ with determinant one. If ϕ is a character with $\phi^1 = |\cdot|^2$, then every local factor $V_{\phi,\ell}$ of $V_{\phi_f} = \otimes'_{\ell<\infty} V_{\phi,\ell}$ contains a unique irreducible invariant subspace of codimension one. We denote by $\widetilde{V}_{\phi_f}$ the subspace in V_{ϕ_f} which is generated by the pure tensors $\otimes'\phi_\ell$ in $\otimes' V_{\phi,\ell}$ which have at least one component ϕ_ℓ in the unique irreducible subspace of $V_{\phi,\ell}$. The space $\widetilde{V}_{\phi_f}$ has codimension one in V_{ϕ_f}. If ϕ is a character with $\phi^1 = 1$, then every local factor $V_{\phi,\ell}$ of V_{ϕ_f} contains a unique invariant one-dimensional subspace generated by some element e_{ϕ_ℓ} and we set $e_{\phi_f} := \otimes' e_{\phi_\ell} \in V_{\phi_f}$. In our situation Theorem 2 in [Har87b] simplifies as follows.

Theorem 2.7. *If the dimension of M is bigger than 1, then the restriction map*

$$\mathrm{res} : H^i(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}}) \longrightarrow H^i(\partial\widetilde{S}_2, \mathcal{M}_{\mathbb{C}})$$

is surjective for $i = 1$ and trivial for $i = 0$.
Let now M be a rational representation of dimension one. Then the image of the restriction map is given by

$$\mathrm{res}(H^1(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}})) = \bigoplus_{\substack{\phi\in S_1,\\ \phi^1\neq|\cdot|^2}} V_{\phi_f} \oplus \bigoplus_{\substack{\phi\in S_1,\\ \phi^1=|\cdot|^2}} \widetilde{V}_{\phi_f}.$$

in degree one and by

$$\mathrm{res}(H^0(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}})) = \bigoplus_{\substack{\phi\in S_0,\\ \phi^1=1}} \mathbb{C}e_{\phi_f},$$

in degree zero.

The main idea in the proof of Theorem 2 in [Har87b] is to construct a map from the image of res to the general cohomology via Eisenstein series. Since we will make use of this map later we want to explain this in more detail.
Let us denote by β the character on $T(\mathbb{A})$ given by $\beta\left(\begin{smallmatrix} t_1 & \\ & t_2 \end{smallmatrix}\right) = |t_1/t_2|$. For s large enough the Eisenstein operator introduced in section 1.3,

$$\mathrm{Eis} : V_{\beta^s\phi} \longrightarrow \mathcal{A}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})),$$

is defined as the absolutely convergent series (1.3) and Theorem 1.16 assures that it can be continued meromorphically to all $s \in \mathbb{C}$.
Let ω' be an element in $\mathrm{res}(H^1(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}}))$ which in the description of Theorem 2.7 lies in a

summand V_{ϕ_f} or $\widetilde{V}_{\phi_f}$ for some character ϕ in S_1. Let ω be a representative of ω' in relative Lie algebra cohomology, i.e. $\omega \in \mathrm{Hom}_{K^\infty}(\Lambda^1(\mathrm{gl}_2/\mathrm{k}), V_\phi \otimes M_{\mathbb{C}})$, where as before $K^\infty = \mathrm{SO}_2(\mathbb{R})Z_2^0(\mathbb{R})$ and $\mathrm{k} = \mathrm{Lie}(K^\infty)$. In the proof of Theorem 2 in [Har87b] it is shown that the mapping

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathrm{Hom}_{K^\infty}(\Lambda^1(\mathrm{gl}_2/\mathrm{k}), \mathcal{C}_\infty(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})) \otimes M_{\mathbb{C}}), \\ s &\longmapsto \mathrm{Eis}(w_s), \end{aligned}$$

where $\{w_s\}_s$ is a carefully constructed family of elements $\omega_s \in \mathrm{Hom}_{K^\infty}(\Lambda^1(\mathrm{gl}_2/\mathrm{k}), V_{\beta^s\phi} \otimes M_{\mathbb{C}})$ with $\omega_0 = \omega$, is holomorphic at $s = 0$ and yields a closed form $\mathrm{Eis}(\omega) := \mathrm{Eis}(\omega_0)$. (This is true even in the case where $\phi = |\cdot|^2$ and Eis is not holomorphic at $s = 0$.) Hence, we obtain a mapping

$$\widetilde{\mathrm{Eis}} : \mathrm{res}(H^1(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}})) \longrightarrow H^1(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}}) \tag{2.12}$$

and in the proof of Theorem 2 in [Har87b] it is shown that $\widetilde{\mathrm{Eis}}$ is actually a section to res. The image of $\widetilde{\mathrm{Eis}}$ is called the Eisenstein cohomology and is denoted by $H^1_{\mathrm{Eis}}(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}})$. Thus, we have

$$H^1(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}}) = \widetilde{H}^1(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}}) \oplus H^1_{\mathrm{Eis}}(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}}),$$

where $\widetilde{H}^1(\widetilde{S}_2, \mathcal{M}_{\mathbb{C}})$ denotes the kernel of the restriction map res. Consulting the exact sequence (2.5) we see that the kernel of the restriction map coincides with the inner cohomology, i.e. the image of the cohomology with compact supports in the general cohomology.

2.6. Cup Product

In this section we explain how a pairing of sheaves gives rise to a pairing in cohomology. Let X and Y be topological spaces and let $\mathcal{F}$ resp. $\mathcal{G}$ be a sheaf of R-modules on X resp. Y for a commutative ring R. We define the exterior tensor product $\mathcal{F}\hat{\otimes}_{\underline{R}}\mathcal{G}$ of the sheaves $\mathcal{F}$ and $\mathcal{G}$ as the sheaf $p_1^*(\mathcal{F}) \otimes_{\underline{R}} p_2^*(\mathcal{G})$, where p_1 resp. p_2 denotes the projection of $X \times Y$ to X resp. Y. Thus, the exterior tensor product is a sheaf on $X \times Y$. If $\mathcal{F}$ or $\mathcal{G}$ is flat (i.e. every stalk is a flat R-module) and admits a flat acyclic resolution, then via spectral sequences one can construct a homomorphism in cohomology

$$\bigoplus_{i+j=n} H^i(X, \mathcal{F}) \otimes H^j(Y, \mathcal{G}) \longrightarrow H^n(X \times Y, \mathcal{F}\hat{\otimes}_{\underline{R}}\mathcal{G}),$$

see for instance [Har08b], section 4.6.7.
Let us now consider the case $X = Y$. Then the diagonal embedding $\Delta : X \to X \times X$ induces a mapping $\Delta^* : H^n(X \times X, \mathcal{F}\hat{\otimes}_{\underline{R}}\mathcal{G}) \to H^n(X, \mathcal{F} \otimes_{\underline{R}} \mathcal{G})$ and combining Δ^* with the mapping above yields a homomorphism

$$\bigoplus_{i+j=n} H^i(X, \mathcal{F}) \otimes H^j(X, \mathcal{G}) \longrightarrow H^n(X, \mathcal{F} \otimes_{\underline{R}} \mathcal{G}). \tag{2.13}$$

Let now M be a manifold of dimension n and let $\mathcal{F}$ and $\mathcal{G}$ be local systems of free R-modules of finite rank on M. We assume that M can be embedded into a oriented compact manifold $\overline{M}$ such that the embedding $i : M \hookrightarrow \overline{M}$ is a homotopy equivalence and $i(M)$ is the interior of $\overline{M}$. Then one has

$$H^i(M, \mathcal{F}) = H^i(\overline{M}, i_*(\mathcal{F})).$$

Similarly, we have according to Proposition 4.7.1 in [Har08b]

$$H_c^i(M,\mathcal{G}) = H^i(\overline{M}, i_!(\mathcal{G})),$$

where $i_!(\mathcal{G})$ denotes the sheaf on $\overline{M}$ obtained from $\mathcal{G}$ by extension by zero. Thus, the mapping (2.13) provides us with a homomorphism

$$H^i(M,\mathcal{F}) \otimes H_c^{n-i}(M,\mathcal{G}) \longrightarrow H^n(\overline{M}, i_*(\mathcal{F}) \otimes_{\underline{R}} i_!(\mathcal{G})). \tag{2.14}$$

We are interested in the case, where we have a pairing of sheaves

$$\mathrm{tr} : \mathcal{F} \otimes_{\underline{R}} \mathcal{G} \longrightarrow \underline{R}. \tag{2.15}$$

First we want to explain that this pairing induces a homomorphism $i_*(\mathcal{F}) \otimes_{\underline{R}} i_!(\mathcal{G}) \to i_!(\underline{R})$ which we also denote by tr. Due to the universal property of sheafification it suffices to construct a morphism of presheaves $\mathcal{B} \to i_!(\underline{R})$ for a presheaf $\mathcal{B}$ corresponding to the sheaf $i_*(\mathcal{F}) \otimes_{\underline{R}} i_!(\mathcal{G})$. Consider the presheaf $\overline{i_!(\mathcal{G})}$ on $\overline{M}$ defined by

$$\overline{i_!(\mathcal{G})}(U) = \begin{cases} \mathcal{G}(U) & \text{if } U \subset M, \\ 0 & \text{else} \end{cases}$$

for U an open subset of $\overline{M}$. One can show that the sheafification of $\overline{i_!(\mathcal{G})}$ coincides with $i_!(\mathcal{G})$. Since sheafification is compatible with the tensor product it suffices to construct a morphism tr from the tensor product of presheaves $i_*(\mathcal{F}) \otimes_{\underline{R}} \overline{i_!(\mathcal{G})}$ to $i_!(\underline{R})$. It is clear how to define this morphism using the pairing of sheaves (2.15), because if U is an open subset of $\overline{M}$, then

$$(i_*(\mathcal{F}) \otimes_{\underline{R}} \overline{i_!(\mathcal{G})})(U) = i_*(\mathcal{F})(U) \otimes_R \overline{i_!(\mathcal{G})}(U)$$

is equal to $\mathcal{F}(U) \otimes_R \mathcal{G}(U)$ if U lies in M and is zero otherwise.
The pairing tr induces a mapping in cohomology

$$H^n(\overline{M}, i_*(\mathcal{F}) \otimes_{\underline{R}} i_!(\mathcal{G})) \longrightarrow H^n(\overline{M}, i_!(\underline{R})) = H_c^n(M, \underline{R}).$$

Combining this with the homomorphism in (2.14) and noting that $H_c^n(M,\underline{R})$ is isomorphic to R we finally obtain a pairing $\widetilde{\mathrm{tr}}$ in cohomology

$$\widetilde{\mathrm{tr}} : H^i(M,\mathcal{F}) \otimes H_c^{n-i}(M,\mathcal{G}) \longrightarrow R. \tag{2.16}$$

In the case where $\mathcal{G}$ is the dual local system $\check{\mathcal{F}} := \mathrm{Hom}(\mathcal{F}, \underline{R})$ of $\mathcal{F}$, this pairing is called the Poincaré-pairing, cf. [Har08b], section 4.8.7.
For the applications we have in mind it is crucial that the pairing

$$H^i(M,\mathcal{F})/\mathrm{Tors} \times H^{n-i}(M,\check{\mathcal{F}})/\mathrm{Tors} \longrightarrow R$$

is non-degenerate if R is a discrete valuation ring, cf. [Har08b], Theorem 4.8.9. Here non-degenerate means the following: For any element ζ which is not a proper multiple of another element one can find an element η such that the pairing between ζ and η equals one.

In the de Rham cohomology we can describe the pairing (2.16) explicitly, cf. [Har08b], Theorem 4.10.5. We keep the notations from above and assume that the manifold M is connected and that $R = \mathbb{C}$, i.e. the sheaves $\mathcal{F}$ and $\mathcal{G}$ are local systems of finite-dimensional $\mathbb{C}$-vector spaces.
Let $[\omega] \in H^i(M,\mathcal{F})$ and $[\eta] \in H_c^{n-i}(M,\mathcal{G})$ and let $\omega \in \Omega_\infty^i(\mathcal{F})$ resp. $\eta \in \Omega_\infty^{n-i}(\mathcal{G})$ be a differential form representing the class $[\omega]$ resp. $[\eta]$. (Here we make use of the notation introduced in section 2.3.) The exterior product $\omega \wedge \eta$ lies in $\mathcal{F}_\infty \otimes_{\underline{\mathbb{C}}} \mathcal{G}_\infty \otimes_{\underline{\mathbb{C}}} \Omega^n$, thus the element $\mathrm{tr}(\omega \wedge \eta)$ is a complex differential form on M. One can show that the pairing (2.16) is given by

$$[\omega] \times [\eta] \longmapsto \int_M \mathrm{tr}(\omega \wedge \eta).$$

3. Twisted Values of L-functions as Mellin Transforms

In this chapter we summarize the notations and results contained in chapters 1, 2 and 3 of [Mah00]. One of the main results here is the construction of a distribution μ whose p-adic Mellin transform evaluated at certain finite characters gives the twisted values of the L-function attached to some cuspidal representation π at some critical integer s_0 of π, i.e.

$$\int_{\mathbb{Z}_p{}^*} \chi_p \, d\mu = L(\pi \otimes \chi, s_0)$$

for idèle class characters $\chi = \prod_\ell \chi_\ell$ of finite order and with level a power of p.
Section 3.3 below will associate this distribution with a certain cohomological pairing.

3.1. Calculation of the Twisted Values of L-functions

The following notations will be valid in the remainder of this thesis.

- Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A})$ such that there exists an even natural number $l_0 > 0$ with

  $$\pi_\infty \cong \mathrm{Ind}_P^{\mathrm{GL}_3(\mathbb{R})}(D(l_0+1), \mathrm{id}),$$

  where $P \leqslant \mathrm{GL}_3(\mathbb{R})$ is the standard parabolic subgroup of type $(2,1)$ and $D(l_0+1)$ is the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest weight l_0+1 and with trivial central character.

- We fix two rational primes $p, q > 3$ such that π_p and π_q are unramified. According to Remark 1.13, there exist unramified characters μ_1, μ_2 and μ_3 on $\mathbb{Q}_p^*$, such that $\pi_p \cong \mathrm{Ind}(\mu_1, \mu_2, \mu_3)$.

- We choose three idèle class characters χ, η, and $\eta' : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ as follows. Let χ be of level $f = p^e$ and with infinity component $\chi_\infty = \mathbb{1}$. Let the two characters η and η' be of finite order and let them satisfy the following conditions
  - $f_\eta = pq$, $f_{\eta'} = p$,
  - $\eta_\infty = \eta'_\infty$,
  - $\eta_p|_{\mathbb{Z}_p^*} = \eta'_p|_{\mathbb{Z}_p^*}$.

  We set $\flat = 0$ if $\eta_\infty = \mathbb{1}$ and $\flat = 1$ if $\eta_\infty = \mathrm{sgn}$ and we define another idèle class character η_0 by $\eta_0 = \eta\eta'^{-1}$. There is a bijection between idèle class characters ζ of finite order and primitive Dirichlet characters $\widetilde{\zeta}$, cf. [Bum97], Proposition 3.1.2. We write $\widetilde{\eta}$, $\widetilde{\eta'}$ and $\widetilde{\eta_0}$ for the Dirichlet characters corresponding to η, η' and η_0.

- For any integer l with $0 < l \leq l_0/2$ and $l \equiv \flat$ modulo 2 let $\Pi(\chi)$ be the induced representation

  $$\Pi(\chi) = \mathrm{Ind}_{B_2(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}(\eta'|\cdot|^{l-1/2}, \eta\chi|\cdot|^{-(l-1/2)}).$$

Remark 3.1. *The discrete series representation $D(l_0+1)$ may be realized as the unique proper irreducible submodule of* $\mathrm{Ind}_{B_2(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\mathrm{sgn}|\cdot|_\infty^{l_0/2}, |\cdot|_\infty^{-l_0/2})$*, cf. [Bum97] section 2.5.*
The assumption on the infinity component of π assures that π is cohomological, i.e. there exists a rational representation (ρ, V) such that $H^\bullet(\mathrm{gl}_3, K_{3,\infty}Z_\infty^0(\mathbb{R}), \pi\otimes\rho)$ is nontrivial, cf. section 3.3. The existence of such representations π follows for example from the fact that the symmetric-square lift transforms cuspidal cohomological representations on GL_2 *to cuspidal cohomological representations on* GL_3*.*
For $p=3$ there does not exist a finite idèle class character η' of level $p=3$ and with infinity component $\eta'_\infty = \mathbb{1}$. To avoid problems we have assumed $p, q > 3$.

The representation $\Pi(\chi)$ is the restricted tensor product of local representations

$$\Pi(\chi) = \otimes'_\ell \, \mathrm{Ind}_{B_2(\mathbb{Q}_\ell)}^{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\eta'_\ell|\cdot|_\ell^{l-1/2}, \eta_\ell\chi_\ell|\cdot|_\ell^{-(l-1/2)})$$

and $\Pi(\chi)$ is generated by products of functions in the local representations. As usual, we denote the component of $\Pi(\chi)$ at the place ℓ by $\Pi_\ell(\chi)$. At infinity we know that there is a proper irreducible submodule Π_∞ in $\Pi_\infty(\chi)$, which is isomorphic to the discrete series representation $D(2l)$, cf. [Bum97] section 2.5. For $\ell < \infty$ the representations $\Pi_\ell(\chi)$ are irreducible if and only if $\eta'_\ell|\cdot|_\ell^{2l-1}\eta_\ell^{-1}\chi_\ell^{-1}$ neither equals $|\cdot|_\ell$ nor $|\cdot|_\ell^{-1}$, cf. [Bum97] section 4.5. Since the characters η', η and χ are of finite order we conclude that $\Pi_\ell(\chi)$ is irreducible for $l > 1$.
We make use of the Eisenstein intertwining operator

$$\begin{aligned} \mathrm{Eis} : \Pi(\chi) &\longrightarrow \mathcal{A}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})), \\ \psi &\longmapsto \sum_{\gamma\in B_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{Q})} \psi(\gamma g). \end{aligned} \tag{3.1}$$

For $l > 1$ this sum converges absolutely and in the case $l = 1$ one defines Eis using an appropriate analytic continuation, cf. section 1.3.
In [Mah00] there is chosen explicitly a pair of automorphic forms $(\phi, \mathrm{Eis}(\psi_\chi))$ in $\pi \times \mathrm{Eis}(\Pi(\chi))$ such that its Mellin transform can be related to the values $L(\pi\otimes\chi\eta, 1-l)$. The elements ϕ and ψ_χ are pure tensors, we write $\phi = \otimes'_\ell\phi_\ell$ and $\psi_\chi = \otimes'_\ell\psi_{\chi,\ell}$. In fact, the elements ϕ and ψ_χ are only fixed up to their infinity components, more precisely ϕ_∞ is an arbitrary element in π_∞ and $\psi_{\chi,\infty}$ is an arbitrary element in Π_∞. We abbreviate $\mathrm{Eis}(\psi_\chi)$ to E_χ.
The Mellin transform of the pair (ϕ, E_χ) is defined as the following Rankin-Selberg integral

$$I(s, \phi, E_\chi) = \int\limits_{\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})} \phi\begin{pmatrix} g & \\ & 1 \end{pmatrix} E_\chi(g)\,|\det g|^{s-1/2}\,dg,$$

which was introduced in section 1.4.1 in the case of cuspidal representations. Although E_χ is not cuspidal, most of the statements in section 1.4.1 remain true, for instance due to the rapid decay of ϕ and the moderate growth of E_χ the integral is still absolutely convergent for $s \in \mathbb{C}$.
The following Theorem gives the promised connection between the values $L(\pi\otimes\chi\eta, 1-l)$ and the Mellin transform of ϕ and E_χ, cf. Theorem 1 in [Mah00]. Instead of $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$ we will sometimes write $X_\mathbb{A}$.

Theorem 3.2. *With the notation introduced above we have for $e \geq 2$*

$$P(1/2)L(\pi\otimes\chi\eta,1-l)=$$

$$\mathbf{A}\sum_{\substack{i\in\left(\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p\right)^*\\ j\in\left(\mathbb{Z}_p/p^{2e-1}\mathbb{Z}_p\right)^*\\ y\in\mathbb{Z}_p/p^e\mathbb{Z}_p}}\chi_p\eta_p^2(j)\eta_p(i)\int\limits_{X_{\mathbb{A}}}\phi\left(\begin{pmatrix}g & \\ & 1\end{pmatrix}\begin{pmatrix}1 & ip/f & y/f\\ & 1 & j/f\\ & & 1\end{pmatrix}\right)E_\chi(g)dg,\quad(3.2)$$

where $P(T)=P_{\phi_\infty,\psi_\infty}(T)\in\mathbb{C}(T)$ is a polynomial, which only depends on the infinity components of the elements ψ_χ and ϕ. The constant $\mathbf{A}$ is defined as follows

$$\mathbf{A}=\mathbf{A}_l'\,f^{l+2}\,\pi^{3l-\flat}\,\zeta^{-e}\,G(\chi_p)^{-1}G(\chi_p\eta_p^2)^{-1}G(\widetilde{\chi}^{-1}\widetilde{\eta}_0^{-1})L(\widetilde{\chi}\widetilde{\eta}_0,1-2l)L_{\{q\}}(\pi\otimes\eta',l)^{-1},$$

where $\zeta=\mu_3\eta'\eta_{0,p}^{-1}(p)$ and the algebraic number $\mathbf{A}_l'\in\bar{\mathbb{Q}}^*$ is independent of χ and $L_{\{q\}}(\pi\otimes\eta',l)$ denotes the L-function of $\pi\otimes\eta'$ with the Euler factor at the place q omitted.

Remark 3.3. *We should point out that the constant $\mathbf{A}$ is nonzero: First, one checks that the Gauss sums occurring in $\mathbf{A}$ are nonzero by computing their absolute values, for instance $|G(\chi_p)|^2=p^e$. Moreover, it is well known that for any Dirichlet character $\widetilde{\xi}$ and any natural number $n\in\mathbb{N}$ the Dirichlet L-function $L(\widetilde{\xi},1-n)$ evaluates to $-B_{n,\widetilde{\xi}}/n$, cf. [Was97], Theorem 4.2, and that for $n>1$ the generalized Bernoulli numbers $B_{n,\widetilde{\xi}}$ are nonzero if and only if the parity of the natural number n coincides with the parity of the Dirichlet character $\widetilde{\xi}$. Since $\widetilde{\chi}\widetilde{\eta}_0$ is an even character we find that $L(\widetilde{\chi}\widetilde{\eta}_0,1-2l)$ is nonzero. Finally, the automorphic L-function $L(\pi\otimes\eta',s)$ is nonzero for* $\mathrm{re}(s)\geq 1$. *(For* $\mathrm{re}(s)>1$ *this follows from the fact that the product defining the L-function $L(\pi\otimes\eta',s)$ is absolutely convergent, cf. Theorem 4.3 in [Cog00], and for* $\mathrm{re}(s)=1$ *the non-vanishing of this L-function is shown in [JS77].) Since by definition the L-function $L_{\{q\}}(\pi\otimes\eta',s)$ multiplied by the Euler factor at the place q equals the L-function $L(\pi\otimes\eta',s)$, we conclude that $L_{\{q\}}(\pi\otimes\eta',s)$ is also nonzero for* $\mathrm{re}(s)\geq 1$.

Let us say a few words about the proof of Theorem 3.2, which takes up the whole first section in [Mah00]. In the notation of section 1.4.1 the integral occurring on the right hand side of equation (3.2) may be written as $I(\frac{1}{2},r_u\phi,E_\chi)$, where we abbreviate

$$u=u(i,j,y;f):=\begin{pmatrix}1 & ip/f & y/f\\ & 1 & j/f\\ & & 1\end{pmatrix}$$

and r_u denotes right translation by u. We have noted in section 1.4.1 that for pure tensors φ resp. φ' in unitary irreducible cuspidal representations π resp. π', the integrals $I(s,\varphi,\varphi')$ factor into local integrals $\Psi_\ell(s,W_{\varphi_\ell},W_{\varphi_\ell'})$. The crucial ingredient in the proof of this factorization is that π and π' are generic. If $\mathrm{Eis}(\Pi(\chi))$ is not irreducible, it is definitely not generic in our sense. Nevertheless we still have a map from $\mathrm{Eis}(\Pi(\chi))$ to the space of Whittaker functions $\mathcal{W}(\tau^{-1})$ which maps an element $E\in\mathrm{Eis}(\Pi(\chi))$ to its Whittaker function W_E defined by

$$W_E(g):=\int\limits_{N_2(\mathbb{Q})\backslash N_2(\mathbb{A})}E(ng)\,\tau(n)\,dn.$$

Since $\psi_\chi=\otimes_\ell\psi_{\chi,\ell}\in\Pi(\chi)$ is a pure tensor, the Whittaker function of $E_\chi=\mathrm{Eis}(\psi_\chi)$ decomposes as follows

$$W_{E_\chi}(g)=\prod_\ell\int\limits_{N_2(\mathbb{Q}_\ell)}\psi_{\chi,\ell}\left(\begin{pmatrix}&-1\\1&\end{pmatrix}ng_\ell\right)\tau_\ell(n)\,dn\qquad(3.3)$$

for $g = (g_\ell)$ in $\mathrm{GL}_2(\mathbb{A})$. For $l > 1$ this follows by replacing E_χ by its defining sum (3.1) in the definition of W_{E_χ}, exploiting the Bruhat decomposition $\mathrm{GL}_2 = B_2 \cup B_2 \left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right) N_2$ and unfolding the integral, cf. [Bum97], page 354. For $l = 1$ this relies on analytic continuation.
Let us denote the local functions on the right hand side of equation (3.3) by $W_{E_\chi,\ell}$, so that $W_{E_\chi}(g) = \prod_\ell W_{E_\chi,\ell}(g_\ell)$. Since ϕ is a pure tensor, its Whittaker function also is the product of local Whittaker functions, we write $W_\phi = \prod_\ell W_{\phi_\ell}$ as in section 1.4.1. These decompositions enable us to apply the proof that $I(s, \varphi, \varphi')$ is Eulerian for φ and φ' cuspidal, which is given in section 2.2.2 of [Cog00], to our situation and we obtain

$$I(s, \phi, E_\chi) = \prod_\ell \int\limits_{N_2(\mathbb{Q}_\ell)\backslash \mathrm{GL}_2(\mathbb{Q}_\ell)} W_{\phi_\ell} \begin{pmatrix} g & \\ & 1 \end{pmatrix} W_{E_\chi,\ell}(g) \, |\det(g)|^{s-1/2} \, dg.$$

Analogously to the cuspidal setting we denote the local factors at the right hand side of the equation above by $\Psi_\ell(s, W_{\phi_\ell}, W_{E_\chi,\ell})$. Thus, we have found that the right hand side of equation (3.2) can be decomposed into the following product

$$\mathbf{A}\ \Psi_p\Big(1/2, \sum_{i,j,y} \chi_p \eta_p^2(j) \eta_p(i) r_{u(i,j,y;f)} W_{\phi_p}, W_{E_\chi,p}\Big) \prod_{\ell \neq p} \Psi_\ell \left(1/2, W_{\phi_\ell}, W_{E_\chi,\ell}\right),$$

where the variables i, j and y run over the same domains as in equation (3.2).
By the definition of the automorphic L-function the left hand side of equation (3.2) also splits into local factors. The strategy of the proof of Theorem 3.2 is to compare the local factors in equation (3.2) place by place.
For instance, if ℓ is a prime not equal to p or q, then the characters χ_ℓ, η_ℓ and η'_ℓ are unramified and $\psi_{\chi,\ell}$ is the spherical vector in $\Pi(\chi)_\ell$. Thus, we can apply the Theorem in section (4.1) of [JPSS81] to see that

$$L(\pi_\ell, s + l - 1/2 + t_{\eta'_\ell}) L(\pi_\ell, s - (l - 1/2) + t_{\eta_\ell} + t_{\chi_\ell}) = W_{E_\chi,\ell}(1)^{-1} \Psi_\ell(s, W_{\phi_\ell}, W_{E_\chi,\ell}), \quad (3.4)$$

where $\eta_\ell = |\cdot|^{t_{\eta_\ell}}$ and analogously for $t_{\eta'_\ell}$ and t_{χ_ℓ}. Looking into the definition of the local L-factors via Rankin-Selberg integrals one checks that $L(\pi \otimes |\cdot|^t, s) = L(\pi, s + t)$ for $t \in \mathbb{C}$. Thus, specifying $s = 1/2$, we obtain for every prime $\ell \neq p, q$

$$L(\pi_\ell \otimes \eta_\ell \chi_\ell, 1 - l) = W_{E_\chi,\ell}(1)^{-1} L(\pi_\ell \otimes \eta'_\ell, l)^{-1} \Psi_\ell(1/2, W_{\phi_\ell}, W_{E_\chi,\ell}).$$

In order to compare the local factors in equation (3.2) at the Archimedean place, in [Mah00] there is in particular computed the local L-function $L(\pi_\infty \otimes \eta_\infty, s)$. We want to repeat this result here since it enables one to determine the set of critical integers for $\pi \otimes \eta$. According to page 260 in [Mah00], we have

$$L(\pi_\infty \otimes \eta_\infty, s) = \begin{cases} 2(2\pi)^{-(s+l_0/2)} \pi^{-s/2} \Gamma(s + l_0/2) \Gamma(s/2) & \text{if } \eta_\infty = \mathbb{1}, \\ 2(2\pi)^{-(s+l_0/2)} \pi^{-(s+1)/2} \Gamma(s + l_0/2) \Gamma((s+1)/2) & \text{if } \eta_\infty = \mathrm{sgn}. \end{cases}$$

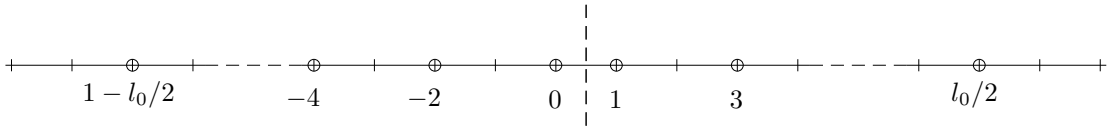
Since $\pi_\infty \otimes \eta_\infty$ is selfdual, the set of critical integers of $\pi \otimes \eta$ consists of all integers n such that $L(\pi_\infty \otimes \eta_\infty, s)$ and $L(\pi_\infty \otimes \eta_\infty, 1 - s)$ do not have poles at $s = n$. Noting that the only poles of the Gamma function are the non-positive integers, we conclude that the set of all critical integers for $\pi \otimes \eta$ is given by the integers

$$l \in \{-l_0/2 + 1, -l_0/2 + 2, \ldots l_0/2\} \quad (3.5)$$

such that

$$l \equiv \begin{cases} \flat & (\text{mod } 2) \quad \text{if } l > 1/2, \\ 1+\flat & (\text{mod } 2) \quad \text{if } l < 1/2, \end{cases} \tag{3.6}$$

cf. Remark 1.6 in [Mah00]. Hence, the integers $1-l$, where $0 < l \leq l_0/2$ and $l \equiv \flat \pmod 2$ are precisely the critical integers on the left hand side of the functional equation. For instance, if $\flat = 1$ and $l_0/2$ is odd, we get the following picture.



3.2. Construction of the Distribution

The right hand side of equation (3.2) already looks like the integral of $\chi_p \eta_p^2$ over $\mathbb{Z}_p^*$ against some distribution μ with $\mu(j + p^e\mathbb{Z}_p) := \sum_{i,y} \eta_p(i) I(1/2, r_{u(i,j,y;p^e)}\phi, E_\chi)$ but the problem is, that μ does not fulfill the distribution relation and that μ still depends on χ. Section 2 in [Mah00] is concerned with modifying μ so that it actually becomes a distribution which does not depend on χ. For instance, getting rid of the dependence of μ on χ is accomplished by replacing E_χ by some Fourier transform. This is what the next few definitions are heading for.

Let $\nu : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ be an idèle class character of level $f_\nu = p^{e_\nu}$ and with infinity component $\nu_\infty = \mathbb{1}$. In particular, ν satisfies the same properties as χ and thus we have elements $\psi_\nu \in \Pi(\nu)$ at our disposal. For any character $\nu \neq 1$ and $e \geq e_\nu$ we define the section ψ_{ν,p^e} by

$$\psi_{\nu,p^e}(g) := p^{-(e-e_\nu)l} \psi_\nu \left(g \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}^{e-e_\nu} \right).$$

In particular, one has $\psi_{\nu,p^{e_\nu}} = \psi_\nu$. In the case $\nu = 1$ we define for $e \geq 1$

$$\psi_{1,p^e}(g) := p^{-(e-1)l} \psi_1 \left(g \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}^{e-1} \right).$$

For $\epsilon \in \mathbb{Z}_p^*$ and $e \geq 1$ we define vectors ψ_{ϵ,p^e} as the Fourier transform

$$\psi_{\epsilon,p^e}(g) := \frac{2}{\phi(p^e)} \sum_\nu \nu_p^{-1}(\epsilon) \psi_{\nu,p^e}(g),$$

where the sum runs over all characters $\nu : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ with conductor $f_\nu | p^e$ and infinity component $\nu_\infty = \mathbb{1}$. We define the Eisenstein series E_{ϵ,p^e} as the image of the vectors ψ_{ϵ,p^e} under Eis. An application of the Fourier inversion formula yields

$$E_\chi = \sum_\epsilon \chi_p(\epsilon) E_{\epsilon,p^e},$$

where ϵ runs over $(\mathbb{Z}_p/p^e\mathbb{Z}_p)^*$.
As suggested at the beginning of this section, we have to modify the integral $I(s, r_{u(i,j,y;p^e)}\phi, E_\chi)$

by replacing E_χ by the Fourier transform E_{1,p^e}: For any $i,j \in \mathbb{Z}_p^*$ and $y \in \mathbb{Z}_p$ we set

$$P(i,j,y;p^e) := \int_{X_{\mathbb{A}}} \phi\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & ip/f & y/f \\ & 1 & j/f \\ & & 1 \end{pmatrix}\right) E_{1,p^e}(g)dg.$$

Lemma 2.2 in [Mah00] shows that the period integral $P(i,j,y;p^e)$ is independent of y and depends on j resp. on i only modulo $p^e\mathbb{Z}_p$ resp. modulo $p\mathbb{Z}$; more precisely we have

$$P(i,jj',y;p^e) = \eta_p^{-1}(i)P(1,j,0;p^e) \tag{3.7}$$

for $y \in \mathbb{Z}_p$ and $i,j,j' \in \mathbb{Z}_p^*$ with $j' \equiv 1 \ (p^e)$. In the following, we will abbreviate $P(1,j,0;p^e)$ to $P(j;p^e)$. Now we are ready to define the desired distribution.

Definition 3.4. *For $e \geq 2$ and l with $0 < l \leq l_0/2$, $l \equiv \flat \mod 2$ we set*

$$\mu^\eta_{l,\phi_\infty,\psi_\infty}(j + p^e\mathbb{Z}_p) := (\eta_p(p)\gamma)^{-e}p^{(4+l)e}P(j;p^e),$$

where $\gamma := \mu_2(p)\mu_3(p^2)p^2$. We abbreviate $\mu^\eta_{l,\phi_\infty,\psi_\infty}$ to μ^η_l or μ_l.

Proposition 2.3 in [Mah00] assures that μ_l indeed satisfies the distribution relation. We can extend μ_l to all cosets $j + p^e\mathbb{Z}_p$ with $e \in \mathbb{N}$ by using the distribution relation.

The next Proposition, cf. Proposition 2.4 in [Mah00], states that for finite characters the Mellin transform of the distribution μ_l actually gives the twisted values of the L-function. Of course, this heavily relies on Theorem 3.2, to prove it one starts with equation (3.2) and then transforms the integral $I(s, r_{u(i,j,y;p^e)}\phi, E_\chi)$ occurring on the right hand side into an expression containing the period integral $P(j;p^e)$ by using the inverse Fourier formula $E_\chi = \sum_\epsilon \chi(\epsilon)E_{\epsilon,p^e}$, several variable transformations and the invariance statement (3.7).

Proposition 3.5. *For any idèle class character χ of conductor $f = p^e$, $e \geq 2$ with $\chi_\infty = \mathbb{1}$ and any integer l with $0 < l \leq l_0/2$ and $l \equiv \flat \mod (2)$ we have*

$$\int_{\mathbb{Z}_p^*} \chi_p\eta_p^2 \, d\mu_l = \mathbf{B}P(1/2)L(\pi \otimes \chi\eta, 1-l)$$

with $\mathbf{B} = \mathbf{A}^{-1}\frac{2p^4}{(p-1)^2}(\eta_p(p)\gamma)^{-e}p^{le} \neq 0$. On the other hand, if χ is an idèle class character with conductor a p-power and with infinity component $\chi_\infty = \mathrm{sgn}$, then the integral $\int_{\mathbb{Z}_p^*} \chi_p\eta_p^2 \, d\mu_l$ vanishes.

We want to modify the distribution μ_l to improve the interpolation property, i.e. we try to get rid of some factors in the constant $\mathbf{B}$. On page 275 in [Mah00] it is shown that

$$\mathbf{B} = \mathbf{C}_0 \, \chi_p(q^{-1})p^{-2e}\hat{\zeta}^e\pi^{\flat-3l}\gamma^{-e}G(\chi_p\eta_p^2)L(\widetilde{\chi}\widetilde{\eta}_0, 1-2l)^{-1},$$

where $\hat{\zeta} := \eta_p^{-2}\eta_{0,q}^{-1}\mu_3(p)$, $\gamma = \mu_2(p)\mu_3(p^2)p^2$ and $\mathbf{C}_0$ is a nonzero constant which does not depend on χ. We know that there exists a bounded distribution $\mu(l)$ with

$$L(\widetilde{\chi}\widetilde{\eta}_0, 1-2l) = \int_{\mathbb{Z}_p^*} \eta_p^2\chi_p \, d\mu(l)$$

for $e \geq 2$. In addition, the Dirac distribution δ_q at q is bounded and satisfies $\chi_p(q) = \int_{\mathbb{Z}_p^*} \chi_p \, d\delta_q$. For two arbitrary distributions μ and ν on $\mathbb{Z}_p^*$ their convolution $\mu * \nu$ is defined as follows

$$\int\limits_{\mathbb{Z}_p^*} \xi \, d(\mu * \nu) := \int\limits_{\mathbb{Z}_p^*} \xi \, d\mu \int\limits_{\mathbb{Z}_p^*} \xi \, d\nu$$

for all characters ξ on $\mathbb{Z}_p^*$ of finite order. This determines the distribution $\mu * \nu$ completely because any characteristic function $\mathbf{ch}_{a+p^e\mathbb{Z}_p}$ can be written as a linear combination of characters of finite order,

$$\mathbf{ch}_{a+p^e\mathbb{Z}_p} = \frac{1}{(p-1)p^{e-1}} \sum_{\xi} \xi^{-1}(a)\xi, \tag{3.8}$$

where the sum runs over all characters ξ on $\mathbb{Z}_p^*$ with conductor dividing p^e. A straightforward calculation shows that $\mu * \nu$ indeed satisfies the distribution relations.
We define $\mu^{\eta,l} := \mu_l^\eta * \mu(l) * \delta_q$. The following Corollary is a consequence of Proposition 3.5.

Corollary 3.6. *Let χ and l be as in Proposition 3.5. If $\chi_\infty = \mathbb{1}$ we have*

$$\int_{\mathbb{Z}_p^*} \chi_p \eta_p^2 \, d\mu^{\eta,l} = \mathbf{C} P(1/2) \pi^{\mathfrak{b}-3l} \hat{\zeta}^e \gamma^{-e} p^{-2e} G(\chi_p \eta_p^2) L(\pi \otimes \chi\eta, 1-l)$$

and if $\chi_\infty = \operatorname{sgn}$ the integral $\int_{\mathbb{Z}_p^*} \chi_p \eta_p^2 \, d\mu^{\eta,l}$ vanishes. The constant $\mathbf{C} \in \mathbb{C}^*$ does not depend on the character χ.

3.3. Cohomological Interpretation

In [Clo90] there is constructed a finite-dimensional representation (ρ, V) of $\mathrm{GL}_3(\mathbb{C})$ such that the cohomology group $H^2(\mathrm{gl}_3, K_{3,\infty} Z_\infty^0(\mathbb{R}), \pi_\infty \otimes \rho)$ is nonzero, to be more precise, it is one-dimensional, cf. [Clo90], Lemma 3.14 and [Mah00], section 3.1. We fix a generator ω_∞ of the space $H^2(\mathrm{gl}_3, K_{3,\infty} Z_\infty^0(\mathbb{R}), \pi_\infty \otimes \rho)$,

$$\omega_\infty = \sum_{i,j,a} \phi_{\infty,i,j,a} \, v_a \otimes \omega_i' \wedge \omega_j',$$

where the $\phi_{\infty,i,j,a}$ lie in π_∞, $\{v_a\}$ is a basis of V and $\{\omega_1', \ldots \omega_5'\}$ is a basis of the dual space $(\mathrm{gl}_3/\mathrm{so}_3 \operatorname{Lie}(\mathrm{Z}_3^0(\mathbb{R})))^*$. Since $\omega_i' \wedge \omega_j'$ is alternating, we can assume that $\phi_{\infty,i,j,a}$ vanishes for $i \geq j$. In section 2.4, cf. equation (2.8), we have mentioned that the following map

$$\begin{aligned} \mathcal{F}_\pi : \pi_f &\longrightarrow H^2_{\mathrm{cusp}}(\widetilde{S}_3, \mathcal{V}), \\ \varphi &\longmapsto \varphi \otimes \omega_\infty \end{aligned}$$

is injective and we set $\omega = \mathcal{F}_\pi(\phi_f)$, where ϕ_f is the finite part of the element $\phi \in \pi$ introduced in section 3.1. As noted in section 2.4, the cuspidal cohomology can be embedded into the cohomology with compact support, thus we can regard ω as an element in $H_c^2(\widetilde{S}_3, \mathcal{V})$.

Now, we turn to the cohomological interpretation of the finite part $\Pi_f(\chi)$ of the representation $\Pi(\chi)$. Let (κ, W) be the $(2l-1)$-dimensional representation of $\mathrm{GL}_2(\mathbb{C})$ with trivial central

character, in the notation of section 2.5 we can write $W = M(2l-2, 1-l)$. The representation $\Pi(\chi)$ is non-unitarily induced from the character

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \longmapsto \left|\frac{t_1}{t_2}\right|^l \eta'(t_1)\chi\eta(t_2)$$

and thus, according to Theorem 2.5, $\Pi_f(\chi)$ appears as a a direct summand in the first cohomology of the boundary $\partial\widetilde{S}_2$ with coefficients in $\mathcal{W}$. As explained in section 2.5 the choice of a generator e^1 of the space $H^1(\mathrm{gl}_2, \mathrm{SO}_2(\mathbb{R})Z_2^0(\mathbb{R}), \Pi_\infty(\chi)\otimes W)$ yields an embedding

$$\begin{aligned} \Pi_f(\chi) &\longhookrightarrow H^1(\partial\widetilde{S}_2, \mathcal{W}), \\ \psi &\longmapsto e^1 \otimes \psi. \end{aligned}$$

A generator e^1 is of the form

$$e^1 = \sum_{i,b} \psi_{\infty,i,b}\, w_b \otimes \omega_i, \tag{3.9}$$

where $\psi_{\infty,i,b}$ lies in $\Pi_\infty(\chi)$, $\{w_b\}$ is a basis of W and $\{\omega_1, \omega_2\}$ is a basis of $(\mathrm{gl}_2/\mathrm{so}_2\,\mathrm{Lie}(\mathrm{Z}_2^0(\mathbb{R})))^*$. For later applications it is convenient to choose the basis $\{\omega_1, \omega_2\}$ compatible with the basis $\{\omega_i'\}$ of $(\mathrm{gl}_3, /\mathrm{so}_3\mathrm{Lie}(\mathrm{Z}_3^0(\mathbb{R})))^*$, which was fixed above. We do not want to specify this here but refer the reader to [Mah00], page 277.
Combining the embedding of $\Pi_f(\chi)$ into the cohomology of the boundary with the Eisenstein operator $\widetilde{\mathrm{Eis}}$ defined in (2.12), we obtain an embedding

$$\mathrm{Eis}^* : \Pi_f(\chi) \longhookrightarrow H^1(\widetilde{S}_2, \mathcal{W})$$

given by

$$\mathrm{Eis}^*(\psi)(g, D) := \sum_{\gamma\in B_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \psi \otimes e^1(\gamma g, D)$$

for $g \in \mathrm{GL}_2(\mathbb{A})$ and $D \in \mathrm{gl}_2/\mathrm{so}_2\,\mathrm{Lie}(Z_2^0(\mathbb{R}))$. Let $\psi_{p^e,f} := \psi_{1,p^e,f}$ be the finite part of the vector ψ_{ϵ,p^e} for $\epsilon = 1$, which was defined in section 3.2. We denote by $\omega_{p^e} := \mathrm{Eis}^*(\psi_{p^e,f})$ the image of $\psi_{p^e,f}$ and, making use of equation (3.9), we find that

$$\omega_{p^e} = \sum_{i=1,2;\, b} E_{p^e,i,b}\, w_b \otimes \omega_i$$

with Eisenstein series $E_{p^e,i,b} := \mathrm{Eis}(\psi_{p^e,f}\psi_{\infty,i,b})$.

In the following, we will make use of the differential forms ω and ω_{p^e} to give a cohomological description of the period integrals $P(\epsilon; p^e)$ introduced in section 3.2.
For any compact open subgroup $K^f \subset \mathrm{GL}_3(\mathbb{A}_f)$ we denote by $K_2^f \subset \mathrm{GL}_2(\mathbb{A}_f)$ the compact open subgroup $K^f \cap \mathrm{GL}_2(\mathbb{A}_f)$, where GL_2 is identified with a subgroup of GL_3 via $g \mapsto \left(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\right)$. Note, that any compact open subgroup $K' \subset \mathrm{GL}_2(\mathbb{A}_f)$ is of the form $K' = K_2^f$ for some compact open subgroup $K^f \subset \mathrm{GL}_3(\mathbb{A}_f)$, cf. Appendix A. Let $i(K^f)$ be the canonical map

$$\begin{aligned} i(K^f): \quad F_2(K_2^f) &\longrightarrow S_3(K^f), \\ \mathrm{GL}_2(\mathbb{Q})gK_2^fK_{2,\infty} &\longmapsto \mathrm{GL}_3(\mathbb{Q}) \begin{pmatrix} g & \\ & 1 \end{pmatrix} K^f K_{3,\infty} Z_3^0(\mathbb{R}). \end{aligned}$$

Since $i(K^f)$ is a proper map, cf. Appendix A, this induces a map on the limit of the cohomology groups with compact support

$$i^* : H_c^2(\widetilde{S}_3, \mathcal{V}) \longrightarrow H_c^2(\widetilde{F}_2, i^*\mathcal{V}) := \varinjlim_{K^f} H_c^2(F_2(K_2^f), i(K^f)^*\mathcal{V}).$$

We recall the construction of the inverse image sheaf $i(K^f)^*\mathcal{V}$ on $F_2(K_2^f)$. Let $i(\overline{K^f})^*\mathcal{V}$ be the presheaf on $F_2(K_2^f)$, whose sections on an open set U are given by

$$i(\overline{K^f})^*\mathcal{V}(U) := \varinjlim_{W} \mathcal{V}(W),$$

where the direct limit is taken over all open sets W in $S_3(K^f)$ containing $i(U)$. One defines $i(K^f)^*\mathcal{V}$ to be the sheafification of $i(\overline{K^f})^*\mathcal{V}$. On the other hand, for any compact open subgroup $K^f \subset \mathrm{GL}_2(\mathbb{A}_f)$ we denote by $\mathcal{V}_{K^f}^{F_2}$ the sheaf induced by $V|_{\mathrm{GL}_2}$ on $F_2(K^f)$ analogously to definition (2.2), where $V|_{\mathrm{GL}_2}$ denotes the restriction of the representation V to GL_2. We want to show that the sheaves $i(K^f)^*\mathcal{V}$ and $\mathcal{V}_{K_2^f}^{F_2}$ are isomorphic: For any $g \in F_2(K^f)$ we can choose an open, connected neighborhood U of g and an open connected neighborhood W of $i(g)$ such that $i(U)$ lies in W and $\mathcal{V}(W) \cong V \cong \mathcal{V}_{K_2^f}^{F_2}(U)$. Then, $i(\overline{K^f})^*\mathcal{V}(U)$ is isomorphic to $\mathcal{V}(W)$ and the mapping

$$\begin{aligned} \mathcal{V}(W) &\longrightarrow \mathcal{V}_{K_2^f}^{F_2}(U), \\ s &\longmapsto \{s' : u \mapsto s(i_0(u))\}, \end{aligned}$$

where we make use of the realization (2.2) of the sheaf $\mathcal{V}$ and the analogous description of $\mathcal{V}_{K_2^f}^{F_2}$ and i_0 denotes the canonical map $\mathrm{GL}_2(\mathbb{A})/K_{2,\infty}K_2^f \to \mathrm{GL}_3(\mathbb{A})/K_{3,\infty}Z_3^0(\mathbb{R})K^f$, induces an isomorphism $i(K^f)^*\mathcal{V} \cong \mathcal{V}_{K_2^f}^{F_2}$.

Analogously, we denote by $p^* : H^1(\widetilde{S}_2, \mathcal{W}) \to H^1(\widetilde{F}_2, p^*\mathcal{W})$ the canonical map induced by the projections $p(K^f) : F_2(K^f) \to S_2(K^f)$ with K^f a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$. Here, $H^1(\widetilde{F}_2, p^*\mathcal{W})$ is defined as the direct limit of the cohomology groups $H^1(F_2(K^f), p(K^f)^*\mathcal{W})$. One can show that the inverse image sheaf $p(K^f)^*\mathcal{W}$ on $F_2(K^f)$ is isomorphic to the sheaf $\mathcal{W}_{K^f}^{F_2}$ induced by (κ, W) on $F_2(K^f)$.

According to Lemma 3.1 in [Mah00], there exists a non-trivial, GL_2-invariant pairing

$$\mathrm{tr} : V|_{\mathrm{GL}_2} \otimes W \longrightarrow \mathbb{C}. \tag{3.10}$$

We know that the representations ρ and κ are defined over $\mathbb{Q}$, i.e. ρ resp. κ acts on a $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$ resp. $W_{\mathbb{Q}}$. The pairing tr is also defined over $\mathbb{Q}$ and induces a pairing of sheaves

$$\mathrm{tr}_{K^f} : \mathcal{V}_{K^f,\mathbb{Q}}^{F_2} \otimes \mathcal{W}_{K^f,\mathbb{Q}}^{F_2} \longrightarrow \underline{\mathbb{Q}}, \tag{3.11}$$

where $K^f \leq \mathrm{GL}_2(\mathbb{A}_f)$ is a compact open subgroup, $\mathcal{V}_{K^f,\mathbb{Q}}^{F_2}$ denotes the sheaf induced by $V_{\mathbb{Q}}|_{\mathrm{GL}_2}$ on $F_2(K^f)$ and similarly for $\mathcal{W}_{K^f,\mathbb{Q}}^{F_2}$. We have seen in section 2.6 that the pairing tr_{K^f} of sheaves induces a pairing $\widetilde{\mathrm{tr}_{K^f}}$ in cohomology

$$\widetilde{\mathrm{tr}_{K^f}} : H_c^2(F_2(K^f), \mathcal{V}_{K^f,\mathbb{Q}}^{F_2}) \times H^1(F_2(K^f), \mathcal{W}_{K^f,\mathbb{Q}}^{F_2}) \longrightarrow \mathbb{Q}.$$

Via the realization of the pairing $\widetilde{\mathrm{tr}_{K^f}}$ in the de Rham cohomology, cf. section 2.6, we check (at least over $\mathbb{C}$) that the rescaled pairings $\mathrm{vol}(K^f)\widetilde{\mathrm{tr}_{K^f}}$ are compatible with the inductive system

on the cohomology groups. Thus, we obtain a pairing $\widetilde{\mathrm{tr}}$ on the inductive limit

$$\begin{array}{ccccc} H_c^2(\widetilde{F}_2, \mathcal{V}_{\mathbb{Q}}^{F_2}) & \times & H^1(\widetilde{F}_2, \mathcal{W}_{\mathbb{Q}}^{F_2}) & \xrightarrow{\widetilde{\mathrm{tr}}} & \mathbb{Q} \\ \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle p^*} & & \\ H_c^2(\widetilde{S}_3, \mathcal{V}_{\mathbb{Q}}) & & H^1(\widetilde{S}_2, \mathcal{W}_{\mathbb{Q}}). & & \end{array} \tag{3.12}$$

Here, $H_c^2(\widetilde{F}_2, \mathcal{V}_{\mathbb{Q}}^{F_2})$ is defined as the direct limit of the cohomology groups $H_c^2(F_2(K^f), \mathcal{V}_{K^f,\mathbb{Q}}^{F_2})$ with respect to the directed set of compact open subgroups $K^f \subset \mathrm{GL}_2(\mathbb{A}_f)$ and analogously for $H^1(\widetilde{F}_2, \mathcal{W}_{\mathbb{Q}}^{F_2})$. We end up with a pairing

$$\langle\,,\,\rangle : \quad H_c^2(\widetilde{S}_3, \mathcal{V}_{\mathbb{Q}}) \times H^1(\widetilde{S}_2, \mathcal{W}_{\mathbb{Q}}) \longrightarrow \mathbb{Q}. \tag{3.13}$$

According to Appendix B, we have $H^1(\widetilde{S}_2, \mathcal{W}_{\mathbb{Q}}) \otimes F = H^1(\widetilde{S}_2, \mathcal{W}_{\mathbb{Q}} \otimes \underline{F})$ for any field F containing the rational numbers and analogously for the cohomology with compact supports. Thus, the pairing $\langle\,,\,\rangle$ extends to a pairing of cohomology groups over F, which we also denote by $\langle\,,\,\rangle$.

We only have to fix a few more notations before we can cite Lemma 3.2 in [Mah00], see Lemma 3.7 below, which will establish the promised connection between the cohomological pairing $\langle\,,\,\rangle$ and the distribution μ_l^η introduced in Definition 3.4. We recall that the distribution μ_l^η still depends on the infinite part ϕ_∞ resp. $\psi_{\chi,\infty}$ of the element $\phi \in \pi$ resp. $\psi_\chi \in \Pi(\chi)$ and define the sum of distributions

$$\mu_{\pi,l}^\eta := \sum_{i,j,k,a,b} \epsilon_{i,j,k} \operatorname{tr}(v_a \otimes w_b)\, \mu_{l,\phi_\infty,i,j,a\psi_{\infty,k,b}}^\eta,$$

where $\epsilon_{i,j,k}$ vanishes unless (i,j,k) is equal to $(2,3,1)$ resp. $(1,3,2)$ in which cases it equals 1 resp. -1. Obviously, $\mu_{\pi,l}^\eta$ defines a distribution on $\mathbb{Z}_p^*$ and Proposition 3.5 shows that integrating $\mu_{\pi,l}^\eta$ against characters gives

$$\int_{\mathbb{Z}_p^*} \chi_p \eta_p^2 \, d\mu_{\pi,l}^\eta = \mathbf{B}\, P_l(1/2)\, L(\pi \otimes \chi\eta, 1-l), \tag{3.14}$$

where $P_l \in \mathbb{C}[T]$ denotes the polynomial $P_l := \sum_{i,j,k,a,b} \epsilon_{i,j,k} \operatorname{tr}(v_a \otimes w_b) P_{\phi_\infty,i,j,a\psi_{\infty,k,b}}$. At the end of section 2.6 we have described the induced pairing in cohomology in the de Rham cohomology, this is the main ingredient in the proof of the following Lemma, cf. Lemma 3.2 in [Mah00].

Lemma 3.7. *For $u = u(1,j,0;p^e) \in N_3(\mathbb{Q}_p)$ and $e \geq 2$ we have*

$$\langle r_u^* \omega, \omega_{p^e} \rangle = p^{-e(4+l)} \eta_p(p^e) \gamma^e \mu_{\pi,l}^\eta(j + p^e \mathbb{Z}_p),$$

where $r_u^* : H_c^2(\widetilde{S}_3, \mathcal{V}) \to H_c^2(\widetilde{S}_3, \mathcal{V})$ is the map induced by right translation with $u \in \mathrm{GL}_3(\mathbb{Q}_p) \subset \mathrm{GL}_3(\mathbb{A}_f)$ as explained in section 2.1.

From Théorème 3.13 in [Clo90] we know that the π_f-isotypical subspace of the cuspidal cohomology is defined over some number field E_π, i.e.

$$H_{\mathrm{cusp}}^2(\widetilde{S}_3, \mathcal{V})(\pi_f) = H_{\mathrm{cusp}}^2(\widetilde{S}_3, \mathcal{V}_{E_\pi})(\pi_f) \otimes \mathbb{C},$$

and in section 3.2 in [Mah00] it is shown that there exists a complex number $\Omega(\pi) \in \mathbb{C}^*$ such that $\Omega(\pi)^{-1}\omega$ lies in $H_{\mathrm{cusp}}^2(\widetilde{S}_3, \mathcal{V}_{E_\pi(\zeta_{q(q-1)})})$. Furthermore, it is shown in [Mah00] that ω_{p^e} is contained in $H^1(\widetilde{S}_2, \mathcal{W}_{\mathbb{Q}(\eta,\eta')})$. Lemma 3.7 now immediately implies the following.

Theorem 3.8. *For all characters $\eta : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of finite order and of conductor $f_\eta = pq$ and all integers $0 < l \leq l_0/2$ with $l \equiv \flat \pmod 2$ the distributions $\mu^\eta_{\pi,l}$ are $\Omega(\pi) \cdot E_\pi(\eta, \eta', \zeta_{q(q-1)})$-valued, i.e. for any compact open subset $U \leq \mathbb{Z}_p^*$ we have*

$$\frac{\mu^\eta_{\pi,l}(U)}{\Omega(\pi)} \in E_\pi(\eta, \eta', \zeta_{q(q-1)}).$$

If $P_l(1/2)$ vanishes, equation (3.14) becomes trivial and the distributions $\mu^\eta_{\pi,l}$ fail to interpolate the automorphic L-function. Hence we make the following

Assumption: $P_l(1/2)$ does not vanish.

According to remark 3.5 in [Mah00], this assumption is fulfilled in the case of trivial coefficients, i.e. if (ρ, V) is the trivial representation. The explicit description of the representation (ρ, V) on page 276 in [Mah00] shows that if (ρ, V) is trivial, then $l_0 = 2$, $l = 1$, $\eta_\infty = \mathrm{sgn}$ and the representation (κ, W) is also trivial. In this case, zero is the only critical value on the left hand side of the functional equation.

Theorem 3.8 and equation (3.14) yield the following.

Corollary 3.9. *Let χ be an idèle class character with infinity component $\chi_\infty = \mathbb{1}$ and of conductor p^e, $e \geq 2$, and let η and l be as in Theorem 3.8. Then, under the above assumption, we have*

$$\frac{L(\pi \otimes \chi\eta, 1-l)}{P_l^{-1}(1/2)\pi^{3l-\flat}\Omega(\pi)\boldsymbol{C}_0^{-1}} \in E_\pi(\eta, \eta', \zeta_{q(q-1)}, \chi).$$

We note that the complex numbers $\hat{\zeta}$ and γ occurring in the definition of the constant $\mathbf{B}$ already lie in E_π. For γ this is shown in [Mah00], page 282 and for $\hat{\zeta}$ it works totally analogously.

3.4. Remark on the Computation of the Local ζ-Integrals

Comparing the results in this chapter with their analogues in [Mah00] one notes that they are slightly different. These differences originate from a minor mistake in Proposition 1.2 in [Mah00]. In this section we want to explain how one can correct the statement of this Proposition and what the influence of this correction on the results stated above is.
Proposition 1.2 in [Mah00] is concerned with the proof of Theorem 3.2 at the place p, it gives the decisive information for computing the local integral

$$\Psi_p\Big(1/2, \sum_{i,j,y} \chi_p\eta_p^2(j)\eta_p(i) r_u W_{\phi_p}, W_{E_\chi,p}\Big)$$

by determining the behavior of the Whittaker function $\sum_{i,j,y} \chi_p\eta_p^2(j)\eta_p(i) r_u W_{\phi_p}$ restricted to GL_2.
More generally, in [Mah00], Proposition 1.2, the following function is introduced

$$w_{\chi_p}(g) = \sum_{\substack{i \in \left(\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p\right)^* \\ j \in (\mathbb{Z}_p/p^e\mathbb{Z}_p)^* \\ y \in \mathbb{Z}_p/p^e\mathbb{Z}_p}} \eta_p(i)\chi_p\eta_p^2(j) w\left(g\begin{pmatrix} 1 & ip/f & y/f \\ 0 & 1 & j/f \\ 0 & 0 & 1 \end{pmatrix}\right), \tag{3.15}$$

where $w \in W(\pi_p, \tau_p)$ is a Whittaker function invariant under the Iwahori subgroup $\mathcal{I} \leq \mathrm{GL}_3(\mathbb{Z}_p)$, as is W_{ϕ_p}. Since

$$u(i, j+f, y; f) = u(i, j, y; f) \begin{pmatrix} 1 & 0 & -ip/f \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the last matrix does not lie in the Iwahori subgroup $\mathcal{I}$ it becomes clear that w_{χ_p} is not well-defined. This problem can be solved by changing the range of j in equation (3.15) from $(\mathbb{Z}_p/p^e\mathbb{Z}_p)^*$ to $\left(\mathbb{Z}_p/p^{2e-1}\mathbb{Z}_p\right)^*$. We again denote the modified well-defined function by w_{χ_p}. Let us shortly summarize the consequences:

- In Proposition 1.2 in [Mah00], there occurs an additional factor p^{e-1}, to be more precise, one has

$$w_{\chi_p}|_{\mathrm{GL}_2}\left(\begin{pmatrix} p^{e-2} & \\ & 1 \end{pmatrix} k\right) = f p^{2e-3} G(\chi_p \eta_p^2) G(\eta_p) w \begin{pmatrix} p^{e-2} & & \\ & 1 & \\ & & 1 \end{pmatrix} \eta_p^{-1}(ad) \chi_p^{-1}(d)$$

  for $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(2, f)$.

- For the local zeta integral we obtain the following result

$$\sum_{\substack{i \in \left(\mathbb{Z}_p/p^{e-1}\mathbb{Z}_p\right)^* \\ j \in \left(\mathbb{Z}_p/p^{2e-1}\mathbb{Z}_p\right)^* \\ y \in \mathbb{Z}_p/p^e\mathbb{Z}_p}} \eta_p(i)\chi_p\eta_p^2(j)\, \Psi_p(s, r_{u(i,j,y;f)} W_{\phi_p}, W_{E_\chi, p}) =$$

$$= f p^{2e-3} G(\chi_p \eta_p^2) G(\eta_p) W_{\phi_p} \begin{pmatrix} p^{e-2} & & \\ & 1 & \\ & & 1 \end{pmatrix} W_{E_\chi, p} \begin{pmatrix} p^{e-2} & \\ & 1 \end{pmatrix} |p^{e-2}|_p^{s-1/2} \int\limits_{K_0(2,f)} dk,$$

  compare [Mah00], page 265.

- The variable j in Theorem 1 in [Mah00] should run over $\left(\mathbb{Z}_p/p^{2e-1}\mathbb{Z}_p\right)^*$ instead of $(\mathbb{Z}_p/p^e\mathbb{Z}_p)^*$. In addition, the constant $\mathbf{A}$ in [Mah00] has to be replaced by $\mathbf{A}p^{e-1}$.

Applying these changes to Theorem 1 in [Mah00] we obtain Theorem 3.2 formulated above. Since Proposition 2.4 in [Mah00] and its Corollary heavily depend on Theorem 1 in [Mah00] one also has to modify the constants stated there. We do not want to go into details since the corrected versions of these results are presented above.
Obviously, Corollary 3 in [Mah00], which finally states the existence of the p-adic L-function in the case where there is only one critical integer on the left hand side, also has to be slightly adapted as follows.

Corollary 3 of [Mah00]. *There is a p-adic analytic function $L_p : X_p \to \mathbb{C}_p$ such that for all characters $\chi : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of conductor a p-power and with infinity component $\chi_\infty = \mathbb{1}$ we have*

$$L_p(\chi_p) = p^{-2e}\hat{\zeta}^e \gamma^{-e} G(\chi_p \eta_p^2) \frac{L(\pi \otimes \chi\eta, 0)}{\Omega(\pi)}.$$

4. The Denominators of Eisenstein Classes

At this point, [Mah00] restricts for simplicity to the case of trivial coefficients, i.e. (l_0, l, η_∞) is assumed to be equal to $(2, 1, \mathrm{sgn})$. The present work aims to get rid of this restriction and doing this, closely follows the exposition in [Mah00].
Most of the remaining part of this thesis is concerned with the investigation of the denominators of $\mu^{\eta}_{\pi,l}/\Omega(\pi)$ in $E_\pi(\eta, \eta', \zeta_{q(q-1)})$. The crucial step is to calculate bounds for the denominators of the Eisenstein classes ω_{p^e}, which will be done in this chapter.
First, we have to introduce some more notations.

4.1. Integral Cohomology

We consider the following $\mathbb{Z}$-module

$$W_{\mathbb{Z}} = \left\{ \sum_{k=0}^{2l-2} a_k \binom{2l-2}{k} X^k Y^{2l-2-k},\ a_k \in \mathbb{Z} \right\}.$$

This is a $\mathrm{GL}_2(\mathbb{Z})$-module: An element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ acts on $P(X,Y) \in W_{\mathbb{Z}}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : P(X,Y) \longmapsto P(aX+cY, bX+dY) \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{1-l}. \tag{4.1}$$

The $\mathbb{Z}$-module dual to $W_{\mathbb{Z}}$ is given by

$$\breve{W}_{\mathbb{Z}} = \left\{ \sum_{k=0}^{2l-2} a_k X^k Y^{2l-2-k},\ a_k \in \mathbb{Z} \right\}$$

with the analogous $\mathrm{GL}_2(\mathbb{Z})$-action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : P(X,Y) \longmapsto P(aX+cY, bX+dY) \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{1-l}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ and $P(X,Y) \in \breve{W}_{\mathbb{Z}}$. A $\mathrm{GL}_2(\mathbb{Z})$-invariant pairing $W_{\mathbb{Z}} \times \breve{W}_{\mathbb{Z}} \to \mathbb{Z}$ is given by $\langle \binom{2l-2}{k} X^k Y^{2l-2-k}, X^j Y^{2l-2-j} \rangle = \delta_{k-l+1, l-1-j}$ for $0 \le k, j \le 2l-2$, cf. [Har87c] section 6.1.
We set $W_{\mathbb{C}} = W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, this is the $\mathbb{C}$-module consisting of the homogeneous polynomials of degree $2l-2$ in two variables with coefficients in $\mathbb{C}$. It is a $\mathrm{GL}_2(\mathbb{C})$-module where the action of $\mathrm{GL}_2(\mathbb{C})$ is given by equation (4.1) for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ and $P(X,Y) \in W_{\mathbb{C}}$.
More generally, for any ring S containing $\mathbb{Z}$ we set $W_S = W_{\mathbb{Z}} \otimes_{\mathbb{Z}} S$ and $\breve{W}_S = \breve{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} S$ and regard these S-modules as $\mathrm{GL}_2(S)$-modules with the obvious action.
In section 2.5 we have defined $M(d, \nu)$ as the rational representation of GL_2 of dimension $d+1$ and with central character $x \mapsto x^{d+2\nu}$. Over $\mathbb{C}$ we find that the $\mathrm{GL}_2(\mathbb{C})$-module $M(2l-2, 1-l)$ is isomorphic to $W_{\mathbb{C}}$. In particular, we obtain an explicit realization of the coefficient system (κ, W) over $\mathbb{C}$ introduced in section 3.3.

In section 2.1 we have described how an arbitrary rational representation (ρ, W) of GL_n induces a sheaf $\mathcal{W}_{\mathbb{Q}}$ on $S_n(K^f)$. Let now the representation (ρ, W) be defined over $\mathbb{Z}$, in particular $W_{\mathbb{Z}}$ is a lattice in $W_{\mathbb{Q}}$. We want to explain how the representation (ρ, W) gives rise to a subsheaf $\mathcal{W}_{\mathbb{Z}}$ in $\mathcal{W}_{\mathbb{Q}}$, cf. [Har08a], page 7.
Let $K^f = \prod_\ell K_\ell$ be a compact open subgroup in $\mathrm{GL}_n(\mathbb{A}_f)$ such that $\prod_{\ell<\infty} W_{\mathbb{Z}} \otimes \mathbb{Z}_\ell$ is invariant under K^f, i.e. $K_\ell(W_{\mathbb{Z}} \otimes \mathbb{Z}_\ell) = W_{\mathbb{Z}} \otimes \mathbb{Z}_\ell$ for any prime ℓ. Then $W_{\mathbb{Z}}$ is invariant under the arithmetic subgroup $\mathrm{GL}_n(\mathbb{Q}) \cap K^f$.
For $g = (g_\ell) \in \mathrm{GL}_n(\mathbb{A}_f)/K^f$ we set $gW_{\mathbb{Z}} = \cap_{\ell<\infty}(g_\ell(W_{\mathbb{Z}} \otimes \mathbb{Z}_\ell) \cap W_{\mathbb{Q}})$. (Here $W_{\mathbb{Q}}$ is regarded as a subspace of $W_{\mathbb{Q}} \otimes \mathbb{Q}_\ell$, so it makes sense to consider the intersection of $g_\ell(W_{\mathbb{Z}} \otimes \mathbb{Z}_\ell)$ and $W_{\mathbb{Q}}$.) Since almost all local components g_ℓ of g are lying in $\mathrm{GL}_n(\mathbb{Z}_\ell)$ and thus keep $W_{\mathbb{Z}} \otimes \mathbb{Z}_\ell$ stable, we can apply Theorem 6.1.1 in [Kit99] and find that $gW_{\mathbb{Z}}$ is a $\mathbb{Z}$-lattice of full rank in $W_{\mathbb{Q}}$. Of course, $gW_{\mathbb{Z}}$ is kept invariant under the arithmetic subgroup $gK^f g^{-1} \cap \mathrm{GL}_n(\mathbb{Q})$. For an open subset $U \subset S_n(K^f)$ we define

$$\mathcal{W}_{\mathbb{Z}}(U) = \{f \in \mathcal{W}_{\mathbb{Q}}(U) \mid f((x_\infty, g_f)) \in g_f W_{\mathbb{Z}} \text{ for all } (x_\infty, g_f) \in \pi^{-1}(U)\}, \tag{4.2}$$

where, as before, $\pi : \mathrm{GL}_n(\mathbb{A})/K^f K_{n,\infty} Z_n^0(\mathbb{R}) \to S_n(K^f)$ denotes the canonical projection. Thus, we obtain a sheaf $\mathcal{W}_{\mathbb{Z}}$ on $S_n(K^f)$. For any number field F with ring of integers $\mathcal{O}_F$ we set $\mathcal{W}_{\mathcal{O}_F} = \mathcal{W}_{\mathbb{Z}} \otimes \underline{\mathcal{O}_F}$ and

$$H^i(S_n(K^f), \mathcal{W}_{\mathcal{O}_F})_{\mathrm{int}} := \mathrm{Im}(H^i(S_n(K^f), \mathcal{W}_{\mathbb{Z}}) \otimes \mathcal{O}_F \to H^i(S_n(K^f), \mathcal{W}_F)).$$

Similarly, we define $\mathcal{W}_{\mathcal{O}_{F,\mathcal{L}}}$ and $H^i(S_n(K^f), \mathcal{W}_{\mathcal{O}_{F,\mathcal{L}}})_{\mathrm{int}}$, where $\mathcal{O}_{F,\mathcal{L}}$ denotes the completion of $\mathcal{O}_F$ at the prime ideal $\mathcal{L}$.

4.2. Basic Idea

In this section, we first want to explain Harder's method of computing the denominators of Eisenstein classes. Then we will describe how one can deduce the denominators of the distributions $\mu^\eta_{\pi,l}$ from the denominators of the Eisenstein classes ω_{p^e} and the denominators of the elements $r^*_{u(i,j,y;p^e)}\Omega(\pi)^{-1}\omega$. We will close the section with a technical remark simplifying the computations in the next sections.

Speaking about denominators of Eisenstein classes one has to mention the work of Harder, [Har87b], which has been pursued for instance by Berger, cf. [Ber05] and Kaiser, cf. [Kai91]. In [Har87b] and [Kai91] there are computed the exact values of the denominators of some Eisenstein classes of fixed level. For the applications that we have in mind it suffices to give a bound for the denominators of the Eisenstein classes ω_{p^e}. The difficulty is that we are dealing with varying level, in other words the bound we are looking for will depend on e.
We will define the denominator not only for Eisenstein classes but for arbitrary elements in the cohomology of arithmetic groups. To this end, let F be an algebraic number field, $\mathcal{O}_F$ its ring of integers and $\mathcal{L}$ a prime ideal in $\mathcal{O}_{\mathcal{L}}$. We denote by $F_{\mathcal{L}}$ resp. $\mathcal{O}_{\mathcal{L}}$ the completion of F resp. $\mathcal{O}$ with respect to $\mathcal{L}$ and choose a uniformizing parameter ϖ in $\mathcal{O}_{F,\mathcal{L}}$. Let M be a rational representation of GL_n defined over $\mathbb{Z}$ and K^f a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$.

Definition 4.1 (Denominator)**.** *The $\mathcal{L}$-denominator of an element ω' in $H^i(S_n(K^f), \mathcal{M}_{F_{\mathcal{L}}})$ resp. in $H^i_c(S_n(K^f), \mathcal{M}_{F_{\mathcal{L}}})$ is defined as the smallest natural number $\delta \geq 0$ such that $\varpi^\delta \omega'$ lies in $H^i(S_n(K^f), \mathcal{M}_{\mathcal{O}_{F,\mathcal{L}}})_{\mathrm{int}}$ resp. in $H^i_c(S_n(K^f), \mathcal{M}_{\mathcal{O}_{F,\mathcal{L}}})_{\mathrm{int}}$.*

The existence of the denominator follows from the fact that $H^i(S_n(K^f), \mathcal{M}_{\mathcal{O}_{F,\mathcal{L}}})_{\mathrm{int}}$ defines a lattice in the finite dimensional vector space $H^i(S_n(K^f), \mathcal{M}_{F_{\mathcal{L}}})$ and that the analogous statement

holds for the cohomology with compact supports, see Appendix B. Note, that for compact open subgroups K^f, L^f of $\mathrm{GL}_n(\mathbb{A}_f)$ with $L^f \subset K^f$ the commutativity of the diagram

$$\begin{array}{ccc} H^i(S_n(K^f), \mathcal{M}_{\mathcal{O}_{F,\mathcal{L}}}) & \longrightarrow & H^i(S_n(L^f), \mathcal{M}_{\mathcal{O}_{F,\mathcal{L}}}) \\ \downarrow & & \downarrow \\ H^i(S_n(K^f), \mathcal{M}_{F_\mathcal{L}}) & \longrightarrow & H^i(S_n(L^f), \mathcal{M}_{F_\mathcal{L}}) \end{array}$$

implies that $H^i(S_n(K^f), \mathcal{M}_{\mathcal{O}_{F,\mathcal{L}}})_{\mathrm{int}}$ lies in $H^i(S_n(L^f), \mathcal{M}_{\mathcal{O}_{F,\mathcal{L}}})_{\mathrm{int}}$ in the direct limit $H^i(\widetilde{S}_n, \mathcal{M}_{F_\mathcal{L}})$; in other words, the $\mathcal{L}$-denominator with respect to the subgroup K^f is greater to or equal than the $\mathcal{L}$-denominator with respect to L^f.
In the following, let F be the algebraic number field $E_\pi(\eta, \eta', \zeta_{q(q-1)})$ and let $\mathcal{L}$ be a prime ideal in its ring of integers $\mathcal{O}$. Sections 4.4 to 4.6 are dedicated to the computation of the denominator of $\omega_{p^e} \in H^1(S_2(K^f), \mathcal{W}_{F_\mathcal{L}})$ for a certain compact open subgroup $K^f \subset \mathrm{GL}_2(\mathbb{A}_f)$. This will be based on the following considerations: Let $\delta_0 \in \mathbb{Z}$ be the smallest number (not necessarily greater than or equal to zero) such that $\varpi^{\delta_0}\omega_{p^e}$ lies in $H^1(S_2(K^f), \mathcal{W}_{\mathcal{O}_\mathcal{L}})_{\mathrm{int}}$. We have mentioned in section 2.6 that one can find a cocycle $\zeta \in H^1_c(S_2(K^f), \widetilde{\mathcal{W}}_{\mathcal{O}_\mathcal{L}})$, such that the Poincaré-pairing between $\varpi^{\delta_0}\omega_{p^e}$ and ζ equals one. Our strategy will be to evaluate ω_{p^e} at a system of generating elements $\{c\}$ in $H^1_c(S_2(K^f), \widetilde{\mathcal{W}}_{\mathcal{O}_\mathcal{L}})$. Then $-\delta_0$ is equal to the minimum of the $\mathcal{L}$-adic orders of the values $\langle \omega_{p^e}, c\rangle$ and of course the denominator δ of ω_{p^e} is equal to δ_0 if δ_0 is greater than or equal to zero and $\delta = 0$ otherwise.
The last section of this chapter, section 4.7, is concerned with the comparatively straightforward computation of the denominators of the elements $r^*_{u(i,j,y;p^e)}\Omega(\pi)^{-1}\omega \in H^2_c(S_3(K^f), \mathcal{V}_{F_\mathcal{L}})$, where K^f is a compact open subgroup of $\mathrm{GL}_3(\mathbb{A}_f)$.

Now, we want to explain how the denominators of ω_{p^e} and of $r^*_{u(i,j,y;p^e)}\omega$ are related to the denominators of $\mu^\eta_{\pi,l}/\Omega(\pi)$, which we are actually interested in. The representation (ρ, V) is defined over $\mathbb{Z}$, let $V_\mathbb{Z}$ be the corresponding lattice. Denote by $\mathrm{tr}_{\mathrm{int}}$ the pairing (3.10) restricted to $V_\mathbb{Z}|_{\mathrm{GL}_2} \times W_\mathbb{Z}$, where the $\mathbb{Z}$-structure $W_\mathbb{Z} \subset W$ has been defined in section 4.1. Since $V_\mathbb{Z}$ resp. $W_\mathbb{Z}$ can be regarded as submodules of $V_\mathbb{Q}$ resp. $W_\mathbb{Q}$ and the pairing tr takes rational values on $V_\mathbb{Q}|_{\mathrm{GL}_2} \times W_\mathbb{Q}$, we find that the pairing $\mathrm{tr}_{\mathrm{int}}$ is $\mathbb{Q}$-valued. Replacing $V_\mathbb{Z}$ by the lattice $aV_\mathbb{Z}$ for a suitable constant $a \in \mathbb{Q}^*$, we can even assume that the pairing $\mathrm{tr}_{\mathrm{int}}$ is $\mathbb{Z}$-valued, i.e. we obtain a non-trivial, $\mathrm{GL}_2(\mathbb{Z})$-invariant pairing

$$\mathrm{tr}_{\mathrm{int}} : V_\mathbb{Z}|_{\mathrm{GL}_2} \times W_\mathbb{Z} \to \mathbb{Z}.$$

Let $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ be a compact open subgroup, which keeps $\prod_\ell V_\mathbb{Z}|_{\mathrm{GL}_2} \otimes \mathbb{Z}_\ell$ and $\prod_\ell W_\mathbb{Z} \otimes \mathbb{Z}_\ell$ stable. Similarly as in section 3.3, we denote by $\mathcal{V}^{F_2}_{K,F_\mathcal{L}}$ the sheaf on $F_2(K)$ induced by the representation $V_{F_\mathcal{L}}|_{\mathrm{GL}_2}$ and, analogously to the last section, we want to construct a subsheaf $\mathcal{V}^{F_2}_{K,\mathcal{O}_\mathcal{L}}$ of $\mathcal{V}^{F_2}_{K,F_\mathcal{L}}$. For any open set $U \subset F_2(K)$ we define similarly to (4.2)

$$\mathcal{V}^{F_2}_{K,\mathcal{O}_\mathcal{L}}(U) = \{f \in \mathcal{V}^{F_2}_{K,F_\mathcal{L}}(U) \mid f((x_\infty, g_f)) \in (g_f V_\mathbb{Z}|_{\mathrm{GL}_2}) \otimes \mathcal{O}_\mathcal{L} \text{ for } (x_\infty, g_f) \in \pi^{-1}(U)\},$$

where π denotes the quotient map $\pi : \mathrm{GL}_2(\mathbb{A})/K_{2,\infty}K \to F_2(K)$. If K_3 is a compact open subgroup of $\mathrm{GL}_3(\mathbb{A}_f)$ with $K_3 \cap \mathrm{GL}_2(\mathbb{A}_f) = K$, then the sheaf $\mathcal{V}^{F_2}_{K,\mathcal{O}_\mathcal{L}}$ coincides with the inverse image sheaf $i(K_3)^*\mathcal{V}_{\mathcal{O}_\mathcal{L}}$, where $\mathcal{V}_{\mathcal{O}_\mathcal{L}}$ denotes the sheaf $\mathcal{V}_\mathbb{Z} \otimes \underline{\mathcal{O}_\mathcal{L}}$ on $S_3(K_3)$. Similarly, we define a subsheaf $\mathcal{W}^{F_2}_{K,\mathcal{O}_\mathcal{L}}$ of the sheaf $\mathcal{W}^{F_2}_{K,F_\mathcal{L}}$ on $F_2(K)$ by

$$\mathcal{W}^{F_2}_{K,\mathcal{O}_\mathcal{L}}(U) = \{f \in \mathcal{W}^{F_2}_{K,F_\mathcal{L}}(U) \mid f((x_\infty, g_f)) \in (g_f W_\mathbb{Z}) \otimes \mathcal{O}_\mathcal{L} \text{ for } (x_\infty, g_f) \in \pi^{-1}(U)\}$$

for every open set U in $F_2(K)$ and find that $\mathcal{W}^{F_2}_{K,\mathcal{O}_\mathcal{L}} \cong p(K)^*\mathcal{W}_{\mathcal{O}_\mathcal{L}}$. In (3.11) we have introduced the pairing tr_K of the sheaves $\mathcal{V}^{F_2}_{K,F_\mathcal{L}}$ and $\mathcal{W}^{F_2}_{K,F_\mathcal{L}}$. We denote its restriction to the sheaves $\mathcal{V}^{F_2}_{K,\mathcal{O}_\mathcal{L}}$ and $\mathcal{W}^{F_2}_{K,\mathcal{O}_\mathcal{L}}$ by $\mathrm{tr}_{\mathrm{int}}$. The next Lemma asserts that the pairing $\mathrm{tr}_{\mathrm{int}}$ is $\mathcal{O}_\mathcal{L}$-valued.

Lemma 4.2. *The image of* $\mathrm{tr}_{\mathrm{int}}$ *lies in* $\underline{\mathcal{O}_\mathcal{L}}$,

$$\mathrm{tr}_{\mathrm{int}} : \mathcal{V}^{F_2}_{K,\mathcal{O}_\mathcal{L}} \otimes \mathcal{W}^{F_2}_{K,\mathcal{O}_\mathcal{L}} \longrightarrow \underline{\mathcal{O}_\mathcal{L}}.$$

Proof. Let $f \in \mathcal{V}^{F_2}_{K,\mathcal{O}_\mathcal{L}}(U)$ and $h \in \mathcal{W}^{F_2}_{K,\mathcal{O}_\mathcal{L}}(U)$ for an open subset U in $F_2(K)$. For $(x_\infty, g_f) \in \pi^{-1}(U)$ we have

$$\big(\mathrm{tr}_{\mathrm{int}}(f,h)\big)(x_\infty, g_f) := \mathrm{tr}\big(f(x_\infty, g_f), h(x_\infty, g_f)\big),$$

where, of course, on the left hand side $\mathrm{tr}_{\mathrm{int}}$ means the pairing of sheaves and on the right hand side tr denotes the pairing of modules. The element $\mathrm{tr}_{\mathrm{int}}(f,h)$ indeed defines a locally constant function on U since f and h are locally constant and the pairing tr of modules is $\mathrm{GL}_2(\mathbb{Q})$-invariant. Let us now show that $\mathrm{tr}_{\mathrm{int}}(f,h)$ is $\mathcal{O}_\mathcal{L}$-valued. We have seen that $f(x_\infty, g_f)$ lies in $(g_f V_\mathbb{Z}|_{\mathrm{GL}_2}) \otimes \mathcal{O}_\mathcal{L}$ and, analogously, one checks that $h(x_\infty, g_f)$ lies in $g_f W_\mathbb{Z} \otimes \mathcal{O}_\mathcal{L}$. Thus, we are done if we can show that the pairing tr of modules is $\mathbb{Z}$-valued on the lattices $g_f V_\mathbb{Z}|_{\mathrm{GL}_2}$ and $g_f W_\mathbb{Z}$. Recalling that the space $g_f W_\mathbb{Z}$ is defined as the intersection $\cap_\ell g_\ell W_{\mathbb{Z}_\ell} \cap W_\mathbb{Q}$, where $g_f = (g_\ell) \in \mathrm{GL}_2(\mathbb{A}_f)$, we obtain an embedding of $g_f W_\mathbb{Z}$ into $g_\ell W_{\mathbb{Z}_\ell}$ for every prime ℓ and analogously for $g_f V_\mathbb{Z}|_{\mathrm{GL}_2}$. Since the pairing tr on the modules $V_\mathbb{Z}|_{\mathrm{GL}_2} \otimes \mathbb{Q}_\ell$ and $W_{\mathbb{Q}_\ell}$ is invariant under $\mathrm{GL}_2(\mathbb{Q}_\ell)$ we obtain the following diagram

$$\begin{array}{ccccc} g_f V_\mathbb{Z}|_{\mathrm{GL}_2} & \times & g_f W_\mathbb{Z} & \xrightarrow{\mathrm{tr}} & \mathbb{Q} \\ \cap & & \cap & & \\ g_\ell(V_\mathbb{Z}|_{\mathrm{GL}_2} \otimes \mathbb{Z}_\ell) & \times & g_\ell W_{\mathbb{Z}_\ell} & \xrightarrow{\mathrm{tr}} & \mathbb{Z}_\ell. \end{array}$$

Consequently, the image of the pairing tr on $g_f V_\mathbb{Z}|_{\mathrm{GL}_2} \times g_f W_\mathbb{Z}$ lies in $\mathbb{Q} \cap \mathbb{Z}_\ell = \mathbb{Z}(\ell)$ for all primes ℓ. Thus, the pairing tr of modules is $\mathbb{Z}$-valued on $g_f V_\mathbb{Z}|_{\mathrm{GL}_2} \times g_f W_\mathbb{Z}$ and we conclude that the pairing $\mathrm{tr}_{\mathrm{int}}$ of sheaves is $\underline{\mathcal{O}_\mathcal{L}}$-valued. □

Let us write $\widetilde{\mathrm{tr}_{\mathrm{int}}}$ for the pairing in cohomology induced by the pairing of sheaves $\mathrm{tr}_{\mathrm{int}}$ according to section 2.6. We obtain the following extension of diagram (3.12)

$$\begin{array}{ccccc} H^2_c(F_2(K), \mathcal{V}^{F_2}_{K,\mathcal{O}_\mathcal{L}}) & \times & H^1(F_2(K), \mathcal{W}^{F_2}_{K,\mathcal{O}_\mathcal{L}}) & \xrightarrow{\widetilde{\mathrm{tr}_{\mathrm{int}}}} & \mathcal{O}_\mathcal{L} \\ \downarrow & & \downarrow & & \downarrow \\ H^2_c(F_2(K), \mathcal{V}^{F_2}_{K,F_\mathcal{L}}) & \times & H^1(F_2(K), \mathcal{W}^{F_2}_{K,F_\mathcal{L}}) & \xrightarrow{\widetilde{\mathrm{tr}_K}} & F_\mathcal{L} \\ {\scriptstyle i(K_3)^*}\uparrow & & {\scriptstyle p(K)^*}\uparrow & & \\ H^2_c(S_3(K_3), \mathcal{V}_{F_\mathcal{L}}) & & H^1(S_2(K), \mathcal{W}_{F_\mathcal{L}}), & & \end{array} \tag{4.3}$$

where K_3 is any compact open subgroup of $\mathrm{GL}_3(\mathbb{A}_f)$ with $K = K_3 \cap \mathrm{GL}_2(\mathbb{A}_f)$, cf. Appendix A. According to Lemma 3.7, the distribution $\mu^\eta_{\pi,l}/\Omega(\pi)$ is up to a factor given by the pairing $\langle \cdot, \cdot \rangle$, which was defined in (3.13), evaluated at the elements $r_u^*\Omega(\pi)^{-1}\omega$ and ω_{p^e}, where we abbreviate $u(i,j,y;p^e)$ to u. Thus, the next corollary shows that bounds for the denominators of ω_{p^e} and the denominators of $r_u^*\Omega(\pi)^{-1}\omega$ will yield bounds for the denominators of the distribution $\mu^\eta_{\pi,l}/\Omega(\pi)$, cf. Theorem 5.1 below.

Corollary 4.3. *Let F be the algebraic number field $E_\pi(\eta,\eta',\zeta_{q(q-1)})$, let $\mathcal{L}$ be a prime ideal in its ring of integers $\mathcal{O}$ and let ϖ be a uniformizing parameter in $\mathcal{O}_\mathcal{L}$. If K is a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ which keeps $\prod_\ell V_\mathbb{Z}|_{\mathrm{GL}_2}\otimes\mathbb{Z}_\ell$, $\prod_\ell W_\mathbb{Z}\otimes\mathbb{Z}_\ell$ and the elements $i^*r_u^*\omega$ and $p^*\omega_{p^e}$ stable, then we have*

$$\left\langle r^*_{u(i,j,y;p^e)}\frac{\omega}{\Omega(\pi)},\omega_{p^e}\right\rangle\in\mathrm{vol}(K)\varpi^{-\delta-\delta'}\mathcal{O}_\mathcal{L},$$

where δ resp. δ' denotes the $\mathcal{L}$-denominator of $r_u^*\Omega(\pi)^{-1}\omega\in H^2_c(S_3(K_3),\mathcal{V}_{F_\mathcal{L}})$ resp. of $\omega_{p^e}\in H^1(S_2(K),\mathcal{W}_{F_\mathcal{L}})$. Here, K_3 is any compact open subgroup of $\mathrm{GL}_3(\mathbb{A}_f)$ which stabilizes $\prod_\ell V_\mathbb{Z}\otimes\mathbb{Z}_\ell$ and $r_u^*\omega$ and satisfies $K\subset K_3\cap\mathrm{GL}_2(\mathbb{A}_f)$.

Proof. Recall that the pairing $\langle\cdot,\cdot\rangle$ is defined as $\widetilde{\mathrm{tr}}(i^*\cdot,p^*\cdot)$ and that $\widetilde{\mathrm{tr}}$ is the inductive limit of the pairings $\mathrm{vol}(K^f)\widetilde{\mathrm{tr}_{K^f}}$, so we have

$$\left\langle r_u^*\frac{\omega}{\Omega(\pi)},\omega_{p^e}\right\rangle=\mathrm{vol}(K)\widetilde{\mathrm{tr}_K}\left(\frac{i^*r_u^*\omega}{\Omega(\pi)},p^*\omega_{p^e}\right).$$

Since $\varpi^\delta r_u^*\Omega(\pi)^{-1}\omega$ lies in $H^2(S_3(K_3),\mathcal{V}_{\mathcal{O}_\mathcal{L}})_{\mathrm{int}}$ and the following diagram is commuting, cf. [Bre67], example II.6.3,

$$\begin{array}{ccccc}
H^2_c(S_3(K_3),\mathcal{V}_{F_\mathcal{L}}) & \xrightarrow{i(K_3)^*} & H^2_c(F_2(K_2),\mathcal{V}^{F_2}_{K_2,F_\mathcal{L}}) & \longrightarrow & H^2_c(F_2(K),\mathcal{V}^{F_2}_{K,F_\mathcal{L}}) \\
\uparrow & & \uparrow & & \uparrow \\
H^2_c(S_3(K_3),\mathcal{V}_{\mathcal{O}_\mathcal{L}}) & \xrightarrow{i(K_3)^*} & H^2_c(F_2(K_2),\mathcal{V}^{F_2}_{K_2,\mathcal{O}_\mathcal{L}}) & \longrightarrow & H^2_c(F_2(K),\mathcal{V}^{F_2}_{K,\mathcal{O}_\mathcal{L}}),
\end{array}$$

where K_2 is the compact open subgroup $K_3\cap\mathrm{GL}_2(\mathbb{A}_f)$ and the second horizontal arrows denote the maps induced by the projection $F_2(K)\to F_2(K_2)$, we find that

$$\varpi^\delta i^*(r_u^*\Omega(\pi)^{-1}(\omega))\in H^2_c(F_2(K),\mathcal{V}^{F_2}_{K,\mathcal{O}_\mathcal{L}})_{\mathrm{int}}.$$

Similarly, one checks that $\varpi^{\delta'}p^*(\omega_{p^e})$ lies in $H^1(F_2(K),\mathcal{W}^{F_2}_{K,\mathcal{O}_\mathcal{L}})_{\mathrm{int}}$. The commutativity of diagram (4.3) now gives the claim of the proposition. □

For the computation of the denominators of ω_{p^e} we make use of the following technical simplification: Consulting Theorem 2.5, we observe that $\mathrm{Ind}(\alpha^{l-1/2},\eta_0\chi\alpha^{-(l-1/2)})=\eta'^{-1}\otimes\Pi(\chi)$ too occurs in the boundary cohomology of S_2 with coefficients in $\mathcal{W}$. We denote by $\psi^0_{p^e,f}\in\eta'^{-1}\otimes\Pi(\chi)$ the finite part of the element $\eta'^{-1}\otimes\psi_{1,p^e}$, where ψ_{1,p^e} is the vector ψ_{ϵ,p^e} with $\epsilon=1$ which was defined in section 3.2. From the definition of ψ_{1,p^e} one checks that $\psi^0_{p^e,f}$ is invariant under $K_1(p^e,q):=K_1(2,p^e)\times K_1(2,q)$.
Let e_0^1 be a generator of $H^1(\mathrm{gl}_2,K_{2,\infty}Z_2^0(\mathbb{R}),\mathrm{Ind}(\alpha_\infty^{l-1/2},\alpha^{-(l-1/2)})\otimes\kappa)=\mathbb{C}$,

$$e_0^1=\sum_{i,b}\psi^0_{\infty,i,b}w_b\otimes\omega_i,$$

where $\psi^0_{\infty,i,b}$ lies in $\eta'^{-1}\otimes\Pi_\infty(\chi)$, $\{w_b\}$ is a basis of W and $\{\omega_1,\omega_2\}$ is a basis of the dual space $(\mathrm{gl}_2/\mathrm{so}_2\,\mathrm{Lie}(\mathrm{Z}_2^0(\mathbb{R})))^*$. According to [Har87b], section 3.4, we can normalize the generator e_0^1 by requiring $e_0^1\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\right)=Y^{2l-2}$. We now set

$$\omega^0_{p^e}(g,D):=\sum_{\gamma\in B_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{Q})}\psi^0_{p^e,f}e_0^1(\gamma g,D)$$

and as in section 3.3 we deduce that $\omega_{p^e}^0$ lies in $H^1(S_2(K_1^f(p^e,q)),\mathcal{W})$, where

$$K_1^f(p^e,q) := K_1(p^e,q)\prod_{\ell\neq p,q,\infty}\mathrm{GL}_2(\mathbb{Z}_\ell).$$

Adapting the considerations at the beginning of section 3.2 in [Mah00] one checks that $\omega_{p^e}^0$ even lies in $H^1(S_2(K_1^f(p^e,q)),\mathcal{W}_{\mathbb{Q}(\eta_0)})$.
Obviously, we have $\omega_{p^e}=\eta'\otimes\omega_{p^e}^0$. For instance by the explicit form of the Poincaré pairing in relative Lie algebra cohomology, cf. section 4.5 below, one checks that the denominators of ω_{p^e} and of $\omega_{p^e}^0$ coincide. Thus, in this chapter we compute the denominators of the Eisenstein classes $\omega_{p^e}^0$ although we are actually interested in the denominators of ω_{p^e}.

4.3. Singular Homology

Here, we will work in the classical instead of the adèlic setting. Let X be either the upper half plane $\mathbb{H}$, its Borel-Serre compactification $\bar{\mathbb{H}}$ or the boundary of the Borel-Serre compactification $\partial\bar{\mathbb{H}}$. The following notations remain valid until the end of this chapter: Γ resp. $\Gamma_1(m)$ denotes the full group $\mathrm{SL}_2(\mathbb{Z})$ resp. its congruence subgroup consisting of matrices $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ satisfying $c\equiv 0$ and $d\equiv 1 \mod m$.
For a Γ-module W, which is also an R-module for some ring R, we will now define the singular homology groups $H_i(\Gamma\backslash X,W)$. As usual, we denote by $\Delta_i:=\{(t_0,\dots t_i)\in\mathbb{R}^{i+1}\mid\sum t_j=1\}$ the i-dimensional standard simplex. Let $C_i(X)$ be the free abelian group of $\mathbb{Z}$-linear combinations of continuous functions $c:\Delta_i\to X$. We have face maps on the simplices

$$\begin{aligned}\tau_q:\qquad &\Delta_{i-1}\longrightarrow\Delta_i,\\ &(t_0,\dots t_{i-1})\longmapsto(t_0,\dots t_{q-1},0,t_q,\dots t_{i-1}).\end{aligned}$$

Now we can define a boundary operator $\partial_i:C_i(X)\to C_{i-1}(X)$ by setting

$$\partial_i(c)=\sum_{q=0}^{i}(-1)^q\,c\circ\tau_q$$

and one can check that this turns

$$\dots\to C_i(X)\to C_{i-1}(X)\to\ \dots\ \to C_0(X)\to 0$$

into a complex $C_\bullet(X)$. If we tensorize every module in this complex by W and extend the boundary map ∂_i to this tensor products by setting $\partial_i(c\otimes w)=\partial_i(c)\otimes w$ for a pure tensor $c\otimes w\in C_i(X)\otimes_{\mathbb{Z}}W$, we again obtain a complex $C_\bullet(X)\otimes W$. The $\mathbb{Z}$-modules $C_i(X)\otimes W$ carry a Γ-module structure defined by

$$\gamma(c(t)\otimes w):=\gamma(c(t))\otimes\gamma w$$

for $\gamma\in\Gamma$, $c:\Delta_i\to X$, $t\in\Delta_i$ and $w\in W$. Obviously, this Γ-action is compatible with the boundary map ∂_i. Thus, we obtain a complex

$$\dots\to(C_i(X)\otimes W)_\Gamma\to(C_{i-1}(X)\otimes W)_\Gamma\to\ \dots\ \to(C_0(X)\otimes W)_\Gamma\to 0,$$

where $(C_i(X)\otimes W)_\Gamma$ denotes the module of Γ-coinvariants. Finally, we define the singular homology groups as the homology of this complex

$$H_i(\Gamma\backslash X,W):=H_i((C_\bullet(X)\otimes W)_\Gamma).$$

In the last section we have seen that we have to evaluate the Poincaré pairing on the Eisenstein class $\omega^0_{p^e}$ and the elements in a generating system of $H^1_c(S_2(K^f_1(p^e,q)),\widetilde{\mathcal{W}}_{\mathcal{O}_\mathcal{L}})$ to determine the denominator of $\omega^0_{p^e}$. Actually, we will make use of the following isomorphism, cf. [Har87c], section E.2,

$$H^1_c(S_2(K^f_1(p^e,q)),\widetilde{\mathcal{W}}_{\mathcal{O}_\mathcal{L}}) = H_1(\Gamma_1(p^eq)\backslash\mathbb{H},\widetilde{W}_{\mathcal{O}_\mathcal{L}}),$$

i.e. we will evaluate the Poincaré pairing on the Eisenstein class $\omega^0_{p^e}$ and the elements in a generating system of $H_1(\Gamma_1(p^eq)\backslash\mathbb{H},\widetilde{W}_{\mathcal{O}_\mathcal{L}})$, which will be constructed in the next section.
Let $\widetilde{\omega}^0_{p^e}$ be a differential form on $\Gamma_1(p^eq)\backslash\mathbb{H} = S_2(K^f_1(p^e,q))$ representing $\omega^0_{p^e}$ and let $c\otimes m$ be a cocycle in $C_1(\mathbb{H})\otimes\widetilde{W}_{\mathcal{O}_\mathcal{L}}$. Then for every $t\in\Delta_1$ the element $\mathrm{tr}(\omega^0_{p^e}(c(t)),m)$ is a scalar-valued 1-form in $c(t)$. We can transport this 1-form via c to a 1-form on Δ_1 and the Poincaré-pairing of $\omega^0_{p^e}$ and $c\otimes m$ is then given by the integral of this form over Δ_1, cf. [Har87c], section E.3.
For further properties of the singular homology groups and their definition in a more general setting we refer to [Har87c], section E.1.

4.4. Special Cycles in the Homology

In the following, we will denote by $\mathcal{O}$ the ring of integers of $\mathbb{Q}(\eta_0)$. Our aim in this section is to construct a system of generating cycles for $H_1(\Gamma_1(p^eq)\backslash\mathbb{H},\widetilde{W}_\mathcal{O})$. We will make use of the Shapiro Isomorphism in homology, cf. [Kai91], page 24,

$$H_i(\Gamma_1(p^eq)\backslash\mathbb{H},\widetilde{W}_\mathcal{O}) \xrightarrow{\sim} H_i(\Gamma\backslash\mathbb{H},\widetilde{W}_\mathcal{O}\otimes_\mathcal{O} M). \tag{4.4}$$

Here, M denotes the induced Γ-module $\mathrm{ind}^\Gamma_{\Gamma_1(p^eq)\langle\pm1\rangle}\mathcal{O}$, where $\mathcal{O}$ is the one-dimensional $\mathcal{O}$-module with trivial $\Gamma_1(p^eq)\langle\pm1\rangle$-action. Explicitly, we have $\mathrm{ind}^\Gamma_{\Gamma_1(p^eq)\langle\pm1\rangle}\mathcal{O} = \{f:\Gamma\to\mathcal{O} | f(\gamma'\gamma) = f(\gamma),\gamma'\in\Gamma_1(p^eq)\langle\pm1\rangle,\gamma\in\Gamma\}$ and Γ acts on this space of functions by right translation. In degree one we can write down the action of the Shapiro Isomorphism explicitly, an element $c\otimes w\otimes\alpha$ on the right hand side in (4.4) is sent to $\sum_{\gamma\in\Gamma_1(p^eq)\langle\pm1\rangle\backslash\Gamma}\alpha(\gamma)\gamma c\otimes\gamma w$, cf. [Kai91], page 24. (Of course, c is a singular 1-chain for $\mathbb{H}$, $w\in\widetilde{W}_\mathcal{O}$ and α lies in M.) The Shapiro Isomorphism holds analogously in relative homology and in homology on the boundary. We are interested in describing a system of generating cycles for $H_1(\Gamma_1(p^eq)\backslash\mathbb{H},\widetilde{W}_\mathcal{O})$, but due to the Shapiro Isomorphism it suffices to find a system of generating cycles for the right hand side in (4.4), $H_1(\Gamma\backslash\mathbb{H},\widetilde{W}_\mathcal{O}\otimes_\mathcal{O} M)$. For this purpose we will make use of the long exact homology sequence

$$\ldots\to H_i(\Gamma\backslash\mathbb{H},\widetilde{W}_\mathcal{O}\otimes_\mathcal{O} M)\xrightarrow{\mathrm{rel}} H_i(\Gamma\backslash\bar{\mathbb{H}},\partial\Gamma\backslash\bar{\mathbb{H}},\widetilde{W}_\mathcal{O}\otimes_\mathcal{O} M)\xrightarrow{\partial} H_{i-1}(\partial\Gamma\backslash\bar{\mathbb{H}},\widetilde{W}_\mathcal{O}\otimes_\mathcal{O} M)\to\ldots. \tag{4.5}$$

The clue is that it can be shown that the first relative homology group consists of the cycles $\mathcal{Z}_{0,\infty}\otimes m\otimes\psi$ with $m\in\widetilde{W}_\mathcal{O}$ and $\psi\in M$, where $\mathcal{Z}_{0,\infty}$ denotes the geodesic in $\mathbb{H}$ running from $\{\infty\}_0$ to $\{0\}_\infty$ as introduced in section 2.2. We want to find out which of the cycles $\mathcal{Z}_{0,\infty}\otimes m\otimes\psi$ are coming from absolute cycles. Since the sequence (4.5) is exact $\mathcal{Z}_{0,\infty}\otimes m\otimes\psi$ is contained in the image of rel if and only if

$$\partial(\mathcal{Z}_{0,\infty}\otimes m\otimes\psi) = \{0\}_\infty\otimes(m\otimes\psi - \left(\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right)(m\otimes\psi)) = 0 \quad \text{in } H_0(\partial\Gamma\backslash\bar{\mathbb{H}},\widetilde{W}_\mathcal{O}\otimes M). \tag{4.6}$$

Let $T = \left(\begin{smallmatrix}1&1\\&1\end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix}&1\\-1&\end{smallmatrix}\right)$. Since $\partial\Gamma\backslash\bar{\mathbb{H}}$ and $N_2(\mathbb{Z})\backslash\mathbb{H}$ are homotopy equivalent, cf. [Har87c], chapter V, page 11, we know that in particular their zeroth homology groups are isomorphic

$$H_0(\partial\Gamma\backslash\bar{\mathbb{H}},\widetilde{W}_\mathcal{O}\otimes M)\cong\left(\widetilde{W}_\mathcal{O}\otimes M\right)/\left((1-T)\widetilde{W}_\mathcal{O}\otimes M\right). \tag{4.7}$$

Here, we have made use of the isomorphism $H_0(N_2(\mathbb{Z})\backslash\mathbb{H}, \widetilde{W}_{\mathcal{O}} \otimes M) \cong \widetilde{W}_{\mathcal{O}} \otimes M/(1-T)\widetilde{W}_{\mathcal{O}} \otimes M$ explained on page 5 in chapter E in [Har87c]. An element $m \otimes \psi$ on the right hand side of (4.7) is mapped to the class of the zero chain $\{1\}_\infty \otimes m \otimes \psi$ lying on the left hand side. Note that $\{1\}_\infty \otimes m \otimes \psi$ and $\{x\}_\infty \otimes m \otimes \psi$ for $x \in \mathbb{R}$ are homologous.
Thus, equation (4.6) is fulfilled if and only if there exist $\psi' \in M$ and $m' \in \widetilde{W}_{\mathcal{O}}$ such that

$$m \otimes \psi - w(m \otimes \psi) = (1-T)(m' \otimes \psi'). \tag{4.8}$$

Since $\{1\}_\infty \otimes (1-T)(m' \otimes \psi') = \partial([0,1]_\infty \otimes m' \otimes \psi')$ we see that $\mathcal{Z}_{0,\infty} \otimes m \otimes \psi$ is mapped to zero by ∂, i.e. equation (4.6) is fulfilled, if and only if there exists ψ' and m' with

$$\partial(\mathcal{Z}_{0,\infty} \otimes m \otimes \psi) = \partial([0,1]_\infty \otimes m' \otimes \psi').$$

Considering the exactness of the homology sequence (4.5) we conclude that the homology group $H_1(\Gamma\backslash\mathbb{H}, \widetilde{W}_{\mathcal{O}} \otimes_{\mathcal{O}} M)$ is generated modulo cycles which are supported on the boundary by the cycles $\mathcal{Z}_{0,\infty} \otimes m \otimes \psi - [0,1]_\infty \otimes m' \otimes \psi'$ where m, ψ, m' and ψ' are related as described in equation (4.8). Applying the Shapiro Isomorphism to this cycle we obtain $\mathcal{Z}_{\psi,m} := \mathcal{Z}^i_{\psi,m} - \mathcal{Z}^b_{\psi,m}$ with

$$\mathcal{Z}^i_{\psi,m} := \sum_\gamma \psi(\gamma)\, \gamma \mathcal{Z}_{0,\infty} \otimes \gamma m \qquad \text{and} \qquad \mathcal{Z}^b_{\psi,m} := \sum_\gamma \psi'(\gamma)\, \gamma [0,1]_\infty \otimes \gamma m',$$

where in both sums γ is running over $\Gamma_1(p^e q)\langle \pm 1\rangle\backslash\Gamma$. Since the Shapiro Isomorphism preserves cycles supported on the boundary we finally conclude that $H_1(\Gamma_1(p^e q)\backslash\mathbb{H}, \widetilde{W}_{\mathcal{O}})$ is generated (modulo cycles supported on the boundary) by the cycles $\mathcal{Z}_{\psi,m}$ with ψ and m satisfying (4.8).
We identify the coset spaces $\Gamma_1(p^e q)\backslash\Gamma = K_1(p^e, q)\backslash \mathrm{GL}_2(\mathbb{Z}_p) \times \mathrm{GL}_2(\mathbb{Z}_q)$. In section 4.6 below we will evaluate the Eisenstein class $\omega^0_{p^e}$ on every summand $\psi(\gamma)\gamma(\mathcal{Z}_{0,\infty} \otimes m)$, $\gamma \in \Gamma_1(p^e q)\backslash\Gamma$ of the relative cycle $\mathcal{Z}^i_{\psi,m}$ separately. Actually, we will only have to evaluate $\omega^0_{p^e}$ on the elements $\gamma(\mathcal{Z}_{0,\infty} \otimes X^k Y^{2l-2-k})$ because the elements $\psi(\gamma)\gamma(\mathcal{Z}_{0,\infty} \otimes m)$ can be built up as $\mathcal{O}$-linear combinations. Since $w\mathcal{Z}_{0,\infty} = -\mathcal{Z}_{0,\infty}$ and $wX^k Y^{2l-2-k} = (-1)^k Y^k X^{2l-2-k}$ we can restrict ourselves to $\gamma \in K_1(p^e, q)\backslash \mathrm{GL}_2(\mathbb{Z}_p) \times \mathrm{GL}_2(\mathbb{Z}_q)/\langle w\rangle$, where w is embedded diagonally into $\mathrm{GL}_2(\mathbb{Z}_p) \times \mathrm{GL}_2(\mathbb{Z}_q)$. The next Lemma, which is proven in [Mah00], page 286, provides us with a system of representatives of this double coset space.

Lemma 4.4. *The union of the following elements forms a system of representatives for the double coset space $K_1(p^e, q)\backslash GL_2(\mathbb{Z}_p) \times GL_2(\mathbb{Z}_q)/\langle w\rangle$:*

$$\begin{pmatrix} 1 & \\ & d \end{pmatrix} w \begin{pmatrix} 1 & p^k \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ & d \end{pmatrix} w \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix},$$

where $d \in \left(\mathbb{Z}/p^{e-k}q\mathbb{Z}\right)^*$, $t \in (\mathbb{Z}/p^e q\mathbb{Z})^*$ and $k = 0, \ldots, e$,

$$\begin{pmatrix} 1 & \\ & d \end{pmatrix} w \begin{pmatrix} 1 & p^k \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ & d \end{pmatrix} w,$$

where $d \in \left(\mathbb{Z}/p^{e-k}q\mathbb{Z}\right)^*$, $t \in (\mathbb{Z}/p^e\mathbb{Z})^*$ and $k = 0, \ldots, e$, and

$$\begin{pmatrix} 1 & \\ & d \end{pmatrix} w \begin{pmatrix} 1 & p^k \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ & d \end{pmatrix},$$

where $d \in \left(\mathbb{Z}/p^{e-k}q\mathbb{Z}\right)^*$, $t \in (\mathbb{Z}/p^e\mathbb{Z})^*$ and $k = 0, \ldots, e$. Here we identify $(\mathbb{Z}/p^{e-k}q\mathbb{Z})^* = (\mathbb{Z}/p^{e-k}\mathbb{Z})^* \times (\mathbb{Z}/q\mathbb{Z})^*$ and in particular for $k = e$ we mean $(\mathbb{Z}/q\mathbb{Z})^* = (\mathbb{Z}/\mathbb{Z})^* \times (\mathbb{Z}/q\mathbb{Z})^*$.
We will refer to the matrices above as $g_{d,k,t}$, $g'_{d,k,t}$ and $g''_{d,k,t}$.

4.5. The Integral on the Boundary

As explained in the last sections, the estimation of the denominators of $\omega^0_{p^e}$ is relying on the evaluation of the Poincaré pairing on $\omega^0_{p^e}$ and the elements $\mathcal{Z}_{\psi,m}$ in $H_1(\Gamma_1(p^eq)\backslash\mathbb{H}, \widetilde{W}_{\mathcal{O}})$ constructed above. This will be done in the next sections. We start with the evaluation of $\omega^0_{p^e}$ on the boundary components $\mathcal{Z}^b_{\psi,m}$.
In the following, we distinguish between cocyles $\omega^0_{p^e}$ in the relative Lie algebra cohomology and the associated cocycles $\widetilde{\omega}^0_{p^e}$ in the de Rham cohomology. We have to integrate $\widetilde{\omega}^0_{p^e}\mid_{\partial\bar{\mathbb{H}}}$ on

$$\mathcal{Z}^b_{\psi,m} = \sum_{\gamma\in\Gamma_1(p^eq)\langle\pm1\rangle\backslash\Gamma} \psi'(\gamma)\gamma([0,1]_\infty\otimes m').$$

As explained in [Kai91], page 11, there is a canonical isomorphism

$$\Gamma_1(p^eq)\backslash\partial\bar{\mathbb{H}} \cong B(\mathbb{Q})\backslash\left(\mathbb{H}^{\pm}_{\infty,\infty}\times G(\mathbb{A}_f)/K_1^f(p^e,q)\right), \tag{4.9}$$

where $\mathbb{H}^{\pm}_{\infty,\infty}$ is the disjoint union of two copies of $\mathbb{H}_{\infty,\infty}$. We recall that $C_i(\partial\bar{\mathbb{H}})$ denotes the abelian group of $\mathbb{Z}$-linear combinations of continuous functions $c:\Delta_i\to\partial\bar{\mathbb{H}}$ and define $C_i(\mathbb{H}^{\pm}_{\infty,\infty}\times G(\mathbb{A}_f)/K_1^f(p^eq))$ analogously. The isomorphism (4.9) induces an isomorphism of modules of coinvariants

$$\left(C_i(\partial\bar{\mathbb{H}})\otimes W_{\mathcal{O}}\right)_{\Gamma_1(p^eq)} \cong \left(C_i(\mathbb{H}^{\pm}_{\infty,\infty}\times G(\mathbb{A}_f)/K_1^f(p^e,q))\otimes W_{\mathcal{O}}\right)_{B(\mathbb{Q})}.$$

Via this isomorphism the element $\gamma([0,1]_\infty\otimes m')$ for $\gamma\in\Gamma_1(p^eq)\langle\pm1\rangle\backslash\Gamma$ corresponds to the element $([0,1]_\infty\times\gamma^{-1})\otimes m'$, where γ^{-1} is embedded diagonally into $G(\mathbb{A}_f)$. For y to infinity the chain $([yi,yi+1]\times\gamma^{-1})\otimes m'$, an element in $(C_1(\mathbb{H}\times G(\mathbb{A}_f)/K_1^f(p^e,q))\otimes W_{\mathcal{O}})_{G(\mathbb{Q})}$, converges to $([0,1]_\infty\times\gamma^{-1})\otimes m'$. Our strategy will be to integrate $\widetilde{\omega}^0_{p^e}$ on these converging inner chains and take the limit, this is exactly how $\widetilde{\omega}^0_{p^e}$ is defined on the boundary, cf. [Kai91], Lemma 2.2.4. The range of summation occurring in the definition of $\mathcal{Z}^b_{\psi,m}$ is described in Lemma 4.4. We have

$$\int_{\mathcal{Z}^b_{\psi,m}} \widetilde{\omega}^0_{p^e}\mid_{\partial\bar{\mathbb{H}}} = \sum_{\gamma\in\Gamma_1(p^eq)\langle\pm1\rangle\backslash\Gamma} \psi'(\gamma)\int_{\gamma([0,1]_\infty\otimes m')} \widetilde{\omega}^0_{p^e}\mid_{\partial\bar{\mathbb{H}}}. \tag{4.10}$$

Let us start with the computation of the summands corresponding to the elements $\gamma=\left(\begin{smallmatrix}1&\\&d\end{smallmatrix}\right)\in K_1(p^e,q)\backslash\mathrm{GL}_2(\mathbb{Z}_p)\times\mathrm{GL}_2(\mathbb{Z}_q)$ with $d\in(\mathbb{Z}/q\mathbb{Z})^*$. We denote by $[\left(\begin{smallmatrix}1&\\&d\end{smallmatrix}\right)]$ the associated element in $\Gamma_1(p^eq)\backslash\Gamma$. During the computations we will see that $\widetilde{\omega}^0_{p^e}$ vanishes on the cycles $\gamma([0,1]_\infty\otimes m')$ for γ not of the form $\gamma=\left(\begin{smallmatrix}1&\\&d\end{smallmatrix}\right)$ with $d\in(\mathbb{Z}/q\mathbb{Z})^*$. As explained above we have

$$\int_{[\left(\begin{smallmatrix}1&\\&d\end{smallmatrix}\right)](([0,1]_\infty\otimes m')} \widetilde{\omega}^0_{p^e}\mid_{\partial\bar{\mathbb{H}}} = \lim_{y\to\infty}\int_{[yi,yi+1]\times1\ldots1\times\left(\begin{smallmatrix}1&\\&d^{-1}\end{smallmatrix}\right)\times1\ldots1\otimes m'} \widetilde{\omega}^0_{p^e},$$

where $f_d := 1\ldots1\times\left(\begin{smallmatrix}1&\\&d^{-1}\end{smallmatrix}\right)\times1\ldots1$ is an element in $\mathbb{A}_f$ where the only nontrivial entry is at the qth position. We parametrize our chain $[yi,yi+1]$ as follows

$$\begin{aligned}\sigma_y : [0,1] &\longrightarrow \mathbb{H},\\ t &\longmapsto yi+t\end{aligned}$$

and lift it into the group $\mathrm{GL}_2(\mathbb{R})$

$$\begin{aligned}\widetilde{\sigma}_y : [0,1] &\longrightarrow \mathrm{GL}_2(\mathbb{R}),\\ t &\longmapsto \left(\begin{smallmatrix} y & t\\ & 1\end{smallmatrix}\right).\end{aligned}$$

According to [Har87c], section E.5, we obtain for our integral

$$\int\limits_{[\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)](([0,1]_\infty \otimes m')} \widetilde{\omega}^0_{p^e} \mid_{\partial\bar{\mathbb{H}}} = \lim_{y\to\infty} \int_0^1 \left\langle \widetilde{\sigma}_y(t)\omega^0_{p^e} \left((\widetilde{\sigma}_y(t), f_d), D_{L_{\left(\begin{smallmatrix} y & t\\ & 1\end{smallmatrix}\right)^{-1}}} \circ D_{\sigma_y}\left(\frac{\partial}{\partial t}\right)\right), m' \right\rangle dt,$$

where $D_{L_{\left(\begin{smallmatrix} y & t\\ & 1\end{smallmatrix}\right)^{-1}}}$ denotes the derivative of the left translation by the element $\left(\begin{smallmatrix} y & t\\ & 1\end{smallmatrix}\right)^{-1}$ on $\mathrm{GL}_2(\mathbb{R})$. Since

$$D_{L_{\left(\begin{smallmatrix} y & t\\ & 1\end{smallmatrix}\right)^{-1}}} \circ D_{\sigma_y}\left(\frac{\partial}{\partial t}\right) = \frac{1}{y}\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix},$$

cf. [Har87c], chapter 6, page 23, and $\omega^0_{p^e}$ is dominated by its constant term $\psi^0_{p^e,f}e^1_0$ for y to infinity, we further get

$$\int\limits_{[\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)](([0,1]_\infty \otimes m')} \widetilde{\omega}^0_{p^e} \mid_{\partial\bar{\mathbb{H}}} = \lim_{y\to\infty} \int_0^1 \psi^0_{p^e,f}(f_d)\frac{1}{y}\left\langle e^1_0\left(\widetilde{\sigma}_y(t), \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\right), \widetilde{\sigma}_y(t)^{-1}m' \right\rangle dt.$$

Now it becomes clear that the summands on the right hand side in equation (4.10) corresponding to the elements $\gamma \in K_1(p^e,q)\backslash \mathrm{GL}_2(\mathbb{Z}_p)\times \mathrm{GL}_2(\mathbb{Z}_q)$ not of the form $\left(\begin{smallmatrix}1 & 0\\ 0 & d\end{smallmatrix}\right)$ with $d \in (\mathbb{Z}/q\mathbb{Z})^*$ vanish. Indeed, Lemma 4.3 in [Mah00] shows that then $\psi^0_{p^e}(\gamma^{-1})$ is zero, where we consider γ^{-1} embedded into $\mathrm{GL}_2(\mathbb{A}_f)$. Furthermore, this Lemma states that $\psi^0_{p^e,f}(f_d) = \eta^{-1}_{0,q}(d)$. At the end of section 4 we have fixed e^1_0 by requiring $e^1_0\left(\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)\right) = Y^{2l-2}$, so we find

$$e^1_0\left(\begin{pmatrix} y & t\\ 0 & 1\end{pmatrix}, \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\right) = |y|^l Y^{2l-2}.$$

Since the expression $\langle Y^{2l-2}, X^kY^{2l-2-k}\rangle$ vanishes for all k smaller than $2l-2$ we are only interested in the coefficient of X^{2l-2} in $\widetilde{\sigma}_y(t)^{-1}m$. For $m' = P(X,Y) = \sum\limits_{k=0}^{2l-2} b_k X^{2l-2-k}Y^k \in W_{\mathcal{O}}$, we obtain

$$\widetilde{\sigma}_y(t)^{-1}m' = y^{1-l}P(1,-t)X^{2l-2} + \left(\text{Terms not containing } X^{2l-2}\right).$$

Finally, we obtain

$$\int\limits_{[\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)](([0,1]_\infty \otimes m')} \widetilde{\omega}^0_{p^e} \mid_{\partial\bar{\mathbb{H}}} = \eta^{-1}_{0,q}(d)\int_0^1 P(1,-t)dt = \eta^{-1}_{0,q}(d)\sum_{k=0}^{2l-2}\frac{b_k(-1)^k}{k+1}.$$

Altogether we have proven

Proposition 4.5. *For all $\psi \in M$, $m \in W_{\mathcal{O}}$ the integral of $\widetilde{\omega}^0_{p^e} \mid_{\partial\bar{\mathbb{H}}}$ on $\mathcal{Z}^b_{\psi,m}$ lies in $\frac{1}{(2l-2)!}\mathcal{O}$.*

4.6. The Inner Integral

To get a bound for the denominators of $\omega^0_{p^e}$ we still have to compute the integral of $\omega^0_{p^e}$ on the inner components

$$\mathcal{Z}^i_{\psi,m} = \sum_{\gamma\in\Gamma_1(p^eq)\langle\pm1\rangle\backslash\Gamma} \psi(\gamma)\gamma(\mathcal{Z}_{0,\infty}\otimes m).$$

We specify $m = X^\mu Y^{2l-2-\mu}$. Recalling the system of representatives for $\Gamma_1(p^eq)\backslash\Gamma$ listed in Lemma 4.4 we see that we have to evaluate $\omega^0_{p^e}$ on the chains $g_{d,k,t}(\mathcal{Z}_{0,\infty}\otimes m)$, $g'_{d,k,t}(\mathcal{Z}_{0,\infty}\otimes m)$ and $g''_{d,k,t}(\mathcal{Z}_{0,\infty}\otimes m)$.
Let T_1 denote the torus in GL(2) consisting of the elements of the form $\left(\begin{smallmatrix} * & \\ & 1\end{smallmatrix}\right)$, so $T_1\cong \mathrm{G}_m$, and let λ be an arbitrary map $\lambda : (\mathbb{Z}/p^eq\mathbb{Z})^* \to \mathcal{O}$. For an element γ in $K_1(p^e,q)\backslash\mathrm{GL}_2(\mathbb{Z}_p)\times\mathrm{GL}_2(\mathbb{Z}_q)$ we denote by $[\gamma]$ the corresponding element in $\Gamma_1(p^eq)\backslash\Gamma$. It turns out that it is convenient to replace the chains $g_{d,k,t}(\mathcal{Z}_{0,\infty}\otimes m)$ by the slightly more general chains

$$\mathcal{Z}_{d,k,\lambda,m} = \sum_{t\in T_1(\mathbb{Z}/p^eq\mathbb{Z})} \lambda(t)[g_{d,k}t](\mathcal{Z}_{0,\infty}\otimes m),$$

where $d\in(\mathbb{Z}/p^eq\mathbb{Z})^*, k = 0,\dots e$ and

$$g_{d,k} = \begin{pmatrix}1 & \\ & d\end{pmatrix}(w,w)\left(\begin{pmatrix}1 & p^k\\ & 1\end{pmatrix},\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\right)\in \mathrm{GL}_2(\mathbb{Z}_p)\times\mathrm{GL}_2(\mathbb{Z}_q).$$

We have to calculate

$$\int_{\mathcal{Z}_{d,k,\lambda,m}} \widetilde{\omega}^0_{p^e} = \sum_{t\in T_1(\mathbb{Z}/p^eq\mathbb{Z})} \lambda(t) \int_{\mathcal{Z}_{0,\infty}\times t^{-1}\mathbf{g}^{-1}_{\mathbf{d},\mathbf{k}}} \langle\widetilde{\omega}^0_{p^e}, m\rangle,$$

where $\mathbf{g}_{d,k} = (g_\ell)\in K_1^f(p^e,q)\backslash\mathrm{GL}_2(\hat{\mathbb{Z}})$ denotes the element where all components outside p,q are trivial and $(g_p,g_q) = g_{d,k}$ and similarly t is regarded as an element (t_ℓ) in $K_1^f(p^e,q)\backslash\mathrm{GL}_2(\hat{\mathbb{Z}})$ with $t_p\equiv t \pmod{p^e}$ and $t_q\equiv t\pmod q$. We parametrize our chain $\mathcal{Z}_{0,\infty}$ as follows

$$\begin{aligned}\sigma : T_1(\mathbb{R})^0 = \mathbb{R}^*_{>0} &\longrightarrow \mathbb{H},\\ t_\infty &\longmapsto \left(\begin{smallmatrix} t_\infty & \\ & 1\end{smallmatrix}\right) i.\end{aligned}$$

This yields

$$\int_{\mathcal{Z}_{0,\infty}\times t^{-1}\mathbf{g}^{-1}_{\mathbf{d},\mathbf{k}}} \langle\widetilde{\omega}^0_{p^e}, m\rangle = \int_{T_1(\mathbb{R})^0} \left\langle t_\infty\omega^0_{p^e}\left((t_\infty, t^{-1}\mathbf{g}^{-1}_{d,k}), D_{L_{t_\infty^{-1}}}\circ D_\sigma(t_\infty\frac{\partial}{\partial t_\infty})\right), m\right\rangle \frac{dt_\infty}{t_\infty},$$

where $D_{L_{t_\infty^{-1}}}$ denotes the derivative of the left translation by t_∞^{-1} and dt_∞ resp. $\frac{\partial}{\partial t_\infty}$ is a Haar measure resp. an invariant tangent vector field on $(\mathbb{R},+)$. As in [Har87c], chapter 6.2, page 28, we see that

$$D_{L_{t_\infty^{-1}}}\circ D_\sigma(t_\infty\frac{\partial}{\partial t_\infty}) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

and so we obtain

$$\int_{\mathcal{Z}_{d,k,\lambda,m}} \widetilde{\omega}^0_{p^e} = \sum_{t\in T_1(\mathbb{Z}/p^eq\mathbb{Z})} \lambda(t) \int_{T_1(\mathbb{R})^0} \left\langle \omega^0_{p^e}\left((t_\infty, t^{-1}\mathbf{g}^{-1}_{d,k}), \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right), t_\infty^{-1}m\right\rangle \frac{dt_\infty}{t_\infty}.$$

Recalling that $m = X^\mu Y^{2l-2-\mu}$ we immediately obtain that $\left(\begin{smallmatrix} t_\infty^{-1} & \\ & 1 \end{smallmatrix}\right) m = |t_\infty|^{l-\mu-1} m$. We adèlize λ as follows. We set

$$T_{1,p^e q} = \{x \in T_1(\hat{\mathbb{Z}}) : x_p \equiv 1 \pmod{p^e},\ x_q \equiv 1 \pmod{q}\},$$

so we have $T_1(\hat{\mathbb{Z}})/T_{1,p^e q} = (\mathbb{Z}/p^e q\mathbb{Z})^*$. Any $\mathbf{t} \in T_1(\mathbb{A})$ uniquely decomposes as $\mathbf{t} = r t_\infty k$ with $r \in T_1(\mathbb{Q})$, $t_\infty \in T_1^0(\mathbb{R}) = \mathbb{R}^*_{>0}$, $k \in T_1(\hat{\mathbb{Z}})$ and we define

$$\widetilde{\lambda} : T_1(\mathbb{Q})\backslash T_1(\mathbb{A}) \longrightarrow \mathcal{O}$$

by $\widetilde{\lambda}(\mathbf{t}) := \lambda(k^{-1} T_{1,p^e q})$. This enables us to adèlize our integral

$$\int\limits_{\mathcal{Z}_{d,k,\lambda,m}} \widetilde{\omega}^0_{p^e} = \mathrm{vol}^{-1} \int\limits_{T_1(\mathbb{Q})\backslash T_1(\mathbb{A})} |\mathbf{t}|^{l-\mu-1} \widetilde{\lambda}(\mathbf{t}) \left\langle \omega^0_{p^e}\left(\mathbf{t}\mathbf{g}^{-1}_{d,k}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), m \right\rangle d\mathbf{t},$$

where vol is the volume of $1 + p^e\mathbb{Z}_p \times 1 + q\mathbb{Z}_q$ and the Haar measure $d\mathbf{t} = \frac{dt_\infty}{|t_\infty|} \prod_{\ell \neq \infty} dt_\ell$ is chosen such that dt_ℓ, $\ell \neq \infty$ is the Haar measure on $\mathbb{Q}^*_\ell$ which assigns volume one to $\mathbb{Z}^*_\ell$.
Next, we replace the $\omega^0_{p^e}$ by its defining sum $\omega^0_{p^e}(\cdot,\cdot) = \sum_{\gamma \in B_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \psi^0_{p^e,f} e^1_0(\gamma\ \cdot, \cdot)$ and split the summation over γ according to the following decomposition of $\mathrm{GL}_2(\mathbb{Q})$ into disjoint $T_1(\mathbb{Q})$-orbits

$$\mathrm{GL}_2(\mathbb{Q}) = B_2(\mathbb{Q}) \ \sqcup\ B_2(\mathbb{Q}) \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right) \ \sqcup\ B_2(\mathbb{Q}) \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right) T_1(\mathbb{Q}).$$

The integrals corresponding to the first two $T_1(\mathbb{Q})$-Orbits vanish, cf. [Kai91], page 29, because for $k_\infty = \mathrm{id}$ and for $k_\infty = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ we have

$$\psi^0_{p^e,f} e^1_0\left(k_\infty \mathbf{t}\mathbf{g}^{-1}_{d,k}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \psi^0_{p^e,f}(\mathbf{g}^{-1}_{d,k}) |\mathbf{t}|^l k_\infty^{-1} e^1_0\left(1, \mathrm{Ad}\,(k_\infty) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

and from the definition of e^1_0, cf. section 4.2 and [Har87b] page 68/69, we see that the last expression is equal to zero. This leaves us with the computation of the integral corresponding to the third $T_1(\mathbb{Q})$-orbit, we find

$$\int\limits_{\mathcal{Z}_{d,k,\lambda,m}} \widetilde{\omega}^0_{p^e} = \mathrm{vol}^{-1} \int\limits_{T_1(\mathbb{Q})\backslash T_1(\mathbb{A})} |\mathbf{t}|^{l-\mu-1} \widetilde{\lambda}(\mathbf{t}) \sum_{a \in T_1(\mathbb{Q})} \left\langle \psi^0_{p^e,f} e^1_0\left(\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} a \mathbf{t}\mathbf{g}^{-1}_{d,k}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), m \right\rangle d\mathbf{t}.$$

Merging the sum and the integral gives

$$\int\limits_{\mathcal{Z}_{d,k,\lambda,m}} \widetilde{\omega}^0_{p^e} = \mathrm{vol}^{-1} \int\limits_{T_1(\mathbb{A})} |\mathbf{t}|^{l-\mu-1} \widetilde{\lambda}(\mathbf{t}) \left\langle \psi^0_{p^e,f} e^1_0\left(\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} \mathbf{t}\mathbf{g}^{-1}_{d,k}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), m \right\rangle d\mathbf{t}.$$

Since any $\lambda : (\mathbb{Z}/p^e q\mathbb{Z})^* \to \mathcal{O}$ can be written as the sum of an even part λ^+ and an odd part λ^- (define $\lambda^\pm(x) = (\lambda(x) \pm \lambda(-x))/2$), we may assume that λ is either even or odd. In this case $\widetilde{\lambda}$ can be split into an infinite and a finite part,

$$\widetilde{\lambda}(\mathbf{t}) = \widetilde{\lambda}_f(t_f)\widetilde{\lambda}_\infty(t_\infty) \quad \text{for } \mathbf{t} = (t_f, t_\infty) \in T_1(\mathbb{A}),$$

where $\widetilde{\lambda}_f(t_f) := \widetilde{\lambda}(t_f, 1)$ and $\widetilde{\lambda}_\infty = \mathbb{1}$ if λ is even and $\widetilde{\lambda}_\infty = \mathrm{sgn}$ if λ is odd. This yields the following decomposition of our adèlic integral

$$\int\limits_{\mathcal{Z}_{d,k,\lambda,m}} \widetilde{\omega}^0_{p^e} = \int\limits_{T_1(\mathbb{R})} |t_\infty|_\infty^{l-\mu-1} \widetilde{\lambda}_\infty(t_\infty) \left\langle e_0^1 \left(\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} t_\infty, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right), m \right\rangle \frac{dt_\infty}{|t_\infty|} \quad \times$$
$$\times \quad \mathrm{vol}^{-1} \int\limits_{T_1(\mathbb{A}_f)} |t_f|_f^{l-\mu-1} \widetilde{\lambda}_f(t_f) \psi^0_{p^e,f} \left(\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} t_f \mathbf{g}_{d,k}^{-1} \right) dt_f.$$

We denote the first resp. second factor by I_∞ resp. I_f.

4.6.1. Computation of I_∞

This works completely analogously to [Kai91], 3.2.3. From there we know that

$$e_0^1 \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \left| \frac{t}{1+t^2} \right|^l \frac{2t}{1+t^2} k_\infty^{-1} Y^{2l-2},$$

where $k_\infty = \frac{1}{\sqrt{1+t^2}} \left(\begin{smallmatrix} 1 & -t \\ t & 1 \end{smallmatrix} \right) \in K_{2,\infty}$. Plugging this into the definition of I_∞ yields

$$I_\infty = \int_{\mathbb{R}^*} |t|^{l-\mu-1} \widetilde{\lambda}_\infty(t) \left| \frac{t}{1+t^2} \right|^l \frac{2t}{1+t^2} \left\langle k_\infty^{-1} Y^{2l-2}, X^\mu Y^{2l-2-\mu} \right\rangle \frac{dt}{|t|}.$$

To determine the expression with the pairing we use the invariance of the pairing and since $\langle Y^{2l-2}, X^k Y^{2l-2-k} \rangle$ vanishes for all k smaller than $2l-2$ we easily get

$$\left\langle k_\infty^{-1} Y^{2l-2}, X^\mu Y^{2l-2-\mu} \right\rangle = \frac{(-t)^{2l-2-\mu}}{(1+t^2)^{l-1}}.$$

We have

$$I_\infty = -2 \int_{\mathbb{R}^*} \widetilde{\lambda}_\infty(t) \frac{|t|^{2l-\mu-1} (-t)^{2l-\mu-1}}{(1+t^2)^{2l}} \frac{dt}{|t|}.$$

If $\widetilde{\lambda}_\infty(-1) = (-1)^{2l-\mu} = (-1)^\mu$ the integrand is odd and the integral vanishes. Otherwise we substitute $1+t^2 = w$ and get

$$\begin{aligned} I_\infty =& (-1)^{2l-2-\mu} 4 \int_0^\infty \frac{t^{2(2l-\mu-1)}}{(1+t^2)^{2l}} \frac{dt}{|t|} = \\ =& 2(-1)^\mu \int_0^1 w^\mu (1-w)^{2l-2-\mu} dw = \\ =& 2(-1)^\mu \frac{\Gamma(\mu+1)\Gamma(2l-1-\mu)}{\Gamma(2l)}. \end{aligned}$$

In summary we can state

$$I_\infty = \begin{cases} 0 & \text{if } \widetilde{\lambda}_\infty(-1) = (-1)^\mu, \\ 2(-1)^\mu \frac{\Gamma(\mu+1)\Gamma(2l-1-\mu)}{\Gamma(2l)} & \text{if } \widetilde{\lambda}_\infty(-1) = (-1)^{\mu+1}. \end{cases}$$

We emphasize that $\int_{\mathcal{Z}_{d.k.\lambda,\mu}} \widetilde{\omega}^0_{p^e}$ vanishes if λ and μ are odd or if λ and μ are even.

4.6.2. Computation of I_f

The last section has shown that we only have to compute I_f in the case where μ and λ have different parities. We want to fix the following convention, if we assume λ to be odd for instance this shall mean that λ is odd and μ is even, analogously for the other cases. We can restrict ourselves to $\lambda = \lambda_\epsilon$, where

$$\lambda_\epsilon(t) = \begin{cases} 1 & t \equiv \epsilon \ (p^e q), \\ (-1)^{\mu+1} & t \equiv -\epsilon \ (p^e q), \\ 0 & \text{else} \end{cases}$$

for ϵ in $(\mathbb{Z}/p^e q\mathbb{Z})^*$. From the unique decomposition $\mathbb{A}_f^* = \mathbb{Q}_{>0}^* \hat{\mathbb{Z}}^*$ we obtain

$$I_f = \sum_{r \in T_{1,>0}(\mathbb{Q})} \sum_{t_f \in T_1(\hat{\mathbb{Z}})/T_{1,p^e q}} |rt_f|_f^{l-\mu-1} \widetilde{\lambda}_{\epsilon,f}(rt_f) \psi^0_{p^e,f}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} rt_f \mathbf{g}_{d,k}^{-1} \right),$$

where $T_{1,>0}(\mathbb{Q})$ denotes the group of elements $\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ with $a \in \mathbb{Q}_{>0}^*$. We can replace $|rt_f|_f^{l-\mu-1}$ by $|r|_\infty^{-l+\mu+1}$ because $|t_f|_f$ is equal to one and $r \in \mathbb{Q}^*$ embedded diagonally into $\mathbb{A}^*$ has trivial idèlic absolute value. Taking into account that $\widetilde{\lambda}_{\epsilon,f}(rt_f) = \lambda_\epsilon(t_f^{-1})$ by definition of $\widetilde{\lambda}$ we find

$$I_f = \sum_{r \in T_{1,>0}(\mathbb{Q})} |r|_\infty^{-l+\mu+1} \sum_{t_f \in T_1(\hat{\mathbb{Z}})/T_{1,p^e q}} \lambda_{\epsilon,f}(t_f^{-1}) \psi^0_{p^e,f}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} rt_f \mathbf{g}_{d,k}^{-1} \right).$$

Recalling the definition of λ_ϵ immediately gives $I_f = I_\epsilon + (-1)^{\mu+1} I_{-\epsilon}$, where I_ϵ is defined as follows

$$I_\epsilon = \sum_{r \in T_{1,>0}(\mathbb{Q})} |r|_\infty^{-l+\mu+1} \psi^0_{p^e,f}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} r \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{g}_{d,k}^{-1} \right).$$

Let a, b be two coprime positive integers with $r = \frac{a}{b}$. There exist integers x, y with $xb - ya = 1$ and we obtain the following Iwasawa decomposition

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} r & \\ & 1 \end{pmatrix} = \begin{pmatrix} a & \\ & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{y}{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ a & b \end{pmatrix}.$$

Plugging this into the definition of I_ϵ and using the behavior of $\psi^0_{p^e,f}$ under elements in $B_2(\mathbb{A}_f)$ we obtain

$$I_\epsilon = \sum_{\substack{a,b \in \mathbb{Z}_{>0} \\ (a,b)=1}} \left|\frac{a}{b}\right|_\infty^{-l+\mu+1} |ab|_\infty^{-l} \psi^0_{p^e,f}\left(\begin{pmatrix} x & y \\ a & b \end{pmatrix} \begin{pmatrix} \epsilon^{-1} & \\ & 1 \end{pmatrix} \times \mathbf{g}_{d,k}^{-1} \right).$$

Consulting the proof of Lemma 4.3 in [Mah00] one checks that the matrix

$$\begin{pmatrix} x & y \\ a & b \end{pmatrix} \begin{pmatrix} \epsilon^{-1} & \\ & 1 \end{pmatrix} \times \mathbf{g}_{d,k}^{-1} = \left(\cdots \begin{pmatrix} y - \epsilon^{-1} x p^k & -d^{-1}\epsilon^{-1} x \\ b - p^k \epsilon^{-1} a & -\epsilon^{-1} d^{-1} a \end{pmatrix} \cdots \begin{pmatrix} y - \epsilon^{-1} x & -d^{-1}\epsilon^{-1} x \\ b - \epsilon^{-1} a & -d^{-1}\epsilon^{-1} a \end{pmatrix} \cdots \right)$$

lies in the support of $\psi^0_{p^e,f}$ if and only if there exists x_0 in $(\mathbb{Z}/q\mathbb{Z})^*$ such that

$$\begin{aligned} &a \equiv x_0, \ b \equiv \epsilon^{-1} x_0 \ (q), \qquad a \equiv d\epsilon, \ b \equiv dp^k \ (p^e) \\ \text{or} \qquad & \qquad (4.11) \\ &a \equiv x_0, \ b \equiv \epsilon^{-1} x_0 \ (q), \qquad a \equiv -d\epsilon, \ b \equiv -dp^k \ (p^e). \end{aligned}$$

The proof of Lemma 4.3 in [Mah00] does not only state the support of $\psi^0_{p^e,f}$ but also describes its values. This yields

$$I_\epsilon = \eta_{0,q}^{-1}(\epsilon d) \sum_{x_0 \in (\mathbb{Z}/q\mathbb{Z})^*} \eta_{0,q}(x_0) \sum_{\substack{a,b \in \mathbb{Z}_{>0},(a,b)=1 \\ a,b \text{ as in (4.11)}}} a^{-2l+\mu+1} b^{-\mu-1}.$$

The next definition provides us with a manageable substitute for the last sum in the above definition. For $x \in (\mathbb{Z}/p^e q\mathbb{Z})^*$, $y \in (\mathbb{Z}/p^{e-k}q\mathbb{Z})^*$, $0 \le k \le e$ and $0 \le \mu \le 2l-2$ we set

$$S_{x,y,k,\mu} = \frac{1}{p^{k(\mu+1)}} \sum \frac{1}{a^{2l-\mu-1} b^{\mu+1}},$$

where the sum runs over all coprime integers $a, b \in \mathbb{Z}$ with $a \equiv x \ (p^e q)$ and $b \equiv y \ (p^{e-k}q)$. Although this notation has become too rough to describe I_ϵ it is still sufficient to describe the integral I_f, which we are actually interested in

$$\begin{aligned} I_f &= I_\epsilon + (-1)^{\mu+1} I_{-\epsilon} = \\ &= \eta_{0,q}^{-1}(\epsilon d) \sum_{x_0 \in (\mathbb{Z}/q\mathbb{Z})^*/\{\pm 1\}} \eta_{0,q}(x_0) \left(S_{(x_0, d\epsilon),(\epsilon^{-1}x_0, d),k,\mu} + S_{(x_0,-d\epsilon),(\epsilon^{-1}x_0,-d),k,\mu} \right). \end{aligned}$$

This is easily proved by noting that $\eta_{0,q}$ is an even character and that $S_{x,y,k,\mu}$ can be split up into the sum of four similar expressions where the summation is restricted to positive integers.

Remark 4.6. *For $l = 0$ neither the sum $S_{x,y,k,0}$ nor the one defining $\omega^0_{p^e}$ converges absolutely and we refer the reader to [Mah00], Remark 4.4 for a justification of our calculations.*
For $l > 0$ the sums $S_{x,y,k,\mu}$ with $0 < \mu < 2l-2$ and the sum occurring in the definition of $\omega^0_{p^e}$ converge absolutely. On page 42 in [Kai91] it is shown that we can do without the sums $S_{x,y,k,0}$ or $S_{x,y,k,2l-2}$. More precisely one can modify the generating system $\mathcal{Z}_{\psi,m}$ of $H_1(\Gamma_1(p^e q)\backslash\mathbb{H}, \widetilde{W_{\mathcal{O}}})$ constructed in section 4.4 in such a way that for all elements $\mathcal{Z}_{\psi,m}$ the polynomial m is of the form $m = \sum X^\mu Y^{2l-2-\mu}$ where μ runs from 1 to $2l-3$. Thus, we only have to compute the integral of $\omega^0_{p^e}$ on the chains $\mathcal{Z}_{d,k,\lambda,m}$ for $m = X^\mu Y^{2l-2-\mu}$ with $0 < \mu < 2l-2$ and we consequently can restrict ourselves to the investigation of the sums $S_{x,y,k,\mu}$ with $0 < \mu < 2l-2$.

The evaluation of $\omega^0_{p^e}$ on the chains $g'_{d,k,t}\mathcal{Z}_{0,\infty}$ and $g''_{d,k,t}\mathcal{Z}_{0,\infty}$ is quite analogous, one simply has to replace the source of λ by $(\mathbb{Z}/p^e\mathbb{Z})^*$. We only state the results:

$$\int_{g'_{d,k,t}\mathcal{Z}_{0,\infty}} \omega^0_{p^e} = \eta_{0,q}^{-1}(d) \sum_{a,b} \frac{\eta_{0,q}(a)}{a^{2l-\mu-1} b^{\mu+1}},$$

where $a, b \in \mathbb{Z}$ run over all pairs satisfying $(a,b) = 1, a \equiv \epsilon d \pmod{p^e}, b \equiv dp^k \pmod{p^e}, q \not| a$ and $q | b$ and

$$\int_{g''_{d,k,t}\mathcal{Z}_{0,\infty}} \omega^0_{p^e} = \eta_{0,q}^{-1}(d) \sum_{a,b} \frac{\eta_{0,q}(b)}{a^{2l-\mu-1} b^{\mu+1}},$$

where $a, b \in \mathbb{Z}$ run over all pairs satisfying $(a,b) = 1, a \equiv \epsilon d \pmod{p^e}, b \equiv dp^k \pmod{p^e}, q \not| b$ and $q | a$. In particular, these integrals are linear combinations of terms of the form

$$S'_{x,y,k,\mu} := \frac{1}{(p^k q)^{\mu+1}} \sum_{a,b} \frac{1}{a^{2l-\mu-1} b^{\mu+1}},$$

where $x \in (\mathbb{Z}/p^e q\mathbb{Z})^*$, $y \in \left(\mathbb{Z}/p^{e-k}\mathbb{Z}\right)^*$ and $a, b \in \mathbb{Z}$ run over all pairs, such that $(a,b) = 1, a \equiv x \pmod{p^e q}, b \equiv y \pmod{p^{e-k}}$ and

$$S''_{x,y,k,\mu} := \frac{1}{p^{k(\mu+1)} q^{2l-\mu-1}} \sum_{a,b} \frac{1}{a^{2l-\mu-1} b^{\mu+1}},$$

where $x \in (\mathbb{Z}/p^e\mathbb{Z})^*$, $y \in \left(\mathbb{Z}/p^{e-k}q\mathbb{Z}\right)^*$ and $a, b \in \mathbb{Z}$ run over all pairs, such that $(a,b) = 1, a \equiv x \pmod{p^e}, b \equiv y \pmod{p^{e-k}q}$.
Again, as explained in Remark 4.6, we can restrict ourselves to the case $0 < \mu < 2l-2$ for $l > 1$ and for $l = 0$ these calculations are justified by the Remark 4.4 in [Mah00].

We may summarize our results so far: for any cycle $\mathcal{Z} \in H_1(\bar{S}_2(K_1^f(p^e, q), \mathcal{W}_\mathcal{O})$ we have

$$\int_{\mathcal{Z}} \omega^0_{p^e} \equiv \mathcal{O}\text{-linear combination of the terms } S_{x,y,k,\mu}, S'_{x,y,k,\mu} \text{ and } S''_{x,y,k,\mu} \pmod{\mathcal{O}}.$$

The following paragraphs are devoted to the investigation of $S_{x,y,k,\mu}$.
We will link $S_{x,y,k,\mu}$ with the quotient of some Dirichlet L-functions. Let $\chi : (\mathbb{Z}/p^e q\mathbb{Z})^* \to \mathbb{C}^*$ be any even Dirichlet character. If λ is odd resp. even, let $\psi : \left(\mathbb{Z}/p^{e-k}q\mathbb{Z}\right)^* \to \mathbb{C}^*$ be any odd resp. even Dirichlet character. We denote by χ' resp. ψ' the primitive character related to χ resp. ψ. For any Dirichlet character ϕ let $L(\phi, s)$ be the L-functions attached to ϕ. Using the example of χ for $e > 0$ we want to point out the difference between the L-function associated to a Dirichlet character and the L-function associated to the corresponding primitive Dirichlet character. Via the Euler product one immediately checks that

$$L(\chi', s) = (1 - \chi'(q)q^{-s})^{-1}(1 - \chi'(p)p^{-s})^{-1} L(\chi, s).$$

For instance, if pq divides the level of χ, then $\chi'(p) = \chi'(q) = 0$ and the two L-functions coincide.
We extend the convention fixed at the beginning of this chapter. Whenever λ, μ and ψ as described above occur, λ and ψ are assumed to be of the same parity whereas λ and μ are of different parity.
We first have to establish the following equation

$$\sum_{\substack{x \in (\mathbb{Z}/p^e q\mathbb{Z})^* \\ y \in \left(\mathbb{Z}/p^{e-k}q\mathbb{Z}\right)^*}} \chi\psi^{-1}(x)\psi(y) p^{k(\mu+1)} S_{x,y,k,\mu} = 4 \frac{L(\chi\psi^{-1}, 2l-\mu-1) L(\psi, \mu+1)}{L(\chi, 2l)}. \tag{4.12}$$

Splitting $S_{x,y,k,\mu}$ into the sum of four similar expressions where the summation only ranges over positive integers and exploiting the different parity of ψ and μ, we obtain that the left hand side is equal to

$$4 \sum_{\substack{x \in (\mathbb{Z}/p^e q\mathbb{Z})^* \\ y \in \left(\mathbb{Z}/p^{e-k}q\mathbb{Z}\right)^*}} \chi\psi^{-1}(x)\psi(y) \sum_{\substack{a,b \in \mathbb{Z}_{>0},\, (a,b)=1, \\ a \equiv x \pmod{p^e q}, \\ b \equiv y \pmod{p^{e-k}q}}} \frac{1}{a^{2l-\mu-1} b^{\mu+1}}.$$

Merging the two sums, this expression is transformed into

$$4 \sum_{\substack{n,m \in \mathbb{Z}_{>0}, \\ (n,m)=1 \\ pq \nmid nm}} \frac{\chi\psi^{-1}(n)}{n^{2l-\mu-1}} \frac{\psi(m)}{m^{\mu+1}}$$

and this is easily seen to coincide with the right hand side of equation (4.12).
Now we derive from equation (4.12) the following identity, in which $S_{x,y,k,\mu}$ is expressed as the linear combination of quotients of L-functions

$$\frac{S_{x,y,k,\mu}}{4} = \frac{1}{\phi(p^e q)\phi(p^{e-k}q)p^{k(\mu+1)}} \sum_{\chi,\psi} \chi^{-1}\psi(x)\psi^{-1}(y) \frac{L(\chi\psi^{-1}, 2l-\mu-1)L(\psi,\mu+1)}{L(\chi,2l)}, \quad (4.13)$$

where the sum runs over all even characters χ and all odd resp. even characters ψ as described above. To verify equation (4.13) we first plug equation (4.12) into the right hand side and obtain

$$\frac{1}{\phi(p^e q)\phi(p^{e-k}q)p^{k(\mu+1)}} \sum_{\chi,\psi} \chi^{-1}\psi(x)\psi^{-1}(y)\frac{1}{4} \sum_{\substack{a\in(\mathbb{Z}/p^e q\mathbb{Z})^* \\ b\in\left(\mathbb{Z}/p^{e-k}q\mathbb{Z}\right)^*}} \chi\psi^{-1}(a)\psi(b)p^{k(\mu+1)} S_{a,b,k,\mu} =$$

$$= \frac{1}{4\phi(p^e q)\phi(p^{e-k}q)} \sum_{\substack{a\in(\mathbb{Z}/p^e q\mathbb{Z})^* \\ b\in\left(\mathbb{Z}/p^{e-k}q\mathbb{Z}\right)^*}} S_{a,b,k,\mu} \sum_{\chi,\psi} \chi(x^{-1}a)\psi(xy^{-1}a^{-1}b).$$

Using the character relation

$$\sum_{\substack{\chi:(\mathbb{Z}/p^e q\mathbb{Z})^*\to\mathbb{C}^* \\ \chi \text{ even}}} \chi(a) = \begin{cases} \frac{\phi(p^e q)}{2} & \text{if } a \equiv \pm 1 \pmod{p^e q}, \\ 0 & \text{else} \end{cases}$$

and the analogous identity for ψ and noting that $S_{ax,by,k,\mu} = (ab)^{\mu+1} S_{x,y,k,\mu}$ for $a, b \in \{\pm 1\}$, we derive equation (4.13).
Let m be an integer of the same parity as ψ. The definition of the L-function yields

$$L(\psi, m) = \frac{1}{(p^e q)^m} \sum_{\epsilon\in(\mathbb{Z}/p^e q\mathbb{Z})^*/\{\pm 1\}} \psi(\epsilon) \sum_{n\in\mathbb{Z}} \frac{1}{(n+\frac{\epsilon}{p^e q})^m}. \quad (4.14)$$

We abbreviate $A_m(x) := \sum_{n\in\mathbb{Z}} \frac{1}{(n+x)^m}$.
Expressing the two L-functions related to ψ in equation (4.13) as linear combinations of A_m's and collecting the ψ's we obtain

$$\frac{S_{x,y,k,\mu}}{4} = \frac{1}{\phi(p^e q)\phi(p^{e-k}q)p^{k(\mu+1)}(p^e q)^{2l}} \sum_{\chi} \frac{\chi^{-1}(x)}{L(\chi,2l)} \times$$

$$\times \sum_{\epsilon\in(\mathbb{Z}/p^e q\mathbb{Z})^*/\{\pm 1\}} \chi(\epsilon) A_{2l-\mu-1}\left(\frac{\epsilon}{p^e q}\right) \sum_{\omega\in(\mathbb{Z}/p^e q\mathbb{Z})^*/\{\pm 1\}} A_{\mu+1}\left(\frac{\omega}{p^e q}\right) \sum_{\psi} \psi(xy^{-1}\epsilon^{-1}\omega).$$

Taking into account the character relations for ψ yields

$$\frac{S_{x,y,k,\mu}}{4} = \frac{1}{2\phi(p^e q)p^{k(\mu+1)}(p^e q)^{2l}} \sum_{\chi} \frac{\chi^{-1}(x)}{L(\chi,2l)} \sum_{\epsilon\in(\mathbb{Z}/p^e q\mathbb{Z})^*/\{\pm 1\}} \chi(\epsilon) A_{2l-\mu-1}\left(\frac{\epsilon}{p^e q}\right) A_{\mu+1}\left(\frac{\epsilon y x^{-1}}{p^e q}\right).$$

Let $\mathbf{f}_\chi$ be the least common multiple of f_χ and pq. Lemma 4.9 below shows that for fixed $\epsilon_0 \in (\mathbb{Z}/\mathbf{f}_\chi\mathbb{Z})^*$ and $\delta \in (\mathbb{Z}/p^e q\mathbb{Z})^*$ the following expression

$$a_{\epsilon_0,\delta,\chi} := \frac{(2l-2-\mu)!(\mu)!\mathbf{f}_\chi}{p^e q \pi^{2l}} \sum_{\substack{\epsilon\in(\mathbb{Z}/p^e q\mathbb{Z})^* \\ \epsilon\equiv\epsilon_0(\mathbf{f}_\chi)}} A_{2l-\mu-1}\left(\frac{\epsilon}{p^e q}\right) A_{\mu+1}\left(\frac{\epsilon\delta}{p^e q}\right)$$

lies in $\mathcal{O}_{\mathbb{Q}(\zeta_{\mathbf{f}_\chi})}$. Our equation above then reads

$$\frac{S_{x,y,k,\mu}}{4} = \frac{1}{(2l-2-\mu)!(\mu)!\phi(p^eq)p^{k(\mu+1)}(p^eq)^{2l-1}} \sum_{\chi} \frac{\chi^{-1}(x)\pi^{2l}}{L(\chi,2l)\mathbf{f}_\chi} \sum_{\epsilon\in(\mathbb{Z}/\mathbf{f}_\chi\mathbb{Z})^*} \chi(\epsilon)a_{\epsilon,yx^{-1},\chi}.$$

We want to apply the functional equation to the remaining L-function

$$\frac{(1-\chi'(p)^{-1}p^{2l-1})(1-\chi'(q)^{-1}q^{2l-1})}{(1-\chi'(p)p^{-2l})(1-\chi'(q)q^{-2l})}(2l-1)!(-1)^lL(\chi,2l) = \frac{L(\bar{\chi},1-2l)G(\chi')}{2}\left(\frac{2\pi}{f_\chi}\right)^{2l}.$$

Note that the quotient of Euler factors on the left hand side equals one whenever the conductor of χ is divisible by pq. Let us denote this quotient by $Q(\chi,l)$. We find

$$\frac{S_{x,y,k,\mu}}{4} = \frac{\binom{2l-2}{\mu}(2l-1)(-1)^l}{2^{2l-1}\phi(p^eq)p^{k(\mu+1)}(p^eq)^{2l-1}} \sum_{\chi} \frac{Q(\chi,l)f_\chi^{2l}\chi^{-1}(x)}{G(\chi')L(\chi^{-1},1-2l)\mathbf{f}_\chi} \sum_{\epsilon\in(\mathbb{Z}/\mathbf{f}_\chi\mathbb{Z})^*} \chi(\epsilon)a_{\epsilon,yx^{-1},\chi}.$$

The identity $G(\chi')G(\chi'^{-1}) = f_\chi$, which can be looked up in [Was97], Lemma 6.1, now gives

$$\frac{S_{x,y,k,\mu}}{4} = \frac{\binom{2l-2}{\mu}(2l-1)(-1)^l}{2^{2l-1}\phi(p^eq)p^{k(\mu+1)}(p^eq)^{2l-1}} \sum_{\chi} \sum_{\epsilon\in(\mathbb{Z}/f_\chi\mathbb{Z})^*} \frac{Q(\chi,l)G(\chi'^{-1})f_\chi^{2l-1}\chi(x^{-1}\epsilon)a_{\epsilon,yx^{-1},\chi}}{L(\chi^{-1},1-2l)\mathbf{f}_\chi}. \tag{4.15}$$

As mentioned before the numbers $a_{\epsilon,yx^{-1},\chi}$ lie in $\mathcal{O}_{\mathbb{Q}(\zeta_{\mathbf{f}_\chi})}$. The Gauss sum $G(\chi')$ also is an algebraic integer in $\mathbb{Q}(\zeta_{f_\chi})$, as one checks from the definition. Using that $Q(\chi,l)$ and $L(\chi^{-1},1-2l)$ are algebraic numbers in $\mathbb{Q}(\chi)$ we finally conclude that $S_{x,y,k,\mu}$ lies in $\mathbb{Q}(\zeta_{p^eq})$.

We want to show that $S_{x,y,k,\mu}$ even lies in $\mathbb{Q}$ so we prove that it is invariant under all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$: Considering the definition of $Q(\chi,l)$ we obviously have $Q(\chi,l)^\sigma = Q(\chi^\sigma,l)$. The fact that the values of the L-function at the negative integers are given by generalized Bernoulli numbers yields a similar relation for the L-function: $L(\chi,-k)^\sigma = L(\chi^\sigma,-k)$ for $k\in\mathbb{Z}_{>0}$.
To determine the action of σ on $a_{\epsilon,yx^{-1},\chi}$ we use that according to the proof of Lemma 4.9 the algebraic integer $a_{\epsilon,yx^{-1},\chi}$ can be written as

$$a_{\epsilon,yx^{-1},\chi} = \sum_{\substack{0\le i\le 2l-\mu-1\\ 0\le j\le \mu+1,\\ i\equiv j\equiv\mu+1 \pmod 2}} c_{i,j} \sum_{\substack{\eta\in(\mathbb{Z}/p^eq\mathbb{Z})^*\\ \eta\equiv\epsilon(\mathbf{f}_\chi)}} \left(\frac{\zeta_{p^eq}^{\eta}+\zeta_{p^eq}^{-\eta}}{\zeta_{p^eq}^{\eta}-\zeta_{p^eq}^{-\eta}}\right)^i \left(\frac{\zeta_{p^eq}^{\eta yx^{-1}}+\zeta_{p^eq}^{-\eta yx^{-1}}}{\zeta_{p^eq}^{\eta yx^{-1}}-\zeta_{p^eq}^{-\eta yx^{-1}}}\right)^j, \tag{4.16}$$

where $c_{i,j}$ are rational numbers independent of ϵ and yx^{-1} and ζ_{p^eq} is given by $\zeta_{p^eq} = e^{i\pi/p^eq}$. Let k be the element in $(\mathbb{Z}/2p^eq\mathbb{Z})^*$ with $\sigma(\zeta_{p^eq}) = \zeta_{p^eq}^k$. With this notation we deduce from equation (4.16) that $a_{\epsilon,yx^{-1},\chi}^\sigma = a_{\epsilon k,yx^{-1},\chi^\sigma}$. Note that $a_{\epsilon,yx^{-1},\chi}$ rather depends on the conductor of χ than on χ itself and we have $f_\chi = f_{\chi^\sigma}$.
We still have to consider the action of σ on the Gauss sum $G(\bar{\chi}') = \sum_{1\le j\le f_\chi}\bar{\chi}'(j)e^{j2i\pi/f_\chi}$. Since f_χ is a divisor of $2p^eq$ we have $\sigma(e^{2i\pi/f_\chi}) = e^{k2i\pi/f_\chi}$. A variable transformation now immediately shows that $G(\bar{\chi}')^\sigma = \chi'^\sigma(k)G(\bar{\chi}'^\sigma)$. Note that χ and the associated primitive character χ' take the same value on k because k is coprime to p^eq.
Collecting all these results we find that

$$\left(\frac{Q(\chi,l)G(\bar{\chi}')f_\chi^{2l-1}\chi(x^{-1}\epsilon)a_{\epsilon,yx^{-1},\chi}}{L(\bar{\chi},1-2l)\mathbf{f}_\chi}\right)^\sigma = \frac{Q(\chi^\sigma,l)G(\bar{\chi}'^\sigma)f_{\chi^\sigma}^{2l-1}\chi^\sigma(x^{-1}\epsilon k)a_{\epsilon k,yx^{-1},\chi^\sigma}}{L(\bar{\chi}^\sigma,1-2l)\mathbf{f}_{\chi^\sigma}}.$$

Since the summation in equation (4.15) is running over all even Dirichlet characters χ and all $\epsilon \in (\mathbb{Z}/p^e q\mathbb{Z})^*$ we conclude that $S_{x,y,k,\mu}$ is invariant under $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and thus is a rational number.

Our aim in this section is to find an upper bound for the p-adic absolute value of $S_{x,y,k,\mu}$. Considering equation (4.15) we are done if we can get hold of the p-adic absolute value of $L(\chi^{-1}, 1-2l)$ and of $Q(\chi, l)$. The next Lemma gives an upper bound for the p-adic absolute value of $Q(\chi, l)$.

Lemma 4.7. *For every prime number ℓ there exists a constant $N_\ell \in \mathbb{Z}$ such that $|Q(\chi,l)|_\ell \leq N_\ell$ for all even Dirichlet characters χ which have domain $(\mathbb{Z}/p^e q\mathbb{Z})^*$ for some $e \in \mathbb{N}$.*

Proof. We recall the definition of $Q(\chi, l)$

$$Q(\chi,l) = \frac{(1-\chi'(p)^{-1}p^{2l-1})(1-\chi'(q)^{-1}q^{2l-1})}{(1-\chi'(p)p^{-2l})(1-\chi'(q)q^{-2l})},$$

where χ' was defined to be the primitive Dirichlet character related to χ. Note that $Q(\chi, l)$ is trivial if the conductor of χ is divisible by pq. Since $l > 0$ the denominator of $Q(\chi, l)$ does not vanish for any possible choice of χ and it is clear, that for proving the existence of N_ℓ, we only have to check all but finitely many values of $|Q(\chi, l)|_\ell$. So we do not care about the finitely many values $|Q(\chi, l)|_\ell$, where $Q(\chi, l)$ contains Euler factors attached to p. (In this case the conductor of χ is dividing q.)
In the remainder of the proof let the conductor of χ be a power of p. As the numerator of $Q(\chi, l)$ is an algebraic integer, its ℓ-adic absolute value is bounded above. This leaves us with finding an upper bound for $|(1-\chi'(q)q^{-2l})|_\ell^{-1}$, where χ' varies over all primitive even Dirichlet characters which have conductor a power of p. Equivalently we will show that there is a lower bound for the ℓ-adic absolute value of $q^{2l} - \chi'(q)$ independent of χ'. Of course, $\zeta := \chi'(q)$ is a primitive ap^kth root of unity for some a dividing $p-1$.
If $\ell = q$ the expression $q^{2l} - \chi'(q)$ has q-adic absolute value one and we are finished.
Let now ℓ be any prime different from q. We denote the order ap^k of the root of unity $\zeta = \chi'(q)$ by $n_{a,k} = n$. As usual, we write ϕ_m, $m \in \mathbb{N}$ for the mth cyclotomic polynomial. In $\mathbb{Q}_\ell(\zeta)$ the polynomial ϕ_n splits into linear factors, $\phi_n(X) = \prod(X - \zeta^j)$. Evaluating ϕ_n at q^{2l} and using the fact that $q^{2l} - \zeta^j$ lies in $\mathbb{Z}_\ell[\zeta]$ for every j we see that the absolute value $|q^{2l} - \zeta|_\ell$ is greater than or equal to the absolute value $|\phi_n(q^{2l})|_\ell$. Thus, it suffices to show that the ℓ-adic absolute value of $\phi_n(q^{2l})$ is bounded below independently of n.
Let now a be fixed. We abbreviate $n_{a,k}$ to n_k. According to [Lan02], page 280, we have $\phi_{n_k}(X) = \phi_{n_{k_0}}(X^{k-k_0})$ for any k_0 with $1 \leq k_0 < k$. We can regard $\phi_{n_{k_0}}$ as a continuous function on $\mathbb{Z}_\ell$ with target set $\mathbb{Z}_\ell$. Since $\mathbb{Z}_\ell^*$ is compact, the image $\phi_{n_{k_0}}(\mathbb{Z}_\ell^*)$ is a compact subset of $\mathbb{Z}_\ell$. As the only roots of unity in $\mathbb{Z}_\ell$ are the $(\ell-1)$th roots of unity it is clear that $\phi_{n_{k_0}}$ is nonzero on $\mathbb{Z}_\ell$ for k_0 large enough. In particular, $\phi_{n_{k_0}}(\mathbb{Z}_\ell^*)$ is a compact subset not containing zero. Hence, its complement is an open neighborhood of zero and contains $B_r = \{x \in \mathbb{Z}_\ell, |x|_\ell < r\}$ for some natural number r. This proves that $|\phi_{n_{k_0}}(\mathbb{Z}_\ell^*)|_\ell$ is bounded below by r. Since $q^{2l(k-k_0)}$ is an ℓ-adic unit we conclude that $|\phi_{n_{k_0}}(q^{2l(k-k_0)})|_\ell$ is bounded below independently of k, where k is any integer larger than k_0. The relation between ϕ_{n_k} and $\phi_{n_{k_0}}$ described above implies that $|\phi_{n_k}(q^{2l})|_\ell$ is larger than some constant for all natural numbers k. Since there are only finitely many choices for a we conclude that the ℓ-adic absolute value of $\phi_{n_{a,k}}$ is bounded below independently of a and k. As explained above this proves the claim of the Lemma. □

The next Lemma takes care of the p-adic absolute value of $L(\chi^{-1}, 1-2l)$. Let L_p be the Kubota-

Leopoldt p-adic L-function, we have

$$L_p(\chi'\omega^{2l}, 1-2l) = (1-\chi'(p)p^{2l-1})L(\chi', 1-2l),$$

where ω denotes the p-adic Teichmuller character. Recalling that the Euler factors at p and q of the L-function associated to χ are trivial we find

$$L_p(\chi'\omega^{2l}, 1-2l)(1-\chi'(q)q^{2l-1}) = L(\chi, 1-2l).$$

It is clear that for varying χ' the p-adic absolute value of $1-\chi'(q)q^{2l-1}$ is bounded above and as in the proof of Lemma 4.7 one checks that it is also bounded below. Hence, the claim of the following Lemma, namely the boundedness of $|L_p(\chi'\omega^{2l}, 1-2l)|_p$ independently of χ', implies that $|L(\chi, 1-2l)|_p$ is bounded for all even Dirichlet characters χ.

Lemma 4.8. *For all primitive even Dirichlet characters χ' whose levels divide p^eq for some $e \in \mathbb{N}$, the absolute values of the p-adic L-function are bounded, i.e. there are constants M_1 and M_2 such that*

$$M_1 < |L_p(\chi', 1-2l)|_p < M_2.$$

This is independent of the embedding i_p.

Proof. We closely follow the proof of Lemma 4.5 in [Mah00].
To simplify notations we will assume throughout the proof that the conductor $f_{\chi'}$ of χ' is of the form p^eq. The case where $f_{\chi'}$ is a power of p works completely analogously. Since we are only interested in the boundedness of the absolute value we do not care about the remaining finitely many characters χ' of level dividing q.
We have the decomposition $(\mathbb{Z}/p^e\mathbb{Z})^* \cong (\mathbb{Z}/p\mathbb{Z})^* \times (1+p\mathbb{Z}_p/1+p^e\mathbb{Z}_p)$ and by the Chinese Remainder Theorem we obtain $(\mathbb{Z}/p^eq\mathbb{Z})^* \cong (\mathbb{Z}/pq\mathbb{Z})^* \times (1+p\mathbb{Z}_p/1+p^e\mathbb{Z}_p)$. This legitimates the following definition, we set $\chi'_0 = \chi'|_{(\mathbb{Z}/pq\mathbb{Z})^*}$. Let u_1 be a topological generator of $1+p\mathbb{Z}_p$. It is proven in Iwasawa theory that for any character $\kappa : (\mathbb{Z}/pq\mathbb{Z})^* \to \mathbb{Z}_p^*$ there is a power series $f(\kappa, T) \in \mathbb{Z}_p[\zeta_{q-1}][[T]]$, such that

$$L_p(\chi', 1-2l) = \begin{cases} f(\chi'_0, \chi'(u_1)u_1^{2l}-1) & \text{if } \chi'_0 \neq \mathbb{1}, \\ \frac{f(\mathbb{1}, \chi'(u_1)u_1^{2l}-1)}{\chi'(u_1)u_1^{2l}-1} & \text{if } \chi'_0 = \mathbb{1}. \end{cases}$$

By the Weierstrass Preparation Theorem (cf. [Was97], Theorem 7.3) there is a factorization

$$f(\kappa, T) = a_\kappa P_\kappa(T) f_\kappa(T), \tag{4.17}$$

where $a_\kappa \in \mathbb{Z}_p[\zeta_{q-1}]$, $P_\kappa(T) \in \mathbb{Z}_p[\zeta_{q-1}][T]$ is a distinguished polynomial, i.e. a monic polynomial where the coefficients of the non-leading terms lie in the maximal ideal, and $f_\kappa(T) = \sum_{i\geq 0} a_{\kappa,i}T^i \in \mathbb{Z}_p[\zeta_{q-1}][[T]]^*$, i.e. $a_{\kappa,0}$ is a unit in $\mathbb{Z}_p[\zeta_{q-1}]$.
We now assume that the level $f_{\chi'}$ of χ' is divisible by p^2, this only excludes finitely many characters. Since u_1 is a generator of $1+p\mathbb{Z}_p$ we see that $\chi'(u_1)$ is a primitive p^{e-1}th root of unity. Let us denote by $\mathcal{P}$ the maximal ideal in the purely ramified field extension $\mathbb{Q}_p(\chi'(u_1))/\mathbb{Q}_p$. It is generated by $1-\chi'(u_1)$ and one easily checks that $\chi'(u_1)u_1^{2l}-1 \in i_p(\mathcal{P})$ has the same p-adic absolute value as $1-\chi'(u_1)$. In particular, $\chi'(u_1)u_1^{2l}-1$ is bounded (below and above) independently of χ' and the Lemma will be verified if we can find a bound for the absolute value of $f(\chi'_0, \chi'(u_1)u_1^{2l}-1)$. We accomplish this by giving a bound for each of the factors in the

decomposition (4.17).
Since $\chi'(u_1)u_1^{2l}-1$ is lying in $i_p(\mathcal{P})$ we obtain

$$f_\kappa(\chi'(u_1)u_1^{2l}-1)\equiv a_{\kappa,0} \pmod{i_p(\mathcal{P})}$$

and thus we have $|f_\kappa(\chi'(u_1)u_1^{2l}-1)|_p=1$.
Let ω be a uniformizing element in $\mathbb{Z}_p[\zeta_{q-1}]$, for example if $p\nmid(q-1)$ we may choose $\omega=p$. Excluding only finitely many characters we now assume that the conductor $f_{\chi'}$ is large enough so that $|\chi'(u_1)u_1^2-1|_p^{\deg(P_\kappa)}>|\omega|_p$ for all characters κ. Consequently, all the coefficients of the non-leading terms in $P_\kappa(T)$ have p-adic absolute value smaller than $|\chi'(u_1)u_1^2-1|_p^{\deg(P_\kappa)}$ and we conclude that

$$|P_\kappa(\chi'(u_1)u_1^2-1)|_p=|(\chi'(u_1)u_1^2-1)^{\deg(P_\kappa)}|_p.$$

Thus, we have found a bound for each of the factors occurring on the right hand side of equation (4.17) and we conclude

$$|\omega|_p\min_\kappa|a_\kappa|_p<|f(\chi_0',\chi'(u_1)u_1^{2l}-1)|_p<\max_\kappa|a_\kappa|_p$$

for all characters χ' having conductor large enough. Taking into account that $1<|(\chi'(u_1)u_1^{2l}-1)^{-1}|_p<|\omega|_p^{-1}$ we finally obtain

$$|\omega|_p\min_\kappa|a_\kappa|_p<|L_p(\chi',-1)|_p<|\omega|_p^{-1}\max_\kappa|a_\kappa|_p$$

for all characters χ' having conductor large enough. This proves the Lemma. □

Since $S_{x,y,k,\mu}$ is a rational number, equation (4.15), Lemma 4.8 and Lemma 4.7 eventually imply that

$$p^{(4l-1)e}M_pS_{x,y,k,\mu}\in\mathbb{Z}_p \tag{4.18}$$

for some constant M_p independent of e and the embedding i_p. (We have used that $k\le e$ and $\mu\le 2l-2$.)

Lemma 4.9. *Let $N\in\mathbb{N}$ be any integer divisible by at least two distinct prime numbers and let $N'|N$ be any divisor which is divisible by the same prime numbers as N. Then for any $\delta,\epsilon_0\in(\mathbb{Z}/N\mathbb{Z})^*$ the following congruence is true:*

$$\frac{(k-1)!(m-1)!}{\pi^{k+m}}\sum_{\substack{\epsilon\in(\mathbb{Z}/N\mathbb{Z})^*\\ \epsilon\equiv\epsilon_0(N')}}A_k\left(\frac{\epsilon}{N}\right)A_m\left(\frac{\epsilon\delta}{N}\right)\equiv 0 \pmod{N/N'\mathcal{O}_{\mathbb{Q}(\zeta_{N'})}},$$

where $k,m>0$ are of the same parity.

Proof. Differentiating the identity $\pi\cot(\pi x)=\sum_{n\in\mathbb{Z}}\frac{1}{n+x}$ $k-1$ times and using $(\cot^a(\pi x))'=-\pi a(\cot^{a-1}(\pi x)+\cot^{a+1}(\pi x))$ for positive integers a, we find

$$A_k\left(\frac{\epsilon}{N}\right)=\sum_{n\in\mathbb{Z}}\frac{1}{\left(n+\frac{\epsilon}{N}\right)^k}=\frac{\pi^k}{(k-1)!}\sum_{j\in I_k}b_{j,k}\cot^j\left(\pi\frac{\epsilon}{N}\right),\qquad b_{j,k}\in\mathbb{Z},$$

where I_k denotes the set of all integers j with $0 \leq j \leq k$ and j is of the same parity as k. So we are finished if we can show, that

$$\sum_{\substack{\epsilon \in (\mathbb{Z}/N\mathbb{Z})^* \\ \epsilon \equiv \epsilon_0(N')}} \cot^j\left(\pi\frac{\epsilon}{N}\right)\cot^n\left(\pi\frac{\delta\epsilon}{N}\right) = \sum_{\substack{\epsilon \in (\mathbb{Z}/N\mathbb{Z})^* \\ \epsilon \equiv \epsilon_0(N')}} \left(i\frac{\zeta_N^{\epsilon}+\zeta_N^{-\epsilon}}{\zeta_N^{\epsilon}-\zeta_N^{-\epsilon}}\right)^j \left(i\frac{\zeta_N^{\epsilon\delta}+\zeta_N^{-\epsilon\delta}}{\zeta_N^{\epsilon\delta}-\zeta_N^{-\epsilon\delta}}\right)^n \tag{4.19}$$

lies in $N/N'\mathcal{O}_{\mathbb{Q}(\chi)}$ for $j, n \in \mathbb{N}$ of the same parity and $\zeta_N = e^{i\pi/N}$, so $\zeta_N^N = -1$. This can be shown analogously to [Mah00]:
Let $p_1, \ldots p_s$ be the prime numbers dividing N, so $s \geq 2$ according to our assumptions. Let $\Phi_d \in \mathbb{Z}[T]$ be the dth cyclotomic polynomial. We want to show that there is a polynomial $P(T) \in \mathbb{Z}[T]$ such that

$$\Phi_{p_1 \cdot \ldots \cdot p_s}(T) - 1 = P(T) \cdot (T-1).$$

It suffices to show, that 1 is a root of the polynomial on the left hand side: Then we can factor this polynomial into $(T-1) \cdot P(T)$ over $\mathbb{C}$ and by comparing the coefficients we see that $P(T)$ even lies in $\mathbb{Z}[T]$. From the explicit form of the cyclotomic polynomial Φ_p we immediately see that $\Phi_p(1) = p$ for any prime p. Using that $\Phi_{pn}(T) = \Phi_n(T^p)/\Phi_n(T)$ for an integer n coprime to p (cf. [Lan02], page 280) we conclude that $\Phi_{p_1 \cdot \ldots \cdot p_s}(1) = 1$ for $s \geq 2$. This proves the existence of $P(T) \in \mathbb{Z}[T]$ as described above.
We set $N_0 = N/(p_1 \cdot \ldots \cdot p_s)$. Substituting $T \mapsto T^{N_0}$ and multiplying the equation by -1 we obtain

$$-\Phi_{p_1 \cdot \ldots \cdot p_s}(T^{N_0}) + 1 = P_N(T) \cdot (T-1),$$

where $P_N(T) = -P(T^{N_0}) \cdot \sum_{i<N_0} T^i \in \mathbb{Z}[T]$. Since $\Phi_{p_1 \cdot \ldots \cdot p_s}(T^{N_0}) = \Phi_N(T)$ (cf. [Lan02], page 280) we see that P_N is the inverse of $T-1$ modulo Φ_N. In particular, we have

$$\frac{1}{\zeta_N^{2\epsilon}-1} = P_N(\zeta_N^{2\epsilon})$$

because $\zeta_N^{2\epsilon}$ is a Nth primitive root of unity for $\epsilon \in (\mathbb{Z}/N\mathbb{Z})^*$. Finally, we conclude that there is a polynomial $Q_N(T) = \sum_i a_{N,i}T^i \in \mathbb{Z}[T]$ with

$$Q_N(\zeta_N^{2\epsilon}) = \left(\frac{\zeta_N^{2\epsilon}+1}{\zeta_N^{2\epsilon}-1}\right)^j \left(\frac{\zeta_N^{2\epsilon\delta}+1}{\zeta_N^{2\epsilon\delta}-1}\right)^n.$$

Since

$$\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_{N'})) = \{\zeta_N \mapsto \zeta_N^{\epsilon} : \epsilon \in (\mathbb{Z}/N\mathbb{Z})^*, \epsilon \equiv 1 \mod N'\}$$

we obtain

$$\begin{aligned}\sum_{\epsilon \equiv \epsilon_0} \left(\frac{\zeta_N^{\epsilon}+\zeta_N^{-\epsilon}}{\zeta_N^{\epsilon}-\zeta_N^{-\epsilon}}\right)^j \left(\frac{\zeta_N^{\epsilon\delta}+\zeta_N^{-\epsilon\delta}}{\zeta_N^{\epsilon\delta}-\zeta_N^{-\epsilon\delta}}\right)^n &= \sum_{\epsilon \equiv \epsilon_0}\sum_i a_{N,i}\left(\zeta_N^{2\epsilon}\right)^i = \\ &= \sum_i a_{N,i} \sum_{\epsilon \equiv 1(N')} \left(\zeta_N^{2\epsilon_0 i}\right)^{\epsilon} = \sum_i a_{N,i}\mathrm{Tr}(\zeta_N^{2\epsilon_0 i}),\end{aligned}$$

where $\mathrm{Tr} = \mathrm{Tr}^{\mathbb{Q}(\zeta_N)}_{\mathbb{Q}(\zeta_{N'})}$. (Note that we can ignore the $i = \sqrt{-1}$ in the brackets in equation (4.19) because j and n are of the same parity.) But for any, not necessarily primitive Nth root of unity ζ we have $\mathrm{Tr}(\zeta) \in N/N'\mathcal{O}_{\mathbb{Q}(\zeta_{N'})}$ (cf. [Gir92] Theorem 1 and Corollary), which gives the claim of the Lemma. □

Lemma 4.10. *Let ℓ be any prime number different from p. There is a constant M_ℓ, which does not depend on the embedding i_ℓ, such that*

$$|L(\chi, 1-2l)|_\ell > M_\ell$$

for all even characters χ of $(\mathbb{Z}/p^e q\mathbb{Z})^*$ where e runs over $\mathbb{Z}_{\geq 0}$.

Proof. In [Mah00], Lemma 4.7, this statement is shown for $l = 1$ and the proof generalizes directly.
We first show that it suffices to verify the claim of the Lemma for primitive characters χ with conductor of the form $p^e q$ or p^e for $e \geq 1$: Let χ be any even character of $(\mathbb{Z}/p^e q\mathbb{Z})^*$ and denote by χ' the associated primitive character. The corresponding L-functions are related as follows

$$L(\chi, 1-2l) = L(\chi', 1-2l)(1-\chi'(p)p^{2l-1})(1-\chi'(q)q^{2l-1}).$$

Roughly speaking, the L-function attached to χ misses the Euler factors at p and q and if these Euler factors are trivial, i.e. the level of χ is divisible by pq, the L-functions coincide. The Euler factor at p is nontrivial only for the finitely many primitive Dirichlet characters χ' having level dividing q. Consequently, for deducing the existence of the constant M_ℓ in the general case from the primitive case we only have to consider Dirichlet characters χ whose level is a power of p. But for these characters one can show in exactly the same manner as in the proof of Lemma 4.7 that the ℓ-adic absolute value of $1-\chi'(q)q^{2l-1}$ is bounded below independently of the level of χ'. This leaves us with proving the claim of the Lemma for primitive characters χ' of level $p^e q$ or p^e for $e \geq 1$. Note that $L(\chi', 1-2l)$ does not vanish since χ' is even and $1-2l$ is odd. We will use the ℓ-adic L-function

$$L_\ell(\chi'\omega^{1-n}, n) = (1-\chi'(\ell)\ell^{-n})L(\chi', n) \tag{4.20}$$

to show that $\ell \nmid L(\chi', 1-2l)$ for almost all primitive Dirichlet characters χ' of level $p^e q$ or p^e. (Here ω denotes the ℓ-adic Teichmuller character.) Since $1-\chi'(\ell)\ell^{2l-1}$ has ℓ-adic absolute value 1 we see that $\ell | L(\chi', 1-2l)$ is equivalent to $\ell | L_\ell(\chi'\omega^{2l}, 1-2l)$. Using [Was97], Corollary 5.13 we further get that for $\chi' \neq 1$ this is equivalent to $\ell | L_\ell(\chi'\omega^{2l}, 0)$. Applying equation (4.20) yields

$$L_\ell(\chi'\omega^{2l}, 0) = (1-\chi'\omega^{2l-1}(\ell))L(\chi'\omega^{2l-1}, 0). \tag{4.21}$$

Next, we show that $1-\chi'\omega^{2l-1}(\ell)$ is not divisible by ℓ for almost all χ'. Since the Teichmuller character ω generates the group of characters on $(\mathbb{Z}/\ell\mathbb{Z})^*$ it is of even order and so ω^{2l-1} is not trivial but has level ℓ.
If $\ell \neq q$ then $\chi'\omega^{2l-1}$ has level $f_{\chi'}\ell$ and vanishes at ℓ. In this case of course $1-\chi'\omega^{2l-1}(\ell) = 1$ is not divisible by ℓ.
If $\ell = q$ then $\chi'\omega^{2l-1}(q)$ is either equal to zero or equal to a $(p-1)p^{e-1}$th root of unity $\zeta_{(p-1)p^{e-1}}$, in which case χ' must have level $p^e q$. So we are reduced to the case that χ' has level $p^e q$ and $\chi'\omega^{2l-1}(q)$ is a (not necessarily primitive) $(p-1)p^{e-1}$th root of unity. We first show that $\chi'\omega^{2l-1}(q)$ is not equal to one if e is large enough. If $\chi'\omega^{2l-1}(q)$ equals one the primitive Dirichlet character $\chi'\omega^{2l-1}$ in particular has level p^e. The primitive Dirichlet characters of level p^e for p an odd prime and $e \geq 2$ are described in [Apo76], Theorem 10.14: Let $g \in \mathbb{N}$ be a primitive root mod p which is also a primitive root mod p^e for all $e \geq 1$, i.e. g generates $(\mathbb{Z}/p^e\mathbb{Z})^*$ for all $e \geq 1$. (Such a root exists according to [Apo76], Theorem 10.6.) For $n \in \mathbb{N}$ with $(n, p) = 1$ let $b(n)$ be the unique integer satisfying $g^{b(n)} \equiv n \pmod{p^e}$ and $0 \leq b(n) < \Phi(p^e)$. For any $h \in \{0, 1, \ldots \Phi(p^e) - 1\}$ we define the Dirichlet character χ_h on $(\mathbb{Z}/p^e\mathbb{Z})^*$ by

$$\chi_h(n) = \begin{cases} e^{\frac{2\pi i h b(n)}{\Phi(p^e)}} & \text{if } p \nmid n, \\ 0 & \text{else.} \end{cases} \tag{4.22}$$

The primitive Dirichlet characters of level p^e are exactly the characters χ_h with $h \in \{0, 1, \dots \Phi(p^e) - 1\}$ coprime to p. Consequently, if $\chi'\omega^{2l-1}(q)$ equals one then $b(q)$ must be divisible by p^{e-1}. But using the definition of $b(q)$ this implies that $q^{p-1} \equiv 1 \pmod{p^e}$. We conclude that $\chi'\omega^{2l-1}(q)$ is not equal to one for e large enough such that $q^{p-1} < p^e$.
We still have to consider the case that $\chi'\omega^{2l-1}(q)$ is a primitive nth root of unity ζ_n for $n > 1$, $n|(p-1)p^{e-1}$. Using again the explicit description of the primitive characters we show that for e large enough, n is divisible by p. Indeed, if n is not divisible by p then (4.22) yields that $b(q)$ has to be divisible by p^{e-1}. But we have already shown above that for e large enough $b(q)$ is not divisible by p^{e-1}. We conclude that n is divisible by p. In particular, n is either a product of at least two different primes or n is equal to a power of p. If n is divisible by two distinct primes, Proposition 2.8 in [Was97] shows that $1-\zeta_n$ is a unit of $\mathbb{Z}[\zeta_n]$ and in particular is not divisible by q. If $n = p^k$ it is well known that $(1-\zeta_{p^k})^{\Phi(p^k)}$ and (p) coincide, where $(1-\zeta_{p^k})$ denotes the principal ideal in $\mathbb{Z}[\zeta_{p^k}]$ generated by $1-\zeta_{p^k}$ and similarly for (p). We conclude that $1-\zeta_{p^k}$ is not divisible by q.
Altogether we conclude that for e large enough $1-\chi'\omega^{2l-1}(\ell)$ is not divisible by ℓ, so $\ell|L_\ell(\chi'\omega^{2l}, 0)$ implies by equation (4.21) that $\ell|L(\chi'\omega^{2l-1}, 0)$. But according to Theorem 4.1 in [Sin87], $L(\chi', 0)$ is a ℓ-adic unit for all but finitely many characters χ', which proves the Lemma. □

Analogously to equation (4.18), Lemma 4.7 and Lemma 4.10 imply that for any prime ℓ different from p there exists a constant $M_\ell \in \mathbb{Z}$ independent of e such that $M_\ell S_{x,y,k,\mu}$ lies in $\mathbb{Z}_\ell$.
Since $\omega^0_{p^e}$ and ω_{p^e} have the same denominators we finally get the following result:

Theorem 4.11. *For any prime number ℓ there is a constant $M_\ell \in \mathbb{Z}$ such that for all $e \in \mathbb{N}$ and any prime ideal $\mathcal{L}$ in the ring of integers $\mathcal{O}$ of $\mathbb{Q}(\eta, \eta')$ lying above ℓ we have*

$$*M_\ell \omega_{p^e} \in H^1(S_2(K^f), \mathcal{W}_{\mathcal{O}_\mathcal{L}})_{\text{int}},$$

where $* = p^{(4l-1)e}$ if $\ell = p$ and $* = 1$ else.

4.7. The Denominators of $r_u^* \omega$

The representation (ρ, V) is defined over $\mathbb{Z}$, we denote by $V_\mathbb{Z}$ the corresponding $\mathbb{Z}$-lattice in $V_\mathbb{Q}$. Let F be the algebraic number field $E_\pi(\eta, \eta', \zeta_{q(q-1)})$, $\mathcal{O}$ its ring of integers and let $\mathcal{L}$ be a prime ideal in $\mathcal{O}$ lying over p. Then, the matrix $u(i, j, y; p^e)$ does not lie in $\mathrm{GL}_3(\mathcal{O}_\mathcal{L})$ and the morphism $r^*_{u(i,j,y;p^e)}$ does not map the integral cohomology $H^2_c(S_3(K^f), \mathcal{V}_{\mathcal{O}_\mathcal{L}})_{\text{int}}$ to itself. In this section we describe the growth of the $\mathcal{L}$-denominators of $r^*_{u(i,j,y;p^e)}\Omega(\pi)^{-1}\omega \in H^2_c(S_3(K^f), \mathcal{V}_{F_\mathcal{L}})$ for e to infinity. The decisive information is contained in the following Lemma, which describes the action of $u(i, j, y; p^e)$ on the lattice $V_\mathbb{Z} \subset V_\mathbb{Q}$.

Lemma 4.12. *We have*

$$u(i, j, y; p^e) \cdot V_\mathbb{Z} \subset p^{-2e(l_0-2)} V_\mathbb{Z},$$

where $V_\mathbb{Z}$ is regarded as a submodul of $V_\mathbb{Q}$.

Proof. Let T be the standard torus in GL_3. The root system Φ relative to T is given by the characters

$$\alpha_{i,j} : \mathrm{diag}(t_1, t_2, t_3) \mapsto t_i/t_j$$

with $i \neq j$. We fix the basis $\{\alpha_{1,2}, \alpha_{2,3}\}$ of the root system, then the positive roots Φ^+ are given by $\{\alpha_{i,j}\}_{i<j}$. Let $U_{\alpha_{i,j}}$ be the unique connected T-stable subgroup of GL_3 with Lie algebra

$\mathrm{gl}_{3,\alpha_{i,j}}$. (Here, $\mathrm{gl}_{3,\alpha_{i,j}} \subset \mathrm{gl}_3$ denotes the weight space corresponding to $\alpha_{i,j}$ with respect to the adjoint action of T on gl_3.) More explicitly, $U_{\alpha_{i,j}}$ consists of the unipotent matrices where the (i,j)-entry is arbitrary and all other entries which are not lying on the diagonal are zero. From [Mah00], page 276, we know that (ρ, V) is the finite dimensional representation of GL_3 with highest weight $\lambda := \alpha_{1,3}^{l_0/2-1}$. Let

$$V = \bigoplus_{\mu} V_\mu$$

be the weight decomposition of V. Here, the weights μ satisfy $w\lambda \leq \mu \leq \lambda$, where w denotes the Weyl element $\left(\begin{smallmatrix} & & 1 \\ & 1 & \\ 1 & & \end{smallmatrix}\right)$, cf. [Hum75], section 31.2. The element $u(i,j,y;f)$ with $f = p^e$ can be written as

$$u(i,j,y;f) = huh^{-1} \quad \text{with } h := \begin{pmatrix} 1 & & \\ & f & \\ & & f^2 \end{pmatrix} \text{ and } u := \begin{pmatrix} 1 & ip & y \\ & 1 & j \\ & & 1 \end{pmatrix}.$$

Let now $v \in V_\mu \cap V_{\mathbb{Z}}$. (Note that $V_{\mathbb{Z}} = \oplus_\mu (V_\mu \cap V_{\mathbb{Z}})$.) By definition, we have $h^{-1} \cdot v = \mu(h^{-1})v$. Since the center of GL_3 acts trivially on V_λ we conclude that the central character of (ρ, V) is trivial. Thus, we can apply Proposition 27.2 in [Hum75] and find that $u \cdot v = \sum_{\mu' \geq \mu} v_{\mu'}$ with $v_{\mu'} \in V_{\mu'}$. Moreover, since u lies in $\mathrm{GL}_3(\mathbb{Z})$ we conclude that $v_{\mu'}$ even lies in $V_{\mu'} \cap V_{\mathbb{Z}}$. Altogether, we obtain

$$huh^{-1} \cdot v = \sum_{\mu' \geq \mu} (\mu' - \mu)(h) v_{\mu'}. \tag{4.23}$$

We check that the p-adic valuation of $(\mu' - \mu)(h)$ is bounded below by the p-adic valuation of $(\lambda - w\lambda)(h)$: First, it follows from $\lambda \geq \mu' \geq \mu \geq w\lambda$ that $\lambda - w\lambda \geq \mu' - \mu$. Using that the p-adic valuation $v_p(\beta(h))$ is negative for any positive root $\beta \in \Phi^+$, we conclude that, indeed,

$$v_p\left((\lambda - w\lambda)(h)\right) \leq v_p((\mu' - \mu)(h)).$$

Making use of the fact that the center of GL_3 lies in the kernel of λ, an explicit computation shows that $w\lambda(h)$ equals $-\lambda(h)$. Thus, the p-adic valuation of $(\mu' - \mu)(h)$ is greater than or equal to $v_p((2\lambda)(h)) = -2e(l_0 - 2)$. Combining this information with equation (4.23) we obtain that $huh^{-1} \cdot v$ lies in $p^{-2e(l_0-2)} V_{\mathbb{Z}}$. Since $v \in V_\mu \cap V_{\mathbb{Z}}$ and μ were chosen arbitrarily this proves the claim of the Lemma. □

It is now a matter of definitions to carry the information about the action of $u(i,j,y;p^e)$ on $V_{\mathbb{Z}}$ contained in Lemma 4.12 over to the cohomological situation.

Proposition 4.13. *Let $\mathcal{L}$ be a prime ideal in $\mathcal{O}$ lying over p and let K^f be a compact open subgroup of $\mathrm{GL}_3(\mathbb{A}_f)$ which stabilizes ω and $\prod_\ell V_{\mathbb{Z}} \otimes \mathbb{Z}_\ell$. Then, there exists an element $c_{\mathcal{L}} \in F_{\mathcal{L}}$ such that*

$$p^{2e(l_0-2)} c_{\mathcal{L}}\, r_u^* \Omega(\pi)^{-1} \omega \in H_c^2(S_3(uK_u^f u^{-1}), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}})_{\mathrm{int}}$$

for any element $u := u(i,j,y;p^e)$ and any compact open subgroup K_u^f of K^f such that K_u^f and $uK_u^f u^{-1}$ keep $\prod_{\ell < \infty} V_{\mathbb{Z}} \otimes \mathbb{Z}_\ell$ stable.

Proof. Our assumptions on the groups K^f and K_u^f imply that the integral cohomology groups $H_c^2(S_3(K^f), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}})$, $H_c^2(S_3(K_u^f), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}})$ and $H_c^2(S_3(uK_u^f u^{-1}), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}})$ exist and that the cohomology class $\Omega(\pi)^{-1}\omega$ lies in $H_c^2(S_3(K^f), \mathcal{V}_{F_{\mathcal{L}}})$. According to the Lemma in Appendix B, there exists

an element $c_{\mathcal{L}} \in F_{\mathcal{L}}$ such that $c_{\mathcal{L}}\,\Omega(\pi)^{-1}\omega$ lies in $H^2_c(S_3(K^f), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}})_{\mathrm{int}}$ and thus also lies in $H^2_c(S_3(K^f_u), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}})_{\mathrm{int}}$. We have seen in section 2.1 that right translation by an element $g \in \mathrm{GL}_n(\mathbb{A}_f)$ induces a morphism r_g^* in cohomology, for the integral cohomology we obtain

$$r_u^* : H^2_c(S_3(K^f_u), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}}) \longrightarrow H^2_c(S_3(uK^f_u u^{-1}), r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}),$$

where $r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}$ denotes the inverse image sheaf on $S_3(uK^f_u u^{-1})$ of the sheaf $\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}$ on $S_3(K^f_u)$. Recall that $r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}$ can be constructed as the sheafification of the presheaf $\overline{r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}}$ with

$$\overline{r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}}(U) := \mathcal{V}_{\mathcal{O}_{\mathcal{L}}}(Uu) = \{f \in \mathcal{V}_{F_{\mathcal{L}}}(Uu) | f((x_\infty, g_f u)) \in g_f u V_{\mathcal{O}_{\mathcal{L}}} \text{ for } (x_\infty, g_f u) \in \pi^{-1}(Uu)\}$$

for $U \subset S_3(uK^f_u u^{-1})$ open, where, as before, $\pi : \mathrm{GL}_3(\mathbb{A})/(uK^f_u u^{-1})K_{3,\infty}Z^0_3(\mathbb{R}) \to S_3(uK^f_u u^{-1})$ denotes the canonical projection. One can check that $\overline{r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}}$ already defines a sheaf, i.e. $\overline{r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}} = r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}$. The morphisms

$$\begin{aligned} i :\ r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}(U) &\longrightarrow \mathcal{V}_{F_{\mathcal{L}}}(U), \\ f &\longmapsto \{f' : (x_\infty, g_f) \mapsto f((x_\infty, g_f u))\} \end{aligned}$$

with $U \subset S_3(uK^f_u u^{-1})$ open, define an embedding $i : r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}} \hookrightarrow \mathcal{V}_{F_{\mathcal{L}}}$ of sheaves on $S_3(uK^f_u u^{-1})$. Taking into account that $uV_{\mathcal{O}_{\mathcal{L}}}$ lies in $p^{-2e(l_0-2)}V_{\mathcal{O}_{\mathcal{L}}}$ by Lemma 4.12, we find that the morphism i even maps into the sheaf $p^{-2e(l_0-2)}\mathcal{V}_{\mathcal{O}_{\mathcal{L}}} \subset \mathcal{V}_{F_{\mathcal{L}}}$ defined by $(p^{-2e(l_0-2)}\mathcal{V}_{\mathcal{O}_{\mathcal{L}}})(U) := p^{-2e(l_0-2)}(\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}(U)) \subset \mathcal{V}_{F_{\mathcal{L}}}(U)$ with the obvious restriction morphisms. Thus, we obtain the following diagram in cohomology

$$\begin{array}{ccc} H^2_c(S_3(K^f_u), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}}) & \xrightarrow{r_u^*} & H^2_c(S_3(uK^f_u u^{-1}), r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}) \\ \downarrow & & \downarrow \\ & & H^2_c(S_3(uK^f_u u^{-1}), p^{-2e(l_0-2)}\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}) \\ & & \downarrow j \\ H^2_c(S_3(K^f_u), \mathcal{V}_{F_{\mathcal{L}}}) & \xrightarrow{r_u^*} & H^2_c(S_3(uK^f_u u^{-1}), \mathcal{V}_{F_{\mathcal{L}}}), \end{array} \tag{4.24}$$

where the two vertical arrows on the right hand side denote the maps in cohomology induced by the embeddings of $r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}$ into $p^{-2e(l_0-2)}\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}$ and of $p^{-2e(l_0-2)}\mathcal{V}_{\mathcal{O}_{\mathcal{L}}}$ into $\mathcal{V}_{F_{\mathcal{L}}}$. Applying Theorem II.15.3 (Universal Coefficient Theorem) in [Bre67] with $\mathcal{A} = \mathcal{V}_{\mathcal{O}_{\mathcal{L}}}$, $L = \mathcal{O}_{\mathcal{L}}$ and the L-module $M = p^{-2e(l_0-2)}\mathcal{O}_{\mathcal{L}}$ we find that

$$p^{-2e(l_0-2)}H^2_c(S_3(uK^f_u u^{-1}), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}}) \longrightarrow H^2_c(S_3(uK^f_u u^{-1}), p^{-2e(l_0-2)}\mathcal{V}_{\mathcal{O}_{\mathcal{L}}})$$

is an isomorphism. (Here, we have used that the torsion product $\mathrm{Tor}(N, p^{-2e(l_0-2)}\mathcal{O}_{\mathcal{L}})$ is trivial for any $\mathcal{O}_{\mathcal{L}}$-module N.) Thus, the image of the morphism j in diagram (4.24) is contained in $p^{-2e(l_0-2)}H^2_c(S_3(uK^f_u u^{-1}), r_u^*\mathcal{V}_{\mathcal{O}_{\mathcal{L}}})_{\mathrm{int}}$. This proves that the element $c_{\mathcal{L}}\,\Omega(\pi)^{-1}r^*_{u(i,j,y;p^e)}\omega$ indeed lies in $p^{-2e(l_0-2)}H^2_c(S_3(uK^f_u u^{-1}), \mathcal{V}_{\mathcal{O}_{\mathcal{L}}})_{\mathrm{int}}$ as was claimed in the Proposition. □

5. The p-adic L-function

This chapter finally contains the result we were heading for in the last two chapters, the existence of the p-adic L-function attached to π and to the critical integers on the left hand side of the functional equation. This mainly relies on the manageable p-adic growth of the distribution $\mu_{\pi,l}$, which we will deduce in the first section from the results in the last chapter, and on the p-adic integration theory developed in [Višˇ76].
The last section in this chapter is concerned with the construction of a p-adic L-function attached to the critical integers on the right hand side of the functional equation, which, similarly as in the complex situation, should be related to the L-function attached to the critical integers on the left hand side by a functional equation.

5.1. The p-adic Growth of the Distribution

We set $u = u(1, \epsilon, 0; p^e)$ for ϵ in $\mathbb{Z}_p^*$. Identity (2.3) on page 270 in [Mah00] implies that the cohomology class ω_{p^e} is invariant under a compact open subgroup $K(\omega_{p^e}) = \prod_\ell K_\ell(\omega_{p^e}) \subset \mathrm{GL}_2(\mathbb{A}_f)$ with $K_p(\omega_{p^e})$ equal to

$$K_p(\omega_{p^e}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : a \equiv 1 \pmod{p}, \quad c \equiv 0,\ d \equiv 1 \pmod{p^e} \right\}.$$

From the construction of the element ϕ on the pages 257 and 258 in [Mah00] we know that ϕ is invariant under the Iwahori subgroup $\mathcal{I}$. Thus, we can find a compact open subgroup $K(\omega) = \prod_\ell K_\ell(\omega) \subset \mathrm{GL}_3(\hat{\mathbb{Z}})$ which keeps the cohomology class ω invariant and whose pth component $K_p(\omega)$ is equal to $\mathcal{I} \cap u^{-1}\mathrm{GL}_3(\mathbb{Z}_p)u$. Therefore, the differential forms $i^* r_u^* \omega$ and $p^* \omega_{p^e}$ are invariant under a subgroup $\mathcal{K}(p^e) = \prod_\ell \mathcal{K}_\ell(p^e) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$, where $\mathcal{K}_p(p^e)$ is the subgroup of $K_p(\omega_{p^e})$ which consists of all elements k satisfying

$$u^{-1} \begin{pmatrix} k & \\ & 1 \end{pmatrix} u \in \mathcal{I}.$$

For $\ell \neq p$ we can choose the local groups $\mathcal{K}_\ell(p^e)$ to be independent of u and such that $\mathcal{K}(p^e)$ lies in $uK(\omega)u^{-1} \cap \mathrm{GL}_2(\mathbb{A}_f)$. In Lemma 5.1 in [Mah00] it is shown that the group $\mathcal{K}_p(p^e)$ does not depend on $\epsilon \in \mathbb{Z}_p^*$ and has volume $p^4(p-1)^{-1}(p^2-1)^{-1}p^{-4e}$ with respect to the normalized Haar measure dt_p on $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence, $\mathcal{K}(p^e)$ has volume $* \, p^{-4e}$ where $*$ is some constant independent of e. Lemma 3.7 states that

$$\frac{\mu_{\pi,l}(\epsilon + p^e\mathbb{Z}_p)}{\Omega(\pi)} = \frac{p^{e(4+l)}}{\eta_p(p^e)\gamma^e} \left\langle \frac{r_u^*\omega}{\Omega(\pi)}, \omega_{p^e} \right\rangle. \tag{5.1}$$

As before, let F be the algebraic number field $E_\pi(\eta, \eta', \zeta_{q(q-1)})$, $\mathcal{L}$ a prime ideal in its ring of integers $\mathcal{O}$ and ϖ a uniformizing parameter in $\mathcal{O}_\mathcal{L}$. The group $\mathcal{K}(p^e)$ satisfies the conditions on K in Corollary 4.3; indeed, it keeps $i^* r_u^* \omega$ and $p^* \omega_{p^e}$ stable by definition and since $\mathcal{K}(p^e)$ lies in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ it also keeps $\prod_\ell V_\mathbb{Z}|_{\mathrm{GL}_2} \otimes \mathbb{Z}_\ell$ and $\prod_\ell W_\mathbb{Z} \otimes \mathbb{Z}_\ell$ stable. Thus, we can apply Corollary 4.3 to the right hand side of equation (5.1) and obtain that

$$\frac{\mu_{\pi,l}(\epsilon + p^e\mathbb{Z}_p)}{\Omega(\pi)} \in \frac{* p^{el}}{\eta_p(p^e)\gamma^e} \varpi^{-\delta(\omega,e)-\delta(\omega_{p^e})} \mathcal{O}_\mathcal{L},$$

where $\delta(\omega, e)$ resp. $\delta(\omega_{p^e})$ denotes the $\mathcal{L}$-denominator of $r_u^*\Omega(\pi)^{-1}\omega \in H_c^2(S_3(uK(\omega)u^{-1}), \mathcal{V}_{F_\mathcal{L}})$ resp. of $\omega_{p^e} \in H^1(S_2(\mathcal{K}(p^e)), \mathcal{W}_{F_\mathcal{L}})$. (Note, that $uK(\omega)u^{-1} \subset \mathrm{GL}_3(\hat{\mathbb{Z}})$ keeps $r_u^*\omega$ and $\prod_\ell V_\mathbb{Z} \otimes \mathbb{Z}_\ell$ stable and that $\mathcal{K}(p^e)$ lies in $uK(\omega)u^{-1} \cap \mathrm{GL}_2(\mathbb{A}_f)$, i.e. $uK(\omega)u^{-1}$ satisfies the conditions on K_3 in Corollary 4.3.) Using Proposition 4.13 (with $K^f = \mathcal{I}\prod_{\ell \neq p} K_\ell(\omega)$ and $K_u^f = K(\omega)$) and Theorem 4.11, we obtain the following final result on the values of $\mu_{\pi,l}$.

Theorem 5.1. *For any $\epsilon \in \mathbb{Z}_p^*$, any integer l with $0 < l \leq l_0/2$ and $l \equiv \flat \pmod 2$ and any embedding $i_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ the following holds*

$$\left| \frac{\mu_{\pi,l}(\epsilon + p^e\mathbb{Z}_p)}{\Omega(\pi)} \right|_p \leq M_p' p^{(3l-5+2l_0)e} |\gamma^{-1}|_p^e,$$

for a constant $M_p' \in \mathbb{Z}$. If ℓ is any prime number different from p and $i_\ell : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_\ell$ is any embedding, then $|\mu_\pi(\epsilon + p^e\mathbb{Z}_p)/\Omega(\pi)|_\ell < M_\ell' |\gamma^{-1}|_\ell^e$ for a constant $M_\ell' \in \mathbb{Z}$.

(Note that η is of finite order and so $|\eta_p(p^e)|_\ell = 1$, where $|\cdot|_\ell$ is the absolute valuation on $\bar{\mathbb{Q}}$ induced by i_ℓ for any prime ℓ.)
Since $\delta_{q^{-1}}$ and $\mu(l)$ are $\mathbb{Z}$-valued the same statement as in Theorem 5.1 is true for the distribution $\delta_{q^{-1}} * \mu(l) * \mu_{\pi,l}/\Omega(\pi)$.

5.2. h-admissible Measures

Obviously, the distribution $\delta_{q^{-1}} * \mu(l) * \mu_{\pi,l}/\Omega(\pi)$ is not a measure in the classical sense, the p-adic absolute value of $\mu_{\pi,l}(\epsilon + p^e\mathbb{Z}_p)/\Omega(\pi)$ is not bounded for e arbitrarily large. Though, Theorem 5.1 shows that our distribution satisfies a growth condition which enables us to apply the p-adic integration theory developed in [Viš76].

For $h \in \mathbb{N}_{>0}$ let $\mathcal{C}^h$ be the vector space of functions $f : \mathbb{Z}_p^* \to \mathbb{C}_p$ which are locally given by polynomials of degree less than h. An h-admissible measure on $\mathbb{Z}_p^*$ is a $\mathbb{C}_p$-linear functional $\widetilde{\mu} : \mathcal{C}^h \to \mathbb{C}_p$, which satisfies the following growth condition

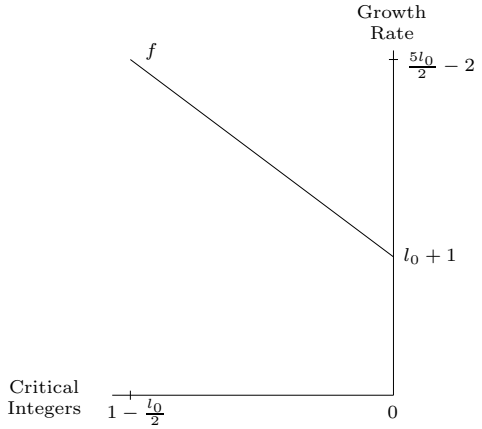
$$\sup_{a \in \mathbb{Z}_p^*} \left| \widetilde{\mu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p} \cdot (x-a)^i) \right|_p = o(|p^e|_p^{i-h}) \tag{5.2}$$

for all $0 \leq i < h$ and $e \to \infty$, cf. [Viš76], section 1.3. Here $\mathbf{ch}_X$ denotes the characteristic function of the set $X \subset \mathbb{Z}_p^*$.
Let $K/\mathbb{Q}_p$ be a local field. We denote by $\mathcal{C}_K^h$ the space of K-valued functions in $\mathcal{C}^h$ and by $\mathcal{M}_K^h$ the K-vector space consisting of the h-admissible measures whose restrictions to $\mathcal{C}_K^h$ are K-valued. An element in $\mathcal{M}_K^h$ is called an h-admissible K-valued measure. Let us further denote by $\mathcal{D}_K^h$ the space of all K-valued distributions μ on $\mathbb{Z}_p^*$ satisfying the following growth condition

$$\sup_{a \in \mathbb{Z}_p^*} |\mu(a + p^e\mathbb{Z}_p)|_p = o(|p^e|_p^{-h}) \quad \text{for } e \to \infty.$$

Example 5.2. *Let $h^* \in \mathbb{Z}$ be the smallest number such that $p^{h^*+2-l_0}\gamma^{-1}$ lies in $\mathcal{O}$. According to Appendix C, equation (C.6), the p-adic valuation of $p^{2-l_0}\gamma^{-1}$ is less than or equal to zero and, in particular, if π is p-ordinary, then $v_p(p^{2-l_0}\gamma^{-1}) = 0$. We conclude that h^* is not only an integral but a natural number and if π is p-ordinary, then h^* equals zero. Theorem 5.1 shows that the distribution $\mu_{\pi,l}/\Omega(\pi)$ lies in $\mathcal{D}_{i_p(F)}^{h^*+3l+l_0-2} \subset \mathcal{D}_{i_p(F)}^{h^*+5l_0/2-2}$ for any integer l with $1 \leq l \leq l_0/2$ and $l \equiv \flat \pmod 2$. In particular, the same is true for $\delta_{q^{-1}} * \mu(l) * \mu_{\pi,l}/\Omega(\pi)$. (As before, F denotes the algebraic number field $E_\pi(\eta, \eta', \zeta_{q(q-1)})$, cf. section 4.2.)*

In the following picture, the map f assigns to each critical integer $1-l$ the growth rate of the distribution $\mu_{\pi,l}/\Omega(\pi)$ under the assumption that $h^*=0$, which, in particular, is the case if π is p-ordinary. This means, the function f is given by $f(k)=3(1-k)+l_0-2$ for $1-l_0/2\le k\le 0$.



We want to assemble the distributions corresponding to the different critical integers on the left hand side of the functional equation in one h-admissible measure. First we need to slightly modify the distributions $\delta_{q^{-1}} * \mu(l) * \mu_{\pi,l}/\Omega(\pi)$.
In Remark 3.5 we have seen that for fixed π the set of critical integers of $\pi\otimes\eta$ depends only on the parity of the character η which we have abbreviated to $\flat$. To be more precise for an odd resp. even character η the critical integers of $\pi\otimes\eta$ on the left hand side of the functional equation are given by the even resp. odd integers in the interval $[1-l_0/2,0]$. Thus, to obtain all critical integers corresponding to π we consider a fixed pair of characters (η_o,η_e), where η_o and η_e are idèle class characters satisfying the same conditions as η and such that the infinity component of η_o is odd, i.e. $\eta_{o,\infty}=\mathrm{sgn}$, and the infinity component of η_e is even, i.e. $\eta_{e,\infty}=\mathbb{1}$. For $k\in\mathbb{Z}$ we mean by η_k the character η_o if k is odd and the character η_e if k is even. The distribution $\mu_{\pi,l}^{\eta_o}$ is only defined for l odd and analogously $\mu_{\pi,l}^{\eta_e}$ has only been defined for l even. In the following $\mu_{\pi,l}$ shall stand for $\mu_{\pi,l}^{\eta_o}$ if l is odd and for $\mu_{\pi,l}^{\eta_e}$ if l is even. With this notation equation (3.14) reads as follows

$$\int\limits_{\mathbb{Z}_p^*}\chi_p\eta_{l,p}^2\,d\mu_{\pi,l}=\mathbf{B}\,P_l(1/2)\,L(\pi\otimes\chi\eta_l,1-l),$$

where l is an integer with $0<l\le l_0/2$ and χ is an idèle class character of level p^e with $e\ge 2$. We define the distribution $\underline{\mu}_{\pi,l}$ by

$$\underline{\mu}_{\pi,l}(a+p^k\mathbb{Z}_p):=\eta_{l,p}^2(a)\,\mu_{\pi,l}(a+p^k\mathbb{Z}_p)$$

for $a\in\mathbb{Z}_p^*$ and $k\ge 1$ and on the whole of $\mathbb{Z}_p^*$ by the distribution relations. Of course, the integral over the p-component of the character χ with respect to the distribution $\underline{\mu}_{\pi,l}$ evaluates to

$$\int\limits_{\mathbb{Z}_p^*}\chi_p\,d\underline{\mu}_{\pi,l}=\mathbf{B}\,P_l(1/2)\,L(\pi\otimes\chi\eta_l,1-l).$$

We recall that

$$\mathbf{B} = \mathbf{C}_0\, \chi_p(q^{-1}) p^{-2e} \hat{\zeta}^e \pi^{\flat - 3l} \gamma^{-e} G(\chi_p \eta_p^2) L(\widetilde{\chi}\widetilde{\eta}_0, 1-2l)^{-1},$$

where $\mathbf{C}_0$ is a constant independent of χ. The Mellin transform of the convolution of distributions

$$\mu^*_{\pi,l} := \frac{1}{\mathbf{C}_0\, P_l(1/2)\Omega(\pi)}\, \delta_{q^{-1}} * \mu(l) * \underline{\mu}_{\pi,l}$$

then satisfies

$$\int\limits_{\mathbb{Z}_p^*} \chi_p\, d\mu^*_{\pi,l} = p^{-2e} \hat{\zeta}^e \pi^{\flat - 3l} \gamma^{-e} G(\chi_p \eta_p^2) \frac{L(\pi \otimes \chi\eta_l, 1-l)}{\Omega(\pi)}. \tag{5.3}$$

This is the final form of the distribution attached to the critical integer $1-l$ and suitable for our aim to assemble all the distributions into one h-admissible measure.

As in the classical setting, i.e. for bounded distributions, an h-admissible measure $\widetilde{\nu}$ canonically gives rise to an h-admissible distribution ν, if we set

$$\nu(a + p^e\mathbb{Z}_p) := \widetilde{\nu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p})$$

for $a \in \mathbb{Z}_p^*$ and $e \geq 0$. This induces a mapping $\mathrm{res} : \mathcal{M}_K^h \to \mathcal{D}_K^h$ and in Lemma 5.3 in [Mah00] it is shown that res is surjective (but for $h > 1$ not injective). More generally, the following statement holds.

Lemma 5.3. *Let $\{\nu_i\}_{0\leq i\leq n}$ be an arbitrary collection of K-valued h-admissible distributions on $\mathbb{Z}_p^*$. Then there exists a K-valued $(h+n)$-admissible measure $\widetilde{\nu}$ satisfying*

$$\widetilde{\nu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p} x^i) = \nu_i(a + p^e\mathbb{Z}_p) \tag{5.4}$$

for $0 \leq i \leq n$. An $(h+n)$-admissible measure $\widetilde{\nu}$ is uniquely defined by condition (5.4) if and only if h is equal to 1. In particular, the mapping res *is bijective for $h = 1$, i.e. there is a one-to-one correspondence between the 1-admissible distributions and the 1-admissible measures.*

Proof (Outline). We recall that a K-valued $(h+n)$-admissible measure $\widetilde{\nu}$ is a K-linear functional on $\mathcal{C}_K^{h+n}$ satisfying certain growth conditions. Since $\mathcal{C}_K^{h+n}$ is generated over K by the functions $\mathbf{ch}_{a+p^e\mathbb{Z}_p} x^i$, where $a \in \mathbb{Z}_p^*$, $e \in \mathbb{N}$ and $0 \leq i < h+n$, in order to define an $(h+n)$-admissible measure it is sufficient to define the value of $\widetilde{\nu}$ on these functions. The requirement (5.4) already determines the values of $\widetilde{\nu}$ on the functions $\mathbf{ch}_{a+p^e\mathbb{Z}_p} x^i$ with $0 \leq i \leq n$ and in this case, i.e. for $0 \leq i \leq n$, an application of the binomial formula shows that condition (5.2) is equivalent to the following condition

$$\sup_{a\in\mathbb{Z}_p^*} \left| \sum_{r=0}^{i} \binom{i}{r} (-a)^{i-r} \nu_r(a + p^e\mathbb{Z}_p) \right| = o(|p^e|^{i-(h+n)}) \qquad \text{for } e \to \infty. \tag{5.5}$$

Since the distributions $\{\nu_i\}_{0\leq i\leq n}$ are h-admissible, condition (5.5) is fulfilled and, using the same arguments as in the proof of Lemma 5.3 in [Mah00], one shows that the values $\widetilde{\nu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p} x^i)$ for $n < i < h+n$ can be chosen such that $\widetilde{\nu}$ defines a K-linear functional on $\mathcal{C}_K^{h+n}$ satisfying condition (5.2) for $0 \leq i < h+n$. The choice of the values $\widetilde{\nu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p} x^i)$ for $n < i < h+n$ is not unique and thus an $(h+n)$-admissible measure $\widetilde{\nu}$ is not uniquely defined by condition (5.4) if h is greater than 1. □

Remark 5.4. *For future reference we state a slight variation of the statement in the Lemma. If $\{\nu_i\}_{0\leq i\leq n}$ is a collection of K-valued bounded distributions, then there exists one and only one K-valued $(n+1)$-admissible measure $\widetilde{\nu}$ satisfying condition (5.4) for $0\leq i\leq n$. In this situation $\widetilde{\nu}$ is already determined on the whole space $\mathcal{C}_K^{n+1}$ by condition (5.4) and one easily verifies the condition (5.2) for the $(n+1)$-admissibility by verifying the equivalent condition (5.5). Note that in this situation we do not have to make use of Lemma 5.3 in [Mah00] because the measure $\widetilde{\nu}$ is already fully determined by condition (5.4). Finally, we want to point out that the situation in this Remark is not covered by the statement in the Lemma: There does not exist an integer h, such that h-admissibility is equivalent to boundedness.*

Next, we want to apply our considerations to the collection $\{\mu^*_{\pi,l}\}_{1\leq l\leq l_0/2}$ of $(h^*+5l_0/2-2)$-admissible distributions. We find that there exists an (h^*+3l_0-3)-admissible measure $\widetilde{\mu}_\pi$ satisfying

$$\widetilde{\mu}_\pi(\mathbf{ch}_{a+p^e\mathbb{Z}_p}x^{l_0/2-l})=\mu^*_{\pi,l}(a+p^e\mathbb{Z}_p) \tag{5.6}$$

for $a\in\mathbb{Z}_p^*$ and for the integers l with $1\leq l\leq l_0/2$. Roughly speaking, the measure $\widetilde{\mu}_\pi$ is given on the monomial functions of degree zero by the distribution $\mu^*_{\pi,l_0/2}$ attached to the most left hand critical value $1-l_0/2$, in other words $\mathrm{res}(\widetilde{\mu}_\pi)=\mu^*_{\pi,l_0/2}$. On the monomial functions of degree $l_0/2-1$ the measure $\widetilde{\mu}_\pi$ is given by the distribution attached to the critical value 0.
We have seen that a measure $\widetilde{\mu}_\pi$ is not uniquely defined by condition (5.6).
For characters χ_p on $\mathbb{Z}_p^*$ of finite order equation (5.6) implies that

$$\int\limits_{\mathbb{Z}_p^*}x^{l_0/2-l}\chi_p\,d\widetilde{\mu}_\pi=\int\limits_{\mathbb{Z}_p^*}\chi_p\,d\mu^*_{\pi,l} \tag{5.7}$$

for l with $1\leq l\leq l_0/2$.

Remark 5.5. *It is clear from the definitions that if $\widetilde{\nu}$ is an h-admissible measure then the distribution $\mathrm{res}(\widetilde{\nu})$ is also h-admissible. Assuming that we are not able to approve our result on the h-admissibility of the distribution $\mu^*_{\pi,l_0/2}$ stated in Example 5.2, the best result we can expect for a measure $\widetilde{\mu}_\pi$ satisfying property (5.6) is, that it is $(h^*+5l_0/2-2)$-admissible.*
Again, the proof of Lemma 5.3 in [Mah00] shows that for proving the existence of an $(h^*+5l_0/2-2)$-admissible measure $\widetilde{\mu}_\pi$ satisfying property (5.6) one only has to verify condition (5.2) in the case $0\leq i\leq l_0/2-1$. We have seen, cf. equation (5.5), that for $0\leq i\leq l_0/2-1$ condition (5.2) is equivalent to the following statement

$$\sup_{a\in\mathbb{Z}_p^*}\left|\sum_{r=0}^{i}\binom{i}{r}(-a)^{i-r}\mu^*_{\pi,l_0/2-r}(a+p^e\mathbb{Z}_p)\right|=o(|p^e|^{i-(h^*+5l_0/2-2)})\qquad\text{for } e\to\infty. \tag{5.8}$$

Unfortunately, we have not managed to adapt our methods in order to prove equation (5.8) and thus have to be content with $\widetilde{\mu}_\pi$ being (h^*+3l_0-3)-admissible.
We want to emphasize that even an $(h^*+5l_0/2-2)$-admissible measure $\widetilde{\mu}_\pi$ would not be uniquely determined by property (5.6) since $h^*+5l_0/2-2$ is bigger than the number of critical integers, which is equal to $l_0/2$.

Eventually, we are ready to define the p-adic L-function, which will be a function on the set X_p of all continuous p-adic characters on $\mathbb{Z}_p^*$,

$$X_p:=\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*,\mathbb{C}_p^*).$$

Definition 5.6. *The p-adic L-function L_p attached to the cuspidal automorphic representation π and to the critical integers on the left hand side of the functional equation is defined by*

$$\begin{aligned} L_p : X_p &\longrightarrow \mathbb{C}_p, \\ \xi &\longmapsto \int\limits_{\mathbb{Z}_p^*} \xi x^{l_0/2-1}\, d\widetilde{\mu}_\pi, \end{aligned}$$

where the integral has to be interpreted in the sense of [Viš76], §1. We will also write $L_p(\xi;\pi,\eta_*)$ or $L_p(\xi;\pi)$ instead of $L_p(\xi)$ if we want to emphasize that L_p depends on π and on the pair of characters (η_o,η_e).

Next, we want to endow X_p with a p-adic analytic structure, which will allow us to speak about analyticity and logarithmic growth of functions on X_p. Recall that we have assumed p to be a prime greater than 3, cf. section 3.1. The basic observation is, that X_p may be identified with a finite disjoint union of open sets in $\mathbb{C}_p$. Let us make this precise. We decompose $\mathbb{Z}_p^* \cong \mu_{p-1} \times (1+p\mathbb{Z}_p)$, where $\mu_{p-1} \cong (\mathbb{Z}/p\mathbb{Z})^*$ is the finite group of $(p-1)$th roots of unity in $\mathbb{Z}_p^*$. Any continuous character $\chi : \mathbb{Z}_p^* \to \mathbb{C}_p^*$ can be decomposed in a unique way as $\chi = \omega\chi'$ where ω is a character of μ_{p-1} and χ' is a character of $1+p\mathbb{Z}_p$. The group $1+p\mathbb{Z}_p$ is topologically generated by $1+p$, so χ' is uniquely defined by the value $\chi'(1+p)$ and we write χ_s for the character on $1+p\mathbb{Z}_p$ sending $1+p$ to $s \in \mathbb{C}_p^*$. This induces an embedding

$$\begin{aligned} \sigma : X_p(1+p\mathbb{Z}_p) &\hookrightarrow \mathbb{C}_p^*, \\ \chi' &\longmapsto \chi'(1+p), \end{aligned}$$

where $X_p(1+p\mathbb{Z}_p)$ denotes the group of continuous p-adic characters on $1+p\mathbb{Z}_p$. Let us denote by $B_r(x)$ for $x \in \mathbb{C}_p$ and $r > 0$ the open ball $\{s \in \mathbb{C}_p : |x-s|_p < r\}$. The continuity of the characters implies that σ maps onto the open ball $B_1(1)$: Since $(1+p)^{p^n}$ tends to 1 for n to infinity we get for a continuous character χ' on $1+p\mathbb{Z}_p$ that $\lim_{n\to\infty} \chi'(1+p)^{p^n} = 1$. Writing $(1-\chi'(1+p))^{p^n} = \sum_k \binom{p^n}{k}(-\chi'(1+p))^k$ and using that the binomial coefficient $\binom{p^n}{k}$ is divisible by p^n for k greater than 0 and less than p^n we deduce that $(1-\chi'(1+p))^{p^n}$ tends to zero for n to infinity. This shows that the image of σ lies in $B_1(1)$. On the other hand, it is clear that any $s \in B_1(1)$ defines a continuous character χ_s. This proves that σ defines an isomorphism

$$X_p(1+p\mathbb{Z}_p) \cong B_1(1).$$

A function $f : X_p \to \mathbb{C}_p$ is called analytic, if it satisfies the following condition, cf. [Viš76], §2: For every character ω of μ_{p-1}, the function $s \mapsto f(\omega\chi_s)$ is given by a power series in s converging on the open ball $\{s \in \mathbb{C}_p^* : |1-s|_p < 1\}$.
If f and g are analytic functions on $B_1(1)$ with values in $\mathbb{C}_p$, then the notation $f = o(g)$ means that

$$\sup_{u\in B_r(1)} |f(u)| = o(\sup_{u\in B_r(1)} |g(u)|) \tag{5.9}$$

for r approaching 1. We are interested in the case, where the function g is the p-adic logarithm. The assertion that an analytic function $f : X_p \to \mathbb{C}_p$ is equal to $o(\log_p^h(\cdot))$ means that for every character ω of μ_{p-1}, the function $s \mapsto f(\omega\chi_s)$ is equal to $o(\log_p^h(\cdot))$. More vaguely, we also say that f is of logarithmic growth.
For a proof of the following fundamental Theorem in the theory of h-admissible measures we refer to [Viš76], Theorem 2.3.

Theorem 5.7. *Let $\widetilde{\mu}$ be an h-admissible measure on $\mathbb{Z}_p^*$. Then the p-adic Mellin transform, $\xi \mapsto \int\limits_{\mathbb{Z}_p^*} \xi \, d\widetilde{\mu}$, $\xi \in X_p$, is a p-adic analytic function on X_p and equal to $o(\log_p^h(\cdot))$.*

For later reference, we note that if f is an analytic function on X_p, then so is the function $f_0 : \xi \mapsto f(\xi\xi_0)$ for any character $\xi_0 \in X_p$. Indeed, the character ξ_0 can be written in the form $\xi_0 = \omega_0\chi_{s_0}$ with ω_0 a character on μ_{p-1} and $s_0 \in B_1(1)$. We have to show that for any character ω on μ_{p-1} the function $s \mapsto f_0(\omega\chi_s) = f(\omega\omega_0\chi_{s_0 s})$ on $B_1(1)$ is given by a converging power series in s. This is true since f is analytic and so $f(\omega\omega_0\chi_{s_0 s})$ is given by a converging power series in $s_0 s$ for $s_0 s \in B_1(1)$. Similarly, one can show that if f is equal to $o(\log_p^h(\cdot))$ then the same is true for the function $\xi \mapsto f(\xi\xi_0)$.
The notation of being of logarithmic growth is crucial if one wants to formulate uniqueness statements. In our situation we can not hope for a uniquely defined L-function because the measure $\widetilde{\mu}_\pi$ is already non-unique. Though, in the next section we will apply the following considerations to construct a unique p-adic epsilon factor. The basic result is Theorem 1.2 in [Kho80]. It states that for any p-adic analytic function $f : X_p \to \mathbb{C}_p$ which is equal to $o(\log_p^h(\cdot))$ resp. bounded there exists an h-admissible resp. bounded measure $\widetilde{\nu}$ such that f is the Mellin transform of $\widetilde{\nu}$. This implies that a p-adic analytic function $f : X_p \to \mathbb{C}_p$ equal to $o(\log_p^h(\cdot))$ is already uniquely determined by its values $f(x^i\chi_p)$ with χ_p a character of finite order on $\mathbb{Z}_p^*$ and $0 \leq i < h$. Indeed, assume that $g : X_p \to \mathbb{C}_p$ is another p-adic analytic function equal to $o(\log_p^h(\cdot))$ and with $g(x^i\chi_p) = f(x^i\chi_p)$ for all characters χ_p of finite order and for $0 \leq i < h$. By Theorem 1.2 in [Kho80] we know that there exist h-admissible measures $\widetilde{\nu}_f$ and $\widetilde{\nu}_g$ such that f resp. g is the Mellin transform of $\widetilde{\nu}_f$ resp. $\widetilde{\nu}_g$. The coincidence of the values of f and g on the functions $\chi_p x^i \in X_p$ implies that

$$\widetilde{\nu}_f(x^i\chi_p) = \widetilde{\nu}_g(x^i\chi_p)$$

for all characters χ_p of finite order and $0 \leq i < h$. Since the characteristic function $\mathbf{ch}_{a+p^e\mathbb{Z}_p}$ may be written as a linear combination of characters of finite order, cf. equation (3.8), we find that the measures $\widetilde{\nu}_f$ and $\widetilde{\nu}_g$ coincide on the functions $\mathbf{ch}_{a+p^e\mathbb{Z}_p}x^i$ for $a \in \mathbb{Z}_p^*$ and $0 \leq i < h$. These functions generate $\mathcal{C}^h$ over $\mathbb{C}_p$, hence the measures $\widetilde{\nu}_f$ and $\widetilde{\nu}_g$ are equal and thus we have $f = g$. Analogously one checks that a bounded p-adic analytic function $f : X_p \to \mathbb{C}_p$ is already uniquely determined by its values $f(\chi_p)$ with χ_p a character of finite order on $\mathbb{Z}_p^*$.

We summarize our results.

Corollary 5.8. *The p-adic L-function $L_p : X_p \to \mathbb{C}_p$ is analytic, equal to $o(\log_p^{h^*+3l_0-3}(\cdot))$ and satisfies the following interpolation property: For all characters $\chi : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of conductor p^e, $e \geq 2$, and with infinity component $\chi_\infty = \mathbb{1}$ and for all integers l with $1 \leq l \leq l_0/2$ we have*

$$L_p(x^{1-l}\chi_p; \pi, \eta_*) = p^{-2e}\hat{\zeta}^e\gamma^{-e}G(\chi_p\eta_{l,p}^2)\frac{L(\pi \otimes \chi\eta_l, 1-l)}{\Omega(\pi)},$$

where $\hat{\zeta}$ is a fixed root of unity. For all characters χ of conductor a p-power and with infinity component $\chi_\infty = \operatorname{sign}$ the left hand side, $L_p(x^{1-l}\chi_p; \pi, \eta_*)$, vanishes.
(Recall that the the pair of characters (η_o, η_e) was fixed at the beginning of this section and that η_l stands for the odd character η_o if l is odd and for the even character η_e if l is even.)

Proof. According to the definition of the p-adic L-function L_p, we have

$$\begin{aligned} L_p(x^{1-l}\chi_p) &= \int\limits_{\mathbb{Z}_p^*} x^{l_0/2-l}\chi_p \, d\widetilde{\mu}_\pi \\ &= \int\limits_{\mathbb{Z}_p^*} \chi_p \, d\mu^*_{\pi,l}, \end{aligned}$$

where the last equality has been verified in equation (5.7). It is now clear from equation (5.3) that L_p interpolates the values of the complex L-function as stated in the Corollary.
Noting that the p-adic L-function L_p is defined as the composition of the p-adic Mellin transform of the $(h^* + l_0 - 1)$-admissible measure $\widetilde{\mu}_\pi$ and the translation by the character $x^{l_0/2} \in X_p$, the statement on the anaylticity and the logarithmic growth of L_p follow from Theorem 5.7 and the discussion afterwards. □

As indicated above, a function L_p satisfying the properties described in the Corollary is not uniquely determined.

5.3. Functional Equation

With regard to the functional equation, it is convenient to change our point of view: In the following, the dual representation of the cuspidal representation investigated in the last chapters will play the principal role. This requires a change of notations.
As before, let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A})$ such that π_∞ is isomorphic to $\mathrm{Ind}(D_{l_0}, \mathrm{id})$ for some $l_0 \in 2\mathbb{N}$ and such that π_p and π_q are spherical; in particular we have $\pi_p \cong \mathrm{Ind}(\mu_1^{-1}, \mu_2^{-1}, \mu_3^{-1})$ for some unramified characters $\mu_i : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$. We also fix a pair of idèle class characters (η_o, η_e) as in section 5.2, i.e. both characters fulfill the same conditions as the character η in section 3.1 and η_o is an odd character whereas η_e is an even character. In this section, we want to construct a p-adic L-function, which interpolates the critical values for π on the right hand side of the functional equation. Furthermore, we aim to establish a functional equation relating this L-function to the p-adic L-function interpolating the critical values on the left hand side.
The starting point for the construction of the p-adic L-function interpolating the critical values on the right hand side of the functional equation is the complex functional equation. Let l be an integer with $1 \leq l \leq l_0/2$. The complex functional equation yields

$$L(\pi \otimes \chi\eta_l, l) = \varepsilon(\pi \otimes \chi\eta_l, l) L(\check{\pi} \otimes \chi^{-1}\eta_l^{-1}, 1-l), \tag{5.10}$$

where, as before, η_l stands for the odd character η_o if l is odd and for η_e if l is even. In the last sections we have constructed a p-adic function L_p interpolating the complex L-function at the critical integers at the left hand side of the functional equation, i.e. interpolating the values of the L-function occurring on the right hand side of equation (5.10). Thus, to construct a function interpolating the left hand side of equation (5.10), i.e. interpolating the complex L-function at the critical integers greater than $1/2$, it suffices to p-adically interpolate the epsilon factor occurring on the right hand side of equation (5.10). In fact, we are not able to interpolate the whole epsilon factor $\varepsilon(\pi \otimes \chi\eta_l, l)$ but instead we will interpolate the epsilon factor with the Euler factor at the place p omitted. This is achieved in the following Lemma.

Lemma 5.9. *There is a unique p-adic analytic function $\varepsilon_p = \varepsilon_p(\quad;\pi) : X_p \to \mathbb{C}_p$ which is equal to $o(\log_p^{l_0/2}(\cdot))$ and satisfies*

$$\varepsilon_p(\chi_p^{-1}x^{-l};\pi) = \varepsilon_{\{p\}}(\pi\otimes\chi\eta_l, l) \tag{5.11}$$

for any $l \in \mathbb{Z}$ with $1 \leq l \leq l_0/2$ and for all idèle class characters χ of conductor p^e and with infinity component $\chi_\infty = \mathbb{1}$. Here $\varepsilon_{\{p\}}(\pi, s)$ denotes the epsilon factor with the Euler factor at the place p omitted, i.e.

$$\varepsilon_{\{p\}}(\pi, s) = \prod_{\ell\neq p}\varepsilon(\pi_\ell, s).$$

Proof. We will show that there exists a collection of bounded distributions $\{\nu_l\}_{1\leq l\leq l_0/2}$ such that

$$\int\limits_{\mathbb{Z}_p^*}\chi_p^{-1}d\nu_l = \varepsilon_{\{p\}}(\pi\otimes\chi\eta_l, l) \tag{5.12}$$

for $1 \leq l \leq l_0/2$ and for all idèle class characters χ of conductor p^e and with infinity component $\chi_\infty = \mathbb{1}$. Then we are finished since we have seen in Remark 5.4 that there exists one (and only one) $(l_0/2)$-admissible measure $\widetilde{\nu}$ satisfying

$$\widetilde{\nu}(\mathbf{ch}_{a+p^e\mathbb{Z}_p}x^{l_0/2-l}) = \nu_l(a+p^e\mathbb{Z}_p)$$

and we can define ε_p as the Mellin transform of $\widetilde{\nu}$ composed with the translation by the character $x^{l_0/2}$, i.e. $\varepsilon_p(\xi) = \int_{\mathbb{Z}_p^*}\xi x^{l_0/2}\,d\widetilde{\nu}$ for $\xi \in X_p$. Indeed, the function ε_p satisfies equation (5.11) since we have

$$\begin{aligned}\varepsilon_p(\chi_p^{-1}x^{-l}) &= \int\limits_{\mathbb{Z}_p^*}\chi_p^{-1}x^{l_0/2-l}\,d\widetilde{\nu}\\ &= \int\limits_{\mathbb{Z}_p^*}\chi_p^{-1}d\nu_l\end{aligned}$$

similarly as in the proof of Corollary 5.8 and, by assumption (5.12) on the distributions ν_l, this last integral is equal to $\varepsilon_{\{p\}}(\pi\otimes\chi\eta_l, l)$. Moreover, ε_p is analytic and equal to $o(\log_p^{l_0/2}(\cdot))$ according to the considerations after Theorem 5.7.
To show the existence of the distributions $\{\nu_l\}_{1\leq l\leq l_0/2}$ we first construct bounded distributions δ_o and δ_e satisfying

$$\varepsilon_{\{p\}}(\pi\otimes\chi\eta_x, s) = \varepsilon_{\{p\}}(\pi\otimes\eta_x, s)\int\limits_{\mathbb{Z}_p^*}\chi_p^{-1}\,d\delta_x \tag{5.13}$$

for $x \in \{o, e\}$, $s \in \mathbb{C}$ and for all idèle class characters χ of conductor p^e and with infinity component $\chi_\infty = \mathbb{1}$.
Let ℓ be any prime number not equal to p and let $x \in \{o, e\}$. The character χ_ℓ is unramified and of finite order, thus $\chi_\ell = |\cdot|_\ell^{it_\ell}$ for some $t_\ell \in \mathbb{R}$. From the definition of the L-function and the epsilon factor via Rankin-Selberg integrals one deduces directly that $\varepsilon(\pi_\ell\otimes\chi_\ell\eta_{x,\ell}, s) = \varepsilon(\pi_\ell\otimes\eta_{x,\ell}, s+it_\ell)$ for $s \in \mathbb{C}$. Denote by $f_{x,\ell}$ the conductor of the representation $\pi_\ell\otimes\eta_{x,\ell}$. It is well known that $\varepsilon(\pi_\ell\otimes\eta_{x,\ell}, s) = c_{x,\ell}\ell^{-sf_{x,\ell}}$ for some constant $c_{x,\ell} \in \mathbb{C}$ and that $\varepsilon(\pi_\ell\otimes\eta_{x,\ell}, s) \equiv 1$ whenever

$\pi_\ell \otimes \eta_{x,\ell}$ is unramified, cf. page 10. Combining these facts and noting that $\chi_\ell(\ell) = \chi_p^{-1}(\ell)$ we find that

$$\varepsilon_{\{p\}}(\pi \otimes \chi\eta_x, s) = \varepsilon_{\{p\}}(\pi \otimes \eta_x, s) \prod_{\ell \in S_x} \chi_p^{-f_{x,\ell}}(\ell),$$

where S_x denotes the finite set of all primes $\ell \neq p$ such that $\pi_\ell \otimes \eta_{x,\ell}$ is ramified. Thus, we may choose for δ_x the Dirac distribution at $f_{x,\pi} := \prod_{S_x} \ell^{f_{x,\ell}} \in \mathbb{Z}_p^*$. If k is an integer, δ_k shall stand for δ_o if k is odd and for δ_e if k is even. In particular, equation (5.13) holds with "x" replaced by "l", where l is any integer with $1 \leq l \leq l_0/2$.
Finally, we define the bounded distributions ν_l by

$$\nu_l(a + p^e\mathbb{Z}_p) = \varepsilon_{\{p\}}(\pi \otimes \eta_l, l)\, \delta_l(a + p^e\mathbb{Z}_p)$$

for $1 \leq l \leq l_0/2$ and $a \in \mathbb{Z}_p^*$. Equation (5.13) implies that the distributions ν_l satisfy condition (5.12) and we have explained at the beginning of the proof that the collection of distributions $\{\nu_l\}_{1 \leq l \leq l_0/2}$ gives rise to the p-adic analytic function ε_p promised in the claim of the Lemma.
We have seen on page 69 that a p-adic analytic function f which is equal to $o(\log_p^h(\cdot))$ is uniquely determined by its values $f(\chi_p x^i)$ for $0 \leq i < h$ and for the characters χ_p of finite order on $\mathbb{Z}_p^*$. Consequently, a p-adic analytic function ε_p equal to $o(\log_p^{l_0/2}(\cdot))$ is uniquely determined by condition (5.11). □

Since the discrete series representation D_{l_0} is selfdual, we can apply the theory developed in the preceding sections to $\check{\pi}$ and obtain a p-adic L-function $L_p(\quad ; \check{\pi}, \eta_*^{-1})$ attached to $\check{\pi}$, to the pair of characters $(\eta_o^{-1}, \eta_e^{-1})$ and to the critical integers on the left hand side of the functional equation. The following Theorem states the existence of an L-function $\boldsymbol{L}_p$ attached to the critical integers greater than $1/2$ and establishes the functional equation promised at the beginning of the section.

Theorem 5.10. *There exists a unique p-adic analytic function $\boldsymbol{L}_p = \boldsymbol{L}_p(\quad ; \pi, \eta_*) : X_p \to \mathbb{C}_p$ such that the following properties hold:*

- *We have*

$$\boldsymbol{L}_p(x^l\chi_p; \pi, \eta_*) = \frac{p^{(3l-2)e}\hat{\zeta}^e \mu_1\mu_2\mu_3(p^e) G(\chi_p^{-1}\eta_{l,p}^{-2})}{\gamma^e \eta_{l,p}(p^{3e}) G(\chi_p^{-1}\eta_{l,p}^{-1})^3 \Omega(\check{\pi})} L(\pi \otimes \chi\eta_l, l) \tag{5.14}$$

  for the integers l with $1 \leq l \leq l_0/2$ and all characters $\chi : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of conductor a p-power p^e, $e \geq 2$, and with infinity component $\chi_\infty = \mathbb{1}$. For all characters χ with conductor a p-power and infinity component $\chi_\infty = \mathrm{sign}$ the left hand side of equation (5.14) vanishes.

- *For $\xi \in X_p$ the following functional equation holds*

$$\boldsymbol{L}_p(\xi; \pi, \eta_*) = \varepsilon_p(\xi^{-1}) L_p(\xi^{-1}x; \check{\pi}, \eta_*^{-1}). \tag{5.15}$$

- *The function $\boldsymbol{L}_p$ is equal to $o(\log_p^{h^*+7l_0/2-3}(\cdot))$.*

Proof. Let $\boldsymbol{L}_p$ be defined by equation (5.15). The analyticity of ε_p and $L_p(\quad ; \check{\pi}, \eta_*^{-1})$ implies the analyticity of $\boldsymbol{L}_p(\quad ; \pi, \eta_*)$. The vanishing of the values $\boldsymbol{L}_p(x^l\chi_p; \pi, \eta_*)$ for the characters χ with infinity component $\chi_\infty = \mathrm{sgn}$ immediately follows from the analogous result for L_p, cf. Corollary 5.8.
As explained at the beginning of the section we will deduce the interpolation statement in (5.14)

from the interpolation result for the p-adic L-function attached to the critical integers less than $1/2$ stated in Corollary 5.8. Plugging this into equation (5.15), we find

$$\boldsymbol{L}_p(x^l\chi_p;\pi,\eta_*) = \varepsilon_p(x^{-l}\chi_p^{-1})p^{-2e}\hat{\zeta}^e\gamma^{-e}G(\chi_p^{-1}\eta_{l,p}^{-2})\frac{L(\check{\pi}\otimes\chi^{-1}\eta_l^{-1},1-l)}{\Omega(\check{\pi})}$$

for $1 \leq l \leq l_0/2$ and for all characters $\chi : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of conductor a p-power p^e and with infinity component $\chi_\infty = \mathbb{1}$. According to Lemma 5.9 we have

$$\varepsilon_p(\chi_p^{-1}x^{-l}) = \varepsilon_{\{p\}}(\pi\otimes\chi\eta_l,l)$$

and in combination with the complex functional equation this yields

$$\boldsymbol{L}_p(x^l\chi_p;\pi,\eta_*) = \frac{p^{-2e}\hat{\zeta}^e\gamma^{-e}G(\chi_p^{-1}\eta_{l,p}^{-2})}{\varepsilon(\pi_p\otimes\chi_p\eta_{l,p},l)\Omega(\check{\pi})}L(\pi\otimes\chi\eta_l,l).$$

To finally obtain equation (5.14) we only have to compute the local epsilon factor $\varepsilon(\pi_p\otimes\chi_p\eta_{l,p},l)$. We know that the p-component of π is given by $\pi_p = \mathrm{Ind}(\mu_1^{-1},\mu_2^{-1},\mu_3^{-1})$, where μ_1,μ_2 and μ_3 are unramified characters of $\mathbb{Q}_p^*$. Using formula (3.1.4) in [Kud94] we find

$$\varepsilon(\pi_p\otimes\chi_p\eta_{l,p},l) = \prod_{i=1,2,3}\varepsilon(\mu_i^{-1}\chi_p\eta_{l,p},l).$$

The local ε-factors occurring on the right hand side are computed in Tate's thesis, see for instance [Kud03b], chapter 6, Prop. 3.8, we have

$$\varepsilon(\mu_i^{-1}\chi_p\eta_{l,p},l) = \mu_i^{-1}\eta_{l,p}(p^e)p^{-le}G(\chi_p^{-1}\eta_{l,p}^{-1}).$$

Note that $\chi_p(p) = 1$. This proves that $\boldsymbol{L}_p$ interpolates the complex L-function at critical integers greater than $1/2$ as stated in equation (5.14).
Finally, we prove that $\boldsymbol{L}_p$ is equal to $o(\log_p^{h^*+7l_0/2-3}(\cdot))$. Recalling the definition of logarithmic growth, cf. equation (5.9), we have to show that

$$\sup_{s\in B_r(1)}|\boldsymbol{L}_p(\chi_s\omega)|_p = o(\sup_{s\in B_r(1)}|\log_p^{h^*+7l_0/2-3}(s)|_p) \qquad \text{for } r \text{ approaching } 1,$$

where, as before, ω is a character of μ_{p-1}, $B_r(1)$ is the open ball with center 1 and radius r and χ_s denotes the unique element in $X_p(1+p\mathbb{Z}_p)$ with $\chi_s(1+p) = s$. From property (5.15) we find for $0 < r < 1$ that

$$\sup_{s\in B_r(1)}|\boldsymbol{L}_p(\chi_s\omega)|_p \leq \sup_{s\in B_r(1)}|\varepsilon_p(\chi_s\omega^{-1})|_p \sup_{s\in B_r(1)}|L_p(\chi_s\omega^{-1})|_p,$$

where we have made use of the fact that the set of characters χ_s with $s \in B_r(1)$ is equal to the set of characters χ_s^{-1} with $s \in B_r(1)$. Using that ε_p is equal to $o(\log_p^{l_0/2}(\cdot))$ and L_p is equal to $o(\log_p^{h^*+3l_0-31}(\cdot))$ and noting that

$$\sup_{s\in B_r(1)}|\log_p^{l_0/2}(s)|_p \sup_{s\in B_r(1)}|\log_p^{h^*+3l_0-3}(s)|_p = \sup_{s\in B_r(1)}|\log_p^{h^*+7l_0/2-3}(s)|_p$$

we conclude that $\boldsymbol{L}_p$ is equal to $o(\log_p^{h^*+7l_0/2-3}(\cdot))$. □

In the following, we will write "the p-adic L-function on the left hand side" instead of "the p-adic L-function attached to the critical integers on the left hand side" and analogously for the right hand side.
One could be surprised that we have obtained a uniqueness statement for the p-adic L-function on the right hand side whereas the p-adic L-function on the left hand side constructed in Corollary 5.8 was not uniquely determined. Though, a closer look at the statement of Theorem 5.10 reveals that the p-adic L-function $\boldsymbol{L}_p$ on the right hand side is only uniquely determined with respect to a choice of a p-adic L-function L_p on the left hand side. Actually, with regard to the interpolation property (5.14), the p-adic L-function $\boldsymbol{L}_p$ on the right hand side is even farer from being uniquely determined than the p-adic L-function on the left hand side. More precisely, $\boldsymbol{L}_p$ interpolates the same amount of values of its complex counterpart as L_p but we could only prove that $\boldsymbol{L}_p$ is equal to $(\log_p^{h^*+7l_0/2-3}(\cdot))$ whereas L_p is equal to $(\log_p^{h^*+3l_0-3}(\cdot))$. (Recall that a p-adic analytic function $f : X_p \to \mathbb{C}_p$ equal to $o(\log_p^h(\cdot))$ is uniquely determined by its values $f(x^i\chi_p)$ with $0 \leq i < h$ and χ_p running over the characters on $\mathbb{Z}_p^*$ of finite order.)
The next results will suggest another choice for a p-adic L-function $\breve{\boldsymbol{L}}_p$ on the right hand which has the same logarithmic growth as L_p, i.e. if L_p is equal to $o(\log_p^h(\cdot))$ then the same is true for $\breve{\boldsymbol{L}}_p$. The only disadvantage of $\breve{\boldsymbol{L}}_p$ is that its interpolation property (an analogous statement to condition (5.14)) is more complicated.

Lemma 5.11. *There exists a unique bounded p-adic analytic function $\varepsilon'_p : X_p \to \mathbb{C}_p$ such that*

$$\varepsilon_{\{p\}}(\pi \otimes \chi\eta_x, s) = \varepsilon'_p(\chi_p^{-1})\varepsilon_{\{p\}}(\pi \otimes \eta_x, s) \tag{5.16}$$

for $x \in \{o, e\}$, for $s \in \mathbb{C}$ and for all idèle class characters χ of conductor p^e and with infinity component $\chi_\infty = \mathbb{1}$.

Proof. As introduced in the proof of Corollary 5.9 we denote by $f_{x,\pi}$ the product $\prod_{S_x} \ell^{f_{x,\ell}} \in \mathbb{Z}_p^*$, where $f_{x,\ell}$ is the conductor of the representation $\pi_\ell \otimes \eta_{x,\ell}$. There we have seen that the Dirac distribution δ_x at $f_{x,\pi}$ satisfies

$$\varepsilon_{\{p\}}(\pi \otimes \chi\eta_x, s) = \varepsilon_{\{p\}}(\pi \otimes \eta_x, s) \int\limits_{\mathbb{Z}_p^*} \chi_p^{-1}\, d\delta_x$$

for $x \in \{o, e\}$, cf. equation (5.13). Recalling that the conductors of the characters η_o and η_e are given by pq and that π_ℓ is spherical for $\ell = p$ and $\ell = q$ we find that $f_{o,\pi} = f_{e,\pi}$ and consequently the distributions δ_o and δ_e coincide. In particular, the Mellin transform of $\delta := \delta_o$ is a bounded p-adic analytic function satisfying equation (5.16).
We have seen on page 69 that a bounded function $f : X_p \to \mathbb{C}_p$ is already uniquely determined by the values $f(\chi_p)$ where χ_p are the characters of finite order on $\mathbb{Z}_p^*$. Since the complex epsilon factor $\varepsilon_{\{p\}}(\pi \otimes \eta_x, s)$ is nonzero this gives the uniqueness statement. □

We fix the notation $f_\pi := f_{x,\pi} = \prod_{S_x} \ell^{f_{x,\ell}} \in \mathbb{Z}_p^*$.

Theorem 5.12. *There exists a unique p-adic analytic function $\breve{\boldsymbol{L}}_p(\quad ; \pi, \eta_*) : X_p \to \mathbb{C}_p$ such that the following properties hold:*

- *We have*

$$\breve{\boldsymbol{L}}_p(x^l\chi_p; \pi, \eta_*) = \frac{p^{(3l-2)e}\hat{\zeta}^e f_\pi^{-l}\mu_1\mu_2\mu_3(p^e)G(\chi_p^{-1}\eta_{l,p}^{-2})}{\gamma^e\eta_{l,p}(p^{3e})\Omega(\breve{\pi})\varepsilon_{\{p\}}(\pi \otimes \eta_l, l)G(\chi_p^{-1}\eta_{l,p}^{-1})^3} L(\pi \otimes \chi\eta_l, l) \tag{5.17}$$

for the integers l with $1 \leq l \leq l_0/2$ and all characters $\chi : \mathbb{Q}^* \backslash \mathbb{A}^* \to \mathbb{C}^*$ of conductor a p-power $p^e,\ e \geq 2$, and with infinity component $\chi_\infty = \mathbb{1}$. For all characters χ with conductor a p-power and infinity component $\chi_\infty = \operatorname{sign}$ the left hand side of equation (5.14) vanishes.

- *For $\xi \in X_p$ the following functional equation holds*

$$\check{\boldsymbol{L}}_p(\xi; \pi, \eta_*) = \varepsilon_p'(\xi^{-1}) L_p(\xi^{-1} x; \check{\pi}, \eta_*^{-1}). \tag{5.18}$$

- *The function $\check{\boldsymbol{L}}_p$ is equal to $o(\log_p^{h^*+3l_0-3}(\cdot))$.*

Proof. The proof works entirely analogously to the proof of Theorem 5.10. One only has to use Lemma 5.11 instead of Lemma 5.9. To deduce interpolation property (5.17) one has to note that

$$\varepsilon_p'(x^i \chi_p^{-1}) = f_\pi^i \, \varepsilon_p'(\chi_p^{-1})$$

for $i \in \mathbb{Z}$ and for a character χ_p of finite order on $\mathbb{Z}_p^*$. □

Appendix

A. Proper Map

In section 3.3 we want to transport cohomology classes with compact support via the following map

$$\begin{aligned} i(K^f): \quad & F_2(K_2^f) \longrightarrow S_3(K^f), \\ & \mathrm{GL}_2(\mathbb{Q}) g K_2^f K_{2,\infty} \longmapsto \mathrm{GL}_3(\mathbb{Q}) \begin{pmatrix} g & \\ & 1 \end{pmatrix} K^f K_{3,\infty} Z_3^0(\mathbb{R}), \end{aligned}$$

where K^f is a compact open subgroup in $\mathrm{GL}_3(\mathbb{A}_f)$ and K_2^f denotes the compact open subgroup $K^f \cap \mathrm{GL}_2(\mathbb{A}_f)$ with GL_2 embedded into GL_3 via $g \mapsto \left(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\right)$. This is legitimated by the following Lemma.

Lemma. *The map $i(K^f)$ is proper.*

Proof. We first consider the case that K^f is a principal congruence subgroup $K(3,N) \leq \mathrm{GL}_3(\mathbb{A}_f)$ of level $N > 1$. Then, the compact open subgroup K_2^f is equal to the principal congruence subgroup $K(2,N)$ and we abbreviate $i(K(3,N))$ to $i(N)$. We change over to the classical setting: Following the considerations on page 106 in [Har82] and using that $\det(K(2,N)) = \det(K(3,N))$ we find that there exists a set of representatives $\{g_i\}$ in $\mathrm{GL}_2(\mathbb{A}_f)$ for the coset space $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_f)/K(2,N)$ such that

$$F_2(K(2,N)) = \bigsqcup_{i=1}^{m} \Gamma_2^{g_i}\backslash\mathrm{GL}_2(\mathbb{R})/K_{2,\infty} \quad \text{and} \quad S_3(K(3,N)) = \bigsqcup_{i=1}^{m} \Gamma_3^{g_i}\backslash\mathrm{GL}_3(\mathbb{R})/K_{3,\infty}Z_3^0(\mathbb{R}),$$

where on the right hand side $g_i \in \mathrm{GL}_2(\mathbb{A}_f)$ is embedded into $\mathrm{GL}_3(\mathbb{A}_f)$ and, similarly as before, $\Gamma_j^{g_i}$ for $j = 2$ or $j = 3$ denotes the congruence subgroup of $\mathrm{GL}_j(\mathbb{Q})$ defined by $\Gamma_j^{g_i} := g_i K(j,N) g_i^{-1} \cap \mathrm{GL}_j(\mathbb{Q})$. In particular, we have $\Gamma_2^{g_i} = \Gamma_3^{g_i} \cap \mathrm{GL}_2(\mathbb{Q})$. Thus, to prove the Lemma it is sufficient to show that the canonical map

$$j : (\Gamma \cap \mathrm{GL}_2(\mathbb{Q}))\backslash\mathrm{GL}_2(\mathbb{R})/K_{2,\infty} \longrightarrow \Gamma\backslash\mathrm{GL}_3(\mathbb{R})/K_{3,\infty}Z_3^0(\mathbb{R})$$

is proper for any congruence subgroup Γ of the form $\Gamma = gK(3,N)g^{-1} \cap \mathrm{GL}_3(\mathbb{Q})$ with $g \in \mathrm{GL}_2(\mathbb{A}_f)$. In the following, we abbreviate $\Gamma \cap \mathrm{GL}_2(\mathbb{Q})$ to Γ_2. Let us first explain, that it suffices to show that the following map

$$i : \Gamma_2\backslash\mathrm{GL}_2(\mathbb{R}) \longrightarrow \Gamma Z_3(\mathbb{R})\backslash\mathrm{GL}_3(\mathbb{R}).$$

is proper, i.e. we assume for the moment that we have already shown that i is proper and explain why this implies that j is proper: The map i can be decomposed into the map $i_0 : \Gamma_2\backslash\mathrm{GL}_2(\mathbb{R}) \to \Gamma Z_3^0(\mathbb{R})\backslash\mathrm{GL}_3(\mathbb{R})$ and the projection map $\Gamma Z_3^0(\mathbb{R})\backslash\mathrm{GL}_3(\mathbb{R}) \to \Gamma Z_3(\mathbb{R})\backslash\mathrm{GL}_3(\mathbb{R})$ and, using Proposition 5 in section I.10.1 in [Bou66], we find that the map i_0 is proper. Furthermore, the projection map $\pi : \Gamma Z_3^0(\mathbb{R})\backslash\mathrm{GL}_3(\mathbb{R}) \to \Gamma Z_3^0(\mathbb{R})\backslash\mathrm{GL}_3(\mathbb{R})/K_{3,\infty}$ is proper. (Note that it has compact fibers and is closed since the product of a closed set and a compact group in $\mathrm{GL}_3(\mathbb{R})$ is closed, see for

instance Corollary 1 in section III.4.1 in [Bou66]. Hence π is proper according to Theorem 1 in section I.10.2 in [Bou66].) Since the composition of proper maps is again proper, we conclude that $\pi \circ i_0$ is proper. Moreover, the map $\pi \circ i_0$ can be written as the composition of the projection map $\Gamma_2 \backslash \mathrm{GL}_2(\mathbb{R}) \to \Gamma_2 \backslash \mathrm{GL}_2(\mathbb{R})/K_{2,\infty}$ and the map j which we are actually interested in. Again consulting Proposition 5 in section I.10.1 in [Bou66], we deduce that j indeed is a proper map.
In order to show that the map i is proper we first check that it is injective. Let h and k be elements in $\mathrm{GL}_2(\mathbb{R})$ with $i(h) = i(k)$, i.e. there exist $\gamma \in \Gamma$ and $z \in Z_3(\mathbb{R})$ with

$$\gamma z \begin{pmatrix} h & \\ & 1 \end{pmatrix} = \begin{pmatrix} k & \\ & 1 \end{pmatrix}. \tag{A.1}$$

We will show that equation (A.1) implies that $\gamma \in \mathrm{GL}_2(\mathbb{Q}) \subset \mathrm{GL}_3(\mathbb{Q})$ and $z = 1$, i.e. $h = k$ in $\Gamma_2 \backslash \mathrm{GL}_2(\mathbb{R})$. Of course, an element $\gamma \in \Gamma$ satisfying equation (A.1) lies in the subgroup

$$L := \begin{pmatrix} \mathrm{GL}_2(\mathbb{Q}) & \\ & \mathbb{Q}^* \end{pmatrix}$$

of $\mathrm{GL}_3(\mathbb{Q})$ and we are done if we can show that the intersection of the groups Γ and L already lies in $\mathrm{GL}_2(\mathbb{Q})$. Recall that there exists $g \in \mathrm{GL}_2(\mathbb{A}_f) \subset \mathrm{GL}_3(\mathbb{A}_f)$ with $\Gamma = gK(3,N)g^{-1} \cap \mathrm{GL}_3(\mathbb{Q})$. The intersection of Γ and L lies in $\mathrm{GL}_2(\mathbb{Q})$ if and only if the intersection of $K(3,N)$ and the conjugated group $g^{-1}Lg$ lies in $\mathrm{GL}_2(\mathbb{A}_f)$. But this is easily verified by taking into account the special structure of the compact open subgroup $K(3,N)$, (recall that we have assumed $N > 1$), and noting that $g^{-1}Lg$ is a subgroup of

$$\begin{pmatrix} \mathrm{GL}_2(\mathbb{A}_f) & \\ & \mathbb{Q}^* \end{pmatrix} \subset \mathrm{GL}_3(\mathbb{A}_f),$$

where $\mathbb{Q}^*$ is embedded diagonally into $\mathbb{A}_f^*$. Thus, we have seen that i is an injective map. Next, we show that i has closed image, hence is proper. This amounts to showing that $\Gamma Z_3(\mathbb{R})\mathrm{GL}_2(\mathbb{R})$ is closed in $\mathrm{GL}_3(\mathbb{R})$. In the proof of Lemma 2.7 in [Ash80] it is shown for a semi-simple algebraic group G over $\mathbb{Q}$, a parabolic $\mathbb{Q}$-subgroup P, a Levi $\mathbb{Q}$-subgroup M of P and an arithmetic subgroup Γ of $G(\mathbb{Q})$, that the set $\Gamma M(\mathbb{R})$ is closed in $G(\mathbb{R})$. Noting that the considerations in [Ash80] also hold for GL_3 instead of a semi-simple algebraic group G and taking the standard parabolic subgroup P of GL_3 of type $(2,1)$ with Levi subgroup $\mathrm{GL}_2 \times \mathrm{GL}_1$, we find that $\Gamma(\mathrm{GL}_2(\mathbb{R}) \times \mathbb{R}^*) = \Gamma Z_3(\mathbb{R})\mathrm{GL}_2(\mathbb{R})$ is closed in $\mathrm{GL}_3(\mathbb{R})$, which proves that i, and thus j and $i(N)$, are proper.
Let now K^f be an arbitrary compact open subgroup of $\mathrm{GL}_3(\mathbb{A}_f)$. The group K^f contains a principal congruence subgroup $K(3,N)$ with $N > 1$ and we obtain the following diagram

$$\begin{array}{ccc} F_2(K(2,N)) & \xrightarrow{i(N)} & S_3(K(3,N)) \\ \downarrow{\scriptstyle \pi_F} & & \downarrow{\scriptstyle \pi_S} \\ F_2(K_2^f) & \xrightarrow{i(K^f)} & S_3(K^f). \end{array} \tag{A.2}$$

We have already seen that the map $i(N)$ is proper. The projection π_S is closed and has compact fibers, hence it is also proper, (cf. above, where we explain that the projection $\pi : \Gamma Z_3^0(\mathbb{R}) \backslash \mathrm{GL}_3(\mathbb{R}) \to \Gamma Z_3^0(\mathbb{R}) \backslash \mathrm{GL}_3(\mathbb{R})/K_{3,\infty}$ is proper). Since the composition of proper maps is proper and diagram (A.2) is commuting, we deduce that the map $i(K^f) \circ \pi_F$ is proper. Using Proposition 5 b) in section I.10.1 in [Bou66] we conclude that $i(K^f)$ is also proper. □

Lemma. *Let K' be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$. Then, there exists a compact open subgroup $K \subset \mathrm{GL}_3(\mathbb{A}_f)$ with*

$$K' = K \cap \mathrm{GL}_2(\mathbb{A}_f),$$

where, as before, GL_2 is identified with a subgroup of GL_3 via $g \mapsto \left(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\right)$.

Proof. We first assume that K' is a subgroup of $K_{\max,2}$, where $K_{\max,n} := \prod_\ell \mathrm{GL}_n(\mathbb{Z}_\ell)$ denotes the standard maximal compact subgroup in $\mathrm{GL}_n(\mathbb{A}_f)$. Let $N \in \mathbb{N}$ be large enough such that the principal congruence subgroup $K(2,N)$ of level N lies in K'. We want to show that the subgroup $K \subset \mathrm{GL}_3(\mathbb{A}_f)$ generated by the elements $\left(\begin{smallmatrix} k & \\ & 1 \end{smallmatrix}\right)$ with $k \in K'$ and the elements $g \in K(3,N)$ satisfies $K \cap \mathrm{GL}_2(\mathbb{A}_f) = K'$. An element $h \in K$ is of the form $h = \left(\begin{smallmatrix} k_1 & \\ & 1 \end{smallmatrix}\right) g_1 \dots \left(\begin{smallmatrix} k_n & \\ & 1 \end{smallmatrix}\right) g_n$ with $k_i \in K'$ and $g_i \in K(3,N)$. Since $K_{\max,3}$ and in particular K' normalize the subgroup $K(3,N)$ we find that h can be written as $h = \left(\begin{smallmatrix} k & \\ & 1 \end{smallmatrix}\right) g$ with $k \in K'$ and $g \in K(3,N)$. If h lies in $\mathrm{GL}_2(\mathbb{A}_f)$, then g lies in $K(3,N) \cap \mathrm{GL}_2(\mathbb{A}_f) = K(2,N) \subset K'$, thus h lies in K'. This proves the claim of the Lemma for compact open subgroups K' lying in the standard maximal compact subgroup $K_{\max,2}$.
Let now K' be an arbitrary compact open subgroup. Since every maximal compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ is conjugated to the standard maximal compact subgroup $K_{\max,2}$ we find that there exists $u \in \mathrm{GL}_2(\mathbb{A}_f)$ with $uK'u^{-1} \subset K_{\max,2}$. According to our considerations, there exists a compact open subgroup $L \subset \mathrm{GL}_3(\mathbb{A}_f)$ with $L \cap \mathrm{GL}_2(\mathbb{A}_f) = uK'u^{-1}$. The subgroup $K := \left(\begin{smallmatrix} u^{-1} & \\ & 1 \end{smallmatrix}\right) L \left(\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right)$ then satisfies $K \cap \mathrm{GL}_2(\mathbb{A}_f) = K'$ as claimed in the Lemma. □

B. Integral Cohomology

Let X be a topological space and let R be a ring. A sheaf $\mathcal{M}$ of R-modules is said to be flat resp. torsion free if it is flat resp. torsion free in every stalk.

Lemma. *Let X be a paracompact manifold, R a principal ideal domain and Q the quotient field of R. For a torsion free sheaf $\mathcal{M}$ of $\underline{R}$-modules on X we have*

$$H_c^q(X, \mathcal{M} \otimes_{\underline{R}} \underline{Q}) \cong H_c^q(X, \mathcal{M}) \otimes_R Q, \qquad q \geq 0.$$

The same identity holds for fields Q and R with $Q \supset R \supset \mathbb{Q}$.

Proof. Let C^* be a fine torsion free resolution of the sheaf $\underline{R}$ on X, see [War83], page 178 for the existence of resolutions of this kind. Since over principal ideal rings the torsion free modules are exactly the flat modules, cf. [Lam99], page 128, we conclude that $C^* \otimes \mathcal{M}$ defines a resolution of $\mathcal{M} \otimes \underline{R} = \mathcal{M}$. Consulting chapter II in [Bre67] we deduce that this resolution is again fine and thus Γ_c-acyclic. Consequently, we get

$$H_c^q(X, \mathcal{M}) = H^q(\Gamma_c(C^* \otimes \mathcal{M})).$$

In the same manner we obtain that $C^* \otimes \mathcal{M} \otimes \underline{Q}$ is a Γ_c-acyclic resolution of $\mathcal{M} \otimes \underline{Q}$. Consequently, we have

$$H_c^q(X, \mathcal{M} \otimes \underline{Q}) = H^q(\Gamma_c(C^* \otimes \mathcal{M} \otimes \underline{Q})).$$

Most of the remaining proof is dedicated to the proof of the following
Claim. *For any torsion free sheaf $\mathcal{A}$ of $\underline{R}$-modules on X we have*

$$\Gamma_c(\mathcal{A} \otimes \underline{Q}) = \Gamma_c(\mathcal{A}) \otimes Q,$$

where as usual Γ_c *denotes the functor of global sections with compact support.*
(Note that this is the point where one gets into trouble trying to carry the proof of the Lemma over to cohomology with compact support instead of ordinary sheaf cohomology: In general, it is not true that $\Gamma(\mathcal{A} \otimes \underline{Q}) = \Gamma(\mathcal{A}) \otimes Q$, where Γ denotes the functor of global sections.)

Proof of the Claim. It is clear from the definitions that $\Gamma_c(\mathcal{A} \otimes \underline{Q}) \supset \Gamma_c(\mathcal{A}) \otimes Q$, so we only have to show the other direction. Using the universal mapping properties of the sheafification and the tensor product one can show that for any presheaf of rings $\mathcal{O}$ and two presheaves $\mathcal{B}, \mathcal{C}$ of $\mathcal{O}$-modules, we have

$$(\mathcal{B} \otimes_{\mathcal{O}} \mathcal{C})^{\#} = \mathcal{B}^{\#} \otimes_{\mathcal{O}^{\#}} \mathcal{C}^{\#},$$

where $^{\#}$ denotes sheafification. This shows that $\mathcal{A} \otimes_{\underline{R}} \underline{Q}$ equals the sheafification of $\mathcal{A} \otimes_R Q$, where Q is the presheaf of Q-valued constant functions on X. For an open set U in X we can write

$$(\mathcal{A} \otimes Q)^{\#}(U) = \left\{ \begin{array}{l} (s_u) \in \prod_{u \in U} (\mathcal{A} \otimes Q)_u : \text{ for every } u \in U, \text{ there exists} \\ \text{an open neighborhood } V \subset U \text{ and } \sigma \in (\mathcal{A} \otimes Q)(V) \\ \text{with } s_v = \sigma_v \text{ for all } v \in V \end{array} \right\}.$$

Let now (s_u) be some element in $\Gamma_c((\mathcal{A} \otimes Q)^{\#})$, i.e. (s_u) lies in $(\mathcal{A} \otimes Q)^{\#}(X)$ and its support $\mathrm{supp}((s_u)) := \{u : s_u \neq 0\}$ is compact. We can find finitely many open sets $V_1, \dots V_n$ such that $\mathrm{supp}(s_u)$ is contained in their union $\cup_i V_i$ and such that for all i there exists $\sigma_i \in (\mathcal{A} \otimes Q)(V_i)$ with $s_v = \sigma_{iv}$ for all $v \in V_i$. Let us denote by V_{n+1} the open set $X - \mathrm{supp}((s_u))$ and we set $\sigma_{n+1} = 0 \in (\mathcal{A} \otimes Q)(V_{n+1})$. Of course, $(V_i)_i$ defines an open cover of X and the family $(\sigma_i)_i$ of sections satisfies

$$\rho^{V_i}_{V_i \cap V_j}(\sigma_i) = \rho^{V_j}_{V_i \cap V_j}(\sigma_j) \tag{B.1}$$

for all $1 \leq i, j \leq n+1$, where ρ^*_* denotes the restriction maps on the presheaf $\mathcal{A} \otimes Q$.
Let first R be a principal ideal domain and let Q be its quotient field. Each of the sections σ_i is of the form $\sum_k m_{k,i} \otimes q_{k,i}$ with $m_{k,i} \in \mathcal{A}(V_i)$ and $q_{k,i} \in Q$. The elements $q_{k,i}$ can be written as $q_{k,i} = \frac{n_{k,i}}{d_{k,i}}$, where $n_{k,i}$ and $d_{k,i}$ are elements in R. Let d be the product of all denominators, i.e. $d := \prod_{i,k} d_{k,i}$. Then, for any element $q_{k,i}$ there exists an element $r_{k,i} \in R$ with $q_{k,i} = r_{k,i} \frac{1}{d}$ and thus, each of the sections σ_i is of the form $m_i \otimes q$ with $m_i \in \mathcal{A}(V_i)$ and $q = \frac{1}{d} \in Q$. Equation (B.1) can then be written as follows

$$\rho^{V_i}_{V_i \cap V_j}(m_i) \otimes q = \rho^{V_j}_{V_i \cap V_j}(m_j) \otimes q, \tag{B.2}$$

where now ρ^*_* denotes the restriction maps on the sheaf $\mathcal{A}$.
By assumption the sheaf $\mathcal{A}$ is torsion free, i.e. its stalks are torsion free. For an open set U in X the module of sections $\mathcal{A}(U)$ can be regarded as a subset of the direct product of the stalks, thus we conclude that $\mathcal{A}(U)$ is also torsion free. Since over R torsion freeness implies flatness we see that by tensoring with $\mathcal{A}(U)$ the embedding $R \hookrightarrow Q$, $r \mapsto rq$ is transformed into an embedding $\mathcal{A}(U) \hookrightarrow \mathcal{A}(U) \otimes Q$, $m \mapsto m \otimes q$. Thus, equation (B.2) yields that $\rho^{V_i}_{V_i \cap V_j}(m_i)$ equals $\rho^{V_j}_{V_i \cap V_j}(m_j)$ for all $1 \leq i, j \leq n+1$. Since $\mathcal{A}$ is a sheaf there exists $m \in \mathcal{A}(X)$ whose restriction to V_i coincides with m_i for all $1 \leq i \leq n+1$. Finally, we conclude that the element $m \otimes q \in \Gamma_c(\mathcal{A}) \otimes Q$ corresponds to $(s_u) \in \Gamma_c((\mathcal{A} \otimes Q)^{\#})$. Thus, we have shown that $\Gamma_c(\mathcal{A} \otimes \underline{Q}) = \Gamma_c(\mathcal{A}) \otimes Q$ for a principal ideal ring R and its quotient field Q.
Let now $Q \supset R \supset \mathbb{Q}$ be arbitrary field extensions of the field of rational numbers. Then each of

the sections σ_i is of the form $\sum_k m'_{k,i} \otimes q_{k,i}$ with $m'_{k,i} \in \mathcal{A}(V_i)$ and $q_{k,i} \in Q$. Let $\{q_1, \dots q_m\}$ be a basis of the R-vector subspace of Q generated by the elements $\{q_{k,i}\}_{k,i}$. Then every section σ_i can be written as $\sum_l m_{l,i} \otimes q_l$ for elements $m_{l,i} \in \mathcal{A}(V)$ and equation (B.1) reads as follows

$$\sum_{l=1}^{m} \rho^{V_i}_{V_i \cap V_j}(m_{l,i}) \otimes q_l = \sum_{l=1}^{m} \rho^{V_j}_{V_i \cap V_j}(m_{l,j}) \otimes q_l.$$

Since the elements q_l are linearly independent over R we conclude that $\rho^{V_i}_{V_i \cap V_j}(m_{l,i})$ equals $\rho^{V_j}_{V_i \cap V_j}(m_{l,j})$ for $1 \leq l \leq m$ and $1 \leq i, j \leq n+1$. Thus, there exist elements $m_l \in \mathcal{A}(X)$ such that their restrictions to V_i coincide with $m_{l,i}$ and the element $\sum_l m_l \otimes q_l \in \Gamma_c(\mathcal{A}) \otimes Q$ corresponds to the element $(s_u) \in \Gamma_c((\mathcal{A} \otimes Q)^{\#})$. We conclude that $\Gamma_c(\mathcal{A} \otimes \underline{Q}) = \Gamma_c(\mathcal{A}) \otimes Q$ also in the case where $Q \supset R \supset \mathbb{Q}$ are field extensions. This proves the Claim.

Let now R and Q be either a principal ideal domain and its quotient ring or field extensions of $\mathbb{Q}$ as described in the Lemma. We know that the sheaves in the resolution C^* and the sheaf $\mathcal{M}$ are torsion free and thus flat over R, i.e. their stalks are torsion free and flat. Since the tensor product of flat modules is flat and the stalks of the tensor product of sheaves $C^i \otimes_{\underline{R}} \mathcal{M}$ equal the tensor products of the stalks of C^i and $\mathcal{M}$ we conclude that $C^i \otimes \mathcal{M}$ is flat and torsion free. Thus, we can apply our considerations to the sheaves $C^i \otimes \mathcal{M}$ and find that $\Gamma_c(C^i \otimes \mathcal{M} \otimes \underline{Q}) = \Gamma_c(C^i \otimes \mathcal{M}) \otimes Q$. Let us denote the map $C^i \otimes \mathcal{M} \to C^{i+1} \otimes \mathcal{M}$ by d_i and the corresponding map in the complex $C^* \otimes \mathcal{M} \otimes Q$ by $d_i \otimes \mathrm{id}$. The exactness of the functor $_ \otimes_R Q$, cf. [Lam99], page 128, implies that for two R-modules A, B with $B \subset A$ we have $A \otimes Q / B \otimes Q = A/B \otimes Q$, $\ker(d_i \otimes \mathrm{id}) = \ker(d_i) \otimes Q$ and similarly for $\mathrm{im}(d_i)$. Consequently, we obtain

$$H^q_c(X, \mathcal{M} \otimes \underline{Q}) = H^q_c(X, \mathcal{M}) \otimes Q.$$

□

We describe the situation in which we want to apply the Lemma. Let $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$ be a finite-dimensional rational representation defined over $\mathbb{Z}$. In chapter 2 we have introduced the manifolds $S_n(K^f)$ and we have constructed sheaves $\mathcal{V}_R$ on them, where R is a field extension of $\mathbb{Q}$ or the ring of integers of a local field. Let now either Q be a local field with ring of integers R or let R be an algebraic number field and $Q = \mathbb{C}$. In these cases the Lemma assures that

$$H^q_c(S_n(K^f), \mathcal{V}_R) \otimes_R Q = H^q_c(S_n(K^f), \mathcal{V}_Q), \quad q \geq 0.$$

We need a similar statement for ordinary cohomology. Since the Lemma does not seem to adapt to ordinary cohomology in this generality we have to make use of the special situation we are in, i.e. we will interpret the sheaf cohomology $H^q(S_n(K^f), \mathcal{V})$ as group cohomology of arithmetic groups and exploit certain finiteness conditions of the cohomology of arithmetic groups.
Let R be a field extension of $\mathbb{Q}$ or the ring of integers of a local field. The decomposition (2.1) of the manifold $S_n(K^f)$ and the considerations in [Har87c], section 2.9, yield

$$H^q(S_n(K^f), \mathcal{V}_R) = \bigoplus_{i=1}^{m} H^q(\Gamma^{g_i}, g_i V_R),$$

where we keep the notation introduced in section 2.1 and $H^q(\Gamma^{g_i}, g_i V_R)$ denotes the q-th cohomology of the Γ^{g_i}-module $g_i V_R := \rho(g_i) V_R$. We assume the compact open subgroup $K^f \subset \mathrm{GL}_n(\mathbb{A}_f)$ to be neat, i.e. the arithmetic subgroups Γ^{g_i} are torsion free. In this case the groups Γ^{g_i} have

type (FL), see the example on page 237 in [Bro74], i.e. the trivial $\mathbb{Z}[\Gamma^{g_i}]$-module $\mathbb{Z}$ admits a finite resolution

$$0 \to P_n \to \ldots \to P_0 \to \mathbb{Z} \to 0$$

of *finitely generated* free $\mathbb{Z}[\Gamma^{g_i}]$-modules P_i. By definition the cohomology groups $H^q(\Gamma^{g_i}, g_i V_R)$ can be computed as the cohomology groups of the following complex, cf. [Har08b] Lemma 2.4.6,

$$0 \to \mathrm{Hom}_{\mathbb{Z}[\Gamma^{g_i}]}(P_0, g_i V_R) \to \mathrm{Hom}_{\mathbb{Z}[\Gamma^{g_i}]}(P_1, g_i V_R) \to \ \ldots \ \to \mathrm{Hom}_{\mathbb{Z}[\Gamma^{g_i}]}(P_n, g_i V_R) \to 0.$$

Since the modules P_j are finitely generated and free, we can write $P_j \cong \mathbb{Z}[\Gamma^{g_i}]^{k_j}$ and obtain

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma^{g_i}]}(P_j, g_i V_R) \cong (g_i V_R)^{k_j}.$$

Let now Q be a local field and let R be its ring of integers. The functor $_ \otimes_R Q$ is exact and thus we conclude that

$$H^q(\Gamma^{g_i}, g_i V_R) \otimes_R Q = H^q(\Gamma^{g_i}, V_Q),$$

where we have used that $g_i V_R$ defines a lattice in V_Q and in particular $g_i V_R \otimes_R Q = V_Q$. In the setting of sheaf cohomology we obtain

$$H^q(S_n(K^f), \mathcal{V}_R) \otimes_R Q = H^q(S_n(K^f), \mathcal{V}_Q). \tag{B.3}$$

The same identity holds if R is an algebraic number field and Q is the field of complex numbers $\mathbb{C}$.

C. Hodge Polygon

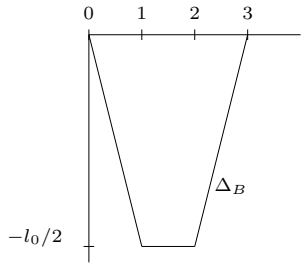
Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A})$ with infinity component $\pi_\infty \cong \mathrm{Ind}(D(l_0+1), \mathrm{id})$ and unramified p-component $\pi_p \cong \mathrm{Ind}(\mu_1, \mu_2, \mu_3)$ as introduced in section 3.1. From [Hid98], Theorem 8.1, we know that the Newton polygon of the Hecke polynomial of π_p always lies on or above the Hodge B-polygon of π, where B denotes the Borel subgroup. In this section we want to explain the consequences for the p-adic absolute values $|\mu_i(p)|_p$, $i = 1, 2, 3$. Consulting equation (8.2) in [Hid98] we find that Δ_B is the convex hull of the following vertices

$$\left\{ (i, \sum_{j=3-i+1}^{3} m_j + 3 - j) \mid \quad i = 0, \ldots 3 \right\}, \tag{C.1}$$

where $\chi = (m_1, m_2, m_3)$ is the cohomological weight of π defined in section 3.2 in [Hid98]. In order to determine the cohomological weight χ we first note that, in the notation of [Kna94], the representation of the Weil group $W_\mathbb{R}$ associated to the infinity component π_∞ by the local Langlands correspondence is given by $(l_0, 0) \oplus (+, 0)$. As in [Hid98], page 673, let $\rho(z)$ with $z \in \mathbb{C}^*$ be the diagonal matrix $\mathrm{diag}(z^{-a_1}\bar{z}^{-a_2}, z^{-a_2}, \bar{z}^{-a_1}, (z\bar{z})^{-a_3})$, where $z \mapsto z^{-a_1}\bar{z}^{-a_2}$ and $z \mapsto z^{-a_2}, \bar{z}^{-a_1}$ are the characters into which the representation $(l_0, 0)$ restricted to the subgroup $\mathbb{C}^* \subset W_\mathbb{R}$ decomposes and $z \mapsto (z\bar{z})^{-a_3}$ is the representation $(+, 0)$ restricted to $\mathbb{C}^* \subset W_\mathbb{R}$; thus we find $a_1 = -l_0/2$, $a_2 = l_0/2$ and $a_3 = 0$. According to equation (3.3) in [Hid98], the cohomological weight χ is then given by

$$\chi = (l_0/2 - 2, -1, -l_0/2).$$

Plugging this into formula (C.1), we find that the Hodge B-polygon Δ_B is the convex hull of the points $\{(0, 0), (1, -l_0/2), (2, -l_0/2), (3, 0)\}$ and we obtain the following image.



On the other hand, let $H_p(T) := \prod_{i=1,2,3}(1 - \mu_i(p)T)$ be the Hecke polynomial of π_p, where as before $\pi_p = \mathrm{Ind}(\mu_1, \mu_2, \mu_3)$. The crucial point is, that the Newton polygon of $H_p(T)$ with respect to an embedding $i_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ lies on or above the Hodge B-polygon Δ_B, cf. Theorem 8.1 in [Hid98]. This means, the values $\alpha_i := \mu_i(p)$ satisfy the following equations

$$v_p(\alpha_1 + \alpha_2 + \alpha_3) \geq -l_0/2, \tag{C.2}$$

$$v_p(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) \geq -l_0/2, \tag{C.3}$$

$$v_p(\alpha_1) + v_p(\alpha_2) + v_p(\alpha_3) \geq 0, \tag{C.4}$$

where v_p denotes the p-adic valuation corresponding to the embedding i_p. Without loss of generality we now assume that $v_p(\alpha_1) \geq v_p(\alpha_2) \geq v_p(\alpha_3)$.

Lemma. *With the notation introduced above we have*

$$v_p(\alpha_2) + 2v_p(\alpha_3) \geq -l_0. \tag{C.5}$$

The Hodge B-polygon Δ_B coincides with the Newton polygon of the Hecke polynomial if and only if $v_p(\alpha_1) = l_0/2$, $v_p(\alpha_2) = 0$ and $v_p(\alpha_3) = -l_0/2$.

Proof. We first show equation (C.5). Distinguish the cases that $v_p(\alpha_2) = v_p(\alpha_3)$ resp. $v_p(\alpha_2) \neq v_p(\alpha_3)$. Let first $v_p(\alpha_2)$ be equal to $v_p(\alpha_3)$. If $v_p(\alpha_1) = v_p(\alpha_2)$ then $v_p(\alpha_i) \geq 0$ for $i = 1, 2, 3$ according to equation (C.4) and equation (C.5) is satisfied. (Recall that we have assumed $p > 3$ throughout this thesis.) If $v_p(\alpha_1) \neq v_p(\alpha_2)$ then $v_p(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)$ equals $v_p(\alpha_2) + v_p(\alpha_3)$ which is greater than or equal to $-l_0/2$ according to equation (C.3). Thus, $v_p(\alpha_2) = v_p(\alpha_3)$ is greater than or equal to $-l_0/4$ and equation (C.5) is fulfilled.
Let now $v_p(\alpha_2)$ be different from $v_p(\alpha_3)$. Then $v_p(\alpha_1 + \alpha_2 + \alpha_3)$ is equal to $v_p(\alpha_3)$, which is greater than or equal to $-l_0/2$ according to equation (C.2). Thus, if $v_p(\alpha_1) = v_p(\alpha_2)$ then we deduce from equation (C.4) that $2v_p(\alpha_2) + 4v_p(\alpha_3) \geq \frac{-3l_0}{2}$, i.e. equation (C.5) is satisfied. If $v_p(\alpha_1) \neq v_p(\alpha_2)$, then $v_p(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)$ is equal to $v_p(\alpha_2) + v_p(\alpha_3)$, which is greater than or equal to $-l_0/2$ by equation (C.3). Consequently, $v_p(\alpha_2) + 2v_p(\alpha_3) \geq -l_0$, i.e. equation (C.5) is satisfied.
Consider the second claim of the Lemma. Of course, if $v_p(\alpha_1) = l_0/2$, $v_p(\alpha_2) = 0$ and $v_p(\alpha_3) = -l_0/2$ then we have equality in the relations (C.2), (C.3) and (C.4) and the Hodge B-polygon indeed coincides with the Newton polygon of the Hecke polynomial. The other direction can be shown in a similar way as equation (C.5). □

Definition. *We call the representation π p-ordinary with respect to the embedding $i_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ if the Hodge B-polygon Δ_B of π coincides with the Newton polygon of the Hecke polynomial $H_p(T)$ of π_p with respect to the embedding i_p.*

According to the Lemma, π is p-ordinary if and only if

$$|\mu_1(p)|_p = p^{-l_0/2}, \quad |\mu_2(p)|_p = 1 \quad \text{and} \quad |\mu_3(p)|_p = p^{l_0/2}.$$

We note that the constant $\gamma = \mu_2(p)\mu_3(p^2)p^2$ has p-adic valuation $v_p(\gamma)$ greater than or equal to $-l_0 + 2$ and equality holds if π is p-ordinary. Equivalently, we can write

$$v_p(p^{2-l_0}\gamma^{-1}) \leq 0 \tag{C.6}$$

with equality if π is p-ordinary.

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Zusammenfassung

Wir konstruieren p-adische L-Funktionen zu kohomologischen cuspidalen automorphen Darstellungen π von GL_3. Für cuspidale Darstellungen π von GL_3, die kohomologisch in Bezug auf die triviale Darstellung sind, wurden in [Mah00] p-adische L-Funktionen, die die Werte der komplexen L-Funktion an den kritischen Stellen auf der linken Seite der Funktionalgleichung interpolieren, konstruiert. Unsere Strategie besteht darin, diesen Zugang auf beliebige kohomologische cuspidale Darstellungen zu verallgemeinern. Das beruht hauptsächlich auf Harders Methode, Schranken für die Nenner bestimmter Eisensteinkohomologieklassen ω_{p^e} in $H^1(S_2(K^f), \mathcal{W})$, wobei $S_2(K^f)$ der adelische symmetrische Raum zu GL_2 und zu einer kompakt offenen Untergruppe $K^f \subset \mathrm{GL}_2(\mathbb{A}_f)$ und $\mathcal{W}$ eine gewisse Garbe auf der Mannigfaltigkeit $S_2(K^f)$ ist, zu berechnen. Wie bei Amice, Manin, Mazur, Velu, Visik und anderen ist die p-adische L-Funktion zu π als p-adische Mellintransformierte eines h-zulässigen Maßes definiert. Schließlich geben wir eine p-adische Funktionalgleichung an. Dazu müssen wir insbesondere die p-adischen L-Funktionen zu cuspidalen Darstellungen π und zu den kritischen Werten auf der rechten Seite der Funktionalgleichung konstruieren.

Abstract

We construct p-adic L-functions attached to cohomological cuspidal automorphic representations π of GL_3. For cuspidal representations π of GL_3 which are cohomological with respect to the trivial representation, p-adic L-functions interpolating the values of the complex L-function at the critical integers on the left hand side of the functional equation have been established in [Mah00]. Our strategy is to generalize this approach to arbitrary cohomological representations. This mainly relies on Harder's method of computing bounds for the denominators of certain Eisenstein cohomology classes ω_{p^e} in $H^1(S_2(K^f), \mathcal{W})$, where $S_2(K^f)$ denotes the adelic symmetric space attached to GL_2 and to a compact open subgroup $K^f \subset \mathrm{GL}_2(\mathbb{A}_f)$ and $\mathcal{W}$ is a certain sheaf on the manifold $S_2(K^f)$. In the spirit of Amice, Manin, Mazur, Velu, Visik and others, the p-adic L-function attached to π will be defined as the p-adic Mellin transform of an h-admissible measure. Finally, we establish a p-adic functional equation. This, in particular, requires the construction of p-adic L-functions attached to cuspidal representations π and the critical integers on the right hand side of the functional equation.