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Polynomials and Fast Fourier Transform

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*To my Mother and Father,
who have been dedicating their lives to their children lovingly*

Abstract

It is the purpose of this thesis to emphasize the connection between general Vandermode matrices and the specific properties of the Fourier matrix, which is interpreted as the Vandermonde matrix for the unit roots of order N . In doing so a number of interesting properties can be derived in an elementary way, and it can be demonstrated that they are in principle consequences of elementary properties of complex numbers. For example, the Fourier matrix is (up to the normalization factor $\sqrt{N}$) a unitary matrix, which follows from the exponential law combined with the formula for finite geometric series. Thesis provides some basic information about Complex Fourier Series for periodic functions and pointwise convergence of Fourier series. Reviewing Euler formula and properties of "odd and even" functions, Fourier series satisfying Dirichlet's conditions can be expressed with complex number coefficients. Further in the thesis some basic properties such as 'linearity, scaling, shifting and modulation' of Fourier transform is introduced along with Plancherel's formula, Convolution property and Shannon's theorem. At the end of the thesis we provide some small applications using MATLAB to the elementary probability theory, large integer multiplication and digital filtering.

Abstrakt

Das Ziel dieser Diplomarbeit besteht darin, den Zusammenhang zwischen allgemeinen Vandermonde Matrizen und spezifischen Eigenschaften der Fourier Matrix zu beschreiben, wobei letztere als Vandermonde Matrix für die Einheitswurzeln der Ordnung N interpretiert wird. Auf diese Weise kann eine Reihe von wichtigen Eigenschaften der Fourier Matrix in elementarer Weise hergeleitet werden. insbesondere wird gezeigt, dass diese im Wesentlichen nichts als Eigenschaften der komplexen Zahlen sind. Beispielsweise ist die Fourier Matrix (bis auf Normalisierung mit $\sqrt{N}$) eine unitäre Matrix, was aus dem Exponentialgesetz und der Summen Formel für endliche geometrische Reihen folgt.

Unter anderem werden in dieser Arbeit grundlegende Eigenschaften komplexer Fourierreihen für periodische Funktionen vorgestellt, es wird auch die punktweise Konvergenz von Fourierreihen charakterisiert. Weiters wird mit Hilfe der Euler Formel und einigen Eigenschaften gerader und ungerader Funktionen hergeleitet, dass Fourierreihen, die die Dirichletbedingungen erfüllen, mit komplexen Koeffizienten dargestellt werden können. Nicht zuletzt werden einige einfache Eigenschaften der Fouriertransformation, wie z.B. Linearität, die Skalierungseigenschaft, Shift und Modulation eingeführt, wie auch die Formel von Plancherel, die Konvolutionseigenschaft und das Theorem von Shannon. Abschließend werden einige Anwendungen in der Wahrscheinlichkeitstheorie, der Multiplikation großer Zahlen und der Theorie der digitalen Filter gegeben.

Summary

The Fast Fourier transform (FFT) is an algorithm to compute the discrete Fourier transform (DFT) and its inverse. In 1965 J. Cooley and J. Tukey published a paper about the algorithm and describing how to perform it conveniently on a computer. The DFT was long known before 1965. However getting benefit from it was limited because of the computational workload. Because the calculation of the DFT of an input sequence of an N length sequence $\{f_n\}$ requires N complex multiplications to compute each on the N values, F_m , for a total of N^2 multiplications. The fast Fourier transform which enables practical fast frequency domain implementation of processing algorithms, revolutionized the digital signal processing.

The aim of this thesis is to emphasize the connection between Vandermonde matrices and some properties of a Fourier matrix which is an interpretation of Vandermonde matrix for the unit roots of order N .

The first chapter gives a brief summary of the representation of polynomials and Vandermonde matrices. Then it is emphasized that the polynomial multiplication is in fact a convolution. Thus, the Cauchy Product of two polynomials is replaced with convolution integral which has the same properties as ordinary multiplication such as bilinearity, commutativity and associativity.

The second chapter is concerned with Discrete Fourier Transform (DFT), whose kernel is the principal root of unity. Further in this chapter the inverse transform and convolution theorem in both time and frequency domains will be introduced. Furthermore some important properties of DFT is explained.

In the third chapter, complex Fourier series for periodic functions is denoted and pointwise convergence of Fourier series is explained. Reviewing Euler formula and properties of "odd and even" functions, Fourier series satisfying Dirichlet's conditions can be expressed with complex number coefficients. Further in this chapter some basic properties such as 'linearity, scaling, shifting and modulation' of Fourier transform are introduced along with Plancherel's formula, the convolution property and Shannon's theorem. In the last Chapter, some applications in probability theory, large integer multiplication and digital filtering are conducted with the help of MATLAB.

Zusammenfassung

Die schnelle Fouriertransformation (FFT) ist ein Algorithmus um die diskrete Fouriertransformation (DFT) und ihre Inverse zu berechnen. Im Jahr 1965 publizierten J. Cooley und J. Tukey ein Paper, das sowohl den Algorithmus selbst als auch eine Möglichkeit beschrieb ihn effizient auf einem Computer zu implementieren. Die diskrete Fouriertransformation war zwar schon lange vor 1965 bekannt, aber aufgrund der hohen notwendigen Rechenleistung konnte man keinen großen Nutzen aus ihr ziehen. Um die DFT einer Input-Folge $\{f_n\}$ der Länge N zu berechnen, benötigt man nämlich N komplexe Multiplikationen für jeden der N Werte F_m , was insgesamt N^2 Multiplikationen ergibt. Die schnelle Fouriertransformation (FFT) hat die digitale Signalverarbeitung revolutioniert, weil sie von vielen schnellen Algorithmen verwendet wird.

Das Ziel dieser Diplomarbeit ist es, den Zusammenhang zwischen Vandermonde Matrizen und einigen Eigenschaften der Fourier Matrix zu beleuchten, die man als Vandermonde Matrix für die Einheitswurzeln der Ordnung N interpretieren kann. Das erste Kapitel gibt einer kurzen Übersicht über die Darstellung von Polynomen und über Vandermonde Matrizen. Es wird dargelegt, dass man die Multiplikation von Polynomen als Konvolution betrachten kann. Im Kontinuierlichen ersetzen wir das Cauchyprodukt von zwei Polynomen durch das Konvolutionsintegral, das dieselben Eigenschaften wie die gewöhnliche Multiplikation aufweist, wie z.B. Bilinearität, Kommutativität und Assoziativität. Das zweite Kapitel beschäftigt sich mit der diskreten Fouriertransformation (DFT), die auf den Eigenschaften auf die Gruppe der n -ten Einheitswurzeln beruht. Es werden die inverse Transformation und das Konvolutionstheorem sowohl im Zeit-, als auch im Frequenzraum eingeführt. Weiters werden einige wichtige Eigenschaften der DFT beschrieben. Im dritten Kapitel werden komplexe Fourierreihen für periodische Funktionen definiert und es wird die punktweise Konvergenz von Fourierreihen beschrieben. Mit Hilfe der Euler Formel und einigen Eigenschaften gerader und ungerader Funktionen wird hergeleitet, dass Fourierreihen, die die Dirichletbedingungen erfüllen, mit komplexen Koeffizienten dargestellt werden können. Weiters werden in diesem Kapitel einige einfache Eigenschaften der Fouriertransformation, wie z.B. Linearität, die Skalierungseigenschaft, Shift und Modulation eingeführt, wie auch die Formel von Plancherel, die Konvolutionseigenschaft und das Theorem von Shannon. Im letzten Kapitel werden einige Anwendungen in der Wahrscheinlichkeitstheorie, der Multiplikation großer Zahlen und der Theorie der digitalen Filter mit Hilfe von MATLAB ausgeführt.

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1 Polynomials

1.1 Basic Notations and Theorems

First we recall some basic notions and mention a few theorems which will be important for the exposition of the topic.

Definition 1.1.1. A polynomial in the variable x is a representation of a function $A(x) = a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$ as a formal sum $A(x) = \sum_{j=0}^{n-1} a_jx^j$.

We call the values $a_0, a_1, \dots, a_{n-1}$ the **coefficients** of the polynomial.

$A(x)$ is said to have **degree** k if its highest nonzero coefficient is a_k . Any integer strictly greater than the degree of a polynomial is a **degree-bound** of that polynomial.

Example 1.1.2. Coefficient representation of the polynomial $p(x) = 6x^3 + 7x^2 - 10x + 9$ is $(9, -10, 7, 6)$.

Evaluating the polynomial $p(x)$ at point x_0 consists of computing the value of $p(x)$ at point x_0 . Numerical evaluation is possible, for instance, via **Horner's Rule**,

$$p(x) = a_0 + x_0(a_1 + x_0(a_2 + \dots + x_0(a_{n-2} + x_0(a_{n-1}))))),$$

although it is costly in terms of time.

1.2 Point Value Representation

Definition 1.2.1. A *point-value representation* of a polynomial $A(x)$ of degree-bound n is a set of n *point-value pairs* $\{(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})\}$. All of the x_k are distinct and $y_k = A(x_k)$.

Definition 1.2.2. [7] (p.199) A determinant is a special number which is associated to any square matrix. I.e., the determinant of an $n \times n$ matrix A having entries from a field F is a scalar in F . It is denoted by $\det(A)$ and can be computed as follows:

1. If A is 1×1 , then $\det(A) = A_{11}$, the single entry of A .
2. If A is $n \times n$ for $n > 1$, then $\det(A) = \sum_{j=1}^n (-1)^{i+j} A_{ij} \det(\bar{A}_{ij})$,

where $\bar{A}_{ij}$ denotes the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j .

Remark 1.2.3. It is one of the well-known results in Linear Algebra that an $n \times n$ -matrix is invertible if and only if $\det(A) \neq 0$.

Definition 1.2.4. A Vandermonde Matrix V is a matrix with terms of a geometric progression in each row ($V_{i,j} = \alpha_i^{j-1}$ for all indices i and j) I.e., it is an $m \times n$ matrix of the form

$$V = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{pmatrix}.$$

Theorem 1. [15](p.10-11) An $n \times n$ Vandermonde matrix V has the following determinant

$$\det(V) = \prod_{1 \leq i < j \leq n} (a_j - a_i)$$

Proof. For $n = 2$ the determinant of V is

$$\det \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} = x_2 - x_1,$$

so the property holds.

Let

$$V_n = \begin{vmatrix} a_1^{n-1} & a_1^{n-2} & \dots & a_1 & 1 \\ a_2^{n-1} & a_2^{n-2} & \dots & a_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2} & \dots & a_n & 1 \end{vmatrix}.$$

For all $n \in \mathbb{N}$, let $P(n)$ be the proposition that $V_n = \prod_{1 \leq i < j \leq n} (a_i - a_j)$. We have showed the basis with the 2×2 matrix. Now we will show that if $P(k)$ is true, where $k \geq 2$, then it follows that $P(k+1)$ is true. So this is our induction hypothesis: $V_k = \prod_{1 \leq i < j \leq k} (a_i - a_j)$, and

$$V_{k+1} = \begin{vmatrix} x^k & x^{k-1} & \dots & x^2 & x & 1 \\ a_2^k & a_2^{k-1} & \dots & & & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{k+1}^k & a_{k+1}^{k-1} & & a_{k+1}^1 & a_{k+1} & 1 \end{vmatrix}.$$

If we expand it in the terms of the first row, we can see it as a polynomial in x whose degree is not greater than k . We denote this polynomial by $f(x)$. If we substitute any a_r for x in the determinant, two of its rows will be the same. If two columns of a matrix are the same, then the determinant of the matrix is 0. Substitution in the determinant is equivalent to substituting a_r for x in $f(x)$. Thus it follows that $f(a_2) = f(a_3) = \dots = f(a_{k+1}) = 0$. So $f(x)$ is divisible by each of the factors $x - a_2, x - a_3, \dots, x - a_{k+1}$. All these factors are distinct, otherwise the original determinant is zero. So $f(x) = C(x - a_2)(x - a_3) \dots (x - a_k)(x - a_{k+1})$. As the degree of $f(x)$ is not greater than k , it follows that C is independent of x . From expansion, we can see that the coefficient of x^k is

$$\begin{vmatrix} a_2^{k-1} & \dots & a_2^2 & a_2 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{k+1}^{k-1} & \dots & a_{k+1}^2 & a_{k+1} & 1 \end{vmatrix}.$$

By the induction hypothesis, this is equal to $\prod_{2 \leq i < j \leq k+1} (a_i - a_j)$. So this has to be

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our value of C . Therefore we obtain

$$f(x) = C(x - a_2)(x - a_3) \dots (x - a_k)(x - a_{k+1}) \prod_{2 \leq i < j \leq k+1} (a_i - a_j).$$

Substituting a_1 for x , we get the proposition $P(k + 1)$. So $P(k) \Rightarrow P(k + 1)$. Therefore $V_n = \prod_{1 \leq i < j \leq n} (a_i - a_j)$. This is equivalent to

$$\det(V) = \prod_{1 \leq i < j \leq n} (a_j - a_i).$$

□

Theorem 2. For any set of n point value pairs (x_i, y_i) with $x_i \neq x_j$ for $i \neq j$, there is a unique order n polynomial $A(x)$ such that $A(x_i) = y_i$ for all pairs.

Proof. We need to solve

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

According to the former theorem, the determinant of the Vandermonde matrix is equal to $\prod_{j < k} (x_k - x_j)$.

Because by assumption the x_i are pairwise distinct, the Vandermonde matrix is nonsingular and the linear system has a unique solution for every right hand side. □

Remark 1.2.5. If we have two polynomials in (the same) point value representation $\{(x_0, y_0^1), (x_1, y_1^1), \dots, (x_n, y_n^1)\}$ and $\{(x_0, y_0^2), (x_1, y_1^2), \dots, (x_n, y_n^2)\}$ the sum of two degree n polynomials in point value representation is computed in $O(n)$ time:

$$\{(x_0, y_0^1 + y_0^2), (x_1, y_1^1 + y_1^2), \dots, (x_{n-1}, y_{n-1}^1 + y_{n-1}^2)\}$$

To compute the product of two degree n polynomials we need an "expanded" point value representation of $2n$ points in order to recover the coefficients.

Given such a representation, the product of two polynomials in point value representation is computed in $O(n)$ ¹ time and it can be written as

$$\left\{ (x_0, y_0^1 y_0^2), (x_1, y_1^1 y_1^2), \dots, (x_{2n-2}, y_{2n-1}^1 y_{2n-1}^2) \right\}.$$

1.3 Polynomial Multiplication

The convolution operation is quite important in Harmonic Analysis. At first sight one can say that it corresponds to the multiplication of polynomials. For the continuous domain one can say that the convolution of f and g , each of them describing the probability density of a random variable (call it $X + Y$ respectively), then $f * g$ describes the probability distribution of $X + Y$ (assuming that X and Y are independent variables). [4]

The convolution of two vectors is like the multiplication of two polynomials in coefficient form. If the coefficient representations of two n degree polynomials are spread out by padding the representation with n zero coefficients as place holders for the higher-order terms, then polynomial multiplication is equivalent to convolution.

Definition 1.3.1. (Polynomial multiplication) If $p(x)$ and $q(x)$ are polynomials of degree-bound n , we say their product $C(x)$ is a polynomial of degree-bound $2n - 1$ such that

$$C(x) = p(x)q(x)$$

for all $x \in F$. There is another way to denote $C(x)$: The well known *Cauchy Product*, which is a discrete convolution of two sequences (in our case, the coefficients of the two polynomials)

$$C(x) = \sum_{j=0}^{2n-2} c_j x^j, \quad \text{where} \quad c_j = \sum_{k=0}^j a_k b_{j-k}.$$

Therefore $\text{degree}(C) = \text{degree}(p) + \text{degree}(q)$, which means if p is a polynomial of degree-bound n and q is a polynomial of degree-bound m , then C is a polynomial of degree-bound $n + m - 1$. To simplify notation we only say that C has a degree-bound

¹Landau's symbol $O(n)$ is used to describe the situation that the duration of the operation is controlled by Cn time units, where $C > 0$.

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$n + m$, since a polynomial of degree-bound $k - 1$ is also a polynomial of degree-bound k .

There is a continuous analogue for the Cauchy-product, which can also be seen as the operation which allows to determine the probability distribution of the sum of two independent random variables with (absolutely) continuous distribution.

Example 1.3.2. [16] Given two polynomials $p(x) = 6x^3 + 7x^2 - 10x + 9$ and $q(x) = -2x^3 + 4x - 5$, their product can be calculated by Matlab as follows

```
a = [9, -10, 7, 6]; for p(x) = 6 * x^3 + 7 * x^2 - 10 * x + 9
b = [-5, 4, 0, -2]; for q(x) = -2 * x^3 + 4 * x - 5
c = conv(a, b)
c = -45, 86, -75, -20, 44, -14, -12
```

That means $p(x) \cdot q(x) = -12x^6 - 14x^5 + 44x^4 - 20x^3 - 75x^2 + 86x - 45$

Polynomial $p(x)$, above, is shown in a coefficient representation as the vector of coefficients $(9, -10, 7, 6)$.

The point-value representation of $p(x)$ can also be described by evaluating the polynomial at $m + 1$ distinct points. If $p(x)$ was evaluated at the points $x = 0, 1, 3, -1$, its point value representation would be $\{(0, 9), (1, 12), (3, 204), (-1, 20)\}$.

The inverse of evaluation is *interpolation*. e.g. using interpolation, one can derive the coefficient representation of a polynomial from a point-value representation. Any set of $m + 1$ point-value pairs (x_i, y_i) such that all x_i values are distinct uniquely defines a polynomial.

Two polynomials described in point-value representation using the same evaluation points can be multiplied by point-wise multiplication.

Though, as the product $C(x)$ of two m -degree polynomials is degree of $2m$, we need to expand the point-value representation of polynomials $p(x)$ and $q(x)$ to $2m + 1$ points in order to be able to interpolate $C(x)$ from the point-wise multiplication of the $2m + 1$ points of $p(x)$ and $q(x)$.

If $p(x)$ and $q(x)$ above are each evaluated at the points $x = -3, -2, -1, 0, 1, 2, 3$, their point-value representations are

$$p = \{(-3, -60), (-2, 9), (-1, 20), (0, 9), (1, 12), (2, 65), (3, 204)\}$$

$$q = \{(-3, 37), (-2, 3), (-1, -7), (0, -5), (1, -3), (2, -13), (3, -47)\}.$$

Given the value of the two polynomials at 7 points one can apply the inverse Vandermonde matrix $\text{inv}(\text{vander}(-3 : 3))$ and obtain the coefficients of the product polynomial $r(x) = p(x)q(x)$ in this way. It is easy to compute the values of product polynomial at the same seven points.

Note however, that in a numerical procedure this computation of the coefficients from the values may be unstable, in the sense that minor errors in the data (the numerical evaluation of the two factor polynomials and subsequent pointwise multiplication) may result in significant deviations between the computed and the true coefficients, if the Vandermonde matrix is not well conditioned (see Chapter 2 for a discussion of condition numbers). As we will see the best choice where this problem of stability is not occurring is to choose unit roots of order n .

Another important fact about Polynomials is the following theorem which states that every polynomial can be written as a product of linear factors.

Theorem 3. [14] *Every polynomial is a product of linear factors, i.e.*

$$p(x) = \prod_{k=1}^n (x - x_i)$$

for a uniquely determined family of complex points $(x_i)_{i=1}^n$ (counted with multiplicities).

1.4 Roots of Unity

Definition 1.4.1. A complex number z is the n -th root of unity if $z^n = 1$.

There are n complex n -th roots of unity given by $e^{2\pi i k/n}$, for $k = 0, \dots, n-1$, where $e^{iu} = \cos(u) + i \sin(u)$ ² and $i = \sqrt{-1}$.

Complex numbers $z = a + ib$ can be represented using their *modulus* $|z| = \sqrt{a^2 + b^2}$ and their *argument*, defined as $\arg z = \arctan \frac{b}{a}$, where the arctangent function is defined so that it takes values in $(-\pi, \pi]$, s.t.

$$z = |z|e^{i \arg z} = |z|(\cos(\arg z) + i \sin(\arg z)).$$

Recall that complex numbers, $z_n = |z|^n e^{in \arg z}$; thus, if we take the primitive n -th root of unity, i.e., $z_n = e^{\frac{2\pi}{n}i}$, since $|z_n| = 1$, we have $|z_n^m| = |z_n|^m$, for all m . Note that $z_n^k = e^{\frac{2\pi k}{n}i}$; thus, all powers of z_n belong to the unit circle and are equally spaced, having arguments which are integer multiples of $\frac{2\pi}{n}$.

In addition to being equally spaced, the roots of unity satisfy the following *cancellation property* $(z_{dn})^{dk} = z_n^k$. Consequently, taking the primitive root of unity of order d times n to the power d times k , is the same as taking the root of unity of order n to the power k . This is demonstrated by the following simple calculation

$$(z_{dn})^{dk} = (e^{\frac{2\pi}{dn}i})^{dk} = (e^{\frac{2\pi}{n}i})^k = (z_n)^k.$$

This has the following consequence:

Lemma 1.4.2 (Halving lemma). *If $n > 0$ is an even number, then the squares of the n -th root of unity are exactly the $\frac{n}{2}$ complex roots of unity of order $\frac{n}{2}$.*

Proof. Using the cancellation property we have

$$(z_n^k)^2 = (z_{2\frac{n}{2}})^{2k} = z_{\frac{n}{2}}^k.$$

□

²Euler's Formula

Thus, the total number of squares of roots of unity of order n is $\frac{n}{2}$. This is in fact very important for the FFT algorithm.

Lemma 1.4.3 (Summation lemma). *For any integer $n > 1$ and non-zero integer k not divisible by n ,*

$$\sum_{j=0}^{n-1} (z_n^k)^j = 0.$$

Proof. The closed form of summation applies to complex values as well as to reals and therefore we have

$$\sum_{j=0}^{n-1} (z_n^k)^j = \frac{(z_n^k)^n - 1}{z_n^k - 1} = \frac{(z_n^n)^k - 1}{z_n^k - 1} = \frac{(1)^k - 1}{z_n^k - 1} = 0.$$

Since k is not divisible by n , the denominator is never 0 ($z_n^k = 1$ happens only when k is divisible by n). □

2 Discrete Fourier Transform

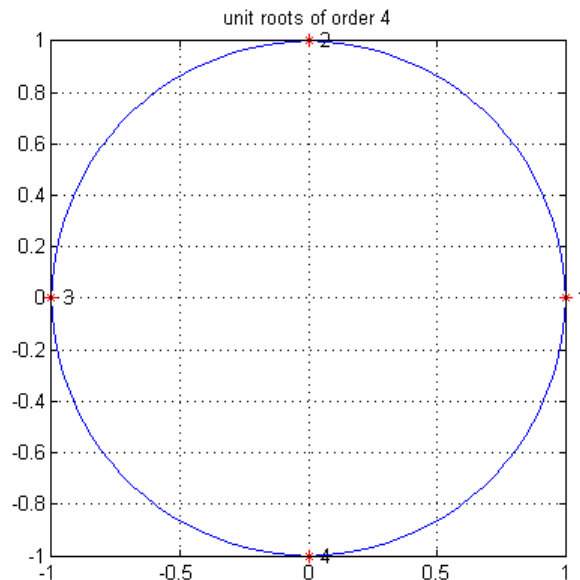
2.1 Motivation

We want to evaluate the polynomial $p(x) = \sum_{j=0}^{n-1} a_j x^j$ of order-bound n at n different points. We use the complex n -th roots of unity $z_n^0, z_n^1, z_n^2, \dots, z_n^{n-1}$ as our evaluation points.

We assume that $p(x)$ is given in coefficient form: $a = (a_0, a_1, \dots, a_{n-1})$.

Definition 2.1.1. Given $k = 0, 1, \dots, n-1$ and $y_k = p(z_n^k) = \sum_{j=0}^{n-1} a_j z_n^{kj}$, the vector $y = (y_0, y_1, \dots, y_{n-1})$ is called the **Discrete Fourier Transform (DFT)** of the coefficient vector $a = (a_0, a_1, \dots, a_{n-1})$.

Example 2.1.2. We have discrete Fourier series $f(x) = \sum_{k=0}^{N-1} a_k e^{ikx}$ and $x = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, \frac{(N-1)\pi}{N}$. For $N = 4$ the unit roots of order 4 looks like this:



2 Discrete Fourier Transform

$N - 1 = 3$, therefore we have $x_4 = x_0 = 0, x_1 = \frac{2\pi}{N}, x_2 = \frac{4\pi}{N}, x_3 = \frac{6\pi}{N}$.

Let $f(x) = y$ and $w = e^{\frac{2\pi i}{N}}$, then

$$\begin{aligned} y_0 &= f(0) = a_0 + a_1 + a_2 + a_3 \\ y_1 &= f\left(\frac{2\pi}{N}\right) = a_0 + wa_1 + w^2a_2 + w^3a_3 \\ y_2 &= f\left(\frac{4\pi}{N}\right) = a_0 + w^2a_1 + w^4a_2 + w^6a_3 \\ y_3 &= f\left(\frac{6\pi}{N}\right) = a_0 + w^3a_1 + w^6a_2 + w^9a_3 \end{aligned}$$

This is equal to:

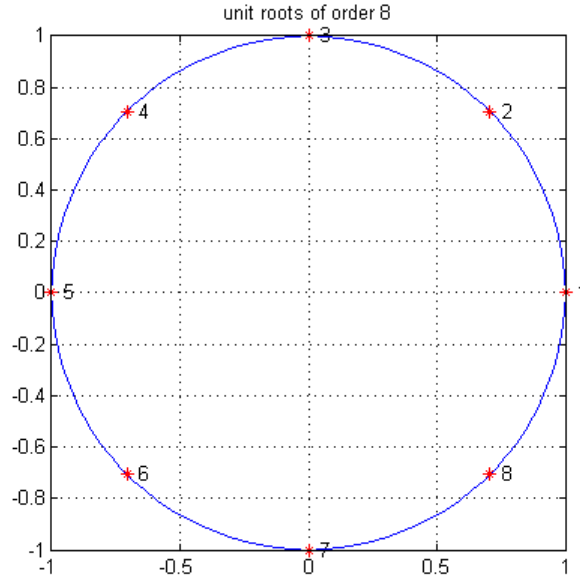
$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{pmatrix}}_{\text{FourierMatrix}} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Definition 2.1.3. [2](p.13) The evaluation of $p(x)$ can also be written as matrix-vector multiplication:

$$\begin{pmatrix} p(1) \\ p(z) \\ p(z^2) \\ \vdots \\ p(z^{n-1}) \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & z & z^2 & \dots & z^{n-1} \\ 1 & z^2 & z^4 & \dots & z^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z^{n-1} & z^{2(n-1)} & \dots & z^{(n-1)(n-1)} \end{pmatrix}$$

This linear mapping is called *discrete Fourier transform of order n* ; the corresponding matrix $\text{DFT}_n := (z^{ij})_{i,j < n}$ is called *the DFT matrix*. The recursive evaluation algorithm which computes this matrix-vector product $O(n \log n)$ operations (as opposed to $O(n^2)$ for standard matrix-vector multiplication) is called the fast Fourier transform (FFT) algorithm.

Example 2.1.4. The coefficients of the DFT always are on the unit circle. We take a look at the complex plane for $N = 8$. Then $w_8 = e^{\frac{2\pi i}{8}}$



The 8×8 Fourier Matrix consists just of powers of i . Therefore

$$\begin{aligned} \sum_{j=0}^7 w^j &= 1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 \\ &= 1 + i + (-1) + (-i) + 1 + i + (-1) + (-i) = 0. \end{aligned}$$

Remark 2.1.5. What happens if $N = 1024$? Observe that $N^2 \approx 10^6$ and the workload takes a lot time. However in FFT $N \rightarrow N \log N$ and this is approximately equals to 10^4 . This means incredible time and effort saving.

Remark 2.1.6. We get our Data from physical space and put it into the frequency space. Then we try to understand what is going on and consequently have to go back to physical space. At this point we encounter the question: How do we get F^{-1} ?

Note that $F_{kj}^{-1} = (\bar{w})^{jk}$, where $\bar{w} = e^{-\frac{2\pi i}{N}}$. In order to illustrate things, we take a look at the product of a 4×4 Fourier Matrix and its inverse:

$$FF^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & 1 & -1 & -1 \\ 1 & i & -1 & i \end{pmatrix} = N \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which implies that $\frac{1}{N}FF^{-1} = I$ and $F' = \frac{\bar{F}}{N}$.

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The columns of the Fourier matrix are pairwise orthogonal and all have the same length, namely $\sqrt{n}$. As we will see this is important for the stability of the forward and inverse Fourier transform. In particular it follows that the Fourier matrix is non-singular.

Definition 2.1.7. Given the N complex numbers, $\{w_j\}_{j=0}^{N-1}$, their N -point DFT is denoted by $\{W_k\}$ where W_k is defined by

$$W_k = \sum_{j=0}^{N-1} h_j e^{-i2\pi jk/N}.$$

Remark 2.1.8. Although the N -point DFT is defined for all integers it is a periodic sequence having period n , hence it is enough to know $W_0, \dots, W_{k-1}$ in order to completely describe this infinite sequence.

Definition 2.1.9. A is unitary if and only if $A' * A = I$ (or $A * A' = I$, or resp. both).

Proposition 2.1.10. *A linear mapping from C^n to C^n is unitary if and only if it preserves the length.*

Proof. Let A be unitary and $x \in V$. Then

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, A * Ax \rangle = \langle x, Ix \rangle = \langle x, x \rangle = \|x\|^2.$$

For the other direction, suppose $\|Ax\| = \|x\|$ for all $x \in V$. Then for all $x, y \in V$, we obtain

$$\|A(x-y)\|^2 = \|Ax - Ay\|^2 = \|Ax\|^2 - 2\langle Ax, Ay \rangle + \|Ay\|^2 = \|x\|^2 - 2\langle Ax, Ay \rangle + \|y\|^2$$

and

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2.$$

Equating $\|A(x-y)\|^2$ and $\|x-y\|^2$ gives $\langle Ax, Ay \rangle = \langle x, y \rangle$. Hence, for all x, y

$$\langle x, (A * A - I)y \rangle = \langle x, A * Ay \rangle - \langle x, y \rangle = \langle Ax, Ay \rangle - \langle x, y \rangle = 0.$$

Thus $A' * A = I$. □

If A is a scalar multiple of a unitary matrix (i.e. the columns are a fixed multiple of an orthonormal system, say γ , then of course the action of A on vectors is just a multiple γId and the operator norm is γ . On the inverse matrix is $(1/\gamma)Id$ and hence has operator norm $1/\gamma$. The condition number is thus the product $\gamma/\gamma = 1$

Proposition 2.1.11. *F is a matrix and for $\alpha > 0$, αF is unitary $\Rightarrow$ the condition number of F $\text{cond}(F) = \|F\| \|F^{-1}\| = 1$*

Proof. αF is unitary, then $\alpha F(\alpha F)' = \alpha^2 F F' = I$ for some $\alpha > 0$

Using Proposition 2.1.10 we can write

$$\|\alpha F(x)\| = \|x\| \Rightarrow \|F(x)\| = \frac{1}{\alpha} \|x\| \Rightarrow \|F\| = \frac{1}{\alpha}$$

$$\|F^{-1}(y)\| = \alpha^2 \|F(y)\| = \alpha^2 \frac{1}{\alpha} \|y\| = \alpha \|y\| \Rightarrow \|F^{-1}\| = \alpha$$

$$\Rightarrow \text{cond}(F) = 1$$

□

Corollary 2.1.12. *Let F be a Fourier Matrix of size N . Then $F' = \text{conj}(F)$. Because $F = F^t$*

Remark 2.1.13. $\|\text{fft}(x)\|_2 = \sqrt{N} \|x\|_2$

Lemma 2.1.14. *Assume that a polynomial $p(x)$ can be expressed as another polynomial (of half degree) in z^2 (i.e., $p(x) = q(x^2)$ or $p(x) = r(x^s)$ for some s in N), then its Fourier transform is a 2-periodic (respectively s -periodic) function on the unit roots of order N (if s is a divisor of N).*

Proof. Similar to halving lemma. □

This brings us to the following conclusion:

Remark 2.1.15. A function has a periodic FFT (with $\frac{N}{s}$ periods) if and only if the coefficients are concentrated on the positions $1, s+1, 2s+1, \dots$ (which is just the sequence of MATLAB coordinates or counting the unit roots of order N). The

general description is the following: Assume that a sequence of coefficients is concentrated on a subgroup, then its Fourier transform is periodic with respect to another group.

2.2 Basic Properties of the DFT

In this section we will mention some basic properties of the DFT. These properties are *linearity*, *periodicity*, *time shifting*, *conjugation*, and *inversion*. The inversion property allows us to define the *inverse* DFT and remove the asymmetry between the original sequence (of length N) and the transformed sequence (of infinite length).

Theorem 4. [12] (p.37-39) Suppose that the sequence $\{w_j\}_{j=0}^{N-1}$ has the N -point DFT $\{W_j\}$ and the sequence $\{g_j\}_{j=0}^{N-1}$ has N -point DFT $\{G_k\}$, then the following properties hold:

(a) *Linearity:* For all complex constants a and b , the sequence $\{aw_j + bg_j\}_{j=0}^{N-1}$ has N -point DFT $\{aW_k + bG_k\}$.

(b) *Periodicity:* For all integers k we have $W_{k+N} = W_k$.

(c) *Inversion:* For $j = 0, 1, \dots, N-1$, $w_j = \frac{1}{N} \sum_{k=0}^{N-1} W_k e^{-i2\pi jk/N}$.

Proof. To prove (b) and (c) we put $w = e^{-\frac{i2\pi}{N}}$ and we use the property $w^N = 1$. The DFT $\{H_k\}$ is defined by

$$H_k = \sum_{j=0}^{N-1} h_j W^{jk},$$

hence

$$\begin{aligned} H_{k+N} &= \sum_{j=0}^{N-1} h_j W^{j(k+N)} = \sum_{j=0}^{N-1} h_j W^{jk} (W^N)^j \\ &= \sum_{j=0}^{N-1} h_j W^{jk} \end{aligned}$$

which proves (b). To prove (c) we note that

$$W^{-1} = e^{\frac{i2\pi}{N}}$$

and then changing both sides of this last equation to the power jk , we get

$$W^{-jk} = e^{\frac{i2\pi jk}{N}}.$$

Then it follows that

$$\begin{aligned} \frac{1}{N} \sum_{k=0}^{N-1} H_k e^{\frac{i2\pi jk}{N}} &= \frac{1}{N} \sum_{k=0}^{N-1} H_k W^{-jk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} h_m W^{mk} \right] W^{-jk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} h_m W^{mk} W^{-jk} \right]. \end{aligned}$$

Now, since $W^{mk} W^{-jk} = W^{(m-j)k}$, by changing the order of sums we obtain

$$\frac{1}{N} \sum_{k=0}^{N-1} H_k e^{\frac{i2\pi jk}{N}} = \frac{1}{N} \sum_{m=0}^{N-1} h_m \left[\sum_{k=0}^{N-1} W^{(m-j)k} \right] (*).$$

For *fixed* j , if $m \neq j$, putting r equal to W^{m-j} gives

$$\begin{aligned} \sum_{k=0}^{N-1} W^{(m-j)k} &= \frac{1 - (W^{m-j})^N}{1 - W^{m-j}} = \frac{0}{1 - W^{m-j}} \\ &= 0, \end{aligned}$$

since $W^{m-j} \neq 1$. If, however, $m = j$, then $W^{(m-j)k} = W^0 = 1$ and $(*)$ becomes

$$\frac{1}{N} \sum_{k=0}^{N-1} H_k e^{\frac{i2\pi jk}{N}} = \frac{1}{N} h_j \sum_{k=0}^{N-1} 1 = h_j$$

and (c) is proved. □

Remark 2.2.1. The most important consequence of the inversion property of DFT is that no two distinct sequences can have the same DFT.

Now we will work on an example which illustrates the complete operation of the DFT:

Example 2.2.2. [8](p.333-335) We define a discrete pulse as following:

$$f_k = \begin{cases} k & (0 \leq k \leq 3) \\ 0 & (otherwise) \end{cases}$$

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In this example our purposes are:

- (a) To transform the given function f_k using the DFT analysis equation, in this way one can produce the DFT line spectrum¹ F_n .
- (b) To invert the line spectrum received in (a) using DFT synthesis equation, in this way one can recreate the original input vector f_k .
- (c) To show that f_k has been rewritten as a linear combination of complex exponentials.

For $N = 4$, the input data calculated from the analytical definition of the function becomes the vector

$$\mathbf{f} = (0, 1, 2, 3) \quad (2.2.1)$$

(a) Expanding $F_n = \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi nk}{N}}$ delivers the following equations:

$$n = 0 : F_0 = f_0 W^0 + f_1 W^0 + f_2 W^0 + f_3 W^0 \quad (2.2.2)$$

$$n = 1 : F_1 = f_0 W^0 + f_1 W^1 + f_2 W^2 + f_3 W^3 \quad (2.2.3)$$

$$n = 2 : F_2 = f_0 W^0 + f_1 W^2 + f_2 W^4 + f_3 W^6 \quad (2.2.4)$$

$$n = 3 : F_3 = f_0 W^0 + f_1 W^3 + f_2 W^6 + f_3 W^9. \quad (2.2.5)$$

Using the values for f from (2.2.1) and substituting the numerical values of the powers of W , these four equations then give us the DFT coefficient as follows:

$$n = 0 : F_0 = 0 + 1 + 2 + 3 = 6 \quad (2.2.6)$$

$$n = 1 : F_1 = 0 - j1 - 2 + j3 = -2 + j2 \quad (2.2.7)$$

$$n = 2 : F_2 = 0 - j1 - 2 + j3 = -2 \quad (2.2.8)$$

$$n = 3 : F_3 = 0 + j1 - 2 - j3 = -2 - j2. \quad (2.2.9)$$

We combine these results to form the DFT spectrum vector

$$\mathbf{F} = (6, -2 + j2, -2, -2 - j2). \quad (2.2.10)$$

¹The Fourier transform is often called 'the spectrum', because large values of the Fourier coefficients at certain frequency implies that the contribution of the pure frequencies in this part of the "musical spectrum" is relevant.

(b) Expanding $f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{\frac{j2\pi nk}{N}}$ gives us the four synthesis equations:

$$k = 0 : f_0 = \frac{1}{4} [F_0 W^0 + F_1 W^0 + F_2 W^0 + F_3 W^0] \quad (2.2.11)$$

$$k = 0 : f_0 = \frac{1}{4} [F_0 W^0 + F_1 W^{-1} + F_2 W^{-2} + F_3 W^{-3}] \quad (2.2.12)$$

$$k = 0 : f_0 = \frac{1}{4} [F_0 W^0 + F_1 W^{-2} + F_2 W^{-4} + F_3 W^{-6}] \quad (2.2.13)$$

$$k = 0 : f_0 = \frac{1}{4} [F_0 W^0 + F_1 W^{-3} + F_2 W^{-6} + F_3 W^{-9}]. \quad (2.2.14)$$

To verify that these four equations in fact give us back the original function f_k , we now substitute the numerical values for the powers of W and use the values for F_n appearing in (2.2.10), obtaining

$$f_0 = \frac{1}{4} [F_0 + F_1 + F_2 + F_3] \quad (2.2.15)$$

$$= \frac{1}{4} [6 + (-2 + j2) + (-2) + (-2 - j2)] = 0 \quad (2.2.16)$$

$$f_1 = \frac{1}{4} [F_0 + jF_1 - F_2 - jF_3] \quad (2.2.17)$$

$$= \frac{1}{4} [6 + j(-2 + j2) - (-2) - j(-2 - j2)] = 1 \quad (2.2.18)$$

$$f_2 = \frac{1}{4} [F_0 - F_1 + F_2 - F_3] \quad (2.2.19)$$

$$= \frac{1}{4} [6 - (-2 + j2) + (-2) - (-2 - j2)] = 2 \quad (2.2.20)$$

$$f_3 = \frac{1}{4} [F_0 - jF_1 - F_2 + jF_3] \quad (2.2.21)$$

$$= \frac{1}{4} [6 - j(-2 + j2) - (-2) + j(-2 - j2)] = 3. \quad (2.2.22)$$

These results can then be assembled to give us the output vector

$$\mathbf{f} = (0, 1, 2, 3), \quad (2.2.23)$$

which is seen to be the same as $\mathbf{f}$ in (2.2.1) that we started out with.

(c) To show that we have been able to rewrite f as a linear combination of complex exponentials, we rewrite (2.2.11) through (2.2.14) using the values of $\mathbf{F}$ from part (a) as follows:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \frac{6}{4} \begin{bmatrix} W^0 \\ W^0 \\ W^0 \\ W^0 \end{bmatrix} + \frac{-2+j2}{4} \begin{bmatrix} W^0 \\ W^{-1} \\ W^{-2} \\ W^{-3} \end{bmatrix} - \frac{2}{4} \begin{bmatrix} W^0 \\ W^{-2} \\ W^{-4} \\ W^{-6} \end{bmatrix} + \frac{-2-j2}{4} \begin{bmatrix} W^0 \\ W^{-3} \\ W^{-6} \\ W^{-9} \end{bmatrix}. \quad (2.2.24)$$

On the LHS(left hand side) we have the vector $\mathbf{f}$, while on the RHS(right hand side) we see four vectors of complex exponentials in a linear combination, with the values of $\mathbf{F}$ as the constants of that combination. This is the discrete counterpart to the two synthesis statements, namely

$$f_p(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad (2.2.25)$$

and

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega. \quad (2.2.26)$$

Three statements in the beginning of this examples are actually the same, in the sense that in each case the given function has been reconstructed as a *linear combination of complex exponentials*.

The vectors on the RHS of the linear combination build an orthogonal set in the sense of linear algebra. The inner product of these vector with each other is zero. The inner product of each with itself is equal to N . Each of these vectors represents the sampling of a *complete complex exponential*. They are the discrete counterparts of the countably infinite set of quantities

$$\dots e^{j0\omega_0 t}, e^{j1\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t}, \dots$$

and of the uncountably infinite set of quantities

$$S = \{e^{j\omega t} \mid \omega \in \mathbb{R}\}$$

which formed the 'bases' for the expansion in Fourier Transforms.

²The continuous Fourier transform will be introduced in Chapter 3.

Time Shifting Property: [5] Let n_0 be any integer. If $x[n]$ is a discrete-time signal of period N , then so is $y[n] = x[n - n_0]$. The k^{th} Fourier coefficient of $y[n]$ is

$$\hat{y}[k] = \frac{1}{N} \sum_{n=0}^{N-1} y[n] e^{-2\pi i \frac{kn}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} x[n - n_0] e^{-2\pi i \frac{kn}{N}}.$$

Here we substitute $m = n - n_0$ in the sum:

$$\hat{y}[k] = \frac{1}{N} \sum_{m=-n_0}^{N-1-n_0} x[m] e^{-2\pi i \frac{k(m+n_0)}{N}} = e^{-2\pi i \frac{kn_0}{N}} \left\{ \frac{1}{N} \sum_{m=-n_0}^{N-1-n_0} x[m] e^{-2\pi i \frac{km}{N}} \right\}.$$

The summand is periodic of period N . Substituting m by $m + N$ has no effect on the summand. So all domains of summation which consist of a single full period give the same sum. Therefore we can replace the sum $\sum_{m=-n_0}^{N-1-n_0}$ by the sum $\sum_{m=0}^{N-1}$ and the sum in parenthesis is exactly $\hat{x}[-k]$. Hence $\hat{y}[k] = e^{-2\pi i \frac{kn_0}{N}} \hat{x}[k]$.

Remark 2.2.3. If $x[n]$ is a discrete-time signal of period N , then so is $y[n] = \overline{x[n]}$.

Conjugation property: [5] The k^{th} Fourier coefficient of $y[n]$ is

$$\hat{y}[k] = \frac{1}{N} \sum_{n=0}^{N-1} y[n] e^{-2\pi i \frac{kn}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} \overline{x[n]} e^{-2\pi i \frac{kn}{N}} = \overline{\frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-2\pi i \frac{(-k)n}{N}}} = \overline{\hat{x}[-k]}.$$

This tells us that the k^{th} Fourier coefficient of the periodic discrete-time signal $\overline{x[n]}$ is $\overline{\hat{x}[-k]}$. In particular, $x[n]$ is real valued if and only if $\overline{x[n]} = x[n]$ for all n , which is true if and only if the Fourier coefficients of $x[n]$ and $y[n] = \overline{x[n]}$ are the same. That is,

$$x[n] \text{ is real for all } n \Leftrightarrow \overline{\hat{x}[-k]} = \hat{x}[-k] \text{ for all } k$$

Waveform decomposition: Any sequence $\{x_n\}$ can always be decomposed into the sum of two sequences, where one is even and the other odd. This is obtained by defining

$$\{x_n\}^{\text{even}} = \frac{\{x_n\} + \{x_{-n}\}}{2} \quad \text{and} \quad \{x_n\}^{\text{odd}} = \frac{\{x_n\} - \{x_{-n}\}}{2}$$

and noting that

$$\{x_n\} = \{x_n\}^{\text{even}} + \{x_n\}^{\text{odd}}.$$

Because of inversion and linearity property, we can write the *waveform decomposition*

as follows

$$DFT[\{x_n\}^{even}]_k = \frac{X_k + X_{-k}}{2} \quad \text{and} \quad DFT[\{x_n\}^{odd}]_k = \frac{X_k - X_{-k}}{2}.$$

The consistency in these relations can be seen via

$$DFT[\{x_n\}]_k = DFT[\{x_n\}^{even} + \{x_n\}^{odd}]_k = F_k.$$

Parseval's Theorem This theorem implies that the sums of the squared magnitudes of the input and the DFT sequences are related by the constant N , the number of samples. That is the signal power can also be computed from the DFT coefficient of the sequence.

Theorem 5. [10](p.90-91) Let $x_n \Leftrightarrow X_k$, $n, k = 0, 1, \dots, N-1$. Then

$$\sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |x_k|^2.$$

Since the squared magnitude can be computed by multiplying a complex number by its conjugate, we can write the left summation as

$$\sum_{n=0}^{N-1} |x_n|^2 = \sum_{n=0}^{N-1} x_n x_n^*.$$

Substituting the corresponding IDFT expressions for x_n and x_n^* , we get

$$\begin{aligned} &= \sum_{n=0}^{N-1} \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} X_k X_m^* W_N^{-n(k-m)} \\ &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} X_k X_m^* \sum_{n=0}^{N-1} W_N^{-n(k-m)} \end{aligned}$$

If $k = m$, the expression becomes

$$\frac{1}{N} \sum_{k=0}^{N-1} X_k X_k^* = \frac{1}{N} \sum_{k=0}^{N-1} |x_k|^2$$

Otherwise, it evaluates to zero due to the orthogonal property.

Example 2.2.4. Consider the DFT pair

$$\{2, 1, 4, 3\} \Leftrightarrow \{10, -2 + j2, 2, -2 - j2\}$$

The sum of the squared magnitude of the data sequence is 30 and that of the DFT coefficients divided by 4 is also 30.

Remark 2.2.5. The generalized form of this theorem applies for two different signals x_n and y_n are given as following:

$$\sum_{n=0}^{N-1} x_n y_n^* = \frac{1}{N} \sum_{k=0}^{N-1} X_k Y_k^*$$

2.3 Discrete Convolution

The linear convolution of two finite series allows us to approximate continuous convolution using sampled function values and the fact that linear convolution is algebraically equivalent to the multiplication of two polynomials. However, for applications in signal processing and system analysis, it is crucial to see the definition as the discrete counterpart of the continuous convolution. In the presentation of time- and frequency-domain convolutions we follow the book of Chu.

Definition 2.3.1. [1](p.282-285) Let $\{A_n\}$ and $\{B_n\}$ be two sequences of the period N . The *periodic convolution of two sequences of period N* is a finite sequence of length N given by the following equation:

$$U_k = \sum_{l=0}^{N-1} A_l B_{k-l}, \quad \text{for } k = 0, 1, \dots, N-1,$$

where $A_l = A_{l+N}$ and $B_{k-l} = B_{k-l+N}$ are satisfied because of periodicity, which ensures that $U_k = U_{k+N}$. Thus, continuing the convolution process beyond one period would simply result in a periodic extension of the first N results. Cyclic convolution of *two sequences of period N* is defined by the following equation:

$$A_n *_N B_n = \sum_{l=0}^{N-1} A_l B_{k-l} \quad \text{for } k = 0, 1, \dots, N-1.$$

Theorem 6. (Time-Domain Cyclic Convolution Theorem)[1] Let the cyclic convolution of sequences $\{x_l\}$ and $\{g_l\}$ of period N be denoted by $\{x_l\} \odot \{g_l\}$. If the discrete Fourier transforms of the two sequences are given by $\{X_r\} = \text{DFT}[\{x_l\}]$ and $\{G_r\} = \text{DFT}[\{g_l\}]$, then

$$\{x_l\} \odot \{g_l\} = N \text{IDFT}[\{X_r G_r\}] \quad (2.3.1)$$

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Proof. By definition, we have $\{u_k\} = \{x_l\} \odot \{g_l\}$ with its elements given by

$$u_k = \sum_{l=0}^{N-1} x_l g_{k-l}, \quad \text{for } k = 0, 1, \dots, N-1, \quad (2.3.2)$$

where $g_{k-l} = g_{k-l+N}$, because of periodicity. Assuming that the DFT coefficients $\{X_r\}$ and $\{G_r\}$ are computed by formula

$$X_r = \frac{1}{N} \sum_{l=0}^{N-1} x_l \omega_N^{-rl}, \quad \text{for } r = 0, 1, \dots, N-1,$$

we use the corresponding IDFT formula

$$x_l = \sum_{r=0}^{N-1} X_r \omega_N^{lr}, \quad \omega_N = e^{\frac{j2\pi}{N}} \quad \text{for } l = 0, 1, \dots, N-1,$$

to express

$$x_l = \sum_{r=0}^{N-1} X_r \omega_N^{lr}, \quad g_{k-l} = \sum_{r=0}^{N-1} G_r \omega_N^{(k-l)r}$$

and we rewrite u_k as

$$\begin{aligned} u_k &= \sum_{l=0}^{N-1} \overbrace{\left[\sum_{m=0}^{N-1} X_m \omega_N^{lm} \right]}^{x_l} \times \overbrace{\left[\sum_{r=0}^{N-1} G_r \omega_N^{(k-l)r} \right]}^{g_{k-l}} \\ &= \sum_{l=0}^{N-1} \left[\sum_{r=0}^{N-1} G_r \omega_N^{kr} \omega_N^{-lr} \right] \times \left[\sum_{m=0}^{N-1} X_m \omega_N^{lm} \right] \\ &= \sum_{l=0}^{N-1} \sum_{r=0}^{N-1} \left[G_r \omega_N^{kr} \sum_{m=0}^{N-1} X_m \omega_N^{l(m-r)} \right] \\ &= \sum_{r=0}^{N-1} G_r \omega_N^{kr} \left[\sum_{m=0}^{N-1} \sum_{l=0}^{N-1} X_m \omega_N^{l(m-r)} \right] \\ &= \sum_{r=0}^{N-1} G_r \omega_N^{kr} \left[\sum_{m=0}^{N-1} X_m \sum_{l=0}^{N-1} \omega_N^{l(m-r)} \right] \\ &= N \sum_{r=0}^{N-1} \{X_r G_r\} \omega_N^{kr}. \end{aligned}$$

In the last step we used the orthogonality property. □

Theorem 7. (*Frequency-domain cyclic convolution theorem*) [1] Let $\{X_r\}$ and $\{G_r\}$ denote two DFT sample sequences of period N . If $\{x_l\} = \text{IDFT}[\{X_r\}]$ and $\{g_l\} = \text{IDFT}[\{G_r\}]$, then

$$\{X_r\} \odot \{G_r\} = \text{DFT}[\{x_l g_l\}].$$

Proof. By periodic convolution we get $\{U_k\} = \{X_r\} \odot \{G_r\}$ with its elements given by

$$U_k = \sum_{r=0}^{N-1} X_r G_{k-r}, \quad \text{for } k = 0, 1, \dots, N-1,$$

where $G_{k-r} = G_{k-r+N}$ because of the periodicity property. Using the DFT formula we express

$$X_r = \frac{1}{N} \sum_{l=0}^{N-1} x_l \omega_N^{-rl}, \quad G_{k-r} = \frac{1}{N} \sum_{l=0}^{N-1} g_l \omega_N^{-(k-r)l}$$

and we rewrite U_k as

$$\begin{aligned} u_k &= \sum_{r=0}^{N-1} \overbrace{\left[\sum_{m=0}^{N-1} x_m \omega_N^{-rm} \right]}^{X_r} \times \overbrace{\left[\sum_{l=0}^{N-1} g_l \omega_N^{-(k-r)l} \right]}^{G_{k-r}} \\ &= \frac{1}{N^2} \sum_{r=0}^{N-1} \left[\sum_{l=0}^{N-1} g_l \omega_N^{-kl} \omega_N^{rl} \right] \left[\sum_{m=0}^{N-1} x_m \omega_N^{-rm} \right] \\ &= \frac{1}{N^2} \sum_{l=0}^{N-1} g_l \omega_N^{-kl} \left[\sum_{m=0}^{N-1} x_m \sum_{r=0}^{N-1} \omega_N^{-r(m-l)} \right] \\ &= \frac{1}{N} \sum_{l=0}^{N-1} \{x_l g_l\} \omega_N^{-kl}. \end{aligned}$$

Thus, we have proved

$$\{U_k\} = \{X_r\} \odot \{G_r\} = \text{DFT}[\{x_l g_l\}].$$

□

These two discrete convolution theorems show that the cyclic convolution of two sequences of length N (in either *time* or *frequency* domain) can be computed via DFT and IDFT.

3 Fast Fourier Transform

3.1 Historical Background

FFT algorithm was actually invented around 1805 by Carl Friedrich Gauss. Gauss used this Method to interpolate the trajectories of the asteroids Pallas and Juno. However, his work was not widely recognized. Besides, He didn't analyse the asymptotic computational time. Throughout the 19th and early 20th centuries many version with restrictions were also rediscovered. However, the FFT became popular in 1965, after James Cooley of IBM and John Tukey of Princeton published a paper reinventing the algorithm and describing how to perform it conveniently on a computer.

“Although the DFT was known for many decades, to get benefit from it was severely limited because of the computational workload. The calculation of the DFT of an input sequence of an N length sequence $\{f_n\}$ requires N complex multiplications to compute each on the N values, F_m , for a total of N^2 multiplications. Early digital computers had neither fixed-point nor floating point hardware multipliers, and multiplication was performed by binary shift-and-add software algorithms. Therefore Multiplication was an ”expensive” and time consuming operation. These problems made the DFT impractical for common usage. After the development of the Fast Fourier Transform ,digital signal processing was revolutionized by allowing practical fast frequency domain implementation of processing algorithms.” [3]

3.2 Fourier Series

Definition 3.2.1. A function on $\mathbb{R}$ is said to be *periodic with period N* (N is a nonzero constant) if we have

$$f(x + N) = f(x) \quad \forall x.$$

Theorem 8. (*Complex Fourier Series for periodic functions*) Let $f_p(n)$ be peri-

3 Fast Fourier Transform

odic with period T_0 . Then it can also be represented by infinite series of complex exponentials

$$f_p(t) = \sum_{N=-\infty}^{\infty} F(n)e^{jn\omega_0 t},$$

where the coefficients $F(n)$ can be found by using $f_p(t)$ as follows

$$F(n) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_p(t)e^{-jn\omega_0 t} \forall x.$$

Example 3.2.2. [8](p.16-17) The waveform $f_p(t)$ is described as:

$$f_p(t) = \begin{cases} 0 & (-2 < t < -1) \\ 1 & (-1 < t < 1) \\ 0 & (1 < t < 2) \end{cases} \quad f_p(t+4) = f_p(t).$$

In order to rewrite $f_p(t)$ as an infinite series of complex exponentials we must find the Fourier coefficients $F(n)$. We use the analysis equation, and for that purpose we note that $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$. Then

$$\begin{aligned} F(n) &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_p(t)e^{-jn\omega_0 t} dt = \frac{1}{4} \\ &= \frac{1}{4} \frac{e^{-jn\pi/2} - e^{-jn\pi/2}}{-jn\pi/2} = \frac{1}{2} \frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2j(n\pi/2)} \\ &= \frac{1}{2} \frac{\sin(n\pi/2)}{n\pi/2}. \end{aligned}$$

For this particular waveform the general expression of the Fourier coefficients is

$$F_n = \frac{1}{2} \frac{\sin(n\pi/2)}{n\pi/2}, \quad n \in \mathbb{Z}.$$

We use some of the coefficients for $F(n)$, Then we have

$$\begin{aligned} f_p(t) &= \sum_{N=-\infty}^{\infty} F(n)e^{jn\pi t/2} \\ &= \frac{1}{\pi} \left[\dots + \frac{1}{5}e^{-j5\pi t/2} - \frac{1}{3}e^{-j3\pi t/2} + \frac{1}{1}e^{-j\pi t/2} + \frac{\pi}{2} \right] \\ &\quad + \frac{1}{1}e^{j\pi t/2} - \frac{1}{3}e^{j3\pi t/2} + \frac{1}{5}e^{j5\pi t/2} + \dots \end{aligned}$$

In this series form we can see precisely what our waveform is comprised of. The average value is $F(0) = \frac{1}{2}$, The first few nonzero harmonics¹ are

$$\begin{aligned} h(1) &= F(1)e^{j\omega_0 t} + F(-1)e^{-j\omega_0 t} \\ &= \frac{1}{\pi}e^{j\pi t/2} + \frac{1}{\pi}e^{-j\pi t/2} = \frac{2}{\pi} \cos \frac{\pi t}{2} \\ h(3) &= \frac{-1}{3\pi}e^{j3\pi t/2} + \frac{-1}{3\pi}e^{-j3\pi t/2} = \frac{-2}{3\pi} \cos\left(\frac{3\pi t}{2}\right) \end{aligned}$$

and so on. All this means that we could synthesize our waveform by combining the outputs of an array of cosine oscillators, using the amplitudes. Thus our waveform could be generated as follows:

$$f_p(t) = \frac{1}{2} + \frac{2}{\pi} \cos \frac{\pi t}{2} - \frac{2}{3\pi} \cos \frac{3\pi t}{2} + \frac{2}{5\pi} \cos \frac{5\pi t}{2} - \dots$$

3.2.1 Pointwise Convergence of Fourier Series

Definition 3.2.3. A function f on a finite interval $[a, b]$ is *piecewise continuous* on $[a, b]$ if the interval $[a, b]$ can be divided into a finite number of subintervals on each of which g is continuous. If g is piecewise continuous on every finite interval, then g is called *piecewise continuous* on the real line.

Theorem 9. [12](p.13-14) Let g be a piecewise continuous function having period P . At each point x where g has a right- and left-hand derivative, the Fourier series for g converges to $[g(x+) + g(x-)]/2$. Thus, we can write

$$\sum_{n=-\infty}^{\infty} c_n e^{i2\pi n x/P} = \frac{1}{2} [g(x+) + g(x-)].$$

If x is also a point of continuity for g , then this result simplifies to

$$\sum_{n=-\infty}^{\infty} c_n e^{i2\pi n x/P} = g(x).$$

Theorem 10. If $\sum_{n=-\infty}^{\infty} |c_n|$ converges, then the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{i2\pi n x/P}$ converges uniformly to a continuous function g with period P .

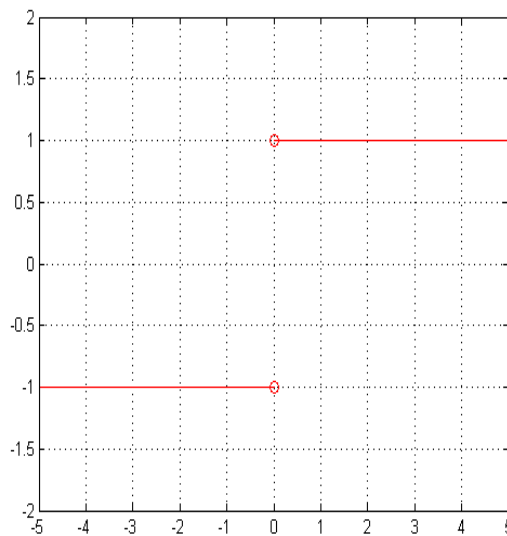
¹A harmonic of a wave is a component frequency of the signal that is an integer multiple of the fundamental frequency. i.e. f is the fundamental frequency, then $2f, 3f, 4f, \dots$ etc are the harmonic frequencies.

Definition 3.2.4. (*Discontinuity*) Let f be a function with real variable x . We define f in a neighbourhood of the point x_0 where f discontinuous is. There appears three possible cases:

1. Left- and right-sided limits exist at x_0 , are equal to $L = L^- = L^+$. If $f(x_0) \neq L$, then x_0 is called a *removable discontinuity*.
2. If the left- and right-sided limits exist and are finite, but not equal, then x_0 is called a *jump discontinuity*.
3. If one or both of the left- and right-sided limits don't exist or are infinite, then x_0 is called an *infinite discontinuity*.

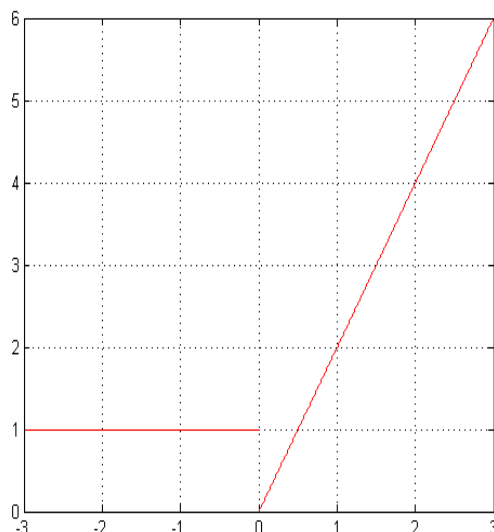
Example 3.2.5. The function $f(x)$ graphed below has a jump discontinuity at $x = 0$.

$$f(x) = \begin{cases} 1 & x > 0, \\ \text{anyvalue} & x = 0 \\ -1 & x < 0 \end{cases}$$



Example 3.2.6. [9] Let us observe where the Fourier series of the function defined and graphed below converge, at $x = -2, x = 0, x = 3, x = 5$, and $x = 6$.

$$f(x) = \begin{cases} 1 & -3 \leq x \leq 0 \\ 2x & 0 < x \leq 3 \end{cases}$$



The first two points are inside the original definition of $f(x)$, so we can just directly consider that instead of having to consider $f_{per}(x)$. The only discontinuity of $f(x)$ occurs at $x = 0$. So at $x = -2$, $f(x)$ is continuous, so the Fourier series will converge to $f(-2) = 1$.

On the other hand, at $x = 0$ we have a jump discontinuity, so the Fourier series will converge to the average of the one-sided limits.

$f(0^+) = \lim_{x \rightarrow 0^+} = 0$ while $f(0^-) = \lim_{x \rightarrow 0^-} = 1$, so the Fourier series will converge to $\frac{1}{2} [f(0^+) + f(0^-)] = \frac{1}{2}$.

For the other three points, we have to consider $f_{per}(x)$ and where it has jump discontinuities. These can only occur either at $x = x_0 + 2lm$ where $-l < x_0 < l$ is a jump discontinuity of $f(x)$ or at endpoints $x = \pm +2lm$, since the periodic extension might not "sync up" at these points, producing a jump discontinuity.

At $x = 3$, we are at one of these "boundary points", and left-sided limit is 6 while the right-sided limit is 1. Therefore the Fourier series will converge here to $\frac{7}{2}$.

$x = 5$, on the other hand, is a point of continuity for $f_{per}(x)$, and so the Fourier series will converge to $f_{per}(5) = f(-1) = 1$.

$x = 6$, though, is a jump discontinuity (corresponding to $x = 0$), and so the Fourier series will converge to $\frac{1}{2}$.

3.2.2 Even and Odd Functions

Definition 3.2.7. A function f is even if and only if $f(-t) = f(t)$ for all t , and it is odd if and only if $f(-t) = -f(t)$ for all t .

Lemma 3.2.8. *Euler's Formula for Fourier series:*

$$c_n e^{2\pi i n x / P} + c_{-n} e^{-2\pi i n x / P} = a_n \cos\left(\frac{2\pi n x}{P}\right) + b_n \sin\left(\frac{2\pi n x}{P}\right)$$

The Fourier series of $f(x)$ on the interval $-P < x < P$, is then defined as

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{P}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{P}\right)$$

Remark 3.2.9. [13] The question of pointwise convergence of the partial sums of Fourier Series is a classical one. The Dirichlet conditions are sufficient conditions for a real-valued, periodic function $f(x)$ to be equal to the sum of its Fourier series at each point where f is continuous. However, the behaviour of the Fourier series at points of discontinuity is determined as well (it is the midpoint of the values of the discontinuity).

The conditions are:

1. $f(x)$ must be absolutely integrable over a period.
2. $f(x)$ must have a finite number of extrema in any given interval, i.e. there must be a finite number of maxima and minima in the interval.
3. $f(x)$ must have a finite number of discontinuities in any given interval, however the discontinuity cannot be infinite.
4. $f(x)$ must be bounded

The following two theorems shows Fourier series of even and odd functions.

Theorem 11. [1](p.51-52) *If $f(t)$ is an even function satisfying the Dirichlet 's conditions, the coefficients in the Fourier series of $f(t)$ are given by the formulas*

$$A_k = \frac{4}{T} \int_0^{T/2} f(t) \cos \frac{2k\pi t}{T} dt, \quad k = 0, 1, 2, \dots$$

$$B_k = 0 \quad k = 1, 2, \dots$$

Proof. Using the Euler-Fourier formula, we obtain

$$\begin{aligned}
A_k &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi kt}{T} dt \\
&= \frac{2}{T} \left[\int_{-T/2}^0 f(t) \cos \frac{2\pi kt}{T} dt + \int_0^{T/2} f(t) \cos \frac{2\pi kt}{T} dt \right] \\
&= \frac{2}{T} \left[- \int_{T/2}^0 f(-s) \cos \frac{2\pi k(-s)}{T} ds + \int_0^{T/2} f(t) \cos \frac{2\pi kt}{T} dt \right] \quad (\text{let } t = -s) \\
&= \frac{2}{T} \left[\int_0^{T/2} f(s) \cos \frac{2\pi ks}{T} ds + \int_0^{T/2} f(t) \cos \frac{2\pi kt}{T} dt \right] \quad (f(-s) = f(s)) \\
&= \frac{2}{T} \int_0^{T/2} f(t) \cos \frac{2\pi kt}{T} dt. \quad (\text{let } s = t)
\end{aligned}$$

Using the Euler-Fourier formula, we obtain

$$\begin{aligned}
B_k &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2\pi kt}{T} dt \\
&= \frac{2}{T} \left[\int_{-T/2}^0 f(t) \sin \frac{2\pi kt}{T} dt + \int_0^{T/2} f(t) \sin \frac{2\pi kt}{T} dt \right] \\
&= \frac{2}{T} \left[- \int_{T/2}^0 f(-s) \sin \frac{2\pi k(-s)}{T} ds + \int_0^{T/2} f(t) \sin \frac{2\pi kt}{T} dt \right] \quad (\text{let } t = -s) \\
&= \frac{2}{T} \left[\int_0^{T/2} f(s) \sin \frac{2\pi k(-s)}{T} ds + \int_0^{T/2} f(t) \sin \frac{2\pi kt}{T} dt \right] \quad (f(-s) = f(s)) \\
&= \frac{2}{T} \left[- \int_0^{T/2} f(s) \sin \frac{2\pi ks}{T} ds + \int_0^{T/2} f(t) \sin \frac{2\pi kt}{T} dt \right] \quad (\sin(-\theta) = -\sin \theta)
\end{aligned}$$

□

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Theorem 12. [1](p.52) *If $f(t)$ is an odd function satisfying the Dirichlet 's conditions, the coefficients in the Fourier series of $f(t)$ are given by the formulas*

$$A_k = 0, \quad k = 0, 1, 2, \dots$$

$$B_k = \frac{4}{T} \int_0^{T/2} f(t) \sin \frac{2k\pi t}{T} dt, \quad k = 1, 2, \dots$$

Proof. (Similar to the proof for Theorem 13)

□

3.3 Fourier Transform

While one can consider the Fourier series expansions of periodic functions on the real line with their natural scalar product (integration over the fundamental domain of the periodic function) as an orthogonal expansion in some Hilbert space the situation is becoming quite a bit different in the setting of the real line or the Euclidean space, where one faces the problem that the so-called pure frequencies, i.e. the complex exponential functions, are not square integrable. As a compensation one has to impose (both in the definition of the Fourier transform or of the convolution) extra properties.

Definition 3.3.1. A function f for which $\|f\|_1 = \int_{-\infty}^{\infty} |f(t)|dt$ is finite, the Fourier transform of f is denoted by $\hat{f}$ and is defined by

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx$$

Theorem 13. [12](p.155-156) The Fourier transform operation $f \xrightarrow{\mathcal{F}} \hat{f}$ has the following properties:

(a)**Linearity:** For all constants a and b ,

$$af + bg \xrightarrow{\mathcal{F}} a\hat{f} + b\hat{g}$$

(b)**Scaling:** For each positive constant ϱ ,

$$f\left(\frac{x}{\varrho}\right) \xrightarrow{\mathcal{F}} \varrho\hat{f}(\varrho u) \quad \text{and} \quad \varrho f(\varrho x) \xrightarrow{\mathcal{F}} \hat{f}\left(\frac{u}{\varrho}\right)$$

(c)**Shifting:** For each real constant c ,

$$f(x - c) \xrightarrow{\mathcal{F}} \hat{f}(u)e^{-2\pi icu}$$

(d)**Modulation:** For each real constant c ,

$$f(x)e^{2\pi icx} \xrightarrow{\mathcal{F}} \hat{f}(u - c)$$

Proof. The proof of (a) is straightforward. To prove (b), we make the change of

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variables $s = \frac{x}{\varrho}$ in the following Fourier transform integral:

$$\begin{aligned} f\left(\frac{x}{\varrho}\right) \mathcal{F} \rightarrow \int_{-\infty}^{\infty} f\left(\frac{x}{\varrho}\right) e^{-i2\pi ux} dx &= \int_{-\infty}^{\infty} f(s) e^{-i2\pi u(\varrho s)} d(\varrho s) \\ &= \varrho \int_{-\infty}^{\infty} f(s) e^{-i2\pi(u\varrho)s} d(s) \end{aligned} \quad (3.3.1)$$

$$= \varrho \hat{f}(\varrho u) \quad (3.3.2)$$

Thus, $f\left(\frac{x}{\varrho}\right) \xrightarrow{\mathcal{F}} \varrho \hat{f}(\varrho u)$. Substituting $\frac{1}{\varrho}$ in place of ϱ , it follows that $f(x) \xrightarrow{\mathcal{F}} \frac{1}{\varrho} \hat{f}\left(\frac{u}{\varrho}\right)$ and (b) is verified by multiplying the resulting equation by ϱ .

To prove (c) we make change of variable $s = x - c$ in the following Fourier transform integral;

$$\begin{aligned} f(x - c) \xrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} f(x - c) e^{-i2\pi ux} dx &= \int_{-\infty}^{\infty} f(s) e^{-i2\pi u(s+c)} ds \\ &= \int_{-\infty}^{\infty} f(s) e^{-i2\pi us} ds e^{-i2\pi uc} \\ &= \hat{f}(u) e^{-i2\pi uc} \end{aligned}$$

Thus (c) holds.

To prove (d), we note that $e^{i2\pi cx} e^{-i2\pi ux} = e^{-i2\pi(u-c)x}$, hence

$$f(x) e^{i2\pi cx} \xrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} f(x) e^{-i2\pi(u-c)x} dx = \hat{f}(u - c)$$

and (d) holds. □

3.3.1 Plancherel's Formula

In the presentation of Plancherel's theorem we follow the book of Vretblad ([11], p.180-181).

We shall indicate an intuitive deduction of a formula that corresponds to the Parseval formula. If the Fourier series arising in Parseval's theorem are written in the "complex" version, we have

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt, \quad \text{where} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

A simple change of variables yields the corresponding formula on the interval $(-P, P)$:
We put

$$c_n = \frac{1}{2P} \int_{-P}^P f(t) e^{-in\pi t/P} dt$$

and thus obtain from Parseval's formula

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2P} \int_{-P}^P |f(t)|^2 dt.$$

Here we introduce the "truncated" Fourier transform

$$\hat{f}(P, \omega) = \int_{-P}^P f(t) e^{-i\omega t} dt,$$

so that $c_n = \frac{1}{2P} \hat{f}(P, n\pi/P)$ takes the form

$$\frac{1}{4P^2} \sum_{n=-\infty}^{\infty} \left| \hat{f}\left(P, \frac{n\pi}{P}\right) \right|^2 = \frac{1}{2P} \int_{-P}^P |f(t)|^2 dt$$

or

$$\int_{-P}^P |f(t)|^2 dt = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left| \hat{f}\left(P, \frac{n\pi}{P}\right) \right|^2 \frac{\pi}{P}$$

We consider the right-hand expression as a Riemannian sum, and if we let $P \rightarrow \infty$ we obtain the following identity for integrals:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{f}(\omega/2\pi) \right|^2 d\omega = \int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

where we have done a substitution in order to obtain the last equality sign. Making use of the the L^2 -norm $\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^{1/2}$ we can reformulate Parseval's equation as

$$\|\hat{f}\|_2 = \|f\|_2 \quad \forall f \in L^2. \quad (3.3.3)$$

In the literature the formula is known as the *Plancherel formula* (sometimes as the *Parseval Formula*). It tells us that the Fourier transform, properly extended to all

3 Fast Fourier Transform

of $L^2(R)$ is a *unitary mapping* on this Hilbert space.

Now we can recall the polarization identity which hold true for arbitrary complex Hilbert spaces $\mathbf{H}$. For any $x, y \in \mathbf{H}$ the scalar product $\langle x, y \rangle_{\mathbf{H}}$ can be expressed as a sum of norms:

$$\langle x, y \rangle_{\mathbf{H}} = \frac{1}{4} \sum_{k=0}^3 i^k \langle (x + i^k y), (x + i^k y) \rangle_{\mathbf{H}} = \frac{1}{4} \sum_{k=0}^3 i^k \|x + i^k y\|^2, \quad (3.3.4)$$

where i denotes to complex unit in C .

It implies that any unitary mapping also preserves scalar products, hence one has for $f, g \in L^2(R)$

$$\langle f, g \rangle_{L^2} = \langle \hat{f}, \hat{g} \rangle_{L^2}. \quad (3.3.5)$$

Example 3.3.2. With the help of Plancherel's formula certain integrals can be computed. If $f(t) = 1$ for $|t| < 1$ and $= 0$ otherwise, then $\hat{f}(\omega/2\pi) = \frac{2 \sin \omega}{\omega}$. Plancherel now gives

$$\begin{aligned} \int_{-1}^1 1 dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4 \sin^2 \omega}{\omega^2} d\omega, \\ \int_{-\infty}^{\infty} \frac{4 \sin^2 t}{t^2} dt &= \pi \end{aligned}$$

This integral is not very easy to compute using other methods.

While random variables with integer values, in particular with values in the natural number, can be modelled by Laurent series resp. even polynomials and hence the addition of independent random variables corresponding to the sum of independent variables of such a type are given by the (discrete) Cauchy-product a similar thing is true for random variable with continuous density distributions. For them the Cauchy product has to be replaced by the continuous convolution integral to be discussed next.

Definition 3.3.3. The convolution integral is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = x(t) * h(t).$$

The function $y(t)$ is said to be the *convolution* of $x(t)$ and $h(t)$.

It is well defined at least if both of the functions are (Lebesgue) integrable. Then the resulting convolution product $y(t)$ is integrable as well. If both factors are probability

distributions (i.e., they are non-negative with integral 1) then so is their convolution product.

It is really difficult to *visualize* the convolution operation. The true meaning of convolution can be developed by graphical analysis

Theorem 14. [6] *Convolution obeys the same algebraic laws as ordinary multiplication. It is a bilinear, commutative and associative relation, i.e.,*

- (i) $f * (ag + bh) = a(f * g) + b(f * h)$ for any constants a, b ,
- (ii) $f * g = g * f$,
- (iii) $f * (g * h) = (f * g) * h$.

Proof. (i) is obvious since integration is a linear operation. For (ii) we make the change of variable $z = x - y$. Then $f * g(x) = \int f(x - y)g(y) dy = \int f(z)g(x - z) dz = g * f(x)$. For (iii), use (ii) and interchange the order of integration:

$$\begin{aligned} (f * g) * h(x) &= \int f * g(x - y)h(y) dy = \iint f(z)g(x - y - z)h(y) dz dy \\ &= \iint f(z)g(x - z - y)h(y) dy dz = \int f(z)g * h(x - z) dz \\ &= f * (g * h)(x) \end{aligned}$$

□

The convolution product of two functions inherits also the smoothness properties of the factors, because differentiation can be applied to each of the factors.

Theorem 15. [6] *Suppose that f is differentiable and convolutions $f * g$ and $f' * g$ are well-defined. Then $f * g$ is differentiable and $(f * g)' = f' * g$. Likewise, if g is differentiable, then $(f * g)' = f * g'$*

Proof. We just need to differentiate under the integral sign:

$$(f * g)'(x) = \frac{d}{dx} \int f(x - y)g(y) dy = \int f'(x - y)g(y) dy = f' * g(x).$$

Since $f * g = g * f$, the same argument works with f and g interchanged. □

The importance of the Fourier transform is partially based on the following convolution theorem.

Theorem 16. *Assume that f and g are integrable functions, then $\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g$.*

3.3.2 The Sampling Theorem

We follow the Fourier analysis and applications book of Vretblad to explain Shannon's sampling theorem.

From Parseval's relation one can derive the so-called Shannon sampling theorem which is of great interest for digital sound processing. It deals with band-limited functions, i.e. with functions which do not contain frequencies above some threshold frequency. The usual standard applied in practice is to use 20 kHz as maximal frequency, because the human ear cannot perceive any pure frequency higher than that. The sound signal can thus be considered to have its frequency spectrum totally within this range. If it is sampled at sufficiently small intervals, and if the sampling is precise enough, it is then possible to recover the sound from the digitalized sample record.

Mathematically speaking we assume that a function $f(t)$ is built up using (angular) frequencies ω satisfying $|\omega| \leq c$. We will explain how to reconstruct the entire signal by sampling it at regular points at distance not larger than $\frac{\pi}{c}$.

Theorem 17. (*Shannon's sampling theorem*)[11](p.187-188) *Suppose that f is continuous on $\mathbf{R}$, that $f \in L^1(\mathbf{R})$ and that $\hat{f}(\omega) = 0$ for $|\omega| > c$. Then*

$$f(t) = \sum_{n \in \mathbf{Z}} f\left(\frac{n\pi}{c}\right) \frac{\sin(ct - n\pi)}{ct - n\pi}$$

where the sum is uniformly convergent on $\mathbf{R}$.

Proof. By the Fourier inversion formula, we have

$$f(t) = \frac{1}{2\pi} \int_{-c}^c \hat{f}(\omega) e^{it\omega} d\omega$$

We shall rewrite this integral. We introduce a function g as follows:

$$g(\omega) = \frac{c}{\pi} \hat{f}(\omega), \quad |\omega| < c.$$

This can be considered as a restriction to the interval $(-c, c)$ of a $2c$ -periodic function with Fourier series

$$g(\omega) \sim \sum_{n \in \mathbf{Z}} c_n(g) e^{i(n\pi/c)\omega},$$

where

$$c_n(g) = \frac{1}{2c} \int_{-c}^c g(\omega) e^{-i(n\pi/c)\omega} d\omega = \frac{1}{2\pi} \int_{-c}^c \hat{f}(\omega) e^{-i(n\pi/c)\omega} d\omega = f\left(-\frac{n\pi}{c}\right)$$

We also consider the function h given by

$$h(\omega) = e^{-it\omega}, \quad |\omega| < c.$$

In a similar way as for g , we compute

$$h(\omega) \sim \sum_{n \in \mathbf{Z}} c_n(h) e^{i(n\pi/c)\omega},$$

from which we can derive

$$\begin{aligned} c_n(h) &= \frac{1}{2c} \int_{-c}^c e^{-it\omega} e^{-i(n\pi/c)\omega} d\omega \\ &= \frac{1}{2c} \left[\frac{e^{-it\omega - i(n\pi/c)\omega}}{-it - i\frac{n\pi}{c}} \right]_{\omega=-c}^{\omega=c} \\ &= \frac{\sin(ct + \pi n)}{ct + \pi n} \end{aligned}$$

We now rewrite the Fourier inversion formula and using the polarized Parseval formula for functions with period $2c$:

$$\begin{aligned} f(t) &= \frac{1}{2c} \int_{-c}^c \frac{c}{\pi} \hat{f}(\omega) \overline{e^{-i\omega t}} d\omega = \frac{1}{2c} \int_{-c}^c g(\omega) \overline{h(\omega)} d\omega \\ &= \sum_{n \in \mathbf{Z}} c_n(g) \overline{c_n(h)} = \sum_{n \in \mathbf{Z}} f\left(-\frac{n\pi}{c}\right) \frac{\overline{\sin(ct + \pi n)}}{ct + \pi n} \\ &= \sum_{n \in \mathbf{Z}} f\left(\frac{n\pi}{c}\right) \frac{\sin(ct + \pi n)}{ct - \pi n} \end{aligned}$$

The convergence of the series is clear, since both g and h are L^2 functions. INdeed, the convergence of symmetric partial sums $s_N = \sum_{-N}^N$ is uniform in t , because estimates of the remainder are uniform. The theorem is proved. $\square$

4 Some FFT Applications

In this last chapter we are going to give some applications of the use of the FFT in probability theory, large integer multiplication and digital filtering. All examples are done in cooperation with Hans G. Feichtinger. We start with some problems from probability theory.

4.1 Applications from Probability Theory

The first observation is the fact, that the complete information about a *random variable* with integer values can be packed into a polynomial resp. Laurent series.

The first and maybe most natural example would be a fair dice, which is assumed to have the possible outcomes 1, 2, 3, 4, 5, 6 with equal probability, hence with probability $1/6$. The corresponding polynomial is then $p_D(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)/6$.

Another example would be flipping the coin to go left or right over the integers. Here one would use the simple Laurent series $q(t) = 0.5/t + 0.5 \cdot t$.

Now the interesting story is the fact, that the addition of independent random variables can be directly translated into multiplication of the corresponding polynomials. Just for illustration, consider two dices. The probability of having a sum equal to 4 is of course $3/36$, because out of the 36 possible outcomes with two independent dices exactly three of them are favorable for our request, namely the three pairs (1, 3), (2, 2) and (3, 1).

But if we look at the computation of $p_D^2(t)$ as it is done at school and we look out for the coefficient of x^4 we find that it arises as the sum of the coefficients

$$x^1/6 \cdot x^3/6 + x^2/6 \cdot x^2/6 + x^3/6 \cdot x^1/6 = 3/36x^4.$$

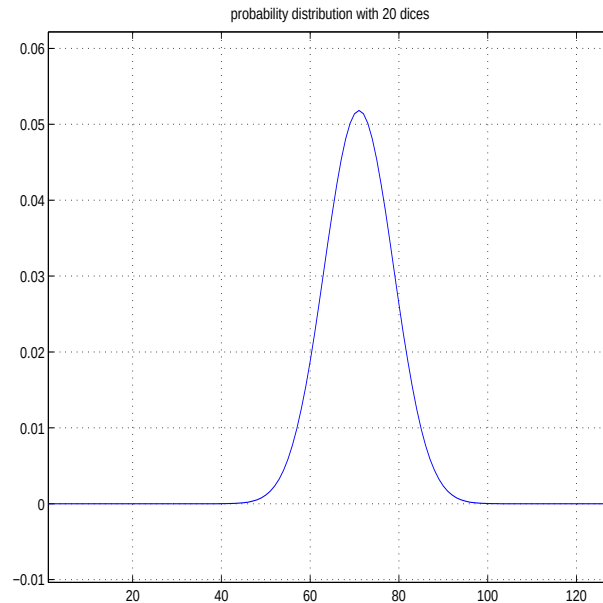
Of course what we have tested for $k = 4$ is valid for general exponents and thus we can claim:

Lemma 4.1.1. *Given K dices (acting independently), the probability distribution for the sum their results is exactly described by the polynomial $p_D^K(t)$.*

4 Some FFT Applications

The connection between FFT and the multiplication of polynomials as described above then allows to compute these coefficients in a fast way.

Example 4.1.2. For the case of illustration of this principle we choose $K = 20$, and display the coefficients of the corresponding probability distribution (which of course has non-zero values only between 20 and 120) as follows:



This shows the probability distribution with 20 dices. Since there is an obvious maximal sum of 120, resp., the degree of the corresponding polynomial $(x + x^2 + x^3 + x^4 + x^5 + x^6)/6$ is exactly 120, hence order 121, it is enough to use an FFT or length $128 = 2^7$.

```
>> N = 256; qgam = zeros(1,N);
>> qgam(N) = 0.2; qgam(2) = 1-0.2;
>> q100 = real( ifft( fft(qgam) .^100));
>> plot(q100); figure(gcf);
% looks phragmented! because no odd outcome is possible!
>> plot(1:2:N,q100(1:2:N)); grid; figure(gcf); e}
>> title(' random walk, probability 20% to go left, otherwise right');
>> xlabel(' 100 iterations')
```

Given this distribution, various probabilities can be computed. e.g one can answer the question: 'What is the probability of moving more than 50 point to the right. In practice this is:

```
>> sum(q100(52:N))
ans =      0.8686
>> 100*ans
      86.8647
```

tells us that the probability of moving more than 50 points to the right is at 86,86 percent.

Lemma 4.1.3. $\forall l \in \mathbb{Z}$, $P(Z = l) = \sum_{k+n=l} P(X = k)P(Y = n)$ is equivalent $P_Z = P_X \cdot P_Y$

Proposition 4.1.4. (*Convolution of probability distributions*)
Let a_k and b_j two different random variables. If

$$\begin{aligned} a_k &\geq 0, \quad \sum a_k = 1 \\ b_j &\geq 0, \quad \sum b_j = 1 \end{aligned}$$

then, the Cauchy product c given as usual by

$$c_l = \sum_{k+n=l} a_k b_l$$

has the same property.

Example 4.1.5. The next example concerns a random walk over the integers. By some random process it is determined whether one takes one step to the left or to the right. For our example we choose an asymmetric version, just for illustration. We define X as a discrete random variable (*random walk*) which takes only two values ($Left, Right$) = $(L, R) = (-1, 1)$ and E as Expectancy value. X is distributed as follows:

$$\begin{aligned} P[X = L] &= 0.2 \text{ defines going left,} \\ P[X = R] &= 0.8 \text{ defines going right} \end{aligned}$$

Now we define a stochastic process S , through the outcomes of random walk. We set $S_0 = 0$ and

$$S = \sum_{k=1}^n X_k.$$

$$E[S_n] = E\left[\sum_{k=1}^n X_k\right] = \sum_{k=1}^n E[X_k]$$

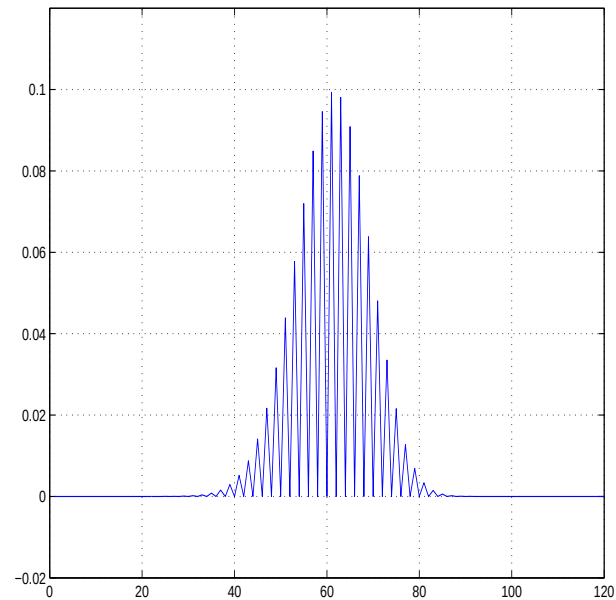
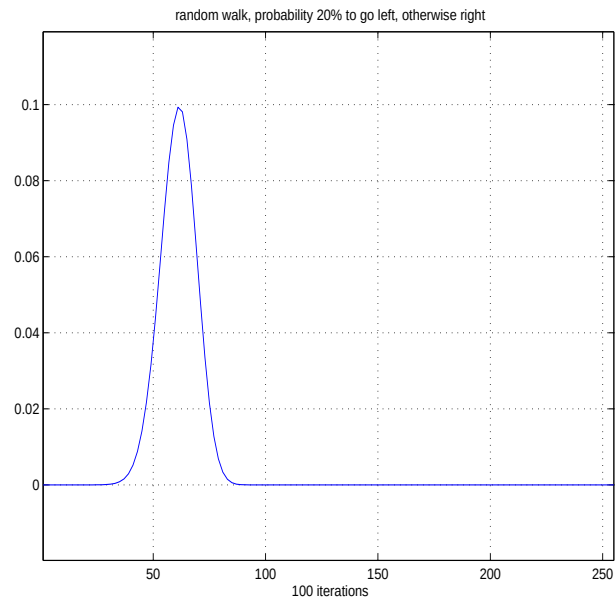
4 Some FFT Applications

$$E[X] = 0.8(1) + 0.2(-1) = 0.6$$

We take $n = 100$ to determine the probability

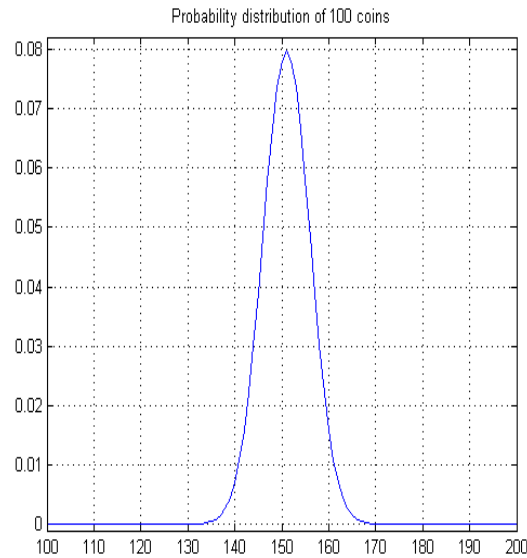
$$E[S_{100}] = E\left[\sum_{k=1}^{100} X_k\right] = \sum_{k=1}^{100} 0.6 = 60$$

The following plots show the probability of the random walk %20 going left and otherwise going right of reading a certain position after 100 steps:



Example 4.1.6. What is the probability distribution of the relative positions after throwing 100 coins?

We describe the computation and use a little application of the fft and ifft, with $n = 256$, because there is a range of 200 possible positions, and 256 is the next power of two to this number. It will be sufficient to identify a Laurent series with at most 201 terms (range -100 to $+100$) from 256 values on the unit circle.



MATLAB CODES:

```
c=zeros(1,N);c(2:3)=.5; the probability for heads or tails is .5
fc=fft(c); take the fft of c
c100=ifft(fc.^100); prob. distr. of 100 coins
subplot(3,1,1); plot(c); axis tight; grid on;
subplot(3,1,2); plot(fc); axis tight; grid on;
subplot(3,1,3);
plot(c100); axis ([100 200 -0.001 0.082]); grid on;
```

4.2 Multiplication of Long Integers Using FFT

Large integer numbers can be written as values of polynomials, typically at $x = 10$ because we are used to the decimal system. Let X and Y be two large numbers. The polynomial representation X and Y which has the powers of 10 as basis:

$$P(z) = \sum_{k=0}^{N-1} x_k z^k, Q(z) = \sum_{l=0}^{N-1} x_l z^l \text{ and } R(z) = P(z) \cdot Q(z)$$

If we want to multiply 2 n numbered integers, we must do n^2 multiplications.

So If x and y have thousands of digits, multiplication would require millions of single-digit multiplications. While the numbers gets larger, the time needed to multiply them becomes enormously long.

Now let us look at multiplication in a different way. Because a number is a sequence of digits, we can consider it as a polynomial with x is 10.

For example: We have the number

$$1234 = 4 + 3.x + 2x^2 + 1.x^3 = 4.10^0 + 3.10^1 + 2.10^2 + 1.10^3$$

We multiply it with itself

$$\begin{aligned} 1234.1234 &= (4.10^0 + 3.10^1 + 2.10^2 + 1.10^3).(4.10^0 + 3.10^1 + 2.10^2 + 1.10^3) \\ &= (4.10^0 + 3.10^1 + 2.10^2 + 1.10^3).4 \\ &\quad + (4.10^0 + 3.10^1 + 2.10^2 + 1.10^3).3.10^1 \\ &\quad + (4.10^0 + 3.10^1 + 2.10^2 + 1.10^3).2.10^2 \\ &\quad + (4.10^0 + 3.10^1 + 2.10^2 + 1.10^3).1.10^3 \\ &= (16 + 12.10^1 + 8.10^2 + 4.10^3) \\ &\quad + (12.10^1 + 9.10^2 + 6.10^3 + 3.10^4) \\ &\quad + (8.10^2 + 6.10^3 + 4.10^4 + 2.10^5) \\ &\quad + (4.10^3 + 3.10^4 + 2.10^5 + 1.10^6) \\ &= 16 + 24.10^1 + 25.10^2 + 20.10^3 + 10.10^4 + 4.10^5 + 10^6 \\ &= 1522756 \end{aligned}$$

In this case $n = 4$ we still need to perform $n^2 = 16$ multiplications. However, we don't need to do that much addition and multiplication. The following example shows how to multiply large integers in a easy and quick way.

Example 4.2.1. Consider the numbers 12345678 and 87654321. Their polynomial forms are

$$x = [8, 7, 6, 5, 4, 3, 2, 1] \quad \text{and} \quad y = [1, 2, 3, 4, 5, 6, 7, 8]$$

where

$$x_0 = 8, x_1 = 7, x_2 = 6, x_3 = 5, x_4 = 4, x_5 = 3, x_6 = 2, x_7 = 1$$

and

$$y_0 = 1, y_1 = 2, y_2 = 3, y_3 = 4, y_4 = 5, y_5 = 6, y_6 = 7, y_7 = 8$$

Both integers have 8 digits. Therefore polynomials are of order 8 and have degree 7. As introduced above, we compute the fast Fourier transform of the vectors x and y to get their FFT coefficients. We multiply x_{val} and y_{val} in the frequency domain and finally take the inverse Fourier transform of the solution in order to return to the time domain. As mentioned in Chapter 1/ Polynomial Multiplication, we pad x and y with 8 zero coefficients as place holders for the higher-order terms. Recall that the polynomial multiplication is in fact a convolution. After finding Fourier coefficients we apply convolution. Zero padding allows us to use a longer FFT, which will produce a longer FFT result vector. Zero padding before FFT is a computationally effective method to interpolate a large number of points.

```
>> xz = [zeros(1,8), 8:-1:1];
>> yz = [zeros(1,8), 1:8];
>> xconvy = real(ifft( fft(xz) .* fft(yz)));
>> polyval(xconvy,10),
>> xz=[0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8];
>> yz=[0 0 0 0 0 0 0 0 8 7 6 5 4 3 2 1];
>> xzval=fft(xz)
xzval =
Columns 1 through 4
36.0000          8.1371 +25.1367i  -4.0000 + 9.6569i  -3.3801 + 7.4830i
Columns 5 through 8
-4.0000 + 4.0000i  -4.2768 + 3.3409i  -4.0000 + 1.6569i  -4.4802 + 0.9946i
Columns 9 through 12
-4.0000          -4.4802 - 0.9946i  -4.0000 - 1.6569i  -4.2768 - 3.3409i
Columns 13 through 16
-4.0000 - 4.0000i  -3.3801 - 7.4830i  -4.0000 - 9.6569i   8.1371 -25.1367i
```

4 Some FFT Applications

```
>> yzval=fft(yz)
yzval =
Columns 1 through 4
36.0000          -17.1371 +20.1094i    4.0000 - 9.6569i   -5.6199 + 5.9864i
Columns 5 through 8
4.0000 - 4.0000i  -4.7232 + 2.6727i    4.0000 - 1.6569i   -4.5198 + 0.7956i
Columns 9 through 12
4.0000          -4.5198 - 0.7956i    4.0000 + 1.6569i   -4.7232 - 2.6727i
Columns 13 through 16
4.0000 + 4.0000i  -5.6199 - 5.9864i    4.0000 + 9.6569i  -17.1371 -20.1094i

>> xconvy = real(ifft( fft(xz) .* fft(yz)));
xconvy =
Columns 1 through 8
8.0000   23.0000   44.0000   70.0000  100.0000  133.0000  168.0000  204.0000
Columns 9 through 16
168.0000  133.0000  100.0000   70.0000   44.0000   23.0000    8.0000    0.0000

>> polyval(xconvy,10)
ans = 1.0822e+016
```

Polynomial representation of the multiplication of two vectors x and y is

$$\begin{aligned}
 x.y &= 0.10^{15} + 8.10^{14} + 23.10^{13} + 44.10^{12} + 70.10^{11} \\
 &\quad + 100.10^{10} + 133.10^9 + 168.10^8 + 204.10^7 + 168.10^6 \\
 &\quad + 133.10^5 + 100.10^{14} + 70.10^3 + 44.10^2 + 23.10^1 + 8.10^0 \\
 &= \sum_{n=0}^{15} z_n 10^n \text{ where } z_n \text{ is the coefficients of the polynomial multiplicaton}
 \end{aligned}$$

In this example, we represented the integers x and y in base 10, using digits(1-8). It is also possible to use this technique with different bases. Especially, choosing a base that is a power of 2 is mostly applied when we wish to do these computations on a computer which represents the integers in binary form.

4.3 Application of DFT: Digital Filtering

Many digital filters are based on the Fast Fourier transform. FFT extracts the frequency spectrum of a signal and allows us to manipulate the spectrum. Then we can convert the modified spectrum back into our fundamental time-series signal.

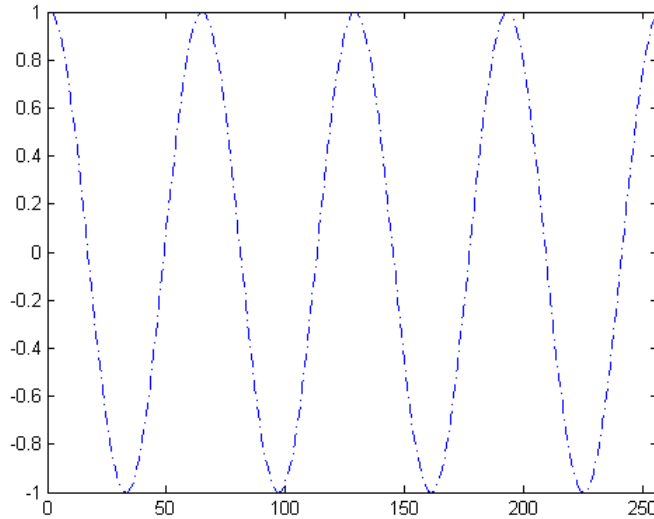
Example 4.3.1. [2] (p.17-20) The discrete Fourier transform has a natural interpretation as a transform of the time (signal) domain of a periodic function into its frequency (spectral) domain:

Let $a : \mathbb{Z} \rightarrow \mathbb{C}$ be an n -periodic function on the discrete time domain $\mathbb{Z}$ given by n consecutive value $(a(0), \dots, a(n-1))$. The Fourier transform of a is the vector $(A(0), \dots, A(N-1))^T := (\omega^{ij})_{i,j < n} \cdot (a(0), \dots, a(n-1))^T$, where $\omega := e^{\frac{2\pi i}{n}}$ describes a primitive n -th root of unity. It is easy to see that for all $t \in \mathbb{Z}$,

$$a(t) = \frac{1}{n} \sum_{f=0}^{n-1} A(f) \omega^{-ft}.$$

In other words, every n -periodic function $a : \mathbb{Z} \rightarrow \mathbb{C}$ can be uniquely written as a linear combination of the n -periodic basis functions $\chi_f : (t \rightarrow \omega^{-ft})$ and the coefficient belonging to frequency f equals $\frac{A(f)}{n}$. Because of this interpretation, the Fourier transform of a function is often called its Fourier *spectrum*. Let us consider a real cosine wave of frequency 4, sampled at 256 equidistant points:

$$a(t) = \cos\left(\frac{4 \cdot 2\pi t}{256}\right), t = 0, \dots, 255$$



4 Some FFT Applications

As

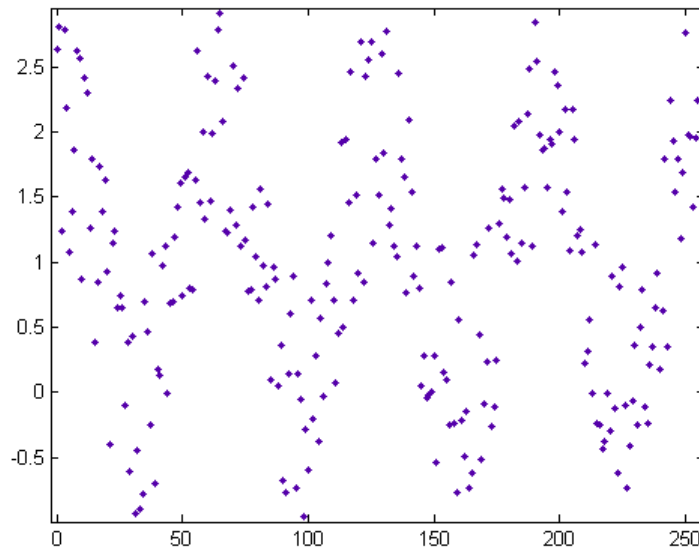
$$\cos\left(\frac{4 \cdot 2\pi t}{256}\right) = \frac{1}{2}e^{i\left(\frac{4 \cdot 2\pi t}{256}\right)} + e^{-i\left(\frac{4 \cdot 2\pi t}{256}\right)} = \frac{1}{2}(\omega^{4t} + \omega^{-4t}),$$

its Fourier spectrum has exactly two non-zero coefficients

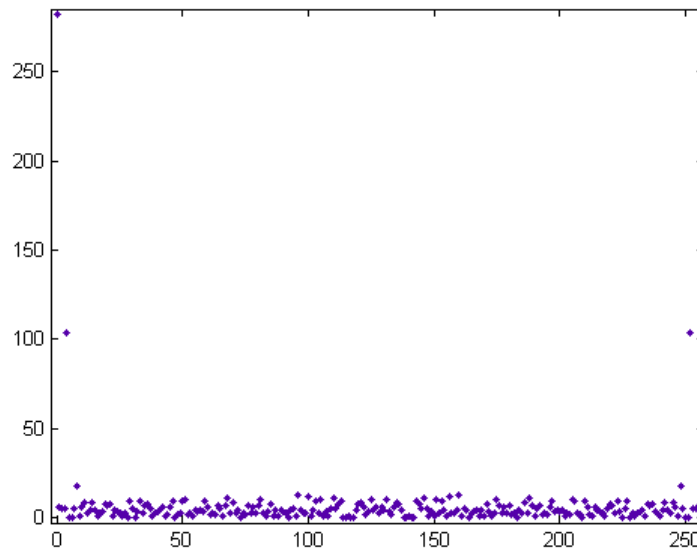
$$A(4) = A(256 - 4) = 128$$

corresponding to the basis functions ω^{-4t} and $\omega^{-(256-4)t} = \omega^{4t}$. Indeed, these functions are complex conjugates and add up to a real cosine function of frequency 4.

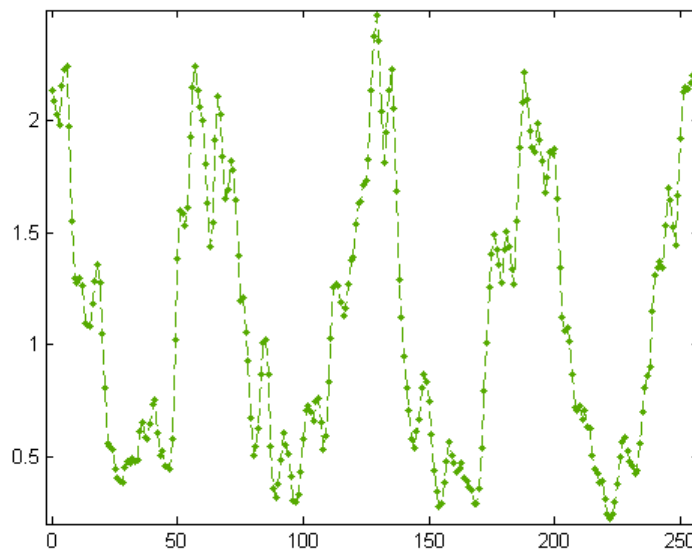
One important application of discrete Fourier transform is *filtering of digital signals*: Suppose that our cosine-wave is an audio signal transmitted over a radio channel and distorted by strong random noise, so the receiver sees something like this:



Note that this signal looks unlikely that the receiver will be able to find out what was actually transmitted. However, looking at the absolute values of the Fourier coefficients of the received signal makes us recognize the peaks corresponding to the original signal:



the receiver may send the signal through a low-pass filter to improve its quality. This filter function removes high-frequency components from the spectrum by pointwise multiplication. Transforming the result back into the time domain, the receiver gets a filtered signal that is much closer to the original:



So the low-pass filter is successful in reducing the noise considerably, and make the signal understandable again.

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Curriculum Vitae

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