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"Sequential barrelledness and (df) spaces"

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Contents

Chapter 1

Introduction

(DF) spaces where first introduced in Grothendieck (1954). As the acronym suggests, (DF) spaces are in some sense generalizations of strong duals of Fréchet spaces and therefore an important part of duality theory, which itself is central to the theory of locally convex spaces. Later several variants of the concept of (DF) spaces were proposed; most notably, the class of generalized (DF) spaces, called (gDF) spaces (cf. Adasch and Ernst, 1973, 1975; Ruess, 1976, 1977), as well as the larger class of (df) spaces (cf. Adasch and Ernst, 1973, 1975; Jarchow, 1981). The main motivation in studying these spaces was to give conditions under which the strong dual of a given space would be a Fréchet space. The notion of (df) space captures this idea.

Following Jarchow (1981), we will base our investigations on (df) spaces – the weakest of the three concepts. Strong duals of (df) spaces are Fréchet spaces, i.e. have the two properties of being metrizable and complete. Correspondingly a (df) spaces has to fulfill two requirements. Firstly, it has to admit a fundamental basis of bounded sets, ensuring that the strong dual is metrizable. Secondly, it has to be quasi-c_0-barrelled to ensure completeness.

The stronger notion of (gDF) spaces ensures that for a (gDF) space E the strong topology on the continuous functionals $\mathcal{L}(E, F)$ to a Fréchet space F are a again Fréchet space (cf. Theorem 4.5). In this sense the the (gDF) property is stronger, since for a (df) space this only holds for $F = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. This can also be seen by the characterization of (gDF) spaces given in Theorem 4.6, characterizing (gDF) spaces as (df) spaces in which strong null sequences in $\mathcal{L}(E, F)$ are equicontinuous – a property analogous to quasi-c_0-barrelledness. Apart from the basic properties, relationships between the different types of (df) spaces and stability properties, we will discuss the connection of these spaces to Schwartz and Montel spaces as well as the close connection of (df) spaces to imbedding spectra.

As already mentioned above, for the strong dual of a locally convex space to be complete, it has to be quasi-c_0-barrelled. More specifically, every strong null sequence in the dual has to be a equicontinuous set. Correspondingly the concept of sequential barrelledness is central to the study of (df) spaces and their generalizations. We therefore discuss different notions of sequential barrelledness as well as the connections to reflexivity of a space in this thesis. In particular, we study (quasi)-c_0-barrelledness and (quasi)-l_∞-barrelledness, which are all variants of sequential barrelledness, as well as quasi-barrelledness and (quasi)-$\aleph_0$-barrelledness, which are weakenings of conventional barrelledness that do not rely on sequences.

Furthermore, a large part of the thesis discusses Schwartz and Montel spaces, which have nice connections to (df) spaces. To be able to formally treat these spaces, we require the notions of continuous convergence as well as the stronger equicontinuous convergence and the associated topologies. I devote Chapter 3 to the study of these concepts.

The thesis is structured as follows: In Chapter 2, I will review selected parts of duality theory in locally convex spaces and introduce the notion of sequential barrelledness. Chapter 3 is devoted to the results on Schwartz and Montel spaces as well as the required technical tools such as continuous convergence and equicontinuous convergence. The last chapter focusses on (df), (DF) and (gDF) spaces. Characterizations, stability properties and relations to Schwartz spaces are discussed.

I started working on my thesis report equipped with basic knowledge about locally convex vector spaces, which I acquired in two excellent courses by Andreas Kriegl. The thesis can be seen as a documentation of my struggle with the more advanced topics of sequential barrelledness, Schwartz spaces, and of course (df) spaces. Correspondingly, the level of explanations given in the text is to be seen relative to my own knowledge at the beginning of the errand: while I use theorems known to me from my functional analysis classes without giving proofs, I try to elaborate on all the results that were new to me when I started writing this report. The reader thus is assumed to be familiar with the basic concepts of functional analysis including the treatment of locally convex spaces (as covered for example in Kriegl (2007)).

Most of the results discussed in this document are directly taken or at least adapted from either Jarchow (1981), Kriegl (2007) or Meise and Vogt (1997). The notation used will be explained along the way and the list of symbols at the end of the thesis will hopefully provide the casual reader with the required orientation.

Chapter 2

Bornologies and sequential barrelledness

This chapter starts by collecting some important definitions and auxiliary results that will be needed later in the thesis. Most notably certain results concerning compactness in locally convex spaces will be proven and a short review on bornologies and their connections to dual topologies will be discussed. After a short intermission dedicated to local convergence, the concept of barrelledness, which is fundamental to this thesis, is discussed. The chapter ends with a section on absorbent and fundamental sequences, which mainly consists of a result which will later be used to define (gDF) spaces (cf. Theorem 2.72).

As a notational convention the abbreviation *lcs* means locally convex space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, which has the additional property of being Hausdorff. If E is a locally convex space but not Hausdorff, it will just be called a *locally convex space*. However, the results contained in this thesis are mostly restricted to Hausdorff spaces, since this is the natural setting for the analysis of df spaces, although some of the results stated may also hold for locally convex spaces which do not have the Hausdorff property.

2.1 Preparations

We will start by the definition of a filter and a characterization of a neighborhood basis in a locally convex space.

Definition 2.1 (Filter)**.** Let E be a topological space and $S \subseteq E$. A system of sets $\mathcal{F} \subseteq 2^S$ in a is called a filter on $S \subseteq E$, if the following conditions are met:

1. $\emptyset \notin \mathcal{F}$.

2. $S \in \mathcal{F}$.

3. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$.

4. $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$.

A filter $\mathcal{F}$ is said to converge to $a \in S$, iff a neighborhood basis of a is contained in $\mathcal{F}$. A filter is a *Cauchy filter*, if for every zero neighborhood U in E there is a $A \in \mathcal{F}$ such that $A - A \in U$. A point $x \in E$ is called a limit point of a filter $\mathcal{F}$, if $x \in \bar{A}$ for all $A \in \mathcal{F}$. A filter $\mathcal{F}$ is called an *ultrafilter*, if there is no filter $\mathcal{F}'$ finer than $\mathcal{F}$. Equivalently $\mathcal{F}$ is an ultrafilter, if for every set $A \subseteq E$, either $A \in \mathcal{F}$ or $(E \setminus A) \in \mathcal{F}$.

Definition 2.2 (Trace of a filter)**.** Let $\mathcal{F}$ be a filter on X and $A \subseteq X$, then the trace of $\mathcal{F}$ on A is defined as

$$\mathcal{F}_A = \{F \cap A : F \in \mathcal{F}\}.$$

Obviously $\mathcal{F}_A$ is a filter on A.

Definition 2.3 (Filter basis)**.** Let E be a topological space and $S \subseteq E$. Then $\mathcal{B} \in 2^S$ is called a filter basis on S, if it has the following properties

1. $A, B \in \mathcal{B} \Rightarrow \exists C \in \mathcal{B} : C \subseteq A \cap B$.

2. $\mathcal{B} \neq \emptyset$ and $\emptyset \notin \mathcal{B}$.

Clearly, not every filter basis has to be a filter. To obtain a filter on $S \subseteq E$ from a filter basis $\mathcal{B}$ on S, we define

$$\mathcal{F} = \{A \subseteq S : \exists B \in \mathcal{B} \text{ with } B \subseteq A\}.$$

Lemma 2.4. Let $\mathcal{F}$ be a filter. Then x is a limit point of $\mathcal{F}$, iff for all closed sets $A \subseteq \mathcal{F}$, $x \in A$.

Proof. Suppose x is a limit point, $A \in \mathcal{F}$ is closed and $x \notin A$. It follows that there is a neighborhood U of x such that $U \cap A = \emptyset$, a contradiction to x being a limit point of $\mathcal{F}$.

Conversely if the above property holds, then $x \in \bigcap_{B \in \mathcal{F}} \bar{B}$. But the latter set are just the limit points of $\mathcal{F}$. $\square$

It is straightforward to define a neighborhood basis in a topological (vector) space, if the topology is already known. The following Theorem describes the other direction, i.e. how to define a system of sets in a vector space, from which a unique locally convex topology can be constructed in such a way that the original system of sets forms a zero neighborhood basis for the generated locally convex topology.

Lemma 2.5 (Theorem 6.5.3, Jarchow (1981))**.** Let E be a vector space and $\mathcal{V} \subseteq 2^E$ be a system of sets which fulfills:

1. $\emptyset \notin \mathcal{V}$.
2. $\forall V_1, V_2 \in \mathcal{V},\ \exists W \in \mathcal{V} : W \subseteq V_1 \cap V_2$.
3. All $V \in \mathcal{V}$ are absolutely convex.
4. All $V \in \mathcal{V}$ are absorbent.
5. $\forall V \in \mathcal{V},\ \forall \rho \in \mathbb{R}^+$ there exists a $V' \in \mathcal{V} : V' \subseteq \rho V$

then $\mathcal{V}$ generates a unique locally convex topology $\mathcal{T}$ on E for which $\mathcal{V}$ is a zero neighborhood basis.

Proof. We start by defining the open sets $O \in \mathcal{T}$ by

$$\mathcal{T} = \{O \subseteq E : \forall x \in O,\ \exists V \in \mathcal{V} \text{ such that } x + V \in O\}.$$

$\mathcal{T}$ is closed with respect to taking unions and for O_1, $O_2 \in \mathcal{T}$ and $x \in O_1 \cap O_2$ there are $V_i : V_i + x \in O_i$ and there is a $W \in \mathcal{V}$ with $W \in V_1 \cap V_2$. Hence $x + W \in O_1 \cap O_2$ and because x was arbitrary $O_1 \cap O_2 \in \mathcal{T}$ and $\mathcal{T}$ is a topology, which is locally convex because of 3. This topology is clearly the only topology with $\mathcal{V}$ as zero neighborhood basis.

Because of 5 and 3 for every $U \in \mathcal{V}$ there is a $V \in \mathcal{V}$ such that $V + V \subseteq U$. This shows that $+ : E \times E \to E$ is continuous, since for $x, y \in E$ and $U \in \mathcal{V}$, we have $(x + V) + (y + V) \subset (x + y) + U$.

The only thing left to show is the continuity of the multiplication $M : \mathbb{K} \times E \to E$. For this fix $(\lambda, x) \in \mathbb{K} \times E$, $U \in \mathcal{V}$ and sets $(V_n)_{n\in\mathbb{N}} \subseteq \mathcal{V}$ with $V_1 = U$ and $V_{n+1} + V_{n+1} \subseteq V_n$. Further choose $n \geq |\lambda|$ and $0 \leq \sigma \leq 1$ such that $\sigma x \in V_{n+2}$ (possible because of 4), $\rho \in \lambda + \sigma\mathbb{D}$ and $y \in x + V_{n+2}$. Then it holds that

$$\begin{aligned}(\rho - \lambda)(y - x) \in \sigma V_{n+2} &\subseteq V_{n+2}\\ \lambda(y - x) \in nV_{n+2} &\subseteq V_3\\ (\rho - \lambda)x \in (\rho - \lambda)\sigma^{-1}V_{n+2} &\subseteq V_{n+2}.\end{aligned}$$

Therefore

$$\rho y - \lambda x = (\rho - \lambda)(y - x) + \lambda(y - x) + (\rho - \lambda)x \in V_{n+2} + V_3 + V_{n+2} \subseteq U$$

and M is continuous at (λ, x), since $M^{-1}(\lambda x + U) \supseteq (\lambda + \sigma\mathbb{D}) \times (x + V_{n+2})$. □

Remark 2.6. Note that the linear topologies in the above Theorem are not necessary Hausdorff.

Theorem 2.7 (Theorem 3.2.4, Jarchow (1981))**.** Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be linear topologies on a vector space E such that $\mathcal{T}_1 \geq \mathcal{T}_2$. Suppose further that $(E, \mathcal{T}_1)$ has a zero basis $\mathcal{U}$ consisting of $\mathcal{T}_2$-closed sets. Let $M \neq \emptyset$ be a subset of E.

1. Let $\mathcal{F}$ be a $\mathcal{T}_1$-Cauchy filter on M. If $\mathcal{F}$ converges to $a \in E$ with respect to $\mathcal{T}_2$, then $\mathcal{F}$ converges to a with respect to $\mathcal{T}_1$.

2. If M is (sequentially) complete for $\mathcal{T}_2$, then it is also (sequentially) complete for $\mathcal{T}_1$.

Proof. 1. Fix $U \in \mathcal{U}$ and choose $A \in \mathcal{F}$ such that $A - A \subseteq U$. Since U is $\mathcal{T}_2$-closed, it follows that $y_0 - \bar{A}^{\mathcal{T}_2} \subseteq U$ for all $y_0 \in A$. Because a is a limit point of $\mathcal{F}$ in $\mathcal{T}_2$, we have $a \in \bar{A}^{\mathcal{T}_2}$ and therefore $y_0 - a \in U$. Since y_0 was arbitrary, we have $A \subseteq a + U$ and $\mathcal{F}$ is finer than the neighborhood filter of a for $\mathcal{T}_1$. Hence, $\mathcal{F} \xrightarrow{\mathcal{T}_1} a$.

2. Is a easy consequence of (1).

□

The preceding Theorem will prove very useful in the context of admissible lcs topologies. Since all topologies $\mathcal{T}$ which are compatible with a dual pairing have the same closed convex sets as the weak topology, every admissible topology has a basis of weakly closed sets. Hence, a Cauchy filter converges in an admissible topology if and only if it already converges in the weak topology.

A direct consequence can be drawn immediately.

Theorem 2.8 (Proposition 3.4.5, Jarchow (1981))**.** Let $\mathcal{T}$ be a topology on E and let $\mathcal{T}_1$ be the initial topology on E defined by a mapping $T \in \mathcal{L}(E, F)$. Suppose further that $(E, \mathcal{T})$ admits a zero neighborhood basis consisting of $\mathcal{T}_1$-closed sets. If T is injective, then so is its extension $\tilde{T} \in \mathcal{L}(\tilde{E}, \tilde{F})$ to the completions.

Proof. Given $z \in \ker(\tilde{T})$, we can find a Cauchy filter $\mathcal{F}$ on $(E, \mathcal{T})$ such that z is the limit of the filter $\tilde{\mathcal{F}}$ generated by $\mathcal{F}$ on $\tilde{E}$. Since $z \in \ker(\tilde{T})$ and $\tilde{T} \in \mathcal{L}(\tilde{E}, \tilde{F})$, $\tilde{T}(\tilde{\mathcal{F}})$ converges to 0 in $\tilde{F}$. Let $A \in \tilde{T}(\tilde{\mathcal{F}})$ be given, then there is a $B \in \mathcal{F}$ such that $T(B) \subseteq A$ and therefore $\tilde{T}(\tilde{\mathcal{F}})$ has a trace $\mathcal{G} \subseteq T(\mathcal{F})$ on F. From $\lim \tilde{T}(\tilde{\mathcal{F}}) = 0$, we get $\lim \mathcal{G} = 0$ in $\mathcal{T}_1$ (because $\mathcal{T}_1$ is the initial topology with respect to T), hence $\lim T(\mathcal{F}) = 0$, which implies $\lim \mathcal{F} = 0$ in $(E, \mathcal{T}_1)$. Since $\mathcal{F}$ is a Cauchy filter in $\mathcal{T}$ and $\mathcal{T}$ has a basis of $\mathcal{T}_1$-closed sets $\mathcal{F}$ converges to zero also in $(E, \mathcal{T})$ by Theorem 2.7. Hence, $z = 0$ which completes the proof. □

2.2 Compactness, precompactness and relative compactness

In this section, we will collect a few facts about the notions of precompactness, relative compactness and compactness. These results will later be helpful when studying (df) spaces.

Definition 2.9 (Precompact)**.** Let E be a topological vector space and $\mathcal{U}$ a zero neighborhood basis of E. A set $C \subseteq E$ is called precompact if for every $U \in \mathcal{U}$ there is a finite set $M \subseteq E$ such that

$$C \subseteq M + U = \bigcup_{x \in M} x + U.$$

Remark 2.10.

1. A set $A \subseteq E$ is relatively countably compact, iff every sequence in A has a cluster point in E.
2. A precompact set is always bounded.
3. The set M can always be chosen such that $M \subseteq C$. To see this choose $V \in \mathcal{U}$ such that $V + V \subseteq U$ and $N \subseteq E$ finite such that $C \subseteq N + V$ and $(x_i + V) \cap C \neq \emptyset$ for all $x_i \in N$. Now choose $y_i \in (x_i + V) \cap C$ and define $M = \{y_i : i = 1, \dots, |N|\}$, then

$$C \subseteq \bigcup_{x_i \in N} (x_i + V) \subseteq \bigcup_{y_i \in M} (y_i + V + V) \subseteq \bigcup_{y_i \in M} (y_i + U).$$

The next Lemma relates precompactness and relative compactness in the completion $\tilde{E}$ of a space E. For a set $A \subseteq E$, we denote by $\tilde{A}$ the set $\bar{A}$, where the closure is taken in $\tilde{E}$.

Lemma 2.11 (Lemma 3.5.1, Jarchow (1981))**.** Let E be a Hausdorff topological vector space. For $C \subseteq E$ the following conditions are equivalent:

1. C is precompact.
2. C is relatively compact in the completion $\tilde{E}$.
3. C is relatively countably compact in the completion $\tilde{E}$.

Proof. **(1)** $\Rightarrow$ **(2)**: Let $\mathcal{U}$ be a zero neighborhood basis consisting of closed and circled sets, then $\tilde{\mathcal{U}} = \left\{\tilde{U} : U \in \mathcal{U}\right\}$ is zero neighborhood basis in $\tilde{E}$ (see 4.10.3 in Kriegl (2007)). Choose an arbitrary $U \in \mathcal{U}$ and a $V \in \mathcal{U}$ such that $V + V \subseteq U$, by our hypothesis there is a finite set M such $C \subseteq M + V$ and since $(x_i + \tilde{V})$ is closed for all $x_i \in V$, $\tilde{C} = (M + \tilde{V}) \cap \tilde{C}$.

If $\mathcal{F}$ is an ultrafilter in $\tilde{C}$, then there exists a $x_i \in M$ such that $(x_i + \tilde{V}) \in \mathcal{F}$. If this would not be the case, then (since $\mathcal{F}$ is a ultra filter) $(x_i + \tilde{V})^c \cap \tilde{C} \in \mathcal{F}$ and therefore

$$\bigcap_{x_i \in M} (x_i + \tilde{V})^c \cap \tilde{C} = \left(\bigcup_{x_i \in M} (x_i + \tilde{V}) \right)^c \cap \tilde{C} = \emptyset \in \mathcal{F}$$

which is a contradiction to $\mathcal{F}$ being a filter. Now we conclude from $(x_i + \tilde{V}) - (x_i + \tilde{V}) = \tilde{V} + \tilde{V} \subseteq \tilde{U}$, that $\mathcal{F}$ is a Cauchy filter and therefore converges in $\tilde{C}$. Hence $\tilde{C}$ is compact.

(2) $\Rightarrow$ **(3)**: Trivial.

(3) $\Rightarrow$ **(1)**: Suppose C is not precompact. Then there is a zero neighborhood U in E such that for any finite set $M \subseteq E$: $C \not\subseteq M + U$. Therefore also $C \not\subseteq M + \tilde{U}$ (otherwise there would be a $y \in C$ such that $y \in \tilde{E} \setminus E$). Now construct a sequence $(x_n)_{n\in\mathbb{N}} \subseteq C$ by requiring $x_{n+1} \notin \bigcup_{i=1}^{n} x_i + \tilde{U}$. Then $x_n - x_m \notin \tilde{U}$ for $m \neq n$. Hence, $(x_n)_{n\in\mathbb{N}}$ can not have a cluster point in $\tilde{E}$ and therefore C fails to be relatively countably compact. □

Corollary 2.12. In a complete Hausdorff topological vector space E, every precompact set is relatively compact.

Theorem 2.13 (Proposition 6.7.1, Jarchow (1981))**.** Let $B \subseteq E$ be precompact in the lcs E, then also the absolutely convex hull $\operatorname{acx}(B)$ is precompact.

Proof. We first show that the circled hull of a precompact set B is again precompact. This follows easily since the circled hull of B in $\tilde{E}$ is the image of $\mathbb{D} \times B$ under the mapping $M_{\tilde{E}}(\lambda, x) := \lambda x$, where $\mathbb{D} = \{x \in \mathbb{K} : |x| \leq 1\}$ and $\mathbb{D} \times B$ is relatively compact in $\mathbb{K} \times \tilde{E}$ (see Theorem 2.11). Since $M_{\tilde{E}}$ is continuous also the circled hull of B in $\tilde{E}$ is relatively compact. Hence, it is precompact in E by Theorem 2.11.

Since $\operatorname{acx}(B)$ is the circled hull of the convex hull $\operatorname{cx}(B)$ we only have to show that the convex hull of a precompact set is again precompact. Choose a zero neighborhood $U \subseteq E$ and define $V := \frac{1}{2}U$. Now choose $M = \{x_1, \ldots, x_m\}$ such that $B \subseteq M + V$ and note that since $\operatorname{cx}(M) + V$ is convex $\operatorname{cx}(B) \subseteq \operatorname{cx}(M) + V$. Because $\operatorname{cx}(M)$ is the image of the compact set $\left\{\alpha \in \mathbb{R}_+^m : \sum_{i=1}^m \alpha_i = 1\right\}$ under the continuous mapping $(\alpha_i)_{i=1}^m \mapsto \sum_{i=1}^m \alpha_i x_i$ it is also compact. From this it follows that there are $y_1, \ldots, y_n$ such that $\operatorname{cx}(M) \subseteq \bigcup_{i=1}^n y_i + V$ and therefore $\operatorname{cx}(B) \subseteq \bigcup_{i=1}^n y_i + U$. It follows that $\operatorname{cx}(B)$ and therefore $\operatorname{acx}(B)$ are precompact sets in E. □

Theorem 2.14 (Proposition 6.7.2, Jarchow (1981))**.** If $K \subseteq E$ is compact, then $\overline{\operatorname{acx}}(K)$ is compact iff it is complete.

Proof. Clearly, if $\overline{\operatorname{acx}}(K)$ is compact, then it is complete. Conversely, note that $\overline{\operatorname{acx}}(K)$ is relatively compact by Theorem 2.13 and if it is complete, then it is also compact by 2.11. □

Lemma 2.15. Let E be a complete metric space and $(x_n)_{n\in\mathbb{N}}$ a null sequence, then $\overline{\operatorname{acx}}\{x_n : n \in \mathbb{N}\}$ is compact in E.

Proof. Since $(x_n)_{n\in\mathbb{N}}$ is a null sequence, $(x_n)_{n\in\mathbb{N}}$ is sequentially compact and because E is a metric space also compact. Hence, it is also precompact and therefore also $\operatorname{acx}\{x_n : n \in \mathbb{N}\}$ is precompact by Theorem 2.13. The completeness of the space together with Lemma 2.11 yields that $\overline{\operatorname{acx}}\{x_n : n \in \mathbb{N}\}$ is compact in E. □

In the last theorem we argued that null sequences in metric spaces have some (relative) compactness property. In the next we will go the other way and show that all precompact sets in metrizable spaces can be *caught* in the closure of the convex hull of a null sequence. We will need this result quite often throughout this document. It will prove essential in Section 3.4 to characterize Schwartz spaces.

Theorem 2.16 (Proposition 9.4.2, Jarchow (1981)). Let F be a dense subspace of a metrizable topological vector space E. If $A \subseteq E$ is precompact, then there is a null sequence $(x_n)_{n\in\mathbb{N}} \subseteq F$, such that $A \subseteq \overline{\operatorname{cx}\{x_n : n \in \mathbb{N}\}}$.

Proof. Fix a zero neighborhood basis $(U_n)_{n\in\mathbb{N}}$ of circled sets with $U_1 = E$ and $U_{n+1} + U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. Now construct a sequence of precompact sets $(A_n)_{n\in\mathbb{N}}$ with $A_0 = A$ and all $A_n \subseteq F$ in the following way: given the set A_{n-1} choose a finite set $N_n \subseteq A_{n-1}$ such that

$$A_{n-1} \subseteq N_n + 2^{-n-1} U_{n+2}.$$

Now choose for every $y \in N_n$ a $x \in F$ such that $x - y \in 2^{-n-1}U_{n+2}$ and denote by M_n the subset obtained in this way. Then

$$A_{n-1} \subseteq M_n + 2^{-n-1} U_{n+1}$$

and we define the precompact set A_n as $A_n := (A_{n-1} - M_n) \cap 2^{-n-1} U_{n+1}$.

Now let $0 =: k_0 < k_1 < k_2 < \ldots$ be integers such that $k_n - k_{n-1} = |M_n|$ and for every $n \in \mathbb{N}$ write

$$2^n M_n = \left\{ x_{k_{n-1}+1}, \ldots, x_{k_n} \right\}.$$

Now since

$$\begin{aligned} M_n &\subseteq N_n + 2^{-n-1}U_{n+2} \subseteq A_{n-1} + 2^{-n-1}U_{n+2} \\ &\subseteq 2^{-n}U_n + 2^{-n-1}U_{n+2} \subseteq 2^{-n}U_{n-1} \end{aligned}$$

we have that $2^n M_n \subseteq U_{n-1}$ for all $n \in \mathbb{N}$, in particular $(x_n)_{n\in\mathbb{N}}$ is a null sequence.

Given a $z \in A$, we can write $z = m_1 + \frac{1}{4}u_2$ with $m_1 \in M_1$ and $u_2 \in U_2$. By definition $z - m_1 \in A_1$ and therefore we can find a $m_2 \in M_2$ and a $u_3 \in U_3$ such that $z = m_1 + m_2 + \frac{1}{8}u_3$. Continuing in this manner, we obtain $m_i \in M_i$ for all $i \in \mathbb{N}$ and $u_{n+1} \in U_{n+1}$ for all $n \in \mathbb{N}$ such that

$$z = \sum_{i=1}^{n} m_i + 2^{-n-1} u_{n+1}$$

holds. Since $(u_n)_{n\in\mathbb{N}}$ is a null sequence, we have that the series is convergent in E and $z = \sum_{i=1}^{\infty} m_i$.

By construction there are $k(i)$ with $k_{i-1} < k(i) \leq k_i$ such that $m_i = 2^{-i}x_{k(i)}$. Now put $\rho_{k(i)} = 2^{-i}$ for all $i \in \mathbb{N}$ and $\rho_j = 0$ for all $j \in \mathbb{N} \setminus \{k(i) : i \in \mathbb{N}\}$. Then we have

$$z = \sum_{i=1}^{\infty} \rho_i x_i$$

and since $\sum_{i=1}^{\infty} \rho_i = 1$, we have $z \in \overline{\mathrm{cx}\,\{x_n : n \in \mathbb{N}\}}$. □

Theorem 2.17 (Proposition 9.4.5, Jarchow (1981))**.** Let F be a closed subspace of the Fréchet space E, and let $Q : E \to E/F$ be the corresponding quotient map. For every compact set K in E/F, there exists a compact set $L \subseteq E$ such that $K \subseteq Q(L)$.

Proof. Since E/F is also a Fréchet space, there is a null sequence $(z_k)_{k\in\mathbb{N}} \subseteq E/F$ such that $K \subseteq \overline{\mathrm{cx}}\,\{z_k : k \in \mathbb{N}\}$ by Theorem 2.16. Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing zero neighborhood basis in E such that $z_k \in Q(U_1)$, $\forall k \in \mathbb{N}$. Further choose $1 =: k_1 < k_2 < k_3 < \cdots$ in $\mathbb{N}$ in such a way that $z_k \in Q(U_n)$, $\forall k \geq k_n$, $\forall n \in \mathbb{N}$. For each n and each $k_n \leq k < k_{n+1}$ select $x_k \in U_n$ such that $Q(x_k) = z_k$. Then $(x_k)_{k\in\mathbb{N}}$ is a null sequence in E, and $L := \overline{\mathrm{cx}}\,\{x_k : k \in \mathbb{N}\}$ is compact in E, since E is Fréchet (cf. Lemma 2.15). By construction $K \subseteq \overline{\mathrm{cx}}\,\{Q(x_k) : k \in \mathbb{N}\} = Q(L)$, where the last equality follows, since E/F is the final topology with respect to Q. □

2.3 Bornologies

Bornologies play an important role in duality theory for lcs in general and in the investigation of (df) spaces in particular. The main reason for that is that bornologies can be used to characterize most of the common topologies on spaces of linear functionals. In this respect bornologies are similar to topologies: while topologies describe which sets are open, bornologies can be used to define open sets in the dual or space of linear functionals on the space in which the topology *lives*. Hence, the notion of bornology can be viewed as a concept which is, in a vague sense, *dual* to the concept of topologies. This viewpoint can be illustrated by the operation $\circ$, i.e. the polarization with respect to a dual pairing $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{K}$. The polar of a zero neighborhood in an admissible topology $\mathcal{T}$ on E is a bounded set for every admissible lcs topology on F and the collection of these sets form a bornology on F. Similarly, taking polars of sets in a bornology in F yields a topology on E. Applying this operation twice in an lcs and starting from a topology (respectively from a bornology) we *get back* the topology (respectively bornology) we started with.

The following theorems will make precise the statements in the last section. Note that for paired spaces E and F with dual pairing $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{K}$ the family of bounded sets in $(E, \mathcal{T})$ is completely determined by the pairing as long as the topology $\mathcal{T}$ is admissible. We will denote these sets by $\mathcal{B}(E)$. Furthermore, will denote by $\mathcal{L}(E, F)$ the linear continuous mappings between two lcs E and F and by $L(E,F)$ the linear mappings between E and F.

Definition 2.18 (Bornology)**.** Let $(E,\mathcal{T})$ be a lcs. A family $\mathcal{B} \neq \emptyset$ of $\sigma(E,E')$-bounded sets is a bornology (relative to the pairing $\langle E,E'\rangle$), if

1. for every B_1, $B_2 \in \mathcal{B}$ there is a $B_3 \in \mathcal{B}$ such that $B_1 \cup B_2 \subseteq B_3$;
2. for every $B \in \mathcal{B}$ and $\rho \in \mathbb{K}$ there is a $C \in \mathcal{B}$ with $\rho B \subseteq C$.

Definition 2.19 (Basis of a bornology)**.** A subset $\mathcal{B}_0$ is called a basis for a bornology $\mathcal{B}$ if every subset $B \in \mathcal{B}$ is contained in some $C \in \mathcal{B}_0$.

Definition 2.20 (Saturated hull, saturated bornologies)**.** The saturated hull $\mathcal{B}^{\text{sat}}$ of a bornology $\mathcal{B}$ is defined by

$$\mathcal{B}^{\text{sat}} = \{C \subseteq B^{\circ\circ} : B \in \mathcal{B}\}.$$

A bornology $\mathcal{B}$ is called saturated if $\mathcal{B} = \mathcal{B}^{\text{sat}}$.

Theorem 2.21 (Topologies from bornologies)**.** Let $\mathcal{B}$ be a bornology on E and let F be a lcs. Let $\mathcal{V}$ be zero neighborhood basis of F consisting of closed, absolutely convex sets, then the sets

$$W_{B,V} = \{T \in \mathcal{L}(E,F) : T(B) \subseteq V\}, \quad \text{with } B \in \mathcal{B},\ V \in \mathcal{V}$$

form a 0-neighborhood basis of a lcs topology on $\mathcal{L}(E,F)$ denoted by $\mathcal{T}_{\mathcal{B}}$. The corresponding seminorms $q_{B,V}$ are given by

$$q_{B,V}(T) := \sup_{x \in B} q_V(Tx)$$

where q_V is the Minkowski-functional of the set $V \in \mathcal{V}$.

Proof. First the conditions of Lemma 2.5 have to be verified to obtain that $W_{B,V}$ is a zero neighborhood basis of a locally convex space. First note that the constant function $T = 0$ is element in all the sets $W_{B,V}$ and hence $W_{B,V} \neq \emptyset$, this shows (1). Let B_1, B_2 be bounded in E, V_1, V_2 be zero neighborhoods in F, then obviously $W_{B_1 \cup B_2, W} \subseteq W_{B_1,V_2} \cap W_{B_2,V_2}$, for $W \in \mathcal{V}$ with $W \subseteq V_1 \cap V_2$. This shows (2), since $\mathcal{B}$ is a bornology and therefore $B_1 \cup B_2 \in \mathcal{B}$. (3) and (4) follow since all $V \in \mathcal{V}$ are absolutely convex and absorbent. (5) follows because $\mathcal{B}$ is a bornology. Hence, the only thing left to show is that the seminorms $q_{B,V}$ are compatible with this topology, this can be shown as follows:

$$\begin{aligned} q_{W_{B,V}}(T) &= \inf\{|\lambda| : \lambda \in \mathbb{K}, T \in \lambda W_{B,V}\} \\ &= \inf\{|\lambda| : \lambda \in \mathbb{K}, T \in \{S \in \mathcal{L}(E,F) : S(B) \subseteq \lambda V\}\} \\ &= \inf\left\{|\lambda| : \lambda \in \mathbb{K}, T \in \left\{S \in \mathcal{L}(E,F) : \sup_{x \in B} q_V(Sx) \leq |\lambda|\right\}\right\} \\ &= \sup_{x \in B} q_V(Tx) = q_{B,V}(T). \end{aligned}$$

□

The following example uses Theorem 2.21 to define some topologies that will be used latter in the text. The topologies generated in this way are topologies of uniform convergence on the sets in the corresponding bornology $\mathcal{B}$. In most cases the space F will be chosen to be $\mathbb{K}$ in which case $\mathcal{L}(E,F)$ becomes the dual E' of E and the polars $B^\circ = \{y \in E' : |y(x)| \leq 1,\ \forall x \in B\}$ form a zero neighborhood bases in the respective topologies. More examples can be found in Jarchow (1981), Chapter 8.4. If E and F are paired lcs and $\mathcal{B}$ is a bornology in E, we denote by $\mathcal{T}_\mathcal{B}$ the topology induced by $\mathcal{B}$ on $\mathcal{L}(E,F)$.

Example 2.22. 1. The topology of simple convergence $\mathcal{T}_s$ on $\mathcal{L}(E,F)$ is generated by the bornology of finite sets. Obviously $\mathcal{T}_s$ is the weakest topology that can be obtained from Theorem 2.21. If $F = \mathbb{K}$, then the topology generated in E' is the weak topology denoted by $\sigma(E',E)$. Note that the weak topology $\sigma(E,E')$ on E can be defined by the bornology of finite sets in E' in the same way.

2. The bornology of all absolutely convex $\sigma(E,E')$-compact sets generates the topology $\mathcal{T}_\mu$ on $\mathcal{L}(E,F)$. In the case of $F = \mathbb{K}$, the topology is called the Mackey topology and denoted by $\mu(E',E)$. By the Theorem of Mackey-Arens (see Jarchow (1981), Theorem 8.5.5) the Mackey topology is the finest topology, which is compatible with the dual pairing $\langle E,E'\rangle$. Analogously the topology $\mu(E,E')$ is the topology of uniform convergence on absolutely continuous $\sigma(E,E')$-compact sets on E.

3. The bornology of all E-compact subsets induces the topology $\tau_c(E',E)$ on E'. Note that $\tau_c(E',E) \leq \mu(E',E)$.

4. The topology generated by the precompact subsets of E on $\mathcal{L}(E,F)$ is denoted by $\mathcal{T}_{pc}$ and in the case of $F = \mathbb{K}$ by τ_{pc}.

5. The bornology of bounded sets in E generates the topology $\mathcal{T}_\beta$ which is the strongest topology that can be obtained in this way. In the case $F = \mathbb{K}$, the respective topology is denoted by $\beta(E',E)$. This topology is usually called the strong topology on E'. Unlike in the case of the weak topology there are two way to define an analogous concept for the space E. The topology generated on E by the $\sigma(E',E)$ bounded sets is denoted by $\beta(E,E')$, while the topology which is defined by the $\beta(E',E)$ bounded sets is denoted by $\beta^*(E,E')$. Clearly, $\beta^*(E,E') \leq \beta(E,E')$.

6. The bornology of Banach disks in E generates the topology $\mathcal{T}_\mathfrak{b}$ on $\mathcal{L}(E,F)$. If $F = \mathbb{K}$, we denote this topology by $\mathfrak{b}(E',E)$.

Remark 2.23. Obviously $\sigma(E',E) \leq \mu(E',E) \leq \tau_{pc}(E',E) \leq \beta(E',E)$ holds.

Definition 2.24 (Total)**.** A subset of a locally convex space E is called total iff its linear span is dense in E. A system of sets $\mathcal{B}$ is called total if its union is total in E.

Theorem 2.25 (Theorem 8.4.2, Jarchow (1981))**.** Let E and F be lcs. The topology $\mathcal{T}_{\mathcal{B}}$ on $\mathcal{L}(E,F)$ generated by the bornology $\mathcal{B}$ in E for a lcs F is Hausdorff iff $\mathcal{B}$ is total in E.

Proof. ($\Rightarrow$): Assume $\mathcal{B}$ is not total. Then $G = \overline{\operatorname{span}\left(\bigcup_{B\in\mathcal{B}} B\right)} \neq E$ and therefore by the theorem of Hahn-Banach there is a linear functional $T : E \to F$ such that $T(x) = 0$ for all $x \in G$ and there is a $x_0 \in E \setminus G$ such that $T(x_0) = 1$. It follows that $q_{W_{B,V}}(T) = 0$ for all choices of $B \in \mathcal{B}$ and V although T is not identical to zero. Hence, $(\mathcal{L}(E,F), \mathcal{T}_{\mathcal{B}})$ can not be Hausdorff.

($\Leftarrow$): If $\mathcal{B}$ is total in E and $q_{W_{B,V}}(T) = 0$ in for all $V \in \mathcal{U}$ and $B \in \mathcal{B}$, then it follows by the separability of F that $T(x) = 0$ for all $x \in \bigcup \mathcal{B}$. Hence, since $\mathcal{B}$ is total and T is continuous $T = 0$ and therefore $(\mathcal{L}(E,F), \mathcal{T}_{\mathcal{B}})$ is Hausdorff.

□

Remark 2.26. Since in this thesis the attention is restricted to Hausdorff spaces, it will be assumed throughout that the considered bornologies are total.

Next we will state some basic facts about the interplay of polarization and taking conjugates of functions. Let E and F be lcs and E^* and F^* be the respective algebraic duals. For a function $T : E \to F$, $T^* : F^* \to E^*$ is defined by $y^* \mapsto y^* \circ T$ for $y^* \in F^*$. The function T^* is called algebraic adjoint of T. The next Lemma clarifies under what conditions T^* induces a map $T' : F' \to E'$. If such a map exists it is called the adjoint to T.

Lemma 2.27 (Proposition 8.6.1, Jarchow (1981))**.** T is $(\sigma(E,E'), \sigma(F,F'))$-continuous, iff $T^*(F') \subseteq E'$. In this case the induced map T' is $(\sigma(F',F), \sigma(E',E))$-continuous.

Proof. If $T^*(F') \subseteq E'$, then $x \mapsto \langle Tx, y'\rangle = \langle x, T^*y\rangle$ is a linear, weakly continuous function in on E. Consider a net $(x_\alpha)_{\alpha\in A} \subseteq E$ with $x_\alpha \xrightarrow{\sigma} x \in E$, i.e. $\langle x_\alpha, y\rangle \to \langle x, y\rangle$ for every $y \in E'$. Now fix an arbitrary $z \in F'$. Since $T^*(z) \in E'$ it follows that

$$\langle T(x_\alpha), z\rangle = \langle x_\alpha, T^*(z)\rangle \to \langle x, T^*(z)\rangle = \langle T(x), z\rangle.$$

Hence, it follows that T is weakly continuous. If on the other side T is weakly continuous, it can be shown in the same way that $T^*(F') \subseteq E'$.

Similarly, if $(z_\alpha)_{\alpha\in A}$ is a net with $z_\alpha \xrightarrow{\sigma} z \in F'$, then from $T^*(F') \subseteq E'$ we get

$$\langle T'(z_\alpha), x\rangle = \langle z_\alpha, T(x)\rangle \to \langle z, T(x)\rangle = \langle T'(z), x\rangle$$

for all $x \in E$. Therefore T' is weakly continuous. □

Lemma 2.28 (Proposition 8.6.2, Jarchow (1981))**.** Let E and F be two lcs and $T : E \to F$ a linear $(\sigma(E, E'), \sigma(F, F'))$-continuous map.

1. For $A \subseteq E$, $T(A)^\circ = (T')^{-1}(A^\circ)$.

2. For $A \subseteq E$ and $B \subseteq F$ with $B^{\circ\circ} = B$ the following equivalence holds:
$$T(A) \subseteq B \Leftrightarrow T'(B^\circ) \subseteq A^\circ.$$

3. $\ker(T') = \{x \in F' : T'(x) = 0\} = T(E)^\circ = (\mathrm{range}(T))^\circ$.

4. If T' is injective, then range(T) is $\sigma(F, F')$-dense in F.

Proof. **(1)**: $f \in T(A)^\circ \subseteq F' \Leftrightarrow |T'(f)(y)| = |f(T(y))| \leq 1,\ \forall y \in A \Leftrightarrow T'(f) \in A^\circ$.

(2), **(⇒)**: $x' \in T'(B^\circ) \Rightarrow\ \exists y' \in B^\circ \subseteq F' : T'(y') = x'$, hence $x'(z) = y'(Tz)$ for all $z \in E$. Since $T(z) \in B$ for all $z \in A$ and $y' \in B^\circ$, we have $|x'(z)| = |y'(Tz)| \leq 1$ for all $z \in A$, i.e. $x' \in A^\circ$.

(⇐): Suppose $T'(B^\circ) \subseteq A^\circ \Rightarrow\ (T'(B^\circ))^\circ \supseteq A^{\circ\circ} \supseteq A$. Therefore we know that if $z \in A$, then $|z(x')| \leq 1,\ \forall x' \in T'(B^\circ)$. This is the same as saying that
$$|y'(Tz)| \leq 1, \quad \forall y' \in B^\circ.$$
Hence, $T(z) \in B^{\circ\circ} = B$ for all $z \in A$.

(3): $x' \in \ker(T') \Leftrightarrow (T'(x'))(y) = x'(T(y)) = 0,\ \forall y \in E \Leftrightarrow x' \in \mathrm{range}(T)^\circ$.

(4): T' is injective, iff $\ker(T') = \{0\}$. By 3, this implies that range(T) is $\sigma(F, F')$-dense. Suppose not, then $\overline{\mathrm{range}(T)} \subset F$ is a closed subspace. Consequently there is a $l \in F'$ such that $l(x) = 0$ for $x \in \overline{\mathrm{range}(T)}$ and $l(y) = 1$ for some $y \in F \setminus \overline{\mathrm{range}(T)}$, by Hahn-Banach. Hence, $l \in \mathrm{range}(T)^\circ$ – a contradiction. □

Theorem 2.29 (Proposition 8.6.4, Jarchow (1981))**.** Let E and F be lcs and $T \in L(E, F)$ be $(\sigma(E, E'), \sigma(F, F'))$-continuous. If $\mathcal{B}$ and $\mathcal{C}$ are bornologies in E' and F' respectively and $T'(C) \in \mathcal{B}^{\mathrm{sat}}$ for all $C \in \mathcal{C}$, then T is $(\mathcal{T}_\mathcal{B}, \mathcal{T}_\mathcal{C})$ continuous.

Proof. We have to show that for all zero neighborhoods U in $(F, \mathcal{T}_\mathcal{C})$, there is a zero neighborhood V in $(E, \mathcal{T}_\mathcal{B})$, such that $T(V) \subseteq U$. By construction $U^\circ \in \mathcal{C}$ and therefore $T'(U^\circ) \in \mathcal{B}^{\mathrm{sat}}$ by assumption, i.e. there is a $B^{\circ\circ} = B \in \mathcal{B}$ such that $T'(U^\circ) \subseteq B$. By Lemma 2.28, this implies $T(B^\circ) \subset U$ and setting $V = B^\circ$ concludes the proof. □

Theorem 2.30 (Proposition 8.6.6, Jarchow (1981))**.** Let E and F be lcs and $T \in L(E, F)$. If T is $(\sigma(E, E'), \sigma(F, F'))$-continuous, then it is also $(\beta(E, E'), \beta(F, F'))$-continuous.

Proof. Since T is weakly continuous, we know from Lemma 2.27, that $T' : F' \to E'$ is weakly continuous and therefore weakly bounded, i.e. for every $B \subseteq F'$ weakly bounded also $T'(B) \subseteq E'$ is weakly bounded. Since $\beta(E, E')$ and $\beta(F, F')$ are the topologies of uniform convergence on these sets, the statement follows from Theorem 2.29. □

Definition 2.31 (Reflexive, semi-reflexive)**.** Every lcs $(E, \mathcal{T})$ can be embedded into its bidual $E'' = (E', \beta(E', E))'$ by means of the evaluation map. If $E = E''$ as vector spaces, then the space E is called *semi-reflexive.* If additionally $(E, \mathcal{T}) = (E'', \beta(E'', E'))$, then the space is called *reflexive.*

Theorem 2.32 (Lemma 23.18, Meise and Vogt (1997))**.** A lcs E is semi-reflexive, iff every bounded set in E is relatively weakly compact.

Proof. By the Mackey-Arens Theorem E is semi-reflexive iff $\beta(E', E) \leq \mu(E', E)$. This is the case iff for each bounded set $B \subseteq E$ there is a absolutely convex, weakly compact set $M \subseteq E$ such that $M^\circ \subseteq B^\circ$, which in turn implies $B \subseteq B^{\circ\circ} \subseteq M^{\circ\circ} = M$ and therefore B is relatively weakly compact. □

In the next two Theorems the topologies on the polars of zero neighborhoods in E are studied.

Theorem 2.33 (Lemma 24.20, Meise and Vogt (1997))**.** For every lcs and each zero neighborhood U in E, $\tau_c(E', E)_{|U^\circ} = \sigma(E', E)_{|U^\circ}$ holds.

Proof. Obviously, $\tau_c(E', E)_{|U^\circ} \geq \sigma(E', E)_{|U^\circ}$. Let $K \subseteq E$ be absolutely convex and compact and let $p_K(y) = \sup_{x \in K} |y(x)|$ be a seminorm in $\tau_c(E', E)$. Then by compactness $K \subseteq \bigcup_{j=1}^n (x_j + \frac{\epsilon}{3} U)$ for some $M = \{x_1, \ldots, x_n\}$. Hence, for every $x \in K$ there is a j_x such that $x \in x_{j_x} + \frac{\epsilon}{3} U$. Choose $y_0 \in U^\circ$ fixed and $y \in U^\circ$ with $p_M(y - y_0) \leq \frac{\epsilon}{3}$, then for all $x \in K$

$$\begin{aligned} |(y - y_0)(x)| &\leq |(y - y_0)(x_{j_x})| + |(y - y_0)(x_{j_x} - x)| \\ &\leq |(y - y_0)(x_{j_x})| + |y(x_{j_x} - x)| + |y_0(x_{j_x} - x)| \\ &\leq p_M(y - y_0) + \frac{\epsilon}{3} + \frac{\epsilon}{3} \leq \epsilon. \end{aligned}$$

Hence, from $p_M(y - y_0) \leq \frac{\epsilon}{3}$ it follows that $p_K(y - y_0) \leq \epsilon$ and therefore $3p_M \geq p_K$, which proves the statement. □

Theorem 2.34 (Lemma 24.21, Meise and Vogt (1997))**.** For a complete lcs $(E, \mathcal{T})$, $\tau_c(E', E)$ is the finest locally convex topology on E' which coincides with $\sigma(E', E)$ on U° for every zero neighborhood U of E.

Proof. By Theorem 2.33 $\tau_c(E', E)$ has this property. If $\mathcal{T}' \geq \sigma(E', E)$ is another locally convex topology on E' which has this property and $z \in (E', \mathcal{T}')'$, then z is γ-continuous by Lemma 3.6. Since E is complete we know from Theorem 3.8 that there is a $x \in E$, such that $z(y) = y(x)$ for all $y \in E'$. Hence, $\mathcal{T}'$ is an admissible topology and $\mathcal{T}' = \mathcal{T}_{\mathcal{M}}$ for $\mathcal{M} = \{U^\circ : U$ zero neighborhood in $(E', \mathcal{T}')\}$ and it is sufficient to show that every $M \in \mathcal{M}$ is compact in E.

Let $M = U^\circ$, then since $(E', \mathcal{T}')' = E'$ and by Theorem 2.33, we have

$$\sigma(E, E')_{|M} = \sigma((E', \mathcal{T}')', E')_{|M} = \tau_c((E', \mathcal{T}')', E')_{|M}.$$

If V is a zero neighborhood in E, then by Alaoğlu-Bourbaki, V° is $\sigma(E', E)$-compact and since $\mathcal{T}'_{|V^\circ} = \sigma(E', E)_{|V^\circ}$, V° is $\mathcal{T}'$-compact. Therefore, $V^{\circ\circ}$ is a $\tau_c((E', \mathcal{T}')', E')$ zero neighborhood and we have

$$\sigma(E, E') \leq \mathcal{T} \leq \tau_c((E', \tau)', E').$$

It follows that $\sigma(E, E')$ and $\mathcal{T}$ coincide on M. Hence, since M is $\sigma(E, E')$-compact, M is $\mathcal{T}$-compact in E, which completes the proof. □

Definition 2.35 (Fundamental sequence)**.** A countable basis $\mathcal{B}_0 = (B_n)_{n \in \mathbb{N}}$ of a bornology $\mathcal{B}$ is called a fundamental sequence. Without loss of generality, we can assume that B_n are absolutely convex and $B_n + B_n \subseteq B_{n+1}$, $\forall n \in \mathbb{N}$.

The notion of a fundamental sequence of bounded sets will play a central role when studying (df) spaces. Note that if a lcs E has a bornology which admits a fundamental sequence of bounded sets, then the strong dual $(E', \beta(E', E))$ has a basis of countably many seminorms and is therefore a metric space.

Theorem 2.36 (Theorem 8.5.1, Jarchow (1981))**.** Let E and F be two lcs. If $H \subseteq \mathcal{L}(E, F)$ is equicontinuous, then it is $\mathcal{T}_\beta$-bounded, and its closure in the product space F^E is contained in $\mathcal{L}(E, F)$ and is also equicontinuous.

Proof. Fix an arbitrary bounded set $B \subseteq E$ as well as a zero neighborhood $V \subseteq F$. Since H is equicontinuous, there is a zero neighborhood $U \subseteq E$ such that $T(U) \subseteq V$ for all $T \in H$. Since B is bounded there is a $\rho > 0$ with $B \subseteq \rho U$ and therefore

$$\sup_{x \in B} p_V(T(x)) \leq \sup_{x \in \rho U} p_V(T(x)) \leq \rho$$

and therefore H is $\mathcal{T}_\beta$ bounded.

Let next $\bar{H}$ be the closure of H in the product space F^E endowed with the product topology (i.e. the topology of pointwise convergence $\mathcal{T}_s$). Let $(S_\alpha)_\alpha \subseteq H$ be a net converging to $S \in \bar{H}$ with respect to $\mathcal{T}_s$. From the linearity of the limit and the functions

S_α, it follows that S is a linear function. Let $V \subseteq F$ be a closed zero neighborhood and choose $U \subseteq E$ to be a neighborhood in E such that $T(U) \subseteq V$ for all $T \in H$. Since $\delta_x : F^E \to F$, with $\delta_x(f) = f(x)$, is continuous, it follows that $\delta_x(\bar{H}) \subseteq V$ for all $x \in U$ (since V is closed). Since this implies $T(U) \subseteq V$ for all $T \in \bar{H}$, we have that $\bar{H} \in \mathcal{L}(E, F)$ and that $\bar{H}$ is equicontinuous. □

2.4 Local Convergence

Definition 2.37 (Disk)**.** Let E and F be paired lcs. A set $B \subseteq F$ is called a *disk*, if it is $\sigma(F, E)$-bounded and absolutely convex. Given a disk B, one can define the normed space $E_B = \operatorname{span}(B)$ with the norm $||x||_B = \inf\{\rho > 0 : x \in \rho B\}$. If this normed space is additionally complete, then B is called a *Banach-Disk*.

Definition 2.38 (Local convergence, local completeness)**.** Let $\mathcal{B}$ be a bornology on a lcs F which has a basis consisting of closed disks. A sequence is called $\mathcal{B}$ locally convergent, if it converges in the normed space F_B for some $B \in \mathcal{B}$. A space is locally complete if every $\mathcal{B}$-local Cauchy sequence is also $\mathcal{B}$-locally convergent, for $\mathcal{B}$ the bornology of bounded sets.

Theorem 2.39 (10.2.4, Jarchow (1981))**.** Let $\langle E, F\rangle$ be a dual pairing and $\mathcal{T}$ a topology on E compatible with this pairing. Then the following statements are equivalent:

1. $(E, \mathcal{T})$ is locally complete.
2. For some (each) $1 \leq p \leq \infty$, we have the following implication: If $B \subseteq E$ is a disk and $(x_n)_{n\in\mathbb{N}}$ is a sequence in E_B, such that $(||x_n||_B) \in \ell_p$ ($\in c_0$ if $p = \infty$), then $\overline{\text{acx}}\,\{x_n : n \in \mathbb{N}\}$ is compact in E_C, for some disk $C \subseteq E$.
3. For some (each) $1 \leq p \leq \infty$, we have the following implication: If $B \subseteq E$ is a disk and $(x_n)_{n\in\mathbb{N}}$ is a sequence in E_B, such that $(||x_n||_B) \in \ell_p$ ($\in c_0$ if $p = \infty$), then $\overline{\text{acx}}\,\{x_n : n \in \mathbb{N}\}$ is compact in $(E, \mathcal{T})$.
4. For some (each) $1 \leq p \leq \infty$, we have the following implication: If $(x_n)_{n\in\mathbb{N}}$ is a sequence in E such that $(\langle y, x_n\rangle)_{n\in\mathbb{N}} \in \ell_p$ ($\in c_0$ if $p = \infty$) holds for every $y \in F$, then $\overline{\text{acx}}\,\{x_n : n \in \mathbb{N}\}$ is compact in $(E, \sigma(E, F))$.

Proof. **(1) ⇒ (2):** Under the conditions of (2), $||x_n||_B \to 0$. Since E_B is a complete normed space by (1), we conclude that $\overline{\text{acx}}\,\{x_n : n \in \mathbb{N}\}$ is compact in E_B by Lemma 2.15.

(2) ⇒ (3): Since the canonical mapping $(E_C, ||.||_C) \hookrightarrow (E, \mathcal{T})$ is continuous, $\overline{\text{acx}}\,\{x_n : n \in \mathbb{N}\}$ is also compact in $(E, \mathcal{T})$.

(4) $\Rightarrow$ (3): Let the hypothesis of (3) hold, i.e. $(||x_n||_B)_{n\in\mathbb{N}} = (\lambda_n)_{n\in\mathbb{N}} \in l_p$ (or c_0 for $p=\infty$) for some disk B. Since B is $\sigma(E,F)$-bounded, for every $y \in F$ there is a $C_y > 0$ such that $|B(y)| = \sup_{x\in B} |\langle x, y\rangle| \leq C_y$. Hence, we have

$$\langle x_n, y\rangle \leq |(\lambda_n B)(y)| = \lambda_n(B(y)) \leq \lambda_n C_y,$$

i.e. $\langle x_n, y\rangle \in l_p$ (or c_0 for $p=\infty$). By (4) it follows that

$$C = \overline{\operatorname{acx}\{x_n : n\in\mathbb{N}\}}^{\sigma}$$

is compact in $\sigma(E,F)$ and therefore $\sigma(E,F)$-complete. It follows from Theorem 2.7 that C is also $\mathcal{T}$-complete. Hence, by Theorem 2.13, we have that

$$\operatorname{acx}(\{x_n : n\in\mathbb{N}\}\cup\{0\})$$

is precompact and therefore, by Lemma 2.11, relatively compact in C. It follows that

$$\overline{\operatorname{acx}\{x_n : n\in\mathbb{N}\}}^{\mathcal{T}} \subseteq C$$

is compact.

(1) $\Rightarrow$ (4): Under the hypothesis of (4), there is a null sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ such that $|\lambda_n^{-1}\langle x_n, y\rangle|$ is bounded. Hence, $(\lambda_n^{-1}x_n)_{n\in\mathbb{N}}$ is weakly bounded and

$$B = \overline{\operatorname{acx}}\left\{(\lambda_n^{-1}x_n)_{n\in\mathbb{N}} : n\in\mathbb{N}\right\}$$

is a bounded set. From $||\lambda_n^{-1}x_n||_B \leq 1$, it follows that $||x_n||_B \leq \lambda_n \to 0$, hence $(x_n)_{n\in\mathbb{N}}$ is a null sequence in E_B. Since E_B is a Banach space $\overline{\operatorname{acx}}\{(x_n)_{n\in\mathbb{N}} : n\in\mathbb{N}\}$ is compact in E_B and since the embedding $E_B \to E$ is continuous also in E.

(3) $\Rightarrow$ (1): Suppose there is a closed Banach disk, such that E_B is not complete. Choose $x \in \tilde{E}_B \setminus E_B$ and iteratively construct a sequence $(z_n)_{n\in\mathbb{N}}$ such that

$$||x - \sum_{i=1}^{n} z_i||_B \leq ((n+1)2^{n+1})^{-1}.$$

This is possible since E_B is dense in $\tilde{E}_B$. Now choose $w_n = 2^n z_n$ and observe that

$$||w_n||_B \leq 2^n\left(||x - \sum_{i=1}^{n} z_i||_B - ||x - \sum_{i=1}^{n-1} z_i||_B\right) \leq 2n^{-1}.$$

Hence, $(||w_n||_B)_{n\in\mathbb{N}} \in l^1$ and $x = \sum_{i=1}^{\infty} 2^{-n}w_n$, in the Banach space $\tilde{E}_B$. Consider the initial topology $\mathcal{T}_B$ with respect to the embedding $\iota : E_B \hookrightarrow E$ (i.e. the topology of the subspace). Since B is closed in E, it is closed in $(E_B, ||.||_B)$ as well as in the subspace topology and therefore $(E_B, ||.||_B)$ has a basis of $\mathcal{T}_B$-closed sets. Since ι is injective, also the completion $\tilde{\iota} : \tilde{E}_B \hookrightarrow \tilde{E}$ is injective by Theorem 2.8. Hence, x can be associated with an unique element in $\tilde{E}$, which we will call x as well. However, by (3) we have that $\overline{\text{acx}}\,\{w_n : n \in \mathbb{N}\}$ is compact in $(E, \mathcal{T})$ and therefore $x \in \overline{\text{acx}}\,\{w_n : n \in \mathbb{N}\} \in E$. Hence, $x \in E_B$ which leads to a contradiction. □

Theorem 2.40 (10.1.4, Jarchow (1981))**.** In a metrizable lcs E, local convergence and convergence are equivalent for sequences.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ converge locally, i.e. $||x_n||_B \to 0$ for some $B \subseteq E$ bounded. Since the embedding $\iota : E_B \to E$ is continuous, $(x_n)_{n\in\mathbb{N}}$ also converges in E.

Conversely let $(x_n)_{n\in\mathbb{N}}$ converge to zero in E. Since E is assumed to be a metrizable space, we know that there is a null sequence $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ such that $x_n\rho_n^{-1}$ is bounded. Hence there is a B such that $(x_n\rho_n^{-1}) \subseteq B$. Therefore

$$||x_n||_B \leq \rho_n \to 0$$

and $(x_n)_{n\in\mathbb{N}}$ converges locally. □

Definition 2.41. Given a bornology $\mathcal{B}$ in E, which has a basis of closed disks, define the system of sets $\mathcal{B}_0$ as bipolars of $\mathcal{B}$ null sequences.

Lemma 2.42. Let $\mathcal{B}$ be a bornology which has a basis of closed disks, then

1. convergence with respect to $\mathcal{B}$ and $\mathcal{B}^{\text{sat}}$ are the same;

2. $\mathcal{B}_0$ is a bornology on E and $\mathcal{B}_0 \subseteq \mathcal{B}^{\text{sat}}$;

Proof. **(1)**: Since $\mathcal{B} \subseteq \mathcal{B}^{\text{sat}}$ a sequence which $\mathcal{B}$-converges also converges with respect to $\mathcal{B}^{\text{sat}}$. If on the other hand $x_n \xrightarrow{\mathcal{B}^{\text{sat}}} 0$, then there is a $B \in \mathcal{B}^{\text{sat}}$ such that $||x_n||_B \to 0$ and by assumption there is a D such that $B \subseteq D = D^{\circ\circ} \in \mathcal{B}$. Hence, $||x_n||_D \to 0$ and therefore $(x_n)_{n\in\mathbb{N}}$ $\mathcal{B}$-converges.

(2): Let C_1, C_2 be two disks. If $B_1 = \{x_n : n \in \mathbb{N}\}^{\circ\circ}$ and $B_2 = \{y_n : n \in \mathbb{N}\}^{\circ\circ}$ with $x_n \xrightarrow{C_1} 0$ and $y_n \xrightarrow{C_2} 0$. Define $z = (x_1, y_1, x_2, y_2, \ldots)$ and notice that $z_n \xrightarrow{C_1 \cup C_2} 0$. Hence, $B_3 = \{z_n : n \in \mathbb{N}\}^{\circ\circ} \in \mathcal{B}_0$ and $B_1 \cup B_2 \subseteq B_3$. Further it is clear that $\rho B_1 = \{\rho x_n : n \in \mathbb{N}\}^{\circ\circ} \in \mathcal{B}$ for $\rho \in \mathbb{K}$, since $\{\rho x_1, \rho x_2, \ldots\}$ is a local null sequence. This proves that $\mathcal{B}_0$ is a bornology.

Since $\mathcal{B}$ has a basis consisting of closed disks, for every null sequence $(x_n)_{n\in\mathbb{N}}$ there is a $B \in \mathcal{B}$ such that $B^{\circ\circ} = B$ and $(x_n)_{n\in\mathbb{N}} \subseteq B$. We then have $\{x_n : n \in \mathbb{N}\}^{\circ\circ} \subseteq B^{\circ\circ} = B \in \mathcal{B}$ and it follows that $\mathcal{B}_0 \subseteq \mathcal{B}^{\text{sat}}$. □

Theorem 2.43 (10.1.2, Jarchow (1981))**.** In the setting of the above Lemma, the $\mathcal{B}$-convergent sequences and the $\mathcal{B}_0$-convergent sequences coincide.

Proof. Since $\mathcal{B}_0 \subseteq \mathcal{B}^{\text{sat}}$, and $\mathcal{B}$ and $\mathcal{B}^{\text{sat}}$-convergence are the same, every $\mathcal{B}_0$ convergent sequence is $\mathcal{B}$-convergent. Conversely if $x_n \xrightarrow{\mathcal{B}} 0$, then there is a sequence $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ such that $\rho_n \to 0$ and $(x_n\rho_n^{-1})_{n\in\mathbb{N}}$ is still a $\mathcal{B}$-null sequence. Now let $A = \{x_n\rho_n^{-1} : n \in \mathbb{N}\}^{\circ\circ} \in \mathcal{B}_0$, then $||x_n\rho_n^{-1}||_A \leq 1$ and therefore $||x_n||_A \leq \rho_n \to 0$. Hence $(x_n)_{n\in\mathbb{N}}$ $\mathcal{B}_0$-converges. □

We will conclude this section by showing that for a locally complete space the topologies $\beta(E', E)$ and $\beta^*(E, E')$ are identical. We start with the following

Theorem 2.44 (Proposition 10.2.1, Jarchow (1981))**.** If $(E, \mathcal{T})$ is locally complete, then

1. every closed disk in E is a Banach disk;

2. $\mathfrak{b}(E', E) = \beta(E', E)$.

Proof. **(1)**: By the definition of local completeness, we know that every $\sigma(E, E')$ closed disk is a Banach disk. Since a disk is convex, $\sigma(E, E')$-closedness and $\mathcal{T}$-closedness are the same and the result follows.

(2): Since the bornology of weakly bounded sets has a basis of closed disks, $\beta(E', E)$ is the convergence on closed disks in E. Since $\mathfrak{b}(E', E)$ is the convergence on Banach disks, we have $\mathfrak{b}(E', E) \leq \beta(E', E)$. By (1) the result follows. □

Theorem 2.45 (Proposition 10.2.2, Jarchow (1981))**.** If E is locally complete, then $\beta^*(E', E) = \beta(E', E)$.

Proof. By the Banach-Mackey Theorem (cf. Theorem 7.4.18, Kriegl (2007)) every barrel absorbs every Banach disk. Let $\mathcal{B}$ be the bornology of $\sigma(E', E)$-bounded sets, then $\{B^\circ : B \in \mathcal{B}\}$ forms a zero neighborhood basis of $\beta(E, E')$ consisting of barrels. Hence, every Banach disk $B \subseteq E$ is bounded in $\beta(E, E')$ and $\mathfrak{b}(E', E) \leq \beta^*(E', E)$. We therefore have

$$\mathfrak{b}(E', E) \leq \beta^*(E', E) \leq \beta(E', E).$$

By local completeness, we know from the last theorem that $\mathfrak{b}(E', E) = \beta(E', E)$ and therefore $\beta^*(E', E) = \beta(E', E)$. □

2.5 Barrelledness

Let me start with the definition of a barrel and a barrelled space.

Definition 2.46 (Barrel, barrelled spaces)**.** A set A which is absolutely convex, absorbing and closed in a lcs E is called a *barrel*. If A is bornivorous instead of merely absorbing, then A is called *bornivorous barrel*. A lcs E in which every barrel is a zero neighborhood is called a *barrelled space*, while a space in which every bornivorous barrel is zero neighborhood is called *quasi-barrelled* space.

The concept of barrelledness is important, because it provides a characterization of the spaces in which the uniform boundedness principle holds: if every barrel is a zero neighborhood, then pointwise boundedness of a set of linear functionals implies uniform boundedness and equicontinuity. Furthermore it is straightforward that the strong topology has a neighborhood basis consisting of barrels (see Lemma 2.47) relating the concept of barrelledness to reflexivity of a space: If an admissible lcs is barrelled, then its topology is finer than the strong topology, which therefore is also admissible. If the space is additionally semi-reflexive, then reflexivity follows (see Theorem 2.50).

In this chapter, I will discuss various concept of barrelledness which are weaker than the above definition. In particular, we will focus on different notions sequential barrelledness, which will become important in the characterizations of (df) spaces in Chapter 4. The idea of sequential barrelledness is to define spaces in which weaker versions of the uniform boundedness principle hold: For a barrelled space E every $\sigma(E', E)$-bounded set A is equicontinuous. In a sequentially barrelled space, this statement is weakened and narrowed down to sets A which are either bounded or convergent sequences in $\sigma(E', E)$ or $\beta(E', E)$.

I will start by examining some results on the connection between barrelledness and reflexivity.

Lemma 2.47. Let E be a lcs.

1. $M \subseteq E'$ is $\sigma(E', E)$-bounded iff M° is a barrel in E. This implies that the barrels in E are a zero neighborhood basis for $\beta(E, E')$ and $\beta(E, E')$ therefore is a barrelled space.

2. $M \subseteq E'$ is $\beta(E', E)$-bounded iff M° is a bornivorous barrel in E. The bornivorous barrels are thus a zero neighborhood basis for $\beta^*(E, E')$ and $\beta^*(E, E')$ therefore is an quasi-barrelled space.

Proof. **(1)**: M is $\sigma(E', E)$-bounded iff for every $x \in E$ there is a $\lambda > 0$ such that $|M(x)| = \lambda < \infty$. This is equivalent to $x \in \lambda M^\circ$, i.e. the fact that M° is absorbing.

(2): M is $\beta(E', E)$-bounded iff for every absolutely convex, closed and bounded set $B \subseteq E$ there is a $\lambda > 0$ such that $M \subseteq \lambda B^\circ$ and therefore $\lambda M^\circ \supseteq B$. Which in turn is equivalent to saying that M° is a bornivorous barrel in E. $\square$

Lemma 2.48 (Lemma 23.19, Meise and Vogt (1997))**.** For every lcs E it holds that $\beta(E, E') \geq \beta^*(E, E') \geq \mu(E, E')$.

Proof. To show $\beta^*(E, E') \geq \mu(E, E')$ it is enough to show that every weakly compact set in E' is also bounded in $\beta(E', E)$. By Theorem 7.4.17 in Kriegl (2007), we have that every absolutely convex $\sigma(E', E)$-compact set is a Banach disk. Since $\beta(E', E)$ has a basis consisting of barrells (see the Lemma 2.47), $\beta^*(E.E') \geq \mu(E, E')$ follows by the Banach-Mackey Theorem. It follows that $\beta(E, E') \geq \beta^*(E, E')$, since every $\beta(E', E)$ bounded set is also $\sigma(E', E)$-bounded. □

Lemma 2.49 (Lemma 23.20, Meise and Vogt (1997))**.** Let E is a quasi-barrelled lcs. If every closed disk in E is a Banach disk, then E is barrelled.

Proof. We show that every barrel is a bornivorous barrel. Since the barrels in E form a zero neighborhood basis for $\beta(E, E')$ by Lemma 2.47, every barrel absorbs every $\beta(E, E')$-bounded set. Therefore we only have to show that every bounded set $B \subseteq E$ is already $\beta(E, E')$-bounded. The set $\overline{\text{acx}}(B)$ is again bounded and by hypothesis a Banach disk. From the Banach-Mackey Theorem it follows that $\overline{\text{acx}}(B)$ and therefore B is $\beta(E, E')$-bounded. □

Theorem 2.50 (Proposition 23.22, Meise and Vogt (1997))**.** For a lcs $(E, \mathcal{T})$ the following statements are equivalent

1. $(E, \mathcal{T})$ is reflexive.
2. $(E, \mathcal{T})$ is semi-reflexive and quasi-barrelled.
3. $(E, \mathcal{T})$ is semi-reflexive and barrelled.

Proof. **(1)** $\Rightarrow$**(2)**: If $(E, \mathcal{T})$ is reflexive, it is semi-reflexive and $\mathcal{T} = \beta^*(E, E')$. It follows from Lemma 2.47 that $(E, \mathcal{T})$ is quasi-barrelled.

(2) $\Rightarrow$ **(1)**: If U is a zero neighborhood in $\beta^*(E, E')$, then by Lemma 2.47 there is a bornivorous barrel B such that $B \subseteq U$ and it follows that $\beta^*(E, E') \leq \mathcal{T}$ (since $(E, \mathcal{T})$ is quasi-barrelled and therefore B is a zero neighborhood in $\mathcal{T}$). By the Lemma 2.48 we get $\mathcal{T} = \beta^*(E, E')$. From this and the fact that E is assumed to be semi-reflexive the result follows.

(2) $\Leftrightarrow$ **(3)**: By Theorem 2.32 absolutely convex, closed and bounded sets are weakly compact and therefore Banach Disks by 7.4.17 in Kriegl (2007). By Lemma 2.49 it follows that E is barrelled. □

Lemma 2.51. If E and F are two paired lcs, then $(F^*, \sigma(F^*, F))$ is the completion of $(E, \sigma(E, F))$, where F^* is the algebraic dual of F.

Proof. Notice that $(F^*, \sigma(F^*, F)) \subseteq \prod_{f \in F} \mathbb{K}$ (closed subspace) and consider the following diagram

$$\begin{tikzcd}
G & & \\
(E, \sigma(E, F)) \arrow[u, "T"] \arrow[r, "\iota"] \arrow[rd, dashed, "pr_f \circ \iota"'] & (F^*, \sigma(F^*, F)) \subseteq \prod_{f \in F} \mathbb{K} \arrow[lu, dashed, "\tilde{T}"'] \arrow[d, "pr_f"] \\
& \mathbb{K}
\end{tikzcd}$$

where $\iota : E \to F^*$ is the canonical mapping given by $\iota(x) = (f \mapsto \langle x, f\rangle)$, G is an arbitrary lcs and $T : E \to G$ is linear and continuous. The composition $(x \mapsto pr_f \circ \iota(x)) = (x \mapsto \langle x, f\rangle)$ is continuous, since E and F are paired. Hence, by the definition of the product topology as initial topology with respect to the projections, $\iota : (E, \sigma(E, F)) \to (F^*, \sigma(F^*, F))$ is continuous. The mapping $\tilde{T}$ is well defined since ι is injective (because the dual pairing is point separating). By the theorem of Hahn-Banach, each linear functional $f' \in F^*$ can be approximated exactly on finitely many points (i.e. in $\sigma(F^*, F)$) by a continuous functional $\iota(e)$ with $e \in E$ and therefore $\iota(E)$ is dense in $(F^*, \sigma(F^*, F))$ and $\tilde{T}$ is unique. Since $\prod_{f \in F} \mathbb{K}$ is complete and because of the continuity of ι as well as $\tilde{T}$ it follows that $(F^*, \sigma(F^*, F))$ is a completion of $(E, \sigma(E, F))$. □

Lemma 2.52 (Corollary 8.1.6, Jarchow (1981))**.** Let E be a lcs, then every bounded subset of $(E, \sigma(E, F))$ is precompact.

Proof. Consider the setting of Lemma 2.51. If $B \subseteq E$ is $\sigma(E, F)$-bounded, then $\iota(B)$ is bounded in $(F^*, \sigma(F^*, F))$. Therefore $\overline{\mathrm{pr}_f(\iota(B))}$ is compact in $\mathbb{K}$ and consequently $\iota(B)$ is compact in $\prod_{f \in F} \mathbb{K}$ and therefore also in $(F^*, \sigma(F^*, F))$. By Lemma 2.51 and Lemma 2.11 the result follows. □

Definition 2.53 (Quasi-complete)**.** A lcs E is said to be quasi-complete, if every closed bounded set is complete.

Theorem 2.54 (Proposition 11.4.1, Jarchow (1981))**.** In a lcs E the following statements are equivalent:

1. E is semi-reflexive.

2. $(E', \mu(E', E))$ is barrelled, i.e. $\mu(E', E) = \beta(E', E)$.

3. $(E, \sigma(E, E'))$ is quasi-complete.

Proof. **(1) ⇔ (2)**: E is semi-reflexive, iff $\beta(E', E) \leq \mu(E', E)$ and therefore $\beta(E', E) = \mu(E', E)$. By Lemma 2.47 the barrels form a zero neighborhood basis of $\beta(E', E)$ and therefore $\mu(E', E)$ is a barrelled space iff $\beta(E', E) = \mu(E', E)$.

(2) ⇒ (3): If $B \subseteq E$ is $\sigma(E, E')$-bounded, closed and absolutely convex, then B° is a barrel in E' by Lemma 2.47 and therefore a zero neighborhood in $\mu(E', E)$. Hence, by Alaoğlu-Bourbaki $B^{\circ\circ} = B$ is compact and therefore complete in $\sigma(E, E')$.

(3) ⇒ (2): If $(E, \sigma(E, E'))$ is quasi-complete, then absolutely convex, $\sigma(E, E')$-closed sets are compact by Lemma 2.52. This in turn implies $\beta(E', E) = \mu(E', E)$ which concludes the proof. □

Theorem 2.55. Every bornological space is quasi-barrelled.

Proof. First note that E bornological iff every seminorm p with $\sup_{x \in B} p(x) < \infty$ for a bounded set $B \subseteq E$ is continuous. (⇐) is trivial, since $|l(x)|$ is such a seminorm for all bounded linear functionals l. (⇒) follows from the fact that if E bornological, it is the final structure with respect to the mappings $j_B : E_B \to E$ with $B \subseteq E$ bounded. Let p be a seminorm that fulfils the above condition, then $p \circ j_B : E_B \to \mathbb{K}$ is bounded and therefore continuous for all bounded sets $B \subseteq E$ (see Kriegl (2007), 4.6.2). Hence, p is continuous by definition of the final topology.

To complete the proof note that the above property is equivalent to the statement that every absolutely convex, bornivorous set is a zero neighborhood. □

Corollary 2.56. Every metrizable space is quasi-barrelled.

Theorem 2.57 (Proposition 11.2.5, Jarchow (1981))**.** If E is quasi-barrelled and locally complete, then it is already barrelled.

Proof. If E is locally complete, we know from Theorem 2.45 that $\beta(E', E)^* = \beta(E', E)$ and thus that every barrel is a bornivorous barrel. Hence, since E is quasi-barrelled it is also barrelled. □

Now we turn to the slightly weaker concept of sequential barrelledness. To see the connection between the two definitions note that barrelledness can also be defined as the property that certain bounded sets in the dual space E' (namely the sets A° for barrells $A \subseteq E$) are equicontinuous. We now weaken this definition by restricting these bounded sets to certain sequences in the dual space, thus effectively generating topologies which admit fewer equicontinuous sets and are therefore coarser. The concepts of sequential barrelledness will be instrumental in the discussion of (df)-spaces: it turns out that sequential barrelledness is the correct concept to characterize space for which the strong dual is complete.

Definition 2.58 (ℓ_∞-barrelled, c_0-barrelled)**.** A lcs E is ℓ_∞-barrelled if every bounded sequence in $(E', \sigma(E', E))$ is equicontinuous, it is c_0-barrelled if every null sequence in $(E', \sigma(E', E))$ is equicontinuous. If the space $(E', \sigma(E', E))$ in the preceding definition is replaced by $(E', \beta(E', E))$, then the spaces are called quasi-ℓ_∞-barrelled and quasi-c_0-barrelled.

Denote by $\mathcal{B}_{\sigma,0}$ the bornology on E' generated by the $\sigma(E', E)$ null sequences and $\mathcal{B}_{\sigma,\infty}$ the bornologies generated by the $\sigma(E', E)$-bounded sequences in E'. Define the bornologies $\mathcal{B}_{\beta,0}$ and $\mathcal{B}_{\beta,\infty}$ correspondingly. Let further $\tau_{\sigma,0}$, $\tau_{\sigma,\infty}$, $\tau_{\beta,0}$ and $\tau_{\beta,\infty}$ the topologies of uniform convergence on the respective bornologies on E.

Remark 2.59. All null sequences (bounded sequences) in $(E, \beta(E', E))$ are null sequences (bounded sequences) in $(E', \sigma(E', E))$. Therefore we have $\mathcal{B}_{\beta,0} \subseteq \mathcal{B}_{\sigma,0}$ and $\mathcal{B}_{\beta,\infty} \subseteq \mathcal{B}_{\sigma,\infty}$. Furthermore it is obvious that $\mathcal{B}_{\sigma,0} \subseteq \mathcal{B}_{\sigma,\infty}$ and $\mathcal{B}_{\beta,0} \subseteq \mathcal{B}_{\beta,\infty}$, therefore

$$\mu(E, E')_{c_0} \le \tau_{\beta,0} \le \left\{ \begin{array}{c} \tau_{\sigma,0} \\ \tau_{\beta,\infty} \end{array} \right\} \le \tau_{\sigma,\infty} \le \beta(E, E').$$

The topology $\mu(E, E')_{c_0}$ is the finest admissible Schwartz topology and will be defined in the section on Schwartz spaces. However, to show the first inequality above, we just have to know that $\mu(E', E)_{c_0}$ is the topology of uniform convergence on the sets $\{a_n : n \in \mathbb{N}\} \subseteq E'$ where $(a_n)_{n\in\mathbb{N}}$ is locally convergent with respect to the bornology induced by the sets U° with U a zero neighborhood in $\mu(E, E')$. If $(a_n)_{n\in\mathbb{N}}$ is such a sequence, then it converges uniformly on U, i.e. for each $\epsilon > 0$ there is a $N(\epsilon) \in \mathbb{N}$ such that $|a_n(x)| \le \epsilon$ for all $n \ge N(\epsilon)$ and all $x \in U$. Since U is a zero neighborhood, we know that for every bounded set $B \subseteq E$ there is a $\lambda_B > 0$ such that $B \subseteq \lambda_B U$. It follows that $|a_n(x)| \le \epsilon$ for all $n \ge N(\frac{\epsilon}{\lambda_B})$ and for all $x \in \lambda_B U \supseteq B$. Hence, a_n converges uniformly on each $B \subseteq E$ bounded and therefore in $\beta(E', E)$. We have thus shown that $(a_n)_{n\in\mathbb{N}}$ is also a null sequence in $\beta(E', E)$ and therefore $\mu(E', E)_{c_0} \le \tau_{\beta,0}$.

Remark 2.60. The topologies $\tau_{\sigma,0}$, $\tau_{\sigma,\infty}$, $\tau_{\beta,0}$ and $\tau_{\beta,\infty}$ are the weakest topologies such that the space has the respective sequential barrelledness property. This is immediately clear, since for every other topology $\mathcal{T}$ which has the property that the respective sets are equicontinuous, the polars of the sets are also zero neighborhoods and therefore $\mathcal{T}$ is finer than the respective topologies.

Theorem 2.61 (Proposition 12.1.1, Jarchow (1981))**.** The topologies $\tau_{\sigma,\infty}$ and $\tau_{\beta,\infty}$ on E are the topologies of uniform convergence on the bounded and separable sets in $(E', \sigma(E', E))$ and $(E', \beta(E', E))$ respectively.

Proof. For every bounded separable set $X \subseteq E'$ a bounded dense sequence $(y_n)_{n\in\mathbb{N}} \subseteq X$ can be found. Consequently

$$\sup_{y\in X} y(x) = \sup_{n\in\mathbb{N}} y_n(x), \quad \forall x \in X$$

and the corresponding seminorms coincide. $\square$

Theorem 2.62 (Proposition 12.1.2, Jarchow (1981))**.** Let E and F be lcs and $T : F \to E$ be linear and continuous. If F is separable and E is (quasi-)ℓ_∞-barrelled, then T is continuous as a map from F to $(E, \beta(E, E'))$ $((E, \beta^*(E, E'))$ respectively).

Proof. Let U be a (bornivorous) barrel in E. Since $F \setminus T^{-1}(U)$ is open it is also separable. Fix a countable dense subset $(y_n)_{n\in\mathbb{N}} \subseteq F \setminus T^{-1}(U)$. Since $T(y_n) \notin U$, $a_n \in U^\circ$ can be found (by Hahn-Banach) with $a_n(T(y_n)) > 1$ for all $n \in \mathbb{N}$. Since $U \subseteq E$ is a (bornivorous) barrel, U° and therefore $(a_n)_{n\in\mathbb{N}}$ is bounded in $\sigma(E', E)$ $(\beta(E', E))$. Hence, by hypothesis $V = \{a_n : n \in \mathbb{N}\}^\circ$ is a zero neighborhood in E.

Since $a_n(T(y_n)) > 1$, we have $T(y_n) \in E \setminus V \subseteq E \setminus \overset{\circ}{V}$ and therefore $y_n \in F \setminus T^{-1}(\overset{\circ}{V})$, where $\overset{\circ}{V}$ is the interior of V. Hence, $F \setminus T^{-1}(U) = \overline{\{y_n : n \in \mathbb{N}\}} \subseteq F \setminus T^{-1}(\overset{\circ}{V})$ and therefore $T^{-1}(\overset{\circ}{V}) \subseteq T^{-1}(U)$. Since V is a zero neighborhood in E, $T^{-1}(U)$ is a zero neighborhood in F.

We have shown that the pre-image of a (bornivorous) barrel under T is a zero neighborhood in F. This in turn implies the claimed continuity properties, since the respective topologies on E have zero neighborhood bases consisting of (bornivorous) barrels (see Lemma 2.47). $\square$

Corollary 2.63. If E is a separable (quasi-)ℓ_∞-barrelled lcs, then it is (quasi-)barrelled.

Proof. Use Theorem 2.61 for the identity $\mathrm{id} : E \to E$ and Lemma 2.47. $\square$

Theorem 2.64 (12.1.4, Jarchow (1981))**.** Let E be a Hausdorff lcs.

1. The topology $(E, \mu(E, E'))$ is c_0-barrelled, iff $(E', \sigma(E', E))$ is locally complete.

2. The topology $(E, \mu(E, E'))$ is quasi-c_0-barrelled, iff $(E', \beta(E', E))$ is locally complete.

Proof. 1. Suppose $(E, \mu(E, E'))$ is c_0-barrelled. If $(a_n)_{n\in\mathbb{N}}$ is a null sequence in $\sigma(E', E)$, then we know by the Theorem of Alaoğlu-Bourbaki that $K = \overline{\mathrm{acx}}\,\{a_n : n \in \mathbb{N}\}$ is $\sigma(E', E)$-compact. An application of Theorem 2.39 (4) $\Rightarrow$ (1) yields that $(E', \sigma(E', E))$ is locally complete.

Conversely suppose $\sigma(E', E)$ is locally complete and $a_n \xrightarrow{\sigma} 0$, then

$$K = \overline{\mathrm{acx}}\,\{a_n : n \in \mathbb{N}\}$$

is compact in $\sigma(E', E)$ by Theorem 2.39, (1) $\Rightarrow$ (4). Therefore $p_K(x) = \sup_{a\in K} |a(x)|$ is a continuous seminorm in $(E, \mu(E, E'))$ and hence (a_n) is equicontinuous. This implies the c_0-barrelledness of $\mu(E, E')$.

2. Suppose $(E, \mu(E, E'))$ is quasi-c_0-barrelled. Let B be a disk in E'_β and $a_n \xrightarrow{E_B} 0$, then $a_n \xrightarrow{\beta} 0$. Similarly to above, we know by the Theorem of Alaoğlu-Bourbaki that
$$K = \overline{\operatorname{acx}\{a_n : n \in \mathbb{N}\}}^{\sigma}$$
is $\sigma(E', E)$-compact and therefore $\sigma(E', E)$-complete. By Theorem 2.7 it follows that K is also $\beta(E', E)$-complete. Since $\{a_n : n \in \mathbb{N}\} \cup \{0\}$ is $\beta(E', E)$-compact, it follows from Theorem 2.14 that
$$\overline{\operatorname{acx}\{a_n : n \in \mathbb{N}\}}^{\beta}$$
is $\beta(E', E)$-compact. Now it follows from Theorem 2.39, (3) $\Rightarrow$ (1) that E'_β is locally complete.

Conversely, suppose $\beta(E', E)$ is locally complete and $a_n \xrightarrow{\beta} 0$, then
$$K = \overline{\operatorname{acx}}\{a_n : n \in \mathbb{N}\}$$
is compact in $\sigma(E', E)$ by Theorem 2.39, (1) $\Rightarrow$ (4). Therefore $p_K(x) = \sup_{a \in K} |a(x)|$ is a continuous seminorm in $(E, \mu(E, E'))$ and hence (a_n) is equicontinuous. This implies the quasi-c_0-barrelledness of $\mu(E, E')$.

□

We can define yet another concept of barrelledness.

Definition 2.65 ($\aleph_0$-barrelledness, quasi-$\aleph_0$-barrelledness)**.** A lcs is E is called $\aleph_0$-barrelled, if every barrel E which can be represented as a countable intersection of closed, absolutely convex zero neighborhoods is itself a zero neighborhood. The space E is called quasi-$\aleph_0$-barrelled, if this property holds for all bornivorous barrels.

The following dual characterization of $\aleph_0$-barrelledness will be useful later on.

Lemma 2.66. E is (quasi-)$\aleph_0$-barrelled, iff every $\sigma(E', E)$-bounded (respectively $\beta(E', E)$-bounded) set which is the countable union of equicontinuous sets is again equicontinuous.

Proof. If $B \subseteq E$ is a barrel with $B = \bigcap_{k \in \mathbb{N}} B_k$, where B_k are closed, absolutely convex zero neighborhoods, then $B_k = A_k^\circ$ for some equicontinuous sets $A_k \subseteq E'$. Now write
$$B = \bigcap_{k \in \mathbb{N}} A_k^\circ = \left(\bigcup_{k \in \mathbb{N}} A_k \right)^\circ .$$
Note that since B is a absorbing, $B^\circ = \bigcup_{k \in \mathbb{N}} A_k$ is $\sigma(E', E)$-bounded and B is a zero neighborhood iff $\bigcup_{k \in \mathbb{N}} A_k$ is equicontinuous. The corresponding characterization of quasi-$\aleph_0$-barrelledness follows analogously. □

Remark 2.67.

1. Obviously every (quasi-)barrelled space is also (quasi-)$\aleph_0$-barrelled.
2. Let E be $\aleph_0$-barrelled and $(l_n)_{n\in\mathbb{N}} \subseteq E'$ be a bounded sequence. Since the $\sigma(E', E)$-bounded set $A = \bigcup_{n\in\mathbb{N}} \{l_n\}$ is a union of equicontinuous sets, A is also equicontinuous. Therefore E is also ℓ_∞-barrelled, i.e. $\aleph_0$-barrelledness implies ℓ_∞-barrelledness. An analogous statement holds for quasi-ℓ_∞-barrelledness and quasi-$\aleph_0$-barrelledness.

The next theorem is analogous to the uniform boundedness theorem for barrelled spaces.

Theorem 2.68 (Theorem 12.2.1, Jarchow (1981))**.** For every lcs the following are equivalent.

1. E is (quasi-)$\aleph_0$-barrelled.
2. For every lcs F, every $\mathcal{T}_s$-bounded ($\mathcal{T}_\beta$-bounded) sequence in $\mathcal{L}(E, F)$ is equicontinuous.
3. For every Banach space F, every $\mathcal{T}_s$-bounded ($\mathcal{T}_\beta$-bounded) sequence in $\mathcal{L}(E, F)$ is equicontinuous.

Proof. **(1) $\Rightarrow$ (2)**: Let $(T_n)_{n\in\mathbb{N}}$ be a $\mathcal{T}_s$ ($\mathcal{T}_\beta$)-bounded sequence in $\mathcal{L}(E, F)$ and V be a absolutely convex, closed zero neighborhood in F. For a finite (bounded) set $B \subseteq E$ there exists a $\rho > 0$ such that $\rho T_n(B) \subseteq V$, $\forall n \in \mathbb{N}$ because of the respective boundedness assumptions. Therefore $U = \bigcap_{n\in\mathbb{N}} T_n^{-1}(V)$ is a (bornivorous) barrel in E, since $\rho B \subseteq U$. Because E is (quasi-)$\aleph_0$-barrelled, we have that U is a zero neighborhood in E. Now we get from $T_n(U) \subseteq V$, $\forall n \in \mathbb{N}$ that $(T_n)_{n\in\mathbb{N}}$ is equicontinuous.
(2) $\Rightarrow$ (3): Trivial.
(3) $\Rightarrow$ (1): Let absolutely convex, closed zero neighborhoods $U_n \subseteq E$ be given such that $U = \bigcap_{n\in\mathbb{N}} U_n$ is absorbent (bornivorous). Furthermore, define the linear space

$$F = \left\{x \in E^{\mathbb{N}} : \exists n_0 : x_n = x_{n_0},\ \forall n \geq n_0\right\}$$

and $V = F \cap \prod_{n=1}^{\infty} U_n$. Then V is absolutely convex and absorbent in F. Let $F_{(V)} = F/\ker(p_V)$ be the normed space which is obtained by factoring out the kernel of p_V. Define furthermore $\Phi_V : F \to \widetilde{F_{(V)}}$ to be the canonical mapping and $T_n : E \to F$ as

$$(T_n(x))_k = \begin{cases} x, & k \leq n \\ 0, & k > n. \end{cases}$$

Since

$$T_n\left(\bigcap_{i=1}^{n} U_i\right) \subseteq V,$$

it follows that $S_n = \Phi_V \circ T_n : E \to \widetilde{F_{(V)}}$ is continuous.

Because U is absorbent (bornivorous) for every finite (bounded) set B there is a $\rho > 0$ such that $B \subseteq \rho U$ and $T_n(B) \subseteq T_n(\rho U) \subseteq \rho V$, $\forall n \in \mathbb{N}$ and therefore $||S_n(B)||_{\widetilde{F_{(V)}}} \leq \rho$. Consequently, $(S_n)_{n\in\mathbb{N}}$ is $\mathcal{T}_s$-bounded ($\mathcal{T}_\beta$-bounded) in $\mathcal{L}(E, \widetilde{F_{(V)}})$. By assumption, we therefore can find a zero neighborhood $W \subseteq E$ such that $T_n(W) \subseteq V$, $\forall n \in \mathbb{N}$. Since $U = \bigcap_{n\in\mathbb{N}} T_n^{-1}(V)$, we know that $W \subseteq U$ and therefore U is a zero neighborhood in E and E is (quasi-)$\aleph_0$-barrelled. □

Theorem 2.69 (Theorem 12.2.4 Jarchow (1981), Theorem 25.6 Meise and Vogt (1997))**.** The strong dual, $F = (E', \beta(E', E))$, of a metrizable lcs is quasi-$\aleph_0$-barrelled.

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a zero neighborhood basis for E. Suppose $V = \bigcap_{n\in\mathbb{N}} V_n$ is a bornivorous barrel in F and all the $V_n \subseteq F$ are closed, absolutely convex zero neighborhoods. We are recursively constructing numbers $\rho_i > 0$ and bounded sets $B_i \subseteq E$ such that, for each $n \in \mathbb{N}$

$$B_i^\circ \subseteq V_i, \qquad \rho_i U_i^\circ \subseteq 2^{-i-1} V \cap B_j^\circ, \qquad \forall 1 \leq i, j \leq n.$$

Start by fixing an arbitrary bounded set $B_1 \subseteq E$ such that $B_1^\circ \subseteq V_1$ and a $\rho_1 > 0$ such that $\rho_1 U_1^\circ \subseteq \frac{1}{4} V \cap B_1^\circ$. Suppose that we already found sets $B_1, \ldots, B_n$ and numbers $\rho_1, \ldots, \rho_n$ complying with the above specifications. Fix a $\lambda > 0$ such that $\lambda U_{n+1}^\circ \subseteq 2^{-(n+1)-1} V \cap \bigcap_{i=1}^n B_i^\circ$ and define the absolutely convex, $\sigma(E', E)$-compact set

$$K = \sum_{i=1}^n \rho_i U_i^\circ + \lambda U_{n+1}^\circ \subseteq \sum_{i=1}^{n+1} 2^{-i-1} V \subseteq \frac{1}{2} V_{n+1}.$$

Let V' be a absolutely convex $\sigma(E', E)$-closed zero neighborhood in F contained in $\frac{1}{2} V_{n+1}$. Now define $B_{n+1} = (V' + K)^\circ$ and note that B_{n+1} is bounded in E (since $V' + K$ is a zero neighborhood in $\beta(E', E)$) and $B_{n+1}^\circ = V' + K$ (since $V' + K$ is $\sigma(E', E)$-closed and absolutely convex). Since $K \subseteq \frac{1}{2} V_{n+1}$ and $V' \subseteq \frac{1}{2} V_{n+1}$, we have $B_{n+1}^\circ \subseteq V_{n+1}$. Now $\rho_{n+1} < \lambda$ can be chosen such that

$$\rho_{n+1} U_{n+1}^\circ \subseteq 2^{-n-2} V \cap B_{n+1}^\circ$$

which completes the recursion.

Set $W = \bigcap_{n\in\mathbb{N}} B_n^\circ$ and note that $W = W^{\circ\circ}$ and W absorbs every U_i° and hence is a barrel in $(E', \sigma(E', E))$. It follows that W° is bounded in E and therefore $W^{\circ\circ} = W$ is a zero neighborhood in F. Since $W \subseteq V$, the assertion follows. □

2.6 Absorbent and Bornivorous Sequences

Definition 2.70 (Absorbent sequence, bornivorous sequence)**.** Let E be a lcs. A sequence of absolutely convex sets $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ is called *absorbent*, if $A_n + A_n \subseteq A_{n+1}$, $\forall n \in \mathbb{N}$

and every finite set is absorbed (and thereby also contained) in some A_n. The sequence $\mathcal{A}$ is called *bornivorous* if the same holds true with finite sets replaced by bounded sets.

Remark 2.71. Trivially every fundamental sequence $\mathcal{B}$ is bornivorous and every bornivorous sequence is absorbent.

Theorem 2.72 (Lemma 12.3.1, Jarchow (1981))**.** Let $(E, \mathcal{T})$ be a lcs, $\mathcal{U}$ be a zero neighborhood basis and $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ be a absorbent sequence in E, then

$$\operatorname{acx}\bigcup_{k=1}^{\infty} A_k \cap U_k,\ U_k \in \mathcal{U}, \quad \bigcap_{k=0}^{\infty} A_k + U_k,\ U_k \in \mathcal{U}$$

form zero neighborhood bases $\mathcal{U}^{\mathcal{A}}$ and $\mathcal{V}^{\mathcal{A}}$ respectively for the finest lcs topology $\mathcal{T}^{\mathcal{A}}$ which coincides with $\mathcal{T}$ on every A_n, $\forall n \in \mathbb{N}$.

Proof. To see that $\mathcal{U}^{\mathcal{A}}$ is indeed a zero neighborhood basis for a lcs, note that 0 is a member of all the sets. Furthermore all the sets are absolutely convex and absorbent, because the sequence $\mathcal{A}$ and every $U_k \in \mathcal{U}$ are absorbent. If $U^1 = \operatorname{acx}\bigcup_{k=1}^{\infty} A_k \cap U_k^1 \in \mathcal{U}^{\mathcal{A}}$ and $U^2 = \operatorname{acx}\bigcup_{k=1}^{\infty} A_k \cap U_k^2 \in \mathcal{U}^{\mathcal{A}}$, then $\exists W_k \in \mathcal{U} : W_k \subseteq U_k^1 \cap U_k^2$ and $\mathcal{U}^{\mathcal{A}} \ni \operatorname{acx}\bigcup_{k=1}^{\infty} A_k \cap W_k \subseteq U_1 \cap U_2$. Similarly, one can show that $\forall \rho > 0, \forall U_1 \in \mathcal{U}^{\mathcal{A}}$ there exists an $U_2 \in \mathcal{U}^{\mathcal{A}}$ such that $U_2 \subseteq \rho U_1$. Hence $\mathcal{U}^{\mathcal{A}}$ generates a unique lcs topology (c.f. Lemma 2.5).

Furthermore, since $E = \bigcup A_k$, choosing $U_k = U$, $\forall k \in \mathbb{N}$ yields that $U \in \mathcal{U}^{\mathcal{A}}$ for all $U \in \mathcal{U}$ and therefore $\mathcal{T}^{\mathcal{A}} \geq \mathcal{T}$. On every set A_n, we have

$$A_n \cap \operatorname{acx}\bigcup_{k=1}^{\infty} A_k \cap U_k \supseteq A_n \cap U_n.$$

Since the latter set is open in the topology induced by $\mathcal{T}$ on A_n, $\mathcal{T}$ and $\mathcal{T}^{\mathcal{A}}$ coincide on all the A_k.

We now show that $\mathcal{T}^{\mathcal{A}}$ is the finest topology which has this property. For this let $\mathcal{T}'$ be a lcs topology that coincides with $\mathcal{T}$ on all the sets A_k. Let V be any absolutely convex zero neighborhood in $(E, \mathcal{T}')$. By assumption, for every $n \in \mathbb{N}$ there is an $U_n \in \mathcal{U}$ such that $A_n \cap U_n = A_n \cap V \subseteq V$. Hence

$$\operatorname{acx}\bigcup_{n=1}^{\infty} A_n \cap U_n \subseteq V$$

establishing that $\mathcal{T}' \leq \mathcal{T}^{\mathcal{A}}$.

We complete the proof by showing that $\mathcal{V}^{\mathcal{A}}$ is a zero neighborhood basis for $\mathcal{T}^{\mathcal{A}}$. Let first $V = \bigcap A_k + U_k \in \mathcal{V}^{\mathcal{A}}$ and $V_k \in \mathcal{U}$ with $V_k^{\circ\circ} \subseteq U_k$ for all $k \in \mathbb{N}$. Next choose

$U'_k \in \mathcal{U}$ such that $U'_k \subseteq \bigcap_{i=1}^{k-1} V_i$, $\forall k \in \mathbb{N}$. Then $A_m \subseteq A_n + V$ for all $n \geq m$ implies $A_m \cap U'_m \subseteq \bigcap_{n=m}^{\infty}(A_n + V_n^{\circ\circ})$. Furthermore we have

$$A_m \cap U'_m \subseteq U'_m \subseteq \bigcap_{k=0}^{m-1} V_i^{\circ\circ} \subseteq \bigcap_{k=0}^{m-1} A_k + V_k^{\circ\circ}$$

and therefore $A_m \cap U'_m \subseteq \bigcap_{n=0}^{\infty} A_n + V_n^{\circ\circ}$. Since the latter set is absolutely convex this implies $\operatorname{acx} \bigcup A_m \cap U'_m \subseteq V$.

Let conversely $U = \operatorname{acx} \bigcup A_m \cap U_m$ be given, then there is a sequence k_n such that $A_{2n} \subseteq 2^{-n} A_{k_n}$ for all $n \in \mathbb{N}$. Choose further $V_n \in \mathcal{U}$ such that $V_n \subseteq 2^{-n} U_{k_n}$ and $2V_{n+1}^{\circ\circ} \subseteq V_n$, $\forall n \in \mathbb{N}$. Then clearly $V = \bigcap_{n=0}^{\infty} A_n + V_n$ is in $\mathcal{V}^{\mathcal{A}}$. We show that $V \subseteq U$ and thereby $\mathcal{T}^{\mathcal{A}} = \mathcal{V}^{\mathcal{A}}$. Let $x \in V$ and $y_n \in A_n$ and $v_n \in V_{n+2}$ such that $x = y_n + v_n$, $\forall n \in \mathbb{N}_0$. Since $A_0 = \{0\}$ we can write

$$x = v_n + \sum_{i=1}^{n} x_i, \text{ where } x_i = y_i - y_{i-1}, \tag{2.1}$$

for $1 \leq i \leq n$. Since we know that $x_i \in A_i + A_{i-1} \subseteq A_{i+1}$ and $x_i = v_{i-1} - v_i \in V_{i+1} - V_{i+2} \subseteq V_i$, it follows that $x_i \in A_{i+1} \cap V_i$ for all $1 \leq i \leq n$. Since (A_n) is a fundamental sequence, there is a n such that $x \in A_n$ and therefore $v_n = x - y_n \in 2A_n \cap V_{n+2} \subseteq A_{n+2} \cap V_{n+1}$ (since x, $y_n \in A_n$ and $v_n \in V_{n+2}$). Therefore by (2.1)

$$x \in \sum_{i=1}^{n} A_{i+1} \cap V_i + A_{n+2} \cap V_{n+1} = \sum_{i=1}^{n+1} A_{i+1} \cap V_i \subseteq \sum_{i=1}^{n+1} 2^{-i}(A_{k_i} \cap U_{k_i}) \subseteq U.$$

□

Chapter 3

Schwartz & Montel spaces

In this Section, we discuss Schwartz and Montel spaces. Both spaces are of interest because of the relation of bounded and compact sets in these spaces. Like in finite dimensional normed spaces, in a Schwartz space every bounded set is precompact and for Montel spaces, it even holds that every bounded set is relatively compact.

To be able to work with Schwartz spaces, we first introduce the notions of continuous convergence and the stronger equicontinuous convergence. Following Jarchow (1981), we define Schwartz spaces as spaces where these two concepts coincide. We will furthermore prove an important characterization of Schwartz spaces in Theorem 3.48 in terms of sequences, which will allow us to relate these spaces and df spaces in Chapter 4.

3.1 Continuous Convergence

In this section we will treat the concepts of continuous convergence. Continuous convergence is interesting by itself, since it allows for a characterization of the completion of a lcs as the continuously convergent linear functionals on E'. This important result relating the dual of a lcs to its completion, is known by the name of *Grothendieck's completeness theorem* and is stated as Theorem 3.8. Furthermore the notion of continuous convergence gives rise to the definition of the two topologies γ^t and γ which are the finest linear and lcs topology in which every continuously convergent filter converges.

Definition 3.1 (Equicontinuous compactology)**.** Let E be a lcs. A compactology in E' is a bornology consisting of $\sigma(E', E)$-compact disks. The system $\mathcal{E}$ of all equicontinuous, $\sigma(E', E)$-closed disks in E' is called the equicontinuous compactology.

Theorem 3.2 (Proposition 8.5.3, Jarchow (1981))**.** If E is a separable lcs, then the sets in $\mathcal{E}$ are separable and metrizable for $\sigma(E', E)$.

Proof. We start by fixing a countable dense subset D in E and setting G to be the linear span of D. Let W be a zero neighborhood in $(E', \sigma(E', G))$. Then there are $y_1, \ldots, y_k \in G$ with $\{y_1, \ldots, y_k\}^\circ \subseteq W$. Since $y_i \in G$, we can find coefficients such that $y_i = \sum_{n=1}^m \rho_n^{(i)} x_n$ with $x_n \in D$ for all $1 \le i \le k$ and $1 \le n \le m$. Choose a rational $\rho \ge \sum_{i,n} |\rho_n^{(i)}|$. Let $y \in \operatorname{acx}(\{y_1, \ldots, y_k\})$, then there are λ_i with $\sum_{i=1}^k |\lambda_i| \le 1$ such that

$$y = \sum_{i=1}^k \lambda_i y_i = \sum_{i=1}^k \lambda_i \sum_{n=1}^m \rho_n^{(i)} x_n = \sum_{n=1}^m x_n \sum_{i=1}^k \lambda_i \rho_n^{(i)}.$$

Since

$$\sum_{n=1}^m \sum_{i=1}^k |\lambda_i \rho_j^{(i)}| \le \sum_{i=1}^k |\lambda_i| \sum_{n=1}^m |\rho_n^{(i)}| \le \sum_{i=1}^k |\lambda_i| \sum_{n,i} |\rho_n^{(i)}| \le \rho \sum_{i=1}^k |\lambda_i| \le \rho,$$

we have $y \in \rho \operatorname{acx}(\{x_1, \ldots, x_m\})$ and therefore

$$\rho^{-1} \{x_1, \ldots, x_m\}^\circ \subseteq \{y_1, \ldots, y_k\}^\circ \subseteq W.$$

Thus the rational multiples of the polars of the finite subsets of D form a zero basis in $(E', \sigma(E', G))$. Since this basis is countable, $(E', \sigma(E', G))$ is metrizable.

Now let $H \in \mathcal{E}$ and p_x be a seminorm in $(E', \sigma(E', E))$. For some $0 < c < 1$ there is a $y \in E$ such that $cp_y > p_x$. For $l \in E'$ there is a $\tilde{y}_l \in G$ such that $l(\tilde{y}_l) > l(y)c$. Define the $\sigma(E', E)$-open set

$$A_l = \{f \in E' : f(\tilde{y}_l) > f(y)c\}.$$

Since

$$H \subseteq \bigcup_{l \in H} A_l$$

is compact there are $l_1, \ldots, l_s$ such that

$$H \subseteq \bigcup_{i=1}^s A_{l_i}.$$

If we define $\tilde{p}(l) = \max_i p_{\tilde{y}_i}(l)$, then

$$\tilde{p}(l) > p_y(l)c > p_x(l), \quad \forall l \in H.$$

Hence, $\sigma(E', G)$ coincides with $\sigma(E', E)$ on all $H \in \mathcal{E}$ and therefore H is metrizable for $\sigma(E', E)$. The separability follows, since H is metrizable and compact in $\sigma(E', E)$. □

Remark 3.3.
1. By Alaoğlu-Bourbaki all the sets in $\mathcal{E}$ are $\sigma(E', E)$-compact (as the name suggests) and therefore also bounded.
2. Let $\mathcal{U}$ be a zero neighborhood basis in E, then $\{U^\circ : U \in \mathcal{U}\}$ is a basis of $\mathcal{E}$.

3. If $H \in \mathcal{E}$, then the normed space E_H is a Banach space (see Kriegl (2007), Lemma 7.4.17).

Definition 3.4 (Continuous Convergence)**.** Let $(E, \mathcal{T})$ be a lcs and $\mathcal{E}$ be the equicontinuous compactology on E'. A filter $\mathcal{F}$ in E' is said to converge continuously to $x' \in E'$, if there is a $H \in \mathcal{E}$ such that $x' \in H$ and the trace of $\mathcal{F}$ on H, $\mathcal{F}_H$, converges to x' in the topology induced by $\sigma(E', E)$ on H.

Definition 3.5 (γ-continuous)**.** Let E be a lcs, X be a topological space, and $M \subseteq E'$. Then $f : M \to X$ is called γ-continuous, iff for every continuously convergent filter $\mathcal{F} \to x \in M$ it follows that $f(\mathcal{F}) \to f(x)$ in X.

Lemma 3.6 (Lemma 9.2.1, Jarchow (1981))**.** A function f is γ-continuous, iff for every $H \in \mathcal{E}$ the restriction $f : M \cap H \to X$ is continuous in the topology induced by $\sigma(E', E)$ on $M \cap H$.

Proof. Let f be γ-continuous and $\mathcal{F}_H \xrightarrow{\sigma} x$ be a filter on $M \cap H$. Then the corresponding filter $\mathcal{F}$ defined on M converges continuously to x and therefore $f(\mathcal{F}) \to f(x)$ and therefore also $F(\mathcal{F}_H) \to f(x)$. Hence, $f : M \cap H \to X$ is weakly continuous.

If, on the other hand side, $f : M \cap H \to X$ is weakly continuous for every H, and $\mathcal{F}$ is a continuously convergent filter then $\mathcal{F}_H \xrightarrow{\sigma} x$ for some x and some $H \in \mathcal{E}$. Consequently $f(\mathcal{F}_H) \to f(x)$ and therefore $f(\mathcal{F}) \to f(x)$. Hence, f is γ-continuous convergent. $\square$

The next Theorem provides a characterization of the completion of lcs in terms of continuous convergence and is known by the name *Grothendieck's completeness theorem*. We first prove the following lemma.

Lemma 3.7 (Proposition 8.1.4, Jarchow (1981))**.** Let F be a subspace of the algebraic dual E^* of a lcs E. The canonical bilinear form $\langle \cdot, \cdot \rangle : F \times E \to \mathbb{K}$ is non-degenerate, iff F is dense in $(E^*, \sigma(E^*, E))$.

Proof. If F is dense in $(E^*, \sigma(E^*, E))$ and $x \in E$ is such that $\langle y, x \rangle = 0$ for all $y \in F$, then $x = 0$ follows from the fact that $y \mapsto \langle y, x \rangle$ is $\sigma(E^*, E)$-continuous. Hence, the pairing is non-degenerate.

If F is not dense and $x^* \in E^* \setminus \bar{F}$, then, by Hahn-Banach, there exists a $x \in E$ such that $\langle y, x \rangle = 0$ for all $y \in F$, but $\langle x^*, x \rangle \neq 0$. This implies that the pairing is not point separating and therefore is degenerate. $\square$

Theorem 3.8 (Theorem 9.2.2, Jarchow (1981))**.** Let $(\tilde{E}, \tilde{\mathcal{T}})$ be a completion of $(E, \mathcal{T})$, then $\tilde{E} \cong \hat{E}$, where $\hat{E}$ is the space of all γ-continuous linear forms on E'.

Proof. Since the dual of E and $\tilde{E}$ coincide, we can consider $\tilde{E}$ as a subspace of $(E')^*$ (the algebraic dual of E'). Since the evaluation at $x \in \tilde{E}$ is a $\sigma(E', \tilde{E})$-continuous linear form on E', every $x \in \tilde{E}$ has a continuous restriction to every $H \in \mathcal{E}$ so that $\tilde{E} \subseteq \hat{E}$ is true by Lemma 3.7.

To prove the converse define the inner product $\langle \cdot, \cdot \rangle : E' \times \hat{E} \to \mathbb{K}$. Let $x,y \in E'$ with $x \neq y$, then there is a $\sigma(E', E)$-continuous function that separates x and y. Since $\sigma(E', E)$-continuity implies γ-continuity, $\langle \cdot, \cdot \rangle$ is non-degenerate. Now consider the identity $\mathrm{id} : E' \to (E', \sigma(E', \hat{E}))$. Because of the definition of the topology $\sigma(E', \hat{E})$ and Lemma 3.7, id is γ-continuous iff all the mappings $f_{H,l}$ are continuous (for $H \in \mathcal{E}$ and $l \in \hat{E}$)

$$\begin{tikzcd} (H, H_{|\sigma}) \arrow[r, "\mathrm{id}"] \arrow[dr, "f_{H,l}"'] & (E', \sigma(E', \hat{E})) \arrow[d, "l"] \\ & \mathbb{K} \end{tikzcd}$$

This is the case, since $l \in \hat{E}$. Hence, id is γ-continuous and therefore all $H \in \mathcal{E}$ are compact in $\sigma(E', \hat{E})$ and $\mathcal{E}$ is a compactology on E' with respect to $\langle E', \hat{E} \rangle$. The topology $\hat{\mathcal{T}}$ induced by $\mathcal{E}$ on $\hat{E}$ is admissible (since $\hat{T} \leq \mu(\hat{E}, E')$) and induces $\tilde{\mathcal{T}}$ on $\tilde{E} \subseteq \hat{E}$ (because of the dual definition of a topology). By Lemma 3.7, $\tilde{E}$ is dense in $((E')^*, \sigma(E'^*, E'))$. Hence, it is also dense in $(\hat{E}, \sigma(\hat{E}, E'))$. Since $\hat{T}$ is compatible with $\langle \hat{E}, E' \rangle$, $\tilde{E}$ is even dense in $(\hat{E}, \hat{\mathcal{T}})$ (since the closures of absolutely convex sets for two admissible lc topologies coincide). Since $\tilde{E}$ is complete, $\tilde{E} = \hat{E}$ follows. □

Definition 3.9 (γ^t-closedness)**.** Let E be a lcs. A subset $A \subseteq E'$ is γ^t-closed, iff every continuously convergent filter $\mathcal{F}$ on E' that contains A, takes its limit in A, i.e. $\mathcal{F} \to x \in A$.

The next two results establish that the *closed sets* defined above actually define a topology η^t on E'. However, note that this topology is not necessarily a locally convex topology.

Lemma 3.10 (Lemma 9.3.1, Jarchow (1981))**.** $A \subseteq E'$ is γ^t-closed, iff $A \cap H$ is closed in $(E', \sigma(E', E))$ for all $H \in \mathcal{E}$.

Proof. ($\Rightarrow$): Fix a $H \in \mathcal{E}$. Let x' be $\sigma(E', E)$-adherent for $A \cap H$. Since H is $\sigma(E', E)$-closed, $x' \in H$. Now let $\mathcal{F}$ be a filter, with $\mathcal{F} \xrightarrow{\sigma} x'$ such that both A and H are in $\mathcal{F}$ (in particular the filter therefore converges continuously). Since $A \in \mathcal{F}$ and A is γ^t closed, we infer that $x' \in A$ and therefore $x' \in A \cap H$.

($\Leftarrow$): Let $\mathcal{F}$ be a filter on E' such that $A \in \mathcal{F}$ and $\mathcal{F} \to x' \in E'$ continuously, i.e. there is a $H \in \mathcal{E}$ such that $H \in \mathcal{F}$. It follows that $A \cap H \in \mathcal{F}$ and since the latter set is $\sigma(E', E)$-closed by our hypothesis, $x' \in A \cap H \subseteq A$. □

Theorem 3.11. The γ^t-closed sets define a topology on E'.

Proof. By Lemma 3.10, $\emptyset$ and E' are γ^t-closed. Furthermore arbitrary intersections and finite unions of γ^t closed sets are closed. Hence, the system of γ^t-closed sets defines a unique topology. □

Let A be $\sigma(E', E)$-closed and note that since $H \in \mathcal{E}$ is also $\sigma(E', E)$-closed, $A \cap H$ is $\sigma(E', E)$-closed and therefore A is γ^t-closed. Hence, it follows that γ^t is a stronger topology than $\sigma(E', E)$.

Theorem 3.12 (Proposition 9.3.2, Jarchow (1981))**.**

1. $\gamma^t(E', E)$ is the finest topology on E' such that every continuously convergent filter on E' converges.

2. $\gamma^t(E', E)$ is the finest topology on E' which coincides with $\sigma(E', E)$ on all the sets $H \in \mathcal{E}$.

3. The scalar multiplication in E' is continuous from $\mathbb{K} \times (E', \gamma^t(E', E)) \to (E', \gamma^t(E', E))$.

Proof. **(1)**: Let $\mathcal{F} \to x'$ be continuously convergent. Then there is a $H \in \mathcal{E}$ such that $H \in \mathcal{F}$. For each A which is γ^t-closed, we have that $A \cap H$ is $\sigma(E', E)$- closed and therefore $\mathcal{F} \to x' \in A \cap H \subseteq A$. Lemma 2.4 establishes that x' is a limit point of $\mathcal{F}$ in $\gamma^t(E', E)$. Since $\gamma^t(E', E)$ is finer than $\sigma(E', E)$, every other limit point of $\mathcal{F}$ in $\gamma^t(E', E)$ also is a limit point for $\mathcal{F}$ for $\sigma(E', E)$. We conclude that x' is the only limit point and therefore $\mathcal{F} \xrightarrow{\gamma^t} x'$.

Let $\mathcal{T}' > \gamma^t(E', E)$ and choose a set A which is closed in $\mathcal{T}'$ but not in $\gamma^t(E', E)$. It follows that there is a $H \in \mathcal{E}$ such that $A \cap H$ is not $\sigma(E', E)$-closed, hence there is a filter $\mathcal{F}$ with $A \cap H \in \mathcal{F}$ and a $x' \in E'$ such that $\mathcal{F} \xrightarrow{\sigma} x' \notin A \cap H$. Since $H \in \mathcal{F}$ the filter converges continuously, but it does not converge in $\mathcal{T}'$, since A is closed in $\mathcal{T}'$.

(2): It is immediate from Lemma 3.10, that $\gamma^t(E', E)$ and $\sigma(E', E)$ coincide on H, $\forall H \in \mathcal{E}$. Furthermore if $\mathcal{T}' > \gamma^t(E', E)$, then there is a set A which is $\mathcal{T}'$-open but not $\gamma^t(E', E)$-open, i.e. there is a set $H \in \mathcal{E}$ such that $H \cap A$ is not $\sigma(E', E)$-open. This in turn means that $\mathcal{T}'$ cannot coincide with $\sigma(E', E)$ on $H \in \mathcal{E}$.

(3): We know from (2) that $\gamma^t(E', E)$ is the final topology with respect to the mappings $H \to E'$. By Kriegl (2000), Remark 1.2.14, $\gamma^t(E', E)$ is a quotient of the co-product $\coprod_{H \in \mathcal{E}} H_\sigma$ and by Kriegl (2000), Theorem 2.2.9. since $\mathbb{K}$ is σ-compact, $\mathbb{K} \times \gamma^t(E', E)$ is a quotient of $\mathbb{K} \times \coprod_{H \in \mathcal{E}} H_\sigma$. Hence, it follows that $\mathbb{K} \times \gamma^t(E', E)$ is the finest topology which coincides with $\mathbb{K} \times \sigma(E', E)$ on sets of the form $K \times H$ with $H \in \mathcal{E}$ and K compact in $\mathbb{K}$. This implies that $\mathbb{K} \times \gamma^t(E', E)$ is the topology resulting from the following final construction:

$$\begin{tikzcd} K \times H|_{\sigma(E',E)} \arrow[r] \arrow[dr, dashed, "\bullet(\cdot,\cdot)"] & \mathbb{K} \times (E', \gamma^t(E',E)) \arrow[d, "\bullet(\cdot,\cdot)"] \\ & \gamma^t(E',E) \end{tikzcd}$$

Hence, it is enough to show continuity of the multiplication $\bullet(\cdot,\cdot) : K \times H|_{\sigma(E',E)} \to (E', \gamma^t(E',E))$. To establish this result, fix $K \subset \mathbb{K}$ compact and $H \in \mathcal{E}$. Since K is compact there is a $\rho > 0$ such that $K \subseteq \rho\mathbb{D}$ and therefore $\bullet(K, H) \subseteq \rho H =: H' \in \mathcal{E}$. Hence, it is enough to verify continuity of $\bullet(\cdot,\cdot) : K \times H|_{\sigma(E',E)} \to (H', \gamma^t(E',E)) = (H', \sigma(E',E))$. Since the multiplication is weakly continuous, this finishes the proof. □

Theorem 3.13 (Corollary 9.3.3, Jarchow (1981))**.** The neighborhood filter of zero in $(E', \gamma^t(E',E))$ has a basis consisting of circled and absorbent sets.

Proof. Let $\mathcal{U}$ be a zero neighborhood basis of $\gamma^t(E',E)$. Since the multiplication is continuous for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ and a $\rho > 0$, such that $\lambda V \subseteq U$ for all $\lambda \in \rho\mathbb{D}$. Now $(\rho\mathbb{D})V \subseteq U$ is circled, hence the statement follows. □

Definition 3.14 (The topology $\gamma(E',E)$)**.** Let $\mathcal{U}$ be a zero neighborhood basis of $\gamma^t(E',E)$ and $\mathcal{W} = \{\mathrm{acx}(U) : U \in \mathcal{U}\}$. By Lemma 2.5, $\mathcal{W}$ forms a zero basis for a locally convex topology, which we denote by $\gamma(E',E)$.

Remark 3.15.

1. $\gamma(E',E)$ is the finest locally convex topology, which is coarser than $\gamma^t(E',E)$. By Theorem 3.12 we also know that it is the finest locally convex topology which coincides with $\sigma(E',E)$ on all $H \in \mathcal{E}$ and that it is the finest locally convex topology such that every continuously convergent filter converges.

2. The topologies $\gamma^t(E',E)$ and $\gamma(E',E)$ need not coincide and $\gamma^t(E',E)$ need not be a linear topology (see Jarchow (1981), Example 9.3.9).

Next we discuss some relations between the two toppologies γ and γ^t as well as consequences of the Grothendieck completeness theorem.

Theorem 3.16 (Proposition 9.3.4, Jarchow (1981))**.** Let E and F be lcs. A linear map $T : E' \to F$ is continuous for $\gamma^t(E',E)$, iff it is continuous for $\gamma(E',E)$. In particular, it follows $\tilde{E} = (E', \gamma)'$.

Proof. Since F is a lcs, T is continuous if $T^{-1}(U)$ is a zero neighborhood in E' for every U in a basis of absolutely convex zero neighborhoods in F. Since U is absolutely convex, also $T^{-1}(U)$ is absolutely convex. Hence, T is continuous for γ^t, iff it is continuous for γ. By Theorem 3.12, every continuously convergent filter in $\gamma^t(E',E)$ converges. Therefore continuity of $T : (E', \gamma(E',E)) \to F$ implies continuous convergence of T. Now, $\tilde{E} = (E', \gamma)'$ follows from Theorem 3.8. □

Theorem 3.17 (Proposition 9.3.5, Jarchow (1981))**.** Let E be a lcs. Then $\gamma^t(E', E)$ and $\gamma(E', E)$ define the same closed hyperplanes in E'.

Proof. Since $\gamma^t(E', E) \geq \gamma(E', E)$, we have to show that every $\gamma^t(E', E)$-closed hyperplane is actually $\gamma(E', E)$-closed. Suppose $0 \neq u \in (E')^*$ is $\gamma^t(E', E)$-continuous and therefore defines a closed hyperplane $\ker(u)$. Fix a $x' \in E'$ and a circled γ^t-zero neighborhood W, such that $(x' + W) \cap \ker(u) = \emptyset$ (this is possible, since u is $\gamma^t(E', E)$-continuous). Note furthermore that $u(x') \notin u(W)$, otherwise 0 would be in $(x' + W)$. This, together with the fact that $u(W)$ is circled, implies that $u(W)$ is bounded. It follows that $u(\text{acx}(W)) = \text{axc}(u(W))$ is bounded and since $\text{acx}(W)$ is a zero neighborhood in $\gamma(E', E)$, u is $\gamma(E', E)$ continuous and therefore $\ker(u)$ is closed in $\gamma(E', E)$. □

Theorem 3.18 (Corollary 9.3.6, Jarchow (1981))**.** E is complete iff every γ^t-closed hyperplane in E' is $\sigma(E', E)$-closed.

Proof. If E is complete, then $(E', \gamma(E', E)) = E$ by Theorem 3.16 and $\gamma(E', E)$ is a permissible topology for $\langle E, E' \rangle$. Therefore the closed hyperplanes in $\gamma(E', E)$ and $\sigma(E', E)$ coincide. By Theorem 3.17 this also holds for $\gamma^t(E', E)$.

Conversely, if the hyperplanes in $\gamma^t(E', E)$ and $\sigma(E', E)$ are the same, then $E = (E', \gamma(E', E))$. By Theorem 3.16 it follows that $\tilde{E} = E$, i.e. that E is complete. □

Although $\gamma^t(E', E)$ is potentially strictly finer than $\gamma(E', E)$ and does not have to be a linear topology, the situation is much easier in metrizable spaces as the next theorem shows.

Theorem 3.19 (Banach-Dieudonné, Theorem 9.4.1, Jarchow (1981))**.** If E is metrizable lcs, then $\gamma^t(E', E) = \gamma(E', E)$.

Proof. Since E is metrizable, we can fix a countable decreasing zero neighborhood basis $(U_n)_{n \in \mathbb{N}}$. Let W be an arbitrary open zero neighborhood in (E', γ^t). Since U_1° is $\sigma(E', E)$-compact and $\gamma^t(E', E)$ coincides with $\sigma(E', E)$ on compact sets, there is a finite set of points $M_1 \subseteq E$ such that $V_1 := M_1^\circ \cap U_1^\circ \subseteq W \cap U_1^\circ \subseteq W$. Suppose that for some $n \in \mathbb{N}$ a finite set $M_n \subseteq E$ has been selected such that $V_n := M_n^\circ \cap U_n^\circ \subseteq W$. Then there is a finite set $N_n \subseteq U_n$ such that $V_{n+1} := M_{n+1}^\circ \cap U_{n+1}^\circ \subseteq W$ for $M_{n+1} = M_n \cup N_n$.

Suppose this would not be the case and define the sets $A_M := M_n^\circ \cap M^\circ \cap (U_{n+1}^\circ \setminus W) \neq \emptyset$ for every finite set $M \subseteq U_n$. Clearly the system $\{A_M : M \subseteq U_n \text{ finite}\}$ is a filter basis in $U_{n+1}^\circ \setminus W$. Since $U_{n+1}^\circ \setminus W$ is $\sigma(E', E)$-compact, the corresponding filter has an accumulation point a. Since all the A_M are closed, $a \in A_M$ for all the sets A_M. Hence, $a \in M_n^\circ \cap U_n^\circ \cap (U_{n+1}^\circ \setminus W)$. But by assumption $M_n^\circ \cap U_n^\circ \cap (U_{n+1}^\circ \setminus W) \subseteq W \cap (U_{n+1}^\circ \setminus W) = \emptyset$. This yields a contradiction.

Having defined the V_n as above ensures $V_n = M_n^\circ \cap U_n^\circ \subseteq M_n^\circ \cap N_n^\circ \cap U_{n+1}^\circ = V_{n+1}$, for all $n \in \mathbb{N}$. Hence, the set $V := \bigcup_n V_n$ is absolutely convex. Since V is a $\gamma^t(E', E)$-zero neighborhood by construction, it is also a $\gamma(E', E)$-zero neighborhood. Since we know that $V \subseteq W$, we are finished. □

Theorem 3.20 (Proposition 9.3.7, Jarchow (1981))**.** For every lcs $(E, \mathcal{T})$ it holds that $\tau_c(E', \tilde{E}) = \gamma(E', E)$ on E', where $\tau_c(E', \tilde{E})$ is the topology of uniform convergence on compact sets in $\tilde{E}$.

Proof. By Theorem 3.8, γ is a permissible topology on E' for $\langle E', \tilde{E}\rangle$, i.e. $\sigma(E', \tilde{E}) \leq \gamma \leq \mu(E', \tilde{E})$. Furthermore we can deduce from Theorem 2.33, that $\tau_c(E', \tilde{E})$ induces the same topology as $\sigma(E', \tilde{E})$ on all sets $H \in \mathcal{E}$. Since γ is the finest lcs topology that has this property by Theorem 3.12, we have $\sigma(E', \tilde{E}) \leq \tau_c(E', \tilde{E}) \leq \gamma \leq \mu(E', \tilde{E})$.

Now let $\mathcal{C}$ be the equicontinuous compactology on $\tilde{E} = (E', \gamma(E', E))'$ induced by $\gamma(E', E)$. By Kriegl (2007), Theorem 7.4.11 the topology $\mathcal{T}_\mathcal{C}$ of uniform convergence on sets in $\mathcal{C}$, equals $\gamma(E', E)$. To show $\mathcal{T}_\mathcal{C} \leq \gamma$, we therefore just have to show that every $C \in \mathcal{C}$ is also $\tilde{\mathcal{T}}$-compact. By assumption, every set $C \in \mathcal{C}$ is $\sigma(\tilde{E}, E')$-compact and by Alaoğlu-Bourbaki also compact for the topology λ of uniform convergence on precompact sets in $(E', \gamma(E', E))$. Since every $H \in \mathcal{E}$ is $\gamma(E', E)$-compact, we deduce $\tilde{\mathcal{T}} \leq \lambda$. Hence, every $C \in \mathcal{C}$ is also $\tilde{\mathcal{T}}$-compact, finishing the proof. □

A similar characterization is available for $\tau_{pc}(E', E)$, which complements the theorem of Alaoğlu-Bourbaki.

Theorem 3.21 (Proposition 9.3.8, Jarchow (1981))**.** $\tau_{pc}(E', E)$ is the finest among all topologies $\mathcal{T}_\mathcal{B}$ on E' which are given by uniform convergence on the sets of a bornology $\mathcal{B}$ on E and which have the property that every $H \in \mathcal{E}$ is $\mathcal{T}_\mathcal{B}$-compact.

Proof. By Alaoğlu-Bourbaki we know that every $H \in \mathcal{E}$ is $\tau_{pc}(E', E)$-compact.

Now let $\mathcal{B}$ be a bornology on E such that every $H \in \mathcal{E}$ is $\mathcal{T}_\mathcal{B}$-compact. Since we know by Theorem 3.20 that $\gamma(E', E) = \tau_c(E', \tilde{E})$ is the finest topology on E' such that every $H \in \mathcal{E}$ is compact, it follows $\mathcal{T}_B \leq \tau_c(E', \tilde{E})$. Denote by $\circ$ and $\bullet$ the polarization with respect to $\langle E, E'\rangle$ and $\langle \tilde{E}, E'\rangle$, respectively. Because $\mathcal{T}_B \leq \tau_c(E', \tilde{E})$, we know that given $B \in \mathcal{B}$ there is a compact set $K \subseteq \tilde{E}$ such that $K^\bullet \subseteq B^\circ$. It follows that $B^{\circ\bullet} \subseteq K^{\bullet\bullet}$ and therefore B is relatively compact in $\tilde{E}$ and precompact in E by Theorem 2.11. It follows that $\mathcal{T}_\mathcal{B} \leq \tau_{pc}(E', E)$. □

Lemma 3.22. For a metric space E, $\tau_c(E', \tilde{E}) = \tau_{pc}(E', E)$.

Proof. If $A \subseteq E$ is precompact, then the closure of A is compact in $\tilde{E}$, cf. Lemma 2.11. Hence $\tau_{pc}(E', E) \leq \tau_c(E', \tilde{E})$. The converse follows since for every compact set $C \in \tilde{E}$, we can find a null sequence $(a_n)_{n\in\mathbb{N}} \subseteq E$ such that

$$C \subseteq \overline{\mathrm{cx}\,\{a_n : n \in \mathbb{N}\}}^{\tilde{\mathcal{T}}}$$

by Theorem 2.16. Note that $\{a_n : n \in \mathbb{N}\}$ is precompact in E by Theorem 2.15 and we have

$$\{a_n : n \in \mathbb{N}\}^\circ = \left(\overline{\mathrm{cx}\,\{a_n : n \in \mathbb{N}\}}^{\tilde{\mathcal{T}}}\right)^\circ \subseteq C^\circ.$$

It follows that $\tau_{pc}(E', E) \geq \tau_c(E', \tilde{E})$. □

Based on the Banach-Dieudonné Theorem we can show the following.

Theorem 3.23 (Corollary 9.4.3, Jarchow (1981))**.** If E is a metrizable lcs, then $\gamma^t(E', E) = \tau_{pc}(E', E)$.

Proof. We know that $\tau_{pc}(E', E) = \tau_c(E', \tilde{E}) = \gamma(E', E) = \gamma^t(E', E)$, where the first equality follows from Lemma 3.22, the second equality follows from Theorem 3.20, and finally the third equality follows from the Banach-Dieudonné Theorem, cf. Theorem 3.19. □

3.2 B-Completeness and Nearly Open Mappings

In this section we will briefly review the notion of B-completeness, which is based on continuous convergence. We are interested in the notion of B-completeness in connection with nearly open mappings. Aim of this section is to prove Theorem 3.31, which states that in a B-complete space every nearly open mapping is already open. This Theorem will enable us to prove a homeomorphism theorem for (df)-spaces in Chapter 3 (see Theorem 4.45).

We know from Theorem 3.18 that a lcs is complete if and only if every $\gamma^t(E', E)$-closed hyperplane in E' is $\sigma(E', E)$-closed. To define B-completeness, we strengthen this criterion in the following way.

Definition 3.24 (B-completeness)**.** An lcs $(E, \mathcal{T})$ is *B*-complete iff every γ^t-closed linear subspace is $\sigma(E', E)$-closed.

The following Theorem summarizes the straightforward implications between the two notions of completeness. Note also the hyperplanes only define the linear subspaces of codimension 1 and therefore B-completeness is stronger than completeness.

Theorem 3.25 (Proposition 9.5.1, Jarchow (1981))**.**

1. Every *B*-complete lcs is complete.

2. If E is a complete lcs and $\gamma^t(E', E) = \gamma(E', E)$ holds on E', then E is B-complete.

Proof. **(1)**: See discussion above.

(2): By hypothesis and Theorem 3.16, $\gamma^t(E', E)$ is compatible with $\langle E, \tilde{E}\rangle$. Therefore $\gamma^t(E', E)$ and $\sigma(E', E)$ define the same closed convex sets on E'. □

The Banach-Dieudonné Theorem yields the following characterization of completeness for metric spaces known as the Krein-Šmulian Theorem.

Theorem 3.26 (Krein-Šmulian Theorem, Proposition 9.5.2, Jarchow (1981))**.** A metrizable lcs E is complete iff every $\gamma^t(E', E)$-closed convex subset of E is $\sigma(E', E)$-closed. In particular, every Fréchet space is B-complete.

Proof. If E is complete then $\gamma^t(E', E) = \gamma(E', E) = \tau_c(E', E)$, where the first equality follows from Theorem 3.19 and the second from Theorem 3.20 and the completeness of E. Hence, $\gamma^t(E', E)$ is compatible with $\langle E, E'\rangle$ and therefore it defines the same closed, convex sets as $\sigma(E', E)$.

If on the other hand side the closed, convex sets of $\gamma^t(E', E)$ and $\sigma(E', E)$ coincide, then $\gamma^t(E', E)$ is compatible with $\langle E, E'\rangle$ and therefore also $\gamma(E', E)$. Then $E = (E', \gamma(E', E)) = \tilde{E}$, where the last equality follows from Theorem 3.16. □

Theorem 3.27 (Proposition 9.5.3, Jarchow (1981))**.** If E is a metrizable lcs and $\mathcal{T}^1$ and $\mathcal{T}^2$ are lcs topologies on E' such that $\tau_{pc}(E', E) \leq \mathcal{T}^1 \leq \mu(E', \tilde{E})$ and $\tau_{pc}(E', E) \leq \mathcal{T}^2 \leq \beta(E', \tilde{E})$, then $(E', \mathcal{T}^1)$ is B-complete and $(E', \mathcal{T}^2)$ is complete.

Proof. The family of polars B° of bounded sets $B \subseteq E$ forms a zero neighborhood basis in $\beta(E', E)$. Since B° is $\mu(E', \tilde{E})$-closed, the statement on $\mathcal{T}^2$ follows by Theorem 2.7 from the statement on $\mathcal{T}^1$.

By Lemma 3.22, $\tilde{E}$ is the dual of $(E, \mathcal{T}^1)$. Let $H \subseteq \tilde{E}$ convex and closed for $\gamma^t(\tilde{E}, (E', \mathcal{T}^1))$. By Alaoğlu-Bourbaki the closure $\bar{H}$ of H with respect to $\sigma(\tilde{E}, E')$ and the Fréchet topology coincide. Hence, for $z \in \bar{H}$ there is a sequence $(z_n)_{n\in\mathbb{N}} \subseteq H$ such that $z = \lim_{n\to\infty} z_n$ in $\tilde{E}$.

Since $\tilde{E}$ is complete, $M := \{z_n : n \in \mathbb{N}\}^{\circ\circ}$ is compact in $\tilde{E}$ by Lemma 2.15. Because $\tau_c(E', \tilde{E}) = \tau_{pc}(E', E)$ by Lemma 3.22, M is $\tau_{pc}(E', E)$-equicontinuous and therefore also $\mathcal{T}^1$-equicontinuous. Since H is closed in $\gamma^t(\tilde{E}, (E', \mathcal{T}^1))$, $H \cap M$ is $\sigma(\tilde{E}, E')$-closed by Lemma 3.10. Therefore $z \in H \cap M \subseteq H$ and H is $\sigma(\tilde{E}, E')$-closed. Hence, $\mathcal{T}^1$ is B-complete. □

Definition 3.28 (Nearly open mapping)**.** Let E and F be lcs. A map $T : E \to F$ is called nearly open, iff for every zero neighborhood $U \in E$, the closure of $T(U)$ is a zero neighborhood in range(T) considered as a subspace of F.

Lemma 3.29 (Proposition 9.5.3, Corollary 8.72, 8.7.3; Jarchow (1981))**.** Let E be a lcs and $F \subseteq E$ be a subspace, $\iota : F \to E$ be the canonical injection, and $R = \iota' : E' \to F' : x' \mapsto x_{|F}$ the *restriction map.*

1. If $\mathcal{B}$ is a bornology (resp. compactology) in E', then $R(\mathcal{B})$ is a bornology (resp. compactology) in F'.

2. $(F, \mathcal{T}_{R(\mathcal{B})})$ is a subspace of $(E, \mathcal{T}_{\mathcal{B}})$.

3. $(F, \sigma(F, F'))$ is a subspace of $(E, \sigma(E', E))$.

4. The equicontinuous sets in F' are exactly the R-images of the equicontinuous sets in E'.

Proof. 1. Obvious, since R is a weakly continuous mapping.

2. Denote by $\bullet$ the polarization in $\langle F, F' \rangle$ and by $\circ$ the polarization in $\langle E, E' \rangle$. From Lemma 2.28, we know that $R(B)^{\bullet} = \iota^{-1}(B^{\circ}) = B^{\circ} \cap F$ and hence that the topology induced by the bornology $R(\mathcal{B})$ on F is the same as the subspace topology induced by $\mathcal{T}_{\mathcal{B}}$.

3. Follow from (2), since if $\mathcal{B}$ is the finite bornology in E' then $R(\mathcal{B})$ is the finite bornology in F' (by Hahn-Banach R is onto).

4. Follows from (2), since the topology on F is induced by the equicontinuous sets in F' and at the same time by $R(\mathcal{E})$ with $\mathcal{E}$ being the equicontinuous sets in E'.

□

Lemma 3.30 (Proposition 9.7.1, Jarchow (1981))**.** If E be B-complete and $F \subseteq E$ is a closed subspace, then E/F is B-complete as well.

Proof. Let $Q : E \to E/F$ and $\bullet$ be the polarization in $(E/F, (E/F)')$. Then the sets $Q(U)^{\bullet}$ are the equicontinuous sets in $(E/F)'$. Since $(E/F)' = F^{\circ}$ (cf. Lemma 4.14), we have $Q' : F^{\circ} \to E'$ is the canonical injection. Hence, $Q(U)^{\bullet} = (Q')^{-1}(U^{\circ}) = F^{\circ} \cap U^{\circ}$, by Lemma 2.28, and these sets are the basis for the equicontinuous compactology in F°. By Lemma 3.29, we have that $\sigma(F^{\circ}, E/F)$ is induced by $\sigma(E', E)$ and therefore every closed subspace H of $(F^{\circ}, \gamma^t(F^{\circ}, E(/F))$ is also closed in $(E', \gamma^t(E', E))$. By B-completeness H is $\sigma(E', E)$-closed and therefore also $\sigma(F^{\circ}, E/F)$-closed, establishing B-completeness of E/F. □

In the following, we show the main theorem in this section, relating B-completeness with the notion of nearly open mappings.

Theorem 3.31 (Proposition 9.7.1, Jarchow (1981))**.** Let F be a lcs. For a B-complete lcs E, every nearly open continuous, linear function $T : E \to F$ is open.

Proof. We show that T is open. With no loss of generality, we can assume that T is bijective. If not, we consider the mapping $\tilde{T}$ with

$$\begin{tikzcd}
E/\ker(T) \arrow[r, "\tilde{T}"] & \mathrm{range}(T) \subseteq F \arrow[d, "\iota"] \\
E \arrow[u, "\pi"] \arrow[r, dashed, "T"'] & F
\end{tikzcd}$$

By Lemma 3.30, $E/\ker(T)$ is B-complete. Clearly, $\tilde{T}$ is bijective and if $\tilde{T}$ is open, so is T, since the embedding in the quotient space is an open mapping.

Fix a zero neighborhood $U = U^{\circ\circ} \subseteq E$. Since T is nearly open, $\overline{T(U)}$ is a zero neighborhood in range(T) and therefore, by Alaoğlu-Bourbaki, $B := T(U)^\circ$ is $\sigma(F', F)$-compact. By Lemma 2.28, $B = (T')^{-1}(U^\circ)$ and since T' is weakly continuous, $T'(B) = U^\circ \cap \mathrm{range}(T')$ is weakly compact and therefore $\sigma(E', E)$-closed. Since this holds for every zero neighborhood $U \subseteq E$, range(T') is γ^t-closed. By B-completeness of E, we have that range(T') is even $\sigma(E', E)$-closed.

Since T is assumed to be injective and range(T') is $\sigma(E', E)$-closed, Lemma 2.28 implies that range$(T') = E'$. Since T is assumed to be nearly open, $V = \overline{T(U)}$ is a zero neighborhood in F with $V = V^{\circ\circ}$. We get $(T')^{-1}(U^\circ) = T(U)^\circ = V^\circ$ by Lemma 2.28 and since range$(T') = E'$ it follows that $U^\circ = T'(V^\circ)$ and therefore $T^{-1}(V) = T^{-1}(V^{\circ\circ}) = (T'(V^\circ))^\circ = U^{\circ\circ} = U$ (again use Lemma 2.28). Hence, we have shown that $V = T(U)$ and therefore T is open. □

3.3 Equicontinuous convergence

For the definition of Schwartz spaces, we will need the notion of equicontinuous convergence. As suggested by the name, equicontinuous convergence is in general stronger than continuous convergence (see Theorem 3.41) and we will see in Section 3.4 that Schwartz spaces are those spaces where these two notions of convergence coincide.

Definition 3.32 (Equicontinuous convergence)**.** Let $(E, \mathcal{T})$ be a lcs and $\mathcal{E}$ be the equicontinuous compactology on E'. Let further $\mathcal{U}$ be a zero basis in E. A filter $\mathcal{F}$ in E' is said to converge equicontinuously to $x' \in E'$, iff there is an $U \in \mathcal{U}$ such that the filters $\langle \mathcal{F}, x\rangle$ converge to $\langle x', x\rangle$ uniformly in $x \in U$.

The next Lemma gives a characterization of equicontinuous convergence that sheds light on the connection with continuous convergence. More specifically, while for continuous convergence weak-convergence of the filter in a local Banach space is sufficient, it requires strong convergence in a local Banach space for equicontinuous convergence.

Lemma 3.33. $\mathcal{F}$ converges equicontinuously to 0, iff there is a $U \in \mathcal{U}$, such that $\mathcal{F}$ contains a filterbasis $\mathcal{G}$ of sets in the Banach space E'_{U° which converges to zero in E'_{U°.

Proof. $\mathcal{F}$ converges equicontinuously to 0 in E', iff for every $\epsilon > 0$ there is a $F \in \mathcal{F}$ such that

$$\sup_{x \in U} |\langle F, x \rangle| < \epsilon$$

and hence $F \subseteq \epsilon U^\circ$. Since the above holds for every $\epsilon > 0$, $\mathcal{F}$ contains a zero neighborhood basis of E'_{U°. Therefore the filter generated by $\{F : F \in \mathcal{F}, F \subseteq \operatorname{span}(U^\circ)\}$ converges to 0 in E'_{U°. □

We will mostly work with the above characterization of equicontinuous convergence. To proceed, we first show a characterization of the dual of E_{U° and then proceed to discuss the basic properties of inductive limits, which will be needed to define the topologies η and η^t corresponding to equicontinuous convergence.

Definition 3.34 (Inductive limit of topological vectors spaces)**.** Let $(E_i)_{i \in J}$ be a family of topological vector spaces and J be directed by the order relation $\leq$. Assume further that for every pair $(i, j) \in J \times J$ for which $i \leq j$ there is a continuous linear map $S_{j,i} : E_i \to E_j$, which fulfills the following conditions

1. $S_{j,j} = I_{E_j}$, $\forall j \in J$
2. $S_{k,i} = S_{k,j} \circ S_{j,i}$, if $i \leq j$ and $j \leq k$.

Then the system of spaces E_j along with all the mappings $S_{j,i}$ is called an inductive system of topological vector spaces, which is denoted by $(E_i, S_{j,i})^{(J,\leq)}$. We want to construct a topological vector space E, for which all the triangles in

$$\begin{tikzcd} \cdots \arrow[r] & E_i \arrow[r, "S_{j,i}"] \arrow[dr, "I_i"'] & E_j \arrow[r, "S_{k,j}"] \arrow[d, "I_j"] & E_k \arrow[r] \arrow[dl, "I_k"] & \cdots \\ & & E & & \end{tikzcd} \tag{3.1}$$

commute for some functions I_j and which has the universal property that if there is any other space F, which is of this type (i.e. all the triangles commute), there is a continuous linear mapping from E to F. Such a space E is called the inductive limit $\operatorname{ind}_{j \in J} E_j$ of the inductive system $(E_i, S_{j,i})^{(J,\leq)}$.

The above principle is quite general and can be applied to different categories. We will be in particular be interested in the categories of topological vector spaces, the category of locally convex spaces and the category of Hausdorff locally convex spaces (called lcs in this document). For all these special cases there exist concrete representations of the above limit, some of which will be discussed in the next Lemma.

Lemma 3.35. Let $E = (E, \mathcal{T})$ be a lcs and U be a zero neighborhood, then $E'_{U^\circ} \cong (E_{(U)})'$.

Proof. Let $\Phi_U : E \to E/\ker(U) = E_{(U)}$ be the quotient map whereby $\ker(U) = p_U^{-1}(0)$. Therefore $\Phi'_U : (E_{(U)})' \to E'$. In fact, if $y \in (E_{(U)})'$, then $(x \mapsto y(\Phi_U(x))) \in E'_{U^\circ}$, since y is continuous on $(E_{(U)}, p_U)$ and therefore $|y(\Phi_U(x))| \leq K$ for every $x \in U$. Hence, $\Phi'_U : (E_{(U)})' \to E'_{U^\circ}$. Furthermore Φ'_U is continuous since Φ_U is continuous and Φ'_U is one to one, because if $y_1 \neq y_2$ for y_1, $y_2 \in (E_{(U)})'$, then

$$(x \mapsto y_1(\Phi_U(x))) = \Phi'_U(y_1) \neq \Phi'_U(y_2) = (x \mapsto \Phi_U(y_2(x))).$$

Consider the following diagram

$$\begin{tikzcd} E \arrow[r, "\Phi_U"] \arrow[d, "y"] & E_{(U)} \arrow[dl, dashed, "l"] \\ \mathbb{K} & \end{tikzcd}$$

with $l(\Phi_U(x)) = y(x)$ and $y \in E'_{U^\circ}$. If $\Phi_U(x) = \Phi_U(z)$, then $(x - z) \in \ker(U)$ and therefore $(x - z) \in \rho U$ for all $\rho > 0$. Since $y \in E'_{U^\circ}$ there is a $K > 0$ such that $|y(a)| \leq K$, $\forall a \in U$. Fix $\epsilon > 0$, then there is a $\rho > 0$ such that $\rho K < \epsilon$ and therefore $|y(a)| < \epsilon$, $\forall a \in \rho U$, hence $|y(x - z)| = 0$ and the mapping l is well defined. Since y is continuous, l is also continuous because of the definition of $E_{(U)}$ as final structure with respect to $\Phi_{(U)}$. Hence, the above diagram commutes, establishing that for every $y \in E'_{U^\circ}$ there is a $l \in (E_{(U)})'$ such that $y = l \circ \Phi_U = \Phi'_U(l)$. Therefore Φ'_U is a continuous bijection and by the open mapping theorem also an homeomorphism. $\square$

Lemma 3.36. 1. If the E_j are topological vector spaces, then the limit in the category of topological vector spaces is

$$E = (\bigoplus_{j \in J} E_j)/L$$

where L is the linear span of the set

$$\bigcup_{i,j \in J,\, i \leq j} \operatorname{range}(I_i - I_j \circ S_{j,i})$$

and the quotient space is endowed with the quotient topology.

2. If the E_j are all locally convex vector spaces (not necessarily Hausdorff), then the limit is given by
$$E = (\bigoplus_{j\in J} E_j)/L$$
with the only difference to (1) being that both constructions are taken in the category of locally convex spaces.

3. If the E_j are all locally convex vector spaces which are additionally Hausdorff, then the limit is given by
$$E = (\bigoplus_{j\in J} E_j)/\bar{L}$$
where again the constructions are taken in the category of locally convex spaces.

Proof. **(1)**: The universal property immediately implies that $E = \operatorname{ind}_{j\in J} E_j$ is the finest topological vector space topology, for which (3.1) commutes. The space $\bigoplus_{j\in J} E_j$ fulfills the requirement that all embeddings $\iota_i : E_i \to \bigoplus_{j\in J} E_j$ are continuous. To make sure that (3.1) commutes, we factor out all the linear subspace where $\iota_i \neq S_{j,i} \circ \iota_j$. This turns out to be exactly L.

Let now F be another topological vector space, such that (3.1) commutes. Consider the following diagram:

$$\begin{tikzcd}
& & F & & \\
\cdots \arrow[r] & E_i \arrow[ur, "f_i"] \arrow[r, "S_{j,i}"] \arrow[dr, "\iota_i"'] & E_j \arrow[u, "f_j"'] \arrow[r, "S_{k,j}"] \arrow[d, "\iota_j"] & E_k \arrow[ul, "f_k"'] \arrow[dl, "\iota_k"] \arrow[r] & \cdots \\
& & \bigoplus_{j\in J} E_j \arrow[d, "q"'] & & \\
& & \bigoplus_{j\in J} E_j/L \arrow[uuu, bend left=90, "h"] & &
\end{tikzcd}$$

where $\iota_i : E_i \to \bigoplus_{j\in J} E_j$ define the direct sum and q the quotient topology. Now define a mapping $h : (\bigoplus_{j\in J} E_j)/L \to F$ by $h(x) = f_i(y)$ for an arbitrary i and a y such that $q \circ \iota_i(y) = x$. Suppose there is another y' such that $q \circ \iota_j(y') = x$, this implies that $\iota_i(y) - \iota_j(y') \in L$ and since the triangles in the upper part of the diagram commute $h(\iota_i(y')) = h(\iota_i(y))$. Hence, h is well defined. Obviously $h \circ q \circ \iota_i$ is continuous for every i and therefore h is continuous because of the definition of $\bigoplus_{j\in J} E_j/L$ as final structure (projective limit). Since the direct sum and the quotient are stable in the category of topological vector spaces, the resulting space is again a topological vector space

(2), (3): By the stability of the direct sum and the quotient in the categories of locally convex spaces and Hausdorff locally convex spaces, (2) and (3) follow in the same way. □

Definition 3.37 (The topology η)**.** Let E by a lcs and $\mathcal{U}$ a basis of zero neighborhoods consisting of closed, absolutely convex sets. For $U, V \in \mathcal{U}$ with $V \subseteq U$ define the embeddings $\Phi_{UV} : E_{(V)} \hookrightarrow E_{(U)}$ and the respective adjoint mappings $\Psi_{VU} = \Phi'_{UV} : E'_{V^\circ} \hookrightarrow E'_{U^\circ}$ (see Lemma 3.35). The system $(E'_{V^\circ}, \Psi_{UV})^{(\mathcal{U},\subseteq)}$ is a inductive system of Banach spaces. We denote the locally convex inductive limit by $\eta(E', E)$. The corresponding inductive limit can be identified with $E' = \bigcup_{U\in\mathcal{U}} E'_{U^\circ}$ as a linear space.

Definition 3.38 (The topology η^t)**.** In the setting of the above definition let $\eta^t(E', E)$ be the finest topology on E', such that all the inclusions $\iota_{U^\circ} : E'_{U^\circ} \hookrightarrow E'$ are continuous.

Remark 3.39. The topology $\eta^t(E', E)$ is the inductive limit of the system $(E'_{V^\circ}, \Phi_{UV})^{(\mathcal{U},\subseteq)}$ in the category of topological spaces. Consequently the topology $\eta^t(E', E)$ need not be a locally convex topology.

Theorem 3.40. The topology $\eta^t(E', E)$ is the finest topology on E', such that every equicontinuous filter $\mathcal{F}$ in E' converges in $(E', \eta^t(E', E))$. Furthermore, $\eta(E', E)$ is the finest locally convex topology on E', such that every equicontinuous filter $\mathcal{F}$ in E' converges.

Proof. We start with the first statement. If $\mathcal{F}$ converges equicontinuously, then $\mathcal{F}$ has a basis $\mathcal{G}$ in some local Banach space E_{U° and the filter generated by $\mathcal{G}$ converges in E_{U° (cf Lemma 3.33). Since the inclusions ι_{U° are continuous, also $\iota_{U^\circ}(\mathcal{G})$ converges in $(E', \eta^t(E', E))$ and therefore $\mathcal{F}$ converges in that space. Conversely, if every equicontinuous filter converges in E', then for an arbitrary convergent filter $\mathcal{G}$ in E'_{U° it holds that $\iota_{U^\circ}(\mathcal{G})$ converges in E' which establishes the continuity of ι_{U°. Since η^t is the finest topology, such that the $\iota_{U_\circ}$ are continuous, we have established the first part of the statement.

The second part follows analogously. □

Theorem 3.41 (Proposition 10.3.1, Jarchow (1981))**.** Equicontinuous convergence implies continuous convergence.

Proof. Suppose $\mathcal{F}$ converges equicontinuously in E'. Then by Lemma 3.33 there is a zero neighborhood U in E, such that $\mathcal{F}$ contains a filter basis $\mathcal{G}$, which converges in E'_{U°. Note that U° is $\sigma(E', E)$-bounded and therefore the identity $\mathrm{id} : E'_{U^\circ} \to (E', \sigma(E', E))$ is continuous. Therefore $\mathrm{id}(\mathcal{G})$ converges in $U^\circ|_{\sigma(E',E)}$ and since $U^\circ \in \mathcal{E}$, the filter converges continuously. □

Theorem 3.42 (Proposition 10.3.3, Jarchow (1981))**.** For every lcs E we have that $\beta(E', E) \leq \eta(E', E)$.

Proof. Let $U \subseteq E$ be a zero neighborhood and $\Phi : E \to E_{(U)}$. Then by Lemma 3.35, $\Phi' : E'_{U^\circ} \to E'$. Further by Lemma 4.19, Φ' is $(\sigma(E'_{U^\circ}, E_{(U)}), \sigma(E', E))$-continuous and therefore $(\beta(E'_{U^\circ}, E_{(U)}), \beta(E', E))$-continuous by Theorem 2.30. Since $\beta(E'_{U^\circ}, E_{(U)})$ is the Banach space topology on E'_{U°, we conclude that by the definition of $\eta(E', E)$, as the finest topology such that all the $\Phi' : E'_{U^\circ} \to E'$ are continuous, $\beta(E', E) \leq \eta(E', E)$. □

3.4 Schwartz Spaces

Having defined continuous and equicontinuous convergence, we can now proceed to study Schwartz spaces, which happen to be spaces where these two notions of convergence coincide and consequently also the topologies η and γ as well as η^t and γ^t are the same.

Definition 3.43 (Schwartz Space)**.** A lcs $(E, \mathcal{T})$ is called a Schwartz space and $\mathcal{T}$ is called a Schwartz topology, if every continuously convergent filter $\mathcal{F}$ on E' is also equicontinuously convergent.

Theorem 3.44. In a Schwartz space E, $\eta^t(E', E) = \gamma^t(E', E)$ and $\eta(E', E) = \gamma(E', E)$.

Proof. Because E is Schwartz, equicontinuous and continuous convergent filters coincide by Theorem 3.41. Therefore $\eta^t(E', E) = \gamma^t(E', E)$ follows from Theorem 3.12 and 3.40. While $\eta(E', E) = \gamma(E', E)$ follows from Remark 3.15 and Theorem 3.40. □

Next we will show an extensive characterization of Schwartz spaces in Theorem 3.48 for which we need the next two technical lemmata.

Lemma 3.45. If K is compact, F Hausdorff, and $f : K \to F$ a continuous bijection, then f is already a homeomorphism.

Proof. Since for $A \subseteq K$ closed, $f(A) \subseteq F$ is compact and therefore closed, f is a closed mapping. Therefore f^{-1} is continuous and f is a homeomorphism as claimed. □

Lemma 3.46. Let E be a lcs and $V \subseteq U$ be two zero neighborhoods and denote by $\circ$ the polarization with respect to $\langle E_{(V)}, E'_{V^\circ}\rangle$, and by $\bullet$ the polarization with respect to $\langle E_{(U)}, E'_{U^\circ}\rangle$. Then for $A \subseteq E_{(U)}$, we have

$$\Phi'_{UV}(A^\bullet) \subseteq \Phi_V(V)^\circ \Leftrightarrow A^{\bullet\bullet} \supseteq \Phi_U(V).$$

Proof. If $\Phi'_{UV}(A^\bullet) \subseteq \Phi_V(V)^\circ$, then

$$A^{\bullet\bullet} \supseteq_1 \Phi''_{UV}(\Phi_V(V)^{\circ\circ}) =_2 \Phi''_{UV}(\Phi_V(V)) = \Phi_{UV}(\Phi_V(V)) =_3 \Phi_U(V)$$

where 1 follows from Lemma 2.28, 2 by the fact that V is absolutely convex and finally 3 by the fact that

$$\begin{tikzcd} E_{(V)} \arrow[r, "\Phi_{UV}"] & E_{(U)} \\ E \arrow[u, "\Phi_V"] \arrow[ur, dashed, "\Phi_U"'] & \end{tikzcd}$$

commutes.

Conversely, if $A^{\bullet\bullet} \supseteq \Phi_U(V) = \Phi_{UV}(\Phi_V(V))$, then again from Lemma 2.28

$$\Phi'_{UV}(A^\bullet) = \Phi'_{UV}(A^{\bullet\bullet\bullet}) \subseteq \Phi(V)^\circ.$$

□

Lemma 3.47. Let E be a lcs and $V \subseteq U$ be two zero neighborhoods. Then $U^\circ \subseteq E'_{V^\circ}$ is compact, iff Φ'_{UV} is $(\tau_{pc}(E'_{U^\circ}, E_{(U)}), \beta(E'_{V^\circ}, E_{(V)}))$-continuous.

Proof. Note that $\Phi'_{UV} : (E'_{U^\circ}, \sigma(E_{U^\circ}, E_{(U)})) \to (E'_{V^\circ}, \sigma(E_{V^\circ}, E_{(V)}))$ is continuous. If U° is compact in E'_{V°, the topology induced by the norm topology of E'_{V° and the topology $\sigma(E'_{V^\circ}, E_{(V)})$ coincide on U° (by Lemma 3.45). Furthermore, since by Alaoğlu-Bourbaki U° is $\tau_{pc}(E', E)$-compact, it again follows from Lemma 3.45 that $\sigma(E_{U^\circ}, E_{(U)})) = \tau_{pc}(E'_{U^\circ}, E_{(U)})$ and therefore ($\Rightarrow$). This also yields ($\Leftarrow$), since U° is compact in $\tau_{pc}(E'_{U^\circ}, E_{(U)})$ and by continuity therefore also in the strong topology in E'_{V°. $\square$

We are now ready to give a characterization of Schwartz spaces. The first four points of the following theorem are taken from Jarchow (1981), while the fifth is from Meise and Vogt (1997).

Theorem 3.48 (Theorem 10.4.1, Jarchow (1981))**.** Let $\mathcal{E}$ be the equicontinuous compactology for a given lcs $(E, \mathcal{T})$ and $\mathcal{U}$ be a zero neighborhood basis consisting of absolutely convex and closed sets. The following statements are equivalent:

1. E is a Schwartz space.
2. Every $U \in \mathcal{U}$ contains a $V \in \mathcal{U}$ such that U° is compact in the Banach space E'_{V°.
3. $\mathcal{T}$ is the topology of uniform convergence on the $\mathcal{E}$-null sequences of E'.
4. Every $U \in \mathcal{U}$ contains a $V \in \mathcal{U}$ such that $\Phi_U(V)$ is precompact in the normed space $E_{(U)}$.
5. For every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $\forall \epsilon > 0$, $\exists x_1, \ldots, x_N$ such that
$$V \subseteq \bigcup_{j=1}^{N} (x_i + \epsilon U).$$

Proof. **(1) $\Rightarrow$ (2)**: Consider the compact space $U^\circ \subseteq (E', \sigma(E', E))$. The neighborhood filter of zero in U° generates a filter $\mathcal{F}$ on E' which converges continuously to zero. By assumption it follows that $\mathcal{F}$ converges to zero equicontinuously, i.e. there is a $V \subseteq U$ such that $\mathcal{F}$ has a basis in E'_{V° converging to zero in the corresponding norm (see Lemma 3.33). Hence, the neighborhood filter of zero in U° is mapped to a convergent filter in E'_{V° and therefore the embedding $\iota : U^\circ \hookrightarrow E'_{V^\circ}$ is continuous. It follows that U° is compact in E'_{V°.

(2) $\Rightarrow$ (3): Let $\mathcal{E}_0$ be the compactology on E' which consists of the bipolars of the local $\mathcal{E}$-null sequences (this is in fact a compactology by Lemma 2.15). Let further $\mathcal{T}_{c_0}$ be the topology induced on E by uniform convergence on the sets $\mathcal{E}_0$. Then for each local $\mathcal{E}$-null sequence $(x'_n)_{n\in\mathbb{N}}$ there is a $H \in \mathcal{E}$ such that $x_n \in H$ for all $n \in \mathbb{N}$ and therefore $\{x'_n : n \in \mathbb{N}\}^{\circ\circ}$ is a compact subset of H and hence $\{x'_n : n \in \mathbb{N}\}^{\circ\circ} \in \mathcal{E}$. Therefore $\mathcal{T}_{c_0} \leq \mathcal{T}$ (since $\mathcal{T}$ is the topology of uniform convergence on sets in $\mathcal{E}$).

To complete the proof, we will show the converse inequality. Let $U \in \mathcal{U}$, then by the hypothesis there is a $V \in \mathcal{U}$ such that $V \subseteq U$ and U° is compact in E'_{V°. By Theorem 2.16 there is a null sequence $(x'_n)_{n\in\mathbb{N}} \subseteq E'_{V^\circ}$ such that $U^\circ \subseteq \{x'_n : n \in \mathbb{N}\}^{\circ\circ}$. This implies that $\mathcal{T} \leq \mathcal{T}_{c_0}$.

(3) ⇒ (1): Suppose $\mathcal{F}$ converges continuously to zero, i.e. there is a $H \in \mathcal{E}$ such that the trace of $\mathcal{F}$ on H converges weakly to zero. Without loss of generality we can assume $H = U^\circ$ and $U^\circ \in \mathcal{F}$ for some $U \in \mathcal{U}$. By our hypothesis, we can find a $V \subseteq U$, $V \in \mathcal{U}$ and a null sequence $(x_n)_{n\in\mathbb{N}} \subseteq E'_{V^\circ}$ such that $U^\circ \subseteq \{x_n : n \in \mathbb{N}\}^{\circ\circ} \subseteq E'_{V^\circ}$. Since $\{x_n : n \in \mathbb{N}\}^{\circ\circ}$ is compact in E'_{V° (by Lemma 2.15) the identity map $\mathrm{id} : (U^\circ, ||.||_{V^\circ}) \to (U^\circ, \sigma(E', E))$ is a homeomorphism (by Lemma 3.45) and therefore the topologies induced by $\sigma(E', E)$ and E'_{V° coincide on U°. This shows that $\mathcal{F}$ converges equicontinuously to zero.

(2) ⇔ (4): From Lemma 3.47 it follows: (2) ⇔ there is a precompact set $A \subseteq E_{(U)}$ with $\Phi'_{UV}(A^\bullet) \subseteq \Phi_V(V)^\circ$ ⇔ (by Lemma 3.46) there is a precompact set $A \subseteq E_{(U)}$ with $\Phi_U(V) \subseteq A^{\bullet\bullet}$ ⇔ $\Phi_U(V)$ is precompact in $E_{(U)}$. This finishes the proof.

(4) ⇔ (5): (Meise & Vogt 24.17) We first argue (⇒). Let $U \in \mathcal{U}$ and V be chosen according to (4), then $\forall \epsilon > 0$ there exist $x_1, \dots, x_N \in E_{(U)}$ with

$$\Phi_U(V) \subseteq \bigcup_{i=1}^{N} (x_i + \epsilon \Phi_U(U))$$

and therefore $V \subseteq V + \ker(p_U) \subseteq \bigcup_{i=1}^N \Phi_U^{-1}(x_i) + \epsilon U = \bigcup_{i=1}^N y_i + \epsilon U$, where $y_i \in \Phi_U^{-1}(x_i)$ and the last inequality holds because $\ker(p_U) \subseteq \epsilon U$ for all $\epsilon > 0$. This shows (⇒). Note that (⇐) is trivial because if V is precompact in E, then $\Phi_U(V)$ is precompact in $E_{(U)}$. □

Remark 3.49. Condition 4 in the above Theorem can be replaced by the condition that for every normed space F and every $A \in L(E, F)$ there is a zero neighborhood $V \subseteq E$ such that $A(V)$ is precompact in F. To see this just choose $U = A^{-1}(\{x \in F : ||x|| < 1\})$ and V according to condition 5 in the above theorem. Then $A(V)$ is precompact in F.

Corollary 3.50 (Lemma 24.19, Meise and Vogt (1997))**.** If E is a complete Schwartz space, then every bounded set in E is relatively compact.

Proof. Let $(||.||_\alpha)_{\alpha\in A}$ be a fundamental system of seminorms for E, then E can be identified with a subspace of $\prod_{\alpha\in A} E_\alpha$, where $E_\alpha = \widetilde{E/ker(\alpha)}$ with $\Phi_\alpha : E \to \widetilde{E/ker(\alpha)}$

the canoncial mapping. Let $B \subseteq E$ be bounded, then $B_\alpha = \overline{\Phi_\alpha(B)}$ is compact in E_α, because $\Phi_\alpha(B)$ is precompact. This is the case, since from Theorem 3.48 it follows that there is a $V \subseteq \{x : ||x||_\alpha \leq 1\} = U$ such that $\Phi_\alpha(V)$ is precompact in $(E_\alpha, ||\cdot||_\alpha)$. Because V is a zero neighborhood, we have $B \subseteq \lambda V$ for some $\lambda > 0$ and therefore $\Phi_\alpha(B) \subseteq \Phi_\alpha(\lambda V)$ is precompact. Since E_α is a Banach space, the closure of every precompact set is compact by Lemma 2.11.

By Tychonov, $\prod_{\alpha\in A} B_\alpha$ is compact in $\prod_{\alpha\in A} E_\alpha$. Since E is complete, it is closed as a subspace of $\prod_{\alpha\in A} E_\alpha$ and therefore B is relatively compact in E. □

Definition 3.51 (Ultra-Bornological)**.** A lcs E is called ultra bornological, if it has the topology of some inductive system $(j_i : E_i \to E)_{i\in I}$ of Banach spaces E_i.

Theorem 3.52 (Proposition 24.23, Meise and Vogt (1997))**.** For every complete Schwartz space E the dual E' is ultra bornological.

Proof. By Corollary 3.50 we know, that $\beta(E', E) = \mu(E', E)$. Now fix a zero neighborhood basis $\mathcal{U}$ of absolutely convex sets in E. Then by Alaoğlu-Bourbaki U° is $\sigma(E', E)$-compact for all $U \in \mathcal{U}$ and therefore by 7.4.17 in Kriegl (2007) is a Banach disk and $(E'_{U^\circ})'$ is a Banach space. The inclusions $j_{U^\circ} : E'_{U^\circ} \to (E', \mu(E', E))$ are continuous and therefore the inductive topology $\mathcal{T}'$ generated by the system $(j_{U^\circ} : E'_{U^\circ} \to E')_{U\in\mathcal{U}}$ is Hausdorff because it is finer than $\mu(E', E)$ (which is Hausdorff).

To see that $\mu(E', E) \geq \mathcal{T}'$, we choose a $U \in \mathcal{U}$ and a $V \in \mathcal{U}$ such that $V \subseteq U$ and U° is compact in E'_{V° (Theorem 3.48, 2). By continuity of j_{V°, U° is compact in $(E, \mathcal{T}')$ and therefore $\mathrm{id} : (U^\circ, \mathcal{T}') \to (U^\circ, \sigma(E', E))$ is a homeomorphism by Lemma 3.45. Hence, $\sigma(E', E)_{|U^\circ} = \mathcal{T}'_{|U^\circ}$ and by Theorem 2.34, $\mathcal{T}' \leq \tau_c(E', E) \leq \mu(E', E)$.

In effect, we have $\beta(E', E) = \mu(E', E) = \mathcal{T}'$ and the latter is a inductive limit of the form required for the space to be ultra bornological. □

The remainder of the section os dedicated to results about the schwartzification $\mathcal{T}_{c_0}$ of a lcs, which was already used in the proof of Theorem 3.48.

Definition 3.53 (The topology $\mathcal{T}_{c_0}$)**.** For a lcs $(E, \mathcal{T})$ we define the topology $\mathcal{T}_{c_0}$ as in the proof of Theorem 3.48, by the topology of uniform convergence on the $\mathcal{E}$-null sequences of E'.

Remark 3.54. Since every finite sequence can be interpreted as $\mathcal{E}$-null sequence (by continuation with zeros), it always holds that

$$\sigma(E', E) \leq \mathcal{T}_{c_0} \leq \mathcal{T}.$$

This shows that $\mathcal{T}_{c_0}$ is compatible with $\langle E, E'\rangle$. We denote $(E, \mathcal{T}_{c_0})$ by E_0.

Theorem 3.55 (Theorem 10.4.4, Jarchow (1981))**.** $\mathcal{T}_{c_0}$ is the finest Schwartz topology on E which is coarser than $\mathcal{T}$.

Proof. Obviously $\mathcal{T}_{c_0} \leq \mathcal{T}$. By Theorem 2.43 the $\mathcal{E}$-null sequences and the $\mathcal{E}_0$-null sequences in E' coincide, therefore $\mathcal{T}_{c_0}$ is the topology of uniform convergence on $\mathcal{E}_0$ sequences. Furthermore by the proof of Theorem 3.48, $\mathcal{E}_0$ is a basis for the $\mathcal{T}_{c_0}$-equicontinuous compactology and therefore $\mathcal{T}_{c_0}$ is a Schwartz topology by Theorem 3.48.

Assume now that $\mathcal{T}_1 \leq \mathcal{T}$ is a Schwartz topology on E, then $E'_1 = (E, \mathcal{T}_1)'$ is contained in E' and $\mathcal{E}$-null sequences in E'_1 are $\mathcal{E}$-null sequences in E' (because of $\mathcal{T}_1 \leq \mathcal{T}$). Therefore $\mathcal{T}_1 \leq \mathcal{T}_{c_0}$. $\square$

Theorem 3.56 (Proposition 10.4.6, Jarchow (1981))**.** If E is a lcs, then $\eta^t(E', E) = \gamma^t(E', E_0)$ and $\eta(E', E) = \gamma(E', E_0)$, where $E_0 = (E, \mathcal{T}_{c_0})$.

Proof. We first prove the the first statement: it follows from Theorem 3.44 and $\mathcal{T}_{c_0} \leq \mathcal{T}$ that $\eta^t(E', E) \leq \eta^t(E', E_0) = \gamma^t(E', E_0)$. To prove the converse consider a null sequence $(x_n)_{n\in\mathbb{N}} \subseteq E'_{U^\circ}$, where U is a zero neighborhood in $(E, \mathcal{T})$. By the definition of $\mathcal{T}_{c_0}$, the set $A = \{x_n : n \in \mathbb{N}\}^{\circ\circ}$ is a zero neighborhood in E_0 and therefore $E'_{A^\circ} \hookrightarrow (E', \eta^t(E', E_0))$ is continuous, which implies that $(x_n)_{n\in\mathbb{N}}$ is a null sequence in $\eta^t(E', E_0)$. This in turn establishes the continuity of the insertation $E'_{U^\circ} \hookrightarrow (E', \eta^t(E', E_0))$ and implies $(E', \eta^t(E', E_0)) \leq (E', \eta^t(E', E))$, since $(E', \eta^t(E', E))$ is the finest topology such that this is the case. The second statement can be proven analogously. $\square$

Theorem 3.57 (Corollary 10.4.7, Jarchow (1981))**.** For every lcs E, the completion $\widetilde{E_0}$ can be identified with the dual of $(E', \eta(E', E))$. Moreover, we have

$$\eta(E', E) = \beta(E', \widetilde{E_0}) = \mu(E', \widetilde{E_0}).$$

Proof. By Theorem 3.56, we know that $\eta(E', E) = \gamma(E', E_0)$, therefore

$$(E', \eta(E', E))' = (E', \gamma(E', E_0))' \cong \widetilde{E_0},$$

where the last equality follows by Theorem 3.8.

We know $\eta(E', E) = \gamma(E', E_0) = \mu(E', \widetilde{E_0}) = \beta(E', \widetilde{E_0})$, where the second equality follows from Theorem 3.20, while the third follows from the fact that E_0 is a Schwartz space and therefore every bounded set in E_0 is precompact and therefore relatively compact in $\widetilde{E_0}$ (by Lemma 2.11). $\square$

3.5 Montel Spaces

In this section we discuss the notion of Montel spaces. These spaces often arise in distribution theory. For exampe, the space of test functions or the space of rapidly decreasing functions as well as their strong dual spaces are Montel spaces. The term Montel space originates from the theorem of Montel, which states that the holomorphic functions on an open connected subset of $\mathbb{C}$ have property that the closure of every bounded subset is compact.

In the context of this thesis we are interested in Montel spaces mainly because a lcs is a (DF) space if and only if its strong dual is a Fréchet-Montel space (see Theorem 4.37). Dually, a space is a (DF) space if it is the strong dual of a Fréchet-Montel space (see Theorem 4.42). Finally, we discuss a useful homomorphism theorem for semi-Montel spaces that are (df) spaces (see Theorem 4.45).

Definition 3.58 (Semi-Montel spaces)**.** A lcs E is called a semi-Montel space, iff every bounded set is relatively compact.

Theorem 3.59 (Proposition 11.5.1, Jarchow (1981))**.** Every semi-Montel space $E = (E, \mathcal{T})$ is semi-reflexive, and $\sigma(E, E')$-convergent sequences are even $\mathcal{T}$-convergent.

Proof. Since in a semi-Montel space every bounded set is relatively compact, it is also relatively $\sigma(E, E')$-compact and therefore the space E is semi-reflexive by Theorem 2.32. Consider a weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$. Then $A = \{x_n : n \in \mathbb{N}\}$ is bounded and therefore $\mathcal{T}$-relatively compact and hence $(x_n)_{n\in\mathbb{N}}$ $\mathcal{T}$ converges. □

Theorem 3.60. For a semi-Montel lcs space E the following conditions are equivalent

1. E is reflexive.

2. E is quasi-barrelled.

3. E is barrelled.

Proof. Since every semi-Montel space is semi-reflexive the equivalence of the three conditions follows from Theorem 2.50. □

Definition 3.61 (Montel spaces)**.** A semi-Montel space which fulfills one of the equivalent conditions of Theorem 3.60 is called a Montel space.

Theorem 3.62 (Proposition 11.5.2, Jarchow (1981))**.** A lcs E is a semi-Montel space, iff it is quasi-complete and every equicontinuous set in E' is relatively compact for $\beta(E', E)$.

Proof. If E is semi-Montel, then every closed and bounded set is compact and therefore complete. Hence, E is quasi-complete. Since every bounded set in E is relatively compact, we know that $\beta(E', E) = \tau_{pc}(E', E)$. By Alaoğlu-Bourbaki equicontinuous sets are therefore relatively $\beta(E', E)$-compact.

Suppose conversely that E is quasi-complete and that every equicontinuous set is relatively $\beta(E', E)$-compact. Then we know by Theorem 3.21 that $\beta(E', E) = \tau_{pc}(E', E)$.

Thus bounded sets are precompact and, by quasi-completeness, even relatively compact (since the closure of these sets is complete). □

Theorem 3.63 (Proposition 11.5.4, Jarchow (1981)). If a lcs E is a Montel space, then also $(E', \beta(E', E))$ is Montel.

Proof. If E is Montel, it is in particular reflexive (Theorem 3.60). Hence, $(E', \beta(E', E))$ is also reflexive and therefore we only have to show that $(E', \beta(E', E))$ is semi-Montel. If $A \subseteq E'$ is an absolutely convex, closed $\beta(E', E)$-bounded set, then for every bounded set $B \subseteq E$ there is a $\lambda_B > 0$ such that $A \subseteq \lambda_B B^\circ$ and therefore $B^{\circ\circ} \subseteq \lambda_B A^\circ$. Hence, A° is a bornivorous barrel in E. Since E is quasi-barrelled (c.f. Theorem 2.55), A° is a zero neighborhood and therefore $A = A^{\circ\circ}$ is equicontinuous. Hence, every $\beta(E', E)$-bounded set is equicontinuous and therefore relatively compact in $\tau_{pc}(E', E)$. Since E is Montel $\tau_{pc}(E', E) = \beta(E', E)$, which finishes the proof. □

Theorem 3.64 (Proposition 11.6.1, Jarchow (1981)). A Fréchet space E is a Montel space, iff $(E', \beta(E', E))$ is a Schwartz space.

Proof. If E is a Montel space, then it is reflexive by Theorem 3.60. Therefore the bornology $\mathcal{B}$ of bounded sets in E is the saturated hull of the equicontinuous compactology associated with $(E', \beta(E', E))$, i.e. the sets A° for $\beta(E', E)$-equicontinuous sets $A \subseteq E$. By Theorem 2.16 every $B \in \mathcal{B}$ is contained in the bipolar of some null sequence $(x_n)_{n\in\mathbb{N}}$ in E. Since E is Fréchet, $(x_n)_{n\in\mathbb{N}}$ is also a $\mathcal{B}$-null sequence by Theorem 2.40. Hence, convergence in $\beta(E', E)$ is the same as convergence on $\mathcal{B}$-null sequences, which implies by Theorem 3.48 that $(E', \beta(E', E))$ is a Schwartz space.

Conversely suppose that $(E', \beta(E', E)$ is Schwartz. Since E is a Fréchet space, $(E', \beta(E', E))$ is complete by Theorem 3.27. By Theorem 3.50, every bounded set in $(E', \beta(E', E))$ is relatively compact and therefore $(E', \beta(E', E))$ is a semi-montel space. If $A \subseteq E'$ is equicontinuous, then A° is a zero neighborhood in E. Hence, for every bounded set $B \subseteq E$, there is a $\lambda_B : B \subseteq \lambda_B A^\circ$. Therefore we have $A \subseteq A^{\circ\circ} \subseteq \lambda_B B^\circ$. Since B° form a zero neighborhood basis of $\beta(E', E)$, A is bounded and therefore relatively $\beta(E', E)$-compact. Therefore $\beta(E', E) \leq \gamma(E', E) = \tau_c(E', E)$, where the first inequality follows from the fact that $\gamma(E', E)$ is the finest lcs topology such that all the equicontinuous sets in E' are compact and the second follows from Theorem 3.20. Hence, $\beta(E', E)$ is a permissible topology and reflexive. By Theorem 3.60 $(E, \beta(E', E))$ is therefore a Montel space. Since $\beta(E', E)$ is admissible and all equicontinuous sets in E' are bounded in $\beta(E', E)$, the strong dual of $(E', \beta(E', E))$ is again E. Hence, by Theorem 3.63 also E is a Montel space. □

Theorem 3.65 (Proposition 11.6.2, Jarchow (1981)). A Fréchet space E is a Montel space, iff it is separable and every $\sigma(E', E)$-convergent sequence in E' converges with respect to $\beta(E', E)$.

Proof. If E is a Fréchet-Montel space, then so is E' with respect to $\beta(E',E) = \mu(E',E)$ by Theorem 3.63. Consequently, by Theorem 3.59, every weakly convergent sequence in E' is strongly convergent.

To show that E is separable, let $\{U_n : n \in \mathbb{N}\}$ be a zero neighborhood basis in E consisting of closed, absolutely convex sets. Let further q_n be the seminorm associated with U_n. By Remark 4.3.4. Kriegl (2007), we know that E can be identified with a subspace of $\prod_{n\in\mathbb{N}} E_{(U_n)}$. Hence, E is separable if all the $E_n = E_{(U_n)}$ are separable. Suppose this would be false. We can then assume w.l.o.g. that E_1 is not separable. Hence, there exists an uncountable set $A_1 \subseteq U_1$ and an $\epsilon > 0$ such that $q_1(x-y) \geq \epsilon$ for all $x \neq y \in A_1$. Because $E = \bigcup_{k\in\mathbb{N}} kU_2$ there is a k_2 such that $A_1 \cap kU_2$ contains a uncountable subset $A_2 \subseteq A_1$. Setting $k_1 = 1$ and continuing this way we get a subsequence $(k_n)_{n\in\mathbb{N}}$ and a decreasing sequence of of sets $(A_n)_{n\in\mathbb{N}}$ such that $A_n \subseteq k_n U_n$, for all $n \in \mathbb{N}$. If the A_n are not strictly decreasing, we can force this property by leaving out some of the A_n.

For each $n \in \mathbb{N}$, choose $x_n \in A_n \setminus A_{n+1}$. Then $q_m(x_n) \leq k_m$, $\forall m, n \in \mathbb{N}$ and $n \geq m$, since $x_n \in A_m$. Hence, $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in E and since E is Fréchet-Montel space $(x_n)_{n\in\mathbb{N}}$ is relatively compact and thus contains a convergent subsequence $(x_{n_i})_{i\in\mathbb{N}}$. In particular we get $q_1(x_{n_i} - x_{n_j}) < \epsilon$ for all i and j big enough. A contradiction to our choice of A_1.

To prove the converse assume that E is a separable Fréchet space such that in E' sequential weak and sequential strong convergence are equivalent. Since E is separable, the polar U° of any zero neighborhood in $U \subseteq E$ is $\sigma(E',E)$-metrizable by Theorem 3.2. Therefore by our second assumption the canonical injection $U^\circ_{|\sigma(E',E)} \hookrightarrow (E', \beta(E',E))$ is continuous so that U° is compact in $\beta(E',E)$. It follows from Theorem 3.62 that E is a semi-Montel space. Since E is Fréchet, it is even a Montel space by Theorem 3.60. □

The following analogous characterization of Fréchet-Schwartz spaces will be needed in the next Chapters to characterize the duals of Fréchet-Schwartz spaces.

Theorem 3.66 (Proposition 11.6.3, Jarchow (1981))**.** A Fréchet space E is a Schwartz space iff it is separable and every $\sigma(E',E)$-convergent sequence in E' converges equicontinuously.

Proof. If E is a Fréchet-Schwartz space, then it is a semi-Montel space by Theorem 3.50. Hence, E is separable by Theorem 3.65. Now let $(u_n)_{n\in\mathbb{N}}$ be a $\sigma(E',E)$-convergent sequence in E'. Then $\forall x \in E$ there is a $\lambda_x > 0$ such that $\{u_n : n \in \mathbb{N}\} \subseteq \lambda_x \{x\}^\circ$ and therefore $\lambda_x \{u_n : n \in \mathbb{N}\}^\circ \supseteq \{x\}^{\circ\circ}$. Hence, $\{u_n : n \in \mathbb{N}\}$ is a barrel and therefore a zero neighborhood in E. It follows from Theorem 3.48 that there is a $U \subseteq \{u_n : n \in \mathbb{N}\}^\circ$ such that $\{u_n : n \in \mathbb{N}\}^{\circ\circ}$ is compact in E_{U° and therefore converges equicontinuously in E'.

Let conversely E be a separable Fréchet space in which every weakly convergent sequence converges equicontinuously. Let U be a zero neighborhood in E. According to

Theorem 3.48 to show that E is Schwartz we have to show that U° is compact in E'_{V° for some zero neighborhood $V \subseteq U$ in E. To do this let $(U_k)_{k\in\mathbb{N}}$ be a zero neighborhood basis in E such that $U_{k+1} \subseteq U_k \subseteq U$ for all $k \in \mathbb{N}$. Since E is a separable Fréchet space, U° is $\sigma(E', E)$-metrizable by Theorem 3.2. Hence, we are done if we can show that every null sequence in U° converges in some Banach space $F_k = E'_{U_k^\circ}$, where $k \in \mathbb{N}$ depends only on U.

If this is not the case, then for all $k \in \mathbb{N}$ we can find a null sequence $(u_n^k)_{n\in\mathbb{N}}$ which does not converge in F_k. Since U° is $\sigma(E', E)$-metrizable, $0 \in U^\circ$ has a countable neighborhood basis $B_0 \supseteq B_1 \supseteq \cdots$. Given $n \in \mathbb{N}$, let m_n be the smallest positive integer so that $u_i^k \in B_n$ $\forall 1 \leq k \leq n$, $\forall i > m_n$. Note that $1 =: m_0 \leq m_1 \leq m_2 \leq \cdots$ and that the finite and possibly empty sequence $(u^1_{m_{n-1}}, \ldots, u^1_{m_n}, \ldots, u^n_{m_{n-1}}, \ldots, u^n_{m_n})$ is contained in B_{n-1}, $\forall n \in \mathbb{N}$. Lining up all these finite subsequences in their natural ordering we thus get a null sequence $(u_n)_{n\in\mathbb{N}}$ in U° which does not converge in any F_k since it contains every $(u_n^k)_{n\in\mathbb{N}}$ as a subsequence. This is a contradiction to the assumption that every weakly convergent sequence converges equicontinuously. $\square$

Chapter 4

df Spaces

After the first three chapters, we can now turn our attention to (df) spaces. We will start by the definitions of the different variants of these class of spaces, their characterizations, inter-relations as well as stability properties in Section 4.1. We proceed with a short section on biduals of Fréchet spaces, where some results about the reflexivity of Fréchet spaces are proven using the theory of (df) spaces developed so far in Section 4.1.

Section 4.3 explores the connection of imbedding spectra of countably many normed spaces and bornological (df) spaces. In particular, we show that every bornological (df) space can be interpreted as a countable imbedding spectrum of normed spaces. Conversely, we show that every weakly compact imbedding spectrum is already a bornological (DF) space.

In the last section of this Chapter, we focus on the various connections between (df) spaces and Schwartz as well as Montel spaces. In particular, we will see that the connection between these two types of spaces can often be made via the dual pairing of two spaces, i.e. that the strong duals of Schwartz or Montel spaces turn out to be Schwartz and or Montel spaces.

4.1 Characterization and stability properties

We start our exposition with a simple result about fundamental sequences of bounded sets, whose existence is necessary for the space to be a (df) space.

Theorem 4.1. The strong dual $\beta(E', E)$ is metrizable, iff E admits a countable basis for the bornology, called a *fundamental sequence* of bounded sets.

Proof. The strong topology on E' is generated by seminorms of the form $p_B(y) = ||y_{|B}||_\infty$ for $B \in \mathcal{B}(E)$.
$(\Rightarrow)$: There exists a countable basis of seminorms $(p_{B_n})_{n\in\mathbb{N}}$. If $B \in \mathcal{B}(E)$, then p_B is

continuous and therefore $\exists n \in \mathbb{N} : p_B \leq p_{B_n} \Rightarrow B \subseteq B_n$.
($\Leftarrow$): If $(B_n)_{n\in\mathbb{N}}$ is a countable basis of the bornology, then $\forall B \in \mathcal{B}(E) \exists B_n : B \subseteq B_n$ and therefore $p_B \leq p_{B_n}$. □

For the rest of the chapter, we will assume without loss of generality that the sets in a fundamental basis $(B_n)_{n\in\mathbb{N}}$ are absolutely convex and that $B_n + B_n \subseteq B_{n+1}$.

The following is an introductory result motivating the notion of (df) spaces (*dual of a Frechét space*). A more general result in this spirit is proven in Theorem 4.5.

Theorem 4.2 (Theorem 12.4.1, Jarchow (1981))**.** The strong dual $(E', \beta(E', E))$ of a lcs E is a Fréchet space, iff

1. E admits a fundamental sequence of bounded sets and
2. $\mu(E, E')$ is quasi-c_0-barrelled.

Proof. By Theorem 4.1 $\beta(E', E)$ is metrizable, iff E admits a fundamental sequence of bounded sets. By Theorem 2.64 local completeness (and therefore completeness by Theorem 2.40) is equivalent to quasi-c_0-barrelledness of $(E, \mu(E, E'))$. □

Remark 4.3. If there exists an admissible topology $\mathcal{T}'$ with $\mathcal{T}' \geq \tau_{\beta,0}$, then condition 2 of the above theorem is fulfilled because $\tau_{\beta,0} \leq \mathcal{T}' \leq \mu(E, E')$ implies that $\mu(E, E')$ is quasi-c_0-barrelled.

Theorem 4.2 motivates the definition of the most general class of (df) spaces, as the class of spaces whose strong dual is a Fréchet space. Note that we need the two conditions in Theorem 4.2 to be true for this to happen: the existence of a fundamental sequence ensures that $(E', \beta(E', E))$ is metrizable, while the sequential barrelledness assumption implies completeness of the dual.

Definition 4.4 ((df), (DF) and (gDF) spaces)**.**
1. Every quasi-c_0-barrelled lcs which admits a fundamental sequence of bounded sets is called a (df) space.
2. A df-space which is quasi-$\aleph_0$-barrelled is called a DF-space.
3. If $(E, \mathcal{T})$ admits a fundamental sequence of bounded sets $\mathcal{A} = (B_n)_n \in \mathbb{N}$ and $\mathcal{T} = \mathcal{T}^{\mathcal{A}}$, then E is called a (gDF) space (*generalized (DF) space*).

The next Theorem is a generalization of 4.2 in the sense that it is shown that for a (gDF) space even the space of linear functions to an arbitrary other Fréchet space is again a Fréchet space.

Theorem 4.5 (Theorem 12.4.2, Jarchow (1981))**.** For a lcs E the following are equivalent:

1. E is a (gDF) space.

2. $(\mathcal{L}(E,F),\mathcal{T}_\beta)$ is a Fréchet space, for every Fréchet space F.

3. $(\mathcal{L}(E,F),\mathcal{T}_\beta)$ is a Fréchet space, for every Banach space F.

Proof. **(1)** $\Rightarrow$ **(2)**: The seminorms that describe the topology of $(\mathcal{L}(E,F),\mathcal{T}_\beta)$ are given by

$$p_{n,m}(T) = \sup_{x\in B_m} q_n(T(x)) \tag{4.1}$$

where B_m is an element of a fundamental sequence of bounded sets $\mathcal{A} = (B_k)_{k\in\mathbb{N}}$ in E and q_n is a seminorm in F. These are countably many and therefore $(\mathcal{L}(E,F),\mathcal{T}_\beta)$ is metrizable.

To show completeness, let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{L}(E,F),\mathcal{T}_\beta)$. Because of (4.1) and the completeness of F, $(T_n)_{n\in\mathbb{N}}$ converges pointwise to some linear map $T \in F^E$. Let V be a absolutely convex zero neighborhood in F. From (4.1) and since $(T_n)_{n\in\mathbb{N}}$ is Cauchy, for each $k\in\mathbb{N}$, we get a $n_k\in\mathbb{N}$ such that

$$(T_m - T_n)(B_k) \subseteq \frac{1}{2}V, \quad \forall n,m \geq n_k$$

and therefore $(T-T_n)(B_k) \subseteq \frac{1}{2}V$, $\forall n \geq n_k$. Since T_{n_k} is continuous there is a zero neighborhood U_k in E such that $T_{n_k}(U_k) \subset \frac{1}{2}V$. Since T and T_{n_k} are linear and V and B_k are absolutely convex, we get

$$T(B_k\cap U_k) \subseteq T_{n_k}(U_k) + (T-T_{n_k})(B_k) \subseteq V, \quad \forall k\in\mathbb{N}$$

and because T is linear

$$T\left(\operatorname{acx}\bigcap_{k\in\mathbb{N}} B_k\cap U_k\right) \subseteq V.$$

By Theorem 2.72 and because $\mathcal{T}^{\mathcal{A}} = \mathcal{T}$, this proves that T is continuous and therefore an element in $(\mathcal{L}(E,F),\mathcal{T}_\beta)$. Since for every $B_k\in\mathcal{A}$ and for every zero neighborhood in V in F there is a $n_{V,k}$ (see above) such that

$$\sup_{x\in B_k} q_V(T-T_n)(x) \leq 1, \quad \forall n \geq n_{V,k},$$

we know that $T_n \to T$ in $(\mathcal{L}(E,F),\mathcal{T}_\beta)$.

(2) $\Rightarrow$ **(3)**: Trivial.

(3) $\Rightarrow$ **(1)**: We fix $F=\mathbb{K}$. Since by hypothesis $(E',\beta(E',E))$ is Fréchet, E has a fundamental sequence of bounded sets $\mathcal{A} = (A_k)_{k\in\mathbb{N}}$. To show that E is a (gDF) space, we have to establish $\mathcal{T}^{\mathcal{A}} \le \mathcal{T}$, i.e. that every zero neighborhood U of $\mathcal{T}^{\mathcal{A}}$ is also a zero neighborhood of $\mathcal{T}$. By Theorem 2.72, we can assume that $U = \bigcap_{k=1}^{\infty} A_k + U_k$ with U_k absolutely convex zero neighborhoods in $(E, \mathcal{T})$. Now define

$$F = \left\{x \in E^{\mathbb{N}} : \exists n_0 : x_n = x_{n_0},\ \forall n \ge n_0\right\}$$

and $V = F \cap \prod_{k=1}^{\infty}(A_k + U_k)$. Then V is absolutely convex and absorbent in F. Define furthermore $\Phi_V : F \to \widetilde{F_{(V)}}$ to be the canonical mapping and $T_n : E \to F$ as

$$(T_n(x))_k = \begin{cases} x, & k \le n \\ 0, & k > n. \end{cases}$$

Since

$$T_n\left(\bigcap_{i=1}^{n} A_i + U_i\right) \subseteq V$$

it follows that $S_n = \Phi_V \circ T_n : E \to \widetilde{F_{(V)}}$ is continuous (since $(\Phi_V(\rho V))_{\rho>0}$ is a zero neighborhood basis) and therefore an element in $\mathcal{L}(E, \widetilde{F_{(V)}})$, $\forall n \in \mathbb{N}$. For every $k \in \mathbb{N}$ and every $n \ge k$, we know because of $A_k + A_k \subseteq A_{k+1}$ that

$$(T_{n+l} - T_n)(A_k) \subseteq 2^{k-n} V,\ \forall l \in \mathbb{N}$$

and therefore $(S_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $(\mathcal{L}(E, \widetilde{F_{(V)}}), \mathcal{T}_\beta)$ and by hypothesis a limit $S = \Phi_V \circ T \in (\mathcal{L}(E, \widetilde{F_{(V)}}), \mathcal{T}_\beta)$ exists. Since $T(x) = (x, x, \ldots)$, we conclude that $T^{-1}(V) = U$ and therefore U is a zero neighborhood in $(E, \mathcal{T})$. □

Theorem 4.6 (Theorem 12.4.3, Jarchow (1981)). Let E be lcs admitting a fundamental basis of bounded sets. The following are equivalent

1. E is a (gDF) space
2. For every lcs F, every null sequence in $(\mathcal{L}(E, F), \mathcal{T}_\beta)$ is equicontinuous.
3. For every Banach space F, every null sequence in $(\mathcal{L}(E, F), \mathcal{T}_\beta)$ is equicontinuous.

Proof. **(1) ⇒ (2)**: Let $(T_n)_{n\in\mathbb{N}}$ be a null-sequence in $(\mathcal{L}(E, F), \mathcal{T}_\beta)$, V be an absolutely convex zero neighborhood in F and $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ a fundamental sequence of bounded sets in E. Then by the definition of the strong topology, for every $k \in \mathbb{N}$ there is an $n_k \in \mathbb{N}$ such that

$$T_n(A_k) \subseteq V,\ \forall n \ge n_k.$$

Now let U_k be a zero neighborhood in E such that $T_n(U_k) \subseteq V$, $\forall n : 1 \le n \le n_k$. Since E is a (gDF) space

$$U = \operatorname{acx} \bigcup_{k\in\mathbb{N}} A_k \cap U_k$$

is a zero neighborhood and because of the definition of the U_k and the choice of the n_k, we have $T_n(U) \subseteq V$, i.e. that $(T_n)_{n\in\mathbb{N}}$ is equicontinuous.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Let $(T_n)_{n\in\mathbb{N}}$ be an arbitrary null-sequence in $(\mathcal{L}(E,F),\mathcal{T}_\beta)$. By hypothesis $(T_n)_{n\in\mathbb{N}}$ and therefore also $\operatorname{acx}\{T_n : n\in\mathbb{N}\}$ is equicontinuous. By Theorem 2.36 also

$$K = \overline{\operatorname{acx}\{T_n : n\in\mathbb{N}\}}^{\mathcal{T}_\beta}$$

is equicontinuous. Every Cauchy sequence $(S_n)_{n\in\mathbb{N}}$ in K converges pointwise F^E to some $S\in F^E$. By Theorem 2.36, $S\in\mathcal{L}(E,F)$ and since $\mathcal{T}_\beta$ has a null basis of $\mathcal{T}_s$ closed sets, we have $S_n \xrightarrow{\mathcal{T}_\beta} S$ and hence $S\in K$, since K is closed. Hence, K is complete.

Since $(T_n)_{n\in\mathbb{N}}$ is $\mathcal{T}_s$ bounded, $\operatorname{acx}\{T_n : n\in\mathbb{N}\}$ is precompact by Lemma 2.52 and Theorem 2.13. It follows from the above and Corollary 2.12 that K is compact.

To conclude the proof note that every null-sequence $(T_n)_{n\in\mathbb{N}}$ in a local Banach space in $(\mathcal{L}(E,F),\mathcal{T}_\beta)$ is also a null-sequence in $(\mathcal{L}(E,F),\mathcal{T}_\beta)$ and therefore $\overline{\operatorname{acx}}\{T_n : n\in\mathbb{N}\}$ is compact by the argument above. Hence, an appeal to Theorem 2.39, 3 establishes that $(\mathcal{L}(E,F),\mathcal{T}_\beta)$ is locally complete. Since $(\mathcal{L}(E,F),\mathcal{T}_\beta)$ is a metrizable space, local completeness implies completeness (see Theorem 2.40) and $(\mathcal{L}(E,F),\mathcal{T}_\beta)$ is a Fréchet space. An application of Theorem 4.5 finishes the proof. □

A similar characterization of (DF) spaces can be given based on Theorem 2.68 (see discussion before Theorem 12.4.3. in Jarchow (1981)).

Theorem 4.7. Let E be lcs admitting a fundamental basis of bounded sets. The following are equivalent

1. E is a (DF) space.
2. For every lcs F, every $\mathcal{T}_s$-bounded ($\mathcal{T}_\beta$-bounded) sequence in $\mathcal{L}(E,F)$ is equicontinuous.
3. For every Banach space F, every $\mathcal{T}_s$-bounded ($\mathcal{T}_\beta$-bounded) sequence in $\mathcal{L}(E,F)$ is equicontinuous.

Proof. Follows directly from Theorem 2.68. □

Next, we show that the existence of a fundamental sequence in a metrizable space already implies that the space is normable. To prove this, we will require the following Lemma.

Lemma 4.8 (Theorem 9.1.3, Jarchow (1981)). Let $(E, \mathcal{T})$ be a lcs and $\mathcal{T}' \leq \beta(E', E)$ be a topology on E' such that the dual pairing $\langle \cdot, \cdot \rangle : (E, \mathcal{T}) \times (E', \mathcal{T}') \to \mathbb{K}$ is continuous, then E is normable.

Proof. Let $U \subseteq E$ and $V \subseteq E'$ be zero neighborhoods such that $U \times V \subseteq (\langle \cdot, \cdot \rangle)^{-1}[-1, 1]$. Then V is equicontinuous since $U \subseteq \bigcap_{y \in V} y^{-1}[-1, 1]$. Hence, V° is a zero neighborhood in $(E, \mathcal{T})$. To show that $(E, \mathcal{T})$ is normable, we show that V° is additionally bounded. Since V is a zero neighborhood in $(E', \mathcal{T}')$ and W° is bounded in $\mathcal{T}'$ for every zero neighborhood $W \subseteq E$ (because $\mathcal{T}' \leq \beta(E', E)$) there is a $\rho > 0$ such that $W^\circ \subseteq \rho V$. It follows that $V^\circ \subseteq \rho W$ and therefore V° is a bounded zero neighborhood in $(E, \mathcal{T})$. □

Theorem 4.9 (Corollary 12.4.4, Jarchow (1981)). A metrizable lcs E admits a fundamental sequence of bounded sets, iff it is normable.

Proof. Since E is metrizable it is quasi-barrelled by Corollary 2.56 and therefore also quasi-c_0-barrelled and hence a (df) space by Theorem 4.2. We therefore know that $\langle \cdot, \cdot \rangle : (E, \mathcal{T}) \times (E', \beta(E', E)) \to \mathbb{K}$ is a bilinear mapping on a product of two metrizable spaces, continuous in every variable. Because $(E', \beta(E', E)$ is a Fréchet space, it is barrelled and hence an application of the uniform boundedness principle (see Kriegl (2007) 5.2.8) yields that $\langle \cdot, \cdot \rangle : (E, \mathcal{T}) \times (E', \beta(E', E)) \to \mathbb{K}$ is a continuous map and $(E, \mathcal{T})$ is normable because of Lemma 4.8. □

Lemma 4.10. Let E be metrizable and $(U_n)_{n \in \mathbb{N}}$ be a neighborhood basis, then $(U_n^\circ)_{n \in \mathbb{N}}$ is a basis for the bornology on $(E', \beta(E', E))$ consisting of Banach disks.

Proof. U_n° are equicontinuous and therefore $\sigma(E', E)$-compact by Alaoğlu-Bourbaki and hence Banach disks by 7.4.17 in Kriegl (2007). An application of the Banach-Mackey theorem (see Kriegl (2007), 7.4.18) yields, that U_n° are bounded in $\beta(E', E)$. Let now B be a bounded set in $\beta(E', E)$, then B° is a bornivorous barrel in E and therefore a zero neighborhood (by Lemma 2.55). Consequently, $\exists n \in \mathbb{N}$, $\exists \epsilon > 0 : \epsilon U_n \subseteq B^\circ \Rightarrow \epsilon B \subseteq U_n^\circ$. □

The next Theorem gives a justification of the name DF (dual of a Fréchet) space.

Theorem 4.11 (Theorem 12.4.5, Jarchow (1981)). The strong dual $F = (E', \beta(E', E))$ of any metrizable lcs E is a complete DF-space.

Proof. By Theorem 2.69, F is quasi-$\aleph_0$-barrelled and since the dual space of any bornological space is complete, F is also complete. By Lemma 4.10, F has a countable fundamental sequence of bounded sets. □

Definition 4.12 (Quasi-normable)**.** Consider a lcs $(E, \mathcal{T})$ with a zero neighborhood basis $\mathcal{U}$ consisting of closed absolutely convex sets. The space E is quasi-normable, if for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$, such that $V \subseteq U$ and the topology induced by E_{V° on U° coincides with the topology induced by $\beta(E', E)$.

Theorem 4.13 (Theorem 12.4.7, Jarchow (1981))**.** Every (gDF) space is quasi-normable.

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a fundamental sequence of bounded sets in E. Given any closed absolutely convex zero neighborhood U in E, define the equicontinuous sets $D_k = kU^\circ \cap A_k^\circ$ for all $k \in \mathbb{N}$ as well as the set $D = \bigcup_{k\in\mathbb{N}} D_k$. Now

$$\begin{aligned}\frac{1}{k}U \cap A_k &= \left(\bigcap_{n\leq k} \frac{1}{n}U\right) \cap \left(\bigcap_{n>k} A_n\right) \cap A_k \subseteq A_k \cap \left(\bigcap_{n\in\mathbb{N}} \frac{1}{n}U \cup A_n\right) \\ &\subseteq A_k \cap D^\circ \subseteq D^\circ.\end{aligned}$$

Where the second last inclusion follows from

$$D^\circ = \left(\bigcup_{n\in\mathbb{N}} D_n\right)^\circ = \bigcap_{n\in\mathbb{N}} D_n^\circ = \bigcap_{n\in\mathbb{N}} \left(n\,(U)^\circ \cap A_n^\circ\right)^\circ \supseteq \bigcap_{n\in\mathbb{N}} \left(\frac{1}{n}U \cup A_n\right).$$

Therefore $\operatorname{acx}\left(\bigcap_{k\in\mathbb{N}} \frac{1}{k}U \cap A_k\right) \subseteq D^\circ$ and by Theorem 2.72 D° is a zero neighborhood in E and D is equicontinuous.

Set $V = D^\circ$. Since V is a zero neighborhood in E, V° is a bounded set in $(E', \beta(E', E))$ and therefore the mapping $E'_{V^\circ} \to (E, \beta(E', E))$ is continuous. To show that E is quasi-normable we therefore only have to show, that $(E, \beta(E', E)) \to U^\circ \subseteq E'_{V^\circ}$ is continuous.

A typical zero neighborhood in E'_{V° is of the form $W = \epsilon V^\circ$. Let $n \in \mathbb{N}$ be such that $\frac{1}{n} \leq \epsilon$, then

$$U^\circ \cap \frac{1}{n}A_n = \frac{1}{n}D_n \subseteq \frac{1}{n}D = \frac{1}{n}V^\circ \subseteq W.$$

Since $U^\circ \cap \frac{1}{n}A_n^\circ$ is a neighborhood of zero for the restriction of $\beta(E', E)$ to U°, this proves the statement. □

The rest of the section is dedicated to stability properties of (df) spaces, which are discussed in Theorem 4.16. To prove this theorem, we need the following two lemmata concerned with the strong topology.

Lemma 4.14. Let E be a lcs and $F \subseteq E$ a closed subspace, then

1. $F^\circ \cong (E/F)'$ as vector spaces.

2. the topology induced on F° by $\beta(E', E)$ is weaker, than $\beta(F^\circ, E/F)$.

Proof. **(1)**: Let Q be the quotient mapping $Q : E \to E/F$, then $Q' : (E/F)' \to E'$. Clearly, from $f \in (E/F)'$ it follows that $(Q'(f))(x) = f(Q(x)) = 0$ for all $x \in F$ and therefore range$(Q') \subseteq F^\circ$. On the other hand for every $f \in F^\circ$, $f(x) = 0$ for all $x \in F$. Hence the the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "Q"] \arrow[d, "f"'] & E/F \arrow[dl, "\tilde{f}"] \\ \mathbb{K} & \end{tikzcd}$$

and $\tilde{f}$ is well defined. Clearly, $Q'(\tilde{f}) = f$ and therefore $\mathrm{im}(Q') = F^\circ$.

(2): The seminorms on F° induced by $\beta(E', E)$ are restrictions to F° of seminorms of the form

$$p_B(y) = \sup_{x \in B} |y(x)|.$$

for $y \in E'$ and $B \subseteq E$ bounded. Since $Q(B)$ is bounded in E/F

$$p_{Q(B)}(y') = \sup_{z \in Q(B)} |y'(z)|$$

is a continuous seminorm in $\beta(F^\circ, E/F)$. For $y' \in (E/F)'$ with $y = Q'(y')$ we have

$$p_{Q(B)}(y') = \sup_{z \in Q(B)} |y'(z)| = \sup_{x \in B} |y'(Q(x))| = \sup_{x \in B} |Q'(y')(x)| = p_B(y).$$

Hence, we know that $\beta(F^\circ, E/F)$ is stronger than $\beta(E', E)$ on F°. $\square$

Lemma 4.15 (Theorem 8.8.10, Jarchow (1981))**.** Let I be an arbitrary index set and E_i lcs for all $i \in I$ and $E = \bigoplus_{i \in I} E_i$, then

$$(E', \beta(E', E)) = \prod_{i \in I} (E_i', \beta(E_i', E_i)).$$

Proof. Since $\bigoplus_{i \in I} E_i$ is the final topology characterized by the inclusions $\mathrm{incl}_j : E_j \to \prod_{i \in I} E_i$, a mapping $y : E \to \mathbb{K}$ is continuous if and only if $y_j = y \circ \mathrm{incl}_j : E_j \to \mathbb{K}$ is continuous. Therefore we can define a mapping $\iota : E' \to \prod_{i \in I} E_i'$ by $\iota(y) = (y_1, y_2, \ldots)$. The inverse is given by

$$\iota^{-1}((y_k))(x) = \sum_k y_k(\mathrm{proj}_k(x)).$$

To see this note that for $x \in E$

$$\begin{aligned} \iota^{-1}(\iota(y))(x) &= \iota^{-1}((y \circ \mathrm{inkl}_k)_k)(x) = \sum_k y \circ \mathrm{inkl}_k \circ \mathrm{proj}_k(x) \\ &= y\left(\sum_k \mathrm{inkl}_k \circ \mathrm{proj}_k(x)\right) = y(x) \end{aligned}$$

Hence, ι is a bijection. To show that it is also a homeomorphism, we consider the following diagram.

$$\begin{tikzcd}
(E', \beta(E', E)) \arrow[r, "\iota"] \arrow[dr, "pr_j \circ \iota"'] & \prod_{i\in I}(E'_i, \beta(E'_i, E_i)) \arrow[d, "pr_j"] & \\
& (E'_j, \beta(E'_j, E_j)) \arrow[r, "p_B"] & \mathbb{K}
\end{tikzcd}$$

To show that ι is continuous it is – by the universal property of the product – enough to show that $pr_j \circ \iota$ is continuous. This in turn can be established by showing that $p_B \circ pr_j \circ \iota$ is a seminorm for every $B \subseteq E_j$ bounded. Since the set $C = \{0\} \times \cdots \times B \times \cdots$ is a bounded set in E and $p_C(y) = (p_B \circ pr_j \circ \iota)(y)$ this follows.

To show that ι^{-1} is continuous it is enough to show that for every bounded set $B \subseteq E$ the function $p_B \circ \iota^{-1}$ is a seminorm in $\prod_{i\in I}(E'_i, \beta(E'_i, E_i))$. This follows since every such B is contained in $\tilde{B} = B_{i_1} \times \cdots \times B_{i_n} \subseteq \prod_{j=1}^{n} E_{i_j}$ with $B_{i_j} \subseteq E_{i_j}$ bounded. From this it follows that

$$(p_B \circ \iota^{-1})(y) \leq p_{\tilde{B}}(\sum_k y_k \circ \mathrm{proj}_k) = \sum_j p_{B_{i_j}}(y_k) \leq n \max_j p_{B_{i_j}}(y)$$

which is a seminorm in $\prod_{i\in I}(E'_i, \beta(E'_i, E_i))$. □

Theorem 4.16 (Theorem 12.4.8, Jarchow (1981))**.** The classes of (df) spaces, (gDF) spaces as well as (DF) spaces are stable with respect to the formation of

1. Hausdorff quotients,
2. countable direct sums,
3. Hausdorff countable inductive limits,
4. completions.

Proof. **(1)**: Let E be a (df) space and $F \subseteq E$ be a closed subspace. Let further $(a_n)_{n\in\mathbb{N}}$ be a $\beta(E', E)$-null sequence in F°. Since E is quasi-c_0-barrelled, $(a_n)_{n\in\mathbb{N}}$ is equicontinuous. Since $a_n \in F^\circ$, we can find $b_n \in (E/F)'$ as $b_n(x + F) = a_n(x)$. By the equicontinuity of $(a_n)_{n\in\mathbb{N}}$, have that there is a absolutely convex zero neighborhood in $U \subseteq E$ such that

$$\bigcap_{n\in\mathbb{N}} b_n^{-1}([-1, 1]) = Q \circ \bigcap_{n\in\mathbb{N}} a_n^{-1}([-1, 1]) \supseteq Q(U)$$

and since $Q : E \to E/F$ is an open mapping, we have that $Q(U)$ is an absolutely convex zero neighborhood in E/F. Therefore $(b_n)_{n\in\mathbb{N}}$ is equicontinuous and hence bounded in $\beta(F^\circ, E/F)$. This in turn implies that the identity map

$$\mathrm{id} : (E', \beta(E', E)) \supseteq F^\circ \to (F^\circ, \beta(F^\circ, E/F))$$

is (sequentially) continuous. By Lemma 4.14 the two topologies coincide and therefore $(F^\circ, \beta(F^\circ, E/F))$ is metrizable and the above argument establishes that E/F is quasi-c_0-barrelled. This shows that E/F is a (df) space, since the fact that $\beta(F^\circ, E/F)$ is metrizable implies the existence of a fundamental sequence of bounded sets in E/F by Theorem 4.1.

Let now E be a (gDF) space (DF space respectively), G be an arbitrary Banach space and $(T_n)_{n\in\mathbb{N}}$ be a null sequence (respectively bounded sequence) in $(\mathcal{L}(E/F, G), \mathcal{T}_\beta)$, then $(T_n \circ Q)_{n\in\mathbb{N}}$ is a null sequence (respectively a bounded sequence) in $(\mathcal{L}(E, G), \mathcal{T}_\beta)$ and therefore by Theorem 4.6 (respectively Theorem 4.7) E/F is (gDF) (respectively DF) space, since the equicontinuity of $(T_n)_{n\in\mathbb{N}}$ follows from the equicontinuity of $(T_n \circ Q)_{n\in\mathbb{N}}$ (see above).

(2): Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of (df) spaces. If $E = \bigoplus_{n\in\mathbb{N}} E_i$, then, by Lemma 4.15, we have that $(E', \beta(E', E))$ is a Fréchet space.

Now suppose $(y_n)_{n\in\mathbb{N}} \to 0$ in $(E', \beta(E', E))$. Since $(E', \beta(E', E))$ is metrizable convergence and local convergence are the same, i.e. there exists $B \subseteq E'$ bounded in $\beta(E', E)$, such that $||y_n||_B \to 0$. This implies that there is a $U \subseteq E$ with $B \subseteq U^\circ$ and a $\rho > 0$ with $\rho\,\{y_n : n \in \mathbb{N}\} \subseteq B$. Therefore $\rho\,\{y_n : n \in \mathbb{N}\} \subseteq U^\circ$ and $(y_n)_{n\in\mathbb{N}}$ are equicontinuous. Hence, E is (df).

Now assume that E is a (gDF) space ((DF) space respectively). Let G be a Banach space and $(T_k)_{k\in\mathbb{N}}$ be a null sequence (respectively a bounded sequence) in $(\mathcal{L}(E, G), \mathcal{T}_\beta)$. Denote by $(T_{kn})_{n\in\mathbb{N}}$ the restrictions of T_k to E_n, which again form a null sequences (bounded sequence) in $(\mathcal{L}(E_n, G), \mathcal{T}_\beta)$. Since E_n is assumed to be (gDF) ((DF) respectively) the sequences $(T_{kn})_{n\in\mathbb{N}}$ are equicontinuous in $\mathcal{L}(E_n, G)$ by Theorem 4.6 (Theorem 4.7 respectively). Since $E = \bigoplus E_n$ carries the final topology with respect to the injections $\mathrm{inj}_n : E_n \to E$ and $T_{kn} = T_k \circ \mathrm{inj}_n$ also $(T_k)_{k\in\mathbb{N}}$ is equicontinuous in $\mathcal{L}(E, G)$. To see this let $V \subseteq G$ be a closed zero neighborhood. By equicontinuity of $(T_{kn})_{n\in\mathbb{N}}$ there is a zero neighborhood $U_n \subseteq E_n$ with

$$U_n \subseteq \bigcap_{k\in\mathbb{N}} (T_k \circ \mathrm{inj}_n)^{-1}(V) = \mathrm{inj}_n^{-1} \bigcap_{k\in\mathbb{N}} T_k^{-1}(V)$$

for each $n \in \mathbb{N}$. Hence, it follows that $\bigcap_{k\in\mathbb{N}} T_k^{-1}(V)$ is a zero neighborhood of E by the definition of the final structure on E. It follows that E is a (gDF) ((DF) respectively) space.

(3): The inductive limit can be written as $E = (\bigoplus_{j\in J} E_j)/\tilde{L}$ where L is the linear span of the set $\bigcup_{i,j\in J,\, i\leq j} range(I_i - I_j \circ S_{j,i})$ and the quotient space is endowed with the quotient topology (see Theorem 3.36). Therefore the result follows from (1) and (2).

(4): Let $(A_n)_{n\in\mathbb{N}}$ be a fundamental sequence in E, then $\tilde{A}_n \subseteq \tilde{E}$ is a fundamental sequence in $\tilde{E}$. Clearly $\tilde{A}_n$ is bounded in $\tilde{E}$. Let further $\iota : E \to \tilde{E}$ be the embedding of E into the completion and $B \subseteq \tilde{E}$ be bounded. Suppose $\nexists \tilde{A}_n : B \subseteq \tilde{A}_n$, then $\nexists \tilde{A}_n : \iota^{-1}(B) \subseteq \iota^{-1}(\tilde{A}_n) \subseteq A_n$ and therefore $\iota^{-1}(B)$ is not bounded in E, i.e. there is a U zero neighborhood in E such that $\nexists \lambda > 0 : \iota^{-1}(B) \subseteq \lambda U$. In this case, $\nexists \lambda > 0 : B \subseteq \iota \circ \iota^{-1}(B) \subseteq \lambda\iota(U)$ but this is a contradiction, since $\iota(U)$ is a zero neighborhood in $\tilde{E}$ and B is bounded.

If $(y_n)_{n\in\mathbb{N}}$ is a null sequence in $(\tilde{E}', \beta(\tilde{E}', \tilde{E}))$, then $y_n \circ \iota$ is a null sequence in $(E', \beta(E', E))$ and hence is equicontinuous. It follows that $(y_n)_{n\in\mathbb{N}}$ is also equicontinuous and therefore $(\tilde{E}', \beta(\tilde{E}', \tilde{E}))$ is a (df) space.

Suppose now that E is (gDF) ((DF) respectively), G is an arbitrary Banach space and $(T_n)_{n\in\mathbb{N}}$ is a null sequence (bounded sequence) in $(\tilde{E}', \beta(\tilde{E}', \tilde{E}))$. Then as above $(T_n \circ \iota)_{n\in\mathbb{N}}$ is a null sequence (bounded sequence) in $(E', \beta(E', E))$ and therefore equicontinuous by Theorem 4.6 (respectively Theorem 4.7). It follows that also $(T_n)_{n\in\mathbb{N}}$ is equicontinuous and therefore $\tilde{E}$ is (gDF) ((DF) respectively). □

4.2 Reflexivity and biduals

In this short section, we will outline a few results on biduals of Fréchet spaces that can be proven using the results on (df) spaces collected so far. In particular, we investigate the reflexivity of Fréchet spaces and subspaces of Fréchet spaces. We will start with a simple Lemma.

Lemma 4.17. 1. The topology of uniform convergence on equicontinuous sets in E' equals the original topology on E.

2. If E is quasi-barrelled, the canonical imbedding $J : E \to E''$ is a homomorphism between E and $J(E) \subseteq E''$ (see also Jarchow (1981), 11.2.2).

Proof. 1. Let $U = U^{\circ\circ}$ be zero neighborhood in E. Then U° is an equicontinuous set and therefore $U = U^{\circ\circ}$ is also a zero neighborhood in the topology of uniform convergence on equicontinuous sets in E'.

Conversely, let $U \subseteq E$ be a zero neighborhood in the topology of uniform convergence on equicontinuous sets in E'. Then there exists a $V \subseteq E'$ equicontinuous such that $V^\circ \subseteq U$. By the equicontinuity of V there is a zero neighborhood W in the topology of E such that $W \subseteq V^\circ$.

2. By (1) the only thing to show is, that the strong topology on E'' is coarser than the topology of uniform convergence on equicontinuous sets in E'. To show this, it is enough to show that every $\beta(E', E)$-bounded set in E' is also equicontinuous.

If V is bounded in $\beta(E', E)$, then $p_B(V) = \sup_{x\in B}\sup_{y\in V}|y(x)| = C_B < \infty$ for all bounded sets B in E. Therefore $\frac{1}{C_B}B \subseteq V^\circ$, i.e. $V^\circ \subseteq E$ is a bornivorous barrel and therefore a zero neighborhood.

□

Theorem 4.18 (Corollary 25.10, Meise and Vogt (1997))**.** For every Fréchet space E, E'' is also a Fréchet space and E is isomorphic to a closed subspace of E''.

Proof. Since E is metrizable it is quasi-barrelled (see Corollary 2.56) and therefore E' is a Fréchet space by Theorem 4.2. It follows from Theorem 4.11 that E' is a (DF) space and therefore E'' is a Fréchet space by another application of Theorem 4.2. By Lemma 4.17, the canonical imbedding $J : E \to J(E) \subseteq E''$ is an homeomorphism and $J(E) \subseteq E''$ is complete, hence a closed subspace of E''. □

Lemma 4.19. Let F be a closed subspace of a lcs E, then $\sigma(E, E')_{|F} = \sigma(F, F')$.

Proof. Since $y_{|F} \in F'$ for every $y \in E'$, we have $\sigma(F, F') \geq \sigma(E, E')$. On the other hand side for every $y \in (F', \sigma(F, F'))$, there is a $\tilde{y} \in \sigma(E', E)$ such that $\tilde{y}_{|F} = y$ by Hahn-Banach. Hence $\sigma(E', E) \geq \sigma(F', F)$ also holds. □

Theorem 4.20 (Proposition 23.26, Meise and Vogt (1997))**.** Every closed subspace F of a reflexive Fréchet space E is reflexive.

Proof. Let $B \subseteq F$ be bounded, By Theorem 2.32, B is $\sigma(E, E')$-relatively weakly compact. Since $B \subseteq F$ and $\sigma(F, F') = \sigma(E, E')_{|F}$, B is also relatively weakly compact in F. Another application of Theorem 2.32 yields that F is semi-reflexive. Since a Fréchet space is barrelled, the result follows from Theorem 2.50. □

Theorem 4.21 (Corollary 25.11, Meise and Vogt (1997))**.** A Fréchet space E is reflexive, iff E' is reflexive.

Proof. If E is reflexive, then $(E')'' = (E'')' \cong E'$. In the same way it follows that if E' is reflexive then also E''. By Theorem 4.18, E is a closed subspace in E''. Since the closed subspace of a reflexive Fréchet space is again reflexive (by Theorem 4.20) we are done. □

4.3 Imbedding spectra

This section is about the relationship of imbedding spectra and weakly compact imbedding spectra to (df) spaces. In particular we will see that every bornological (df) space is a imbedding spectrum of normed spaces E_n and that the converse also holds, if the imbedding spectrum is actually a lcs. It will be possible to characterize the fundamental sequence of bounded sets in (df) spaces by the bounded sets of the spaces E_n.

Definition 4.22 (Imbedding spectra)**.** A final topology on E defined by countably many maps $\iota_n : E_n \to E$ is called imbedding spectrum, iff

1. E_n is a linear subspace in E and $\iota_n : E_n \to E$ is the inclusion.

2. $E_n \subseteq E_{n+1}$ and the inclusion $E_n \hookrightarrow E_{n+1}$ is continuous.

We denote the imbedding spectrum by $(\iota_n : E_n \to E)_n$.

Remark 4.23. Note that the above definition does not ensure that the resulting topology on E is a locally convex topology. We will see in Lemma 4.27 that this is the case for weakly compact imbedding spectra. We will say that the inductive topology $\mathrm{ind}_{n\to} E_n$ exists if it is a locally convex topology.

Theorem 4.24 (Proposition 25.16, Meise and Vogt (1997))**.**

1. If E is a bornological (df) space, then it is a inductive limit of an imbedding spectrum $(\iota_n : E_n \to E)_{n\in\mathbb{N}}$ of normed spaces.

2. Let $(\iota_n : E_n \to E)_{n\in\mathbb{N}}$ be an imbedding spectrum of normed spaces E_n, whose inductive topology $\mathcal{T}$ exists. If B_n denotes the closed unit ball of E_n, then $(\bar{B}_n^{\mathcal{T}})_{n\in\mathbb{N}}$ is a fundamental system of bounded sets in $(E, \mathcal{T})$ and $(E, \mathcal{T})$ is a bornological (DF) space.

Proof.

1. Every bornological lcs is the inductive limit w.r.t. the imbeddings $j_B : E_B \to B$, where B is a bounded set in E and $E_B = \mathrm{span}(B)$ (see Kriegl (2007), Remark 4.6.2). Since in the case of a (df) space the bornology of E has a fundamental sequence $(B_k)_{k\in\mathbb{N}}$, the statement follows.

2. Note that $\mathcal{M} = \{\lambda B_n : n \in \mathbb{N}, \lambda \in \mathbb{K}\}$ is a bornology and we can therefore construct a topology $\mathcal{T}_{\mathcal{M}}$ on E' using these sets (see Theorem 2.21). To show that $(\bar{B}_n^{\mathcal{T}})_{n\in\mathbb{N}}$ is a fundamental system of bounded sets, we first show that $\mathcal{T}_{\mathcal{M}} = \beta(E', E)$. Since B_n are bounded, we only have to show $\mathcal{T}_{\mathcal{M}} \geq \beta(E', E)$: Let $Q \subseteq E'$ be a $\mathcal{T}_{\mathcal{M}}$ bounded set, then $\forall n \in \mathbb{N}$, $\exists \lambda_n$ such that $Q \subseteq \lambda_n B_n^\circ$. Therefore we have
$$B_n \subseteq B_n^{\circ\circ} \subseteq \lambda_n Q^\circ \subseteq E, \quad \forall n \in \mathbb{N}$$
and consequently Q° is a zero neighborhood in E (because of the definition of the inductive limit as final structure). By the Theorem of Alaoğlu-Bourbaki $Q^{\circ\circ}$ is $\sigma(E', E)$-compact. By Theorem 7.4.17 in Kriegl (2007), $Q^{\circ\circ}$ is a Banach disk and by the Theorem of Banach-Mackey also $\beta(E', E)$-bounded. Since $\mathcal{T}_{\mathcal{M}}$ is metrizable and therefore bornological, this implies that the identity
$$\mathrm{id} : (E', \mathcal{T}_{\mathcal{M}}) \to (E', \beta(E', E))$$

is continuous and $\beta(E', E) \leq \mathcal{T}_{\mathcal{M}}$.

Therefore for every bounded set B in E there is an $\epsilon > 0$ and a B_n such that $\epsilon B_n^\circ \subseteq B^\circ$. By the bipolar theorem, we get

$$B \subseteq B^{\circ\circ} \subseteq \frac{1}{\epsilon} B_n^{\circ\circ} = \frac{1}{\epsilon} \bar{B}_n^{\mathcal{T}}.$$

Therefore $(\bar{B}_n^{\mathcal{T}})_{n\in\mathbb{N}}$ is a fundamental system of bounded sets. Since $(E, \mathcal{T})$ is metrizable, it is bornological and also quasi-barrelled (Theorem 2.56) and therefore a (DF) space.

□

Lemma 4.25 (Lemma 25.17, Meise and Vogt (1997))**.** Let E, F and G be normed spaces, $A \in \mathcal{L}(E, F)$ be weakly compact and $B \in \mathcal{L}(F, G)$ be injective. Then for each $y \in F'$ and each $\epsilon > 0$ there is a $z \in G'$, such that

$$||A'y - (B \circ A)'z|| < \epsilon.$$

Proof. Let $U = \{x \in E : ||x|| \leq 1\}$ be the unit ball in E and $K = \overline{A(U)}^{\sigma(F,F')}$. By hypothesis K is a weakly compact absolutely convex set in F. Therefore $p_K : F' \to \mathbb{K}$ defined by $p_K(y) = \sup\{|y(x)| : x \in K\}$ is a Mackey continuous seminorm on F'. We have

$$||A'(y)|| = \sup_{x\in U} |A'(y)(x)| = \sup_{x\in U} |y(Ax)| = \sup_{\xi\in A(U)} |y(\xi)| \leq p_K(y)$$

and hence $A' : (F', \mu(F', F)) \to E'$ is continuous and $(A')^{-1}(A'(y) + \epsilon V)$ is a $\mu(F', F)$-neighborhood for all $y \in F'$, $\epsilon > 0$ and V the unit ball in E'.

Since $\{0\} = \ker(B) = \mathrm{range}(B')^\circ$ (cf. Lemma 2.28), we have that $\mathrm{range}(B')$ is dense in $(F', \mu(F', F))$. This follows easily, since

$$E = \{0\}^\circ = \ker(B)^\circ = \mathrm{range}(B')^{\circ\circ} = \overline{\mathrm{acx}}\,\{\mathrm{range}(B')\},$$

where the closure coincides in every admissible topology. It follows that there is a $z \in G'$ such that $B'(z) \in (A')^{-1}(A'y + \epsilon V)$ and $(B \circ A)'(z) = A' \circ B'(z) \in (A'y + \epsilon V)$. □

Definition 4.26 ((Weakly) compact imbedding spectrum)**.** An imbedding spectrum $(\iota_n : E_n \to E)_{n\in\mathbb{N}}$ of normed spaces E_n is called (weakly) compact, if for each $n \in \mathbb{N}$ there is a $k > n$ such that the inclusion $E_n \hookrightarrow E_k$ is (weakly) compact.

Lemma 4.27 (Lemma 25.18, Meise and Vogt (1997))**.** For every weakly compact imbedding spectrum $(\iota_n : E_n \to E)_{n\in\mathbb{N}}$ of normed spaces E_n, the inductive limit $\mathrm{ind}_{n\to} E_n$ is a lcs.

Proof. By assumption there exists a subsequence $(n_k)_{k\in\mathbb{N}}$, such that the inclusions $E_{n_k} \hookrightarrow E_{n_{k+1}}$ are weakly compact. Note that the inductive limit for the system $(\iota_n : E_n \hookrightarrow E)_{n\in\mathbb{N}}$ and $(\iota_{n_k} : E_{n_k} \to E)_{k\in\mathbb{N}}$ coincide (this is the case since the continuity of $\iota_m : E_m \hookrightarrow E$ follows from the continuity of $\iota_n : E_n \hookrightarrow E$ if $m \leq n$). Therefore we can w.l.o.g. assume that the inclusions $E_n \hookrightarrow E_{n+1}$ are weakly compact. Since $E_n \hookrightarrow E_{n+1}$ is continuous, we can further assume w.l.o.g. that $||\cdot||_{n+1}|_{E_n} \leq ||\cdot||_n$.

All that there is to show is that the inductive limit is a Hausdorff locally convex space, i.e. that the corresponding seminorms are point separating. We will establish this by showing, that for every $0 \neq x \in E$, there is a $y \in E'$ such that $y(x) \neq 0$ (which yields the required result since $|y|$ is a continuous seminorm). To show this let $x \in E$ be arbitrary and w.l.o.g. contained in E_1 with $||x||_1 = 1$. Further choose $y_2 \in E_2'$ such that $y_2(x) = 1$. Now define elements $y_k \in E_k'$ recursively for $k \geq 2$, which fulfill

$$||y_k|_{E_{k-1}} - y_{k+1}|_{E_{k-1}}||_{k-1} \leq 2^{-k}, \quad \forall k \geq 2. \tag{4.2}$$

To see that this is possible assume that y_k has already be found. An application of Lemma 4.25 for $A : E_{k-1} \hookrightarrow E_k$ and $B : E_k \hookrightarrow E_{k+1}$, $\epsilon = 2^{-k}$ and $y = y_k$ yields a $y_{k+1} \in E_{k+1}'$ such that (4.2) holds (since $A'(y) = y|_{E_{k-1}}$ and $B'(y) = y|_{E_k}$).

Since we assumed that the norms in E_k are decreasing the reverse holds for the dual norms and $||y|_{E_k}||_k \leq ||y||_{k+1}$. Therefore it follows that for $k < n < m$

$$\begin{aligned} ||y_n|_{E_{k-1}} - y_m|_{E_{k-1}}||_{k-1} &\leq \sum_{j=n}^{m-1} ||y_j|_{E_{k-1}} - y_{j+1}|_{E_{k-1}}||_{k-1} \\ &\leq \sum_{j=n}^{m-1} ||y_j|_{E_{j-1}} - y_{j+1}|_{E_{j-1}}||_{j-1} \leq \sum_{j=n}^{m-1} \frac{1}{2^j} \\ &< \frac{1}{2^{n-1}}. \end{aligned}$$

Hence, $(y_n|_{E_{k-1}})_{n\geq k}$ is a Cauchy sequence in (the complete space) E_{k-1}' for all $k \in \mathbb{N}$, and therefore there is a linear functional $y : E \to \mathbb{K}$ defined by $y_{|E_k}(x) = \lim_{n\to\infty} y_n(x)$, $\forall x \in E_k$. Because of the final structure on E and the fact that $y \circ \iota_n$ is continuous we get that $y \in E'$. Using the reverse triangle inequality and the fact that $1 = ||x||_1 \geq ||x||_k$ for $k \in \mathbb{N}$, it follows

$$\begin{aligned} |y(x)| &= \left| y_2(x) - \sum_{k=2}^{\infty} (y_{k+1}(x) - y_k(x)) \right| \geq |y_2(x)| - \sum_{k=2}^{\infty} |y_{k+1}(x) - y_k(x)| \\ &\geq 1 - \sum_{k=2}^{\infty} ||y_{k+1}|_{E_{k-1}} - y_k|_{E_{k-1}}||_{k-1} \geq 1 - \sum_{k=2}^{\infty} \frac{1}{2^k} = \frac{1}{2} \end{aligned}$$

establishing the required result. ∎

Theorem 4.28 (Theorem 25.19, Meise and Vogt (1997))**.** If $(\iota_n : E_n \to E)$ is a weakly compact imbedding spectrum of normed spaces, then

1. $\mathrm{ind}_{n\to} E_n$ exists and is a complete, bornological (DF) space

2. The sequence $(B_n)_{n\in\mathbb{N}}$ of closed unit balls B_n of E_n is a fundamental sequence of bounded sets in $\mathrm{ind}_{n\to} E_n$.

Proof. By Lemma 4.27 the inductive topology exists and by Theorem 4.24 $(E, \mathcal{T}) = \mathrm{ind}_{n\to} E_n$ is a bornological (DF) space in which the sequence $(\bar{B}_n^{\mathcal{T}})_{n\in\mathbb{N}}$ is a fundamental system of bounded sets.

To show (2), we have to show that $\forall n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ and $\lambda_k > 0$ such that $\bar{B}_n^{\mathcal{T}} \subseteq \lambda_k B_k$. If we fix $n \in \mathbb{N}$, then by weak compactness there is a $k \in \mathbb{N}$ such that $E_n \hookrightarrow E_k$ is weakly compact. Hence, $Q = \bar{B}_n^{\sigma(E_k, E_k')}$ is compact in $(E_k, \sigma(E_k, E_k'))$ and by the Banach-Mackey Theorem is bounded in E_k. This in turn implies that there is a $\lambda_k > 0$ with $Q \subseteq \lambda_k B_k$. By Lemma 4.19, Q is also $\sigma(E, E')$ compact and therefore closed in $\mathcal{T}$. This implies $\bar{B}_n^{\mathcal{T}} \subseteq Q \subseteq \lambda_k B_k$, which establishes (2).

It remains to show that E is complete. Note that by the above for every bounded B there exists $\lambda > 0$ and B_n such that $B \subseteq \lambda_n B_n$ and $\bar{B}_n^{\mathcal{T}} \subseteq Q$ with Q $\sigma(E, E')$-compact, i.e. every bounded set is relatively weakly compact. By Theorem 2.32, $(E, \mathcal{T})$ is therefore semi-reflexive and by Theorem 2.50 and the fact that $(E, \mathcal{T})$ is bornological, we get that $(E, \mathcal{T})$ is reflexive. By Theorem 3.2.7. in Kriegl (2007) it follows that E, as the strong dual, is complete. $\square$

4.4 Relations to Schwartz Spaces

Schwartz spaces are characterized by locally strongly convergent null sequences in E' (see Theorem 3.48), while (df) spaces are characterized by null sequences in $\beta(E', E)$. Hence, it is obvious that there will be various connection between the two concepts. The purpose of this section is to investigate a few of these results. We start by characterizing the spaces whose dual is a Fréchet-Schwartz space as a subclass of (DF) spaces in Theorem 4.30. In Theorem 4.31, which is essential for most of the results in this section, we establish that in a (df) space the topology $\tau_{\beta,0}$ coincides with the topology $\mathcal{T}_{c_0}$, i.e. the finest Schwartz topology that is coarser than $\mathcal{T}$. It turns out that in this case $\mathcal{T}_{c_0}$ is also the finest Schwartz topology compatible with the dual pairing. We further show that a complete (df)-Schwartz space is already a (gDF) space (cf. Remark 4.35) and give various characterizations of (DF)-Schwarts spaces in terms of the topology $\mathcal{T}_{c_0}$ in Theorem 4.31.

The results in this section are also used to obtain auxiliary results such as the equivalence of quasi-completeness and completeness for (df) spaces in Theorem 4.33

(which also proves useful in proofs of other theorems in this section) and Theorem 4.41 characterizing semi-reflexivity in (df) spaces. The section concludes with a homeomorphism theorem stating that a continuous mapping between a semi-Montel and a (df) space for which the pre-images of bounded sets are again bounded is an open map (see Theorem 4.45).

Lemma 4.29. Let $\Phi : E \to F$ be a compact mapping, then $\tilde{\Phi} : E \to \tilde{F}$ is compact as well.

Proof. Let $\iota : F \to \tilde{F}$ be the imbedding of F into it's completion. By hypothesis there is a zero neighborhood $U \subseteq E$ such that $\Phi(U) \subseteq F$ is relatively compact. By Lemma 2.11, also $\iota(\Phi(U))$ is relatively compact and therefore $\tilde{\Phi} = \iota \circ \Phi : E \to \tilde{F}$ is a compact operator. $\square$

Theorem 4.30 (Theorem 25.20, Meise and Vogt (1997))**.** The following statements are equivalent for a lcs $(E, \mathcal{T})$.

1. There is a Fréchet-Schwartz space F, so that E is isomorphic to F'.

2. E is a bornological (DF)-space and to each bounded set $A \subseteq E$ there exists a bounded Banach disk B, so that A is relatively compact in E_B.

3. There exists a compact imbedding spectrum $(\iota_n : E_n \to E)_{n\in\mathbb{N}}$ of Banach spaces with $E = \operatorname{ind}_{n\to} E_n$.

Proof. **(1)** $\Rightarrow$ **(2)**: Since E is isomorphic to the dual of a Fréchet space it is a (DF) space (by Theorem 4.11). By Theorem 3.52, F' is ultra bornological and therefore also bornological. If $A \subseteq F'$ is bounded, then A° is a bornivorous barrel in F and therefore a zero neighborhood. By Theorem 3.48, 2, there is a absolutely convex zero neighborhood $V \subseteq A^{\circ\circ}$ such that A° is relatively compact in $(F')_{V^\circ}$. Therefore also A is relatively compact in $(F')_{V^\circ}$. Since $B = V^\circ$ is closed in $\sigma(F', F)$ it is $\sigma(F', F)$ compact by Alaoğlu-Bourbaki and therefore a Banach disk by Theorem 7.4.17 in Kriegl (2007) and the statement follows.

(2) $\Rightarrow$ **(3)**: Let $(A_n)_{n\in\mathbb{N}}$ be a countable fundamental system of bounded sets in $(E, \mathcal{T})$. By assumption, we can find sets B_n recursively such that $B_1 \supseteq A_1$ and $\bigcup_{k=1}^{n}(A_k \cup B_k)$ is relatively compact in $E_{B_{n+1}}$ for all $n \in \mathbb{N}$. Then $(\iota_n : E_{B_n} \to E)_{n\in\mathbb{N}}$ is a compact embedding spectrum and therefore the limit is a locally convex topology whose topology is finer than $\mathcal{T}$ (by Theorem 4.28). To show that the identity $\mathrm{id} : (E, \mathcal{T}) \to \operatorname{ind}_{n\to} E_{B_n}$ is continuous, note that if B is bounded in $(E, \mathcal{T})$, then there is a A_n such that $B \subseteq \lambda A_n$. Therefore B is bounded in E_{B_n} and also in $\operatorname{ind}_{n\to} E_{B_n}$. Since $(E, \mathcal{T})$ is bornological the continuity of the identity follows and the two topologies coincide.

(3) $\Rightarrow$ **(1)**: By the proof of Theorem 4.28, E is a reflexive, bornological (DF) space. By Theorem 4.2, $F := E'$ is a Fréchet space and since E is reflexive $F' = E'' \cong E$. It remains to show that F is Schwartz space. This will be established by showing that for each normed space X and $A \in \mathcal{L}(F, X)$, there is a zero neighborhood $V \subseteq F$ such that $A(V)$ is precompact in X (see Theorem 3.48 and Remark 3.49). Since A is continuous and X is a Banach space, $A' : X' \to F' \cong E$ is continuous and therefore maps the closed unit ball $U' \subseteq X'$ to a bounded set. By Theorem 4.28, 2, there is a $n \in \mathbb{N}$ such that $A'(U')$ is bounded in E_n and since E is a compact imbedding spectrum, there is a $m > n$ such that $A'(U')$ is relatively compact in E_m. Thus $\Phi : X' \to E_m$ defined by $\Phi(x') = A'(x')$ with $A' = \iota_m \circ \Phi$ is a compact map. Since $\iota'_m : F \to E'_m$ is continuous, there is a zero neighborhood $V \subseteq F$ such that, such that $\iota'_m(V)$ is contained in the unit ball of E'_m. Since Φ is also compact as a mapping from X' to $\tilde{E}_m$ (cf. Lemma 4.29), we get by Schauder's Theorem (cf. Meise and Vogt, 1997, Theorem 15.3), that also the conjugate mapping $\Phi' : E'_m = \tilde{E}'_m \to X''$ is compact. We have $A(V) = A''(V) = \Phi'(I'_m(V))$ and therefore $A(V)$ is relatively compact in X'' and hence precompact in X, since X can be embedded into X'' (see Lemma 2.11). $\square$

Theorem 4.31 (Proposition 12.5.1, Jarchow (1981))**.** Let $(E, \mathcal{T})$ be a (df)-space, then:

1. $\tau_{\beta,0}$ equals the Schwartz topology $\mathcal{T}_{c_0}$;
2. $\tau_{\beta,0}$ is the finest Schwartz topology on E compatible with $\langle E, E' \rangle$, i.e. $\tau_{\beta,0} = \mu(E, E')_{c_0}$;
3. $\beta(E', E) = \eta^t(E', E) = \eta(E', E)$ so that E'' coincides with the completion $\tilde{E}_0$ of the space $E_0 = (E, \tau_{\beta,0})$ as a vector space.

Proof. **(1) & (2)**: Let $(a_n)_{n\in\mathbb{N}}$ be a null sequence in $(E', \beta(E', E))$. Since $(E', \beta(E', E))$ is Fréchet by Theorem 4.2, we get from Theorem 2.40, that $(a_n)_{n\in\mathbb{N}}$ converges locally in $(E', \beta(E', E))$, i.e. converges with respect to the bornology $\mathcal{B}$ of all bounded sets in $(E', \beta(E', E))$. By Theorem 2.43 local convergence is the same as $\mathcal{B}_0$ convergence, which in turn implies there is a set $A = \{x_n : (x_n)_{n\in\mathbb{N}}\ \mathcal{B} - \text{converges to } 0\}^{\circ\circ}$, such that $a_n \to 0$ in E'_A. Since $U = \{x_n : n \in \mathbb{N}\}^\circ$ is a $\tau_{\beta,0}$ zero neighborhood, we have that $a_n \to 0$ in E'_{U° for a $\tau_{\beta,0}$ zero neighborhood.

By definition the topology $\tau_{\beta,0}$ is the topology of uniform convergence on $(E', \beta(E', E))$-null sequences. The above establishes that it is actually also the topology of uniform convergence on $\mathcal{E}_{\tau_{\beta,0}}$-null sequences (because every $(E', \beta(E', E))$ null sequence is also a $\mathcal{E}_{\tau_{\beta,0}}$-null sequence), where $\mathcal{E}_{\tau_{\beta,0}}$ is the compactology in E' assigned to $\tau_{\beta,0}$. This together with Theorem 3.48 implies that $\tau_{\beta,0}$ is a Schwartz topology.

Since we know from Remark 2.59 that $\mu(E,E')_{c_0} \leq \tau_{\beta,0} \leq \mathcal{T}$, it follows that $\mu(E,E')_{c_0} = \tau_{\beta,0} = \mathcal{T}_{c_0}$ and hence $\tau_{\beta,0}$ is the finest Schwartz topology compatible with $\langle E, E'\rangle$.

(3): Since convergence and local convergence are the same in $(E', \beta(E',E))$, we know that for $x_n \xrightarrow{\beta} 0$, there is a zero neighborhood $U \subseteq E$ such that $x_n \to 0$ in the local Banach space $(E'_{U^\circ}, ||.||_{U^\circ})$. Since η^t is the finest topology such that all the inclusions $\iota_{U^\circ} : (E'_{U^\circ}, ||.||_{U^\circ}) \to (E', \eta^t(E',E))$ are continuous, we get $\iota_{U^\circ}(x_n) = x_n \to 0$ in $(E', \eta^t(E',E))$ and hence the mapping $\mathrm{id} : (E', \beta(E',E)) \to (E', \eta^t(E',E))$ is sequentially continuous and therefore also continuous (because $(E', \beta(E',E))$ is metrizable). We therefore obtain $\beta(E',E) = \eta^t(E',E) = \eta(E',E)$ by Theorem 3.42.

Since we know from 3.57 that $(E', \eta(E',E))' = \widetilde{(E, \mathcal{T}_{c_0})}$, we conclude by (1) that $(E', \beta(E',E))'$ can be identified with $E_0 = \widetilde{(E, \mathcal{T}_{c_0})}$. □

Definition 4.32. A topological vector space is quasi-complete, if every closed and bounded subset is complete.

Theorem 4.33 (Corollary 12.5.2, Jarchow (1981))**.** A (df) space E is complete, iff it is quasi-complete.

Proof. By Theorem 4.31 and Theorem 3.57, we know that $\beta(E',E) = \eta(E',E) = \beta(E', \widetilde{E_0})$. Since the topology of $\tilde{E}_0$ is coarser than that of $\tilde{E}$, we therefore have $\beta(E', \tilde{E}) \leq \beta(E', \tilde{E}_0) = \beta(E',E)$. The reverse inequality is trivial, since there is a continuous mapping $\iota : E \to \tilde{E}$ and therefore $\iota(B) \subseteq \tilde{E}$ is a bounded set for every bounded set $B \subseteq E$. Hence, $\tilde{E}$ has more bounded sets than E and $\beta(E',E) \leq \beta(E', \tilde{E})$. Therefore we have $\beta(E',E) = \beta(E', \tilde{E})$.

Assume that E is quasi-complete and consider an absolutely convex, closed, and bounded set $B \subseteq \tilde{E}$, because $\beta(E',E) = \beta(E', \tilde{E})$ there exists a closed disk $C \subseteq E$, such that $p_B \leq p_C$ and therefore $B^\bullet \supseteq C^\circ$ where $\circ$ and $\bullet$ are the polarization with respect to the dual pairing $\langle \tilde{E}, E'\rangle$ and $\langle E, E'\rangle$ respectively. Hence, we have

$$B = B^{\bullet\bullet} \subseteq C^{\circ\bullet} \subseteq \tilde{E}.$$

Since by assumption $C^{\circ\circ}$ is complete, we have $C^{\circ\bullet} = C^{\circ\circ}$ and therefore $B \subseteq C^{\circ\circ} \subseteq E$, hence $E = \tilde{E}$. □

Theorem 4.34 (Proposition 12.5.3, Jarchow (1981))**.** If in a (df) space $(E, \mathcal{T})$ every bounded subset is precompact, then it is a gDF-Schwartz space.

Proof. Let $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$ be a fundamental sequence of bounded sets in $(E, \mathcal{T})$. We show that $\mathcal{T}^{\mathcal{A}} \leq \tau_{\beta,0}$ and since we know that $\tau_{\beta,0} \leq \mathcal{T} \leq \mathcal{T}^{\mathcal{A}}$ this establishes that $\mathcal{T} = \mathcal{T}^{\mathcal{A}} = \tau_{\beta,0}$

and therefore that E is a (gDF) space. Since $\mathcal{T}_{c_0} = \tau_{\beta,0}$ by Theorem 4.31, we also know that E is Schwartz.

Let U be a absolutely convex, closed zero neighborhood in $(E, \mathcal{T}^{\mathcal{A}})$. Since $\mathcal{T}$ and $\mathcal{T}^{\mathcal{A}}$ coincide on the A_n and therefore define the same precompact sets in E (since precompact sets are bounded), U° is relatively $(E', \tau_{pc}(E', E))$ compact. Since E is Schwartz, every bounded set in E is precompact (cf. Corollary 3.50), and we further have that $\tau_{pc}(E', E) = \beta(E', E)$. Hence, τ_{pc} is metrizable. By Theorem 2.16, there is a $\beta(E', E)$ null sequence such that $U^\circ \subseteq \{a_n : n \in \mathbb{N}\}^{\circ\circ}$. It follows that $\{a_n : n \in \mathbb{N}\}^\circ \subseteq U$ and U is a zero neighborhood in $(E, \tau_{\beta,0})$. Hence we have established $\mathcal{T}^{\mathcal{A}} \leq \tau_{\beta,0} \leq \mathcal{T}$. □

Remark 4.35. This in particular implies that every complete (df)-Schwartz space $(E, \mathcal{T})$ is a (gDF) space by Corollary 3.50, In this case $(E, \mathcal{T})$ is also the dual of a Fréchet-Montel space (see Theorem 4.42).

Lemma 4.36. Let E be a metrizable space, $B \subseteq E$ separable and $A \subseteq B$ then A is separable as well.

Proof. B contains points $(y_n)_{n\in\mathbb{N}}$ which are dense. For every pair $n,\ k \in \mathbb{N}$ choose a point in

$$\left\{x \in A : d(x, y_n) \leq \frac{1}{k}\right\}$$

if the set is not empty. The resulting points $(z_m)_{m\in\mathbb{N}}$ are dense in A, since for every $x \in A$ and every $\epsilon > 0$, there is a y_n such that $d(x, y_n) \leq \frac{\epsilon}{2}$ and z_m such that $d(z_m, y_n) \leq \frac{\epsilon}{2}$. An application of the triangle inequality yields the required result. □

Theorem 4.37 (Proposition 12.5.4, Jarchow (1981))**.** If E is a quasi-l_∞-barrelled (df) space, then $(E, \tau_{\beta,\infty})$ is a (DF) space. If E is additionally separable, then E is a quasi-barrelled (DF) space.

Proof. Since $(E, \mathcal{T})$ has a fundamental basis of bounded sets $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ and $\sigma(E, E') \leq \tau_{\beta,\infty} \leq \mathcal{T}$, we conclude that $\mathcal{A}$ is a basis of bounded sets in $(E, \tau_{\beta,\infty})$ (since $\tau_{\beta,\infty}$ is admissible). Hence, $(E, \tau_{\beta,\infty})$ is a quasi-l_∞-barrelled (df) space.

Let $B \subseteq E'$ be $\beta(E', E)$-bounded and $B = \bigcup_{n\in\mathbb{N}} B_n$ with $\tau_{\beta,\infty}$-equicontinuous sets $B_n \subseteq E'$. Since B_n is $\tau_{\beta,\infty}$-equicontinuous, we know that there is a $\beta(E', E)$-bounded sequence $(y_k)_{k\in\mathbb{N}}$ such that $\{y_k : k \in \mathbb{N}\}^\circ \subseteq B_n^\circ$ and consequently

$$\begin{aligned}
B_n &\subseteq B_n^{\circ\circ} \subseteq \{y_k : k \in \mathbb{N}\}^{\circ\circ} = \overline{\operatorname{acx}}\,\{y_k : k \in \mathbb{N}\} \\
&= \overline{\left\{\sum_{k=1}^{N} \lambda_k y_k : N \in \mathbb{N},\ \sum_{k=1}^{N} |\lambda_k| \leq 1\right\}} \\
&= \overline{\left\{\sum_{k=1}^{N} \lambda_k y_k : N \in \mathbb{N},\ \lambda_k \in \mathbb{Q},\ \sum_{k=1}^{N} |\lambda_k| \leq 1\right\}}.
\end{aligned}$$

Hence, every B_n is contained in some separable, bounded set in $\beta(E', E)$ and therefore is separable (by Lemma 4.36). Therefore B is separable as well. By Theorem 2.61, B is therefore $\tau_{\beta,\infty}$-equicontinuous. Now it follows from Lemma 2.66, that $(E, \tau_{\beta,\infty})$ is quasi-$\aleph_0$-barrelled and therefore a (DF) space. Quasi-barrelledness follows from Corollary 2.63. □

Theorem 4.38 (Theorem 12.5.5, Jarchow (1981))**.** For every (df) space $(E, \mathcal{T})$, the following statements are equivalent

1. $(E, \mathcal{T})$ is a DF-Schwartz space.
2. $(E, \mathcal{T}_{c_0})$ is quasi-ℓ_∞-barrelled.
3. $(E, \mathcal{T}_{c_0})$ is a DF-space.
4. $(E', \beta(E', E))$ is a Fréchet-Montel space.
5. $(E, \mathcal{T}_{c_0})$ is quasi-barrelled.

Proof. **(1) ⇒ (2)**: Since $(E, \mathcal{T})$ is Schwartz, $\mathcal{T} = \mathcal{T}_{c_0}$ and since it is (DF), $\mathcal{T} \geq \tau_{\beta,\infty}$. Therefore $\mathcal{T}_{c_0} \geq \tau_{\beta,\infty}$ and the conclusion follows.

(2) ⇒ (3): By Theorem 4.31, we have $\mathcal{T}_{c_0} = \tau_{\beta,0} \leq \tau_{\beta,\infty}$ and our hypothesis implies $\mathcal{T}_{c_0} \geq \tau_{\beta,\infty}$ which together with Theorem 4.37 proves the statement.

(3) ⇒ (4): Since $\mathcal{T}$ is a (df) space, $(E', \beta(E', E))$ is a Fréchet space. By Theorem 4.31, $\mathcal{T}_{c_0} = \tau_{\beta,0}$, and by Remark 2.67, $\mathcal{T}_{c_0} \geq \tau_{\beta,\infty}$. Hence, $\mathcal{T}_{c_0} = \tau_{\beta,0} = \tau_{\beta,\infty}$. Consequently, for every $\beta(E,', E)$-bounded sequence $(a_n)_{n\in\mathbb{N}}$, there is a $\beta(E', E)$-null sequence $(b_n)_{n\in\mathbb{N}}$ such that

$$\{a_n : n \in \mathbb{N}\} \subseteq \{a_n : n \in \mathbb{N}\}^{\circ\circ} \subseteq \{b_n : n \in \mathbb{N}\}^{\circ\circ} = \overline{\mathrm{acx}}\,\{b_n : n \in \mathbb{N}\}\,.$$

Since $\overline{\mathrm{acx}}\,\{b_n : n \in \mathbb{N}\}$ is $\beta(E', E)$-compact (cf. Lemma 2.15), we know that $\{a_n : n \in \mathbb{N}\}$ is relatively compact. Since $\mathcal{T}_{c_0} = \tau_{\beta,\infty}$ is a permissible topology, it has the same bounded sets as $\mathcal{T}$. Hence, $\beta(E', E)$ is the strong dual of $\mathcal{T}_{c_0}$. If $\mathcal{U}$ is a zero neighborhood basis in $\tau_{\beta,\infty}$, then $(V^\circ)_{V\in\mathcal{U}}$ is a basis of bounded sets in $\beta(E', E)$. For every $V \in \mathcal{U}$ there is a $\beta(E', E)$-bounded sequence $(a_n)_{n\in\mathbb{N}}$ such that $\{a_n : n \in \mathbb{N}\}^\circ \subseteq V$. Consequently, $V^\circ \subseteq \{a_n : n \in \mathbb{N}\}^{\circ\circ}$ and we conclude by the above, that every $\beta(E', E)$-bounded set is relatively $\beta(E', E)$-compact, hence $(E', \beta(E', E))$ is a semi-Montel space. Since $(E', \beta(E', E))$ is a Fréchet space (and therefore bornological) it is a Montel space.

(4) $\Rightarrow$ (5): Because $(E', \beta(E', E))$ is a Montel space all bounded sets are relatively compact. Hence, $\beta(E'', E')$, the topology of uniform convergence on bounded sets of $(E', \beta(E', E))$, and the topology of uniform convergence on the compact sets of $(E', \beta(E', E))$, $\tau_c(E'', E')$ coincide on E''. Since $\beta(E'', E')$ induces $\beta^*(E, E')$ on E and $\tau_c(E'', E')$ induces $\mathcal{T}_{c_0}$, it follows that $\mathcal{T}_{c_0} = \beta^*(E, E')$. Since $\beta^*(E, E')$ is quasi-barrelled by Lemma 2.47, the result follows.

To see that the topology $\tau_c(E'', E')$ in fact induces $\mathcal{T}_{c_0}$ on E, observe that $\mathcal{T}_{c_0}$ is the topology of uniform convergence on $\mathcal{E}$-null sequences, which are relatively $\beta(E', E)$-compact by Lemma 2.15. Hence, $\tau_c(E'', E') \geq \mathcal{T}_{c_0}$. To show the converse, note that by Theorem 2.16 for every relatively compact set A, there is a $\beta(E', E)$-null sequence $(a_n)_{n\in\mathbb{N}}$ (which by Theorem 2.40 is also a $\mathcal{E}$-null sequence) such that $A \subseteq \overline{\operatorname{acx}}\{a_n : n \in \mathbb{N}\}$ and therefore $\{a_n : n \in \mathbb{N}\}^\circ \subseteq A^\circ$.

(5) $\Rightarrow$ (1): If $(E, \mathcal{T}_{c_0})$ is quasi-barrelled, then it is quasi-$\aleph_0$-barrelled and therefore also $\mathcal{T}$ is quasi-$\aleph_0$-barrelled and hence a (DF) space. Since $\mathcal{T}$ is quasi-barrelled, $\mathcal{T} = \tau_{\beta,0} = \mathcal{T}_{c_0}$ (by Theorem 4.31), hence $\mathcal{T}$ is a Schwartz space. □

Remark 4.39. The previous Theorem shows that not every (df)-Schwartz space is already a (DF) space. To see this, let E be a infinite dimensional normable lcs. Then $(E', \beta(E', E))$ is also a normed space and therefore $\overline{\{a_n : n \in \mathbb{N}\}}^{\beta(E',E)}$ is compact in $\beta(E', E)$ for every sequence $a_n \xrightarrow{\beta} 0$. Hence, the set is compact in $\sigma(E', E)$ and $\tau_{\beta,0} \leq \mu(E', E)$. Therefore, the bounded sets in $\tau_{\beta,0}$ are the same as in $(E, \mathcal{T})$ and there is a fundamental sequence of bounded sets in $(E, \tau_{\beta,0})$. Consequently, $(E, \tau_{\beta,0}) = (E, \mathcal{T})$ is a (df) space. By Theorem 4.31, $(E, \tau_{\beta,0}) = (E, \mathcal{T}_{c_0})$ and therefore $(E, \tau_{\beta,0})$ is a (df)-Schwartz space.

If $(E, \tau_{\beta,0})$ would be a (DF) space, then $(E', \beta(E', E))$ would be a Montel space by Theorem 4.38. This leads to a contradiction, since $(E', \beta(E', E))$ is a infinite dimensional normed space and therefore the unit ball can not be compact, as it is the case in a Montel space.

Theorem 4.40 (Theorem 12.5.6, Jarchow (1981))**.** If E is a metrizable lcs, then $(E', \gamma(E', E))$ is a (gDF)-Schwartz space and $\gamma(E', E) = \mu_0(E', \tilde{E})$.

Proof. By Theorem 3.16, $\tilde{E}$ is the dual of (E', γ) and therefore γ is consistent with the dual pairing $\langle E', \tilde{E}\rangle$. By Theorem 3.20, we know that $\gamma(E', E) = \tau_c(E', \tilde{E})$. Now, if $(a_n)_{n\in\mathbb{N}}$ is a null sequence in $(\tilde{E}, \tilde{\mathcal{T}}) = (E, \gamma(E', E))'$, then $\{a_n : n \in \mathbb{N}\}^{\circ\circ}$ is compact, since $(\tilde{E}, \tilde{\mathcal{T}})$ is metrizable. On the other hand, for every compact set $C \subseteq \tilde{E}$ there is a null sequence $(a_n)_{n\in\mathbb{N}}$ such that $C \subseteq \{a_n : n \in \mathbb{N}\}^{\circ\circ}$. Hence, $\tau_c(E', \tilde{E})$, and by the above $\gamma(E', E)$, is the topology of uniform convergence on $\mathcal{T}$-null sequences. Since $(\tilde{E}, \tilde{\mathcal{T}})$ is the strong dual of $\gamma(E', E)$, $\gamma(E', E)$ is quasi-c_0-barrelled.

Since $\gamma(E', E)$ is the strong dual of $(\tilde{E}, \tilde{\mathcal{T}})$, one gets a fundamental basis of bounded sets by polarization of a suitable zero neighborhood basis. These sets are compact by Alaoğlu-Bourbaki, since $\gamma(E', E) = \tau_c(E', E) \leq \tau_{pc}(E', \tilde{E})$. Applying Theorem 4.34 yields that $\gamma(E', E)$ is a (gDF)-Schwartz space. By Theorem 4.31, 2, $\gamma(E', E) = \mu_0(E', \tilde{E})$ since $\gamma(E', E)$ is the topology of uniform convergence on null sequences in $(\tilde{E}, \tilde{\mathcal{T}})$. □

Theorem 4.41 (Theorem 12.5.7, Jarchow (1981))**.** For every (df) space E, the following statements are equivalent:

1. E is semi-reflexive.
2. $E_0 = (E, \tau_{\beta,0})$ is quasi-complete.
3. $E_0 = (E, \tau_{\beta,0})$ is B-complete.

In that case, E is also B-complete.

Proof. **(1) ⇒ (2)**: By Theorem 2.54, $(E, \sigma(E, E'))$ is quasi-complete, and by Theorem 2.7, this also holds for $\tau_{\beta,0}$ (since $\tau_{\beta,0}$ is compatible with $\langle E, E' \rangle$). By Theorem 4.31, $\mathcal{T}c_0 = \tau_{\beta,0}$ and the result follows.

(2) ⇒ (3): From Theorem 4.33, we know that E_0 is complete. By Theorem 4.31, 3, it follows that $(E', \beta(E', E))' = E'' = \tilde{E}_0 = E_0$. Hence, $(E', \beta(E', E))$ is a Fréchet space with dual $E = E_0$ (as vector spaces). We know that $\gamma(E, E') = \gamma^t(E, E') = \tau_{pc}(E, E')$, where the first equality follows from Theorem 3.19, while the second follows from Theorem 3.23. Since $\gamma(E, E') = \tau_{pc}(E, E')$, we conclude from Theorem 3.27 that every topology $\mathcal{T}'$ on E with $\gamma(E, E') \leq \mathcal{T}' \leq \mu(E, \tilde{E}')$ is B-complete. By Theorem 4.40, this even holds for every $\mathcal{T}'$ with $\mu_0(E, \tilde{E}') \leq \mathcal{T}' \leq \mu(E, \tilde{E}')$. Since the topology on E_0, by Theorem 4.31, is $\tau_{\beta,0} = \mu_0$ the assertion follows.

(3) ⇒ (1): If $E_0 = (E, \mathcal{T}_{c_0})$ is B-complete, then it is complete and therefore $E_0 = E''$ by Theorem 4.31, i.e. E_0 is semi-reflexive. Since $\mathcal{T}_{c_0}$ is admissible for $\langle E, E' \rangle$, also $(E, \mathcal{T})$ is semi-reflexive.

The last remark follows, since $\mathcal{T}_{c_0} \leq \mathcal{T}$ and therefore $\mathcal{T}$ is also B complete. □

Theorem 4.42 (Theorem 12.5.8, Jarchow (1981))**.** A lcs $(E, \mathcal{T})$ is the strong dual of a Fréchet-Montel space F, iff it is a complete (DF)-Schwartz space. In this case, E is even barrelled.

Proof. Suppose $E = (F', \beta(F', F))$ for some Fréchet-Montel space F. Then E is a Schwartz space by Theorem 3.64 and additionally a complete DF-space by Theorem 4.11.

By Theorem 4.38 and since $\mathcal{T} = \mathcal{T}c_0$, $\mathcal{T}$ is quasi-barrelled. Since E is complete (and thus locally complete), it follows from Theorem 2.57 that E is barrelled.

If on the other hand side E is a complete DF-Schwartz space, then we have by Theorem 3.50 that every bounded set is relatively compact, i.e. E is a semi-Montel space. By Theorem 4.38, E is quasi-barrelled and therefore a Montel space. By Theorem 3.63, $F = (E', \beta(E', E))$ is also a Montel space, and since E is reflexive by Theorem 3.60, it is the strong dual of a Montel space. □

The following Corollary sums up some remarks after Jarchow (1981), Corollary 12.5.9.

Corollary 4.43. A lcs E is the strong dual of a Fréchet-Montel space, iff it is quasi-$\aleph_0$-barrelled and admits a fundamental sequence of compact disks.

Proof. In view of the previous theorem, we only have to show that E is a complete (DF)-Schwartz space, iff it is quasi-$\aleph_0$-barrelled and admits a fundamental sequence of compact disks.

A complete (DF)-Schwartz space E is quasi-$\aleph_0$-barrelled and has a fundamental sequence of bounded sets. Since the space is Schwartz every bounded set is precompact and since it is complete relatively compact. Hence, E has a fundamental sequence of compact disks.

Now assume the converse. If E has a fundamental sequence of compact disks $(K_n)_{n\in\mathbb{N}}$, then it already has a fundamental sequence of bounded sets. To see this, choose K_{k_n} such that $nK_n \subseteq K_{k_n}$ and assume that there is a bounded set B, which is not in any of the $(K_{k_n})_{n\in\mathbb{N}}$. Then we can find a sequence $(x_n)_{n\in\mathbb{N}} \subseteq B$ such that $n^{-1}x_n \notin K_n$ for all $n \in \mathbb{N}$. But since $(n^{-1}x_n)_{n\in\mathbb{N}}$ converges to zero, $\{n^{-1}x_n\}^{\circ\circ}$ is a compact disk and therefore contained in some K_m – a contradiction. Hence, E is a (DF) space. By our assumption every bounded set is precompact and therefore E is additionally a Schwartz space by Theorem 4.34. Since every closed and bounded set C is contained in a compact disk, C is also compact and therefore complete. Hence, E is quasi-complete and therefore complete by Theorem 4.33. □

Corollary 4.44 (Corollary 12.5.9, Jarchow (1981))**.** A lcs E is the strong dual of a Fréchet-Schwartz space, iff it is a complete (DF)-Schwartz space in which every null sequence converges locally.

Proof. We first assume that $(E, \mathcal{T}) \cong (F', \beta(F', F))$ for a Fréchet-Schwartz space F. Since F is a complete Schwartz space, it is a semi-Montel space by Corollary 3.50. Since F is additionally metrizable, it is even a Montel space by Theorem 3.60. We therefore know from Theorem 4.42 that E is a complete (DF)-Schwartz space. It remains to show that every null sequence converges locally. To show this, let $(x_n)_{n\in\mathbb{N}} \subseteq E$ be a null sequence and note that $A = \{x_n : n \in \mathbb{N}\}$ is bounded and therefore $U = A^\circ$ is a bornivorous barrel in F. Since F is Fréchet and therefore barrelled, U is a zero neighborhood in F. Because F is Schwartz there exists a zero neighborhood $V \subseteq U$ such that U° is compact in E_{V° by Theorem 3.48. Therefore $(x_n)_{n\in\mathbb{N}}$ has a limit point in V°. Since the mapping $E_{V^\circ} \hookrightarrow E$ is continuous and one to one $(x_n)_{n\in\mathbb{N}}$ converges to a unique limit in E_{V°. Hence, we have shown that $(x_n)_{n\in\mathbb{N}}$ converges locally.

To show the converse, we assume E is a complete DF-Schwartz space in which every null sequence converges locally. By Theorem 4.42, we know that E is the strong dual of a Fréchet-Montel space F. The topology on F is the topology of uniform convergence on equicontinuous sets $U^\circ \subseteq E$ for zero neighborhoods U in F. The sets U° are $\sigma(E,F)$-compact and $\sigma(E,F)$-metrizable, where the latter follows from Theorem 3.2. Hence by Theorem 2.16, there is a $\sigma(E,F)$-convergent sequence $(x_n)_{n\in\mathbb{N}}$ such that $\{x_n : n \in \mathbb{N}\}^{\circ\circ} \supseteq U^\circ$. Since F is Montel, we know that $\beta(E,F) = \tau_{pc}(E,F) = \tau_c(E,F) = \gamma(E,F)$ where the last equality follows from Theorem 3.20 and the second last from the fact that F is complete. Since $\gamma(E,F)$ is the finest locally convex topology which coincides with $\sigma(E,F)$ on sets of the form U° by Remark 3.15, we know that $\beta(E,F)$ coincides with $\sigma(E,F)$ on U°. Hence, the sequence constructed above also converges in $(E,\mathcal{T}) = (E,\beta(E,F))$ and we can say that the topology on F is the topology of uniform convergence on biduals of null sequences in E. Since every such null sequence is also locally convergent by assumption, the topology on F is the topology of uniform convergence on biduals of locally convergent sequences in E. This however implies that $\mathcal{T} = \mathcal{T}_{c_0}$, i.e. that E is a Schwartz space. $\square$

Theorem 4.45 (Theorem 12.5.10, Jarchow (1981))**.** Let E be a semi-Montel space and F a (df) space. Suppose that $T \in \mathcal{L}(E,F)$ has the property that $T^{-1}(B)$ is bounded in E for every bounded set $B \subseteq F$. Then T is open and injective. Moreover, E is a complete gDF-Schwartz space in this case.

Proof. Since $\ker(T) = T^{-1}(0)$ and $\ker(T)$ is a subspace of E, it follows that $\ker(T) = \{0\}$ and T is injective. By Theorem 2.30, $T' : (F',\beta(F',F)) \to (E',\beta(E',E))$ is continuous. The fact that T is injective implies that range(T') is $\sigma(E',E)$-dense, cf Lemma 2.28. Since E is semi-Montel, we have $\beta(E',E) = \mu(E',E)$ and since the closures of convex sets coincide in every admissible lc topology, we have that range(T') is also dense in $\beta(E',E)$.

For a bounded set $B = B^{\circ\circ} \subseteq F$, we have $\overline{T'(B^\circ)} = T'(B^\circ)^{\circ\circ} = T^{-1}(B)^\circ$, where the second equality follows from Lemma 2.28. Since $T^{-1}(B)$ is bounded by assumption, $T^{-1}(B)^\circ$ is a $(E',\beta(E',E))$-zero neighborhood. The set B° is a zero neighborhood in $(F',\beta(F',F)$, and therefore T' is a nearly open mapping. As $(F',\beta(F',F))$ is a Fréchet space, it is B-complete (cf. Theorem 3.26) and T' is open by Theorem 3.31. Therefore, we have range$(T') \cong (F',\beta(F',F))/\ker(T')$, and since $\ker(T')$ is closed and $(F',\beta(F',F))/\ker(T')$ is a Fréchet space, we have that range$(T') = E'$ and that $(E',\beta(E',E))$ is Fréchet.

Let $U = U^{\circ\circ}$ be a zero neighborhood in E. Then U° is $\beta(E',E)$-compact by Theorem 3.62. By Theorem 2.17, there is a compact set K in $(F',\beta(F',F))$ such that $U^\circ \subseteq$

$T'(K)$. By Theorem 2.16, there exists a null sequence in $(F', \beta(F', F))$ such that $K \subseteq \overline{\operatorname{acx}}\{x_n : n \in \mathbb{N}\}$. Since F is quasi-c_0-barrelled, $V := K^\circ \supseteq \{x_n : n \in \mathbb{N}\}^\circ$ is a zero neighborhood in F. From $U^\circ \subseteq T'(V^\circ)$, we get $(T')^{-1}(U^\circ) \subseteq V^\circ$ and by Lemma 2.28 $T(U)^\circ \subseteq V^\circ$, which implies $T(U) \supseteq V$. Hence we have shown that T is in fact an open mapping.

Let $(x_n)_{n\in\mathbb{N}}$ be a null sequence in $(E', \beta(E', E))$. Similar as above we get: since $\{x_n : n \in \mathbb{N}\}$ is compact, there is a $\beta(F', F)$-compact set K such that $\{x_n : n \in \mathbb{N}\} \subseteq T'(K)$. We get that $V = K^\circ$ is a zero neighborhood in F and that $T(\{x_n : n \in \mathbb{N}\}^\circ) \subseteq V$ or $\{x_n : n \in \mathbb{N}\}^\circ \subseteq T^{-1}(V) \subseteq E$. Hence, $\{x_n : n \in \mathbb{N}\}$ is equicontinuous and E is quasi-c_0-barrelled. If $(B_n)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets in F, then $T^{-1}(B_n)$ is a fundamental sequence of bounded sets in E because T is one to one. Consequently E is a (df) space. Since E is semi-Montel, the sets $T^{-1}(B_n)$ are even relatively compact and E is a (gDF)-Schwartz space by Theorem 4.34. The space E is quasi-complete by Theorem 3.62 and therefore complete by Theorem 4.33. □

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Notation

Appendices

Zusammenfassung

Diese Arbeit befasst sich mit dem Konzept der Folgentonnelliertheit und mit sogenannten df-Räumen. Bei Folgentonneliertheit handelt es sich um eine Abschwächung des gängigen Konzepts der Tonniertheit, welches die lokalkonvexen Räume charakterisiert, in denen der Satz von Banach-Steinhaus gilt. Für die Definition der Folgentonnelliertheit schwächt man die Definition der Tonnelliertheit insofern ab, als dass nicht alle Tonnen Nullumgebungen sein müssen, sondern die Bipolaren von (schwach) konvergenten bzw beschränkten Folgen im entsprechenden Dualraum. Die Arbeit geht auf unterschiedliche Konzepte der Folgentonnelliertheit ein und beschreibt die Verbindungen zu Eigenschaften der Dualräume. Aufbauend hierauf werden df-Räume definiert, welche gerade jene lokalkonvexen Räume sind, deren starke Dualräume Fréchetraume sind. Es werden zahlreiche Varianten von df-Räumen, deren Vererbungseigenschaften sowie Beziehungen zu Schwartz und Montel Räumen sowie zu sogennanten *imbedding spectra* diskutiert.

Abstract

This thesis discusses the notions sequential barrelledness and df-spaces. Sequential barrelledness is a weakening of the well known barrelledness property of a locally convex space, which is used to characterize the spaces in which the uniform boundedness principle holds. For sequential barrelledness, instead of requiring that all barrels are zero neighborhoods, one restricts the attention to bipolars of (weakly) convergent or bounded sequences in the dual space. We study the different notions of sequential barrelledness and the implications these properties have for the respective dual spaces. We use the concepts of barrelledness to characterize so called df-spaces, which are those spaces whose strong duals are Fréchet spaces. We study several variations of the concept of df-spaces, stability properties, and their connections to Schwartz and Montel spaces, as well as to imbedding spectra.

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