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DISSERTATION

Titel der Dissertation

"On the Geometric Construction of Cohomology Classes for Cocompact Discrete Subgroups of the Real and Complex Special Linear Group"

Verfasserin

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Contents

Introduction

Arithmetic subgroups of algebraic groups have been an object of extensive research over the last few decades. Although these groups are interesting in many respects, a particularly important aspect from the number theoretical point of view is their connection to the theory of automorphic forms. When studying this relation, the cohomology of arithmetic groups is an important tool. Understanding the cohomology of an arithmetic group is a very difficult task—in many cases it is not even known whether the cohomology of such a group in a given degree vanishes or not.

The present thesis contributes to the research on cohomology of arithmetic groups by providing a nonvanishing-result for the cohomology of families of arithmetic groups that can be embedded as cocompact discrete subgroups into the the real or complex special linear group. The approach is via a geometric argument: We construct nontrivial cohomology classes that are induced by pairs of complementary dimensional special cycles with nonvanishing intersection number.

Geometric Construction of Cohomology Classes. Let us explain how geometric methods come into play. Consider a reductive algebraic $\mathbb{Q}$-group $\mathbf{G}$ and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. The group Γ can be considered as a discrete subgroup of the Lie group $G := \mathbf{G}(\mathbb{R})$ and as such it acts smoothly and properly on the symmetric space $X = K\backslash G$ associated with G, where K denotes a maximal compact subgroup of G. If Γ is in addition torsion-free, this action is free and X/Γ is a $K(\Gamma, 1)$-space. In particular, one has $H^*(\Gamma, \mathbb{C}) = H^*(X/\Gamma; \mathbb{C})$, i. e. the group cohomology of Γ w. r. t. the trivial Γ-module $\mathbb{C}$ equals the singular cohomology of X/Γ with complex coefficients.

Now X/Γ is a Riemannian manifold, even a locally symmetric space, so one can use geometric methods to gain information about its cohomology. The space X/Γ is always of finite invariant volume; however, one has to distinguish between compact and noncompact quotients. For example, torsion-free arithmetic subgroups of the algebraic group $\mathbf{SL}_n$ over $\mathbb{Q}$ are never cocompact. In this case, one can study the cohomology with the help of the Borel-Serre compactification of X/Γ, a compact manifold with corners that is homotopy equivalent to X/Γ.

On the other hand, a connected reductive Lie group always has a cocompact discrete subgroup Γ that arises as an arithmetic subgroup of a suitable algebraic group ([**5**], [**8**]). In this situation, and under the additional assumption that X/Γ is orientable, one approach to understand the cohomology of X/Γ is to find suitable oriented submanifolds of X/Γ and show that their fundamental classes induce nontrivial cohomology classes in $H^*(X/\Gamma; \mathbb{C})$. This method goes back to the work of Millson in his 1976 article [**44**] on the nonvanishing of the first Betti number of certain compact hyperbolic manifolds. The main idea of his paper is the construction of an oriented hypersurface Y in the manifold in question, such that the fundamental class of Y contributes nontrivially to the cohomology in degree one via Poincaré duality. It was only a few years later, that Millson and Raghunathan pursued this idea further and generalized it within the framework of algebraic groups, to submanifolds of arbitrary codimension ([**45**]): With a reductive subgroup $\mathbf{H}$ of $\mathbf{G}$, one associates the locally symmetric space X_H/Γ_H, where $X_H := K_H\backslash\mathbf{H}(\mathbb{R})$ for a maximal compact subgroup K_H of $\mathbf{H}(\mathbb{R})$ and $\Gamma_H := \mathbf{H}(\mathbb{Q}) \cap \Gamma$. Then (possibly after passing to a subgroup of finite index in Γ), X/Γ_H can be considered as a totally geodesic submanifold of X/Γ, to be called a *geometric cycle*. The approach of Millson and Raghunathan is based on finding two such geometric cycles of complementary dimension in X/Γ that intersect transversally and with positive multiplicity in all points of intersection. Under this assumption, the fundamental classes of the two submanifolds have nontrivial intersection number and hence they contribute nontrivially to the cohomology of X/Γ.

In the method of Millson and Raghunathan, it is essential that the two cycles intersect transversally, i. e. that their intersection is a set of points. This is, however, not necessarily the case for the intersection of two arbitrary submanifolds of complementary dimension. In 1993, Rohlfs and Schwermer found a way to generalize the method in such a way that it also applies to more complicated intersections. In their work, they concentrate on the case of *special cycles*, i. e. geometric cycles arising from fixed point groups of a $\mathbb{Q}$-rational automorphism of $\mathbf{G}$ of finite order. For two such special cycles of complementary dimension, Schwermer and Rohlfs describe the connected components of their intersection with the help of nonabelian Galois cohomology and show that under certain conditions the intersection number of such manifolds is nontrivial. Their result relies on an assumption on the orientability properties of actions of the involved Lie groups, which can be understood as an analogue to the condition of positive intersection multiplicity in the approach of Millson and Raghunathan.

Using these geometric methods, compact arithmetic quotients attached to several classical and exceptional Lie groups have been studied. Millson and Raghunathan applied their result to certain families of quotients of the real Lie groups $SO(p,q)$, $SU(p,q)$ and $Sp(p,q)$. In his thesis, Waldner addressed the case of the exceptional group G_2 and even came up with a further generalization of the method of Millson and Raghunathan that is applicable to geometric cycles that are not special, see [**68**]. Furthermore, results have been obtained for the exceptional group F_4 and the real Lie group $SU^*(2n)$, which is the special linear group over Hamilton's quaternions, see [**59**] and [**60**].

This thesis deals with the application of the above method to the real and complex special linear group. These groups arise (up to compact factors) as groups of real points of certain algebraic groups that are associated with special unitary groups over suitable algebraic number fields. For an appropriate choice of such an algebraic group $\mathbf{G}$, we show that sufficiently small arithmetic subgroups of $\mathbf{G}$ give rise to cocompact discrete subgroups Γ of $SL_n(\mathbb{R})$ or $SL_n(\mathbb{C})$ having nontrivial cohomology induced from special cycles. More precisely, we will prove a result of the following form:

THEOREM. *Let $n \in \mathbb{N}$, $n \geq 2$.*

(1) Let $X := SO(n)\backslash SL_n(\mathbb{R})$ denote the symmetric space attached to the real Lie group $SL_n(\mathbb{R})$. If n is even, there exists a discrete cocompact arithmetically defined subgroup $\Gamma \subset SL_n(\mathbb{R})$ such that $H^k(X/\Gamma;\mathbb{C})$ contains nontrivial cohomology classes for all k of the form

$$k = pq, \quad \text{and} \quad k = \frac{p^2+q^2+n}{2} - 1,$$

where p and q are positive integers with $p+q=n$ and, if $n \neq 2$, for

$$k = \frac{n^2+2n}{4} \quad \textit{and} \quad k = \frac{n^2}{4} - 1.$$

(2) *Let $X := SU(n)\backslash SL_n(\mathbb{C})$ denote the symmetric space attached to the real Lie group $SL_n(\mathbb{C})$. There exists a discrete cocompact arithmetically defined subgroup $\Gamma \subset SL_n(\mathbb{C})$ such that $H^k(X/\Gamma;\mathbb{C})$ contains nontrivial cohomology classes for all k of the form*

$$k = 2pq \quad \textit{and} \quad k = p^2 + q^2 - 1,$$

where p and q are positive integers with $p+q=n$, and for

$$k = \frac{n^2-n}{2} \quad \textit{and} \quad k = \frac{n^2+n}{2} - 1.$$

Moreover, if n is even and $n \neq 2$, there are nontrivial cohomology classes in the degrees

$$k = \frac{n^2+n}{2}, \quad k = \frac{n^2-n}{2} - 1, \quad k = \frac{n^2}{2} - 1 \quad \textit{and} \quad k = \frac{n^2}{2}.$$

When interpreting $H^*(X/\Gamma,\mathbb{C})$ as the cohomology of the de Rham complex $\Omega^*(X/\Gamma,\mathbb{C})$, the constructed classes are not represented by $SL_n(\mathbb{R})$- or $SL_n(\mathbb{C})$-invariant differential forms on X.

For a more thorough version of this result, see Theorems 6.2 and 6.4. There, we will also specify the algebraic groups from which the discrete subgroups Γ originate.

Applications to Automorphic Forms. Our result has the following application in the theory of automorphic forms: For a compact quotient X/Γ, the cohomology can also be expressed in terms of irreducible unitary representations of the group G, via Matsushima's formula:

$$H^*(X/\Gamma;\mathbb{C}) = \bigoplus_{\pi \in \widehat{G}} m(\pi,\Gamma) H^*(\mathfrak{g}, K; H^\infty_{\pi,K}).$$

Here, the sum on the right-hand side is a finite sum over equivalence classes of irreducible unitary representations of G with nontrivial $(\mathfrak{g},K)$-cohomology, and the $m(\pi,\Gamma)$ are the multiplicities with which π occurs in the decomposition of $L^2(G/\Gamma)$ into irreducible representations. It is a consequence of Matsushima's formula that the existence of a nontrivial cohomology class for X/Γ which is not represented by an invariant differential form implies the nonvanishing of at least one multiplicity $m(\pi,\Gamma)$ for a nontrivial representation π. In other words, π is an *automorphic representation of G with respect to* Γ.

To get a more precise statement, one studies the representations that may possibly contribute to the cohomology on the right-hand side of Matsushima's formula. They have been classified by the work of Vogan-Zuckerman [**67**] (for connected semisimple real Lie groups) and Enright [**17**] (for simply connected complex groups). In the second part of the thesis we will determine up to unitary equivalence all irreducible unitary representations of $SL_n(\mathbb{C})$ with nontrivial $(\mathfrak{g},K)$-cohomology and compare the occurring degrees in which X/Γ may possibly have nontrivial cohomology with those detected by special cycles.

Content of the Chapters. Let us briefly describe the content of the different chapters of this thesis: In Chapter I, we set up the notation and recall some basics from the theory of algebraic groups, Lie groups and hermitian forms. Chapter II gives an overview of the method of Rohlfs and Schwermer [**53**]. We explain the notion of geometric and special cycles and define the intersection number of two such cycles of complementary dimension. Then we state the main theorem of [**53**], which gives precise conditions under which two special cycles contribute nontrivially to the cohomology of the arithmetic quotient X/Γ. In the last section of this chapter, following [**8**], we show how to construct cocompact discrete subgroups of certain Lie groups that arise as the groups of real or complex points of simply connected simple algebraic groups over $\mathbb{R}$ or $\mathbb{C}$.

In Chapter III, we prove our main results. The first three sections of this chapter are devoted to the construction of cocompact discrete subgroups of the real semisimple Lie groups $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$. Given an algebraic number field F, a quadratic extension E/F, a central division algebra D/E with an involution σ of the second kind, and a σ-hermitian form h on D^m for some natural number m, we define the algebraic F-group $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$. We will see that for appropriate choices of the involved number fields and the hermitian form, the group of real points of the $\mathbb{Q}$-group $\mathbf{G}$ obtained from $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ by restriction of scalars is isomorphic to $SL_n(\mathbb{R})$ or $SL_n(\mathbb{C})$ up to compact factors and that arithmetic subgroups of $\mathbf{G}$ give rise to cocompact discrete subgroups Γ of these real Lie groups. In this setting, we can apply the method of Rohlfs and Schwermer. In Sections 4–7, we construct several special cycles that are associated with the fixed points of $\mathbb{Q}$-rational automorphisms of order two on $\mathbf{G}$ and show that many of these cycles contribute nontrivially to the cohomology of X/Γ. It turns out that the construction of the special cycles involves the existence of involutions of the first kind on the division algebra D. This reduces our analysis to two cases: Either D equals the number field E and σ is the nontrivial Galois automorphism of E/F or D is a quaternion division algebra over E, and σ is the unique involution of the second kind. For both cases, we perform a detailed analysis of the occurring cycles and the degrees in the cohomology of X/Γ to which they contribute. The results are stated in Theorem 6.2 for the real special linear group and in Theorem 6.4 for the complex case.

Finally, in Chapter IV, we deal with the relations to representation theory via Matsushima's formula. In general, the existence of a nontrivial cohomology class for X/Γ that is not represented by an invariant differential form implies the existence of a nontrivial automorphic representation. We point out that for the groups we consider, this is not a discrete series representation (unless we are in the case of $SL_2(\mathbb{R})$).

To make this result more precise, we study the representations occurring on the right hand side of Matsushima's formula. We concentrate on the case of the complex special linear group. Using results of Delorme [**14**] and Enright [**17**], we classify all irreducible unitary representations of $SL_n(\mathbb{C})$ with nontrivial $(\mathfrak{g}, K)$-cohomology up to unitary equivalence and determine the degrees in which these representations may possibly contribute to the cohomology of X/Γ. For small values of n, we can identify specific automorphic representations of G with respect to Γ that are detected by the nontrivial cohomology classes we have constructed in Chapter III.

CHAPTER I

Notation and Preliminaries

1. Notation

- For an algebraic number field k, we let $V = V(k)$ and $V_\infty = V_\infty(k)$ denote its set of places and archimedean places, respectively. For a place $v \in V$, we denote by k_v the completion of k at v. Let ℓ/k be a finite extension. For a place $w \in V(\ell)$, we write $w|v$ if the corresponding valuation $|\cdot|_w$ extends the valuation $|\cdot|_v$ of a place $v \in V(k)$. If $[\ell : k] = 2$ we say that v is *decomposed* in ℓ if there are exactly two places $w \in V(\ell)$ such that $w|v$, and *nondecomposed* otherwise.
- We will use the following notations for categories: $\mathfrak{Alg}_k$: category of commutative, associative k-algebras, $\mathfrak{Set}$: category of sets, $\mathfrak{Grp}$: category of groups.
- All algebraic groups are assumed to be linear, i. e. they can be considered as smooth affine algebraic group schemes. We denote algebraic groups by bold letters ($\mathbf{G}$, $\mathbf{H}$, $\dots$). For an algebraic group $\mathbf{G}$ defined over a number field k, we set $\mathbf{G}_\infty := \prod_{v\in V_\infty} \mathbf{G}(k_v)$.
- Lie groups are denoted by standard Roman letters (G, H, $\dots$). Whenever we speak of a semisimple Lie group, we assume that it has finite center and finitely many connected components.[1] We use the notion of a *reductive* Lie group as in [**34**, VII.2].
- Lie algebras are denoted by small German letters ($\mathfrak{g}$, $\mathfrak{h}$, $\dots$) and can be real or complex depending on context. If $\mathfrak{g}$ is a real Lie algebra, we will denote by $\mathfrak{g}_\mathbb{C}$ its complexification and if $\mathfrak{g}$ is complex we write $\mathfrak{g}_\mathbb{R}$ for the real Lie algebra underlying $\mathfrak{g}$. In general, we denote the Lie algebra of a Lie group G by $\mathfrak{g}$ and consider it as a real or complex Lie algebra depending on whether G is a real or a complex group.

  We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $Z(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$.
- If $\mathfrak{g}$ is a complex reductive Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, we denote by $\Phi(\mathfrak{g},\mathfrak{h})$, $\Phi^+(\mathfrak{g},\mathfrak{h})$ and $\Delta(\mathfrak{g},\mathfrak{h})$ the set of roots, a system of positive roots and the associated set of simple roots, respectively, and we write $W := W(\mathfrak{g},\mathfrak{h})$ for the Weyl group of $\Phi(\mathfrak{g},\mathfrak{h})$.

  If $\mathfrak{g}$ is semisimple, we can identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the restriction of the Killing form κ to $\mathfrak{h}\times\mathfrak{h}$. This induces a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ that we denote by $\langle\cdot,\cdot\rangle$. An element $\lambda \in \mathfrak{h}^*$ is said to be an *integral weight* if it satisfies $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for all roots $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$ and it is said to be *regular integral* if these numbers are in addition nonzero. For a fixed system of positive

[1] This is to ensure that the semisimple groups are also reductive in the sense of Knapp [**34**]. Lie groups arising as the groups of real or complex points of semisimple algebraic groups will always have this property.

roots Φ^+, we say that λ is *dominant integral* if $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is nonnegative for all roots $\alpha \in \Phi^+$. We denote the set of integral (resp. dominant integral) weights by $\mathcal{P}(\mathfrak{h}^*)$ (resp. $\mathcal{P}^+(\mathfrak{h}^*)$).

- Let k be a field. The term k-algebra stands for a finite-dimensional associative algebra over k with unit element 1. For a k-algebra A we denote its opposite algebra[2] by A^{op} and the elements of A^{op} are labelled with a superscript.

  The Brauer group of k, i. e. the group of Brauer equivalence classes of central simple k-algebras, is denoted by $\mathrm{Br}(k)$ and for a central simple k-algebra A we write $[A]$ for its class in $\mathrm{Br}(k)$. We denote by $\deg(A)$, $\mathrm{ind}(A)$ and $\exp(A)$ the *degree*, *index* and *exponent* of A, respectively.
- Let R be a ring and $n \in \mathbb{N}$. We denote by I_n the $n\times n$ unity matrix in $M_n(R)$ and by $I_{p,q}$ the matrix $\mathrm{diag}(I_p, -I_q) \in M_n(R)$, for $p+q=n$. For even n, we set $J_n := \left(\begin{smallmatrix} 0 & -I_{n/2} \\ I_{n/2} & 0 \end{smallmatrix}\right)$. With this notation, we can define some classical Lie subgroups of $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$ as follows:

  For $p, q \in \mathbb{N}$ with $p+q=n$, we set

$$\begin{aligned} S(GL_p(\mathbb{R})\times GL_q(\mathbb{R})) &:= \{x \in SL_n(\mathbb{R}) \mid I_{p,q} x I_{p,q} = x\} \\ S(GL_p(\mathbb{C})\times GL_q(\mathbb{C})) &:= \{x \in SL_n(\mathbb{C}) \mid I_{p,q} x I_{p,q} = x\} \\ SO(p,q) &:= \{x \in SL_n(\mathbb{R}) \mid I_{p,q} x^t I_{p,q} x = I_n\} \\ SU(p,q) &:= \{x \in SL_n(\mathbb{C}) \mid I_{p,q} \bar{x}^t I_{p,q} x = I_n\}. \end{aligned}$$

  For even n, we define

$$\begin{aligned} Sp(n,\mathbb{R}) &:= \{x \in SL_n(\mathbb{R}) \mid J_n x^t J_n^{-1} x = I_n\} \\ Sp(n,\mathbb{C}) &:= \{x \in SL_n(\mathbb{C}) \mid J_n x^t J_n^{-1} x = I_n\} \\ Sp(n) &:= \{x \in Sp(n,\mathbb{C}) \mid x\bar{x}^t = I_n\} \\ GL^{(1)}_{n/2}(\mathbb{C}) &:= \{x \in GL_{n/2}(\mathbb{C}) \mid |\det(x)| = 1\} \\ &\cong \{x \in SL_n(\mathbb{R}) \mid J_n x J_n^{-1} = x\} \\ SU^*(n) &:= \{x \in SL_n(\mathbb{C}) \mid J_n \bar{x} J_n^{-1} = x\} \cong SL_{n/2}(\mathbb{H}). \end{aligned}$$

  Here the isomorphisms in the second-last row is induced by the embedding

$$\phi : GL_{n/2}(\mathbb{C}) \to GL_n(\mathbb{R}), \quad X = A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

  for $A, B \in M_{n/2}(\mathbb{R})$, and the isomorphism in the last row is induced by the embedding

$$\psi : GL_{n/2}(\mathbb{H}) \to GL_n(\mathbb{C}), \quad X = A + jB \mapsto \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix},$$

  for $A, B \in M_{n/2}(\mathbb{R}) \oplus iM_{n/2}(\mathbb{R}) \subset M_{n/2}(\mathbb{H})$.[3]

[2] The opposite algebra of A is the k-algebra with the same underlying vector space as A but endowed with the multiplication $x^{op} * y^{op} := (yx)^{op}$.

[3] See e. g. Sections 1.10.2 and 1.10.3 in [**49**].

2. Algebraic Groups

For the sake of completeness, we recall some basic notions of the theory of algebraic groups. Throughout this section, let k be a field of characteristic 0.

2.1. We will use the functorial approach to algebraic groups, i. e. a *linear algebraic group* over k is a functor $\mathbf{G}\colon \mathfrak{Alg}_k \to \mathfrak{Grp}$ from commutative k-algebras to groups that is representable[4] as a functor to $\mathfrak{Set}$ by a finitely generated k-algebra. Let $\mathbf{G}, \mathbf{H}$ be algebraic k-groups. A *morphism* from $\mathbf{G}$ to $\mathbf{H}$ is a natural transformation $\boldsymbol{\psi}\colon \mathbf{G} \to \mathbf{H}$ such that $\boldsymbol{\psi}(R)\colon \mathbf{G}(R) \to \mathbf{H}(R)$ is a group homomorphism for any commutative k-algebra R.

For a morphism $\boldsymbol{\psi}\colon \mathbf{G} \to \mathbf{H}$ of algebraic k-groups, the functor

$$\ker(\boldsymbol{\psi})\colon \mathfrak{Alg}_k \to \mathfrak{Grp}, \quad \ker(\boldsymbol{\psi})(R) := \ker(\boldsymbol{\psi}(R)) \subset \mathbf{G}(R)$$

defines an algebraic subgroup of $\mathbf{G}$ (cf. [**71**, 2.1]). For $\mathbf{G} = \mathbf{H}$, one can also show that the fixed point group $\mathrm{Fix}(\boldsymbol{\psi}, \mathbf{G})$ of $\boldsymbol{\psi}$ is an algebraic subgroup of $\mathbf{G}$, by interpreting it as a fibre product, cf. [**71**, 1.4].

2.2. Restriction of scalars. Let ℓ be a finite extension of k. There exists a functor $\mathrm{Res}_{\ell/k}$ from algebraic ℓ-groups to algebraic k-groups that assigns to any ℓ-group $\mathbf{G}$ the k-group $\mathrm{Res}_{\ell/k}(\mathbf{G})$ given by

$$\mathrm{Res}_{\ell/k}(\mathbf{G})(R) := \mathbf{G}(R \otimes_k \ell)$$
$$\mathrm{Res}_{\ell/k}(\mathbf{G})(f) := \mathbf{G}(\mathrm{id} \otimes f)$$

for any commutative k-algebra R and any homomorphism of commutative k-algebras $f\colon R \to R'$; and to an ℓ-morphism $\boldsymbol{\varphi}\colon \mathbf{G} \to \mathbf{H}$ of algebraic ℓ-groups the morphism $\mathrm{Res}_{\ell/k}(\boldsymbol{\varphi})\colon \mathrm{Res}_{\ell/k}(\mathbf{G}) \to \mathrm{Res}_{\ell/k}(\mathbf{H})$, given by

$$\mathrm{Res}_{\ell/k}(\boldsymbol{\varphi})(R) := \boldsymbol{\varphi}(R \otimes_k \ell).$$

The functor is called *Weil restriction* or *Restriction of scalars.*

EXAMPLE. Let $\mathbf{G}_m$ be the multiplicative group over $\mathbb{C}$, i.e. $\mathbf{G}_m(R) = R^\times$ for any commutative $\mathbb{C}$-algebra R. Then we can apply restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$ to obtain an algebraic $\mathbb{R}$-group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbf{G}_m)$ that satisfies

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbf{G}_m)(\mathbb{R}) = \mathbf{G}_m(\mathbb{C}) = \mathbb{C}^\times$$
$$\text{and } \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbf{G}_m)(\mathbb{C}) = \mathbf{G}_m(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times.$$

Moreover, we can see that the functor $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbf{G}_m)$ is represented by the finitely generated $\mathbb{R}$-algebra $A = \mathbb{R}[x, y, z]/((x^2 + y^2)z - 1)$.

REMARK. The Weil restriction commutes with taking kernels and fixed points. More precisely, for a morphism $\boldsymbol{\psi} : \mathbf{G} \to \mathbf{H}$ of algebraic ℓ-groups, we have

$$\ker(\mathrm{Res}_{\ell/k} \boldsymbol{\psi}) = \mathrm{Res}_{\ell/k}(\ker \boldsymbol{\psi})$$

and in case $\mathbf{H} = \mathbf{G}$, we also have

$$\mathrm{Fix}(\mathrm{Res}_{\ell/k} \boldsymbol{\psi}, \mathrm{Res}_{\ell/k} \mathbf{G}) = \mathrm{Res}_{\ell/k}(\mathrm{Fix}(\boldsymbol{\psi}, \mathbf{G})).$$

[4] A functor $F\colon \mathfrak{Alg}_k \to \mathfrak{Set}$ is called *representable* if there is a k-algebra A such that F is naturally equivalent to $\mathrm{Hom}_k(A, \cdot)$.

3. Group Actions on Manifolds

In this section, we recall some well-known facts about Lie group actions on manifolds. The given propositions and examples are for use in chapters II and III. For most of the proofs, the reader is referred to [**39**].

3.1. Let M be a smooth manifold and let G be a Lie group acting on M continuously. The action is called *proper*, if the map $G \times M \to M \times M$, $(g, m) \mapsto (g.m, m)$ is proper, i. e. if the inverse image of any compact set is compact. An equivalent condition is that for any compact subset $K \subset M$, the set

$$G_K := \{g \in G : (g.K) \cap K \neq \emptyset\}$$

is compact (see [**39**, Thm. 9.12]).

Example 3.2. Let G be a Lie group, $K \subset G$ a maximal compact subgroup and $H \subset G$ a closed Lie subgroup.

(1) The action of G on itself by left/right translations is proper.
(2) If G is compact, any action of G on a manifold M is proper.
(3) If G acts properly on a manifold M, then H acts properly on M.

Proposition 3.3. *Let M be a connected smooth manifold and let G be a Lie group acting smoothly and properly on M.*

(1) If G acts freely on M, the orbit space M/G is a topological manifold of dimension $\dim M - \dim G$ and has a unique smooth structure such that $p\colon M \to M/G$ is a smooth submersion.
(2) If G is a discrete group that acts freely on M, p is even a smooth normal[5] covering map.
(3) For $m_0 \in M$, let $G_{m_0} := \{g \in G \mid g.m_0 = m_0\}$ denote the stabilizer subgroup of m_0. Then the map $\alpha\colon G/G_{m_0} \to M$, $gG_{m_0} \mapsto g.m_0$ is a smooth embedding.

Proof. For (1) and (2) see [**39**, Thm. 9.16 and Thm. 9.19]. (3): It follows from the proof of [**39**, Thm. 9.24] that α is an injective immersion. Since the action of G is proper, α is a proper map and thus a smooth embedding with closed image (cf. e.g. [**39**, Prop. 7.4]). □

Example 3.4. Let G be a reductive[6] Lie group, $K \subset G$ a maximal compact subgroup and $\Gamma \subset G$ a discrete subgroup.

(1) By Example 3.2, K acts properly on G by left translations. The orbit space $X := K\backslash G$ is thus a smooth manifold of dimension $\dim G - \dim K$. One can show that this is a *Riemannian symmetric space* (cf. [**26**, IV, §3])[7] and that the action of G on X by right translations is proper.
(2) If Γ is torsion-free, it acts smoothly, freely and properly on X by right translations: As a subgroup of G, the group Γ acts smoothly and properly on X. Let $x \in X$ and $\Gamma_x := \{\gamma \in \Gamma \mid x.\gamma = x\}$ be the stabilizer subgroup of x. The

[5]A covering map $\pi\colon Y \to X$ of topological spaces is called *normal* if the group of decktransformations acts transitively on the fibres of π.

[6]Recall that we use the term *reductive Lie group* in the sense of Knapp [**34**, VII.2].

[7]Strictly speaking, Helgason [**26**] only treats Riemannian symmetric spaces attached to connected Lie groups but one can easily generalize to the case of reductive Lie groups (that have finitely many connected components by definition).

set $\{x\}$ is compact, so by the criterion in 3.1, the group Γ_x is compact and hence finite as a compact subgroup of a discrete group. But since Γ is torsion-free, this means $\Gamma_x = \{1\}$, i. e. Γ acts freely on X.

Now, by Proposition 3.3 (2), the quotient X/Γ is a smooth manifold of dimension $\dim X$ and $p\colon X \to X/\Gamma$ is a smooth normal covering map. One can even show that X/Γ is a *Riemannian locally symmetric space* (cf. [**31**, 4.9]).

(3) Let $H \subset G$ be a closed subgroup. The action of H on X by right translations is smooth and proper by (1). Let K_H denote a maximal compact subgroup of H. Then K_H acts on X via isometries and by [**26**, Thm. 13.5] there exists $x \in X$ such that $x.h = x$ for all $h \in K_H$. It is easy to see that

$$H_x = \{h \in H \mid x.h = x\} = K_H,$$

and so by Proposition 3.3 (3), the map $\alpha\colon H_x\backslash H \to X$ is a closed embedding of symmetric spaces.

3.5. Let us now consider the situation where M is a connected oriented manifold of dimension n, i. e. there exists a nowhere-vanishing differential n-form $\omega_M \in \Omega^n(M)$. We say that the action of G on M is *orientation-preserving* if the diffeomorphism

$$\lambda_g\colon M \to M, \quad m \mapsto g.m$$

is orientation-preserving for each $g \in G$. Then we have the following result:

PROPOSITION 3.6. *Let G be a discrete group acting smoothly on M. Then the smooth manifold M/G is orientable if and only if G acts orientation-preserving on M.*

PROOF. Assume that G acts orientation-preserving on M. Then, for all $g \in G$, we have $\lambda_g^*\omega_M = f\cdot\omega_M$ for a positive function f on M. Note that since G is discrete, M/G is of dimension n and the differential of the projection map $p\colon M \to M/G$ is an isomorphism. Therefore, we can define a nowhere-vanishing n-form $\omega_{M/G}$ on M/G by choosing one representative $m \in \bar{m} = mG$ in each G-orbit of M and setting $\omega_{M/G}(\bar{m}) = \omega_M(m)$. For different choices of representatives, the n-forms will only differ by a positive factor and thus define the same orientation on M/G.

Conversely, let us assume that there is an orientation on M/G represented by an n-form $\omega_{M/G} \in \Omega(M/G)$. Then we can take the pullback of $\omega_{M/G}$ under the projection map $p\colon M \to M/G$. For any $g \in G$, $m \in M$ and vector fields $\xi_1, \dots \xi_n \in T_m(M)$, we have

$$\begin{aligned}\lambda_g^*(p^*(\omega_{M/G}))(m, \xi_1, \dots, \xi_n) &= \omega_{M/G}((p\circ\lambda_g)(m), d(p\circ\lambda_g)(\xi_1), \dots, d(p\circ\lambda_g)(\xi_n)))\\ &= p^*(\omega_{M/G})(m, \xi_1, \dots, \xi_n)\end{aligned}$$

since $p \circ \lambda_g = p$, and thus G acts orientation-preserving. $\square$

To apply the proposition, one needs to know whether the discrete group acts orientation-preserving. This is in particular the case if it is a subgroup of the connected component of the identity of a Lie group:

LEMMA 3.7. *Let G be a Lie group acting on the oriented manifold M. If G is connected, the action of G on M is orientation-preserving.*

PROOF. Note that a Lie group is connected if and only if it is path-connected (cf. [**39**, Prop. 1.8]). Assume that G is connected, let $g \in G$ and let $\gamma\colon [0,1] \to G$ be a path connecting g with the identity element e in G. Then we can define a homotopy

$$H : [0,1] \times M \to M, \quad H(t,m) := \lambda_{\gamma(t)}(m)$$

that connects $H_0 = \lambda_g$ and $H_1 = \lambda_e = \mathrm{id}$ in such a way that all intermediate maps H_t are diffeomorphisms of M. Clearly, $H_1 = \mathrm{id}$ preserves the orientation of M. Then $H_0 = \lambda_g$ must be orientation-preserving as well, see [**23**, 3.22, Example 5]. $\square$

Corollary 3.8. *Let G be a Lie group with finitely many connected components $X_1, \dots, X_r$ and assume that G acts smoothly on a connected oriented manifold M. The action is orientation-preserving if and only if in each connected component X_i there exists an element g_i such that λ_{g_i} acts orientation-preserving.*

Proof. If G acts orientation-preserving, the implication is trivial.

For the other direction, let $g \in G$ and assume that g lies in the component X_i. Then $g_i g^{-1} \in G^0$, the connected component of the identity. By Lemma 3.7, $\lambda_{g_i g^{-1}} = \lambda_{g_i} \circ \lambda_{g^{-1}}$ is orientation-preserving. Since λ_{g_i} preserves orientation by assumption, and the composition of two diffeomorphisms is orientation-preserving if and only if either both are orientation-preserving or both are orientation-reversing, we conclude that $\lambda_{g^{-1}}$ preserves the orientation of M. $\square$

3.9. In this section we deal with Lie group actions on orientable symmetric spaces and the question of whether these actions are orientation-preserving. Let G be a real reductive Lie group with maximal compact subgroup K and corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then there is a diffeomorphism $K \times \mathfrak{p} \to G, (k, Y) \mapsto k \exp(Y)$ that identifies $\mathfrak{p}$ with $K\backslash G$ via $\pi \circ \exp$, where $\pi\colon G \to K\backslash G$ denotes the canonical projection. In particular, the symmetric space $K\backslash G$ is diffeomorphic to Euclidean space and hence orientable.

For $k \in K$, let $\tau_k\colon K\backslash G \to K\backslash G$, $Kg \mapsto Kgk$ denote right translation by k. This defines a right action of K on $K\backslash G$. On the other hand, K acts on $\mathfrak{g}$ via the adjoint representation of G and $\mathfrak{p}$ is invariant under this action, i. e. we have $\mathrm{Ad}(k)\mathfrak{p} \subset \mathfrak{p}$ for all $k \in K$ (see [**1**, Prop. 6.4]). This defines a left action of K on $\mathfrak{p}$.

Lemma 3.10. *Let $k \in K$ be fixed. Under the identification of $K\backslash G$ with $\mathfrak{p}$, the diffeomorphism τ_k goes over to $Ad(k^{-1})$. In other words, to following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\mathrm{Ad}(k^{-1})|_{\mathfrak{p}}} & \mathfrak{p} \\ {\scriptstyle \pi\circ\exp}\downarrow & & \downarrow{\scriptstyle \pi\circ\exp} \\ K\backslash G & \xrightarrow{\tau_k} & K\backslash G \end{array}$$

Proof. This is an easy calculation: Let $Y \in \mathfrak{p}$. Then we have

$$\begin{aligned} \tau_k((\pi \circ \exp)(Y)) &= K \cdot \exp(Y)k = K \cdot (k^{-1} \exp(Y)k) \\ &= K \cdot \exp(\mathrm{Ad}(k^{-1})Y) = (\pi \circ \exp)(\mathrm{Ad}(k^{-1})Y), \end{aligned}$$

where the third equality follows from the fact that $g(\exp Y)g^{-1} = \exp(\mathrm{Ad}(g)Y)$ for $Y \in \mathfrak{g}$ and $g \in G$, cf. [**34**, I.10, (1.90)]. $\square$

3.11. With the help of Lemma 3.10, it is often easier to determine whether or not a real reductive Lie group G acts orientation-preserving on the symmetric space $K\backslash G$ by right translations. By [**34**, Prop. 7.19], a reductive Lie group has finitely many connected components that are all met by the maximal compact subgroup. Therefore, in view of Corollary 3.8, one can find a finite number of elements $k_1, \dots, k_r \in K$ such that G acts orientation-preserving on $K\backslash G$ if and only if τ_{k_i} is an orientation-preserving diffeomorphism of $K\backslash G$ for $i = 1, \dots, r$. By Lemma 3.10, this is equivalent to the requirement that $\mathrm{Ad}(k_i^{-1})|_{\mathfrak{p}} \colon \mathfrak{p} \to \mathfrak{p}$ be orientation-preserving for $i = 1, \dots, r$.

EXAMPLE. Let $p, q \in \mathbb{N}$ with $p \neq 0 \neq q$.

a) Consider $G = SO(p,q)$ and $K = S(O(p) \times O(q))$. The group K has two connected components that are distinguished by the determinant of the upper left $(p \times p)$-block and that are represented by the elements I_{p+q} and $k := \mathrm{diag}(-1, I_{p-1}, -1, I_{q-1})$. As we have seen above, G acts orientation-preserving on $K\backslash G$ if and only if both $\tau_{I_{p+q}}$ and τ_k are orientation-preserving diffeomorphisms.

   Clearly, the trivial diffeomorphism $\tau_{I_{p+q}}$ is orientation-preserving on $K\backslash G$. To see if τ_k is orientation-preserving, it suffices to consider the map $\mathrm{Ad}(k^{-1})|_{\mathfrak{p}}$. Note that $\mathfrak{g} = \mathfrak{so}(p,q)$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ where

$$\mathfrak{p} = \{X \in \mathfrak{so}(p,q) : X = X^t\}.$$

   A basis of $\mathfrak{p}$ is given by the matrices $B_{ij} := E_{ij} + E_{ji}$ for $1 \leq i \leq p$ and $p+1 \leq j \leq p+q$. Here we denote by $E_{ij} \in M_{p+q}(\mathbb{R})$ the matrix with 1 at the (i,j)-th entry and 0 everywhere else.

   Let us compute $\mathrm{Ad}(k^{-1})(B_{ij}) = k^{-1} B_{ij} k$: We have $\mathrm{Ad}(k^{-1})(B_{ij}) = B_{ij}$ if $i = j = 1$ or $i > 1$ and $j > 1$, and $\mathrm{Ad}(k^{-1})(B_{ij}) = -B_{ij}$ otherwise. This implies

$$\det(\mathrm{Ad}(k^{-1})|_{\mathfrak{p}}) = (-1)^{p-1+q-1} = (-1)^{p+q}.$$

   Therefore, $\mathrm{Ad}(k^{-1})|_{\mathfrak{p}}$ is an orientation-preserving diffeomorphism of $\mathfrak{p}$ if and only if $p+q$ is even.

b) Let now $G = S(GL_p(\mathbb{R}) \times GL_q(\mathbb{R}))$ and $K = S(O(p) \times O(q))$. As above, K (and hence G) has two connected components that are represented by I_{p+q} and k. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} := \{X \in \mathfrak{sl}_{p+q}(\mathbb{R}) : I_{p,q} X = X I_{p,q} \text{ and } X = X^t\}$. A basis of this space is given by the matrices

$$\begin{aligned} &B_{ij} := E_{ij} + E_{ji}, \quad \text{for } j > i \text{ and } 1 \leq i,j \leq p \text{ or } p+1 \leq i,j \leq p+q, \\ \text{and} \quad &B_i := E_{i,i} - E_{i+1,i+1} \quad \text{for } 1 \leq i < p+q. \end{aligned}$$

   We have $\mathrm{Ad}(k^{-1})(B_i) = B_i$ for all $1 \leq i < p+q$, $\mathrm{Ad}(k^{-1})(B_{ij}) = B_{ij}$ if $i \neq 1$ and $i \neq p+1$ and $\mathrm{Ad}(k^{-1})(B_{ij}) = -B_{ij}$ for $i = 1$ and $i = p+1$. This implies

$$\det(\mathrm{Ad}(k^{-1})|_{\mathfrak{p}}) = (-1)^{p-1+q-1} = (-1)^{p+q},$$

   so as above, the group G acts orientation-preserving on $K\backslash G$ if and only if $p+q$ is even.

4. Parabolic Subgroups of Real and Complex Lie Groups

In this section we will recall some basics about parabolic subalgebras of semisimple Lie algebras and the corresponding parabolic subgroups.

4.1. Complex Case. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Recall that we denote by $\Phi := \Phi(\mathfrak{g}, \mathfrak{h})$ the set of roots of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$. We fix a system of positive roots $\Phi^+ := \Phi^+(\mathfrak{g}, \mathfrak{h})$ in Φ and denote by Δ the corresponding set of simple roots. For a subset $I \subseteq \Delta$, we write $\langle I \rangle$ for the span of I in $\Phi(\mathfrak{g}, \mathfrak{h})$, that is, the set of roots in $\Phi(\mathfrak{g}, \mathfrak{h})$ that are linear combinations of elements of I. Let $\{X_\alpha \mid \alpha \in \Phi\} \cup \{H_\alpha \mid \alpha \in \Delta\}$ be a Chevalley basis of $\mathfrak{g}$.

The Borel subalgebra associated with Φ^+ is of the form $\mathfrak{q}_0 := \mathfrak{h} \oplus \mathfrak{n}_0$, where $\mathfrak{n}_0 := \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$. A *standard parabolic subalgebra* w. r. t. $\mathfrak{q}_0$ is a subalgebra $\mathfrak{q} \subset \mathfrak{g}$ that contains $\mathfrak{q}_0$. More generally, a *parabolic subalgebra of* $\mathfrak{g}$ is a subalgebra that is conjugate under $\mathrm{Int}(\mathfrak{g})$[8] to a standard parabolic subalgebra.

It is a well-known result that standard parabolic subalgebras can be classified by the subsets of the set of simple roots Δ:

PROPOSITION 4.2. *Let $\mathfrak{q} \supset \mathfrak{q}_0$ be a standard parabolic subalgebra of $\mathfrak{g}$. Then there exists a set $I \subseteq \Delta$ such that*

$$\mathfrak{q} = \mathfrak{q}_I := \mathfrak{h} \oplus \bigoplus_{\alpha\in\langle I\rangle\cup\Phi^+} \mathfrak{g}_\alpha.$$

Moreover, for any subset $I \subseteq \Delta$, the associated algebra $\mathfrak{q}_I$ is a standard parabolic subalgebra.

PROOF. See [**34**, Prop. 5.90]. □

REMARK. The Borel subalgebra $\mathfrak{q}_0$ is the parabolic subalgebra associated with $I = \emptyset$. It is also called a minimal parabolic subalgebra.

Let $\mathfrak{q}$ be a standard parabolic subalgebra of $\mathfrak{g}$. Then the proposition implies that $\mathfrak{q} = \mathfrak{q}_I$ for a suitable subset $I \subseteq \Delta$ of simple roots, and so we can define

$$\mathfrak{l} := \mathfrak{h} \oplus \bigoplus_{\alpha\in\langle I\rangle} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n} := \bigoplus_{\alpha\in\Phi^+\setminus\langle I\rangle} \mathfrak{g}_\alpha.$$

Then we have $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ and $\mathfrak{l}$ and $\mathfrak{n}$ have the following properties:

LEMMA 4.3. *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be a parabolic subalgebra with $\mathfrak{l}$ and $\mathfrak{n}$ as above. Then $\mathfrak{l}$ and $\mathfrak{n}$ are subalgebras of $\mathfrak{q}$, $\mathfrak{n}$ is nilpotent and $\mathfrak{l}$ is reductive. We can further decompose $\mathfrak{l}$ into its semisimple part $\mathfrak{s} := [\mathfrak{l}, \mathfrak{l}]$ and its center $Z_\mathfrak{l}$. Then we have $Z_\mathfrak{l} = \bigcap_{\alpha\in\langle I\rangle} \ker\alpha \subset \mathfrak{h}$ and $\mathfrak{s} = (\mathfrak{h} \cap \mathfrak{s}) \oplus \bigoplus_{\alpha\in\langle I\rangle} \mathfrak{g}_\alpha$, where $\mathfrak{h} \cap \mathfrak{s} = \bigoplus_{\alpha\in I} \mathbb{C}H_\alpha$.*

PROOF. See [**34**, Cor. 5.94]. □

We call the subalgebra $\mathfrak{l}$ of $\mathfrak{q}$ the *Levi factor* of $\mathfrak{q}$ and the subalgebra $\mathfrak{n}$ the *nilpotent radical* of $\mathfrak{q}$. Note that $\mathfrak{s}$ is a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h} \cap \mathfrak{s}$ and root system $\Phi_\mathfrak{s} := \Phi(\mathfrak{s}, \mathfrak{h} \cap \mathfrak{s})$ that can be identified with the set

$$\{\alpha \in \Phi \mid \alpha|_{Z_\mathfrak{l}} = 0\} = \langle I\rangle = \{\alpha \in \Phi \mid X_\alpha \in \mathfrak{s}\}.$$

EXAMPLE 4.4. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. The subalgebra $\mathfrak{h} := \{X \in \mathfrak{sl}_n(\mathbb{C}) \mid X = \mathrm{diag}(x_1, \dots, x_n)\}$ of diagonal matrices is a Cartan subalgebra of $\mathfrak{g}$. We define linear functionals

$$\varepsilon_j \colon \mathfrak{h} \to \mathbb{C}, \quad X \mapsto x_j$$

for all $1 \le j \le n$ and set $\alpha_{jk} := \varepsilon_j - \varepsilon_k$. Then the root system $\Phi(\mathfrak{g}, \mathfrak{h})$ is given by $\Phi = \{\alpha_{jk} \mid 1 \le j, k \le n \text{ and } j \ne k\}$, the set $\Delta := \{\alpha_{j,j+1} \mid 1 \le j \le n-1\}$ defines a basis for Φ, and the corresponding system of positive roots is of the form

$$\Phi^+ = \{\alpha_{jk} \in \Phi \mid 1 \le j \le n-1 \text{ and } k > j\}.$$

With respect to this positive system, the Borel subalgebra $\mathfrak{q}_0$ is the subalgebra of upper triangular matrices in $\mathfrak{sl}_n(\mathbb{C})$. Then the standard parabolic subalgebras of $\mathfrak{g}$

[8] $\mathrm{Int}(\mathfrak{g})$ denotes the group of inner automorphisms of $\mathfrak{g}$.

are in bijective correspondence with the set of compositions[9] of n; with a composition $n = \ell_1 + \ldots + \ell_m$ we associate the parabolic subalgebra

$$\mathfrak{q} = \mathfrak{q}_{\ell_1,\ldots \ell_m} = \{X \in \mathfrak{g} \mid X = \begin{pmatrix} X_1 & * & * \\ & \ddots & * \\ 0 & & X_m \end{pmatrix}, \text{ where } X_j \in GL_{\ell_j}(\mathbb{C}),\ 1 \le j \le m\}.$$

The Levi component of $\mathfrak{q}$ is given by the subalgebra of block diagonal matrices

$$\mathfrak{l} = \{X \in \mathfrak{q} \mid X = \mathrm{diag}(X_1, \ldots, X_m)\}$$

and it decomposes into its semisimple part

$$\mathfrak{s} = \{X \in \mathfrak{l} \mid \mathrm{tr}(X_j) = 0 \text{ for all } 1 \le j \le m\}$$

and its center

$$Z_{\mathfrak{l}} = \{X \in \mathfrak{l} \mid X_j = x_j \cdot I_{\ell_j} \text{ for some } x_j \in \mathbb{C},\ 1 \le j \le m\}.$$

This follows from Proposition 4.2 and the way that we have chosen the positive roots.

4.5. Assume that $\mathfrak{g}$ is the Lie algebra of a complex connected semisimple Lie group G. Then we denote the unique connected subgroup of G with Lie algebra $\mathfrak{q}_0$ by Q_0. A *standard parabolic subgroup of* G is a connected Lie subgroup of G containing Q_0. This means that the standard parabolic subgroups of G are the connected subgroups of G corresponding to the standard parabolic subalgebras of $\mathfrak{g}$.

Let Q be a parabolic subgroup. We denote by L, N and S the connected subgroups of Q with Lie algebras $\mathfrak{l}$, $\mathfrak{n}$ and $\mathfrak{s}$, respectively. Then we have a decomposition of Q as a (semidirect) product $Q = LN$. Moreover, one can show that if G is simply connected, the semisimple subgroup S of Q is also simply connected.[10]

4.6. Real case. Now let $\mathfrak{g}$ denote a real semisimple Lie algebra with Cartan involution θ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}_0$ be a maximal abelian subalgebra of $\mathfrak{p}$. We denote by $\Phi(\mathfrak{g}, \mathfrak{a}_0)$ the set of *restricted roots* of $\mathfrak{g}$ w. r. t. $\mathfrak{a}_0$, that is, the set of nonzero functionals $\lambda \in \mathfrak{a}_0^*$ such that

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [A, X] = \lambda(A)X \text{ for all } A \in \mathfrak{a}_0\} \neq 0.$$

This is a root system for $\mathfrak{a}_0^*$, and we fix a set of positive restricted roots $\Phi^+(\mathfrak{g}, \mathfrak{a}_0)$ and denote the associated set of simple restricted roots by $\Delta := \Delta(\mathfrak{g}, \mathfrak{a}_0)$. Set $\mathfrak{m}_0 := Z_{\mathfrak{k}}(\mathfrak{a}_0)$, $\mathfrak{n}_0 := \bigoplus_{\lambda \in \Phi^+(\mathfrak{g},\mathfrak{a}_0)} \mathfrak{g}_\lambda$ and $\mathfrak{q}_0 := \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$. Then $\mathfrak{q}_0$ is the Borel subalgebra w. r. t. the choice of positive roots. A *standard parabolic subalgebra* of $\mathfrak{g}$ is a subalgebra that contains $\mathfrak{q}_0$. As above, we define a parabolic subalgebra to be a subalgebra that is conjugate under $\mathrm{Int}(\mathfrak{g})$ to a standard parabolic one.

As in the complex case, the standard parabolic subalgebras of a real semisimple Lie algebra are parametrized by the set of subsets $I \subseteq \Delta$:

[9] A *composition* of an integer n is a way of writing n as the sum of a sequence of positive integers. Two compositions that differ in the order of their summands are considered different, i. e. the order of the terms does matter.

[10] To see this, note first that a corresponding statement holds in terms of algebraic groups, cf. [**63**, Exercise 8.4.6.(6)]. Then the result follows from the fact that any complex Lie group can be realized as the group of complex points of a complex algebraic group and this Lie group is simply connected if and only if the algebraic group is simply connected, see [**21**, Remark 6.3.2.1] and [**64**, Thm. 13].

PROPOSITION 4.7. *Let $\mathfrak{q}$ be a standard parabolic subalgebra of $\mathfrak{g}$. Then there exists a subset $I \subseteq \Delta$ such that $\mathfrak{q}$ is of the form*

$$\mathfrak{q}_I := \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \bigoplus_{\lambda \in \langle I \rangle \cup \Phi^+(\mathfrak{g}, \mathfrak{a}_0)} \mathfrak{g}_\lambda.$$

Moreover, for any subset $I \subseteq \Delta$, the subalgebra $\mathfrak{q}_I$ is a standard parabolic subalgebra of $\mathfrak{g}$.

PROOF. See [**34**, Prop. 7.76]. □

REMARK. As in the complex case, we have $\mathfrak{q}_0 = \mathfrak{q}_\emptyset$ and we say that $\mathfrak{q}_0$ is a *minimal parabolic subalgebra* of $\mathfrak{g}$.

Let us look at the structure of a parabolic subalgebra $\mathfrak{q}_I$ more closely. We set

$$\mathfrak{a}_I := \bigcap_{\lambda \in I} \ker \lambda \subset \mathfrak{a}_0,$$

$$\mathfrak{m}_I := \mathfrak{a}_I^\perp \oplus \mathfrak{m}_0 \oplus \bigoplus_{\lambda \in \langle I \rangle} \mathfrak{g}_\lambda,$$

$$\text{and} \quad \mathfrak{n}_I := \bigoplus_{\lambda \in \Phi^+ \setminus \langle I \rangle} \mathfrak{g}_\lambda,$$

where $\mathfrak{a}_I^\perp$ denotes the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{a}_0$ w. r. t. the restriction of the Killing form. These are Lie subalgebras of $\mathfrak{q}_I$ and we have $\mathfrak{q}_I = \mathfrak{m}_I \oplus \mathfrak{a}_I \oplus \mathfrak{n}_I$. We call this decomposition of $\mathfrak{q}_I$ the *Langlands decomposition*. Note that the minimal parabolic subalgebra decomposes as $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ by definition and this is exactly the Langlands decomposition of $\mathfrak{q}_0$.

The Lie subalgebras of $\mathfrak{q}_I$ occurring in the Langlands decomposition have the following properties: $\mathfrak{a}_I$ is abelian, $\mathfrak{n}_I$ is nilpotent and $\mathfrak{m}_I$ is reductive. If we assume that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_0$ (which exists by [**34**, Prop. 6.47]), then $\mathfrak{b}_I := \mathfrak{m}_I \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{m}_I$ and we have a decomposition $\mathfrak{h} = \mathfrak{a}_I \oplus \mathfrak{b}_I$ that is orthogonal w. r. t. the Killing form (cf. [**4**, Section 2.4]). In particular, there is an isomorphism $\mathfrak{h}_\mathbb{C}^* = (\mathfrak{a}_I)_\mathbb{C}^* \oplus (\mathfrak{b}_I)_\mathbb{C}^*$ and we can identify $(\mathfrak{a}_I)_\mathbb{C}^*$ (resp. $(\mathfrak{b}_I)_\mathbb{C}^*$) with the space of linear functionals on $\mathfrak{h}_\mathbb{C}$ that vanish on $(\mathfrak{b}_I)_\mathbb{C}$ (resp. on $(\mathfrak{a}_I)_\mathbb{C}$).

REMARK. To simplify notation, we will denote the Langlands decomposition of an arbitrary standard parabolic subgroup $\mathfrak{q}$ by $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and the corresponding Cartan subalgebra of $\mathfrak{m}$ by $\mathfrak{b}$.

EXAMPLE 4.8. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, considered as a real Lie algebra.

The map $\theta \colon X \mapsto -\overline{X}^t$ is a Cartan involution on $\mathfrak{g}$ that gives rise to a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} := \mathfrak{su}(n) = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ is the subalgebra of skew-hermitian matrices and $\mathfrak{p} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}$ is the subalgebra of hermitian matrices. A maximal abelian subalgebra of $\mathfrak{p}$ is given by the set of real diagonal matrices

$$\mathfrak{a}_0 = \{X \in \mathfrak{sl}_n(\mathbb{R}) \mid X = \operatorname{diag}(x_1, \dots, x_n)\},$$

and its centralizer in $\mathfrak{k}$ has the form

$$\mathfrak{m}_0 = \{X \in \mathfrak{sl}_n(\mathbb{C}) \mid X = \operatorname{diag}(x_1, \dots, x_n) \text{ and } \overline{X} = -X\},$$

which is the subalgebra of purely imaginary diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$.

Let us look at the system of restricted roots of $\mathfrak{g}$ w. r. t. $\mathfrak{a}_0$: We denote by $\delta_j : \mathfrak{a}_0 \to \mathbb{R}$ the linear functional given by $\delta_j(X) = x_j$. Then the restricted roots are of the form $\lambda_{jk} = \delta_j - \delta_k$ for $1 \le j, k \le n$ and $j \ne k$, and we have a positive system

$$\Phi^+(\mathfrak{g}, \mathfrak{a}_0) = \{\lambda_{jk} \mid j < k \text{ and } 1 \le j \le n-1\}$$

and the corresponding set of simple roots

$$\Delta(\mathfrak{g}, \mathfrak{a}_0) = \{\lambda_{j,j+1} \mid 1 \le j \le n-1\}.$$

Note that the restricted root spaces $\mathfrak{g}_\lambda$ are of (real) dimension 2 in this case. The minimal parabolic subalgebra $\mathfrak{q}_0$ w. r. t. this positive system of roots is the subalgebra of upper triangular matrices in $\mathfrak{sl}_n(\mathbb{C})$.

Now we consider an arbitrary standard parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$. By Proposition 4.7, such an algebra is of the form

$$\mathfrak{q} = \mathfrak{q}_I = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \bigoplus_{\lambda \in \langle I \rangle \cup \Phi^+(\mathfrak{g}, \mathfrak{a}_0)} \mathfrak{g}_\lambda$$

for a suitable subset $I \subseteq \Delta$. Noting that the (complex) Lie algebra $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ coincides with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_n(\mathbb{C})$ from Example 4.4 and that the restricted roots of $\mathfrak{sl}_n(\mathbb{C})$ w. r. t. $\mathfrak{a}_0$ are exactly the restrictions to $\mathfrak{a}_0$ of the roots of $\mathfrak{sl}_n(\mathbb{C})$ w. r. t. $\mathfrak{h}$, we see that the real parabolic subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ coincide with the complex parabolic subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ described in Example 4.4. In particular, the (real) standard parabolic subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ are the real subalgebras of the form

$$\mathfrak{q}_{\ell_1,\dots\ell_m} = \{X \in \mathfrak{g} \mid X = \begin{pmatrix} X_1 & * & * \\ & \ddots & * \\ 0 & & X_m \end{pmatrix}, \text{ where } X_j \in GL_{\ell_j}(\mathbb{C}),\ 1 \le j \le m\},$$

for a composition $n = \ell_1 + \ldots + \ell_m$.

Let $\mathfrak{q} = \mathfrak{q}_{\ell_1,\dots\ell_m}$ be such a real parabolic subalgebra. Then $\mathfrak{q}$ has a Langlands decomposition of the form $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where

$$\mathfrak{m} = \{X \in \mathfrak{q} \mid X = \operatorname{diag}(X_1, \ldots, X_m) \text{ and } \operatorname{tr}(X_j) \in i \cdot \mathbb{R} \text{ for all } 1 \le j \le m\},$$

$$\mathfrak{a} = \{X \in \mathfrak{q} \mid X = \operatorname{diag}(X_1, \ldots, X_m) \text{ and } X_j = x_j I_{\ell_j} \text{ for some } x_j \in \mathbb{R},\ 1 \le j \le m\}$$

$$\text{and } \mathfrak{n} = \{X \in \mathfrak{q} \mid X = \begin{pmatrix} 0 & * & * \\ & \ddots & * \\ 0 & & 0 \end{pmatrix}, \text{where the } j\text{-th 0-diagonal block is of size } \ell_j \times \ell_j\}.$$

4.9. Let us now assume that G is a connected semisimple real Lie group with Lie algebra $\mathfrak{g}$. A subgroup of G is said to be *parabolic* if it is the normalizer in G of a parabolic subalgebra of $\mathfrak{g}$. If $\mathfrak{q} \subset \mathfrak{g}$ is a parabolic subalgebra, we will write $Q = N_G(\mathfrak{q})$ for the associated parabolic subgroup of G. The Lie algebra of a parabolic subgroup is a parabolic subalgebra, since we have $N_{\mathfrak{g}}(\mathfrak{q}) = \mathfrak{q}$ for any parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$, cf. [**34**, Prop. 7.83]. However, unlike in the complex case, a parabolic subgroup is not necessarily connected.

Let $\mathfrak{q}$ be a standard parabolic subalgebra of $\mathfrak{g}$ having a Langlands decomposition $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then this decomposition gives rise to a similar decomposition of the parabolic subgroup Q (cf. [**34**, Prop. 7.82 and 7.83]): We let A and N denote the connected subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then there exists a subgroup M of G such that $Z_G(A) = M \times A$ and the parabolic subgroup Q decomposes as a (semidirect) product $Q = MAN$. Note that M is a reductive subgroup of G that is not necessarily connected and that M has Lie algebra $\mathfrak{m}$ (see [**34**, Prop. 7.82]).

The minimal parabolic subgroup Q_0 has the Langlands decomposition $Q_0 = M_0A_0N_0$, where the reductive group M_0 is of the form $Z_K(\mathfrak{a}_0)$ by [**34**, Prop. 7.27]. The groups A_0 and N_0 are simply connected (cf. [**34**, Thm. 6.46]) and since they are also nilpotent Lie groups, the exponential map yields diffeomorphisms $\mathfrak{a}_0 \xrightarrow{\cong} A_0$ and $\mathfrak{n}_0 \xrightarrow{\cong} N_0$. This property passes to the groups A and N occurring in the Langlands decomposition of an arbitrary standard parabolic subgroup $Q = MAN$: Since A and N are connected subgroups of the connected, simply connected nilpotent groups A_0 and N_0, respectively, we may apply [**34**, Cor. 1.111] to obtain the desired result.

4.10. We have seen in Example 4.8 that the (real) parabolic subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ coincide with its complex parabolic subalgebras. This is not a special property of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ but is true for any complex semisimple Lie algebra $\mathfrak{g}$: Such a Lie algebra is in particular a real semisimple Lie algebra, and we emphasize this by writing $\mathfrak{g}_{\mathbb{R}}$ whenever we consider $\mathfrak{g}$ with its real structure. Let θ be a Cartan involution of $\mathfrak{g}_{\mathbb{R}}$ and denote the corresponding Cartan decomposition by $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$. By [**34**, Cor. 6.22], $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$ and θ is the conjugation w. r. t. this real form.[11] This implies $\mathfrak{p} = i\mathfrak{k}$. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}$ and set $\mathfrak{m}_0 := Z_{\mathfrak{k}}(\mathfrak{a}_0)$.

LEMMA 4.11. *The algebra* $\mathfrak{h} := \mathfrak{a}_0 \oplus \mathfrak{m}_0$ *is a complex Cartan subalgebra of* $\mathfrak{g}$.

PROOF. We note that since $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}$, we have $Z_{\mathfrak{p}}(\mathfrak{a}_0) = \mathfrak{a}_0$. Now let $X \in \mathfrak{m}_0 = Z_{\mathfrak{k}}(\mathfrak{a}_0) = \{X \in \mathfrak{k} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{a}_0\}$. Since $\mathfrak{k} = i\mathfrak{p}$, we can write $X = iX'$ for some $X' \in \mathfrak{p}$. Then we have

$$\begin{aligned} X \in \mathfrak{m}_0 &\Leftrightarrow [iX', Y] = 0 \text{ for all } Y \in \mathfrak{a}_0 \\ &\Leftrightarrow [X', Y] = 0 \text{ for all } Y \in \mathfrak{a}_0 \Leftrightarrow X' \in Z_{\mathfrak{p}}(\mathfrak{a}_0) = \mathfrak{a}_0. \end{aligned}$$

This implies $\mathfrak{m}_0 = i\mathfrak{a}_0$ and so the subalgebra $\mathfrak{h} := \mathfrak{a}_0 \oplus \mathfrak{m}_0$ is complex. Moreover, $\mathfrak{m}_0$ is abelian and hence by [**34**, Prop. 6.47], $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. □

Let $\Phi(\mathfrak{g}, \mathfrak{h})$ denote the root system of $\mathfrak{g}$ w. r. t. $\mathfrak{h}$ and fix a positive system $\Phi^+(\mathfrak{g}, \mathfrak{h})$ and the corresponding set of simple roots $\Delta(\mathfrak{g}, \mathfrak{h})$. From the special structure of $\mathfrak{h} = \mathfrak{a}_0 \oplus i\mathfrak{a}_0$, we can deduce that the restricted roots of $\mathfrak{g}_{\mathbb{R}}$ w. r. t. $\mathfrak{a}_0$ are exactly the restrictions to $\mathfrak{a}_0$ of the elements in $\Phi(\mathfrak{g}, \mathfrak{h})$ and any root of $\mathfrak{g}$ w. r. t. $\mathfrak{h}$ arises from an element in $\Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_0)$ via complex linear extension. Thus we can set

$$\Phi^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_0) := \{\lambda \in \Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_0) \mid \lambda = \alpha|_{\mathfrak{a}_0} \text{ for some } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})\}$$

and the corresponding set of simple roots is given by

$$\Delta(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_0) = \{\lambda \in \Phi^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_0) \mid \lambda = \alpha|_{\mathfrak{a}_0} \text{ for some } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h})\}.$$

Moreover, if $\lambda \in \Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_0)$ is of the form $\lambda = \alpha|_{\mathfrak{a}_0}$ for some $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$, the restricted root space $\mathfrak{g}_\lambda$ coincides with the root space $\mathfrak{g}_\alpha$.

PROPOSITION 4.12. *The subalgebra* $\mathfrak{q}_0 := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_\alpha$ *of* $\mathfrak{g}$ *is a minimal complex parabolic subalgebra of* $\mathfrak{g}$ *and—considered as a real Lie algebra—a minimal real parabolic*

[11] Let $\mathfrak{g}$ be a complex Lie algebra with real form $\mathfrak{g}_0$, i. e. we have $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ as real Lie algebras. The *conjugation of* $\mathfrak{g}_{\mathbb{R}}$ *w. r. t.* $\mathfrak{g}_0$ is the $\mathbb{R}$-linear map on $\mathfrak{g}_{\mathbb{R}}$ that is 1 on $\mathfrak{g}_0$ and -1 on $i\mathfrak{g}_0$. This is a Lie algebra isomorphism of $\mathfrak{g}_{\mathbb{R}}$.

subalgebra of $\mathfrak{g}_{\mathbb{R}}$*. Furthermore, the standard complex parabolic subalgebras of* $\mathfrak{g}$ *w. r. t.* $\mathfrak{q}_0$ *coincide with the standard real parabolic subalgebras of* $\mathfrak{g}_{\mathbb{R}}$ *w. r. t.* $(\mathfrak{q}_0)_{\mathbb{R}}$*.*

PROOF. This follows from Lemma 4.11, Propositions 4.2 and 4.7 and the correspondence between the roots and root spaces of $(\mathfrak{g}, \mathfrak{h})$ and the restricted roots and root spaces of $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_0)$ described above. □

4.13. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{q}$ has a decomposition $\mathfrak{q} = Z_{\mathfrak{l}} \oplus \mathfrak{s} \oplus \mathfrak{n}$ if considered as a complex parabolic subalgebra and the Langlands decomposition $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ if considered as a real parabolic subalgebra. Let us take a closer look at the relation between these decompositions. Clearly, we have $Z_{\mathfrak{l}} \oplus \mathfrak{s} = \mathfrak{a} \oplus \mathfrak{m}$. Looking at the structure of these algebras given in Lemma 4.3 and Section 4.6, one can see that $Z_{\mathfrak{l}} = \mathfrak{a} \oplus i\mathfrak{a}$ and $\mathfrak{s} = [\mathfrak{m}, \mathfrak{m}]$. The decompositions restrict to the Cartan subalgebra $\mathfrak{h}$ as $\mathfrak{h} = Z_{\mathfrak{l}} \oplus (\mathfrak{h} \cap \mathfrak{s}) = \mathfrak{a} \oplus \mathfrak{b}$, where $\mathfrak{b} = \mathfrak{h} \cap \mathfrak{m}$. Since $\mathfrak{g}$ is a complex Lie algebra, we have seen in Lemma 4.11 above that $\mathfrak{m}_0 \subset \mathfrak{h}$. This implies $\mathfrak{b} = \mathfrak{a}^{\perp} \oplus \mathfrak{m}_0 = \mathfrak{a}^{\perp} \oplus i\mathfrak{a} \oplus i\mathfrak{a}^{\perp}$ and hence $\mathfrak{h} \cap \mathfrak{s} = \mathfrak{a}^{\perp} \oplus i\mathfrak{a}^{\perp}$.

4.14. The identification of real and complex parabolic subalgebras also transfers to the group level: Let G be a complex connected Lie group with Lie algebra $\mathfrak{g}$. The complex parabolic subgroups of G are the unique connected subgroups of G whose Lie algebras are parabolic subalgebras of $\mathfrak{g}$. The real parabolic subgroups of G are the normalizers of the parabolic subalgebras of $\mathfrak{g}$. However, since G has a complex structure, the groups $N_G(\mathfrak{q})$ are all connected (see [**18**, Lemma 1.4.6]) and hence coincide with the complex parabolic subgroups of G.

Let us look at the minimal parabolic subgroup Q_0. Considered as a real Lie group, it has a Langlands decomposition $Q_0 = M_0 A_0 N_0$, where A_0 and N_0 are connected and simply connected. Since G is also a complex Lie group, the group M_0 is connected as well, i. e. it is a maximal torus of the maximal compact subgroup K of G.[12]

Let $H = Z_G(\mathfrak{h})$. The group H is a so-called *Cartan subgroup of* G. Note that since $\mathfrak{h} = \mathfrak{a}_0 \oplus i\mathfrak{a}_0$, we have $Z_G(\mathfrak{h}) = Z_G(\mathfrak{a}_0) = M_0 A_0$ and hence H is connected.

5. Representation Theory of Real Reductive Lie Groups

In this section, we will review the most important notions from the representation theory of real reductive Lie groups, such as admissible and unitary representations, $(\mathfrak{g}, K)$-modules and infinitesimal characters. Standard references are Knapp [**33**] and Wallach [**69**].

5.1. We denote by G a real reductive Lie group and by $\mathfrak{g}$ its Lie algebra. In this section, we assume that G is in the *Harish-Chandra class.* This means that G is reductive and the connected semisimple subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ has finite center. Clearly, any connected, semisimple Lie group with finite center is of this type. We let K denote a maximal compact subgroup of G and we write $\mathfrak{k}$ for its Lie algebra.

[12] This can be seen as follows: The Lie algebra $\mathfrak{m}_0$ is a maximal abelian subalgebra of $\mathfrak{k}$. If we let M_0' denote the connected subgroup of K with Lie algebra $\mathfrak{m}_0$, this is by definition a maximal torus of K. By [**34**, Cor. 4.52], we have $Z_K(M_0') = M_0'$. This implies

$$M_0' = Z_K(M_0') = Z_K(\mathfrak{m}_0) = Z_K(i\mathfrak{a}_0) = Z_K(\mathfrak{a}_0) = M_0,$$

where we use that $M_0 = Z_K(\mathfrak{a}_0)$, cf. Section 4.9.

A (Hilbert space) *representation* of G is a pair (π, H_π) consisting of a Hilbert space H_π and a homomorphism $\pi\colon G \to \mathrm{Aut}(H_\pi)$ such that the map

$$G \times H_\pi \to H_\pi, \quad (g, v) \mapsto \pi(g)(v)$$

is continuous. We also say that H_π is a (topological) *G-module.* A representation (π, H_π) of G is *irreducible* if the only closed G-invariant subspaces of H_π are $\{0\}$ and H_π itself. Given two G-modules H_{π_1} and H_{π_2}, we say that they are *equivalent* if there exists a continuous linear operator $L\colon H_{\pi_1} \to H_{\pi_2}$ such that $L(\pi_1(g)(v)) = \pi_2(g)(L(v))$ for all $g \in G$ and $v \in H_{\pi_1}$ and L has a continuous linear inverse.

A *representation* of the Lie algebra $\mathfrak{g}$ is a pair (π, V_π) consisting of a complex vector space V_π and a Lie algebra homomorphism $\pi\colon \mathfrak{g} \to \mathrm{End}(V_\pi)$. The space V_π is also called a *$\mathfrak{g}$-module.* As above, it is said to be *irreducible* if the only $\mathfrak{g}$-invariant subspaces are $\{0\}$ and V_π.

A $(\mathfrak{g}, K)$-module is a $\mathfrak{g}$-module (π, V_π) endowed with a K-module structure[13], such that the following conditions are satisfied:

- V_π is *locally finite* as a K-module, i.e. each $v \in V_\pi$ is contained in a finite-dimensional K-stable subspace of V_π;
- For any K-stable finite-dimensional subspace $W \subset V_\pi$, the representation of K on W is differentiable and has $\pi|_{\mathfrak{k}}$ as its differential;
- We have $k \cdot (\pi(X)(v)) = \pi(\mathrm{Ad}(k)X)(k \cdot v)$, for all $k \in K$, $X \in \mathfrak{g}$ and $v \in V_\pi$.

We say that two $(\mathfrak{g}, K)$-modules V_{π_1} and V_{π_2} are equivalent if they are equivalent as K-modules via an operator $L\colon V_{\pi_1} \to V_{\pi_2}$ that is compatible with the $\mathfrak{g}$-module structure, that is, L satisfies $L(\pi_1(X)(v)) = \pi_2(X)(L(v))$ for all $X \in \mathfrak{g}$ and $v \in V_{\pi_1}$.

5.2. Let (π, H_π) be a representation of G. An element $v \in H_\pi$ is called *smooth* if the map $\pi_v\colon G \to H_\pi$, $\pi_v(g) := \pi(g)(v)$ is smooth. We denote by H_π^∞ the space of smooth vectors. The representation π can be restricted to H_π^∞ and by differentiation, this yields representations of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ on H_π^∞ that we will also denote by π (cf. [**33**, Prop. 3.9]).[14] A vector $v \in H_\pi$ is called *K-finite* if it is contained in a finite-dimensional subspace stable under K. Stated differently, this means that the set $\{\pi(k)(v) \mid k \in K\}$ spans a finite-dimensional subspace of H_π. We denote the set of K-finite vectors by $H_{\pi,K}$. Set $H_{\pi,K}^\infty := H_{\pi,K} \cap H_\pi^\infty$. This space is stable under the action of $\mathfrak{g}$, and one can show that it is even a $(\mathfrak{g}, K)$-module, the so-called *Harish-Chandra module* of the representation (π, H_π). Two representations of G are said to be *infinitesimally equivalent* if their underlying $(\mathfrak{g}, K)$-modules of K-finite smooth vectors are equivalent in the above sense.

The representation (π, H_π) is said to be *admissible* if the restriction $\pi|_K$ of π to K is unitary (cf. 5.4) and the multiplicity of any irreducible representation of K in $\pi|_K$ is finite. It turns out that irreducibility of an admissible representation implies the irreducibility of the corresponding $(\mathfrak{g}, K)$-module (which is not the case for arbitrary representations, cf. [**3**, §1, Example]):

---

[13]Here, the term K-module is to be understood only with respect to the group structure of K and not regarding the topology of K.

[14]By a representation of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ on H_π^∞ we mean a homomorphism $\mathcal{U}(\mathfrak{g}_\mathbb{C}) \to \mathrm{End}(H_\pi^\infty)$ of $\mathbb{C}$-algebras. It arises as the extension to $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ of the complexification of the $\mathfrak{g}$-action on H_π^∞, via the universal property of the universal enveloping algebra, cf. [**34**, Prop. 3.3].

PROPOSITION 5.3. *Let (π, H_π) be an admissible representation of G. Then (π, H_π) is irreducible if and only if the underlying $(\mathfrak{g}, K)$-module $H^\infty_{\pi,K}$ is irreducible. Moreover, any irreducible $(\mathfrak{g}, K)$-module is the Harish-Chandra module of an irreducible admissible representation of G.*

PROOF. See [**3**, Cor. 1 and Thm. 7]. □

5.4. Let (π, H_π) be a representation of G and denote by $\langle \cdot, \cdot \rangle_\pi$ the inner product on H_π. The representation is called *unitary* if we have

$$\langle \pi(g)(v), w \rangle_\pi = \langle v, \pi(g)(w) \rangle_\pi \quad \text{for all } v, w \in H_\pi, g \in G.$$

Two unitary representations (π_1, H_{π_1}) and (π_2, H_{π_2}) of G are said to be *unitarily equivalent* if they are equivalent via an operator L that is in addition unitary (i. e. that satisfies $\langle L(v), L(w) \rangle_{\pi_2} = \langle v, w \rangle_{\pi_1}$ for all $v, w \in H_{\pi_1}$). The set of unitary equivalence classes of irreducible unitary representations of G is called the *unitary dual of* G and is denoted by $\widehat{G}$.

It is a well-known result of Harish-Chandra that irreducible unitary representations are admissible (see [**69**, Thm. 3.4.10]) and that for such representations unitary equivalence is the same as infinitesimal equivalence:

PROPOSITION 5.5. *Two irreducible unitary representations of G are unitarily equivalent if and only if they are infinitesimally equivalent.*

PROOF. See [**3**, Thm. 6], or [**25**, Thm. 8] for the original proof. □

5.6. Let (π, H_π) be a representation of G. Recall that we denote by $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ the universal enveloping algebra of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$ and by $Z(\mathfrak{g}_\mathbb{C})$ its center. We have seen in paragraph 5.2 that π gives rise to a representation $\pi\colon \mathcal{U}(\mathfrak{g}_\mathbb{C}) \to \mathrm{End}(H^\infty_\pi)$. We say that π has *infinitesimal character* χ if there exists a homomorphism $\chi\colon Z(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$ such that

$$\pi(X) = \chi(X) \cdot \mathrm{id}_{H^\infty_\pi} \qquad \text{for each } X \in Z(\mathfrak{g}_\mathbb{C}).$$

We will denote the infinitesimal character of a representation π of G by $\chi(\pi)$.

LEMMA 5.7. *If (π, H_π) is an irreducible admissible representation, it has an infinitesimal character.*

PROOF. It follows from Proposition 5.3 that the Harish-Chandra module $H^\infty_{\pi,K}$ of π is irreducible as a $(\mathfrak{g}, K)$-module. In particular, $\pi\colon \mathcal{U}(\mathfrak{g}_\mathbb{C}) \to \mathrm{End}(H^\infty_{\pi,K})$ is an irreducible representation of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$. By a generalization of Schur's Lemma ([**34**, Prop. 5.19]), for each element $X \in Z(\mathfrak{g}_\mathbb{C})$, the map $\pi(X)\colon H^\infty_{\pi,K} \to H^\infty_{\pi,K}$ is $\mathcal{U}(\mathfrak{g})$-linear and hence of the form $\lambda_X \cdot \mathrm{id}_{H^\infty_{\pi,K}}$ for some $\lambda_X \in \mathbb{C}$. Since $H^\infty_{\pi,K}$ is a dense subspace of H^∞_π (cf. [**3**, Thm. 2]), $Z(\mathfrak{g}_\mathbb{C})$ acts by scalars on all of H^∞_π. Then the map

$$\chi\colon Z(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}, \quad \chi(X) := \lambda_X$$

is the desired infinitesimal character of π. □

REMARK. This result holds in particular for irreducible unitary representations because such representations are admissible, as we have seen above.

We will briefly recall Harish-Chandra's parametrization of infinitesimal characters by the Weyl group orbits of the dual of a Cartan subalgebra $\mathfrak{h}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$, as to be found in [**35**, Section IV.7 and IV.8]: Let $\mathfrak{h}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$ be a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ and denote by

$W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the Weyl group of the root system of $\mathfrak{g}_\mathbb{C}$ w.r.t. $\mathfrak{h}_\mathbb{C}$. With each element $\mu \in \mathfrak{h}_\mathbb{C}^*$, we can associate a character $\chi_\mu \colon Z(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$ by extending μ to $\mathcal{U}(\mathfrak{h}_\mathbb{C})$ and composing it with the Harish-Chandra homomorphism.[15] These characters have the property that $\chi_\mu = \chi_{\mu'}$ if and only if $\mu' = w \cdot \mu$ for some $w \in W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Moreover, they exhaust all characters of $Z(\mathfrak{g}_\mathbb{C})$:

PROPOSITION 5.8. *Every homomorphism $\chi \colon Z(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$ is of the form $\chi = \chi_\mu$ for some $\mu \in \mathfrak{h}_\mathbb{C}^*$.*

PROOF. See e.g. [**35**, Thm. 4.115]. □

In other words, the characters of $Z(\mathfrak{g}_\mathbb{C})$ are in one-to-one correspondence with the Weyl group orbits of $\mathfrak{h}_\mathbb{C}^*$.

EXAMPLE 5.9. Let (π, V) be an irreducible finite dimensional representation of G. We fix a system of positive roots $\Phi^+ := \Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and set $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. Furthermore, we denote by w_0 the longest element of the Weyl group $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

(1) If π is of highest weight $\lambda \in \mathfrak{h}_\mathbb{C}^*$, it has infinitesimal character $\chi(\pi) = \chi_{\lambda+\rho}$.
(2) Let $\tilde{\pi}$ denote the contragredient representation to π.[16] Then we have
$$\chi(\tilde{\pi}) = \chi_{-w_0(\lambda)+\rho} = \chi_{-(\lambda+\rho)}.$$
Here, we use the fact that the highest weight of $\tilde{\pi}$ is $-w_0(\lambda)$ (see [**12**, Cor. 3.2.14]) and that $w_0(\rho) = -\rho$.

5.10. Principal Series Representations. We will now define an important class of unitary representations of G that arise via induction from a parabolic subgroup. These are the so-called *(generalized) Principal series representations.*

We fix a minimal parabolic subgroup $Q_0 \subset G$ with Lie algebra $\mathfrak{q}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \mathfrak{n}_0$ and we denote the corresponding system of positive restricted roots by $\Phi^+(\mathfrak{g}, \mathfrak{a}_0)$. As in Section 4.6, we let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$ that contains $\mathfrak{a}_0$. Then we can choose a positive system of roots $\Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ in $\Phi(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ that is compatible with $\Phi^+(\mathfrak{g}, \mathfrak{a}_0)$, i.e. the restriction to $\mathfrak{a}_0$ of an element of $\Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is positive. As above, we set $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$.

Let $Q \supset Q_0$ be a parabolic subgroup of G with Langlands decomposition $Q = MAN$. Recall that A is simply connected and $\exp : \mathfrak{a} \to A$ is a diffeomorphism. For an element $a \in A$ we write $\log(a)$ to denote the inverse of a under exp. Given an irreducible unitary representation $\sigma : M \to \mathrm{Aut}(H_\sigma)$ and an element $\nu \in \mathfrak{a}_\mathbb{C}^*$, we define a representation $(\sigma \otimes (\nu + \rho|_{\mathfrak{a}_\mathbb{C}}) \otimes 1)$ of Q on H_σ by
$$(\sigma \otimes (\nu + \rho|_{\mathfrak{a}_\mathbb{C}}) \otimes 1)(man) := e^{(\nu+\rho|_{\mathfrak{a}_\mathbb{C}})(\log a)}\sigma(m).$$
By parabolic induction, we obtain a representation $(\pi_{Q,\sigma,\nu}, I_{Q,\sigma,\nu})$ of G, where
$$I_{Q,\sigma,\nu} := \{f \in C^\infty(G; H_\sigma) \mid f(man \cdot g) = e^{(\nu+\rho|_{\mathfrak{a}_\mathbb{C}})(\log a)}\sigma(m)f(g)\}$$
and G acts by right translations. The representations $I_{Q,\sigma,\nu}$ are called *principal series representations.*

[15]The *Harish-Chandra homomorphism* is an algebra homomorphism of $Z(\mathfrak{g}_\mathbb{C})$ into $\mathcal{U}(\mathfrak{h}_\mathbb{C})$. See [**35**, Section IV.7] for the precise definition and properties.

[16]The *contragredient* or *dual* representation to π is the representation $\tilde{\pi}$ of G on V^* given by $\tilde{\pi}(g)(\varphi) := \varphi \circ \pi(g^{-1})$ for all $\varphi \in V^*$.

Recall from Section 4.6 that $\mathfrak{b} := \mathfrak{h} \cap \mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{m}$. Then, by Lemma 5.7, an irreducible unitary representation of M has an infinitesimal character which is parametrized by an element of $\mathfrak{b}_{\mathbb{C}}^*$.

LEMMA 5.11. *Let $\sigma : M \to \mathrm{Aut}(H_\sigma)$ be an irreducible unitary representation and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. Let $\mu_\sigma \in \mathfrak{b}_{\mathbb{C}}^*$ be chosen such that $\chi(\sigma) = \chi_{\mu_\sigma}$. Then the following hold:*

- *The principal series representation $\pi_{Q,\sigma,\nu}$ is unitary if and only if $\nu \in i \cdot \mathfrak{a}^*$, i. e. ν is purely imaginary;*
- *The representation $\pi_{Q,\sigma,\nu}$ has infinitesimal character $\chi(\pi_{Q,\sigma,\nu}) = \chi_{\mu_\sigma + \nu}$.*

PROOF. The first statement follows from the fact that $(\sigma \otimes \nu \otimes 1)$ is a unitary representation of Q if and only if ν is purely imaginary. For the second statement, see [**4**, Lemma 2.5]. $\square$

6. Relative Lie Algebra Cohomology

In this section, we will recall the basic definitions from relative Lie algebra cohomology. We retain the notation of the previous section.

6.1. Let V be a $(\mathfrak{g}, K)$-module. Set $C^q(\mathfrak{g}, K; V) := \mathrm{Hom}_K(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), V)$, where K acts on $\mathfrak{g}/\mathfrak{k}$ by the adjoint representation. We define a differential $d^q : C^q(\mathfrak{g}, K; V) \to C^{q+1}(\mathfrak{g}, K; V)$ by

$$\begin{aligned}(d^q f)(X_0, \dots X_q) = &\sum_{i=0}^{q} X_i \cdot f(X_0, \dots \widehat{X}_i, \dots X_q) \\ &+ \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots X_q),\end{aligned}$$

where the $\widehat{\ }$ over an argument of f means that this argument is to be omitted. One can easily verify that $d^q \circ d^{q-1} = 0$. So $(C^*(\mathfrak{g}, K; V), d^*)$ is a cochain complex, whose cohomology is as usual given by

$$H^q(\mathfrak{g}, K; V) := \ker(d^q)/\operatorname{im}(d^{q-1}).$$

It is called the *relative Lie algebra cohomology* or $(\mathfrak{g}, K)$*-cohomology* of the $(\mathfrak{g}, K)$-module V. For a representation (π, H_π) of G, we have defined in 5.2 the associated $(\mathfrak{g}, K)$-module $H_{\pi,K}^\infty$ of K-finite, smooth vectors. We will occasionally use the notation $H^*(\mathfrak{g}, K; \pi)$ or $H^*(\mathfrak{g}, K; H_\pi)$ to refer to the $(\mathfrak{g}, K)$-cohomology of $H_{\pi,K}^\infty$.

For later reference, we give some properties of the $(\mathfrak{g}, K)$-cohomology groups. For $q > \dim \mathfrak{g}/\mathfrak{k}$, we have $\bigwedge^q \mathfrak{g}/\mathfrak{k} = 0$ and hence $H^q(\mathfrak{g}, K; V) = 0$. Moreover, one can easily see from the definitions that

$$\begin{aligned}H^0(\mathfrak{g}, K; V) &= \ker(d^0) \\ &= V^{\mathfrak{g},K} = \{v \in V \mid X.v = 0 \text{ for all } X \in \mathfrak{g} \text{ and } k.v = v \text{ for all } k \in K\}.\end{aligned} \tag{6.1}$$

Finally, let us state a vanishing result by Wigner that is essential for the classification of representations with nontrivial $(\mathfrak{g}, K)$-cohomology:

LEMMA 6.2 (Wigner). *Let U and V be two $(\mathfrak{g}, K)$-modules that have infinitesimal characters $\chi(U)$ and $\chi(V)$, respectively, and assume that U is finite-dimensional. Let $\tilde{U}$ denote the contragredient $(\mathfrak{g}, K)$-module*[17] *to U. If $\chi(\tilde{U}) \neq \chi(V)$, then*

$$H^q(\mathfrak{g}, K; U \otimes V) = 0$$

for all q.

PROOF. The result is originally due to Wigner. In the form stated here, it can be found in [**10**, Thm. 5.3(ii)]. □

6.3. The cohomology of the trivial representation. We conclude this section by studying the $(\mathfrak{g}, K)$-cohomology of the trivial representation, i. e. the cohomology groups $H^*(\mathfrak{g}, K; \mathbb{C})$. In many cases, this relative Lie algebra cohomology can be expressed as the cohomology of a compact symmetric manifold. To state the result, we need to introduce the notions of the compact dual group and the compact dual symmetric space.

DEFINITION 6.4. Let $G = \mathbf{G}(\mathbb{R})$ be the group of real points of a connected reductive algebraic $\mathbb{R}$-group $\mathbf{G}$. Then G is a real reductive Lie group and the group $G_{\mathbb{C}} := \mathbf{G}(\mathbb{C})$ is a complex connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, to be called the *complexification of G w. r. t. $\mathbf{G}$*.

The Lie algebra $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$. Let U denote the unique connected Lie subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{u}$. Then U is a maximal compact subgroup of $G_{\mathbb{C}}$ and there exists an embedding $K \hookrightarrow U$.[18] We say that U is the *compact dual group of G*. Set $X := K\backslash G$ and $X_u := K\backslash U$. Endowed with a suitable Riemannian metric, the quotient X_u is a symmetric space of compact type, the so-called *compact dual symmetric space of X*.

EXAMPLE 6.5.
(1) Let $\mathbf{G} = \mathbf{SL_n}$. Then we have $G = SL_n(\mathbb{R})$, $K = SO(n)$ and an associated Cartan decomposition $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R}) = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p} = \{X \in \mathfrak{sl}_n(\mathbb{R}) \mid X^t = X\}$. We have $G_{\mathbb{C}} = SL_n(\mathbb{C})$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_n(\mathbb{C})$. Then $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p} = \{X \in \mathfrak{sl}_n(\mathbb{C}) \mid \bar{X}^t = -X\}$ is the Lie algebra of the compact connected real Lie group $U := SU(n)$ and
$$X_u := SO(n)\backslash SU(n)$$
is the compact dual symmetric space of $X = SO(n)\backslash SL_n(\mathbb{R})$.
(2) Let $\mathbf{G} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{SL_n}$. Then $G = \mathbf{G}(\mathbb{R}) = SL_n(\mathbb{C})$, considered as a real Lie group. We have $K = SU(n)$, $G_{\mathbb{C}} \cong SL_n(\mathbb{C}) \times SL_n(\mathbb{C})$ and $U \cong SU(n) \times SU(n)$, where $K \hookrightarrow U$ is given by $x \mapsto (x, x)$. Thus,
$$X_u := SU(n)\backslash(SU(n) \times SU(n)) \cong SU(n)$$
is the compact dual symmetric space of $X = SU(n)\backslash SL_n(\mathbb{C})$.

[17] Let U be a $(\mathfrak{g}, K)$-module. The contragredient $(\mathfrak{g}, K)$-module $\tilde{U}$ is the space of K-finite vectors in the dual space U^* of U, where $\mathfrak{g}$ and K act on U^* via the usual contragredient representations.

[18] Since K is not necessarily connected, the existence of an embedding $K \hookrightarrow U$ is not completely obvious. A priori, only the connected component of the identity K^0 of K lies in U. However, the group $K' := U \cap G$ is a maximal compact subgroup of G with $K'^0 = K^0$. Then the result follows from the fact that for any maximal compact subgroup K of a reductive subgroup G, one has $N_G(K^0) = K$, which can be proved using the Cartan decomposition.

For a real reductive Lie group G arising as the group of real points of a real reductive algebraic group as in the definition above, we can now characterize the $(\mathfrak{g}, K)$-cohomology of the trivial representation as follows:

Proposition 6.6. *Let $G = \mathbf{G}(\mathbb{R})$ be the group of real points of a connected reductive algebraic group defined over $\mathbb{R}$ and denote by U the compact dual group of G. Then we have*

$$H^*(\mathfrak{g}, K; \mathbb{C}) = H^*(K\backslash U, \mathbb{C}).$$

Proof. See [**66**, Thm. 2.10]. □

Remark. In terms of symmetric spaces, the statement of the Proposition is equivalent to saying that the $(\mathfrak{g}, K)$-cohomology of the trivial representation is given by the cohomology of the compact dual symmetric space X_u. The problem is thus reduced to the determination of the cohomology of a symmetric space of compact type. This is well-known and can be found e. g. in [**24**, Ch.XI, §7].

Looking at our examples, this means that

$$H^*(\mathfrak{sl}_n(\mathbb{R}), SO(n); \mathbb{C}) = H^*(SO(n)\backslash SU(n); \mathbb{C})$$

and

$$H^*(\mathfrak{sl}_n(\mathbb{C}), SU(n); \mathbb{C}) = H^*(SU(n); \mathbb{C}).$$

We will come back to these examples in Sections IV.3 and IV.4.

7. Algebras with Involution and Hermitian Forms

In this section, we recall some notions and results from the theory of central simple algebras and hermitian forms. Standard references are [**36**] and [**55**].

7.1. An *involution* on a k-algebra A is a homomorphism $\sigma\colon A \to A$ of k-vector spaces such that $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2 = \mathrm{id}$. We say σ is an *involution of the first kind* if $\sigma|_{Z(A)} = \mathrm{id}$, and an *involution of the second kind* otherwise. Let k' be a splitting field of A and $\varphi\colon A \otimes_k k' \to M_r(k')$ the splitting isomorphism. For an involution σ of the first kind, the map $\varphi \circ (\sigma \otimes \mathrm{id}) \circ \varphi^{-1}$ on $M_r(k')$ is of the form $x \mapsto ux^tu^{-1}$ for some $u \in GL_r(k')$ with $u = \pm u^t$. Then σ is called *of symplectic type* if $u = -u^t$ and *of orthogonal type* if $u = u^t$. This notion is independent of the chosen splitting.

Let A be a k-algebra and σ an involution on A. The pair (A, σ) is called a *central simple algebra with involution* if it is one of the following:

- A is central simple over k and σ is an involution of the first kind,
- A is simple, $\ell := Z(A)$ is a quadratic extension of k with nontrivial Galois automorphism ι, and σ is an involution of the second kind with $\sigma|_\ell = \iota$,
- $A = A_1 \oplus A_2$, where A_i is central simple over k, $Z(A) \cong k \oplus k$ and σ is an involution of the second kind such that $\sigma|_{Z(A)}(k_1, k_2) = (k_2, k_1)$.

Remark. The notion of a central simple algebra with involution is a bit misleading, as such an algebra need not be simple nor central over k. However, this notion is commonly used in the literature, cf. [**29**] or [**36**].

Example 7.2. Let $Q = Q(a, b|F)$ be a quaternion algebra[19] over a field F.

[19] The *quaternion algebra* $Q(a, b|F)$ for $a, b \in F^\times$ is the four-dimensional F-algebra with basis 1, i, j, k and multiplication given by $i^2 = a$, $j^2 = b$ and $ij = -ji = k$.

(1) There is a canonical involution of the first kind and symplectic type, the so-called *conjugation*, on Q:

$$\tau_c\colon x_0 + ix_1 + jx_2 + kx_3 \mapsto x_0 - ix_1 - jx_2 - kx_3.$$

Moreover, with each of the basis elements i, j and k we associate an involution of orthogonal type, the *reversion* w. r. t. i, j and k:

$$\tau_i\colon x_0 + ix_1 + jx_2 + kx_3 \mapsto x_0 - ix_1 + jx_2 + kx_3,$$
$$\tau_j\colon x_0 + ix_1 + jx_2 + kx_3 \mapsto x_0 + ix_1 - jx_2 + kx_3,$$
$$\tau_k\colon x_0 + ix_1 + jx_2 + kx_3 \mapsto x_0 + ix_1 + jx_2 - kx_3.$$

(2) Let σ be an involution of the second kind on Q, such that $F_0 := Fix(\sigma, F)$ is a subfield of F of index 2. By a theorem of Albert ([**36**, Prop. 2.22]), there exists a unique quaternion algebra Q_0 over F_0 such that

$$(Q, \sigma) \cong (Q_0 \otimes_{F_0} F, \tau_{c,0} \otimes \iota),$$

where ι denotes the nontrivial Galois automorphism of F over F_0 and $\tau_{c,0}$ is the conjugation on Q_0.

Note, however, that not every quaternion algebra admits an involution of the second kind (for an example, see [**40**, Example 2.4]). In general, the existence of involutions (of the first or second kind) on a central simple algebra requires certain conditions, to be found e.g. in [**36**, §I.3]. We will come back to this question below, when we treat the case of central simple algebras over number fields.

When defining the notion of symplectic and orthogonal type, we have used the fact that any involution of the first kind on $M_r(k')$ is of the form ux^tu^{-1} for a symmetric or skew-symmetric element u. This is a special case of the following lemma:

LEMMA 7.3. *Let A be a central simple k-algebra with involution σ. If σ' is another involution on A that coincides with σ on $Z(A)$, there exists an element $u \in A^\times$, uniquely determined up to a factor in $k^\times$, such that*

$$\sigma' = \operatorname{Int} u \circ \sigma, \tag{7.1}$$

where $\sigma(u) = \pm u$ if σ is of the first kind and $\sigma(u) = u$ if σ is of the second kind.

Conversely, every map σ' defined as in (7.1) *is an involution on A of the same kind as σ. Moreover, for involutions of the first kind, σ' and σ are of the same type if and only if $\sigma(u) = u$.*

PROOF. See [**36**, Prop. 2.7 and Prop. 2.18]. □

7.4. Let (A, σ) be a central simple k-algebra with involution and let M be a right A-module. A *σ-hermitian form* on M is a map $h\colon M \times M \to A$ such that

(i) $h(x_1 + x_2\lambda, y) = h(x_1, y) + \sigma(\lambda)h(x_2, y)$,
(ii) $h(x, y_1 + y_2\mu) = h(x, y_1) + h(x, y_2)\mu$,
(iii) $\sigma(h(x, y)) = h(y, x)$,

for all $x, x_1, x_2, y, y_1, y_2 \in M$ and $\lambda, \mu \in A$. We will occasionally also need the notion of a *σ-skew-hermitian form.* This is a map $h :: \times M \to A$ that satisfies conditions (i), (ii) and (iii)': $\sigma(h(x, y)) = -h(y, x)$.

A (skew-)hermitian form h on M is called *nondegenerate* if the map

$$M \to \operatorname{Hom}_A(M, A), \quad x \mapsto h(x, \cdot\,)$$

is an isomorphism of A-modules. In case M is a free A-module of finite rank, this is equivalent to the familiar condition $h(x, y) = 0$ for all $y \in M$ implies $x = 0$. Unless explicitly stated otherwise, all our forms will be nondegenerate.

7.5. A special situation in which we will mostly be interested in is the case where A is a division algebra and M is a free A-module of finite rank. In this setting, many results are analogous to the theory of symmetric bilinear forms over fields.

Let (D, σ) be a central simple k-division algebra with involution, let M be a free right D-module of rank m and let $h\colon M \times M \to D$ be a σ-hermitian (or σ-skew-hermitian) form. The number m is called the *dimension* of h. Let $\mathcal{E} := \{e_1, \dots e_m\}$ be a basis of M over D. Then we can define the *matrix of h w. r. t. $\mathcal{E}$* to be the matrix $H := (h(e_i, e_j))_{i,j} \in M_m(D)$. Clearly, H is invertible if h is nondegenerate. For $x, y \in M$, one can easily see that

$$h(x, y) = \sigma(x)^t H y,$$

where $\sigma(x)$ should be understood as applying σ to each entry of x.

In the case of hermitian forms, we have a nice result concerning the structure of this matrix:

LEMMA 7.6. *Let h be a hermitian form on M. There exists a basis $\mathcal{E}$ of M over D such that the matrix H of h w. r. t. $\mathcal{E}$ is of the form $H = \operatorname{diag}(h_1, \dots, h_m)$, for some $h_i \in D$ satisfying $\sigma(h_i) = h_i$ for $1 \le i \le m$.*

PROOF. See [**55**, Ch. 7, Thm. 6.3]. □

7.7. Central Simple Algebras over Algebraic Number Fields. The theory of central simple algebras gets particularly interesting if we let k be a number field. We will give a very rough overview of the most important results. Details and proofs can be found in [**52**] and [**55**].

Let k be a number field and A a central simple algebra over k. For a place $v \in V(k)$ we set $A_v := A \otimes_k k_v$. This yields a map of the Brauer groups

$$\operatorname{Br}(k) \to \operatorname{Br}(k_v), \quad [A] \mapsto [A_v].$$

We say that A is *split* at v if $[A_v]$ is the trivial element in $\operatorname{Br}(k_v)$ and *ramified* at v otherwise. The completions k_v are local fields, whose Brauer groups are well-known:

LEMMA 7.8. *Let k be an algebraic number field and $v \in V(k)$. There exists a homomorphism $\operatorname{inv}_v\colon \operatorname{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, called the local invariant map. For a non-archimedean place v, the local invariant map is an isomorphism, so we have $\operatorname{Br}(k_v) \cong \mathbb{Q}/\mathbb{Z}$. At the archimedean places, we have*

$$\operatorname{Br}(k_v) \cong \begin{cases} \operatorname{Br}(\mathbb{R}) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z} & \text{if } v \text{ is a real place,} \\ \operatorname{Br}(\mathbb{C}) \cong 0 & \text{if } v \text{ is a complex place.} \end{cases}$$

PROOF. See [**52**, Thm. 31.8]. □

With the help of this lemma, we can associate with each place $v \in V(k)$ an element $\operatorname{inv}_v(A) := \operatorname{inv}_v([A_v]) \in \mathbb{Q}/\mathbb{Z}$, to be called the *local Hasse-invariant at v*. Obviously, we have $\operatorname{inv}_v(A) = 0$ at the complex places and $\operatorname{inv}_v(A) \in \{0, \frac{1}{2}\}$ at the real places. Moreover, one can show that $\operatorname{inv}_v(A) = 0$ for almost all $v \in V(k)$.

The relation between the Brauer group of an algebraic number field and the Brauer groups of the local fields is expressed in the following exact sequence:

THEOREM 7.9. *Let k be an algebraic number field. Then the sequence*

$$0 \to \mathrm{Br}(k) \xrightarrow{[A]\mapsto([A_v])_v} \bigoplus_{v\in V(k)} \mathrm{Br}(k_v) \xrightarrow{\sum_{v\in V(k)} \mathrm{inv}_v([A_v])} \mathbb{Q}/\mathbb{Z} \to 0$$

is exact.

PROOF. See [**32**, Thm. 8.26(2)]. □

COROLLARY 7.10. *A central simple algebra over a number field k is completely determined by its local Hasse-invariants. More precisely, given $\epsilon_v \in \mathbb{Q}/\mathbb{Z}$ for each $v \in V(k)$ such that $\epsilon_v = 0$ for almost all places including all complex places, $\epsilon_v \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ at the real places, and $\sum_{v\in V(k),\epsilon_v\neq 0} \epsilon_v = 0$, there exists a central simple k-algebra A with $\mathrm{inv}_v(A) = \epsilon_v$. Moreover, two central simple k-algebras are Brauer equivalent if and only if they have the same Hasse-invariants. In particular, A is split if and only if $\mathrm{inv}_v(A) = 0$ for all $v \in V(k)$.*[20]

7.11. Now we consider involutions on central simple algebras over number fields. Let ℓ/k be a quadratic extension and let B denote a central simple algebra over ℓ with an involution σ of the second kind. For a place $v \in V(k)$, set $B_v := B \otimes_k k_v$ and $\sigma_v := \sigma \otimes \mathrm{id}_{k_v}$. Then B_v is a k_v-algebra with center $\ell \otimes_k k_v$ and σ_v is an involution of the second kind on B_v.

LEMMA 7.12. *Let $v \in V(k)$ be a place of k that is decomposed in ℓ and denote by w_1 and w_2 the two places of ℓ lying above v. Then there exists an isomorphism*

$$(B_v, \sigma_v) \cong ((B_{w_1} \oplus B_{w_1}^{op}), (x, y^{op}) \mapsto (y, x^{op}))$$

of k_v-algebras.

PROOF. We have $\ell \otimes_k k_v = \bigoplus_{w|v} \ell_w = \ell_{w_1} \oplus \ell_{w_2} \cong k_v \oplus k_v$ (cf. [**51**, 1.1.2]) and hence

$$B_v \cong B \otimes_k k_v \cong B \otimes_\ell (\ell \otimes_k k_v) \cong (B \otimes_\ell \ell_{w_1}) \oplus (B \otimes_\ell \ell_{w_2}) = B_{w_1} \oplus B_{w_2}$$
$$\text{and} \quad Z(B_v) \cong k_v \oplus k_v.$$

Now it follows from the proof of [**36**, Prop. 2.14] that

$$(B_v, \sigma_v) \cong (B_{w_1} \oplus B_{w_1}^{op}, \varepsilon)$$

as k_v-algebras, where ε denotes the *exchange involution* $(x, y^{op}) \mapsto (y, x^{op})$. □

For an algebra B as in the lemma, to admit an involution of the second kind it is thus necessary to satisfy $B_{w_2} \cong B_{w_1}^{op}$ at all decomposed places. In terms of the Hasse-invariants, this means $\mathrm{inv}_{w_2}(B) = -\mathrm{inv}_{w_1}(B)$, since $[B_{w_1}^{op}]$ is the inverse element of $[B_{w_1}]$ in the Brauer group of $\ell_{w_1} \cong k_v$. In particular, B_{w_2} splits if and only if B_{w_1} splits.

The exact conditions for the existence of involutions of the first or second kind on central simple algebras over number fields are given in the following proposition:

PROPOSITION 7.13 (Existence of involutions). *Let k be a number field.*

(1) *If A is a central simple k-algebra, it admits an involution of the first kind if and only if either $A \cong M_r(k)$ or $A \cong M_r(D)$, for some $r \in \mathbb{N}$ and a quaternion division algebra D.*

[20] The last statement, which is actually the injectivity in the above sequence, is what was originally proved as the Brauer-Hasse-Noether Theorem.

(2) *Let ℓ/k be a quadratic extension and B a central simple algebra over ℓ. Then B admits an involution of the second kind if and only if*

$$\begin{aligned} \mathrm{inv}_w(B) = 0 &\quad \text{for all nondecomposed places } v \text{ of } k \text{ and } w|v, \\ \mathrm{inv}_{w_1}(B) + \mathrm{inv}_{w_2}(B) = 0 &\quad \text{for all decomposed places } v \text{ of } k \text{ and } w_1|v,\ w_2|v. \end{aligned}$$

PROOF. (1): Clearly, there exist involutions of the first kind on both $M_r(k)$ and $M_r(D)$, where D is a quaternion division algebra. For the other direction, note that an involution of the first kind is in particular an isomorphism of k-algebras $A \to A^{op}$. Therefore, the class of A is of order at most two in the Brauer group. If $[A]$ is of order one, we have $A \cong M_r(k)$. Thus, assume that $[A]$ is of order two. Over number fields, any algebra of order two is already Brauer equivalent to a quaternion division algebra, i. e. it is of the form $M_r(D)$ for a quaternion division algebra D. This follows from the fact that over a number field we have $\mathrm{ind}(A) = \exp(A)$, where $\exp(A)$ denotes the order of $[A]$ in the Brauer group (see [**52**, Thm. 32.19]). Therefore, we have $A \cong M_r(D)$ for a division algebra D of degree two, which is necessarily a quaternion algebra (this follows easily from the theory of central simple algebras or can be found explicitly in [**37**, Thm. 5.1]).

For (2), see [**55**, Ch. 10, Thm. 2.4]. □

The proposition implies that for an algebra with involution of the second kind, only the behaviour at decomposed places is of interest. The next lemma deals with this behaviour in the case of archimedean decomposed places. We will denote the nontrivial element of $\mathrm{Br}(\mathbb{R})$ by $\mathbb{H}$ – this is the quaternion algebra $Q(-1,-1|\mathbb{R})$, also called *Hamilton's quaternions.*

LEMMA 7.14. *Let ℓ/k be a quadratic extension and let B be a central simple algebra over ℓ with an involution σ of the second kind. Let v be a decomposed archimedean place of k and w_1, w_2 the two places of ℓ lying above v. Set*

$$\mathbb{K} := \begin{cases} \mathbb{C} & \text{if } v \text{ is complex,} \\ \mathbb{R} & \text{if } v \text{ is real and } B \text{ splits at } w_1 \text{ and } w_2, \\ \mathbb{H} & \text{if } v \text{ is real and } B \text{ ramifies at } w_1 \text{ and } w_2, \end{cases}$$

set $r := \deg(B_{w_1})\,\mathrm{ind}(B_{w_1})^{-1}$, and let $\omega = \mathrm{id}$ if $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\omega = \tau_c \colon \mathbb{H} \to \mathbb{H}$ if $\mathbb{K} = \mathbb{H}$.

Then there is an isomorphism of k_v-algebras with involution of the second kind

$$(B_v, \sigma_v) \cong (M_r(\mathbb{K}) \oplus M_r(\mathbb{K}), (x,y) \mapsto (\omega(y)^t, \omega(x)^t)).$$

PROOF. By Lemma 7.12, we have an isomorphism

$$(B_v, \sigma_v) \cong (B_{w_1} \oplus B_{w_1}^{op}, (x, y^{op}) \mapsto (y, x^{op})).$$

The definition of r and $\mathbb{K}$ implies that $B_{w_1} \cong M_r(\mathbb{K})$ as k_v-algebras. Moreover, the map $M_r(\mathbb{K})^{op} \to M_r(\mathbb{K}), x \mapsto \omega(x)^t$ is an isomorphism of k_v-algebras. Putting everything together, we get an isomorphism

$$B_v \to B_{w_1} \oplus B_{w_1}^{op} \to M_r(\mathbb{K}) \oplus M_r(\mathbb{K})^{op} \to M_r(\mathbb{K}) \oplus M_r(\mathbb{K}),$$

under which σ_v is identified with the involution $(x,y) \mapsto (\omega(y)^t, \omega(x)^t)$.

□

7.15. Let us consider the case of division algebras. For a k-division algebra D with involution σ, the center $Z(D) =: \ell$ is a field that equals k if σ is of the first kind and is a quadratic extension of k if σ is of the second kind. In this section, we will assume that σ is of the second kind.

Let $v \in V(k)$ be a place of k. Note that we have

$$D_v = D \otimes_\ell (\ell \otimes_k k_v) = D \otimes_\ell (\bigoplus_{w|v} \ell_w) = \bigoplus_{w|v} D_w$$

and D_v is a (not-necessarily simple) algebra over k_v with center $\ell \otimes_k k_v$ and an involution $\sigma_v := \sigma \otimes \mathrm{id}_{k_v}$ of the second kind. Moreover, for a right D-module M of rank m and a σ-hermitian form $h : M \times M \to D$ we set $M_v := M \otimes_k k_v$ and h extends to a σ_v-hermitian form $h_v : M_v \times M_v \to D_v$. We fix this notation for the rest of the section.

We will now define certain invariants of σ-hermitian forms and see that these give rise to a complete classification of σ-hermitian forms over D.

Definition 7.16. The *determinant* of the hermitian form h is defined by

$$\det(h) := \mathrm{Nrd}_{M_m(D)/\ell}(H) N_{\ell/k}(\ell^\times) \in k^\times / N_{\ell/k}(\ell^\times),$$

where $H \in M_m(D)$ is a matrix representation of h and $\mathrm{Nrd}_{M_m(D)/\ell}$ denotes the reduced norm of the central simple ℓ-algebra $M_m(D)$.[21]

Since $\det(h)$ takes values in k, we can look at the image $\det(h)_v$ under the embedding $k \hookrightarrow k_v$ for any place $v \in V(k)$. If D splits at v, we have $\det(h)_v = \det(H_v)$ by the definition of the reduced norm. At a real place v, the sign of $\det(h)_v$ is independent of the chosen coset representative.

Definition 7.17. Denote by d the degree of D over ℓ. An involution σ of the second kind on D is called *definite*, if for every real non-decomposed place v of k, we have an isomorphism $(D, \sigma)_v \cong (M_d(\mathbb{C}), {}^*)$, where $x \mapsto x^* = \bar{x}^t$ denotes the conjugate-transpose involution on $M_d(\mathbb{C})$.

In general, an involution of the second kind need not be definite. However, if there exists an involution of the second kind on D, there is also a definite one and when classifying σ-hermitian forms on D^m, it is no loss of generality to assume that the involution σ is definite:

Proposition 7.18. *Let σ be an arbitrary involution of the second kind on D. Then there exists an element $T \in D^\times$ with $T = \sigma(T)$ such that $\sigma_T = \mathrm{Int}(T) \circ \sigma$ is a definite involution. If h is a hermitian form w. r. t. σ, then the form Th is hermitian w. r. t. σ_T and this defines a functorial bijection between the category of σ-hermitian forms and the category of σ_T-hermitian forms.*

Proof. See [**55**, Ch. 10, Rem. 6.11]. ☐

For quaternion algebras, involutions of the second kind are of a special form due to Albert's theorem, cf. Example 7.2. For definite involutions, this yields an interesting result:

[21] This is well-defined: For any matrix representation H of h, we have $\sigma(H)^t = H$. This implies $\mathrm{Nrd}_{M_m(D)/\ell}(H) = \mathrm{Nrd}_{M_m(D)/\ell}(\sigma(H)^t) = \sigma(\mathrm{Nrd}_{M_m(D)/\ell}(H))$, and therefore $\mathrm{Nrd}_{M_m(D)/\ell}(H) \in k^\times$. Moreover, the coset is independent of the chosen matrix representation.

PROPOSITION 7.19. *Let ℓ/k be a quadratic field extension and let D be a quaternion division algebra over ℓ with a definite involution σ of the second kind. By Albert's Theorem, we can find $a, b \in k$ such that $(D, \sigma) \cong (Q(a, b|k) \otimes_k \ell, \tau_{c,0} \otimes \iota)$. Then the algebra $D_0 = Q(a, b|k)$ ramifies at any nondecomposed real place v of k.*

PROOF. Let v be a nondecomposed real place of k, denote by w the complex place of ℓ with $w|v$ and write a_v and b_v for the images of a and b under the embedding $k \hookrightarrow k_v \cong \mathbb{R}$. If D_0 splits at v, exactly one of the numbers a_v, b_v, $-a_v b_v$ is negative. Without loss of generality, let us assume that a_v and b_v are positive. Then we can define the following splitting isomorphism $D_0 \otimes_k k_v \to M_2(k_v)$:

$$\begin{aligned}
1 &\mapsto I_2 \\
i \otimes 1 &\mapsto \begin{pmatrix} \sqrt{a_v} & 0 \\ 0 & -\sqrt{a_v} \end{pmatrix} \\
j \otimes 1 &\mapsto \begin{pmatrix} 0 & \sqrt{b_v} \\ \sqrt{b_v} & 0 \end{pmatrix} \\
k \otimes 1 &\mapsto \begin{pmatrix} 0 & \sqrt{a_v b_v} \\ -\sqrt{a_v b_v} & 0 \end{pmatrix}.
\end{aligned}$$

By k-linear extension, this yields an isomorphism

$$D \otimes_k k_v \cong (D_0 \otimes_k k_v) \otimes_k \ell \to M_2(k_v) \otimes_k \ell \cong M_2(\ell_w) \cong M_2(\mathbb{C})$$

of k-algebras under which the involution $\tau_{c,0} \otimes \iota$ is mapped to $x \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) x^* \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since the signature occurring here is independent of the chosen splitting isomorphism, this contradicts the fact that σ is definite. $\square$

For a hermitian form h on D^m w. r. t. a definite involution, we can now define the *signature* at a nondecomposed real place v of k:

$$\operatorname{sign}_v(h) := \operatorname{sign}(H_v),$$

where $H_v \in M_{dm}(\mathbb{C})$ is a hermitian matrix representing h_v and sign denotes the ordinary signature[22] of hermitian matrices.

Now hermitian forms over D w. r. t. σ can be classified as follows:

THEOREM 7.20. *Let ℓ/k be a quadratic extension of number fields and let D be a central division algebra of degree d over ℓ endowed with a definite involution σ of the second kind.*

(1) Two hermitian forms h and h' over D are isomorphic if and only if $\dim(h) = \dim(h')$, $\det(h) = \det(h')$ and $\operatorname{sign}_v(h) = \operatorname{sign}_v(h')$ for all nondecomposed real places $v \in V(k)$.

(2) Let $m \in \mathbb{N}$, an element $\delta \in k^\times$ and integers s_v for each $v \in V(k)$ with v real and nondecomposed be given. Then there exists a hermitian form h over D with $\dim(h) = m$, $\det(h) = \delta \cdot N_{\ell/k}(\ell^\times)$ and $\operatorname{sign}_v(h) = s_v$ if and only if

- $|s_v| \le dm$,
- $s_v - dm \equiv 0 \mod 4$ *if* $\delta_v > 0$,
- $s_v - dm \equiv 2 \mod 4$ *if* $\delta_v < 0$,

at all nondecomposed real places v of k, and

- $\delta_v > 0$

[22] Let H be an invertible hermitian matrix with p positive and q negative eigenvalues. The *signature* of H is defined as $\operatorname{sign}(H) := p - q$. Depending on context, we may also denote the signature by the pair (p, q).

at all decomposed real places v such that D does not split at $w_1|v$ and $w_2|v$.

PROOF. See [**55**, Ch. 10, Cor. 6.6 and Thm. 6.9]. □

REMARK. The first condition in (2) is obvious, and the next two are equivalent to saying that if $\det(H_v)$ is positive (negative) then H_v has an even (odd) number of negative eigenvalues. The third condition is due to the Hasse-Schilling-Maass Norm Theorem, cf. [**52**, Thm. 33.15].

CHAPTER II

Geometric Cycles in Compact Locally Symmetric Spaces and How They Contribute to Cohomology

In this chapter, we give an overview of the method of Rohlfs and Schwermer for the geometric construction of nontrivial cohomology classes as developed in [**53**]. We start by defining the notions of geometric and special cycles in certain locally symmetric spaces. In Sections 2 and 3, we deal with orientability questions and define the intersection number of two compact geometric cycles of complementary dimension. This leads to the statement of the main result of Rohlfs and Schwermer in Theorem 3.7. In Section 4, we discuss the role of the compact dual symmetric space and in this context the non-G-invariance of the constructed cohomology classes. Finally, Section 5 is devoted to the construction of cocompact discrete subgroups of real or complex Lie groups, using the ideas of [**8**].

1. Geometric Cycles

1.1. Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{Q}$ and write G for its group of real points $\mathbf{G}(\mathbb{R})$. Then G is a real reductive Lie group with a maximal compact subgroup $K \subset G$ and we can form the associated symmetric space $X := K\backslash G$. Let now $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Then Γ is a discrete subgroup of G and it acts on the symmetric space X by right translations. By Example I.3.4 (2), we know that if Γ is a torsion-free arithmetic subgroup, the action on X is smooth, proper and free, and the quotient X/Γ is a Riemannian locally symmetric space.

From now on, let us assume that Γ is torsion-free. For a reductive $\mathbb{Q}$-subgroup $\mathbf{H}$ of $\mathbf{G}$, we can in the same way consider the real Lie group $H := \mathbf{H}(\mathbb{R})$ and the symmetric space $X_H := K_H\backslash H$, where K_H denotes a maximal compact subgroup of H. By Example I.3.4 (3), there exists an element $x_H \in X$ that is stable under the action of K_H on X by right translations such that the map $X_H \to X$, $K_H h \mapsto x_H h$ is a closed embedding that identifies X_H with a totally geodesic submanifold of X. Setting $\Gamma_H := \mathbf{H}(\mathbb{Q}) \cap \Gamma$ we also get a map

$$j_{H|\Gamma} : X_H/\Gamma_H \to X/\Gamma.$$

It is known that in case $\mathbf{H}$ is connected, this map is proper and - possibly after passing to a subgroup of finite index in Γ - an injective immersion ([**58**, 6.1, 6.5]). Thus, we have the following result:

Lemma 1.2. *Let $\mathbf{G}$ be a connected reductive algebraic $\mathbb{Q}$-group, $\mathbf{H} \subset \mathbf{G}$ a connected reductive subgroup, and Γ a torsion-free arithmetic subgroup of $\mathbf{G}$. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index, such that the map $j_{H|\Gamma'} : X_H/\Gamma'_H \to X/\Gamma'$ is a closed embedding.*

In particular, we may assume without loss of generality that X_H/Γ_H is an embedded, totally geodesic submanifold of X/Γ which we call a *geometric cycle*.

1.3. Let now $\boldsymbol{\mu}\colon \mathbf{G} \to \mathbf{G}$ be a $\mathbb{Q}$-rational automorphism of finite order of $\mathbf{G}$. Then $\mathbf{H} := \mathrm{Fix}(\boldsymbol{\mu}, \mathbf{G})$ is a reductive $\mathbb{Q}$-subgroup of $\mathbf{G}$ (see [**50**, Ch. 4, § 4.8]). Although $\mathbf{H}$ is not necessarily connected, the injectivity of the map $j_{H|\Gamma}$ is not an issue in this case (see [**58**, 6.4]). A geometric cycle arising from such a group $\mathbf{H}$ of fixed points is called a *special cycle* and will be denoted by $C(\boldsymbol{\mu}, \Gamma)$ or just $C(\boldsymbol{\mu})$.

A setting in which special cycles naturally occur is the analysis of the fixed point set of a finite abelian group of $\mathbb{Q}$-rational automorphisms of $\mathbf{G}$, acting on X/Γ. More precisely, let Θ be a finite group of commuting $\mathbb{Q}$-rational automorphisms of $\mathbf{G}$ of finite order and assume that K and Γ are invariant under Θ.[1] Then the group Θ acts on X and on X/Γ and the connected components of $\mathrm{Fix}(\Theta, X/\Gamma)$ are parametrized by the first nonabelian Galois cohomology set[2] of Θ in Γ: For any cocycle γ of Θ in Γ, there is a γ-twisted Θ-action on X, $\mathbf{G}$ and Γ (cf. Appendix C), and we denote the fixed point sets of this action by $X(\gamma)$, $\mathbf{G}(\gamma)$ and $\Gamma(\gamma)$, respectively. Note that we have $X(\gamma) \cong K(\gamma)\backslash \mathbf{G}(\gamma)(\mathbb{R})$. The natural map $X(\gamma)/\Gamma(\gamma) \to X/\Gamma$ is injective and its image, to be called $F(\gamma)$, is a nonempty, connected, totally geodesic submanifold of X/Γ. It can be shown that $F(\gamma)$ only depends on the class in $H^1(\Theta, \Gamma)$ represented by γ and that there is the following decomposition (see e.g. [**53**, 1.1] or [**58**, 6.4]):

$$\mathrm{Fix}(\Theta, X/\Gamma) = \bigsqcup_{[\gamma]\in H^1(\Theta,\Gamma)} F(\gamma).$$

If $\Theta = \langle \boldsymbol{\mu} \rangle$, the group generated by a single automorphism $\boldsymbol{\mu}$ of $\mathbf{G}$ of finite order, each of the $F(\gamma)$ is a special cycle, associated with the nontrivial automorphism obtained from twisting $\boldsymbol{\mu}$ with γ. The component $F(\gamma)$ corresponding to the trivial element in $H^1(\Theta, \Gamma)$ coincides with the above defined special cycle $C(\boldsymbol{\mu})$ arising from the group of fixed points of $\boldsymbol{\mu}$ in $\mathbf{G}$.

1.4. Let us now assume that Θ is the group generated by two commuting $\mathbb{Q}$-rational automorphisms $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ of $\mathbf{G}$ of finite order and that K and Γ are Θ-invariant. We denote by $\mathrm{res}_i\colon H^1(\Theta, \Gamma) \to H^1(\langle \boldsymbol{\tau}_i \rangle, \Gamma)$ the usual restriction maps in Galois cohomology, for $i \in \{1, 2\}$. The intersection of the associated special cycles $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ has finitely many connected components that can be parametrized by a subset of $H^1(\Theta, \Gamma)$ as in [**53**, Prop. 1.2]:

LEMMA. *For any* $[\gamma] \in \ker(\mathrm{res}_1 \times \mathrm{res}_2)$*, the set* $F(\gamma) \subset \mathrm{Fix}(\Theta, \Gamma)$ *is a connected component of* $C(\boldsymbol{\tau}_1) \cap C(\boldsymbol{\tau}_2)$ *and the map* $[\gamma] \mapsto F(\gamma)$ *yields a bijective correspondence between* $\ker(\mathrm{res}_1 \times \mathrm{res}_2)$ *and the connected components of* $C(\boldsymbol{\tau}_1) \cap C(\boldsymbol{\tau}_2)$.

2. Orientability

The symmetric spaces X and X_H are naturally oriented manifolds, since they are homeomorphic to Euclidean space. In this section, we study the orientability of the locally symmetric space X/Γ and of the embedded geometric cycles. Orientability questions will be essential for the definition and determination of the intersection number of two special cycles (cf. Section 3).

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[1] Such a choice of K and Γ is possible: For K, this follows from [**26**, Thm. 13.5]. For Γ, set $\Gamma' := \bigcap_{\theta\in\Theta} \theta(\Gamma)$. Then Γ' is of finite index in Γ and stable under Θ.

[2] See Appendix C for details on nonabelian Galois cohomology and twisting.

2.1. We have seen in Proposition I.3.6 and Lemma I.3.7 that for a discrete subgroup Γ of a Lie group G, acting on the symmetric space $K\backslash G$, the quotient $K\backslash G/\Gamma$ will be oriented if Γ is contained in G^0, the connected component of the identity of G. In our situation, this can be achieved by passing to a subgroup of finite index of Γ, as shown by Schwermer and Rohlfs in [**53**]. To state the result, we need the notion of a congruence subgroup: Let Γ be an arithmetic subgroup of an algebraic $\mathbb{Q}$-group $\mathbf{G}$. A subgroup $\Gamma' \subset \Gamma$ is called a *congruence subgroup of* Γ, if there exist a finite set S of primes and an open compact subgroup $U \subset \prod_{p \in S} G(\mathbb{Q}_p)$ such that $\Gamma' = \Gamma \cap U$ under the diagonal embedding.

Lemma 2.2. *Let $\mathbf{G}$ be an algebraic $\mathbb{Q}$-group and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Then there exists a congruence subgroup Γ' of Γ such that $\Gamma' \subset \mathbf{G}(\mathbb{R})^0$.*

Proof. See [**53**, Prop. 2.2]. □

For reductive subgroups $\mathbf{H}_1, \ldots, \mathbf{H}_k$ of $\mathbf{G}$ we can thus find congruence subgroups Γ_i of Γ such that $\Gamma_i \subset \mathbf{H}_i(\mathbb{R})^0$ for all $i = 1, \ldots, k$. If these groups are defined by open compact subsets U_i, the congruence subgroup Γ' of Γ, associated with $\cap_{i=1}^k U_i$, will satisfy $\Gamma' \subset \mathbf{H}_i(\mathbb{R})^0$ for all i. In other words, for a finite number of geometric cycles X_{H_i}/Γ_{H_i} in X/Γ, we can always find an arithmetic subgroup $\Gamma' \subset \Gamma$, such that X/Γ' and all cycles X_{H_i}/Γ'_{H_i} are orientable.

2.3. Now we concentrate on the case of special cycles. For the convenience of the reader, we recall the proof of the following lemma (cf. [**53**, Prop. 2.3]):

Lemma 2.4. *Let Θ be a finite group of $\mathbb{Q}$-rational automorphisms of $\mathbf{G}$ and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a Θ-stable arithmetic subgroup. Then there exists a normal, Θ-stable, torsion-free arithmetic subgroup Γ' of Γ such that for all cocycles $\gamma \in Z^1(\Theta, \Gamma')$ one has $\Gamma'(\gamma) \subset \mathbf{G}(\gamma)(\mathbb{R})^0$.*

Proof. Since $H^1(\Theta, \Gamma)$ is finite, we can choose a finite set of cocycles $\gamma^1, \ldots, \gamma^t$ representing the classes in $H^1(\Theta, \Gamma)$. Using Lemma 2.2 and the remark afterwards, we find a congruence subgroup $\Gamma_0 \subset \Gamma$ such that

$$\Gamma_0 \cap \mathbf{G}(\gamma^i)(\mathbb{Q}) \subset \mathbf{G}(\gamma^i)(\mathbb{R})^0, \quad \text{for } i = 1, \ldots, t.$$

Moreover, Γ_0 can be chosen torsion-free (see e. g. [**51**, Lemma 4.19]). Now we set $\Gamma_1 := \bigcap_{s \in \Theta} s(\Gamma_0)$ and $\Gamma' := \bigcap_{x \in \Gamma/\Gamma_1} x \Gamma_1 x^{-1}$. Then Γ' is a Θ-stable, torsion-free, normal subgroup of Γ of finite index, and in particular an arithmetic subgroup of $\mathbf{G}$.

Let now $\gamma \in Z^1(\Theta, \Gamma')$ be a cocycle for Γ'. We need to show $\Gamma'(\gamma) \subset \mathbf{G}(\gamma)(\mathbb{R})^0$. Since γ is also a cocycle for Γ, we find an $i \in \{1, \ldots, t\}$ such that γ is equivalent to γ^i, i. e. there exists $\alpha \in \Gamma$ such that $\gamma^i_s = \alpha^{-1} \gamma_s {}^s\alpha$ for all $s \in \Theta$. Then the map $\mathrm{Int}(\alpha)$ yields an isomorphism $\mathbf{G}(\gamma^i)(\mathbb{R})^0 \cong \mathbf{G}(\gamma)(\mathbb{R})^0$ that maps the subgroup $\Gamma' \cap \mathbf{G}(\gamma^i)(\mathbb{Q}) \subset \mathbf{G}(\gamma^i)(\mathbb{R})^0$ onto $\Gamma'(\gamma)$. Hence the claim follows. □

For two commuting $\mathbb{Q}$-rational automorphisms $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ of finite order of $\mathbf{G}$ and the associated cycles $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$, we can now draw the following corollary:

Corollary 2.5. *Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$-stable arithmetic subgroup of $\mathbf{G}$. Then there exists a torsion-free, $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$-stable subgroup $\Gamma' \subset \Gamma$ of finite index such that X/Γ', $C(\boldsymbol{\tau}_1, \Gamma')$, $C(\boldsymbol{\tau}_2, \Gamma')$ and all connected components of $C(\boldsymbol{\tau}_1, \Gamma') \cap C(\boldsymbol{\tau}_2, \Gamma')$ are orientable.*

PROOF. We use Lemma 2.2 and Lemma 2.4 together with the fact that the connected components of $C(\boldsymbol{\tau}_1, \Gamma') \cap C(\boldsymbol{\tau}_2, \Gamma')$ are of the form $F(\gamma) \cong X(\gamma)/\Gamma(\gamma)$ for certain cocycles $\gamma \in Z^1(\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle, \Gamma')$. $\square$

3. Intersection Numbers and Cohomology

3.1. In this section, we recall the general theory of intersection numbers of compact oriented manifolds. Let M be a compact, connected oriented manifold of dimension m and M_1 and M_2 connected oriented submanifolds of dimensions m_1 and m_2 respectively, such that $m_1 + m_2 = m$. We denote by $j_i \colon M_i \to M$ the inclusion maps and by $(j_i)_* \colon H_*(M_i) \to H_*(M)$ the induced homomorphisms in homology, for $i \in \{1, 2\}$. Furthermore, we write $[M_i]$ and $(j_i)_*[M_i]$ for the fundamental homology class of M_i in $H_{m_i}(M_i; \mathbb{Z})$ and its image under $(j_i)_*$, respectively. Since M is compact, we have the Poincaré-duality isomorphism $D \colon H^k(M; \mathbb{Z}) \xrightarrow{\sim} H_{m-k}(M; \mathbb{Z})$.

Now we can define the *intersection number* of $[M_1]$ and $[M_2]$ as follows:

$$[M_1][M_2] := (j_1)_*[M_1](j_2)_*[M_2] := \langle D^{-1}((j_1)_*[M_1]) \cup D^{-1}((j_2)_*[M_2]), [M] \rangle.$$

Here, $\langle \cdot, [M] \rangle \colon H^m(M; \mathbb{Z}) \to \mathbb{Z}$ denotes the evaluation of a cohomology class at the fundamental homology class $[M] \in H_m(M; \mathbb{Z})$.

EXAMPLE 3.2. Two submanifolds M_1 and M_2 of M as above are said to intersect *transversally* if for each $x \in M_1 \cap M_2$, we have $T_x M_1 + T_x M_2 = T_x M$. In particular, if $M_1 \cap M_2 \neq \emptyset$, such submanifolds intersect in a finite number of points since they are of complementary dimension in M. In this case, one has the following formula for the intersection number:

$$[M_1][M_2] = \sum_{x \in M_1 \cap M_2} \varepsilon(x),$$

where $\varepsilon(x) := \begin{cases} +1 & \text{if } T_x M_1 \oplus T_x M_2 = T_x M \text{ as oriented vector spaces,} \\ -1 & \text{else.} \end{cases}$

This follows from the definition of the intersection number, the fact that for transversal intersections one has

$$D^{-1}((j_1)_*[M_1]) \cup D^{-1}((j_2)_*[M_2]) = D^{-1}((j_1)_*[M_1 \cap M_2]),$$

and the knowledge of the fundamental class of a finite set of points.

In general, two submanifolds of complementary dimension need not intersect transversally and the determination of their intersection number is a more complicated issue. In the case of two special geometric cycles, Rohlfs and Schwermer have carried out a detailed analysis in [**53**, Section 3]. We briefly summarize their main results.

3.3. As above, let M_1 and M_2 be connected oriented submanifolds of the compact, connected oriented manifold M that are of complementary dimension and assume that their intersection is a connected orientable submanifold N of M of dimension n. We say that M_1 and M_2 *intersect perfectly*, if $TN = TM_1|_N \cap TM_2|_N$. Here, $TM_1|_N$ and $TM_2|_N$ are considered as subbundles of $TM|_N$, the restriction of the tangent bundle of M to N. If the manifolds intersect perfectly, we can define an n-dimensional vector bundle η over N, the so-called *excess bundle of* N, by the following exact sequence of bundles:

$$0 \to TM_1|_N + TM_2|_N \to TM|_N \to \eta \to 0.$$

Here, $TM_1|_N + TM_2|_N$ denotes the bundle whose fibre over a point $x \in N$ consists of the span of T_xM_1 and T_xM_2 in T_xM. Note that if M_1 and M_2 intersect transversally, the excess bundle will be the zero bundle.

In case that the intersection is not transverse, i. e. $n \geq 1$, we can choose an orientation on N and thus make η an oriented vector bundle in a natural way. We denote its *Euler class* by $e(\eta) \in H^n(N;\mathbb{Z})$ and its *Euler number* by $\langle e(\eta), [N]\rangle \in \mathbb{Z}$. In case $n = 0$, we set $\langle e(\eta), [N]\rangle := \varepsilon(x)$, where $N = \{x\}$ and $\varepsilon(x)$ is defined as in Example 3.2. Then we can express the intersection number of M_1 and M_2 as follows:

PROPOSITION 3.4. *Let M, M_1, M_2 and $N := M_1 \cap M_2$ be as above; in particular assume that M_1 and M_2 are of complementary dimension and intersect perfectly. Then we have*

$$[M_1][M_2] = \langle e(\eta), [N]\rangle.$$

PROOF. See [**53**, Prop. 3.3]. □

REMARK. Note that this coincides with Example 3.2 in case of a transversal intersection.

3.5. Let $\mathbf{G}$ be an algebraic $\mathbb{Q}$-group, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ two commuting $\mathbb{Q}$-rational automorphisms of finite order of $\mathbf{G}$ and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ a torsion-free, $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle$-stable arithmetic subgroup that is cocompact, i. e. such that $\mathbf{G}(\mathbb{R})/\Gamma$ is compact. Then, by Corollary 2.5, we may assume that the associated cycles $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are connected, oriented totally geodesic submanifolds of the compact oriented locally symmetric space X/Γ, such that all connected components of $C(\boldsymbol{\tau}_1) \cap C(\boldsymbol{\tau}_2)$ are oriented. By [**53**, Lemma 1.4] the cycles intersect perfectly. If we moreover suppose that $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are of complementary dimension in X/Γ, we can apply the above considerations to each connected component of the intersection of $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$. Recall that these are parametrized by the set $\ker(\mathrm{res}_1 \times \mathrm{res}_2) \subset H^1(\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle, \Gamma)$ (cf. 1.4). For each component $F(\gamma)$ we denote the excess bundle by $\eta(\gamma)$ and get the following result ([**53**, Prop. 3.5]):

PROPOSITION 3.6. *Let $\mathbf{G}$, $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ and Γ be as above and assume that $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are of complementary dimension in X/Γ. Then we have*

$$[C(\boldsymbol{\tau}_1)][C(\boldsymbol{\tau}_2)] = \sum_{\gamma \in \ker(\mathrm{res}_1 \times \mathrm{res}_2)} \langle e(\eta(\gamma)), [F(\gamma)]\rangle.$$

In other words, the intersection number of $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ is the sum of the Euler numbers of the connected components of the intersection. In [**53**], Rohlfs and Schwermer have analyzed the contribution of each connected component and have shown that the intersection number is nontrivial under certain assumptions. In this context, one has to deal with deep orientability questions.

THEOREM 3.7. *Let $\mathbf{G}$ be a reductive algebraic $\mathbb{Q}$-group, let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be $\mathbb{Q}$-rational automorphisms of $\mathbf{G}$ of finite order, and let Γ be a torsion-free, $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle$-stable, cocompact arithmetic subgroup of $\mathbf{G}$ that is chosen as in Corollary 2.5. Suppose that the associated cycles $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are of complementary dimension. Assume that*

(i) the real Lie groups $\mathbf{G}(\mathbb{R})$, $\mathrm{Fix}(\boldsymbol{\tau}_1, \mathbf{G})(\mathbb{R})$ and $\mathrm{Fix}(\boldsymbol{\tau}_2, \mathbf{G})(\mathbb{R})$ act orientation-preserving on X, $X(\boldsymbol{\tau}_1)$ and $X(\boldsymbol{\tau}_2)$ respectively,
(ii) the group $\mathrm{Fix}(\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle, \mathbf{G})(\mathbb{R})$ is compact.

Then there exists a $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle$-stable normal subgroup $\Gamma' \subset \Gamma$ of finite index such that

$$[C(\boldsymbol{\tau}_1, \Gamma')][C(\boldsymbol{\tau}_2, \Gamma')] \neq 0.$$

Proof. See [**53**, Thm. 4.11]. □

Remark.
- Condition (i) ensures that all fixed point components contribute with the same sign. In fact, this requirement is quite restricting and we will later see natural choices for $\mathbf{G}$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, where it is not met. Note that the condition is satisfied if $\mathbf{G}(\mathbb{R})$, $\mathrm{Fix}(\boldsymbol{\tau}_1, \mathbf{G})(\mathbb{R})$ and $\mathrm{Fix}(\boldsymbol{\tau}_2, \mathbf{G})(\mathbb{R})$ are connected.
- Condition (ii) is equivalent to saying that $X(\boldsymbol{\tau}_1)$ and $X(\boldsymbol{\tau}_2)$ intersect in exactly one point. Note that the intersection number in this point can be chosen to be positive by orienting the symmetric spaces in a suitable way. Below we will see a setting in which the condition is always satisfied.

The intersection number of two homology classes α, β is nontrivial if and only if $\alpha \neq 0$ and $\beta \neq 0$. Applied to our situation, this means that the nonvanishing of the intersection number implies the existence of certain (co-)homology classes of the arithmetic quotient:

Corollary 3.8. *In the situation of Theorem 3.7 there exists a $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$-stable normal subgroup $\Gamma' \subset \Gamma$ of finite index such that*

$$H^k(X/\Gamma', \mathbb{C}) \neq 0,$$

for $k \in \{\dim X(\boldsymbol{\tau}_1), \dim X(\boldsymbol{\tau}_2)\}$.

Proof. This is a direct consequence of Theorem 3.7 and Poincaré duality. □

Remark. These results carry over to finite index subgroups: Let $\boldsymbol{\mu} \in \{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$. For Γ' as in the Theorem, we have $(j_{\Gamma'})_*[C(\boldsymbol{\mu}, \Gamma')] \neq 0$. Let $\Delta \subset \Gamma'$ be a $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$-stable normal subgroup of Γ' of finite index r. Then we also have $(j_\Delta)_*[C(\boldsymbol{\mu}, \Delta)] \neq 0$. To see this, note that $p\colon X/\Delta \to X/\Gamma'$ is a normal covering map of degree r that restricts to a covering map $p|_{C(\boldsymbol{\mu})}\colon C(\boldsymbol{\mu}, \Delta) \to C(\boldsymbol{\mu}, \Gamma')$. In particular,

$$p_*([X/\Delta]) = r \cdot [X/\Gamma'] \quad \text{and} \quad (p|_{C(\boldsymbol{\mu})})_*([C(\boldsymbol{\mu}, \Delta)]) = r \cdot [C(\boldsymbol{\mu}, \Gamma')].$$

From the commutative diagram

$$\begin{array}{ccc} H_*(C(\boldsymbol{\mu}, \Delta)) & \xrightarrow{(j_\Delta)_*} & H_*(X/\Delta) \\ {\scriptstyle (p|_{C(\boldsymbol{\mu})})_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ H_*(C(\boldsymbol{\mu}, \Gamma')) & \xrightarrow{(j_{\Gamma'})_*} & H_*(X/\Gamma') \end{array}$$

we deduce

$$p_*((j_\Delta)_*[C(\boldsymbol{\mu}, \Delta)]) = (j_{\Gamma'} \circ p|_{C(\boldsymbol{\mu})})_*([C(\boldsymbol{\mu}, \Delta)]) = r \cdot (j_{\Gamma'})_*[C(\boldsymbol{\mu}, \Gamma')] \neq 0,$$

and hence $(j_\Delta)_*[C(\boldsymbol{\mu}, \Delta)] \neq 0$.

3.9. We conclude this section by describing conditions on $\mathbf{G}$, Γ, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ such that the corresponding cycles are of complementary dimension and satisfy assumption (ii) in Theorem 3.7.

Let $\mathbf{G}$ denote a reductive algebraic $\mathbb{Q}$-group and write $G := \mathbf{G}(\mathbb{R})$ for its group of real points. We assume that G is of the form $G = G_c \times G_{nc}$, where G_c is a compact Lie group and G_{nc} is a noncompact Lie group.

Lemma 3.10. *Let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be commuting $\mathbb{Q}$-rational morphisms of order two on $\mathbf{G}$ such that $\boldsymbol{\theta} := \boldsymbol{\tau}_1 \circ \boldsymbol{\tau}_2$ induces a Cartan involution θ on G_{nc}. Let Γ be a torsion-free, $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$-stable, cocompact arithmetic subgroup of $\mathbf{G}$ chosen as in Corollary 2.5. Then the* *associated cycles $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are of complementary dimension in X/Γ and satisfy condition (ii) of Theorem 3.7.*

Proof. We denote by τ_1 and τ_2 the induced automorphisms on G. Recall that the maximal compact subgroup K of G was chosen to be $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle$-stable. Then K is of the form $K = K' \times G_c$, where K' is the maximal compact subgroup of G_{nc} associated with θ. Let $g \in \operatorname{Fix}(\langle \tau_1, \tau_2\rangle, G)$. Then g is of the form $g = (g_1, k)$ for some $g_1 \in G_{nc}$ and $k \in G_c$. We have $\theta(g_1) = (\tau_1 \circ \tau_2)(g_1) = g_1$, so $g_1 \in K'$ and therefore $g \in K$.

Let $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ denote the Lie algebras of the real Lie groups G_{nc}, $\operatorname{Fix}(\tau_1, G_{nc})$ and $\operatorname{Fix}(\tau_2, G_{nc})$, respectively. Since τ_1 and τ_2 commute, $\mathfrak{g}_i$ is stable under $d\theta$ for $i \in \{1, 2\}$ and $d\theta$ is a Cartan involution on $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$. We denote the corresponding Cartan decompositions by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ for $i \in \{1, 2\}$. To deal with the dimensions, note that we have $X = K\backslash G \cong K'\backslash G_{nc}$ and $X(\boldsymbol{\tau}_i) \cong \operatorname{Fix}(\tau_i, K')\backslash \operatorname{Fix}(\tau_i, G)$. Hence, we know that $\dim X = \dim \mathfrak{p}$, $\dim X(\boldsymbol{\tau}_1) = \dim \mathfrak{p}_1$ and $\dim X(\boldsymbol{\tau}_2) = \dim \mathfrak{p}_2$, and it suffices to show that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$: For $X \in \mathfrak{p}$, we have $d\theta(X) = (d\tau_1 \circ d\tau_2)(X) = -X$. Since τ_1 and τ_2 are involutions, we either have $d\tau_1(X) = X$ and $d\tau_2(X) = -X$, so $X \in \mathfrak{p}_1$ and $X \notin \mathfrak{p}_2$, or vice versa. This proves the claim. $\square$

4. Non-G-Invariance of the Cohomology Classes

Recall the notion of the compact dual symmetric space X_u of X from Definition I.6.4. For cocompact arithmetic quotients, it is well-known that there exists an injection from the cohomology of the compact dual symmetric space into the cohomology of X/Γ. In this section we will see that the classes constructed with Theorem 3.7 are *new* in the sense that they do not lie in the image of this map.

4.1. Let Γ denote a torsion-free, discrete subgroup of G. We are interested in the relationship between the cohomology of X_u and the cohomology of X/Γ. Note that there is an isomorphism $H^*(X_u, \mathbb{C}) \cong \Omega^*(X; \mathbb{C})^G$, where $\Omega^*(X; \mathbb{C})^G$ denotes the G-invariant differential forms on X (see [**7**, 1.8.4]). A G-invariant form on X is clearly Γ-invariant, and thus we have an inclusion

$$\Omega^*(X; \mathbb{C})^G \hookrightarrow \Omega^*(X; \mathbb{C})^\Gamma \cong \Omega^*(X/\Gamma, \mathbb{C}).$$

Moreover, since G-invariant forms are harmonic ([**10**, Remark II.3.2]) and hence closed, they represent certain de Rham cohomology classes. In particular, the above inclusion defines a homomorphism $\Omega^*(X; \mathbb{C})^G \to H^*(\Omega(X/\Gamma, \mathbb{C}))$. Using the de Rham Theorem, this yields a homomorphism

$$\beta_\Gamma^*\colon H^*(X_u, \mathbb{C}) \cong \Omega^*(X; \mathbb{C})^G \to H^*(\Omega(X/\Gamma), \mathbb{C}) \cong H^*(X/\Gamma, \mathbb{C}).$$

Proposition 4.2. *Let X/Γ be compact. Then the homomorphism*

$$\beta_\Gamma^*\colon H^*(X_u, \mathbb{C}) \to H^*(X/\Gamma, \mathbb{C})$$

is injective.

Proof. Looking at the construction of the homomorphism β_Γ^*, it suffices to show that the homomorphism $\Omega^*(X; \mathbb{C})^G \to H^*(\Omega(X/\Gamma, \mathbb{C}))$ is injective. This is the case since

X/Γ is compact, and a nontrivial harmonic form on a compact manifold can never be a coboundary.[3] □

REMARK. The identifications made above show why classes in the image of β_Γ^* are often referred to as classes that are represented by a G-invariant differential form on X.

4.3. As we will see later, the cohomology of the compact dual is well-known. Therefore, in the cohomology of X/Γ, we are particularly interested in such classes that do not lie in the image of β_Γ^*, i. e. are not represented by a G-invariant differential form on X.

Fortunately, there is a result by Millson and Raghunathan showing that classes obtained as in Theorem 3.7 originating from morphisms of order two have this property, cf. [**45**, Thm. 2.1] and [**58**, Thm. K]:

THEOREM 4.4. *Let* $\mathbf{G}$*,* $\boldsymbol{\tau}_1$*,* $\boldsymbol{\tau}_2$ *and* Γ *satisfy the assumptions of Theorem 3.7 and suppose moreover that* $\boldsymbol{\tau}_1$ *and* $\boldsymbol{\tau}_2$ *are of order two. Then there exists a* $\langle\boldsymbol{\tau}_1,\boldsymbol{\tau}_2\rangle$*-stable subgroup* Γ'' *of* Γ' *of finite index such that the nontrivial cohomology classes defined by* $(j_1)_*[C(\boldsymbol{\tau}_1,\Gamma'')]$ *and* $(j_2)_*[C(\boldsymbol{\tau}_2,\Gamma'')]$ *via Poincaré-duality are not in the image of* $\beta_{\Gamma''}^*$*.*

REMARK. One can even show that for any $\langle\boldsymbol{\tau}_1,\boldsymbol{\tau}_2\rangle$-stable subgroup $\Delta\subset\Gamma''$ of finite index the cohomology classes defined by $j_{1*}[C(\boldsymbol{\tau}_1,\Delta)]$ and $j_{2*}[C(\boldsymbol{\tau}_2,\Delta)]$ (that are nontrivial by the remark following Corollary 3.8) are not in the image of β_Δ^*. This follows from a reformulation of the non-G-invariance condition in terms of invariance under decktransformations, cf. [**68**, Sec. 4].

5. The Existence and Construction of Cocompact Discrete Subgroups

Let $\mathbf{G}$ be a semisimple algebraic $\mathbb{Q}$-group and $\Gamma\subset\mathbf{G}(\mathbb{Q})$ an arithmetic subgroup. The quotient $\mathbf{G}(\mathbb{R})/\Gamma$ is always of finite volume with respect to an invariant Haar measure (cf. [**51**, Thm. 4.13]) but not necessarily compact. For example, if $\mathbf{G}=\mathbf{SL_n}$ and $n\geq 2$, there is no arithmetic subgroup $\Gamma\subset SL_n(\mathbb{Q})$ that gives rise to a compact quotient $SL_n(\mathbb{R})/\Gamma$.

However, there are results by Borel [**5**] and Borel-Harder [**8**] that ensure the existence of a discrete cocompact subgroup in many cases:

THEOREM 5.1. *Let* ℓ *be a local field of characteristic 0 and let* $\mathbf{H}$ *be a reductive algebraic group defined over* ℓ*. Then the group* $\mathbf{H}(\ell)$ *contains a discrete cocompact subgroup.*

PROOF. See [**8**, Thm. A]. □

In the proof, the main work is to show that one can restrict to the following situation: $\mathbf{H}$ is an absolutely almost simple algebraic group and there exists an algebraic number field k such that $k_v\cong\ell$ for some place (finite or infinite) v of ℓ and an anisotropic semisimple algebraic group $\mathbf{G}$ defined over k, such that $\mathbf{G}(k_v)\cong\mathbf{H}(\ell)$ and $G(k_{v'})$ is compact for all $v'\in V_\infty$, $v'\neq v$. By a well-known argument, one can then show that any arithmetic subgroup of $\mathbf{G}$ leads to a cocompact discrete subgroup of $\mathbf{H}$. This is based on a compactness criterion for arithmetic quotients, that goes back to the work of Borel and Harish-Chandra [**9**] as well as Mostow and Tamagawa [**47**].

[3] This follows from Hodge Theory on compact manifolds: Let M be a compact manifold, let $d\colon\Omega^k(M)\to\Omega^{k+1}(M)$ and $d^*\colon\Omega^k(M)\to\Omega^{k-1}(M)$ denote the differential and codifferential, respectively, and let $\langle\cdot,\cdot\rangle\colon\Omega^k(M)\times\Omega^k(M)\to\mathbb{C}$ denote the inner product. A form $\omega\in\Omega^k(M)$ is harmonic if and only if $\Delta(\omega):=dd^*(\omega)+d^*d(\omega)=0$ if and only if $d\omega=0$ and $d^*\omega=0$. Therefore, if $\omega=d\eta$ is also a coboundary, we have $\langle\omega,\omega\rangle=\langle d\eta,\omega\rangle=\langle\eta,d^*\omega\rangle=0$, which implies $\omega=0$.

5.2. For the case of an archimedean local field, we give the argument in detail, as we will need this construction in the next chapter: Let us assume that $\ell \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathbf{H}$ is a simply connected absolutely almost simple algebraic group over ℓ. Moreover, we suppose that we are given a number field k with at least two archimedean places, an archimedean place $v \in V_\infty$ such that $k_v \cong \ell$ and $k_{v'} \cong \mathbb{R}$ for all $v' \in V_\infty \setminus \{v\}$, and a semisimple algebraic k-group $\mathbf{G}$ such that $\mathbf{G}(k_v) \cong \mathbf{H}(\ell)$ and $\mathbf{G}(k_{v'})$ is compact for all $v' \in V_\infty \setminus \{v\}$.

Let $\Gamma \subset \mathbf{G}(k)$ be an arithmetic subgroup. Then the image of Γ in $\mathbf{G}_\infty = \prod_{v \in V_\infty} \mathbf{G}(k_v)$ under the diagonal embedding (still denoted by Γ) is a discrete subgroup. To see that Γ is even a cocompact group, we use the above mentioned compactness criterion in the formulation for semisimple algebraic groups:

THEOREM 5.3. *Let $\mathbf{G}$ be a semisimple algebraic group defined over an algebraic number field k and let $\Gamma \subset \mathbf{G}(k)$ be an arithmetic subgroup. The quotient $\mathbf{G}_\infty/\Gamma$ is compact if and only if $\mathbf{G}$ is anisotropic over k.*

PROOF. See e.g. [**51**, Thm. 4.17(3)]. □

In our case, since $\mathbf{G}(k_{v'})$ is compact for at least one place $v' \in V_\infty$, the group $\mathbf{G}$ is k-anisotropic and thus the quotient $\mathbf{G}_\infty/\Gamma$ is compact. Now we let $p_v \colon \mathbf{G}_\infty \to \mathbf{G}(k_v)$ denote the projection onto the factor of $\mathbf{G}_\infty$ corresponding to the place v and write $\overline{p_v} \colon \mathbf{G}_\infty/\Gamma \to \mathbf{G}(k_v)/p_v(\Gamma)$ for the induced map on the quotients. Being the image of a discrete subgroup under a surjective morphism of locally compact groups with compact kernel, $p_v(\Gamma)$ is a discrete subgroup of $\mathbf{G}(k_v)$ (cf. [**65**, Cor. 4.10]). Moreover, $\mathbf{G}(k_v)/p_v(\Gamma)$ is compact as the image of a compact manifold under the smooth map $\overline{p_v}$. Therefore, $p_v(\Gamma)$ is the desired cocompact discrete subgroup of $\mathbf{H}(\ell) \cong \mathbf{G}(k_v)$.

REMARK. We have seen that the k-anisotropy of $\mathbf{G}$ follows from the fact that k has at least two archimedean places, i.e. at least one archimedean place v' such that $k_{v'} \cong \mathbb{R}$ and $\mathbf{G}(k_{v'})$ is compact. In our construction, we require this property of k to *ensure* the k-anisotropy of $\mathbf{G}$. However, if we are given an algebraic group over $\mathbb{Q}$ or over an imaginary quadratic number field, of which we know for some reason that it is anisotropic, the compactness criterion applies as well and we get a similar result.

EXAMPLE 5.4. Let k be either $\mathbb{Q}$ or an imaginary quadratic field and let Q be a quaternion division algebra over k that splits over $\mathbb{R}$ if $k = \mathbb{Q}$. The algebraic group $\mathbf{G}$ associated with $SL_1(Q)$ (cf. Section III.1.1) is an anisotropic[4] algebraic k-group with

$$\mathbf{G}_\infty \cong \begin{cases} SL_2(\mathbb{R}) & \text{if } k = \mathbb{Q}, \\ SL_2(\mathbb{C}) & \text{if } k \text{ is imaginary quadratic.} \end{cases}$$

So by Theorem 5.3, any arithmetic subgroup $\Gamma \subset \mathbf{G}(k)$ gives rise to a cocompact discrete subgroup of $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$.

[4]For a quaternion algebra Q defined over an algebraic number field k we have: $\mathbf{SL_1(Q)}$ is anisotropic over k if and only if Q is a division algebra, see e.g. [**51**, Prop. 2.12].

CHAPTER III

Application to the Real and Complex Special Linear Group

In this chapter, we apply the theory developed in Chapter II to the case where G is isomorphic to the real or complex special linear group up to compact factors. Using the method of Rohlfs and Schwermer, we prove our main result about the nonvanishing of the cohomology of certain discrete cocompact subgroups of G.

In the first section, we give a precise definition of the algebraic group associated with the special unitary group over a division algebra D with respect to an involution of the first or second kind. In Section 2, we study this group if defined over an algebraic number field and in particular determine the various real and complex Lie groups that arise as groups of points at the archimedean places. In the third section, we see how arithmetic subgroups of these algebraic groups give rise to cocompact discrete subgroups of $SL_n(\mathbb{R})$ or $SL_n(\mathbb{C})$. In Theorem 3.2, we give precise conditions on the underlying number field and the division algebra that imply the existence of such cocompact discrete subgroups.

Sections 4 and 5 are devoted to the construction of geometric cycles in the locally symmetric spaces attached to $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$, respectively. In both Sections, we deal with the somewhat simple case, where D is an algebraic number field, as well as with the more complicated situation, where D is a quaternion division algebra. Finally, in Section 6, we present our main results, Theorem 6.2 for the real case and Theorem 6.4 for the complex case. Section 7 addresses questions of linear independence of the constructed cohomology classes.

1. The Algebraic Group Associated with the Special Unitary Group over a Division Algebra

In this section we define what we call the *special unitary group over a division algebra* and the associated algebraic group.

1.1. Let A be a central simple algebra over a field k. Recall that there always exists a finite extension k' of k such that $A \otimes_k k' \overset{\varphi}{\cong} M_d(k')$ for some natural number d. Then for any $a \in A$ the element $\det(\varphi(a \otimes 1))$ lies in k and is moreover independent of the choice of the splitting field k'. The homomorphism $A \to k$, $a \mapsto \det(\varphi(a \otimes 1))$ is denoted by $\mathrm{Nrd}_{A/k}$ and called the *reduced norm of A over k*. Having chosen a basis of A over k, the reduced norm is a polynomial with coefficients in k and indeterminates the coordinates with respect to that basis ([**11**, §12, Prop. 11]). In particular, we can evaluate the polynomial Nrd at any element of $A \otimes_k R$ for a commutative k-algebra R.[1]

[1] Let $e_1, \dots e_{d^2}$ be a basis of A over k. Then $e_1 \otimes 1, \dots, e_{d^2} \otimes 1$ is a basis of $A \otimes_k R$ over R and any element $a \otimes r$ can be written in the form $\sum r_i(e_i \otimes 1)$. Then we set $\mathrm{Nrd}(a \otimes r) := \mathrm{Nrd}(r_1, \dots, r_{d^2})$.

Let us now consider the matrix algebra $M_m(A)$ for a natural number m, which is again a central simple k-algebra. We can define the *special linear group over* A to be the group

$$SL_m(A) := \{x \in M_m(A) \mid \mathrm{Nrd}_{M_m(A)/k}(x) = 1\}.$$

There exists an algebraic k-group associated with $SL_m(A)$:

PROPOSITION 1.2. *The functor* $\mathbf{SL_m(A)}\colon \mathfrak{Alg}_k \to \mathfrak{Grp}$ *from commutative k-algebras to groups, assigning to a k-algebra R the group*

$$SL_m(A \otimes_k R) := \{x \in M_m(A) \otimes R \mid \mathrm{Nrd}(x) = 1\}$$

defines an algebraic k-group. The k-rational points of this algebraic group coincide with the group $SL_m(A)$.

PROOF. Set $s := \dim_k M_m(A) = d^2m^2$ and let $e_1, \dots, e_s$ be a basis of $M_m(A)$ over k. Then the functor $\mathbf{SL_m(A)}$ is, as a functor to $\mathfrak{Set}$, represented by the finite dimensional k-algebra $B := k[x_1, \dots, x_s]/(\mathrm{Nrd}(x_1, \dots, x_s) - 1)$: For any commutative k-algebra R, a k-algebra homomorphism $\varphi\colon B \to R$ is uniquely determined by the values $r_1 := \varphi(x_1), \dots, r_s := \varphi(x_s) \in R$. These elements satisfy $\mathrm{Nrd}(r_1, \dots r_s) = 1$ and the map sending the tuple $(r_1, \dots, r_s)$ to the element $\sum r_i(e_i \otimes 1) \in M_m(A) \otimes R$ defines the required natural equivalence of functors. $\square$

1.3. Let now D be a simple division algebra over k and let σ be an involution on D such that the pair (D, σ) is a central simple algebra with involution (cf. I.7.1). This means that D is either central simple over k and σ is of the first kind, or D is central simple over its center $Z(D)$ (which is a field extension of degree two of k) and σ is of the second kind. To simplify notation, we set $\ell := Z(D)$. Let h be a σ-hermitian (or σ-skew-hermitian) form on D^m and define the *special unitary group of rank m over D* as follows:

$$SU_m(h, D, \sigma) := \{x \in SL_m(D) \mid h(xv, xw) = h(v, w) \text{ for all } v, w \in D^m\}.$$

Note that when choosing a basis of D^m over D and denoting by H the matrix of h w.r.t. this basis (cf. I.7.5), we can define a k-linear antiautomorphism[2] ρ of $M_m(D)$ by $\rho(x) := H^{-1}\sigma(x)^tH$, where $\sigma(x)$ should be understood as applying σ to each entry of the matrix x. By [**15**, §22, Lemma 5], we have

$$\mathrm{Nrd}_{M_m(D)/\ell}(\rho(x)) = \rho(\mathrm{Nrd}_{M_m(D)/\ell}(x)) \quad \text{for all } x \in M_m(D),$$

and thus ρ restricts to an antiautomorphism of the group $SL_m(D)$. Therefore, we can restate the above definition as

$$SU_m(h, D, \sigma) = \{x \in SL_m(D) \mid \rho(x)^{-1} = x\}. \tag{1.1}$$

EXAMPLE. Let us consider $D = \mathbb{C}$ and the two involutions $\sigma_1 = \mathrm{id}$ and $\sigma_2\colon x \mapsto \bar{x}$ on D. Then (D, σ_1) is a central simple algebra over $\mathbb{C}$ with involution of the first kind and (D, σ_2) is a central simple $\mathbb{R}$-algebra with involution of the second kind and $Z(D) = \mathbb{C}$. For the trivial hermitian form, the corresponding special unitary groups are well-known classical groups:

$$SU_m(h, D, \sigma_1) = \{x \in SL_m(\mathbb{C}) \mid x^tx = I_m\} \cong SO(m, \mathbb{C}),$$
$$SU_m(h, D, \sigma_2) = \{x \in SL_m(\mathbb{C}) \mid \bar{x}^tx = I_m\} \cong SU(m, \mathbb{C}).$$

[2]By an *antiautomorphism* of $M_m(D)$ we mean an automorphism φ of $M_m(D)$ as k-vector space that fulfills $\varphi(xy) = \varphi(y)\varphi(y)$ for all $x, y \in M_m(D)$.

For even m and the form h represented by $H := J_m = \begin{pmatrix} 0 & -I_{m/2} \\ I_{m/2} & 0 \end{pmatrix} \in M_m(D)$, we get the group

$$SU_m(h, D, \sigma_1) = \{x \in SL_m(\mathbb{C}) \mid J^{-1}x^tJx = I_m\} \cong Sp(m, \mathbb{C}).$$

Note that h is a skew-hermitian form in this case.

1.4. We will now see how to define an algebraic group $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ defined over k that is associated with $SU_m(h, D, \sigma)$. Recall that we have set $\ell := Z(D)$ and that we have $\ell = k$ if σ is of the first kind and ℓ is a quadratic extension of k if σ is of the second kind.[3]

Let us start by looking at the functor $\mathbf{X}'\colon \mathfrak{Alg}_\ell \to \mathfrak{Grp}$, given by $\mathbf{X}'(S) := M_m(D) \otimes_\ell S$ for any commutative ℓ-algebra S. This is an (additive) algebraic group defined over ℓ, but we need only consider it as an affine algebraic variety. If we denote by $\mathbf{G}_a$ the additive group over ℓ, the reduced norm defines a natural transformation of functors $\mathbf{Nrd}\colon \mathbf{X}' \to \mathbf{G}_a$, i. e. an ℓ-morphism of affine varieties. Note that we have an embedding $\mathbf{SL_m}(\mathbf{D}) \hookrightarrow \mathbf{X}'$ of affine varieties, since $\mathbf{SL_m}(\mathbf{D})(S) = \{x \in \mathbf{X}'(S) \mid \mathrm{Nrd}(x) = 1\}$ for any commutative ℓ-algebra S.

We can now apply the restriction of scalars functor (for affine varieties) and obtain an affine k-variety $\mathbf{X} = \mathrm{Res}_{\ell/k}\,\mathbf{X}'$ and a k-morphism

$$\mathrm{Res}_{\ell/k}\,\mathbf{Nrd}\colon \mathbf{X} \to \mathrm{Res}_{\ell/k}\,\mathbf{G}_a,$$

of affine varieties, that is given for any commutative k-algebra R by

$$(\mathrm{Res}_{\ell/k}\,\mathbf{Nrd})(R) = \mathbf{Nrd}(R \otimes_k \ell)\colon \mathbf{X}(R) = \mathbf{X}'(R \otimes_k \ell) \to \mathbf{G}_a(R \otimes_k \ell).$$

It is now easy to see that $\mathrm{Res}_{\ell/k}\,\mathbf{SL_m}(\mathbf{D})$ is the affine subvariety of $\mathbf{X}$ whose R-points are given by the set $\{x \in \mathbf{X}(R) \mid (\mathrm{Res}_{\ell/k}\,\mathbf{Nrd})(R)(x) = 1\}$.

On the other hand, we can define a natural transformation $\boldsymbol{\rho}$ on $\mathbf{X}$ as follows: For any commutative k-algebra R, we have

$$\mathbf{X}(R) = \mathbf{X}'(R \otimes_k \ell) = M_m(D) \otimes_\ell (R \otimes_k \ell) = M_m(D) \otimes_k R$$

and we set

$$\boldsymbol{\rho}(R) := \rho \otimes \mathrm{id}\colon M_m(D) \otimes_k R \to M_m(D) \otimes_k R.$$

Since ρ is a k-linear map on $M_m(D)$, this defines a morphism on $\mathbf{X}$. Restriction to the center ℓ of $M_m(D)$ also yields a morphism $\boldsymbol{\rho}|_{\mathrm{Res}_{\ell/k}\,\mathbf{G}_a}\colon \mathrm{Res}_{\ell/k}\,\mathbf{G}_a \to \mathrm{Res}_{\ell/k}\,\mathbf{G}_a$.

LEMMA 1.5. *The natural transformations* $\mathrm{Res}_{\ell/k}\,\mathbf{Nrd}$ *and* $\boldsymbol{\rho}$ *commute, i. e. they satisfy the identity*

$$(\mathrm{Res}_{\ell/k}\,\mathbf{Nrd}) \circ \boldsymbol{\rho} = \boldsymbol{\rho}|_{\mathrm{Res}_{\ell/k}\,\mathbf{G}_a} \circ (\mathrm{Res}_{\ell/k}\,\mathbf{Nrd}), \tag{1.2}$$

as morphisms of affine varieties $\mathbf{X} \to \mathrm{Res}_{\ell/k}\,\mathbf{G}_a$.

PROOF. Note that we have $\mathbf{X}(k) = M_m(D)$ and $\mathrm{Res}_{\ell/k}\,\mathbf{G}_a(k) = \ell$. Then an application of [**15**, §22, Lemma 5] yields

$$\mathrm{Nrd}_{M_m(D)/\ell}(\rho(x)) = \rho|_\ell(\mathrm{Nrd}_{M_m(D)/\ell}(x))$$

[3] To avoid case distinctions, we allow $\ell = k$ such that we can handle involutions of the first and the second kind simultaneously. We are aware that if $\ell = k$ the restriction of scalars functor $\mathrm{Res}_{\ell/k}$ applied in the following paragraph is trivial, so many of the arguments are redundant in this case.

for all $x \in M_m(D)$, i. e. (1.2) holds on the k-rational points $\mathbf{X}(k)$. However, by [**6**, 18.3], we know that $\mathbf{X}(k)$ is Zariski-dense in $\mathbf{X}$ and thus by [**6**, 6.2], the identity (1.2) holds on $\mathbf{X}$. □

Using this result, we can now define the algebraic group $\mathbf{SU_m(h, D, \boldsymbol{\sigma})}$. The lemma implies that the natural transformation $\boldsymbol{\rho}$ of $\mathbf{X}$ can be restricted to the algebraic group $\operatorname{Res}_{\ell/k} \mathbf{SL_m(D)}$. We will still denote this restriction by $\boldsymbol{\rho}$. It is, however, not a morphism of the algebraic group, since $\boldsymbol{\rho}(R)$ is not a group automorphism (but an antiautomorphism) on the R-rational points. Let **inv** denote the natural transformation on $\operatorname{Res}_{\ell/k} \mathbf{SL_m(D)}$ that induces the group inversion on $\operatorname{Res}_{\ell/k} \mathbf{SL_m(D)}(R)$ for any commutative k-algebra R. Then the composition $\boldsymbol{\psi} := (\mathbf{inv} \circ \boldsymbol{\rho})$ is a k-morphism of the algebraic group $\operatorname{Res}_{\ell/k} \mathbf{SL_m(D)}$.

PROPOSITION 1.6. *Let* $\mathbf{SU_m(h, D, \boldsymbol{\sigma})} := \operatorname{Fix}(\boldsymbol{\psi}, \operatorname{Res}_{\ell/k} \mathbf{SL_m(D)})$ *be the group of fixed points of* $\boldsymbol{\psi}$ *in* $\operatorname{Res}_{\ell/k} \mathbf{SL_m(D)}$. *This is an algebraic group defined over* k, *whose* k-*rational points coincide with the group* $SU_m(h, D, \sigma)$. *If* σ *is an involution of the second kind, the group* $\mathbf{SU_m(h, D, \boldsymbol{\sigma})}$ *is a simply connected simple algebraic group.*

PROOF. By I.2.1, the fixed points of a k-morphism define an algebraic k-group. Let us look at the k-rational points of $\mathbf{SU_m(h, D, \boldsymbol{\sigma})}$:

$$\begin{aligned}\mathbf{SU_m(h, D, \boldsymbol{\sigma})}(k) &= \{x \in \operatorname{Res}_{\ell/k} \mathbf{SL_m(D)}(k) \mid \boldsymbol{\psi}(k)(x) = x\} \\ &= \{x \in \mathbf{SL_m(D)}(\ell) \mid \rho(x)^{-1} = x\} \\ &= \{x \in SL_m(D) \mid \rho(x)^{-1} = x\} \overset{(1.1)}{=} SU_m(h, D, \sigma).\end{aligned}$$

For the last statement, see [**51**, Prop. 2.15]. □

2. Over Number Fields: The Points at Archimedean Places

In this section, we consider special unitary groups that are defined over number fields and in particular examine the points at the archimedean places. Our focus is on involutions of the second kind; however, we will start by studying groups with respect to involutions of the first kind for later use.

2.1. Let us once and for all fix the following notation: D denotes a central division algebra of degree d over an algebraic number field E and σ is an involution of the first or second kind on D. If σ is of the second kind, we denote by F the subfield of E of index 2 such that $\sigma|_F = \mathrm{id}$ and $\sigma|_E = \iota$, where ι is the nontrivial Galois automorphism of E over F. Moreover, for $m \in \mathbb{N}$, we are given a σ-hermitian (or σ-skew-hermitian) form $h\colon D^m \times D^m \to D$ whose matrix representation is denoted by H. Set $n := dm$.

The following result deals with the points at archimedean places of the group $\mathbf{SL_m(D)}$ and will be useful for later reference. As before, we write $\mathbb{H}$ to denote Hamilton's quaternions.

PROPOSITION 2.2. *Let* $w \in V_\infty(E)$ *be an archimedean place. Then we have*

$$\mathbf{SL_m(D)}(E_w) \cong \begin{cases} SL_n(\mathbb{C}) & \text{if } w \text{ is a complex place,} \\ SL_n(\mathbb{R}) & \text{if } w \text{ is a real place and } D \text{ splits at } w, \\ SL_{n/2}(\mathbb{H}) & \text{if } w \text{ is a real place and } D \text{ ramifies at } w. \end{cases}$$

REMARK. If w is a real place of E and D ramifies at w, we have $D \otimes_E E_w \cong M_k(\mathbb{H})$ for some $k \in \mathbb{N}$. This implies that $2 = \deg \mathbb{H}$ divides $d = \deg D$, thus d is even and the expression $SL_{n/2}(\mathbb{H}) = SL_{dm/2}(\mathbb{H})$ in the proposition makes sense.

PROOF. Recall from the definition of $\mathbf{SL_m}(\mathbf{D})$ that

$$\mathbf{SL_m}(\mathbf{D})(E_w) = \{x \in M_m(D \otimes_E E_w) \mid \mathrm{Nrd}(x) = 1\}.$$

If D splits at w, we have $M_m(D \otimes_E E_w) \cong M_n(E_w)$ and the reduced norm polynomial goes over to the determinant under this isomorphism. Therefore, we get

$$\mathbf{SL_m}(\mathbf{D})(E_w) \cong \{x \in M_n(E_w) \mid \det(x) = 1\} = SL_n(E_w),$$

which gives the result for the complex and the split real case.

If w is a real place and D ramifies at w, we have $M_m(D \otimes_E E_w) \cong M_{n/2}(\mathbb{H})$. Under this identification, the polynomial function Nrd coincides with the map $\mathrm{Nrd}_{M_{n/2}(\mathbb{H})/\mathbb{R}}$ on $M_{n/2}(\mathbb{H})$.[4] This yields

$$\mathbf{SL_m}(\mathbf{D})(E_w) \cong \{x \in M_{n/2}(\mathbb{H}) \mid \mathrm{Nrd}_{M_{n/2}(\mathbb{H})/\mathbb{R}}(x) = 1\} = SL_{n/2}(\mathbb{H}). \qquad \square$$

2.3. We proceed by considering the group $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ for an involution σ of the first kind. This is an algebraic group defined over E. Recall that when defining the group, we have defined a map $\rho = \mathrm{Int}(H^{-1}) \circ \sigma^t$ on $M_m(D)$.[5] By Lemma I.7.3 and the properties of the matrix H, this is an involution of the first kind on $M_m(D)$ that is of the same type as σ if h is hermitian, and of the opposite type if h is skew-hermitian.

PROPOSITION 2.4. *Let σ be an involution of the first kind on D and assume that D splits at all real places of E. Let $w \in V_\infty(E)$ be an archimedean place of E. Then there are the following possibilities:*

- *If w is a complex place, we have*
$$\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(E_w) \cong \begin{cases} SO(n, \mathbb{C}) & \text{if } \rho \text{ is of orthogonal type} \\ Sp(n, \mathbb{C}) & \text{if } \rho \text{ is of symplectic type.} \end{cases}$$
- *If w is a real place, we have*
$$\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(E_w) \cong \begin{cases} SO(p, q) & \text{if } \rho \text{ is of orthogonal type,} \\ Sp(n, \mathbb{R}) & \text{if } \rho \text{ is of symplectic type,} \end{cases}$$
for suitable nonnegative integers p and q with $p + q = n$.

PROOF. By the definition of $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$, Proposition 2.2 and the fact that D splits at w, we have

$$\begin{aligned} \mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(E_w) &= \{x \in \mathbf{SL}_m(D)(E_w) \mid \boldsymbol{\psi}(E_w)(x) = x\} \\ &= \{x \in SL_n(E_w) | (\boldsymbol{\rho}(E_w)(x))^{-1} = x\}, \end{aligned}$$

where $\boldsymbol{\psi}$ and $\boldsymbol{\rho}$ are as in Section 1.4. By definition, $\boldsymbol{\rho}(E_w) = \rho \otimes \mathrm{id}$ on the group of E_w-rational points $M_m(D) \otimes_E E_w \cong M_n(E_w)$. This is an involution of the first kind which is of the form $x \mapsto ux^tu^{-1}$ for some $u \in M_n(E_w)$ with $u^t = \pm u$ by Lemma I.7.3.

[4]This follows from the fact that both maps are polynomial functions on $M_{n/2}(\mathbb{H})$ that are mapped to the determinant under any splitting $M_{n/2}(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \to M_n(\mathbb{C})$, and such a function is unique.

[5]For an involution σ on D we will use the notation σ^t for the involution on $M_m(D)$ defined by $\sigma^t(x) = \sigma(x)^t$ for $x \in M_m(D)$.

Let ρ be of symplectic type. Then the element u is skew-symmetric and by a suitable choice of the splitting isomorphism $M_m(D) \otimes_E E_w \xrightarrow{\sim} M_n(E_w)$, we may assume that $u = J_n$. This implies

$$\begin{aligned}\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(E_w) &= \{x \in SL_n(E_w) \mid J_n (x^t)^{-1} J_n^{-1} = x\} \\ &\cong \begin{cases} Sp(n, \mathbb{R}) & \text{if } E_w \cong \mathbb{R}, \\ Sp(n, \mathbb{C}) & \text{if } E_w \cong \mathbb{C}. \end{cases}\end{aligned}$$

If ρ is of orthogonal type, the element u is symmetric. In case $E_w \cong \mathbb{C}$, we can find an element $b \in M_n(\mathbb{C})$ such that $u = bb^t$. Therefore, after suitable conjugation of the splitting isomorphism, we may assume that $u = I_n$ and conclude

$$\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(E_w) = \{x \in SL_n(E_w) \mid (x^t)^{-1} = x\} \cong SO(n, \mathbb{C}).$$

If $E_w \cong \mathbb{R}$, we have to deal with signatures: Any symmetric element $u \in M_n(\mathbb{R})$ can be written in the form $bI_{p,q}b^t$, for some nonnegative integers p, q with $p + q = n$. As above, we may assume $u = I_{p,q}$ and get

$$\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(E_w) = \{x \in SL_n(E_w) \mid I_{p,q} (x^t)^{-1} I_{p,q} = x\} \cong SO(p, q). \qquad \square$$

Remark. (1) As we have seen in Proposition I.7.13, the only division algebras over a number field admitting an involution of the first kind are the number field itself or quaternion algebras. If D is a quaternion algebra, it has degree $d = 2$ which implies that n is even and all groups occurring in the proposition are defined. Note that there is no symplectic involution on the field E. Therefore, if $D = E$, the involution ρ can only be symplectic if h is skew-hermitian. However, on E^m a nondegenerate skew-hermitian form can only exist if m is even, so also in this case the groups $Sp(n, \mathbb{R})$ and $Sp(n, \mathbb{C})$ are well-defined.
(2) If D ramifies at w, we get in a similar way subgroups of the quaternionic special linear group. We omit this case as it is not needed in the following.

2.5. In this paragraph, we consider the situation where σ is an involution of the second kind. Here we have to deal with an additional algebraic number field F, where E/F is a quadratic extension. In this case, the algebraic group $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ is defined over F and we can look at the F_v-rational points of $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ for any archimedean place $v \in V_\infty(F)$. We will now discuss the different possibilities for the group $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F_v)$ depending on the type of the place v (real or imaginary), the behaviour of v w. r. t. the extension E (decomposed or nondecomposed), as well as the behaviour of D at v (split or ramified). We restrict to the case where h is a σ-hermitian form.

Recall that for a quadratic extension of number fields E/F, there are at most two places of E lying over a place v of F. For an archimedean place v, this allows the following possibilities: If v is a complex place, then v is decomposed, i. e. there exist exactly two places $w_1, w_2 \in V_\infty(E)$ such that $w_1|v$ and $w_2|v$ and these places are complex. If v is real, it is either decomposed, i. e. there are two real places $w_1, w_2 \in V_\infty(E)$ that lie above v, or it is nondecomposed, i. e. there is one complex place w with $w|v$. For the completions, this means if $F_v = \mathbb{C}$, we have $E_{w_1} = E_{w_2} = \mathbb{C}$ and if $F_v = \mathbb{R}$, we have either $E_{w_1} = E_{w_2} = \mathbb{R}$ or $E_w = \mathbb{C}$.

Proposition 2.6. *Let σ be an involution of the second kind on D and consider an archimedean place $v \in V_\infty(F)$. There are the following possibilities:*

- *v is a complex place. Then $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F_v) \cong SL_n(\mathbb{C})$.*

- *v is a nondecomposed real place. Then*

$\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F_v) \cong SU(p,q)$ for nonnegative integers p, q with $p+q=n$.

- *v is a decomposed real place and $w_1|v$ and $w_2|v$ are the real places of E lying above v. Then we have*

$$\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F_v) \cong \begin{cases} SL_n(\mathbb{R}) & \text{if } D \text{ splits at } w_1 \text{ and } w_2, \\ SL_{n/2}(\mathbb{H}) & \text{if } D \text{ ramifies at } w_1 \text{ and } w_2. \end{cases}$$

Proof. By the definitions of $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ and $\mathbf{SL_m}(\mathbf{D})$, we have:

$$\begin{aligned} \mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F_v) &= \{x \in \operatorname{Res}_{E/F} \mathbf{SL_m}(\mathbf{D})(F_v) \mid \boldsymbol{\psi}(F_v)(x) = x\} \\ &= \{x \in \mathbf{SL_m}(\mathbf{D})(E \otimes_F F_v) \mid (\boldsymbol{\rho}(F_v)(x))^{-1} = x\}. \end{aligned}$$

The further treatment depends on the number of places above v. If v is nondecomposed, which means v is real and there is one complex place w above v, we have $E \otimes_F F_v = E_w \cong \mathbb{C}$ and hence

$$\mathbf{SL_m}(\mathbf{D})(E \otimes_F F_v) = \mathbf{SL_m}(\mathbf{D})(E_w) \cong SL_n(\mathbb{C})$$

by Proposition 2.2. Now $\rho \otimes \mathrm{id}$ is an involution of the second kind on $M_n(\mathbb{C})$ and thus of the form $x \mapsto g\bar{x}^t g^{-1}$ for some $g \in M_n(\mathbb{C})$ with $\bar{g}^t = g$ by Lemma I.7.3. This implies

$$\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F_v) \cong \{x \in SL_n(\mathbb{C}) \mid g(\bar{x}^t)^{-1} g^{-1} = x\} \cong SU(p,q),$$

for some nonnegative integers p and q with $p+q=n$ that depend on the signature of the hermitian matrix g.

If v is decomposed, we have two places w_1 and w_2 lying above v. Using the notation and results of Lemma I.7.14, we have an isomorphism of algebras with involution of the second kind:

$$(M_m(D) \otimes_F F_v, \rho \otimes \mathrm{id}) \xrightarrow{\cong} (M_r(\mathbb{K}) \oplus M_r(\mathbb{K}), (x,y) \mapsto (\omega(y)^t, \omega(x)^t)).$$

Moreover, we have

$$\begin{aligned} \mathbf{SL_m}(\mathbf{D})(E \otimes_F F_v) &= \mathbf{SL_m}(\mathbf{D})(E_{w_1} \oplus E_{w_2}) \\ &\cong \mathbf{SL_m}(\mathbf{D})(E_{w_1}) \times \mathbf{SL_m}(\mathbf{D})(E_{w_2}) \\ &\cong SL_r(\mathbb{K}) \times SL_r(\mathbb{K}) \end{aligned}$$

by Proposition 2.2.[6] This yields

$$\begin{aligned} \mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F_v) &\cong \{(x,y) \in SL_r(\mathbb{K}) \times SL_r(\mathbb{K}) \mid (x,y) = (\omega(y)^t, \omega(x)^t)\} \\ &\cong \{(x,y) \in SL_r(\mathbb{K}) \times SL_r(\mathbb{K}) \mid y = \omega(x)^t\} \cong SL_r(\mathbb{K}). \end{aligned}$$

Replacing $\mathbb{K}$ and r with the respective objects depending on the nature of the place v and the splitting behaviour of D (cf. Lemma I.7.14) leads to the desired result. □

[6]Here we also use that the functor $\mathbf{SL_m}(\mathbf{D})$ preserves products since it is a representable functor.

2.7. Finally, we want to analyze the integers p and q occurring in the nondecomposed situation, i. e. the signature of the special unitary group $SU(p,q)$. Suppose that σ is a definite involution. Under this assumption, we can obtain any combination of signatures at the nondecomposed real places by a suitable choice of the hermitian form h:

PROPOSITION 2.8. *Let E, F and D be as above and let us assume that the involution σ on D is definite. Given $m \in \mathbb{N}$ and a pair of nonnegative integers (p_v, q_v) with $p_v + q_v = n$ for each nondecomposed real place v of F, there exists a σ-hermitian form $h\colon D^m \times D^m \to D$ such that*

$$\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F_v) \cong SU(p_v, q_v)$$

for each nondecomposed real place $v \in V_\infty(F)$.

PROOF. This is basically an application of the classification theorem for hermitian forms. Set $s_v := p_v - q_v$ for all nondecomposed real places of F. By the Weak Approximation Theorem for number fields, we can find an element $\delta \in F^\times$ such that m, δ and the s_v satisfy the conditions of Theorem I.7.20(2). The precise argument is as follows: For each real place $v \in V_\infty(F)$, we choose an element $\alpha_v \in F_v$ such that $\alpha_v = -1$ if v is a nondecomposed real place and $s_v - n \equiv 2 \mod 4$, and $\alpha_v = 1$ for all other real places. Then for each real number $\epsilon > 0$, the Weak Approximation Theorem for number fields[7] (applied to the finite set of real places of F) implies the existence of an element $\delta \in F$ such that $|\delta_v - \alpha_v| < \epsilon$. In particular, one can find an element $\delta \neq 0$ in F such that δ_v is positive at all decomposed real places and at the nondecomposed real places where $s_v - n \equiv 0 \mod 4$, and δ_v is negative at the nondecomposed real places where $s_v - n \equiv 2 \mod 4$. This shows that the last three conditions of Theorem I.7.20(2) are satisfied. The first one is obvious by our choice of the s_v. Hence, there exists a σ-hermitian form h with $\dim(h) = m$, $\det(h) = \delta \cdot N_{E/F}(E^\times)$ and $\operatorname{sign}_v(h) = s_v = p_v - q_v$ at all nondecomposed real places.

Let us look at the group $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ w. r. t. this hermitian form and let $v \in V_\infty(F)$ be a fixed nondecomposed real place. Proposition 2.6 implies that $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F_v)$ is of the form $SU(p,q)$ for suitable $p, q \in \mathbb{N}$. Recall that these integers depend on the signature of the matrix g occurring in the proof of Proposition 2.6. Let $\varphi_v\colon M_m(D) \otimes_F F_v \cong M_n(\mathbb{C})$ denote the isomorphism at the place v, and let H_v denote the image of $H \otimes 1$ under this isomorphism. Then it is not hard to see that the involution $\rho \otimes \mathrm{id}$ is identified under φ_v with the involution $x \mapsto H_v^{-1} \bar{x}^t H_v$ (here we use that σ is definite) and therefore the matrix g in the proof of Proposition 2.6 coincides with H_v. Thus we have

$$p - q = \operatorname{sign}(g) = \operatorname{sign}(H_v) = \operatorname{sign}_v(h) = s_v = p_v - q_v$$

and $p + q = p_v + q_v = n$ which implies $p = p_v$ and $q = q_v$ and proves the claim. □

3. Construction of Cocompact Discrete Subgroups of $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$

We fix a natural number n. We are now going to use the results from Section II.5 to construct cocompact discrete subgroups of $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$ that are arithmetically defined.

[7] We use the Weak Approximation Theorem for number fields as stated in [**51**, Thm. 1.4]: Let F be a number field and $S \subset V(F)$ a finite set of places. Set $F_S := \prod_{v \in S} F_v$. Then the image of the diagonal embedding $F \hookrightarrow F_S$ is dense in F_S.

3.1. Let ℓ denote an archimedean local field, that is either $\mathbb{R}$ or $\mathbb{C}$, and set $\mathbf{H} := \mathbf{SL}_n$. It is well-known that $\mathbf{H}$ is a simply connected absolutely almost simple algebraic group over ℓ and that $\mathbf{H}(\ell) = SL_n(\ell)$. Recall from Section II.5.2 that for the construction of discrete cocompact subgroups in $\mathbf{H}(\ell)$ we need a number field F with at least two archimedean places and a semisimple algebraic group $\mathbf{G}$ over F subject to the following conditions:

(1) There exists an archimedean place $v \in V_\infty(F)$ such that $F_v \cong \ell$.
(2) $F_{v'} \cong \mathbb{R}$ for all $v' \in V_\infty(F) \setminus \{v\}$.
(3) $\mathbf{G}(F_v) \cong \mathbf{H}(\ell) = SL_n(\ell)$.
(4) $\mathbf{G}(F_{v'})$ is compact for all $v' \in V_\infty(F) \setminus \{v\}$.

We will see in the next theorem that for certain choices of an algebraic number field F, a quadratic extension E, a division algebra D with involution σ of the second kind, a hermitian form h and a natural number m, the algebraic group $\mathbf{SU_m(h, D, \boldsymbol{\sigma})}$ satisfies these conditions.

THEOREM 3.2. *Let ℓ be an archimedean local field and let $n \in \mathbb{N}$ be fixed.*

(1) If $\ell = \mathbb{R}$, let F be a totally real number field with $[F : \mathbb{Q}] \geq 2$, let E/F be a quadratic extension such that there is exactly one place $v \in V_\infty(F)$ that is decomposed in E, and let D be a division algebra of degree $d|n$ over E that splits at the places w_1 and w_2 of E lying above v. Moreover, assume that there is a definite involution σ of the second kind on D.

(2) If $\ell = \mathbb{C}$, let F be an algebraic number field with $[F : \mathbb{Q}] \geq 3$ that has exactly one complex place v, and let E/F be a quadratic extension such that all real places of F are nondecomposed in E. Let D be a division algebra of degree $d|n$ over E that admits a definite involution σ of the second kind.

Let $m \in \mathbb{N}$ such that $dm = n$. Then one can choose a σ-hermitian form h on D^m such that any arithmetic subgroup $\Gamma \subset \mathbf{SU_m(h, D, \boldsymbol{\sigma})}(F)$ gives rise to a discrete cocompact subgroup of $SL_n(\ell)$.

PROOF. Set $(p_{v'}, q_{v'}) := (dm, 0)$ for each nondecomposed real place v' of F. Then, by Proposition 2.8, we can find a hermitian form h on D^m such that

$$\mathbf{SU_m(h, D, \boldsymbol{\sigma})}(F_{v'}) \cong SU(dm, 0) = SU(n)$$

for all nondecomposed real places of F, in particular these groups are compact. Moreover, the algebraic F-group $\mathbf{SU_m(h, D, \boldsymbol{\sigma})}$ is semisimple by Proposition 1.6. Thus, the number field F and the algebraic group $\mathbf{SU_m(h, D, \boldsymbol{\sigma})}$ satisfy the conditions (1)-(4) from above (for (3) use Proposition 2.6 and the fact that D is split at w_1 and w_2 in the real case). Then the result follows from the argumentation in Section II.5.2. $\square$

REMARK. (i) The proof will still work if we allow $(p_v, q_v) = (0, dm)$ at some of the real nondecomposed places: The group $SU(0, dm)$ is isomorphic to $SU(n)$ and hence compact.

(ii) The question arises of whether it is possible to choose objects D, E and F as in the assumptions of the theorem. A number field F with these properties clearly exists, as one can take for example any real quadratic extension of $\mathbb{Q}$ in case $\ell = \mathbb{R}$ and a cubic extension such as $\mathbb{Q}(\sqrt[3]{2})$ which has one real and one complex place in case $\ell = \mathbb{C}$. Having chosen such an F, by the weak approximation theorem for number fields (cf. 7, p. 44) one can find an element $\alpha \in F$ such that α_v is positive at one place $v \in V_\infty(F)$ and $\alpha_{v'}$ is negative at any other archimedean place v' in the real case ($\ell = \mathbb{R}$) and such that $\alpha_{v'}$ is negative at all real places v' in the complex case ($\ell = \mathbb{C}$). Then by setting $E := F(\sqrt{\alpha})$ one gets the desired quadratic extension.

The last issue to be considered is the existence of a division algebra D with involution of the second kind and degree dividing n that splits at the places w_1 and w_2 of E in the real case. Of course, taking $D = E$ will always be sufficient, since then D is split (at all places) and the nontrivial Galois automorphism of E/F is an involution of the second kind on D.

However, it is also possible to find such division algebras of higher degree: Recall that division algebras over E are completely determined by their local invariants and that the existence of an involution of the second kind imposes certain conditions on these invariants (cf. I.7.13). Given a natural number $d|n$, we can choose a finite decomposed place[8] v' of F and let D be the unique division algebra over E with local invariants $\mathrm{inv}_{w'_1}(D) = \mathrm{inv}_{w'_2}(D) = \frac{1}{d}$, where w'_1 and w'_2 are the places of E lying above v', and $\mathrm{inv}_w(D) = 0$ at all other places of E. Then D is of degree d (the degree of a division algebra is the least common multiple of the local degrees, see [**52**, 32.3, 32.17]), satisfies the conditions of I.7.13 (2), i. e. it admits an involution of the second kind, and splits at all archimedean places of E (in particular at w_1 and w_2 in the real case).

Example 3.3. Consider the number fields $F = \mathbb{Q}(\sqrt{2})$ and $E = F(\sqrt[4]{2})$. Then F has two real places v and v' and E has two real places $w_1|v$ and $w_2|v$ and one complex place $w'|v'$. In other words, v' is nondecomposed and v is decomposed in E.

(1) Set $D = E$ and let σ be the nontrivial Galois automorphism of E over F. Then σ is a definite involution since $\sigma_{v'}$ can only be complex conjugation on $E_{v'} \cong \mathbb{C}$. Let $n \in \mathbb{N}$ be arbitrary. Then, by Theorem 3.2(1), we can find a hermitian form h on E^n such that any arithmetic subgroup $\Gamma \subset \mathbf{SU_n}(\mathbf{h}, \mathbf{E}, \boldsymbol{\sigma})(F)$ gives rise to a discrete cocompact subgroup of $SL_n(\mathbb{R})$.

   In fact, we can even describe the hermitian form in this special case: Any diagonal matrix $H = \mathrm{diag}(h_1, \dots h_n) \in M_n(F)$ such that all $(h_i)_{v'}$ are of the same sign in $F_{v'}$ represents a suitable hermitian form.

(2) Consider the quaternion algebra $Q := Q(5, 73|E)$. Then Q is a division algebra[9] that splits at all the archimedean places of E and is moreover of the form $Q = Q(5, 73|F) \otimes_F E$. Set $D := Q$. As we have seen in Example I.7.2, there exists an involution of the second kind on D, and thus by Proposition I.7.18 we can assume that we have a definite involution σ on D. Then, for each even number $n = 2m \in \mathbb{N}$, we can find a hermitian form h on D^m such that any arithmetic subgroup $\Gamma \subset \mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})(F)$ gives rise to a cocompact discrete subgroup of $SL_n(\mathbb{R})$.

4. Construction of Geometric Cycles for $SL_n(\mathbb{R})$

Let F be a totally real number field of degree $r \geq 2$ and let E/F be a quadratic extension such that there is exactly one archimedean place $v \in V_\infty(F)$ that is decomposed in E, and all other archimedean places are nondecomposed. Let us denote the nontrivial

[8] The existence of such a place (even of infinitely many) is provided by Chebotarev's density Theorem, see [**51**, Thm. 1.2].

[9] A proof that Q is indeed a division algebra over $\mathbb{Q}(\sqrt[4]{2})$ can be found in Appendix A.

Galois automorphism of E/F by ι. Let D be a central division algebra of degree d over E with a definite involution of the second kind σ. Let $m \in \mathbb{N}$ be arbitrary and set $n := dm$.

We have seen in Theorem 3.2 that under certain conditions one can find a hermitian form $h\colon D^m \times D^m \to D$ such that all arithmetic subgroups Γ of the algebraic group $\mathbf{G}' := \mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ give rise to cocompact discrete subgroups of $SL_n(\mathbb{R})$. In this section, we discuss different choices of D and σ for which Theorem 3.2(1) applies. Moreover, in these cases we define morphisms on the algebraic group $\mathbf{G}'$ and study the corresponding special cycles.

4.1. Case 1: $\mathbf{D} = \mathbf{E}$. Let us first consider the case $D = E$. The nontrivial Galois automorphism ι of E/F is the only involution of the second kind on E, thus we set $\sigma := \iota$. Obviously, σ is a definite involution. Let $m \in \mathbb{N}$ be an arbitrary natural number. Then n equals m and D, E, F, σ and m satisfy the conditions in (1) of Theorem 3.2. Therefore, we can find a hermitian form h on D^m such that any arithmetic subgroup $\Gamma \subset \mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ gives rise to a cocompact discrete subgroup of $SL_n(\mathbb{R})$. In this case, H can be chosen as a diagonal matrix with entries in F. For technical reasons, we choose H such that it is positive definite at the decomposed real place v, i. e. such that it has only positive eigenvalues under the embedding corresponding to v. If m is even, we assume in addition that H is a symplectic matrix, that is, it commutes with the matrix J_m. Such a choice of H clearly exists, take the identity matrix, for example.

Now we can define certain maps on $SL_m(D)$ of which we will later show that they give rise to morphisms of the algebraic group $\mathbf{G}'$. Set

$$\begin{aligned}
\theta\colon SL_m(D) &\to SL_m(D), \quad x \mapsto H^{-1}(x^t)^{-1}H \\
\nu_{p,q}\colon SL_m(D) &\to SL_m(D), \quad x \mapsto H^{-1}I_{p,q}(x^t)^{-1}I_{p,q}H, \quad \text{for } p,q \in \mathbb{N}\setminus\{0\},\ p+q=m, \\
\mu\colon SL_m(D) &\to SL_m(D), \quad x \mapsto H^{-1}J_m(x^t)^{-1}J_m^{-1}H, \qquad \text{if } m \text{ is even and } m > 2.
\end{aligned}$$

These are automorphisms of $SL_m(D)$ of order two. For $m = 2$, the map μ coincides with the identity automorphisms, so we assume $m > 2$ for μ. Note that θ commutes with any of the other automorphisms.

4.2. Case 2: D is a Quaternion Algebra. Now we let D be a quaternion division algebra over E that admits an involution of the second kind σ and splits at the places w_1 and w_2 of E lying above v. By Proposition I.7.18, we may assume that σ is definite. By Albert's Theorem (cf. I.7.2) we can suppose that D is of the form

$$D = Q(a, b \,|\, F) \otimes_F E = Q(a, b \,|\, E)$$

for some $a, b \in F^\times$ and $\sigma = \tau_{c,0} \otimes \iota$, where $\tau_{c,0}$ denotes the conjugation on the quaternion algebra $Q(a, b \,|\, F)$.

Let $m \in \mathbb{N}$ be arbitrary and set $n := 2m$. Then D, E, F, σ and m satisfy the conditions of Theorem 3.2 (1) and we can find a hermitian form h such that arithmetic subgroups of $\mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ give rise to discrete cocompact subgroups of $SL_n(\mathbb{R})$. Again we assume for technical reasons that the matrix H of h is a diagonal matrix in $M_m(F)$ that is positive definite under the embedding corresponding to the decomposed place v.

We aim to define certain maps on $SL_m(D)$, analogous to the situation where D is a field. However, for the definition of θ and $\nu_{p,q}$ we need to make some technical preparations. The reason lies in the desired behaviour at the archimedean places that these maps will have as morphisms of the algebraic groups and will become apparent below.

Recall the definition of the orthogonal involutions τ_i, τ_j and τ_k on a quaternion algebra from Example I.7.2.

PROPOSITION 4.3. *Let E be a number field and $Q := Q(a,b|E)$ a quaternion algebra that splits at a real place w of E. Then there exist orthogonal involutions τ and $\tau_{(1,-1)}$ in $\{\tau_i, \tau_j, \tau_k\}$ and an isomorphism $Q(a,b|E_w) \to M_2(\mathbb{R})$ such that*

$$(Q(a,b\,|\,E_w), \tau \otimes \mathrm{id}) \cong (M_2(\mathbb{R}),\, x \mapsto x^t)$$
$$\textit{and } (Q(a,b\,|\,E_w), \tau_{(1,-1)} \otimes \mathrm{id}) \cong (M_2(\mathbb{R}),\, x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) x^t \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)).$$

PROOF. Let a_w and b_w denote the images of a and b under the embedding corresponding to w. Since D splits at w, exactly one of the elements a_w, b_w and $-a_w b_w$ is negative, i. e. there exists exactly one basis element $e_0 \in \{i,j,k\}$ such that e_0^2 is negative. Denote the remaining nontrivial basis elements by e_1 and e_2. We set $\tau := \tau_{e_0}$ and $\tau_{(1,-1)} := \tau_{e_1}$.

It remains to show the existence of a splitting isomorphism

$$\varphi\colon Q(a,b\,|\,E_w) \to M_2(\mathbb{R})$$

that maps $\tau_{e_0} \otimes \mathrm{id}$ onto $x \mapsto x^t$ and $\tau_{e_1} \otimes \mathrm{id}$ onto $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) x^t \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Let φ be the $\mathbb{R}$-linear map given on the basis of $Q(a,b\,|\,E_w)$ by

$$\begin{aligned} 1 &\mapsto I_2 \\ e_0 \otimes 1 &\mapsto \begin{pmatrix} 0 & \sqrt{-e_0^2} \\ -\sqrt{-e_0^2} & 0 \end{pmatrix} \\ e_1 \otimes 1 &\mapsto \begin{pmatrix} 0 & \sqrt{e_1^2} \\ \sqrt{e_1^2} & 0 \end{pmatrix} \\ e_2 \otimes 1 &\mapsto \begin{pmatrix} \sqrt{e_2^2} & 0 \\ 0 & -\sqrt{e_2^2} \end{pmatrix}. \end{aligned}$$

Then $\varphi\colon Q(a,b\,|\,E_w) \to M_2(\mathbb{R})$ is a well-defined isomorphism under which $\tau_{e_0} \otimes \mathrm{id}$ goes over to $x \mapsto x^t$ and $\tau_{e_1} \otimes \mathrm{id}$ goes over to $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) x^t \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. □

REMARK. Note that the two orthogonal involutions τ and $\tau_{(1,-1)}$ commute and that we have $\tau \circ \tau_{(1,-1)} = \tau_{(1,-1)} \circ \tau = \mathrm{Int}(e_2)$. Moreover, τ commutes with the conjugation τ_c of Q and we have $\tau \circ \tau_c = \mathrm{Int}(e_0)$.

Let us return to our specific choice of a division algebra $D = Q(a,b|F) \otimes_F E$ as described above. Applying Proposition 4.3 to D, we get the existence of orthogonal involutions τ and $\tau_{(1,-1)}$ that are mapped to $x \mapsto x^t$ and $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) x^t \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ under a suitable splitting isomorphism at the real place w_1.

Using these involutions, we can now define the following automorphisms of order two on $SL_m(D)$:

$$\begin{aligned} \theta\colon SL_m(D) &\to SL_m(D), & x &\mapsto H^{-1}(\tau(x)^t)^{-1}H, \\ \mu\colon SL_m(D) &\to SL_m(D), & x &\mapsto H^{-1}(\tau_c(x)^t)^{-1}H, & &\text{for } m > 1, \\ \nu_{p,q}\colon SL_m(D) &\to SL_m(D), & x &\mapsto H^{-1}I_{p/2,q/2}(\tau(x)^t)^{-1}I_{p/2,q/2}H, \\ \nu_{n/2,n/2}\colon SL_m(D) &\to SL_m(D), & x &\mapsto H^{-1}(\tau_{(1,-1)}(x)^t)^{-1}H, \end{aligned}$$

for positive, even integers p and q such that $p+q=n$. Again, note that θ commutes with any of the other automorphisms.

4.4. In both cases $D = E$ and $D = Q(a, b|E)$, we have thus defined the maps θ, $\nu_{p,q}$ and μ on $SL_m(D)$. In the following paragraphs, we will treat both cases simultaneously whenever possible. To avoid case distinctions, we put in place the following general assumptions: Whenever we deal with the maps $\nu_{p,q}$ it should be understood that the parameters p and q are nonzero natural numbers satisfying $p + q = n$. Moreover, if D is a quaternion algebra, we assume that both p and q are even or that $p = q = n/2$. Furthermore, statements involving the map μ are only applicable in case n is even and $n > 2$.

The maps θ, $\nu_{p,q}$ and μ are basically built out of E-linear maps (involutions of the first kind on $SL_m(D)$) and the group inversion, so they define E-rational morphisms $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$ and $\boldsymbol{\mu}$ on the algebraic E-group $\mathbf{SL_m(D)}$. By I.2.2, these give rise to F-rational automorphisms on $\mathrm{Res}_{E/F}\,\mathbf{SL_m(D)}$ that can be restricted to the group $\mathbf{G}'$ if and only if they commute with the morphism $\boldsymbol{\psi}$ whose fixed points in $\mathrm{Res}_{E/F}\,\mathbf{SL_m(D)}$ define $\mathbf{G}'$.

LEMMA 4.5. *The F-rational morphisms $\mathrm{Res}_{E/F}\,\boldsymbol{\theta}$, $\mathrm{Res}_{E/F}\,\boldsymbol{\nu_{p,q}}$ and $\mathrm{Res}_{E/F}\,\boldsymbol{\mu}$ defined on $\mathrm{Res}_{E/F}\,\mathbf{SL_m(D)}$ commute with $\boldsymbol{\psi}$. In particular, they can be restricted to the group $\mathrm{Fix}(\boldsymbol{\psi}, \mathrm{Res}_{E/F}\,\mathbf{SL_m(D)}) = \mathbf{G}'$.*

PROOF. As in the proof of Lemma 1.5, it suffices to show that the morphisms commute on the F-rational points of $\mathrm{Res}_{E/F}\,\mathbf{SL_m(D)}$, i. e. on $SL_m(D)$. We claim the following: *Let γ be an antiautomorphism of order two on $SL_m(D)$ such that $\gamma(H) = H$ and γ commutes with σ^t. Then the map $x \mapsto H^{-1}\gamma(x)^{-1}H$ commutes with $\boldsymbol{\psi}(F)$.*

Recall that for $x \in SL_m(D)$, we have $\boldsymbol{\psi}(F)(x) = \rho(x) = H^{-1}\sigma(x^t)^{-1}H$. Then the claim is proved by a straightforward calculation, where one has to use $H = H^t$, $H = \sigma(H) = \gamma(H)$ and $\sigma(\gamma(x))^t = \gamma(\sigma(x)^t)$ at the right places: Let $x \in SL_m(D)$, then

$$\begin{aligned}\boldsymbol{\psi}(F)(H^{-1}\gamma(x)^{-1}H) =&\rho(H^{-1}\gamma(x)^{-1}H)^{-1} = H^{-1}\sigma((H^{-1}\gamma(x)^{-1}H)^t)^{-1}H\\ =&H^{-1}\sigma(H)\sigma(\gamma(x))^t\sigma(H^{-1})H = H^{-1}\gamma(H)\gamma((\sigma(x)^t)^{-1})^{-1}\gamma(H^{-1})H\\ =&H^{-1}\gamma(H^{-1}(\sigma(x)^t)^{-1}H)^{-1}H = H^{-1}\gamma(\rho(x)^{-1})^{-1}H\\ =&H^{-1}\gamma(\boldsymbol{\psi}(F)(x))^{-1}H.\end{aligned}$$

With the help of this claim, we can easily prove the lemma: Note that the maps θ, $\nu_{p,q}$ and μ are of the form $x \mapsto H^{-1}\gamma(x)^{-1}H$ for a suitable antiautomorphism

$$\begin{aligned}\gamma &\in \{x \mapsto x^t,\, x \mapsto I_{p,q}x^tI_{p,q},\, x \mapsto J_m x^t J_m^{-1}\} && \text{if } D = E\\ \text{and } \gamma &\in \{\tau^t,\, \mathrm{Int}(I_{p,q}) \circ \tau^t,\, \tau^t_{(1,-1)},\, \tau^t_c\} && \text{if } D = Q(a, b|E).\end{aligned}$$

Since H is a diagonal matrix with entries in F that is in addition symplectic if $D = E$ and n even, we have $\gamma(H) = H$ for all choices of γ. Moreover, it is straightforward to check that γ commutes with σ^t in all cases (here we use that the involutions τ and $\tau_{(1,-1)}$ are of the form τ_e for a basis element e of $Q(a, b\,|\,E)$, cf. proof of Proposition 4.3.) Therefore, these choices of γ satisfy the conditions of the claim and the maps θ, $\nu_{p,q}$ and μ commute with $\boldsymbol{\psi}(F)$. □

The lemma implies the existence of F-rational morphisms $\mathrm{Res}_{E/F}\,\boldsymbol{\theta}$, $\mathrm{Res}_{E/F}\,\boldsymbol{\nu_{p,q}}$ and $\mathrm{Res}_{E/F}\,\boldsymbol{\mu}$ on $\mathbf{G}'$. The fixed points of these morphisms define algebraic subgroups of $\mathbf{G}'$ whose F_v-rational points are certain Lie subgroups of $SL_n(\mathbb{R})$. We are now going to determine these subgroups. Recall the definition of $GL_r^{(1)}(\mathbb{C}) := \{g \in GL_r(\mathbb{C}), |\det(g)| = 1\}$ for a natural number r.

PROPOSITION 4.6. *Let F be a totally real number field. For $\mathbf{G}'$ defined as above, we have $\mathbf{G}'(F_v) \cong SL_n(\mathbb{R})$ and the fixed points of the morphisms $\operatorname{Res}_{E/F} \boldsymbol{\theta}$, $\operatorname{Res}_{E/F} \boldsymbol{\nu}_{p,q}$, $\operatorname{Res}_{E/F}(\boldsymbol{\nu}_{\boldsymbol{p},\boldsymbol{q}} \circ \boldsymbol{\theta})$, $\operatorname{Res}_{E/F} \boldsymbol{\mu}$ and $\operatorname{Res}_{E/F}(\boldsymbol{\mu} \circ \boldsymbol{\theta})$ define the following subgroups of $SL_n(\mathbb{R})$:*

$$\begin{aligned}
\operatorname{Fix}(\operatorname{Res}_{E/F} \boldsymbol{\theta}, \mathbf{G}')(F_v) &\cong SO(n),\\
\operatorname{Fix}(\operatorname{Res}_{E/F} \boldsymbol{\nu}_{p,q}, \mathbf{G}')(F_v) &\cong SO(p,q),\\
\operatorname{Fix}(\operatorname{Res}_{E/F}(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta}), \mathbf{G}')(F_v) &\cong S(GL_p(\mathbb{R}) \times GL_q(\mathbb{R})),\\
\operatorname{Fix}(\operatorname{Res}_{E/F} \boldsymbol{\mu}, \mathbf{G}')(F_v) &\cong Sp(n,\mathbb{R}),\\
\operatorname{Fix}(\operatorname{Res}_{E/F}(\boldsymbol{\mu} \circ \boldsymbol{\theta}), \mathbf{G}')(F_v) &\cong GL^{(1)}_{n/2}(\mathbb{C}).
\end{aligned}$$

In particular, $Res_{E/F}\boldsymbol{\theta}$ induces a Cartan involution on $SL_n(\mathbb{R})$.

PROOF. We start with a general observation: Let $\boldsymbol{\varphi}$ denote an E-rational morphism of $\mathbf{SL_m}(\mathbf{D})$ such that $\operatorname{Res}_{E/F} \boldsymbol{\varphi}$ commutes with $\boldsymbol{\psi}$. Then $\boldsymbol{\varphi}$ can be restricted to $\mathbf{G}'$ and we have

$$\operatorname{Fix}(\operatorname{Res}_{E/F} \boldsymbol{\varphi}, \mathbf{G}') = \operatorname{Fix}(\operatorname{Res}_{E/F} \boldsymbol{\varphi}, \operatorname{Res}_{E/F} \mathbf{SL_m}(\mathbf{D})) \cap \mathbf{G}'$$

as a subgroup of $\operatorname{Res}_{E/F} \mathbf{SL_m}(\mathbf{D})$. At the F_v-rational points, there exists an isomorphism $\operatorname{Res}_{E/F}(\mathbf{SL_m}(\mathbf{D}))(F_v) \cong \mathbf{SL_m}(\mathbf{D})(E_{w_1} \oplus E_{w_2}) \cong SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$, cf. Proposition 2.2. Moreover,

$$\begin{aligned}
&\operatorname{Fix}(\operatorname{Res}_{E/F} \boldsymbol{\varphi}, \operatorname{Res}_{E/F} \mathbf{SL_m}(\mathbf{D}))(F_v) = \operatorname{Res}_{E/F} \operatorname{Fix}(\boldsymbol{\varphi}, \mathbf{SL_m}(\mathbf{D}))(F_v)\\
&= \operatorname{Fix}(\boldsymbol{\varphi}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \times \operatorname{Fix}(\boldsymbol{\varphi}, \mathbf{SL_m}(\mathbf{D}))(E_{w_2}) \subset SL_n(\mathbb{R}) \times SL_n(\mathbb{R}).
\end{aligned}$$

On the other hand, the defining condition of $\mathbf{G}'$ identifies the two copies of $SL_n(\mathbb{R})$ (cf. Proposition 2.6). Thus we can restrict to one of the components (we choose the first one without loss of generality) and get

$$\begin{aligned}
\operatorname{Fix}(\operatorname{Res}_{E/F} \boldsymbol{\varphi}, \mathbf{G}')(F_v) &= \operatorname{Fix}(\operatorname{Res}_{E/F} \boldsymbol{\varphi}, \operatorname{Res}_{E/F} \mathbf{SL_m}(\mathbf{D}))(F_v) \cap \mathbf{G}'(F_v)\\
&\cong \operatorname{Fix}(\boldsymbol{\varphi}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \subset SL_n(\mathbb{R}).
\end{aligned}$$

Let us now specify $\boldsymbol{\varphi}$ to be one of the above morphisms. For $\boldsymbol{\varphi} = \boldsymbol{\theta}$ or $\boldsymbol{\varphi} = \boldsymbol{\nu}_{\boldsymbol{p},\boldsymbol{q}}$, the group $\operatorname{Fix}(\boldsymbol{\varphi}, \mathbf{SL_m}(\mathbf{D}))$ is a special unitary group w. r. t. a hermitian form and an orthogonal involution. Therefore, by Proposition 2.4, we have $\operatorname{Fix}(\boldsymbol{\varphi}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \cong SO(p', q')$ for suitable $p', q' \in \mathbb{N}$. Since H is positive definite at the place v, it does not influence the signature (p', q'). In case $D = E$ it is clear from the definitions of θ and $\nu_{p,q}$ that the signature comes from the matrices $I_{p,q}$. In case $D = Q(a, b|E)$, we can conclude from Proposition 4.3 that under a suitable splitting at the place w_1, the involution τ^t is mapped to $x \mapsto x^t$ on $M_n(\mathbb{R})$ and $\tau^t_{(1,-1)}$ is mapped to $x \mapsto \operatorname{Int}(I_{n/2,n/2})(x^t)$. Moreover, the matrices $I_{p/2,q/2}$ are mapped to $I_{p,q}$ at the place w_1. Therefore, for both choices of D, we get

$$\operatorname{Fix}(\operatorname{Res}_{E/F}(\boldsymbol{\theta}), \mathbf{G}')(F_v) = \operatorname{Fix}(\boldsymbol{\theta}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \cong SO(n)$$

and

$$\operatorname{Fix}(\operatorname{Res}_{E/F}(\boldsymbol{\nu}_{p,q}), \mathbf{G}')(F_v) = \operatorname{Fix}(\boldsymbol{\nu}_{p,q}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \cong SO(p,q).$$

In particular, $\operatorname{Res}_{E/F} \boldsymbol{\theta}$ induces a Cartan involution on $SL_n(\mathbb{R})$, as the group of F_v-rational points of its fixed point group is isomorphic to $SO(n)$, a maximal compact subgroup of $SL_n(\mathbb{R})$.

For $\boldsymbol{\varphi} = \boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta}$ one can easily see from the definition of $\nu_{p,q}$ that

$$\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta} = \begin{cases} \mathrm{Int}(I_{p,q}) & \text{if } D = E, \\ \mathrm{Int}(I_{p/2,q/2}) & \text{if } D = Q(a, b|E) \text{ and } p, q \text{ even}, \\ \mathrm{Int}(\mathrm{diag}(e_2, \dots, e_2)) & \text{if } D = Q(a, b|E) \text{ and } p = q = n/2. \end{cases}$$

Here, the statement in the last line follows from the remark following Proposition 4.3. Under a suitable splitting isomorphism at the place w_1 of E, these morphisms go over to $\mathrm{Int}(I_{p,q})$ on $SL_n(\mathbb{R})$ (to see this in the third case, use the isomorphism given in Proposition 4.3). Therefore, we have

$$\begin{aligned} \mathrm{Fix}(\mathrm{Res}_{E/F}(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta}), \mathbf{G}')(F_v) &= \mathrm{Fix}(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \\ &\cong \{x \in SL_n(\mathbb{R}) \mid I_{p,q} x I_{p,q} = x\} \\ &\cong S(GL_p(\mathbb{R}) \times GL_q(\mathbb{R})). \end{aligned}$$

Let now $\boldsymbol{\varphi} = \boldsymbol{\mu}$: The group $\mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{SL_m}(\mathbf{D}))$ is either a special unitary group with respect to a skew-hermitian form and an orthogonal involution (in case $D = E$, the matrix $H^{-1} J_m$ occurring in the definition of μ is skew-symmetric and hence it describes a skew-hermitian form over E) or a special unitary group with respect to a hermitian form and a symplectic involution (in case D is a quaternion algebra, H is a diagonal matrix with entries in F and thus τ_c-invariant). In both cases, Proposition 2.4 implies

$$\mathrm{Fix}(\mathrm{Res}_{E/F}(\boldsymbol{\mu}), \mathbf{G}')(F_v) = \mathrm{Fix}(\boldsymbol{\mu}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \cong Sp(n, \mathbb{R}).$$

Finally, we consider $\boldsymbol{\varphi} = \boldsymbol{\mu} \circ \boldsymbol{\theta}$. If $D = E$, we have $(\boldsymbol{\mu} \circ \boldsymbol{\theta})(E) = \mathrm{Int}\, J_m$. In the case $D = Q(a, b \,|\, E)$, we note that $(\boldsymbol{\mu} \circ \boldsymbol{\theta})(E) = \tau_c \circ \tau = \mathrm{Int}\,\mathrm{diag}(e_0, \dots, e_0)$ on $M_m(D)$, by the remark following Proposition 4.3. Under a suitable splitting isomorphism, $\mathrm{diag}(e_0, \dots, e_0)$ is mapped to J_n at the place w_1 of D (see Proposition 4.3). Therefore, for both choices of D, we have

$$\begin{aligned} \mathrm{Fix}(\mathrm{Res}_{E/F}(\boldsymbol{\mu} \circ \boldsymbol{\theta}), \mathbf{G}')(F_v) &= \mathrm{Fix}(\boldsymbol{\mu} \circ \boldsymbol{\theta}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \\ &\cong \{x \in SL_n(\mathbb{R}) \mid J_n x J_n^{-1} = x\} \\ &\cong GL^{(1)}_{n/2}(\mathbb{C}). \end{aligned}$$ □

4.7. Now that we have defined certain F-rational morphisms on $\mathbf{G}'$ and studied their fixed point groups, we are ready to define the corresponding special cycles. To do this, we pass to the algebraic $\mathbb{Q}$-group $\mathbf{G} := \mathrm{Res}_{F/\mathbb{Q}}\, \mathbf{G}'$. By I.2.2, $\mathbf{G}$ is an algebraic $\mathbb{Q}$-group with $\mathbf{G}(\mathbb{Q}) \cong \mathbf{G}'(F)$ and

$$\mathbf{G}(\mathbb{R}) \cong \mathbf{G}'(\mathbb{R} \otimes_{\mathbb{Q}} F) = \prod_{v' \in V_\infty(F)} \mathbf{G}'(F_{v'}) = SL_n(\mathbb{R}) \times \prod_{v' \in V_\infty, v' \neq v} SU(n).$$

Moreover, we have $\mathbb{Q}$-rational morphisms $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$ and $\boldsymbol{\mu}$ of order two on $\mathbf{G}$ that are induced from the corresponding morphisms of $\mathbf{G}'$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup of $\mathbf{G}$. The image of Γ under the isomorphism $\mathbf{G}(\mathbb{Q}) \cong \mathbf{G}'(F)$ is an arithmetic subgroup of $\mathbf{G}'$ and thus it gives rise to a discrete cocompact subgroup of $SL_n(\mathbb{R})$ that we will still denote by Γ for simplicity of notation.[10] Let K' denote a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ and $X := K' \backslash \mathbf{G}(\mathbb{R})$ the symmetric space attached to $\mathbf{G}(\mathbb{R})$. Since $\mathbf{G}(\mathbb{R})$ is a product of

[10] To be precise, the arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ can be regarded as a subgroup of $\mathbf{G}(\mathbb{R}) = SL_n(\mathbb{R}) \times$ compact factors, and the discrete cocompact subgroup is in fact the projection of Γ to the noncompact factor of $\mathbf{G}(\mathbb{R})$, see also II.5.2.

$SL_n(\mathbb{R})$ and compact factors, we have $X \cong SO(n)\backslash SL_n(\mathbb{R})$ and Γ acts on X by right translations. Note that X is a symmetric space of dimension

$$\dim X = \dim(SL_n(\mathbb{R})) - \dim(SO(n)) = n^2 - 1 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2} - 1.$$

Let now $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a torsion-free arithmetic subgroup of $\mathbf{G}$ and assume that Γ and K' are invariant under the morphisms $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$ and $\boldsymbol{\mu}$.[11]. Then the morphisms induce certain special geometric cycles in X/Γ, as explained in Section II.1.

THEOREM 4.8. *The morphisms $\boldsymbol{\nu}_{p,q}$ and $\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta}$ of $\mathbf{G}$ induce a family of pairs of special geometric cycles $C(\boldsymbol{\nu}_{p,q})$, $C(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta})$ in X/Γ, for positive integers p and q with $p+q=n$ if $\mathbf{G}$ comes from a special unitary group over an algebraic number field, and for positive integers p and q with $p+q=n$ and p and q even or $p=q=n/2$ if $\mathbf{G}$ comes from a special unitary group over a quaternion algebra. In case n is even and $n>2$, the morphisms $\boldsymbol{\mu}$ and $\boldsymbol{\mu} \circ \boldsymbol{\theta}$ induce a pair of geometric cycles $C(\boldsymbol{\mu})$, $C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$ in X/Γ. The properties of these cycles are summarized in Table 1.*

$C = C(\boldsymbol{\varphi})$	$\mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{G})(\mathbb{R}) \cong$	$\dim C$
$C(\boldsymbol{\nu}_{p,q})$	$SO(p,q)$	pq
$C(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta})$	$S(GL_p(\mathbb{R}) \times GL_q(\mathbb{R}))$	$\frac{p^2+q^2+n}{2} - 1$
$C(\boldsymbol{\mu})$	$Sp(n,\mathbb{R})$	$\frac{n^2+2n}{4}$
$C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$	$GL^{(1)}_{n/2}(\mathbb{C})$	$\frac{n^2}{4} - 1$

TABLE 1. Geometric cycles in $SO(n)\backslash SL_n(\mathbb{R})/\Gamma$. The isomorphism in the second column is up to compact factors and the lower half of the table is only applicable if n is even and $n>2$.

PROOF. The existence of the cycles is clear from Section II.1.3. The isomorphisms in the second column of the table follow from Proposition 4.6 and the fact that $\mathbf{G}(\mathbb{R}) \cong \mathbf{G}'(F_v)$ up to compact factors. The dimensions of the cycles can be computed as the dimensions of the associated symmetric spaces (cf. I.3.4 (2)), using the dimensions of the occurring real Lie groups and their maximal compact subgroups (for a list of dimensions of classical Lie groups, see e.g. [**26**, Table IV, p.516]). Note that both $SO(p,q)$ and $S(GL_p(\mathbb{R}) \times GL_q(\mathbb{R}))$ have maximal compact subgroup $S(O(p) \times O(q))$ and both $Sp(n,\mathbb{R})$ and $GL^{(1)}_{n/2}(\mathbb{C})$ have maximal compact subgroup isomorphic to $U(n/2)$.[12] $\square$

REMARK. Note that in the case where $\mathbf{G}$ comes from a special unitary group over a quaternion algebra, we do not get special cycles related to the real Lie groups $SO(p,q)$ with p and q odd (unless $n = 2 \mod 4$ and $p=q=n/2$). To construct such cycles

[11]Such a choice of K' is possible by [**26**, Thm. 13.5]

[12]To see this for the symplectic group, consider the embedding $\phi : GL_{n/2}(\mathbb{C}) \to GL_n(\mathbb{R})$ that maps $X = A + iB$ to $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$, for $A, B \in GL_{n/2}(\mathbb{R})$. Then

$$\phi(U(n/2)) = \{x \in SL_n(\mathbb{R}), xx^t = I_n, J_n(x^t)^{-1}J_n^{-1} = x\} = Sp(n,\mathbb{R}) \cap SO(n).$$

for p and q odd, one would have to define an orthogonal involution on $M_m(D)$ that has signature (p,q) at the real place w_1 of E, i. e. that is mapped to the involution $I_{p,q}x^tI_{p,q}$ under a suitable splitting isomorphism. Such an involution would be of the form $\mathrm{Int}\, A\circ\tau^t$, where τ is the orthogonal involution defined as in Proposition 4.3 and $A\in M_m(D)$ is a suitable τ^t-invariant element whose image under the splitting isomorphism at the place w_1 is a symmetric matrix of signature (p,q). Moreover, for this involution to define a morphism of the algebraic group $\mathbf{G}$ that commutes with $\boldsymbol{\theta}$, such an element A would have to satisfy $\sigma(A^t)=A$ and $A^2=Id_{M_m(D)}$. It is not clear whether or not such an element exists.

5. Construction of Geometric Cycles for $SL_n(\mathbb{C})$

We will work in the following general setting: Let F_0 be a totally real number field and let E/F_0 be a totally complex biquadratic extension.[13] Assume that there is a quadratic extension F/F_0 such that F is a subfield of E with exactly one complex place v and denote by v_0 the real place of F_0 with $v|v_0$. Moreover, let L and L' denote the other two intermediate fields of the extension E/F_0. Then L is a quadratic extension of F_0 that has two real places v_1,v_2 lying above v_0 and only complex archimedean places otherwise, and L' is a totally complex quadratic extension of F_0 (see Corollary B.3). This is to say, we have $F=F_0(\sqrt{D_1})$, $L=F_0(\sqrt{D_2})$ and $L'=F_0(\sqrt{D_1D_2})$ for D_1 and D_2 in F_0 such that neither of D_1, D_2 and D_1D_2 is a square in F_0 and $(D_1)_{v_0}<0$, $(D_2)_{v_0}>0$ and $(D_1)_{v'}>0>(D_2)_{v'}$ for $v'\in V_\infty(F_0)$ and $v'\neq v_0$. We write ι and ω for the nontrivial Galois automorphisms of E/F and E/L, respectively. Then ι and ω are both trivial on F_0 and are hence elements of the Galois group of E/F_0. In fact, they generate this group and the third nontrivial element $\iota\circ\omega=\omega\circ\iota$ is the nontrivial Galois automorphism of E over L'. Moreover, we note that $\iota|_L$ is the nontrivial Galois automorphism of the quadratic extension L/F_0, $\omega|_F$ is the one of F/F_0 and $\iota|_{L'}=\omega|_{L'}$ is the one of L'/F_0. The field extension E/F_0, its intermediate subfields and the nontrivial Galois automorphisms corresponding to each extension are illustrated in the following diagram:

$$\begin{tikzcd}
& E \arrow[dl, dash, "\iota"'] \arrow[d, dash, "\omega"] \arrow[dr, dash, "\iota\circ\omega=\omega\circ\iota"] & \\
F \arrow[dr, dash, "\omega|_F"'] & L \arrow[d, dash, "\iota|_L"] & L' \arrow[dl, dash, "\omega|_{L'}=\iota|_{L'}"] \\
& F_0 &
\end{tikzcd}$$

Now we let D be a division algebra of degree d over E, let $m\in\mathbb{N}$ be arbitrary and set $n:=dm$.

In this section, we study examples of division algebras D and involutions σ of the second kind on D such that condition (2) in Theorem 3.2 is satisfied. This yields the existence of a hermitian form h on D^m such that any arithmetic subgroup of the group $\mathbf{G}':=\mathbf{SU_m(h,D,\boldsymbol{\sigma})}$ gives rise to a discrete, cocompact subgroup of $SL_n(\mathbb{C})$. Analogous to the real case, the two classes of examples are the field E itself and quaternion division algebras over E. In both cases, we will define certain morphisms of $\mathbf{G}'$ and study the corresponding special cyles.

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[13]A *biquadratic extension* of a number field F_0 is an extension of degree 4 with Galois group $\mathrm{Gal}(E/F_0)\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. For details about biquadratic extensions, see Appendix B.

5.1. Case 1: $\mathbf{D} = \mathbf{E}$. Set $D := E$ and $\sigma := \iota$. Then σ is clearly a definite involution on D. Let $m \in \mathbb{N}$ be arbitrary. Then D, E, F, σ and m satisfy the conditions of Theorem 3.2 (2) and we can choose h accordingly. Let H denote a diagonal matrix representation of h such that $H \in M_m(F_0)$ and H is symplectic in case m is even.[14] Moreover, for technical reasons, we assume that H is positive definite at the place v_0 of F_0. Now we can define the following automorphisms of order two on $SL_m(D)$:

$$\begin{aligned}
\theta\colon SL_m(D) &\to SL_m(D), \quad x \mapsto H^{-1}(\omega(x)^t)^{-1}H,\\
\nu_{p,q}\colon SL_m(D) &\to SL_m(D), \quad x \mapsto H^{-1}I_{p,q}(\omega(x^t))^{-1}I_{p,q}H,\\
\eta\colon SL_m(D) &\to SL_m(D), \quad x \mapsto H^{-1}(x^t)^{-1}H,
\end{aligned}$$

for positive integers p and q with $p + q = m$. Moreover, if m is even and $m > 2$, we set

$$\mu\colon SL_m(D) \to SL_m(D), \quad x \mapsto H^{-1}J_m(x^t)^{-1}J_m^{-1}H.$$

The map θ commutes with any of the other automorphisms.

5.2. Case 2: D is a Quaternion Algebra. Now we consider the case where D is a quaternion algebra. We have F_0, F, L, L' and E as above. Let D' over L' be a quaternion division algebra that does not split over E and admits an involution γ of the second kind (w. r. t. the subfield $F_0 \subset L'$). By Proposition I.7.18, we may assume that γ is definite. Moreover, by Albert's Theorem, we find a quaternion division algebra $D_0 = Q(a, b|F_0)$ over F_0 with $a, b \in F_0^\times$ such that $(D', \gamma) \cong (D_0 \otimes_{F_0} L', \tau_{c,0} \otimes \omega|_{L'})$, where $\tau_{c,0}$ denotes the conjugation on D_0.

Lemma 5.3. *In the above situation, the involutions $\tau_{c,0} \otimes \omega|_F$ and $\tau_{c,0} \otimes \iota|_L$ are definite involutions on the algebras $D_0 \otimes_{F_0} F$ and $D_0 \otimes_{F_0} L$, respectively.*

Proof. We prove the statement for $(D_0 \otimes_{F_0} F, \tau_{c,0} \otimes \omega|_F)$. The second case follows analogously.

Recall that F has only one complex place v, i. e. v_0 is the only real place of F_0 that is nondecomposed in F. We write F_{0,v_0} for the completion of F_0 at v_0 and set $D_{0,v_0} := D_0 \otimes_{F_0} F_{0,v_0}$. To see that $\tau_{c,0} \otimes \omega|_F$ is definite, we need to show the existence of an isomorphism $((D_0 \otimes_{F_0} F) \otimes_{F_0} F_{0,v_0}, (\tau_{c,0} \otimes \omega|_F) \otimes \mathrm{id}) \to (M_2(\mathbb{C}), {}^*)$ of F_{0,v_0}-algebras with involution of the second kind (cf. Definition I.7.17). For simplicity of notation, we denote both the complex place of F and the complex place of L' lying above v_0 by v. Then we have the following isomorphisms of F_{0,v_0}-algebras:

$$\begin{aligned}
(D_0 \otimes_{F_0} F) \otimes_{F_0} F_{0,v_0} &\cong D_0 \otimes_{F_0} F_v \cong D_0 \otimes_{F_0} (F_{0,v_0} \otimes_{F_{0,v_0}} F_v)\\
&\cong D_{0,v_0} \otimes_{F_{0,v_0}} F_v \cong D_{0,v_0} \otimes_{F_{0,v_0}} L'_v\\
&\cong D_0 \otimes_{F_0} (F_{0,v_0} \otimes_{F_{0,v_0}} L'_v) \cong D_0 \otimes_{F_0} L'_v\\
&\cong (D_0 \otimes_{F_0} L') \otimes_{F_0} F_{0,v_0}
\end{aligned}$$

The involution $(\tau_{c,0} \otimes \omega|_F) \otimes \mathrm{id}$ is mapped to $(\tau_{c,0} \otimes \omega|_{L'}) \otimes \mathrm{id}$ under these isomorphisms. The latter goes over to $x \mapsto x^*$ under a suitable splitting $(D_0 \otimes_{F_0} L') \otimes_{F_0} F_{0,v_0} \to M_2(\mathbb{C})$ because $\tau_{c,0} \otimes \omega|_{L'}$ is definite by assumption, hence the claim follows. □

Set $D := D_0 \otimes_{F_0} E = D' \otimes_{L'} E$. By our choice of D', this a quaternion division algebra over E that admits two involutions $\sigma := \tau_{c,0} \otimes \iota$ and $\sigma' := \tau_{c,0} \otimes \omega$ of the second

[14]Note that it follows from the proof of Theorem 3.2 that such a choice is always possible (take e. g. the trivial hermitian form).

kind. Note that σ is trivial on the subfield F of E, and σ' is trivial on the subfield L of E.

LEMMA 5.4. *The involutions σ and σ' are definite involutions on D.*

PROOF. We will prove the result only for σ', the other case will follow analogously. Recall that L has exactly two real places v_1 and v_2 and these places lie above v_0. Since E is totally complex, v_i is nondecomposed in E and we denote by w_i the place of E such that $w_i|v_i|v_0$, for $i \in \{1,2\}$. On the other hand, we have a complex place $v|v_0$ of F that must be decomposed in E and the two places lying above v are again w_1 and w_2. Then for $i \in \{1,2\}$, we have isomorphisms

$$\begin{aligned}(D \otimes_L L_{v_i}, \sigma' \otimes \mathrm{id}) &\cong (D_0 \otimes_{F_0} (E \otimes_L L_{v_i}), \tau_{c,0} \otimes (\omega \otimes \mathrm{id}))\\ &\cong (D_0 \otimes_{F_0} E_{w_i}, \tau_{c,0} \otimes {}^-) \cong (D_0 \otimes_{F_0} F_v, \tau_{c,0} \otimes {}^-)\\ &\cong (D_0 \otimes_{F_0} (F \otimes_{F_0} F_{0,v_0}), \tau_{c,0} \otimes (\omega|_F \otimes \mathrm{id})) \cong (M_2(\mathbb{C}), {}^*),\end{aligned}$$

of L_{v_i}-algebras with involution, where $^-$ denotes the complex conjugation. We have used that $E_{w_i} \cong F_v$ as F_0-algebras for $i \in \{1,2\}$ for the third isomorphism and Lemma 5.3 in the last step. □

Let $m \in \mathbb{N}$ be arbitrary and set $n := 2m$. Then D, E, F, σ and m satisfy the conditions of Theorem 3.2 (2), and we can choose a hermitian form h on D^m such that any arithmetic subgroup of $\mathbf{G}' := \mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ gives rise to a cocompact discrete subgroup of $SL_n(\mathbb{C})$. As above, h can be chosen such that its diagonal representation H is an element of $M_m(F_0)$ and we will restrict to this situation for technical reasons. Moreover, we suppose that H is positive definite at the real place v_0 of F_0.

We define the following maps on $SL_m(D)$:

$$\begin{aligned}\theta &\colon SL_m(D) \to SL_m(D), \quad x \mapsto H^{-1}(\sigma'(x)^t)^{-1}H,\\ \nu_{p,q} &\colon SL_m(D) \to SL_m(D), \quad x \mapsto H^{-1}I_{p/2,q/2}(\sigma'(x)^t)^{-1}I_{p/2,q/2}H,\\ \eta &\colon SL_m(D) \to SL_m(D), \quad x \mapsto H^{-1}(\tau_k(x)^t)^{-1}H,\\ \mu &\colon SL_m(D) \to SL_m(D), \quad x \mapsto H^{-1}(\tau_c(x)^t)^{-1}H, \quad \text{if } m > 1\end{aligned}$$

for positive, even integers p and q with $p + q = n$.

5.5. We have studied two situations in which arithmetic subgroups of the group $\mathbf{G}' = \mathbf{SU_m}(\mathbf{h}, \mathbf{D}, \boldsymbol{\sigma})$ give rise to cocompact discrete subgroups of $SL_n(\mathbb{C})$. In both cases $D = E$ and $D = Q(a,b|E)$, we have defined the maps $\theta, \nu_{p,q}$, η and μ on $SL_m(D)$. As in paragraph 4.4, we will assume from now on that p and q are positive integers such that $p + q = n$ and that p and q are both even, whenever we deal with the case where D is a quaternion algebra. Statements involving the map μ will again only be applicable if n is even and $n > 2$.

As in the real case, we would like the automorphisms from above to define morphisms of the algebraic group $\mathbf{G}'$. The maps η and μ are built out of E-linear maps (involutions of the first kind on $SL_m(D)$) and the group inversion, and hence they define E-rational morphisms $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ on the algebraic E-group $\mathbf{SL_m}(\mathbf{D})$, as expected. However, the maps θ and $\nu_{p,q}$ involve involutions of the second kind with respect to the subfield L of E, and therefore they only define L-rational morphisms $\boldsymbol{\theta}$ and $\boldsymbol{\nu}_{p,q}$ of the algebraic L-group $\mathrm{Res}_{E/L}\,\mathbf{SL_m}(\mathbf{D})$. On the other hand, the morphism $\boldsymbol{\psi}$ defining the algebraic group $\mathbf{G}'$ is an F-rational morphism of the group $\mathrm{Res}_{E/F}\,\mathbf{SL_m}(\mathbf{D})$. To work with all of these morphisms simultaneously, we need to pass to an algebraic group over the common subfield F_0 of F, L and E.[15]

Set $\mathbf{G}'' := \mathrm{Res}_{F/F_0} \mathbf{G}' = \mathrm{Fix}(\mathrm{Res}_{F/F_0} \boldsymbol{\psi}, \mathrm{Res}_{E/F_0} \mathbf{SL_m}(\mathbf{D}))$. By I.2.2, the morphisms $\boldsymbol{\eta}$, $\boldsymbol{\mu}$, $\boldsymbol{\theta}$ and $\boldsymbol{\nu}_{p,q}$ give rise to F_0-rational morphisms on $\mathrm{Res}_{E/F_0} \mathbf{SL_m}(\mathbf{D})$ via restriction of scalars. To see that they can be restricted to $\mathbf{G}''$, one needs to show that they commute with $\mathrm{Res}_{F/F_0} \boldsymbol{\psi}$:

Lemma 5.6. *The morphisms* $\mathrm{Res}_{E/F_0} \boldsymbol{\eta}$, $\mathrm{Res}_{E/F_0} \boldsymbol{\mu}$, $\mathrm{Res}_{L/F_0} \boldsymbol{\nu_{p,q}}$, *and* $\mathrm{Res}_{L/F_0} \boldsymbol{\theta}$ *of the algebraic F_0-group* $\mathrm{Res}_{E/F_0} \mathbf{SL_m}(\mathbf{D})$ *commute with* $\mathrm{Res}_{F/F_0} \boldsymbol{\psi}$. *In particular, they can be restricted to the group* $\mathbf{G}''$.

Proof. First of all, we note that as in the proof of Lemma 1.5, it suffices to show that the morphisms commute on the group of F_0-rational points of $\mathrm{Res}_{E/F_0} \mathbf{SL_m}(\mathbf{D})$, i. e. on $SL_m(D)$. We have $\mathrm{Res}_{F/F_0}(\boldsymbol{\psi})(F_0) = \boldsymbol{\psi}(F)$, therefore we can use the claim from Lemma 4.5. The maps $\nu_{p,q}$, η, μ and θ are of the form $H^{-1}\gamma(x)H$ for suitable antiautomorphisms γ of order two of $SL_m(D)$ that satisfy $\gamma(H) = H$ by our choice of H and commutes with σ^t. This can be verified by straightforward calculations (we use that the Galois automorphisms ω and ι of E/F_0 and the involutions τ_c and τ_k on a quaternion algebra commute). Therefore, as in Lemma 4.5, we can deduce that the associated morphisms commute with $\mathrm{Res}_{F/F_0} \boldsymbol{\psi}$. □

The lemma implies the existence of F_0-rational morphisms $\mathrm{Res}_{E/F_0} \boldsymbol{\eta}$, $\mathrm{Res}_{E/F_0} \boldsymbol{\mu}$, $\mathrm{Res}_{L/F_0} \boldsymbol{\nu_{p,q}}$, and $\mathrm{Res}_{L/F_0} \boldsymbol{\theta}$ on $\mathbf{G}''$. One can easily verify that these morphisms are of order two, that $\mathrm{Res}_{L/F_0} \boldsymbol{\theta}$ commutes with any of the other morphisms and that $\mathrm{Res}_{E/F_0} \boldsymbol{\eta}$ and $\mathrm{Res}_{E/F_0} \boldsymbol{\mu}$ commute. Their fixed points define algebraic subgroups of $\mathbf{G}''$ whose F_{0,v_0}-rational points are certain Lie subgroups of $SL_n(\mathbb{C})$. We determine these subgroups in the next Proposition:

Proposition 5.7. *The algebraic F_0-group $\mathbf{G}''$ satisfies $\mathbf{G}''(F_{0,v_0}) \cong SL_n(\mathbb{C})$. The fixed points of (certain compositions of) the above defined F_0-rational morphisms define the following subgroups of $SL_n(\mathbb{C})$:*

$$\begin{aligned}
\mathrm{Fix}(\mathrm{Res}_{L/F_0} \boldsymbol{\theta}, \mathbf{G}'')(F_{0,v_0}) &\cong SU(n),\\
\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\eta}, \mathbf{G}'')(F_{0,v_0}) &\cong SO(n,\mathbb{C})\\
\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\eta} \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta}, \mathbf{G}'')(F_{0,v_0}) &\cong SL(n,\mathbb{R})\\
\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\mu}, \mathbf{G}'')(F_{0,v_0}) &\cong Sp(n,\mathbb{C}),\\
\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\mu} \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta}, \mathbf{G}'')(F_{0,v_0}) &\cong SU^*(n),\\
\mathrm{Fix}(\mathrm{Res}_{L/F_0} \boldsymbol{\nu}_{p,q}, \mathbf{G}'')(F_{0,v_0}) &\cong SU(p,q),\\
\mathrm{Fix}(\mathrm{Res}_{L/F_0}(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta}), \mathbf{G}'')(F_{0,v_0}) &\cong S(GL_p(\mathbb{C}) \times GL_q(\mathbb{C})),\\
\mathrm{Fix}(\mathrm{Res}_{E/F_0}(\boldsymbol{\eta} \circ \boldsymbol{\mu}), \mathbf{G}'')(F_{0,v_0}) &\cong S(GL_{n/2}(\mathbb{C}) \times GL_{n/2}(\mathbb{C})),\\
\mathrm{Fix}(\mathrm{Res}_{E/F_0}(\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta}, \mathbf{G}'')(F_{0,v_0}) &\cong SU(n/2, n/2).
\end{aligned}$$

In particular, $\mathrm{Res}_{L/F_0} \boldsymbol{\theta}$ *induces a Cartan involution on* $SL_n(\mathbb{C})$.

[15] The reason for this additional complication is that we want the map $\boldsymbol{\theta}$ to define a Cartan involution of $SL_n(\mathbb{C})$. Unlike in the real case, this involves complex conjugation and can hence not be defined by an E-rational morphism.

PROOF. We have $\mathbf{G}''(F_{0,v_0}) = \mathbf{G}'(F_v) \cong SL_n(\mathbb{C})$ by construction of the algebraic group $\mathbf{G}'$. To determine the fixed points of the morphisms, we need to study each map separately. We start with the morphism $\boldsymbol{\theta}$: The F_0-group $\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\theta}, \mathbf{G}'')$ is defined by the equations $\theta(x) = x = \psi(x)$ on $SL_m(D) = \mathrm{Res}_{E/F_0}\mathbf{SL_m}(\mathbf{D})(F_0)$. Let $D = E$. Then we have

$$\begin{aligned}\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\theta}, \mathbf{G}'')(F_0) =& \{x \in SL_m(E) \mid \theta(x) = x = \psi(x)\}\\ =& \{x \in SL_m(E) \mid H^{-1}(\omega(x)^t)^{-1}H = x = H^{-1}(\iota(x)^t)^{-1}H\}\\ =& \{x \in SL_m(E) \mid (\omega \circ \iota)(x) = x \text{ and } x = H^{-1}(\iota(x)^t)^{-1}H\}\\ =& \{x \in SL_m(L') \mid x = H^{-1}(\iota|_{L'}(x)^t)^{-1}H\},\end{aligned}$$

which implies $\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\theta}, \mathbf{G}'') = \mathbf{SU_m}(\mathbf{h}|_{\mathbf{L}'}, \mathbf{L}', \boldsymbol{\iota}|_{\mathbf{L}'})$.

For $D = Q(a, b|E)$ we get

$$\begin{aligned}&\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\theta}, \mathbf{G}'')(F_0) = \{x \in SL_m(D) \mid \theta(x) = x = \psi(x)\}\\ &= \{x \in SL_m(D) \mid H^{-1}((\tau_{c,0} \otimes \omega)(x)^t)^{-1}H = x = H^{-1}((\tau_{c,0} \otimes \iota)(x)^t)^{-1}H\}\\ &= \{x \in SL_m(D) \mid (\mathrm{id} \otimes (\omega \circ \iota))(x) = x \text{ and } x = H^{-1}((\tau_{c,0} \otimes \iota)(x)^t)^{-1}H\}\\ &= \{x \in SL_m(D_0 \otimes_{F_0} L') \mid x = H^{-1}((\tau_{c,0} \otimes \iota|_{L'})(x)^t)^{-1}H\},\end{aligned}$$

which yields $\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\theta}, \mathbf{G}'') = \mathbf{SU_m}(\mathbf{h}|_{\mathbf{D}'}, \mathbf{D}', \boldsymbol{\tau}_{\mathbf{c,0}} \otimes \boldsymbol{\iota}|_{\mathbf{L}'})$, where we have used that $D' = D_0 \otimes_{F_0} L'$ by definition.

In both cases, Proposition 2.6 implies

$$\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\theta}, \mathbf{G}'')(F_{0,v_0}) \cong SU(n),$$

since the real place v_0 of F_0 is nondecomposed in L', the maps $\iota|_{L'}$ and $\tau_{c,0} \otimes \iota|_{L'}$ are definite involutions on L' and D', respectively, and the matrix $H \in M_m(F_0)$ is chosen positive definite at the place v_0. In particular, this shows that $\mathrm{Res}_{L/F_0}\boldsymbol{\theta}$ induces a Cartan involution on $SL_n(\mathbb{C})$ because the group of F_{0,v_0}-rational points of its fixed points is isomorphic to a maximal compact subgroup of $SL_n(\mathbb{C})$.

Next, we consider the maps $\mathrm{Res}_{L/F_0}\boldsymbol{\nu}_{p,q}$ and $\mathrm{Res}_{L/F_0}(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta})$. On $SL_m(D)$ we have $\nu_{p,q} = \mathrm{Int}(I_{p,q}) \circ \theta$ and $\nu_{p,q} \circ \theta = \mathrm{Int}(I_{p,q})$ if $D = E$, and $\nu_{p,q} = \mathrm{Int}(I_{p/2,q/2}) \circ \theta$ and $\nu_{p,q} \circ \theta = \mathrm{Int}(I_{p/2,q/2})$ if $D = Q(a, b|E)$.[16] However, in the latter case, the matrices $I_{p/2,q/2}$ are mapped to $I_{p,q}$ under a suitable splitting of $M_m(D) \otimes \mathbb{C} \to M_n(\mathbb{C})$, and therefore these maps induce the groups

$$\begin{aligned}\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\nu}_{p,q}, \mathbf{G}'')(F_{0,v_0}) &\cong \{x \in SL_n(\mathbb{C}) \mid I_{p,q}(x^*)^{-1}I_{p,q} = x\} \cong SU(p,q)\\ \text{and } \mathrm{Fix}(\mathrm{Res}_{L/F_0}(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta}), \mathbf{G}'')(F_{0,v_0}) &\cong \{x \in SL_n(\mathbb{C}) \mid I_{p,q}xI_{p,q} = x\}\\ &\cong S(GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))\end{aligned}$$

for both choices of D.

To deal with the maps $\mathrm{Res}_{E/F_0}\boldsymbol{\eta}$, $\mathrm{Res}_{E/F_0}\boldsymbol{\mu}$ and $\mathrm{Res}_{E/F_0}(\boldsymbol{\eta} \circ \boldsymbol{\mu})$ we proceed as in the proof of Proposition 4.6. In fact, for any E-rational morphism $\boldsymbol{\varphi}$ on $\mathbf{SL_m}(\mathbf{D})$ such that $\mathrm{Res}_{E/F}\boldsymbol{\varphi}$ commutes with $\boldsymbol{\psi}$, we have that

$$\begin{aligned}\mathrm{Fix}(\mathrm{Res}_{E/F_0}\boldsymbol{\varphi}, \mathbf{G}'')(F_{0,v_0}) &= \mathrm{Fix}(\mathrm{Res}_{E/F}\boldsymbol{\varphi}, \mathbf{G}')(F_v)\\ &\cong \mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \subset SL_n(\mathbb{C}).\end{aligned}$$

[16] Recall that in case of a quaternion algebra the maps $\nu_{p,q}$ are only defined for even p and q.

Here, the isomorphism in the second row is chosen as in the proof of Proposition 4.6. However, since v is now a complex place of F, we obtain a subgroup of $SL_n(\mathbb{C})$ instead of $SL_n(\mathbb{R})$.

For $\boldsymbol{\varphi} = \boldsymbol{\eta}$, the group $\mathrm{Fix}(\boldsymbol{\eta}, \mathbf{SL_m}(\mathbf{D}))$ is a special unitary group w. r. t. a hermitian form and an orthogonal involution. Therefore, by Proposition 2.4, the E_{w_1}-rational points are isomorphic to $SO(n, \mathbb{C})$ and we get

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0}\, \boldsymbol{\eta}, \mathbf{G}'')(F_{0,v_0}) \cong SO(n, \mathbb{C}).$$

For $\boldsymbol{\varphi} = \boldsymbol{\mu}$, we can argue exactly as in the proof of Proposition 4.6. However, since now w_1 is a complex place of E, Proposition 2.4 implies

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0}\, \boldsymbol{\mu}, \mathbf{G}'')(F_{0,v_0}) \cong \mathrm{Fix}(\boldsymbol{\mu}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \cong Sp(n, \mathbb{C}).$$

Finally, we consider $\boldsymbol{\varphi} = \boldsymbol{\eta} \circ \boldsymbol{\mu}$. Note that on $SL_m(D)$, $\eta \circ \mu = \mathrm{Int}(J_m)$ in case $D = E$ and $\eta \circ \mu = \tau_k \circ \tau_c = \mathrm{Int}(\mathrm{diag}(k, \dots, k))$ if $D = Q(a, b|E)$. It is not hard to see that the images of J_m (if $D = E$) and $\mathrm{diag}(k, \dots, k)$ (if $D = Q(a, b|E)$) under suitable splitting isomorphisms $M_m(D) \otimes \mathbb{C} \to M_n(\mathbb{C})$ are of the form

$$\mathrm{diag}(\lambda i I_{n/2}, -\lambda i I_{n/2}) = \lambda i \cdot I_{n/2,n/2} \in GL_n(\mathbb{C})$$

for some $\lambda \in \mathbb{R}$, where $\lambda = 1$ in case $D = E$. This yields:

$$\begin{aligned}\mathrm{Fix}(\mathrm{Res}_{E/F_0}(\boldsymbol{\eta} \circ \boldsymbol{\mu}), \mathbf{G}'')(F_{0,v_0}) &\cong \mathrm{Fix}(\boldsymbol{\eta} \circ \boldsymbol{\mu}, \mathbf{SL_m}(\mathbf{D}))(E_{w_1}) \\ &\cong \{x \in SL_n(\mathbb{C}) \mid \lambda i \cdot I_{n/2,n/2}\, x\, (-\lambda^{-1} i) I_{n/2,n/2} = x\} \\ &= \{x \in SL_n(\mathbb{C}) \mid I_{n/2,n/2} x I_{n/2,n/2} = x\} \\ &\cong S(GL_{n/2}(\mathbb{C}) \times GL_{n/2}(\mathbb{C})).\end{aligned}$$

Next we treat the morphisms $\mathrm{Res}_{E/F_0}\, \boldsymbol{\eta} \circ \mathrm{Res}_{L/F_0}\, \boldsymbol{\theta}$ and $\mathrm{Res}_{E/F_0}\, \boldsymbol{\mu} \circ \mathrm{Res}_{L/F_0}\, \boldsymbol{\theta}$. We start with the case $D = E$: We have

$$\begin{aligned}\mathrm{Fix}(\mathrm{Res}_{E/F_0}\, \boldsymbol{\eta} \circ \mathrm{Res}_{L/F_0}\, \boldsymbol{\theta}, \mathbf{G}'')(F_0) =& \{x \in SL_m(E) \mid (\eta \circ \theta)(x) = x = \psi(x)\} \\ =& \{x \in SL_m(E) \mid \omega(x) = x = H^{-1}(\iota(x)^t)^{-1} H\} \\ =& \{x \in SL_m(L) \mid x = H^{-1}(\iota|_L(x)^t)^{-1} H\},\end{aligned}$$

which implies $\mathrm{Fix}(\mathrm{Res}_{E/F_0}\, \boldsymbol{\eta} \circ \mathrm{Res}_{F/F_0}\, \boldsymbol{\theta}, \mathbf{G}'') = \mathbf{SU_m}(\mathbf{h}|_\mathbf{L}, \mathbf{L}, \iota|_\mathbf{L})$ and hence

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0}\, \boldsymbol{\eta} \circ \mathrm{Res}_{F/F_0}\, \boldsymbol{\theta}, \mathbf{G}'')(F_{0,v_0}) = \mathbf{SU_m}(\mathbf{h}|_\mathbf{L}, \mathbf{L}, \iota|_\mathbf{L})(F_{0,v_0}) \cong SL_n(\mathbb{R}),$$

since the real place v_0 of F_0 is decomposed in L and $\iota|_L$ is an involution of the second kind on L. In particular, this means that ω induces the complex conjugation in the group $\mathbf{G}''(F_{0,v_0}) \cong SL_n(\mathbb{C})$ under a suitable isomorphism.

Furthermore, we clearly have $(\mu \circ \theta)(x) = \mathrm{Int}(J_m) \circ (\eta \circ \theta) = \mathrm{Int}(J_m) \circ \omega$ on $SL_m(E)$. Since ω induces complex conjugation and J_m doesn't change when passing to the F_{0,v_0}-rational points, it follows that

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0}\, \boldsymbol{\mu} \circ \mathrm{Res}_{L/F_0}\, \boldsymbol{\theta}, \mathbf{G}'')(F_{0,v_0}) \cong \{x \in SL_n(\mathbb{C}) \mid J_n \bar{x} J_n^{-1} = x\} \cong SU^*(n).$$

The case $D = Q(a, b|E)$ is slightly more complicated. Recall that $D = D_0 \otimes_{F_0} E$ and that by Proposition I.7.19, D_0 ramifies at all archimedean places since the involution $\gamma = \tau_{c,0} \otimes \omega|_{L'}$ on $D_0 \otimes_{F_0} L'$ is definite and all real places of F_0 are nondecomposed in L'. In particular, we have $a_{v_0} < 0$ and $b_{v_0} < 0$. Moreover, recall that $F = F_0(\sqrt{D_1})$ for some square-free element $D_1 \in F_0$ such that D_1 is negative under the embedding corresponding to the place v_0 of F_0. Therefore, the quaternion algebra $Q_0 := Q(D_1 a, D_1 b|F_0)$ is a division algebra that splits at the place v_0 of F_0. Note that $x \in (Q_0 \otimes_{F_0} L)$ if and only if

$x \in D$ and $(\tau_k \circ \tau_c)(x) = (\mathrm{id} \otimes \omega)(x)$. With the help of these observations, we can describe the fixed points of $\eta \circ \theta$ in $\mathbf{G}''(F_0)$ as follows:

$$\begin{aligned}
&\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\eta} \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta}, \mathbf{G}'')(F_0) \\
&= \{x \in SL_m(D) \mid (\eta \circ \theta)(x) = x = \psi(x)\} \\
&= \{x \in SL_m(D) \mid (\tau_r \circ (\tau_{c,0} \otimes \omega))(x) = x = H^{-1}((\tau_{c,0} \otimes \iota)(x)^t)^{-1} H\} \\
&= \{x \in SL_m(D) \mid (\tau_r \circ \tau_c)(x) = (\mathrm{id} \otimes \omega)(x) \text{ and } x = H^{-1}((\tau_{c,0} \otimes \iota)(x)^t)^{-1} H\} \\
&= \{x \in SL_m(Q_0 \otimes_{F_0} L) \mid x = H^{-1}((\tau_{c,Q_0} \otimes \iota|_L)(x)^t)^{-1} H\},
\end{aligned}$$

where τ_{c,Q_0} denotes the canonical symplectic involution of Q_0. This implies

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\eta} \circ \mathrm{Res}_{F/F_0} \boldsymbol{\theta}, \mathbf{G}'') = \mathbf{SU_m}(\mathbf{Q_0} \otimes_{\mathbf{F_0}} \mathbf{L}, \mathbf{h}|_{\mathbf{Q_0} \otimes_{\mathbf{F_0}} \mathbf{L}}, \boldsymbol{\tau}_{\mathbf{c},\mathbf{Q_0}} \otimes \boldsymbol{\iota}|_{\mathbf{L}}),$$

and hence by Proposition 2.6

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\eta} \circ \mathrm{Res}_{F/F_0} \boldsymbol{\theta}, \mathbf{G}'')(F_{0,v_0}) \cong SL_n(\mathbb{R})$$

since the real place v_0 of F_0 is decomposed in L and Q_0 splits at v_0.

Next, we consider the fixed points of $\mu \circ \theta$ in $\mathbf{G}''(F_0)$:

$$\begin{aligned}
&\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\mu} \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta}, \mathbf{G}'')(F_0) \\
&= \{x \in SL_m(D) \mid (\mu \circ \theta)(x) = x = \psi(x)\} \\
&= \{x \in SL_m(D) \mid (\tau_c \circ (\tau_{c,0} \otimes \omega))(x) = x = H^{-1}((\tau_{c,0} \otimes \iota)(x)^t)^{-1} H\} \\
&= \{x \in SL_m(D) \mid (\mathrm{id} \otimes \omega)(x) = x \text{ and } x = H^{-1}((\tau_{c,0} \otimes \iota)(x)^t)^{-1} H\} \\
&= \{x \in SL_m(D_0 \otimes_{F_0} L) \mid x = H^{-1}((\tau_{c,0} \otimes \iota|_L)(x)^t)^{-1} H\}.
\end{aligned}$$

This implies $\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\mu} \circ \mathrm{Res}_{F/F_0} \boldsymbol{\theta}, \mathbf{G}'') = \mathbf{SU_m}(\mathbf{D_0} \otimes_{\mathbf{F_0}} \mathbf{L}, \mathbf{h}|_{\mathbf{D_0} \otimes_{\mathbf{F_0}} \mathbf{L}}, \boldsymbol{\tau}_{\mathbf{c},\mathbf{0}} \otimes \boldsymbol{\iota}|_{\mathbf{L}})$ and hence by Proposition 2.6

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\mu} \circ \mathrm{Res}_{F/F_0} \boldsymbol{\theta}, \mathbf{G}'')(F_{0,v_0}) \cong SL_{n/2}(\mathbb{H}) \cong SU^*(n)$$

since the the real place v_0 of F_0 is decomposed in L and D_0 ramifies at v_0.

Finally, we deal with the morphism $\mathrm{Res}_{E/F_0}(\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta}$: As we have seen above, $\mathrm{Res}_{E/F_0}(\boldsymbol{\eta} \circ \boldsymbol{\mu})$ induces conjugation by the matrix $I_{n/2,n/2}$ on the F_{0,v_0}-rational points of $\mathbf{G}''$ and $\boldsymbol{\theta}$ induces a Cartan involution. Therefore, we can deduce

$$\begin{aligned}
&\mathrm{Fix}(\mathrm{Res}_{E/F_0}(\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta})(F_{0,v_0}) \\
&\cong \{x \in SL_n(\mathbb{C}) \mid I_{n/2,n/2}(\bar{x}^t)^{-1} I_{n/2,n/2} = x\} \cong SU(n/2, n/2).
\end{aligned}$$

□

5.8. In this paragraph, we study the geometric cycles defined by the various morphisms on $\mathbf{G}''$. To do this, we pass to the algebraic $\mathbb{Q}$-group $\mathbf{G} := \mathrm{Res}_{F_0/\mathbb{Q}} \mathbf{G}''$. This is an algebraic group over $\mathbb{Q}$ with $\mathbf{G}(\mathbb{Q}) \cong \mathbf{G}''(F_0)$ and

$$\mathbf{G}(\mathbb{R}) \cong \mathbf{G}''(\mathbb{R} \otimes_{\mathbb{Q}} F_0) = \mathbf{G}'(\mathbb{R} \otimes_{\mathbb{Q}} F) = \prod_{v' \in V_\infty(F)} \mathbf{G}'(F_{v'}) = SL_n(\mathbb{C}) \times \prod_{v' \in V_\infty, v' \neq v} SU(n).$$

Moreover, we have $\mathbb{Q}$-rational morphisms $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$, $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ of order two on $\mathbf{G}$ that are induced from the corresponding morphisms of $\mathbf{G}''$.

Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup of $\mathbf{G}$. In analogy to Section 4.7, Γ gives rise to a discrete cocompact subgroup of $SL_n(\mathbb{C})$ that we still denote by Γ for simplicity of notation. Let K' denote a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ and $X := K' \backslash \mathbf{G}(\mathbb{R})$ the symmetric space attached to $\mathbf{G}(\mathbb{R})$. Since $\mathbf{G}(\mathbb{R})$ is a product of $SL_n(\mathbb{C})$ and compact factors, we have $X \cong SU(n)\backslash SL_n(\mathbb{C})$ and Γ acts on X by right translations. Note that X is a symmetric space of real dimension

$$\dim X = \dim(SL_n(\mathbb{C})) - \dim(SU(n)) = 2n^2 - 2 - (n^2 - 1) = n^2 - 1.$$

Let now $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a torsion-free arithmetic subgroup of $\mathbf{G}$ and assume that Γ and K' are invariant under $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$, $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$. Then these morphisms induce certain special geometric cycles in X/Γ, as explained in Section II.1:

THEOREM 5.9. *The pair of morphisms $(\boldsymbol{\eta}, \boldsymbol{\eta} \circ \boldsymbol{\theta})$ and—if n is even and $n > 2$—the pairs $(\boldsymbol{\mu}, \boldsymbol{\mu} \circ \boldsymbol{\theta})$ and $(\boldsymbol{\eta} \circ \boldsymbol{\mu}, (\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \boldsymbol{\theta})$ induce pairs of special geometric cycles $C(\boldsymbol{\eta})$, $C(\boldsymbol{\eta} \circ \boldsymbol{\theta})$, $C(\boldsymbol{\mu})$, $C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$ and $C(\boldsymbol{\eta} \circ \boldsymbol{\mu})$, $C((\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \boldsymbol{\theta})$ in X/Γ. Moreover, the morphisms $\boldsymbol{\nu}_{p,q}$ and $\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta}$ induce a family of pairs of special geometric cycles $C(\boldsymbol{\nu}_{p,q})$, $C(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta})$ in X/Γ, for positive integers p and q with $p + q = n$ if $\mathbf{G}$ is induced from a special unitary group over an algebraic number field, and for positive, even integers p and q with $p + q = n$ if $\mathbf{G}$ is induced from a special unitary group over a quaternion algebra. The properties of these cycles are summarized in Table 2.*

$C = C(\boldsymbol{\varphi})$	$\mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{G})(\mathbb{R}) \cong$	$\dim C$
$C(\boldsymbol{\nu}_{p,q})$	$SU(p,q)$	$2pq$
$C(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta})$	$S(GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))$	$p^2 + q^2 - 1$
$C(\boldsymbol{\eta})$	$SO(n, \mathbb{C})$	$\frac{n^2-n}{2}$
$C(\boldsymbol{\eta} \circ \boldsymbol{\theta})$	$SL_n(\mathbb{R})$	$\frac{n^2+n}{2} - 1$
$C(\boldsymbol{\mu})$	$Sp(n, \mathbb{C})$	$\frac{n^2+n}{2}$
$C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$	$SU^*(n)$	$\frac{n^2-n}{2} - 1$
$C(\boldsymbol{\eta} \circ \boldsymbol{\mu})$	$S(GL_{n/2}(\mathbb{C}) \times GL_{n/2}(\mathbb{C}))$	$\frac{n^2}{2} - 1$
$C((\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \boldsymbol{\theta})$	$SU(n/2, n/2)$	$\frac{n^2}{2}$

TABLE 2. Geometric cycles in $SU(n)\backslash SL_n(\mathbb{C})/\Gamma$: The isomorphism in the second column is up to compact factors and the bottom half of the table is only applicable if n is even and $n > 2$.

PROOF. This is proved completely analogously to Theorem 4.8. To compute the dimensions of the cycles, note that both $SU(p,q)$ and $S(GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))$ have maximal compact subgroup $S(U(p) \times U(q))$, both $SL_n(\mathbb{R})$ and $SO(n, \mathbb{C})$ have maximal compact subgroup $SO(n)$ and both $Sp(n, \mathbb{C})$ and $SU^*(n)$ have maximal compact subgroup $Sp(n)$. $\square$

6. The Construction of Nontrivial Cohomology Classes

We will now use the results from the previous sections to construct nontrivial cohomology classes in cocompact arithmetic quotients of $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$. The idea is to apply Theorem II.3.7 to the algebraic group $\mathbf{G}$ and the special cycles defined in Theorems 4.8 and 5.9. For a $\mathbb{Q}$-rational morphism $\boldsymbol{\varphi}$ on $\mathbf{G}$ of finite order, we denote by $X(\boldsymbol{\varphi}) := \mathrm{Fix}(\boldsymbol{\varphi}, K)\backslash \mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{G})(\mathbb{R})$ the associated symmetric space (where K is a $\boldsymbol{\varphi}$-stable maximal compact subgroup of $\mathbf{G}(\mathbb{R})$).

6.1. Let $X \cong SO(n)\backslash SL_n(\mathbb{R})$ denote the symmetric space attached to $SL_n(\mathbb{R})$ and $X_u \cong SO(n)\backslash SU(n)$ the compact dual symmetric space.

THEOREM 6.2. *Let $n \in \mathbb{N}$ be even.*

(1) *There exists a cocompact discrete subgroup Γ_1 of $SL_n(\mathbb{R})$ that arises from an arithmetic subgroup of a special unitary group over an algebraic number field, such that $H^k(X/\Gamma_1, \mathbb{C})$ contains nontrivial cohomology classes for*

$$k = pq \quad \textit{and} \quad k = \frac{p^2 + q^2 + n}{2} - 1,$$

where p and q are positive integers with $p + q = n$, and if $n \neq 2$, for

$$k = \frac{n^2 + 2n}{4} \quad \textit{and} \quad k = \frac{n^2}{4} - 1.$$

(2) *There exists a cocompact discrete subgroup Γ_2 of $SL_n(\mathbb{R})$ that arises from an arithmetic subgroup of a special unitary group over a quaternion algebra, such that $H^k(X/\Gamma_2, \mathbb{C})$ contains nontrivial cohomology classes for*

$$k \ = pq \quad \textit{and} \quad k = \frac{p^2 + q^2 + n}{2} - 1,$$

where p and q are positive, even integers with $p + q = n$ or $p = q = n/2$, and if $n \neq 2$, for

$$k = \frac{n^2 + 2n}{4} \quad \textit{and} \quad k = \frac{n^2}{4} - 1.$$

In both cases, these classes are not in the image of the respective injective map

$$\beta^*_{\Gamma_i} : H^*(X_u, \mathbb{C}) \to H^*(X/\Gamma_i, \mathbb{C}),$$

i. e. they are not represented by $SL_n(\mathbb{R})$-invariant forms on X.

PROOF. We give a detailed proof of (1), then (2) follows analogously.

Let $\mathbf{G}$ denote the algebraic $\mathbb{Q}$-group whose real points are isomorphic to $SL_n(\mathbb{R})$ up to compact factors that is defined via a special unitary group over an algebraic number field (cf. Section 4.1). Set $\Psi := \{\boldsymbol{\nu}_{p,q} \mid p + q = n,\, p \neq 0 \neq q\} \cup \{\boldsymbol{\mu}\}$, for $\boldsymbol{\nu}_{p,q}$ and $\boldsymbol{\mu}$ as defined in paragraph 4.7, and let Γ be a torsion-free arithmetic subgroup of $\mathbf{G}$ that is stable under the group generated by $\Psi \cup \{\boldsymbol{\theta}\}$. As in Section 4.7, the group Γ gives rise to a discrete cocompact subgroup Γ of $SL_n(\mathbb{R})$ via the embedding $\mathbf{G}(\mathbb{Q}) \hookrightarrow \mathbf{G}(\mathbb{R})$ and projection to the noncompact factor. Let $\boldsymbol{\tau}_1 \in \Psi$ and set $\boldsymbol{\tau}_2 := \boldsymbol{\tau}_1 \circ \boldsymbol{\theta}$. Then $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are $\mathbb{Q}$-rational morphisms of order two on $\mathbf{G}$ that commute with each other, and $\boldsymbol{\tau}_1 \circ \boldsymbol{\tau}_2 = \boldsymbol{\theta}$ induces a Cartan involution on $SL_n(\mathbb{R})$ (which is the noncompact factor of $\mathbf{G}(\mathbb{R})$) by Proposition 4.6. The pair $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ defines a pair of geometric cycles on X/Γ, whose properties are given in Theorem 4.8. By Corollary II.2.5, we may assume that the cycles and the connected components of their intersection are orientable. Moreover, by Lemma II.3.10, the cycles $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are of complementary dimension and satisfy condition (ii) of Theorem II.3.7.[17]

To apply Theorem II.3.7, it remains to check condition (i). Recall that for any of the cycles $C(\boldsymbol{\varphi})$ in Table 1, we have that $\mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{G})(\mathbb{R})$ is isomorphic to a reductive subgroup H_φ of $SL_n(\mathbb{R})$ up to compact factors. Let K_{H_φ} denote a maximal compact subgroup of H_φ. Then we have an isomorphism $X(\boldsymbol{\varphi}) \cong (K_{H_\varphi}\backslash H_\varphi) \times 1 \times \ldots \times 1 \cong K_{H_\varphi}\backslash H_\varphi$,

[17]This can also be read off from Table 1 on page 52.

and therefore $\mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{G})(\mathbb{R})$ acts orientation-preserving on $X(\boldsymbol{\varphi})$ if and only if H_φ acts orientation preserving on $K_{H_\varphi}\backslash H_\varphi$.

For $\boldsymbol{\tau}_1 = \boldsymbol{\mu}$, we have $\mathrm{Fix}(\boldsymbol{\tau}_1, \mathbf{G})(\mathbb{R}) \cong Sp(n, \mathbb{R})$ and $\mathrm{Fix}(\boldsymbol{\tau}_2, \mathbf{G})(\mathbb{R}) \cong GL^{(1)}_{n/2}(\mathbb{C})$ up to compact factors (cf. Table 1). These are connected Lie subgroups of $SL_n(\mathbb{R})$ (see e. g. [**26**, Lemma X.2.4]) and hence they act orientation-preserving on the respective symmetric spaces by Lemma I.3.7. For $\boldsymbol{\tau}_1 = \boldsymbol{\nu}_{p,q}$, we have $\mathrm{Fix}(\boldsymbol{\tau}_1, \mathbf{G})(\mathbb{R}) \cong SO(p, q)$ and $\mathrm{Fix}(\boldsymbol{\tau}_2, \mathbf{G})(\mathbb{R}) \cong S(GL_p(\mathbb{R}) \times GL_q(\mathbb{R}))$ up to compact factors, where $p+q = n$. Since n is even, it follows from example I.3.11 that these groups act orientation-preserving on their associated symmetric spaces.

We conclude that for any choice of $\boldsymbol{\tau}_1 \in \Psi$, the pair $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ meets all assumptions of Theorem II.3.7. Therefore, for each such $\boldsymbol{\tau}_1$, we can find a normal, $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$-stable subgroup $\Gamma_{\tau_1} \subset \Gamma$ of finite index such that $H^k(X/\Gamma_{\tau_1}, \mathbb{C}) \neq 0$ for $k \in \{\dim C(\boldsymbol{\tau}_1), \dim C(\boldsymbol{\tau}_2)\}$ by Corollary II.3.8. Moreover, Γ_{τ_1} can be chosen such that the nontrivial cohomology classes detected by $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are not represented by $SL_n(\mathbb{R})$-invariant differential forms on X, as follows from Theorem II.4.4. Set

$$\Gamma' := \bigcap_{\boldsymbol{\tau}_1 \in \Psi} \Gamma_{\tau_1} \quad \text{and} \quad \Gamma_1 := \bigcap_{\boldsymbol{\tau}_1 \in \Psi} \boldsymbol{\tau}_1(\Gamma') \cap \boldsymbol{\tau}_2(\Gamma').$$

Then Γ_1 is a cocompact discrete subgroup of $SL_n(\mathbb{R})$ that is of finite index in each Γ_{τ_1} and $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$-stable for each $\boldsymbol{\tau}_1 \in \Psi$. By the remarks following Corollary II.3.8 and Theorem II.4.4, the group Γ_1 admits nontrivial cohomology classes in all degrees $k \in \{\dim C(\boldsymbol{\tau}_1), \dim C(\boldsymbol{\tau}_2)\}$ for possible pairs $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ with $\boldsymbol{\tau}_1 \in \Psi$, and these classes are not represented by $SL_n(\mathbb{R})$-invariant differential forms on X. The exact dimensions can be read off from Table 1.

To prove (2), let $\mathbf{G}$ be the algebraic $\mathbb{Q}$-group whose real points are isomorphic to $SL_n(\mathbb{R})$ up to compact factors that is defined via a special unitary group over a quaternion algebra (cf. Section 4.2). Set

$$\Psi := \{\boldsymbol{\nu}_{p,q} \mid p+q = n,\ p, q \neq 0 \text{ and } p \text{ and } q \text{ even or } p = q = n/2\} \cup \{\boldsymbol{\mu}\}.$$

Then a completely analogous argument as in case (1) yields a discrete, cocompact subgroup Γ_2 of $SL_n(\mathbb{R})$ with the desired properties. □

Remark.
(1) We do not get any result in case n is odd. The morphism $\boldsymbol{\mu}$ is not defined in this case, so we are left with the cycles $C(\boldsymbol{\nu}_{p,q})$ and $C(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta})$. These are indeed of complementary dimension and $\mathrm{Fix}(\langle \boldsymbol{\nu}_{p,q}, \boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta} \rangle, \mathbf{G})(\mathbb{R})$ is compact. However, the cycles do not satisfy condition (i) in Theorem II.3.7 (see Example I.3.11). Therefore, the result of Rohlfs and Schwermer is not applicable and we cannot deduce any statement about the intersection number of $C(\boldsymbol{\nu}_{p,q})$ and $C(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta})$. It is still an open question whether or not this number is nontrivial.
(2) In their work [**2**], Ash and Ginzburg show a part of the result in Theorem 6.2(1) to use it in the proof of their Lemma 5.4.2. More precisely, in the case where n is even and $\mathbf{G}$ is the algebraic group associated with a special unitary group over a number field (cf. Section 4.1), they construct the pair of special cycles $C(\boldsymbol{\nu}_{n/2,n/2})$, $C(\boldsymbol{\nu}_{n/2,n/2} \circ \boldsymbol{\theta})$ (in our notation). Then, using the result of Rohlfs and Schwermer, they show that the intersection number of these cycles is nonzero and deduce the existence of a nonvanishing homology class.

6.3. Let us now consider the complex case. We set $X := SU(n)\backslash SL_n(\mathbb{C})$, the symmetric space attached to $SL_n(\mathbb{C})$ and denote by $X_u \cong SU(n)$ the compact dual symmetric space.

THEOREM 6.4. *Let $n \in \mathbb{N}$ be arbitrary.*

(1) There exists a cocompact discrete subgroup Γ_1 of $SL_n(\mathbb{C})$ that arises from an arithmetic subgroup of a special unitary group over an algebraic number field, such that $H^k(X/\Gamma_1, \mathbb{C})$ contains nontrivial cohomology classes for

$$k = 2pq \quad \text{and} \quad k = p^2 + q^2 - 1,$$

where p and q are positive integers with $p + q = n$, and for

$$k = \frac{n^2 - n}{2} \quad \text{and} \quad k = \frac{n^2 + n}{2} - 1.$$

Moreover, if n is even and $n \neq 2$, there are nontrivial cohomology classes in the degrees

$$k = \frac{n^2 + n}{2}, \quad k = \frac{n^2 - n}{2} - 1, \quad k = \frac{n^2}{2} - 1 \quad \text{and} \quad k = \frac{n^2}{2}.$$

(2) If n is even, there exists a discrete, cocompact subgroup Γ_2 of $SL_n(\mathbb{C})$ that arises from an arithmetic subgroup of a special unitary group over a quaternion algebra, such that $H^k(X/\Gamma_2, \mathbb{C})$ contains nontrivial cohomology classes for

$$k = 2pq \quad \text{and} \quad k = p^2 + q^2 - 1$$

where p and q are positive, even integers with $p + q = n$, and for

$$k = \frac{n^2 - n}{2} \quad \text{and} \quad k = \frac{n^2 + n}{2} - 1.$$

Moreover, if $n \neq 2$, there exist nontrivial cohomology classes in the degrees

$$k = \frac{n^2 + n}{2}, \quad k = \frac{n^2 - n}{2} - 1, \quad k = \frac{n^2}{2} - 1 \quad \text{and} \quad k = \frac{n^2}{2}.$$

In both cases these classes are not in the image of the respective injective map

$$\beta^*_{\Gamma_i} : H^*(X_u, \mathbb{C}) \to H^*(X/\Gamma_i, \mathbb{C}),$$

i. e. they are not represented by $SL_n(\mathbb{C})$-invariant forms on X.

PROOF. The proof is mostly analogous to the proof of Theorem 6.2, therefore we will only sketch the argument.

For $n \in \mathbb{N}$, let $\mathbf{G}$ denote the algebraic $\mathbb{Q}$-group defined in Section 5.8 whose $\mathbb{Q}$-rational points are isomorphic to $SL_n(\mathbb{C})$ up to compact factors. This group is obtained by restriction of scalars from a special unitary group over a central simple algebra D, that is either a number field or a quaternion algebra. We define a set Ψ of morphisms of $\mathbf{G}$:

$$\mathbf{\Psi} := \begin{cases} \{\boldsymbol{\nu}_{p,q} \mid p + q = n\} \cup \{\boldsymbol{\eta}\} & \text{if } D \text{ is a field and } n \text{ is odd,} \\ \{\boldsymbol{\nu}_{p,q} \mid p + q = n\} \cup \{\boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\eta} \circ \boldsymbol{\mu}\} & \text{if } D \text{ is a field and } n \text{ is even,} \\ \{\boldsymbol{\nu}_{p,q} \mid p + q = n,\ p, q \text{ even}\} \cup \{\boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\eta} \circ \boldsymbol{\mu}\} & \text{if } D \text{ is a quaternion algebra.} \end{cases}$$

Recall that we always assume $p \neq 0 \neq q$.

As in the proof of Theorem 6.2, for each pair $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ where $\boldsymbol{\tau}_1 \in \Psi$ and $\boldsymbol{\tau}_2 := \boldsymbol{\tau}_1 \circ \boldsymbol{\theta}$, we can find a suitable torsion-free, cocompact, $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$-stable arithmetic subgroup Γ_{τ_1}

of $\mathbf{G}$ such that $H^k(X/\Gamma_{\tau_1}, \mathbb{C}) \neq 0$ for $k \in \{\dim C(\boldsymbol{\tau}_1), \dim C(\boldsymbol{\tau}_2)\}$ and such that the nontrivial cohomology classes detected by the cycles are not represented by $SL_n(\mathbb{C})$-invariant differential forms on X. Here, conditions (i) and (ii) of Theorem II.3.7 are satisfied due to Lemma II.3.10 and the fact that all groups $\mathrm{Fix}(\boldsymbol{\tau}_i, \mathbf{G})(\mathbb{R})$, are (up to compact factors) connected Lie subgroups of $SL_n(\mathbb{C})$, for $i \in \{1, 2\}$ (cf. Table 2).

From this family of arithmetic subgroups we can construct a group Γ_1 (resp. Γ_2) with all the desired properties exactly as in the proof of Theorem 6.2. $\square$

Let us look at some examples:

EXAMPLE 6.5. (1) First we consider the case $SL_n(\mathbb{R})$. Set $X := SO(n)\backslash SL_n(\mathbb{R})$ and let Γ be a cocompact discrete subgroup of $SL_n(\mathbb{R})$ chosen as in Theorem 6.2 (1) or (2). In the following table we give an overview of the occurring cycles, the associated subgroups of $SL_n(\mathbb{R})$ and the degrees in the cohomology of X/Γ to which these cycles contribute,[18] for some choices of n. The last two columns indicate if the respective cycle exists for the choice of Γ as in Theorem 6.2 (1) or (2).

	$\dim X/\Gamma$	Cycle	Subgroup of $SL_n(\mathbb{R})$	Contributing to degree	Occurs for $\Gamma = \Gamma_1$	$\Gamma = \Gamma_2$
$n = 2$	2	$C(\boldsymbol{\nu}_{1,1})$	$SO(1,1)$	1	$\times$	$\times$
		$C(\boldsymbol{\nu}_{1,1} \circ \boldsymbol{\theta})$	$S(GL_1 \times GL_1)$	1	$\times$	$\times$
$n = 4$	9	$C(\boldsymbol{\nu}_{1,3} \circ \boldsymbol{\theta})$	$S(GL_1 \times GL_3)$	3	$\times$	
		$C(\boldsymbol{\mu})$	$Sp(4, \mathbb{R})$	3	$\times$	$\times$
		$C(\boldsymbol{\nu}_{2,2} \circ \boldsymbol{\theta})$	$S(GL_2 \times GL_2)$	4	$\times$	$\times$
		$C(\boldsymbol{\nu}_{2,2})$	$SO(2,2)$	5	$\times$	$\times$
		$C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$	$GL_2^{(1)}(\mathbb{C})$	6	$\times$	$\times$
		$C(\boldsymbol{\nu}_{1,3})$	$SO(1,3)$	6	$\times$	

TABLE 3. Real Case: Degrees in $H^*(X/\Gamma)$ in which we have nontrivial cohomology classes coming from special cycles.

(2) We have a similar table for the case $SL_n(\mathbb{C})$, see Table 4 on page 65. Here, $X := SU(n)\backslash SL_n(\mathbb{C})$ is the symmetric space attached to $SL_n(\mathbb{C})$, and Γ denotes a cocompact discrete subgroup of $SL_n(\mathbb{C})$ chosen as in Theorem 6.4 (1) or (2).

REMARK. Looking at these examples, the question arises of whether the degrees in which we have constructed nontrivial cohomology classes are all degrees in the cohomology of X/Γ in which there is cohomology that is not coming from the compact dual symmetric space. In general, this is not the case, as we will see in Chapter IV using methods from representation theory. For certain compact arithmetic quotients X/Γ attached to $SL_n(\mathbb{C})$, this can also be seen using the Euler characteristic: Recall that the Euler characteristic

[18] Note that these degrees are not the dimensions of the cycles but their complements since we are looking at the cohomology classes obtained via Poincaré duality.

	$\dim X/\Gamma$	Cycle	Subgroup of $SL_n(\mathbb{C})$	Contributing to degree	Occurs for $\Gamma = \Gamma_1$	$\Gamma = \Gamma_2$
$n=2$	3	$C(\boldsymbol{\nu}_{1,1})$	$SU(1,1)$	1	$\times$	
		$C(\boldsymbol{\eta} \circ \boldsymbol{\theta})$	$SL_2(\mathbb{R})$	1	$\times$	$\times$
		$C(\boldsymbol{\nu}_{1,1} \circ \boldsymbol{\theta})$	$S(GL_1 \times GL_1)$	2	$\times$	
		$C(\boldsymbol{\eta})$	$SO(2,\mathbb{C})$	2	$\times$	$\times$
$n=3$	8	$C(\boldsymbol{\eta} \circ \boldsymbol{\theta})$	$SL_3(\mathbb{R})$	3	$\times$	
		$C(\boldsymbol{\nu}_{1,2})$	$SU(1,2)$	4	$\times$	
		$C(\boldsymbol{\nu}_{1,2} \circ \boldsymbol{\theta})$	$S(GL_1 \times GL_2)$	4	$\times$	
		$C(\boldsymbol{\eta})$	$SO(3,\mathbb{C})$	5	$\times$	
$n=4$	15	$C(\boldsymbol{\mu})$	$Sp(4,\mathbb{C})$	5	$\times$	$\times$
		$C(\boldsymbol{\eta} \circ \theta)$	$SL_4(\mathbb{R})$	6	$\times$	$\times$
		$C(\boldsymbol{\nu}_{1,3} \circ \boldsymbol{\theta})$	$S(GL_1 \times GL_3)$	6	$\times$	
		$C(\boldsymbol{\nu}_{2,2})$	$SU(2,2)$	7	$\times$	$\times$
		$C((\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \boldsymbol{\theta})$	$SU(2,2)$	7	$\times$	$\times$
		$C(\boldsymbol{\eta} \circ \boldsymbol{\mu})$	$S(GL_2 \times GL_2)$	8	$\times$	$\times$
		$C(\boldsymbol{\nu}_{2,2} \circ \boldsymbol{\theta})$	$S(GL_2 \times GL_2)$	8	$\times$	$\times$
		$C(\boldsymbol{\nu}_{1,3})$	$SU(1,3)$	9	$\times$	
		$C(\boldsymbol{\eta})$	$SO(4,\mathbb{C})$	9	$\times$	$\times$
		$C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$	$SU^*(4)$	10	$\times$	$\times$
$n=5$	24	$C(\boldsymbol{\nu}_{1,4} \circ \boldsymbol{\theta})$	$S(GL_1 \times GL_4)$	8	$\times$	
		$C(\boldsymbol{\eta} \circ \boldsymbol{\theta})$	$SL_5(\mathbb{R})$	10	$\times$	
		$C(\boldsymbol{\nu}_{2,3} \circ \boldsymbol{\theta})$	$S(GL_2 \times GL_3)$	12	$\times$	
		$C(\boldsymbol{\nu}_{2,3})$	$SU(2,3)$	12	$\times$	
		$C(\boldsymbol{\eta})$	$SO(5,\mathbb{C})$	14	$\times$	
		$C(\boldsymbol{\nu}_{1,4})$	$SU(1,4)$	16	$\times$	

TABLE 4. Complex Case: Degrees in $H^*(X/\Gamma)$ in which we have nontrivial cohomology classes coming from special cycles.

of a topological space with finitely generated cohomology is defined to be the alternating sum over the Betti numbers, i.e. in our situation we have

$$\chi(X/\Gamma) := \sum_{k=0}^{\dim X/\Gamma} (-1)^k \dim H^k(X/\Gamma, \mathbb{C}).$$

It is a consequence of the Gauss-Bonnet formula, that for compact quotients X/Γ, where $X = SU(n)\backslash SL_n(\mathbb{C})$ and $n \geq 2$, the Euler characteristic of X/Γ is always 0. This implies that the sum over the Betti numbers in even degrees equals the sum over the Betti numbers in odd degrees.

Now assume that $n \geq 2$ and $n \equiv 1 \mod 4$. Then n is odd, so by Theorem 6.4(1) we get nontrivial cohomology classes in degrees $2pq$ and $n^2 - 1 - 2pq$ for all pairs (p, q) of positive integers such that $p+q = n$ and in degrees $\frac{n^2-n}{2}$ and $\frac{n^2+n}{2}-1$. By our assumption on n, one can easily verify that all these degrees are even. Therefore, the vanishing of the Euler characteristic implies the existence of at least one nontrivial cohomology class in an odd degree. Moreover, we know that this class does not lie in the image of the cohomology of the compact dual symmetric space, since the Euler characteristic of the compact dual symmetric space is trivial as well.

The smallest n meeting our assumptions is $n = 5$. For this case, we can see in Table 4 that the cycles do indeed only contribute to even degrees in the cohomology of X/Γ.

7. Linear Independence of the Constructed Cohomology Classes

We have seen in the above examples that different geometric cycles may give rise to nontrivial cohomology classes for X/Γ in the same degree. It is a natural question to ask whether such cohomology classes are linearly independent.

We will freely use the notation of the preceding sections and of Chapter II. Let X denote the symmetric space attached to the group of real points of a connected reductive algebraic $\mathbb{Q}$-group $\mathbf{G}$, and let $\Gamma \subset \mathbf{G}(\mathbb{R})$ be a cocompact discrete subgroup. We start with a general observation:

Lemma 7.1. *Set $r := \dim X/\Gamma$ and let $C(\boldsymbol{\tau}_1)$, $C(\boldsymbol{\tau}_2)$ and $C(\boldsymbol{\tau}'_1)$, $C(\boldsymbol{\tau}'_2)$ denote two pairs of special geometric cycles of complementary dimension in X/Γ such that we have $k := \dim C(\boldsymbol{\tau}_1) = \dim C(\boldsymbol{\tau}'_1)$ and $r-k = \dim C(\boldsymbol{\tau}_2) = \dim C(\boldsymbol{\tau}'_2)$. Assume that X/Γ and the special cycles are oriented and that $[C(\boldsymbol{\tau}_1)][C(\boldsymbol{\tau}_2)] \neq 0$ and $[C(\boldsymbol{\tau}'_1)][C(\boldsymbol{\tau}'_2)] \neq 0$. Then the cycles induce nontrivial homology classes in the respective degrees of $H_*(X/\Gamma, \mathbb{C})$, and both the classes induced by $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}'_1)$ in $H_k(X/\Gamma, \mathbb{C})$ and by $C(\boldsymbol{\tau}_2)$ and $C(\boldsymbol{\tau}'_2)$ in $H_{r-k}(X/\Gamma, \mathbb{C})$ are linearly independent if $[C(\boldsymbol{\tau}_1)][C(\boldsymbol{\tau}'_2)] = 0$ or $[C(\boldsymbol{\tau}'_1)][C(\boldsymbol{\tau}_2)] = 0$.*

Proof. This follows directly from the observation that if the intersection number of two homology classes α, β is zero, then the intersection number of α' and β must be zero for any class α' in the same degree as α such that α and α' are linearly independent. □

Corollary 7.2. *Let the geometric cycles $C(\boldsymbol{\tau}_1)$, $C(\boldsymbol{\tau}_2)$ and $C(\boldsymbol{\tau}'_1)$, $C(\boldsymbol{\tau}'_2)$ be as in Lemma 7.1, and assume that $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}'_2$ are commuting morphisms of finite order of $\mathbf{G}$. Set $\Theta := \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}'_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}'_2 \rangle$. If $X(\boldsymbol{\tau}_1)$ and $X(\boldsymbol{\tau}'_2)$ intersect in an odd-dimensional manifold, there exists a normal, Θ-stable subgroup $\Delta \subset \Gamma$ of finite index such that $[C(\boldsymbol{\tau}_1, \Delta)]$ and $[C(\boldsymbol{\tau}'_1, \Delta)]$ are linearly independent in $H_k(X/\Delta, \mathbb{C})$ and $[C(\boldsymbol{\tau}_2, \Delta)]$ and $[C(\boldsymbol{\tau}'_2, \Delta)]$ are linearly independent in $H_{r-k}(X/\Delta, \mathbb{C})$.*

Proof. By Lemma 7.1, it suffices to show the existence of a suitable subgroup Δ such that $[C(\boldsymbol{\tau}_1, \Delta)][C(\boldsymbol{\tau}'_2, \Delta)] = 0$. Recall from Proposition II.3.4 that the intersection number of two special cycles that are of complementary dimension and whose defining morphisms commute, can be computed as the sum of the Euler numbers of the excess bundles of all connected components. Following the arguments in [**54**, § 2], we may pass to a $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}'_2 \rangle$-stable, normal subgroup $\Delta \subset \Gamma$ of finite index such that all connected components of $C(\boldsymbol{\tau}_1, \Delta) \cap C(\boldsymbol{\tau}'_2, \Delta)$ are of the same dimension modulo 4. Recall from Section II.1.3 that the connected components of $C(\boldsymbol{\tau}_1, \Delta) \cap C(\boldsymbol{\tau}'_2, \Delta)$ are parametrized by the set $\ker(\mathrm{res}_1 \times \mathrm{res}'_2) \subset H^1(\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}'_2 \rangle, \Delta)$ and that the component corresponding to the trivial cocycle is of dimension $\dim X(\boldsymbol{\tau}_1) \cap X(\boldsymbol{\tau}'_2)$. Since this is an odd-dimensional manifold by assumption, it follows that all connected components of $C(\boldsymbol{\tau}_1, \Delta) \cap C(\boldsymbol{\tau}'_2, \Delta)$ are of odd dimension and thus the excess bundle is an odd-dimensional oriented vector bundle for each connected component. However, the Euler number of such an odd-dimensional bundle is always zero (see e. g. [**46**, Property 9.4]). Now the claim follows from the observation that we may assume that Δ is a normal, Θ-stable subgroup of finite index in Γ, by an argument as in the proof of Lemma II.2.4. □

REMARK. Using Poincaré duality, we also get linear independence results for the associated cohomology classes.

7.3. With the help of this general result, we can now analyze the specific situations for quotients of the groups $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$.

PROPOSITION 7.4. *Let n be even and set $X := SO(n)\backslash SL_n(\mathbb{R})$. Let $\Gamma = \Gamma_1$ be a cocompact discrete subgroup of $SL_n(\mathbb{R})$ chosen as in Theorem 6.2 (1).*

- *For $n = 2$ the special cycles $C(\boldsymbol{\nu}_{1,1})$ and $C(\boldsymbol{\nu}_{1,1} \circ \boldsymbol{\theta})$ both contribute to degree 1 in the homology of X/Γ and the two homology classes are linearly independent.*
- *For $n > 2$, the special cycles $C(\boldsymbol{\nu}_{(n/2)-1,(n/2)+1})$ and $C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$ both contribute to homology in degree $\frac{n^2}{4} - 1$ and the special cycles $C(\boldsymbol{\mu})$ and $C(\boldsymbol{\nu}_{(n/2)-1,(n/2)+1} \circ \boldsymbol{\theta})$ both contribute to homology in degree $\frac{n^2+2n}{4}$.*

These are the only possibilities that cycles from Theorem 4.8 can contribute to the same degree in the homology of X/Γ.

PROOF. By a straightforward calculation, one sees that the cycles given in the theorem contribute to the same degrees and that this cannot occur in any other cases. It remains to show the linear independence in case $n = 2$. We set $\boldsymbol{\tau}_1 = \boldsymbol{\tau}'_2 = \boldsymbol{\nu}_{1,1}$ and $\boldsymbol{\tau}_2 = \boldsymbol{\tau}'_1 = \boldsymbol{\nu}_{1,1} \circ \boldsymbol{\theta}$. By Corollary 7.2, to see that the corresponding homology classes are linearly independent, it suffices to show that $X(\boldsymbol{\tau}_1) \cap X(\boldsymbol{\tau}'_2) = X(\boldsymbol{\tau}_1)$ is odd-dimensional. This is the case since $\dim X(\boldsymbol{\tau}_1) = \dim(S(O(1) \times O(1))\backslash SO(1,1)) = 1$. Note that here, we do not even have to pass to a subgroup of finite index, since $C(\boldsymbol{\tau}_1) \cap C(\boldsymbol{\tau}'_2)$ is connected. □

REMARK.

- A similar result also holds for Γ as in Theorem 6.2 (2) for those cycles that are defined in that case.
- Unfortunately, for $n > 2$ we do not get a result on the linear independence of the cycles that contribute to the same degree in homology. Corollary 7.2 is not applicable since the defining morphisms may not commute or, if they do, the intersection of the associated symmetric spaces is of even dimension. Of course, this does not necessarily mean that the corresponding homology classes are linearly dependent.

The situation is more complicated for $SL_n(\mathbb{C})$ since we have more special cycles and thus there are more possibilities for two of them to contribute to the same degree:

PROPOSITION 7.5. *Let $X = SU(n)\backslash SL_n(\mathbb{C})$ and let $\Gamma = \Gamma_1$ be the discrete cocompact subgroup of $SL_n(\mathbb{C})$ from Theorem 6.4 (1).*

- *For $n = 2$, the cycles $C(\boldsymbol{\nu}_{1,1} \circ \boldsymbol{\theta})$ and $C(\boldsymbol{\eta})$ contribute to homology in degree* 1 *and $C(\boldsymbol{\nu}_{1,1})$ and $C(\boldsymbol{\eta} \circ \boldsymbol{\theta})$ contribute to homology in degree* 2*. The two classes in each degree are linearly independent (after passing to a subgroup of finite index in Γ if necessary).*

- *If n is a square, the cycles $C(\boldsymbol{\nu}_{p,q})$ for $p = \frac{n-\sqrt{n}}{2}$ and $q = \frac{n+\sqrt{n}}{2}$ and $C(\boldsymbol{\eta})$ both contribute to the homology of X/Γ in degree $\frac{n^2-n}{2}$ and their counterparts of complementary dimension both contribute in degree $\frac{n^2+n}{2} - 1$. If n is even and $\sqrt{n} \equiv 2 \mod 4$, the respective homology classes are linearly independent after passing to a subgroup of finite index in Γ.*
- *If n is odd, the dimension of X/Γ is even. The cycle $C(\boldsymbol{\nu}_{(n-1)/2,(n+1)/2})$ and its counterpart $C(\boldsymbol{\nu}_{(n-1)/2,(n+1)/2} \circ \boldsymbol{\theta})$ both contribute to the middle degree $\frac{n^2-1}{2}$ in the homology of X/Γ.*
- *Let n be even and $n > 2$. The cycles $C((\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \boldsymbol{\theta})$ and $C(\boldsymbol{\nu}_{n/2,n/2})$ contribute to homology in degree $\frac{n^2}{2}$ and their counterparts of complementary dimension contribute in degree $\frac{n^2}{2} - 1$. If $n \equiv 2 \mod 4$, these classes are linearly independent after passing to a subgroup of finite index in Γ.*

  Moreover, if $n+2$ is a square, the cycles $C(\boldsymbol{\nu}_{p,q})$ where $p = \frac{n-\sqrt{n+2}}{2}$, $q = \frac{n+\sqrt{n+2}}{2}$ and $C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$ both contribute in degree $\frac{n^2-n}{2} - 1$ and their counterparts of complementary dimension contribute in degree $\frac{n^2+n}{2}$.

These are the only possibilities that cycles from Theorem 5.9 can contribute to the same degree in the homology of X/Γ.

Proof. As above, we only prove the linear independence statements and leave the straightforward calculations to the reader. In case $n = 2$, we set $\boldsymbol{\tau}_1 = \boldsymbol{\nu}_{1,1} \circ \boldsymbol{\theta}$, $\boldsymbol{\tau}'_1 = \boldsymbol{\eta}$, $\boldsymbol{\tau}_2 = \boldsymbol{\nu}_{1,1}$ and $\boldsymbol{\tau}'_2 = \boldsymbol{\eta} \circ \boldsymbol{\theta}$. The morphisms $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}'_2$ commute. Moreover, we have

$$
\begin{aligned}
(\mathrm{Fix}(\boldsymbol{\tau}_1, \mathbf{G}) \cap \mathrm{Fix}(\boldsymbol{\tau}'_2, \mathbf{G}))(\mathbb{R}) &\cong S(GL_1(\mathbb{C}) \times GL_1(\mathbb{C})) \cap SL_2(\mathbb{R}) \\
&= S(GL_1(\mathbb{R}) \times GL_1(\mathbb{R})).
\end{aligned}
$$

The associated symmetric space $X(\boldsymbol{\tau}_1) \cap X(\boldsymbol{\tau}'_2)$ is thus of dimension 1 and the linear independence follows from Corollary 7.2.

Now let n be a square such that $\sqrt{n} \equiv 2 \mod 4$. We set $\boldsymbol{\tau}_1 = \boldsymbol{\nu}_{p,q}$ and $\boldsymbol{\tau}'_2 = \boldsymbol{\eta} \circ \boldsymbol{\theta}$, for p and q as given in the proposition. These morphisms commute, and the intersection of the associated symmetric spaces is the symmetric space corresponding to the real Lie group $SO(\frac{n-\sqrt{n}}{2}, \frac{n+\sqrt{n}}{2})$ which is of odd dimension by our choice of n. As above, the result follows from Corollary 7.2.

The last case is a bit more complicated. We let n be even, $n > 2$ and such that $n \equiv 2 \mod 4$. Set $\boldsymbol{\tau}_1 = (\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \boldsymbol{\theta}$ and $\boldsymbol{\tau}'_2 = \boldsymbol{\nu}_{n/2,n/2} \circ \boldsymbol{\theta}$. We start by considering the intersection $\mathrm{Fix}(\boldsymbol{\tau}_1, \mathbf{G}) \cap \mathrm{Fix}(\boldsymbol{\tau}'_2, \mathbf{G})$ on the algebraic group level. The $\mathbb{Q}$-rational points are given by the group of matrices

$$\{x \in \mathbf{G}(\mathbb{Q}) | x = \mathrm{diag}(x_1, \theta(x_1)) \text{ for } x_1 \in GL_{m/2}(E)\}.$$

Up to compact factors, the real points of this group are isomorphic to

$$\{x \in SL_n(\mathbb{C}) \mid x = \mathrm{diag}(x_1, (x_1^*)^{-1}) \text{ for } x_1 \in GL_{n/2}(\mathbb{C})\}.$$

This group has dimension $\frac{n^2}{2} - 1$ as a real Lie group, and the associated symmetric space is of dimension $\frac{n^2}{4}$. By our choice of n, this is an odd-dimensional space and we can again apply the Corollary. □

Remark. • An analogous statement also holds for the group Γ_2 (that comes from a special unitary group over a quaternion algebra) from (2) in Theorem 6.4, for all cycles that are defined in this case. It is an interesting point that whenever we were able to show the linear independence of two homology classes, one of the involved cycles is not defined in the quaternion algebra case.

- As in the real situation, Corollary 7.2 does not apply to the other cases where two cycles contribute to the same degree in homology. Either the morphisms defining these special cycles do not commute or, if they do, the intersection of the corresponding symmetric spaces is even-dimensional.

Example 7.6. We recall from Example 6.5 (2) that for $n = 2$ there are two cohomology classes in each of the degrees 1 and 2. The first statement of the above Proposition shows that these classes are linearly independent. For $n = 4$ we have two classes in each of the degrees 6, 7, 8 and 9. Since $\sqrt{n} = 2 \equiv 2 \mod 4$ we can apply the second statement of the Proposition to deduce that the two classes in each of the degrees 6 and 9 are linearly independent.

However, it remains an open question whether in case $n = 3$ the two classes in degree 4, in case $n = 4$ the two classes in each of the degrees 7 and 8, and in case $n = 5$ the two classes in degree 12 are linearly independent.

CHAPTER IV

Matsushima's Formula and Representations with Non-Vanishing $(\mathfrak{g}, K)$-Cohomology

1. Matsushima's Formula

Let G be a connected semisimple Lie group (with finite center), K a maximal compact subgroup, $X := K\backslash G$ the associated symmetric space and $\Gamma \subset G$ a discrete, cocompact subgroup. Recall from Section I.5 that we denote by $\widehat{G}$ the unitary dual of G and given an element $\pi \in \widehat{G}$, we write $H^\infty_{\pi,K}$ for the associated Harish-Chandra module of K-finite, smooth vectors.

Theorem 1.1 (Matsushima). *The cohomology of X/Γ decomposes as a finite algebraic sum*

$$H^*(X/\Gamma, \mathbb{C}) = \bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, K; H^\infty_{\pi,K}),$$

where the $m(\pi, \Gamma)$ are nonnegative integers and π ranges over the finite set of equivalence classes of unitary representations with trivial infinitesimal character. Moreover, $m(\mathbb{C}, \Gamma) = 1$, i. e. there is an injection of the $(\mathfrak{g}, K)$-cohomology of the trivial representation into $H^*(X/\Gamma, \mathbb{C})$.

Proof. This result is originally due to Matsushima [**43**]. The version stated here can be found in [**10**, Thm. VII.5.2]. □

Remark. In case G is the group of real points of a semisimple algebraic $\mathbb{R}$-group $\mathbf{G}$, the fact that $m(\mathbb{C}, \Gamma) = 1$ can also be deduced from Proposition II.4.2 and Proposition I.6.6.

The unitary representations contributing to the right hand side of Matsushima's formula are called *unitary representations with nonvanishing $(\mathfrak{g}, K)$-cohomology.* They are classified by the work of Enright [**17**] (for complex groups) and Vogan-Zuckerman [**67**] (for real groups). Note that irreducible unitary representations always have an infinitesimal character (cf. Lemma I.5.7). Therefore, by Lemma I.6.2, the representations π with $H^*(\mathfrak{g}, K; \mathbb{C} \otimes H^\infty_{\pi,K}) \neq 0$ are only those representations whose infinitesimal character coincides with the one of the contragredient representation of $\mathbb{C}$; in other words, those with trivial infinitesimal character.

1.2. The multiplicities occurring in Matsushima's formula originate as the multiplicities of the respective representations in the decomposition of the space $L^2(G/\Gamma)$ as a direct sum of irreducible unitary representations. Representations occurring with a nontrivial multiplicity are so-called *automorphic representations of G with respect to* Γ. In general, given an irreducible unitary representation π of G, it is still an open question whether the corresponding multiplicity $m(\pi, \Gamma)$ is nontrivial or not. For groups admitting discrete series representations, there are nonvanishing results by DeGeorge and Wallach [**20**], Wallach [**70**], Langlands [**38**], and others, cf. [**57**]. We point out that for our cases of interest (i. e. $G = SL_n(\mathbb{R})$ or $G = SL_n(\mathbb{C})$), there is no discrete series except for the case $G = SL_2(\mathbb{R})$.

Against this background, the nonvanishing of the cohomology on the left hand-side of Matsushima's formula can be interpreted as a result in the theory of automorphic representations: The existence of a nontrivial cohomology class for X/Γ that is not induced from the cohomology of the compact dual symmetric space (i. e. is not related to the $(\mathfrak{g}, K)$-cohomology of the trivial representation) implies the nonvanishing of at least one multiplicity $m(\pi, \Gamma)$ for an irreducible unitary representation $\pi \neq \mathbb{C}$ of G. In other words, the representation π is a nontrivial automorphic representation of G with respect to Γ.

Note, however, that this is so far only an existence result, without identifying one (or several) automorphic representations explicitly. To make a more precise statement, it is helpful to classify all representation that may possibly contribute to the right hand side of Matsushima's formula and determine their cohomology. Then, by comparing the degrees in which these representations have cohomology with those in which there are nontrivial cohomology classes in $H^*(X/\Gamma, \mathbb{C})$ (for example coming from cycles), one can in some cases identify a single automorphic representation of G with respect to Γ. With this goal in mind, we will devote the next two sections to the study of irreducible unitary representations of complex simply connected Lie groups and the determination of their cohomology.

2. Irreducible Unitary Representations with Nontrivial $(\mathfrak{g}, K)$-Cohomology for Simply Connected Complex Lie groups

Let G denote a connected, simply connected complex Lie group. The irreducible unitary representations with regular infinitesimal character[1] were classified in 1979 by Enright [**17**]. We will make this result explicit for the case of trivial infinitesimal character. First we will need some preparatory results.

2.1. Let G be a complex simply connected semisimple Lie group with complex Lie algebra $\mathfrak{g}$. As in Section I.4.10, we write $\mathfrak{g}_\mathbb{R}$ whenever we consider $\mathfrak{g}$ as a real Lie algebra. Then $\mathfrak{g}_\mathbb{R}$ has a Cartan involution θ and a corresponding Cartan decomposition $\mathfrak{g}_\mathbb{R} = \mathfrak{k} \oplus \mathfrak{p}$. Recall from Section I.4.10 that $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$ and θ is the conjugation with respect to this real form. This implies $\mathfrak{p} = i\mathfrak{k}$ and θ is given by $\theta(X + iY) = X - iY$. Let $\mathfrak{h}_\mathbb{R}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$. We can write $\mathfrak{h}_\mathbb{R} = \mathfrak{m}_0 \oplus \mathfrak{a}_0$, where $\mathfrak{m}_0 = \mathfrak{h}_\mathbb{R} \cap \mathfrak{k}$ and $\mathfrak{a}_0 = \mathfrak{h}_\mathbb{R} \cap \mathfrak{p}$. One can show (cf. [**34**, VI.11, p.365]) that $\mathfrak{a}_0 = i\mathfrak{m}_0$ and hence $\mathfrak{h}_\mathbb{R}$ is both maximally compact and maximally noncompact, i. e. $\mathfrak{a}_0$ and $\mathfrak{m}_0$ are maximal abelian subalgebras of $\mathfrak{p}$ and $\mathfrak{k}$, respectively. Moreover, it follows that $\mathfrak{h}_\mathbb{R} = \mathfrak{m}_0 \oplus i\mathfrak{m}_0$ is the real Lie algebra underlying a complex subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{h}$ is a (complex) Cartan subalgebra of $\mathfrak{g}$.

For simplicity of notation, we denote by $\mathfrak{g}_\mathbb{C}$ the complexification $\mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ of the real Lie algebra $\mathfrak{g}_\mathbb{R}$.

Lemma 2.2. *There is an isomorphism*

$$\mathfrak{g}_\mathbb{C} \xrightarrow{\cong} \mathfrak{g} \times \mathfrak{g} \tag{2.1}$$

[1] A representation has *regular* infinitesimal character, if the infinitesimal character is parametrized by the Weyl group orbit of a regular element of $\mathfrak{h}^*$.

of complex Lie algebras that identifies the real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{C}}$ with the real form

$$\{(X, \theta(X)) \in \mathfrak{g} \times \mathfrak{g} \mid X \in \mathfrak{g}_{\mathbb{R}}\}$$

of $\mathfrak{g} \times \mathfrak{g}$.

PROOF. See [**48**, §2, Prop. 3 and Example 4]. □

We denote by j the inclusion $\mathfrak{g}_{\mathbb{R}} \hookrightarrow \mathfrak{g} \times \mathfrak{g}$, $X \mapsto (X, \theta(X))$. For the Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$, we have in the same way a complexification $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{h} \times \mathfrak{h}$ and $\mathfrak{h}_{\mathbb{R}}$ injects into $\mathfrak{h} \times \mathfrak{h}$ via $j|_{\mathfrak{h}_{\mathbb{R}}}$.

EXAMPLE 2.3. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then $\theta\colon X \mapsto -\overline{X}^t$ is a Cartan involution on $\mathfrak{g}_{\mathbb{R}}$ and the real Lie algebra $\mathfrak{k} := \mathfrak{su}(n) = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ is a compact real form of $\mathfrak{g}$. We have a Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}$.

Let $\mathfrak{h} := \{X \in \mathfrak{g} \mid X = \operatorname{diag}(x_1, \dots, x_n)\}$ denote the subalgebra of diagonal matrices. Considered as a real Lie algebra, this is a θ-stable Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Then the real Lie algebras $\mathfrak{m}_0$ and $\mathfrak{a}_0$ are the subalgebras of skew-hermitian and real diagonal matrices, respectively:

$$\begin{aligned} \mathfrak{m}_0 &= \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{k} = \{X \in \mathfrak{sl}_n(\mathbb{C}) \mid X = \operatorname{diag}(x_1 \dots x_n) \text{ and } \overline{X} = -X\}, \\ \mathfrak{a}_0 &= \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{p} = \{X \in \mathfrak{sl}_n(\mathbb{R}) \mid X = \operatorname{diag}(x_1 \dots x_n)\}. \end{aligned}$$

In particular, $\mathfrak{a}_0 = i\mathfrak{m}_0$.

We have $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C})$ and the isomorphism identifies $\mathfrak{sl}_n(\mathbb{C})$ with a subalgebra of $\mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C})$ via $X \mapsto (X, -\overline{X}^t)$.

2.4. The Dual of $\mathfrak{h}_{\mathbb{C}}$. By definition, $\mathfrak{h}_{\mathbb{C}}$ is a complex Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and we are interested in the root system of $\mathfrak{g}_{\mathbb{C}}$ w. r. t. $\mathfrak{h}_{\mathbb{C}}$ and in the dual $\mathfrak{h}_{\mathbb{C}}^*$ of $\mathfrak{h}_{\mathbb{C}}$, i. e. the space of linear functionals $\alpha\colon \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$. We follow the presentation in [**16**, I.1.4].

There are two ways to consider the dual of the complex Lie algebra $\mathfrak{h}_{\mathbb{C}}^*$: On the one hand, we have an isomorphism $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{h} \times \mathfrak{h}$ and therefore $\mathfrak{h}_{\mathbb{C}}^* \cong \mathfrak{h}^* \times \mathfrak{h}^*$. For $X \in \mathfrak{h}_{\mathbb{C}}$, we denote by (X_1, X_2) the pair of elements in $\mathfrak{h} \times \mathfrak{h}$ corresponding to X under the isomorphism (2.1). For two elements $\mu, \lambda \in \mathfrak{h}^*$, we define the linear functional $(\mu, \lambda) \in \mathfrak{h}_{\mathbb{C}}^*$ by

$$(\mu, \lambda)(X) := \mu(X_1) + \lambda(X_2) \quad \text{for all } X \in \mathfrak{h}_{\mathbb{C}}.$$

On the other hand, we have seen above that $\mathfrak{h}_{\mathbb{R}} = \mathfrak{m}_0 \oplus \mathfrak{a}_0$ and this implies $\mathfrak{h}_{\mathbb{C}} = (\mathfrak{m}_0)_{\mathbb{C}} \oplus (\mathfrak{a}_0)_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}^* = (\mathfrak{m}_0)_{\mathbb{C}}^* \oplus (\mathfrak{a}_0)_{\mathbb{C}}^*$. For each $X \in \mathfrak{h}_{\mathbb{C}}$, we write $X = X_{\mathfrak{m}} + X_{\mathfrak{a}}$ for the corresponding decomposition of X, where $X_{\mathfrak{m}} \in (\mathfrak{m}_0)_{\mathbb{C}}$ and $X_{\mathfrak{a}} \in (\mathfrak{a}_0)_{\mathbb{C}}$. Then, given $\xi \in (\mathfrak{m}_0)_{\mathbb{C}}^*$ and $\nu \in (\mathfrak{a}_0)_{\mathbb{C}}^*$, we define the linear functional $(\xi \oplus \nu) \in \mathfrak{h}_{\mathbb{C}}^*$ by

$$(\xi \oplus \nu)(X) := \xi(X_{\mathfrak{m}}) + \nu(X_{\mathfrak{a}}) \quad \text{for all } X \in \mathfrak{h}_{\mathbb{C}}.$$

Note that that the inclusions $\mathfrak{a}_0 \hookrightarrow \mathfrak{h}$ and $\mathfrak{m}_0 \hookrightarrow \mathfrak{h}$ give rise to isomorphisms $(\mathfrak{a}_0)_{\mathbb{C}} \cong \mathfrak{h}$ and $(\mathfrak{m}_0)_{\mathbb{C}} \cong \mathfrak{h}$. Therefore, we may identify ξ and ν with elements in $\mathfrak{h}^*$. With this identification, we can easily switch between the two ways to represent $\mathfrak{h}_{\mathbb{C}}^*$:

LEMMA 2.5. *Let $\chi \in \mathfrak{h}_{\mathbb{C}}^*$ be such that $(\mu, \lambda) = \chi = (\xi \oplus \nu)$ for suitable elements $\mu, \lambda, \xi, \nu \in \mathfrak{h}^*$. Then we have $\xi = \mu + \lambda$ and $\nu = \mu - \lambda$.*

PROOF. It suffices to show the equality on the real subalgebra $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}_{\mathbb{C}}$. Let $X \in \mathfrak{m}_0$. Then X is mapped to the element $(X, \theta(X)) = (X, X)$ under the inclusion $j : \mathfrak{h}_{\mathbb{R}} \to \mathfrak{h} \times \mathfrak{h}$. This implies $(\mu, \lambda)(X) = \mu(X) + \lambda(X) = (\mu + \lambda)(X)$. On the other hand, an element

$X \in \mathfrak{a}_0$ is mapped to $(X, \theta(X)) = (X, -X)$ under the above inclusion. Hence we have $(\mu, \lambda)(X) = \mu(X) + \lambda(-X) = (\mu - \lambda)(X)$.

Now, for $X \in \mathfrak{h}_{\mathbb{R}}$, we can write $X = X_{\mathfrak{m}_0} + X_{\mathfrak{a}_0}$, where $X_{\mathfrak{m}_0} \in \mathfrak{m}_0$ and $X_{\mathfrak{a}_0} \in \mathfrak{a}_0$. Then we have

$$(\mu, \lambda)(X) = (\mu, \lambda)(X_{\mathfrak{m}_0}) + (\mu, \lambda)(X_{\mathfrak{a}_0}) = (\mu + \lambda)(X_{\mathfrak{m}_0}) + (\mu - \lambda)(X_{\mathfrak{a}_0}),$$

which implies $\xi = \mu + \lambda$ and $\nu = \mu - \lambda$. □

Remark. Clearly, the lemma also implies $\mu = \frac{1}{2}(\xi + \nu)$, $\lambda = \frac{1}{2}(\xi - \nu)$.

2.6. Let us fix some notation concerning root systems and Weyl groups. We denote by $\Phi(\mathfrak{g}, \mathfrak{h})$ the set of roots of $\mathfrak{g}$ w. r. t. $\mathfrak{h}$ and fix a system $\Phi^+(\mathfrak{g}, \mathfrak{h})$ of positive roots and the corresponding set of simple roots $\Delta(\mathfrak{g}, \mathfrak{h})$. Let

$$\{X_\alpha \mid \alpha \in \Phi\} \cup \{H_\alpha \mid \alpha \in \Delta\}$$

be a Chevalley basis of $\mathfrak{g}$. Under the correspondence (2.1), the root system of $\mathfrak{g}_{\mathbb{C}}$ w. r. t. $\mathfrak{h}_{\mathbb{C}}$ is given by

$$\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = (\Phi(\mathfrak{g}, \mathfrak{h}) \times \{0\}) \cup (\{0\} \times \Phi(\mathfrak{g}, \mathfrak{h})).$$

In $\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ we can define a system of positive roots by

$$\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) := (\Phi^+(\mathfrak{g}, \mathfrak{h}) \times \{0\}) \cup (\{0\} \times -\Phi^+(\mathfrak{g}, \mathfrak{h})),$$

then we have $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = (\Delta(\mathfrak{g}, \mathfrak{h}) \times \{0\}) \cup (\{0\} \times -\Delta(\mathfrak{g}, \mathfrak{h}))$ for the associated set of simple roots. Similarly, the Weyl group of $\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is given by $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = W(\mathfrak{g}, \mathfrak{h}) \times W(\mathfrak{g}, \mathfrak{h})$.

Let H denote the connected subgroup of G with Lie algebra $\mathfrak{h}$. We have seen in I.4.14 that H is a (θ-stable) Cartan subgroup of G, so we can define the Weyl group $W(G, H) := N_G(H)/Z_G(H)$. It is a subgroup of $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ that can be identified with the set

$$\{w' \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \mid w'(\mathfrak{a}_0) = \mathfrak{a}_0\} = \{(w, w) \in W(\mathfrak{g}, \mathfrak{h}) \times W(\mathfrak{g}, \mathfrak{h})\}, \tag{2.2}$$

by [**35**, Lemma 4.41(d)]. We set

$$\delta := \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \alpha \quad \text{and} \quad \rho := (\delta, -\delta) = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \alpha$$

and we denote by w_0 the longest element of the Weyl group $W(\mathfrak{g}, \mathfrak{h})$.

Let $\mathfrak{q}_0$ denote the minimal parabolic subalgebra of $\mathfrak{g}$ associated with the system of positive roots Φ^+ and let $\mathfrak{q} \supset \mathfrak{q}_0$ be a standard parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{l}$ denote the Levi factor of $\mathfrak{q}$ and $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$ its derived Lie algebra. As we have already seen in Section I.4.1, $\mathfrak{h} \cap \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{s}$ and we can identify the root system $\Phi_{\mathfrak{s}}$ of $\mathfrak{s}$ w. r. t. $\mathfrak{h} \cap \mathfrak{s}$ with the set of roots in Φ that are trivial on the center $Z_{\mathfrak{l}}$ of $\mathfrak{l}$. We will usually not distinguish between considering the roots in $\Phi_{\mathfrak{s}}$ as elements of $(\mathfrak{h} \cap \mathfrak{s})^*$ and as elements of Φ. We set $\Phi_{\mathfrak{s}}^+ := \Phi_{\mathfrak{s}} \cap \Phi^+$ and $\Delta_{\mathfrak{s}} := \Phi_{\mathfrak{s}} \cap \Delta$. Then $\Phi_{\mathfrak{s}}^+$ is a system of positive roots for $\Phi_{\mathfrak{s}}$ and $\Delta_{\mathfrak{s}}$ is the associated set of simple roots. We denote by $\mathcal{P}_{\mathfrak{s}}^+$ the set of dominant integral weights of $\mathfrak{h} \cap \mathfrak{s}$ w. r. t. $\Phi_{\mathfrak{s}}^+$ and set

$$\delta_Q := \frac{1}{2} \sum_{\alpha \in \Phi_{\mathfrak{s}}^+(\mathfrak{g}, \mathfrak{h})} \alpha.$$

Lemma 2.7. *We have* $\delta|_{\mathfrak{h} \cap \mathfrak{s}} = \delta_Q$.

PROOF. By [**27**, Lemma 13.3.A], we can write $\delta = \sum_{j=1}^{|\Delta|} \lambda_j$, where the λ_j are the fundamental dominant weights relative to $\Delta = \{\alpha_1, \dots, \alpha_{|\Delta|}\}$, i.e. the elements in $\mathfrak{h}^*$ such that $\lambda_j(H_{\alpha_i}) = \delta_{ij}$ for all $1 \le i, j \le |\Delta|$, cf. [**27**, §13.1]. Recall from Lemma I.4.3 that $\mathfrak{h} \cap \mathfrak{s}$ is the complex vector space generated by the H_α for $\alpha \in \Delta_\mathfrak{s}$. Let $\alpha = \alpha_i \in \Delta_\mathfrak{s}$. Then we have

$$\delta(H_{\alpha_i}) = \lambda_i(H_{\alpha_i}) = 1.$$

In particular, this implies $\delta|_{\mathfrak{h}\cap\mathfrak{s}} = \sum_{\{i|\alpha_i \in \Delta_s\}} \lambda_i = \delta_Q$, where we consider the λ_i for $\alpha_i \in \Delta_\mathfrak{s}$ as elements of $(\mathfrak{h} \cap \mathfrak{s})^*$ by restriction. □

Let us denote by $W_\mathfrak{s}$ the Weyl group of $\Phi_\mathfrak{s}$ and by w_Q its longest element. Then $W_\mathfrak{s}$ is the subgroup of $W(\mathfrak{g}, \mathfrak{h})$ generated by the set of simple reflections $\{s_\alpha \mid \alpha \in \Delta_\mathfrak{s}\}$.

LEMMA 2.8. *We have* $w_Q(\Phi_\mathfrak{s}^+) = -\Phi_\mathfrak{s}^+$ *and* $w_Q(\Phi^+ \setminus \Phi_\mathfrak{s}^+) = \Phi^+ \setminus \Phi_\mathfrak{s}^+$.

PROOF. See [**42**, Cor. A.26]. □

REMARK. Note that it is immediately clear from the Lemma that $w_Q(\delta_Q) = -\delta_Q$ and $w_Q(\delta - \delta_Q) = \delta - \delta_Q$.

Finally, we consider $\mathfrak{q}$ as a real parabolic subgroup and denote by $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ its Langlands decomposition. We have seen in I.4.6 that $\mathfrak{b} = \mathfrak{h} \cap \mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{m}$. Let us denote by $\Phi(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$ the root system of the pair $(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$. We can identify the elements in $\Phi(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$ with the roots in $\Phi(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ that are trivial on $\mathfrak{a}$, cf. I.4.6 and set $\Delta(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C}) := \Phi(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C}) \cap \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Note that since $\mathfrak{m}_\mathbb{C}$ is reductive, its roots can be identified with the roots of its derived Lie algebra $\mathfrak{s}_\mathbb{C}$, where an element of $(\mathfrak{h} \cap \mathfrak{s})_\mathbb{C}^*$ is extended trivially to $\mathfrak{b}_\mathbb{C}^*$.

2.9. Let us denote by K, M_0, A_0 and H the connected subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{m}_0$, $\mathfrak{a}_0$ and $\mathfrak{h}$, respectively. Since G is a complex Lie group, we have $H = M_0 \times A_0$, where A_0 is simply connected and M_0 is a maximal torus of K (cf. Section I.4.14). Furthermore, K is simply connected since G has this property and K is a deformation retract of G via the Cartan decomposition, cf. [**34**, Thm. 6.31(c)].

Denote by $\mathfrak{X}(H)$ the character group of H. Then we have $\mathfrak{X}(H) = \mathfrak{X}(M_0) \times \mathfrak{X}(A_0)$. Since $\exp : \mathfrak{a}_0 \to A_0$ is an isomorphism, we can identify $\mathfrak{X}(A_0)$ with $(\mathfrak{a}_0)_\mathbb{C}^*$. Moreover, by representation theory of compact Lie groups and the fact that K is simply connected, we can identify $\mathfrak{X}(M_0)$ with the weight lattice

$$\mathcal{P}((\mathfrak{m}_0)_\mathbb{C}^*) = \{\xi \in (\mathfrak{m}_0)_\mathbb{C}^* \mid \frac{2\langle \xi, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \xi(H_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi(\mathfrak{k}_\mathbb{C}, (\mathfrak{m}_0)_\mathbb{C})\},$$

see e.g. [**34**, §IV.7 and §V.8]. Note that under the isomorphism $(\mathfrak{m}_0)_\mathbb{C}^* \cong \mathfrak{h}^*$, this is just the weight lattice $\mathcal{P} = \mathcal{P}(\mathfrak{h}^*)$ of $\mathfrak{h}^*$. Thus, in view of Lemma 2.5, we get

$$\mathfrak{X}(H) = (\mathfrak{a}_0)_\mathbb{C}^* \oplus \mathcal{P}((\mathfrak{m}_0)_\mathbb{C}^*) \cong \{(\mu, \lambda) \in \mathfrak{h}^* \times \mathfrak{h}^* \mid \mu + \lambda \in \mathcal{P}\}.$$

2.10. A Result of Želobenko. We will recall an important result of Želobenko about the classification of irreducible admissible representations of G up to infinitesimal equivalence. We use the following notation (cf. [**17**, §4]): Let Q_0 denote the minimal parabolic subgroup of G w.r.t. the choice of positive roots $\Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Recall from Section I.4.10 that there is no need to distinguish between real and complex parabolic subgroups of G. In particular, Q_0 has a Langlands decomposition $Q_0 = M_0A_0N_0$, where the groups M_0 and A_0 are as above, and N_0 is the connected subgroup of G corresponding to the Lie algebra $\mathfrak{n}_0$ from the Langlands decomposition of the Lie algebra $\mathfrak{q}_0$ of Q_0. Since M_0 is compact, any irreducible unitary representation of M_0 is one-dimensional, i. e. a unitary character of M_0. Thus we have $\widehat{M_0} = \mathfrak{X}(M_0) \cong \mathcal{P}((\mathfrak{m}_0)^*_{\mathbb{C}})$ by the result of the previous paragraph. For $\sigma \in \widehat{M_0}$ and $\nu \in (\mathfrak{a}_0)^*_{\mathbb{C}}$, we denote the principal series representation $(\pi_{Q_0,\sigma,\nu}, I_{Q_0,\sigma,\nu})$ simply by $\pi_{\xi,\nu}$ where ξ is the element of $(\mathfrak{m}_0)^*_{\mathbb{C}}$ obtained from σ by differentiation.

Let us denote by ${}^0\pi_{\xi,\nu}$ the Harish-Chandra module of K-finite, smooth vectors of $\pi_{\xi,\nu}$, considered as a representation of $\mathcal{U}(\mathfrak{g})$. By [**16**, I.3.4], there exists a unique irreducible subquotient ${}^0\hat{\pi}_{\xi,\nu}$ of ${}^0\pi_{\xi,\nu}$ that contains with multiplicity one the $\mathfrak{k}$-module with highest weight $\tilde{\xi}$, where $\tilde{\xi}$ denotes the unique dominant integral weight in the Weyl group orbit of ξ. By Proposition I.5.3, there exists an irreducible admissible representation $\hat{\pi}_{\xi,\nu}$ of G that has ${}^0\hat{\pi}_{\xi,\nu}$ as its Harish-Chandra module. Želobenko has shown that these representations classify the irreducible admissible representations of G up to infinitesimal equivalence:

THEOREM 2.11. *Each irreducible admissible representation of G is infinitesimally equivalent to a representation $\hat{\pi}_{\xi,\nu}$, for some $\xi \in \mathcal{P}((\mathfrak{m}_0)^*_{\mathbb{C}})$ and $\nu \in (\mathfrak{a}_0)^*_{\mathbb{C}}$. Moreover, two such representations $\hat{\pi}_{\xi,\nu}$ and $\hat{\pi}_{\xi',\nu'}$ are infinitesimally equivalent if and only if there exists an element $w \in W(G,H)$ such that $w(\xi) = \xi'$ and $w(\nu) = \nu'$.*

PROOF. See [**16**, Thm. I.4.5]. □

Recall that we may represent the element $\xi \oplus \nu$ by a pair $(\mu, \lambda) \in \mathfrak{h}^* \times \mathfrak{h}^*$ via the identification in Lemma 2.5. If $\xi \oplus \nu = (\mu, \lambda)$, we write $V(\mu, \lambda)$ for the module ${}^0\hat{\pi}_{\xi,\nu}$.

LEMMA 2.12. *Let $\mu, \mu', \lambda, \lambda' \in \mathfrak{h}^*$ such that $\mu + \lambda$ and $\mu' + \lambda' \in \mathcal{P}$. Then the following hold:*

- *$V(\mu, \lambda)$ and $V(\mu', \lambda')$ are equivalent if and only if there exists a Weyl group element $w \in W(\mathfrak{g}, \mathfrak{h})$ such that $(w, w)(\mu, \lambda) = (\mu', \lambda')$.*
- *$V(\mu, \lambda)$ and $V(\mu', \lambda')$ have the same infinitesimal character if and only if there exist elements $w_1, w_2 \in W(\mathfrak{g}, \mathfrak{h})$ such that $w_1(\mu) = \mu'$ and $w_2(\lambda) = \lambda'$.*

PROOF. The first statement follows from Lemma 2.5, Theorem 2.11 and the fact that we can identify the Weyl group $W(G, H)$ with the set

$$\{(w, w) \mid w \in W(\mathfrak{g}, \mathfrak{h})\} \subset W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}),$$

cf. (2.2). For the second statement, see [**16**, II.1.9]. □

LEMMA 2.13. *Under the correspondence described in Theorem 2.11, the trivial representation corresponds to the representation $\hat{\pi}_{0,-\rho|_{(\mathfrak{a}_0)_{\mathbb{C}}}}$, i. e. ${}^0\hat{\pi}_{0,-\rho|_{(\mathfrak{a}_0)_{\mathbb{C}}}} = V(\delta, -\delta)$ is the trivial $(\mathfrak{g}, K)$-module.*

PROOF. Consider the principal series representation $\pi_{0,-\rho|_{(\mathfrak{a}_0)_{\mathbb{C}}}}$. Since $\xi = 0$ corresponds to the trivial representation σ on M_0, this is the space of functions

$$\{f \in C^\infty(G; \mathbb{C}) \mid f(mang) = e^{(-\rho|_{(\mathfrak{a}_0)_{\mathbb{C}}} + \rho|_{(\mathfrak{a}_0)_{\mathbb{C}}})(\log a)} f(g) = f(g)\}$$

with the action of G given by right translations.

Clearly, the constant functions form a one-dimensional subspace of $\pi_{0,-\rho|_{(\mathfrak{a}_0)_{\mathbb{C}}}}$ that is G-invariant and on which G acts trivially. In fact, this means that the trivial representation occurs as an irreducible subrepresentation of $\pi_{0,-\rho|_{(\mathfrak{a}_0)_{\mathbb{C}}}}$ (and also of ${}^0\pi_{0,-\rho|_{(\mathfrak{a}_0)_{\mathbb{C}}}}$). The trivial representation has highest weight 0 as a $\mathfrak{k}$-module, and this implies ${}^0\hat{\pi}_{0,-\rho|_{(\mathfrak{a}_0)_{\mathbb{C}}}} \cong \mathbb{C}$.

The identification ${}^0\hat{\pi}_{0,-\rho|_{(\mathfrak{a}_0)_\mathbb{C}}} = V(\delta, -\delta)$ follows from Lemma 2.5, Lemma 2.12 and the fact that $w_0(\delta) = -\delta$. $\square$

2.14. Irreducible Finite-dimensional Representations of a Parabolic Subgroup. Let $Q \supset Q_0$ be a standard parabolic subgroup of G with Levi decomposition $Q = LN$. Let us briefly recall the classification of irreducible finite-dimensional representations of Q as given in [**14**, I.2]. Note that we consider Q as a real Lie group.

PROPOSITION 2.15. *The irreducible finite-dimensional representations of Q are classified by pairs of elements $(\mu, \lambda) \in \mathfrak{h}^* \times \mathfrak{h}^*$ such that $\mu + \lambda \in \mathcal{P}$ and $(\mu|_{\mathfrak{h}\cap\mathfrak{s}}, \lambda|_{\mathfrak{h}\cap\mathfrak{s}}) \in \mathcal{P}_\mathfrak{s}^+ \times (-\mathcal{P}_\mathfrak{s}^+)$.*

PROOF. An irreducible finite-dimensional representation of Q is trivial on the unipotent radical N. Therefore, it suffices to classify the irreducible finite-dimensional representations of the real reductive Lie group L. Note that H is a connected abelian subgroup of L and so the restriction of any such representation to H is an element of $\mathfrak{X}(H)$. On the other hand, the restriction of such a representation to the semisimple subgroup S of L is an irreducible finite-dimensional representation of S. Since S is simply connected (cf. I.4.5), its irreducible finite-dimensional (real analytic) representations are in one-to-one correspondence with the complex linear representations of its complexified Lie algebra. Thus they can be identified with the dominant integral weights in $(\mathfrak{h} \cap \mathfrak{s})^*_\mathbb{C}$ with respect to the system of positive roots $\Phi^+(\mathfrak{s}_\mathbb{C}, (\mathfrak{h}\cap\mathfrak{s})_\mathbb{C})$, by highest weight theory for complex semisimple Lie algebras. Under the identifications in 2.6, these are given by the subset

$$\mathcal{P}_\mathfrak{s}^+ \times (-\mathcal{P}_\mathfrak{s}^+) \subset (\mathfrak{h} \cap \mathfrak{s})^* \times (\mathfrak{h} \cap \mathfrak{s})^*.$$

Combining this with the characterization of elements of $\mathfrak{X}(H)$ in terms of pairs $(\mu, \lambda) \in \mathfrak{h}^* \times \mathfrak{h}^*$ such that $\mu + \lambda \in \mathcal{P}$, cf. Section 2.9, gives the desired result. $\square$

REMARK. (1) We say that an irreducible finite-dimensional representation of Q has highest weight $(\mu, \lambda) \in \mathfrak{h}^* \times \mathfrak{h}^*$, if it is associated with this pair via Proposition 2.15.
(2) If $(\mu, \lambda) \in \mathfrak{h}^* \times \mathfrak{h}^*$ is chosen such that $\mu + \lambda \in \mathcal{P}$ and $\mu|_{\mathfrak{h}\cap\mathfrak{s}} = 0 = \lambda|_{\mathfrak{h}\cap\mathfrak{s}}$, the irreducible representation of Q with highest weight (μ, λ) is a character of Q. More precisely, it is the trivial extension to Q of the character of H that corresponds to (μ, λ) under the identification in paragraph 2.9.

2.16. Recall that the objective of this section is a classification of all irreducible unitary representations of G with trivial infinitesimal character. We will proceed as follows: First, we define certain unitary representations π_Q of G that are induced representations from a unitary character of a standard parabolic subgroup Q. We will see that these representations are irreducible and have trivial infinitesimal character. These results are essentially due to Delorme [**14**]. In a second step, following the presentation in Enright [**17**], we show that any irreducible unitary representation of G with trivial infinitesimal character is equivalent to one of the π_Q. Finally, we will give a formula for the $(\mathfrak{g}, K)$-cohomology of the representations π_Q.

Let $Q \supset Q_0$ denote a standard parabolic subgroup of G. We use the notation of the preceding paragraphs. Recall that we have defined an element $\delta_Q = \frac{1}{2}\sum_{\alpha\in\Phi_\mathfrak{s}^+} \alpha$ in $(\mathfrak{h}\cap\mathfrak{s})^*$ such that $\delta|_{\mathfrak{h}\cap\mathfrak{s}} = \delta_Q$. Let us consider the pair $(\delta_Q - \delta, \delta_Q - \delta) \in \mathfrak{h}^* \times \mathfrak{h}^*$. The restriction of the element $\delta_Q - \delta$ to $\mathfrak{h} \cap \mathfrak{s}$ is trivial, and we have $2(\delta_Q - \delta) \in \mathcal{P}$. Hence, it follows from Proposition 2.15 and the remark that there exists an irreducible one-dimensional representation of Q of highest weight $(\delta_Q - \delta, \delta_Q - \delta)$. We denote this representation by σ_Q.

LEMMA 2.17. *The representation σ_Q is a unitary character of Q.*

PROOF. Recall that σ_Q is basically a character of H that is extended trivially to a one-dimensional representation of Q. The restriction of σ_Q to A_0 is the element of $\mathfrak{X}(A_0)$ corresponding to the linear functional $(\delta_Q - \delta) - (\delta_Q - \delta) = 0$ in $(\mathfrak{a}_0)^*_{\mathbb{C}}$ via the exponential map. Therefore, σ_Q is trivial on A_0 and must be unitary as a character of the compact group M_0. □

Let us define the representation π_Q to be the representation of G obtained from σ_Q by unitary[2] induction from Q to G.

PROPOSITION 2.18. *The representations π_Q are irreducible unitary representations of G that have trivial infinitesimal character. Moreover, for two standard parabolic subgroups $Q \neq Q'$ of G, the associated representations π_Q and $\pi_{Q'}$ are not equivalent.*

PROOF. This is proved in [**14**, Prop. II.4]. However, as the notation in [**14**] differs from our notation concerning the embedding of $\mathfrak{g}_{\mathbb{R}}$ into $\mathfrak{g} \times \mathfrak{g}$, we will give an outline of the proof for the convenience of the reader.

Let $Q \supset Q_0$ be a standard parabolic subgroup. The representation π_Q is unitary as representation obtained via unitary induction from a unitary character. To show irreducibility of π_Q, we use [**13**, Cor. 2.13], which adapted to our situation says: *If for all $\alpha \in \Phi^+ \setminus \Phi^+_{\mathfrak{s}}$ the numbers $(2\delta_Q - \delta)(H_\alpha)$ and $-\delta(H_\alpha)$ are both integers of the same sign, the representation π_Q is irreducible.* Now δ is a dominant integral weight, so we have $-\delta(H_\alpha) < 0$ for all $\alpha \in \Phi^+$. Furthermore, by the remark following Lemma 2.8, we have

$$w_Q(\delta) = w_Q((\delta - \delta_Q) + \delta_Q) = \delta - 2\delta_Q, \tag{2.3}$$

and hence $2\delta_Q - \delta = -w_Q(\delta)$. This implies $(2\delta_Q - \delta)(H_\alpha) = -w_Q(\delta)(H_\alpha) < 0$ for all $\alpha \in \Phi^+ \setminus \Phi^+_{\mathfrak{s}}$ by elementary properties of the element w_Q.[3] Thus the assumptions of the quoted corollary are fulfilled and irreducibility of π_Q follows.

To complete the proof of the first statement, it remains to show that π_Q has trivial infinitesimal character. By [**13**, Prop. 2.6], the Harish-Chandra module $H^\infty_{\pi_Q,K}$ of π_Q is isomorphic to an irreducible submodule of ${}^0\pi_{\xi,\nu}$ containing the irreducible $\mathfrak{k}$-module with highest weight $\tilde{\xi}$, for elements $\xi \in \mathcal{P}((\mathfrak{m}_0)^*_{\mathbb{C}})$ and $\nu \in (\mathfrak{a}_0)^*_{\mathbb{C}}$ such that $\xi \oplus \nu = (-\delta, 2\delta_Q - \delta)$. However, ${}^0\hat{\pi}_{\xi,\nu}$ is the unique irreducible subquotient of ${}^0\pi_{\xi,\nu}$ with this property (cf. 2.10) and since a submodule is naturally also a subquotient, this implies $H^\infty_{\pi_Q,K} = {}^0\hat{\pi}_{\xi,\nu} = V(-\delta, 2\delta_Q - \delta)$. Now using Lemma 2.12 and (2.3), we get

$$H^\infty_{\pi_Q,K} = V(-\delta, -w_Q(\delta)) = V(-w_Q(\delta), -\delta).$$

Moreover, $H^\infty_{\pi_Q,K}$ has the same infinitesimal character as $V(w_0(-\delta), w_Q^2(-\delta)) = V(\delta, -\delta)$, i. e. the trivial infinitesimal character, cf. Lemma 2.13.

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[2]When we speak of *unitary* induction, we include a normalization factor to ensure that the induced representation of a unitary representation is again unitary, cf. I.5.10.

[3]More precisely, this can be seen as follows: We have

$$-w_Q(\delta)(H_\alpha) = -\frac{2\langle w_Q(\delta), \alpha\rangle}{\langle \alpha, \alpha\rangle} = -\frac{2\langle \delta, w_Q(\alpha)\rangle}{\langle \alpha, \alpha\rangle} = -\delta(H_{w_Q(\alpha)}).$$

For $\alpha \in \Phi^+ \setminus \Phi^+_{\mathfrak{s}}$, $w_Q(\alpha)$ is still a positive root by Lemma 2.8 and hence $-\delta(H_{w_Q(\alpha)}) < 0$.

For the second statement, assume we are given two distinct standard parabolic subalgebras Q and Q'. As we have just seen, the Harish-Chandra modules of π_Q and $\pi_{Q'}$ are given by $V(w_Q(-\delta), -\delta)$ and $V(w_{Q'}(-\delta), -\delta)$, respectively. These modules cannot be isomorphic, as this would imply the existence of a nontrivial element $w \in W$ such that $w(\delta) = \delta$ by Lemma 2.12, which is not possible (cf. [**27**, Lemma 13.2.B and Lemma 13.3.A]). In particular, π_Q and $\pi_{Q'}$ cannot be equivalent. $\square$

Now that we have constructed a class of irreducible unitary representations of G with trivial infinitesimal character, it remains to show that these are indeed all representations of G with these properties, up to unitary equivalence. We will prove this by making explicit a result of Enright [**17**].

PROPOSITION 2.19. *Let (π, H_π) be an irreducible unitary representation of G with trivial infinitesimal character. Then there exists a standard parabolic subgroup Q of G such that π is equivalent to the representation π_Q as defined above.*

PROOF. This is a special case of a Theorem of Enright. The proof is analogous to the proof in [**17**].

By Theorem 2.11, the Harish-Chandra module $H_{\pi,K}^\infty$ is isomorphic to a module ${}^0\hat{\pi}_{\xi,\nu}$ for suitable $\xi \in \mathcal{P}((\mathfrak{m}_0)_\mathbb{C}^*)$ and $\nu \in (\mathfrak{a}_0)_\mathbb{C}^*$ that are determined up to the action of the Weyl group. Using the identification in Lemma 2.5, we write ${}^0\hat{\pi}_{\xi,\nu} = V(\mu, \lambda)$ for suitable $\mu, \lambda \in \mathfrak{h}^*$. Since π has trivial infinitesimal character, it follows from Lemma 2.12 and Lemma 2.13 that there exist elements $w_1, w_2 \in W(\mathfrak{g}, \mathfrak{h})$ such that $w_1\mu = \delta$ and $w_2\lambda = -\delta$. Thus, by Lemma 2.12, we may assume without loss of generality that $\lambda = -\delta$, so $H_{\pi,K}^\infty$ is equivalent to $V(\mu, -\delta)$ for a suitable $\mu \in \mathfrak{h}^*$. Note that $-\delta$ is Φ-integral, regular and $-\Phi^+$-dominant (cf. [**27**, Lemma 13.3.A]).

We can now use the results from Enright's proof in [**17**, Section 6] to specify the element μ. Since π is unitary, we can apply Lemma 6.2 in [**17**] which gives us the existence of a unique element $w \in W$ such that $w^2 = 1$ and $\mu = -w(\delta)$. We set

$$\Phi_w := \{\alpha \in \Phi \mid \langle -w(\delta) - \delta, \alpha \rangle = 0\}$$

and denote by W_w the subgroup of W generated by the set of reflections $\{s_\alpha \mid \alpha \in \Phi_w\}$. Moreover, we set $\Phi_w^+ := \Phi_w \cap \Phi^+$ and $\delta_w := \sum_{\alpha \in \Phi_w^+} \alpha$. Now by [**17**, Lemma 6.3 and Prop. 6.4], w has the following properties: The element w is the longest element of the Weyl group W_w, i.e. the unique element such that $w(\Phi_w^+) = -\Phi_w^+$. Moreover, we have $\Delta \cap \Phi_w = \Delta_w$, where Δ_w denotes the set of simple roots in Φ_w^+ and $\delta - w(\delta) = 2\delta_w$.

Now Δ_w is a subset of Δ and hence we can associate a standard parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Δ_w as explained in Section I.4.1. Denote the associated parabolic subgroup of G by Q. The algebra $\mathfrak{q}$ is of the form $\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_w \cup \Phi^+} \mathfrak{g}_\alpha$. It has a Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ and as above, we denote by $\mathfrak{s} := [\mathfrak{l}, \mathfrak{l}]$ the semisimple part of $\mathfrak{l}$. Recall that $\mathfrak{s}$ is of the form $\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi_w} \mathfrak{g}_\alpha$ and that we may identify Φ_w with the root system $\Phi_\mathfrak{s}$ of $\mathfrak{s}$ by restriction of the elements in Φ_w to $\mathfrak{h} \cap \mathfrak{s}$. In particular, this implies that $w = w_Q$, the longest element of the Weyl group $W_\mathfrak{s}$ of $\Phi_\mathfrak{s}$.

Combining this with the results of the first part of the proof, we see that the Harish-Chandra module $H_{\pi,K}^\infty$ of π is isomorphic to $V(-w_Q(\delta), -\delta)$. However, this is also the Harish-Chandra module of the representation π_Q, as we have seen in Proposition 2.18, in other words π and π_Q are infinitesimally equivalent. Since they are both irreducible unitary representations of G, they are also unitarily equivalent by Proposition I.5.5. $\square$

Propositions 2.18 and 2.19 give us a complete characterization of the equivalence classes of irreducible unitary representations of G with trivial infinitesimal character. Moreover we are able to determine the $(\mathfrak{g}, K)$-cohomology of these representations. For a parabolic subalgebra $\mathfrak{q}$, let $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Langlands decomposition of $\mathfrak{q}$ considered as a real Lie algebra and recall that $\mathfrak{b} := \mathfrak{h} \cap \mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{m}$. Set

$$W^Q := \{w \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \mid w^{-1}(\alpha) > 0 \text{ for all } \alpha \in \Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})\},$$

where we consider elements of $\Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ as elements of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ that are trivial on $\mathfrak{a}_{\mathbb{C}}$.

THEOREM 2.20. *Let G be a connected, simply connected complex Lie group. The correspondence $Q \longleftrightarrow \pi_Q$ is a bijective correspondence between the standard parabolic subgroups $Q \supset Q_0$ of G and the set of equivalence classes of irreducible unitary representations of G with trivial infinitesimal character.*

The relative Lie algebra cohomology of the representations π_Q is given by

$$H^{k+d_Q}(\mathfrak{g}, K; H^{\infty}_{\pi_Q,K}) = \bigoplus_{r+s=k} (H^r(\mathfrak{m}, K_Q; \mathbb{C}) \otimes \bigwedge^s \mathfrak{a}_{\mathbb{C}}), \tag{2.4}$$

where $K_Q := K \cap Q$ and $d_Q := |\Phi^+(\mathfrak{g}, \mathfrak{h})| - |\Phi^+_{\mathfrak{s}}|$.[4]

For the proof of this theorem we will need the following result about the $(\mathfrak{g}, K)$-cohomology of induced representations (see [**10**, Thm. 3.3]):[5]

THEOREM 2.21. *Let (σ, H_σ) be a unitary representation of M and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. For a dominant integral weight $\mu \in \mathfrak{b}_{\mathbb{C}}$, denote by E_μ the irreducible M-module with highest weight μ. Then the following hold:*

(i) If $H^*(\mathfrak{g}, K; I_{Q,\sigma,\nu}) \neq 0$, there exists an element $w \in W^Q$ such that
(1) $w(\rho)|_{\mathfrak{a}_{\mathbb{C}}} + \nu = 0$,
(2) $\chi(\sigma) = \chi_{-w(\rho)|_{\mathfrak{b}_{\mathbb{C}}}}$,
and such an element w is unique.

(ii) If $w \in W^Q$ satisfies (1) and (2), then we have

$$H^{k+\ell(w)}(\mathfrak{g}, K; I_{Q,\sigma,\nu}) = \bigoplus_{r+s=k} (H^r(\mathfrak{m}, K_Q; H_\sigma \otimes E_{(w(\rho)-\rho)|_{\mathfrak{b}_{\mathbb{C}}}}) \otimes \bigwedge^s \mathfrak{a}_{\mathbb{C}}).$$

PROOF OF THEOREM 2.20. The existence of a bijective correspondence between the standard parabolic subgroups of G and the equivalence classes of irreducible unitary representations of G with trivial infinitesimal character is simply a restatement of Propositions 2.18 and 2.19. To compute the cohomology of the representations π_Q, we use Theorem 2.21. A representation π_Q arises by unitary induction from Q to G, so it is a principal series representation as defined in Section I.5.10. The parameters ν and σ of this principal series representation are given by the restriction of σ_Q to A and M. We have seen that σ_Q is a one-dimensional representation of Q, and the restriction of its differential to $\mathfrak{h}_{\mathbb{C}}$ is given by the pair $(\delta_Q - \delta, \delta_Q - \delta)$. Using Lemma 2.5, we see that $(\delta_Q - \delta, \delta_Q - \delta)|_{(\mathfrak{a}_0)_{\mathbb{C}}} = 0$. Since $\mathfrak{a} \subset \mathfrak{a}_0$, this implies $\nu = 0$. Therefore, we have $\pi_Q = I_{Q,\sigma_Q|_M,0}$.

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[4] Note that $K_Q = K \cap Q \subset M$ and is a maximal compact subgroup of M by [**10**, §0.1.6], so taking the relative Lie algebra cohomology of $\mathfrak{m}$ w. r. t. K_Q is defined.

[5] We are stating the theorem from [**10**] for $\lambda = 0$, F_λ the trivial representation.

To apply Theorem 2.21, we need to find a suitable element $w \in W^Q$ that satisfies conditions (1) and (2) of this theorem. Set $w = (1, w_Q w_0) \in W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. We claim that $w \in W^Q$ and that w has the desired properties: Recall that we can identify $\Delta(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$ with the set $(\Delta(\mathfrak{s}, \mathfrak{h} \cap \mathfrak{s}) \times \{0\}) \cup (\{0\} \times \Delta(\mathfrak{s}, \mathfrak{h} \cap \mathfrak{s})) \subset \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, cf. 2.6. For any simple root $\alpha \in \Delta(\mathfrak{s}, \mathfrak{h} \cap \mathfrak{s})$, we have $w_Q(\alpha) \in -\Delta(\mathfrak{s}, \mathfrak{h} \cap \mathfrak{s})$ and hence $w_0 w_Q(\alpha) = (w_Q w_0)^{-1}(\alpha)$ is a positive simple root. This implies $w \in W^Q$.

Next, we look at condition (1). Using (2.3), we get

$$w(\rho) = (\delta, w_Q w_0(-\delta)) = (\delta, w_Q(\delta)) = (\delta, \delta - 2\delta_Q).$$

Lemma 2.5 implies $w(\rho)|_{(\mathfrak{a}_0)_\mathbb{C}} = \delta - \delta + 2\delta_Q = 2\delta_Q$ and since we have $\mathfrak{a} \subset \mathfrak{a}_0$, we obtain $w(\rho)|_{\mathfrak{a}_\mathbb{C}} = 2\delta_Q|_{\mathfrak{a}_\mathbb{C}} = 0$ by the definition of δ_Q.

To show condition (2), let us denote by $\xi \in (\mathfrak{m}_\mathbb{C})^*$ the element obtained from $\sigma_Q|_M$ by differentiation. Recall that the Cartan subalgebra $\mathfrak{b}_\mathbb{C}$ of $\mathfrak{m}_\mathbb{C}$ is of the form $\mathfrak{b}_\mathbb{C} = \mathfrak{a}_\mathbb{C}^\perp \oplus (\mathfrak{m}_0)_\mathbb{C}$. Then $\sigma_Q|_M$ has highest weight

$$\xi|_{\mathfrak{b}_\mathbb{C}} = (\delta_Q - \delta, \delta_Q - \delta)|_{\mathfrak{b}_\mathbb{C}} = 0 \oplus 2(\delta_Q - \delta),$$

and hence by Example I.5.9 its infinitesimal character $\chi(\sigma_Q|_M)$ is parametrized by the Weyl group orbit of

$$\xi|_{\mathfrak{b}_\mathbb{C}} + \rho|_{\mathfrak{b}_\mathbb{C}} = 2\delta|_{\mathfrak{a}_\mathbb{C}^\perp} \oplus 2(\delta_Q - \delta) = 2\delta_Q \oplus 2(\delta_Q - \delta),$$

where the last equality follows from the fact that $\delta|_{\mathfrak{a}_\mathbb{C}^\perp} = \delta|_{\mathfrak{h} \cap \mathfrak{s}} = \delta_Q$, cf. Section I.4.13 and Lemma 2.7. Moreover, we note that

$$w(\rho)|_{\mathfrak{b}_\mathbb{C}} = 2\delta_Q \oplus 2(\delta - \delta_Q).$$

It follows from a straightforward computation that the element $(w_Q, w_Q) \in W(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$ maps $\xi|_{\mathfrak{b}_\mathbb{C}} + \rho|_{\mathfrak{b}_\mathbb{C}}$ to $-w(\rho)|_{\mathfrak{b}_\mathbb{C}}$, and hence these elements define the same infinitesimal character. This proves assumption (2), so we may apply Theorem 2.21 (ii).

Set $d_Q := \ell(w) = \ell(w_Q w_0)$. By [**28**, 1.8, (2)], and since $(w_Q w_0)^{-1} = w_0 w_Q$, we get $d_Q = \ell(w_0) - \ell(w_Q) = |\Phi^+(\mathfrak{g}, \mathfrak{h})| - |\Phi_\mathfrak{s}^+|$. Finally, we look at the coefficient module $H_\sigma \otimes E_{(w(\rho)-\rho)|_{\mathfrak{b}_\mathbb{C}}}$. Because H_σ is a one-dimensional representation of highest weight $\xi|_{\mathfrak{b}_\mathbb{C}}$, the tensor product is an irreducible representation of M of highest weight

$$\xi|_{\mathfrak{b}_\mathbb{C}} + w(\rho)|_{\mathfrak{b}_\mathbb{C}} - \rho|_{\mathfrak{b}_\mathbb{C}} = (0 \oplus 2(\delta_Q - \delta)) + (2\delta_Q \oplus 2(\delta - \delta_Q)) - (2\delta_Q \oplus 0) = 0,$$

that is, the trivial representation. Now, applying the formula in Theorem 2.21 (ii) to the element w will yield the desired description of the cohomology. □

3. Classification of Irreducible Unitary Representations with Nontrivial $(\mathfrak{g}, K)$-Cohomology for $SL_n(\mathbb{C})$

We will now apply the results of the previous section to the group $G = SL_n(\mathbb{C})$. With the help of Theorem 2.20, we will determine (up to equivalence) all irreducible unitary representations of $SL_n(\mathbb{C})$ with trivial infinitesimal character and compute their $(\mathfrak{g}, K)$-cohomology. Using Matsushima's formula, this gives us all the degrees in which the arithmetic quotients X/Γ from Theorem III.6.4 can possibly have nontrivial cohomology. In special cases, we will deduce the existence of certain automorphic representations in the spectrum of $SL_n(\mathbb{C})/\Gamma$.

3.1. Let us fix a minimal parabolic subgroup Q_0 of $G := SL_n(\mathbb{C})$. Recall from Section 2.16 that we have defined an irreducible unitary representation π_Q of G with trivial infinitesimal character for each standard parabolic subgroup $Q \supset Q_0$. Below, we will explicitly determine the $(\mathfrak{g}, K)$-cohomology of these representations. To be able to describe these cohomology spaces, we introduce the notion of the Poincaré polynomial of a graded vector space:

Let $X = \bigoplus_{j=1}^{n} X_j$ be a finite-dimensional graded vector space. Then the *Poincaré polynomial of* X is the polynomial

$$P_X(t) := \sum_{j=1}^{n} \dim(X_j) \cdot t^j.$$

If we are given two finite-dimensional graded vector spaces X and Y, we can define a gradation on the tensor product $X \otimes Y$ by setting $(X \otimes Y)_j := \bigoplus_{k+l=j} X_k \otimes Y_l$. Then it is easy to see that $P_{X\otimes Y} = P_X \cdot P_Y$, i. e. the Poincaré polynomial is multiplicative with respect to tensor products.

THEOREM 3.2. *The equivalence classes of irreducible unitary representations of $SL_n(\mathbb{C})$ with trivial infinitesimal character are in one-to-one correspondence with the standard parabolic subgroups $Q \supset Q_0$ of G; a standard parabolic subgroup corresponds to the induced representation π_Q. The $(\mathfrak{g}, K)$-cohomology of the representations π_Q is given as follows:*

(1) The Poincaré polynomial of $H^*(\mathfrak{g}, K; H^\infty_{\pi_{Q_0},K})$ is of the form

$$P_{H^*(\mathfrak{g},K;H_{\pi_{Q_0},K})}(t) = t^{\frac{n(n-1)}{2}} \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} t^k.$$

(2) For $Q \neq Q_0$, let m denote the number of blocks of Q, let ℓ_j denote the length of the j-th block and set $d := \frac{n(n-1)}{2} - \sum_{j=1}^{m} \frac{\ell_j(\ell_j-1)}{2}$, cf. Example I.4.4. Then the Poincaré polynomial of $H^*(\mathfrak{g}, K; H^\infty_{\pi_Q,K})$ is given by

$$P_{H^*(\mathfrak{g},K;H^\infty_{\pi_Q,K})}(t) = t^d \cdot \left(\sum_{k=0}^{m-1} \binom{m-1}{k} t^k \right) \cdot \prod_{j=1,\ell_j\neq 1}^{m} \prod_{k=2}^{\ell_j} (1 + t^{2k-1}).$$

PROOF. The first part of the theorem is a direct application of Theorem 2.20 to the connected simply connected complex Lie group $SL_n(\mathbb{C})$.

Now let Q be a standard parabolic subgroup of $SL_n(\mathbb{C})$, denote its Lie algebra by $\mathfrak{q}$ and the corresponding Langlands decomposition by $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Recall that we have studied the structure of the Lie algebras $\mathfrak{q}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ and their corresponding Lie groups in Example I.4.8. To compute the cohomology of the representations π_Q, we use the formula from Theorem 2.20. Note that in terms of Poincaré polynomials, this formula says

$$P_{H^*(\mathfrak{g},K;H^\infty_{\pi_Q,K})}(t) = t^{d_Q} \cdot P_{H^*(\mathfrak{m},K_Q;\mathbb{C})}(t) \cdot P_{\bigwedge(\mathfrak{a}_\mathbb{C})}(t). \tag{3.1}$$

Therefore, it suffices to determine the number d_Q and the Poincaré polynomials of $H^*(\mathfrak{m}, K_Q; \mathbb{C})$ and $\bigwedge(\mathfrak{a}_\mathbb{C})$.

(1) *The Poincaré polynomial of* $\bigwedge(\mathfrak{a}_\mathbb{C})$: Let V be a complex vector space of finite dimension $\dim V = v$. Then the exterior algebra $\bigwedge(V) = \bigoplus_{k=0}^{v} \bigwedge^k V$ is a finite-dimensional

graded vector space, where the summand $\bigwedge^k V$ is of dimension $\binom{v}{k}$ for $0 \leq k \leq v$. In particular, the Poincaré polynomial of $\bigwedge(V)$ is of the form

$$P_{\bigwedge(V)}(t) = \sum_{k=0}^{v} \binom{v}{k} t^k.$$

Let us apply this to the complex vector space $\mathfrak{a}_{\mathbb{C}}$: As we have seen in Example I.4.8, the Lie algebra $\mathfrak{a}$ consists of the real diagonal matrices with trace zero that are of the form $\operatorname{diag}(x_1 I_{\ell_1}, \ldots, x_m I_{\ell_m})$. This implies that $\mathfrak{a}_{\mathbb{C}}$ has complex dimension $m-1$ and

$$P_{\bigwedge(\mathfrak{a}_{\mathbb{C}})}(t) = \sum_{k=0}^{m-1} \binom{m-1}{k} t^k.$$

(2) *The Poincaré polynomial of* $H^*(\mathfrak{m}, K_Q; \mathbb{C})$: We start with the case $Q = Q_0$. The group Q_0 has the Langlands decomposition $Q_0 = M_0 A_0 N_0$, where M_0 is compact. This implies $K_{Q_0} = K \cap M_0 = M_0$, so in fact we consider the relative Lie algebra cohomology $H^*(\mathfrak{m}_0, M_0; \mathbb{C})$. By I.6.1, this is one-dimensional in degree 0 and since $\dim(\mathfrak{m}_0/\mathfrak{m}_0) = 0$, it is trivial in all the other degrees. In particular,

$$P_{H^*(\mathfrak{m}_0, K_{Q_0}; \mathbb{C})}(t) = 1.$$

Now let $Q \neq Q_0$. The Lie algebra $\mathfrak{m}$ is reductive and has semisimple part $\mathfrak{s}$ and center $Z_{\mathfrak{m}} \subset \mathfrak{k}$, cf. Section I.4.13. Using the Künneth rule (see [**10**, I.1.3]) and the fact that K_Q and $K_Q \cap S$ are connected, we obtain $H^*(\mathfrak{m}, K_Q; \mathbb{C}) = H^*(\mathfrak{s}, K_Q \cap S; \mathbb{C})$, so we can restrict to the semisimple part. Recall from Example I.4.4 that $S \cong \prod_{j=1}^m SL_{\ell_j}(\mathbb{C})$. This is clearly the group of real points of a reductive algebraic $\mathbb{R}$-group, so by Proposition I.6.6 and Example I.6.5, $H^*(\mathfrak{s}, K_Q \cap S; \mathbb{C})$ equals the cohomology of the compact symmetric space $\prod_{j=1}^m SU(\ell_j)$. For $\ell_j > 2$, the Poincaré polynomial of $H^*(SU(\ell_j); \mathbb{C})$ is given by $P_{H^*(SU(\ell_j);\mathbb{C})}(t) = \prod_{k=2}^{\ell_j}(1+t^{2k-1})$, see [**24**, Thm. VI.X]. For $\ell_j = 1$, we have $SU(1) = S^1$, so the Poincaré polynomial is given by $P_{H^*(SU(1), \mathbb{C})}(t) = 1$. Putting everything together, we obtain the formula

$$P_{H^*(\mathfrak{m}, K_Q; \mathbb{C})}(t) = \prod_{j=1}^{m} P_{H^*(SU(\ell_j), \mathbb{C})}(t) = \prod_{j=1, \ell_j \neq 1}^{m} \prod_{k=2}^{\ell_j} (1 + t^{2k-1}),$$

where we have used the Künneth rule for singular cohomology in the first step to deduce $P_{H^*(\prod_{j=1}^m SU(\ell_j), \mathbb{C})}(t) = \prod_{j=1}^m P_{H^*(SU(\ell_j), \mathbb{C})}(t)$.

(3) *Determination of* d_Q: It remains to determine the number d_Q that is the shift in the cohomology on the left hand side of equation (2.4). Recall from Theorem 2.20 that $d_Q = |\Phi^+| - |\Phi_{\mathfrak{s}}^+|$. The cardinality of Φ^+ equals $\frac{n(n-1)}{2}$, as can be seen from the explicit set of positive roots given in Example I.4.4. By definition of $\Phi_{\mathfrak{s}}$, a root $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})$ lies in Φ_s^+ if and only if $X_\alpha \in \mathfrak{s}$. These are exactly the X_α lying in the upper triangle of any block of $\mathfrak{s}$, so we conclude $|\Phi_{\mathfrak{s}}^+| = \sum_{j=1}^m \frac{\ell_j(\ell_j-1)}{2}$ and $d_Q = \frac{n(n-1)}{2} - \sum_{j=1}^m \frac{\ell_j(\ell_j-1)}{2}$.

Now, putting everything together and using equation (3.1), we obtain the formulas for the $(\mathfrak{g}, K)$-cohomology of π_Q stated in the theorem. $\square$

3.3. Examples. Finally, we look at some explicit examples. For certain values of n we will give an overview of the occurring representations and their cohomology.

Example 3.4. Let us make the results of Theorem 3.2 explicit for small values of n. As a minimal parabolic subgroup of $SL_n(\mathbb{C})$ we fix the subgroup Q_0 of upper triangular matrices.

Case $n = 2$: In $SL_2(\mathbb{C})$ we only have two standard parabolic subgroups, corresponding to the compositions $2 = 1 + 1$ and $2 = 2$ of 2. These are the minimal parabolic subgroup $Q = Q_0$ and the whole group $Q = G$. The associated representations are the representation π_{Q_0} that is induced from the one-dimensional representation of Q_0 with highest weight $(\delta, -\delta)$ and the trivial representation π_G. An application of Theorem 3.2 gives us the Poincaré polynomials of the $(\mathfrak{g}, K)$-cohomology of these representations:

$$P_{H^*(\mathfrak{g},K;\pi_{Q_0})}(t) = t + t^2,$$
$$P_{H^*(\mathfrak{g},K;\pi_G)}(t) = 1 + t^3.$$

For $n = 3$ and $n = 4$ the situation is more complicated and we will give the results in Table 1 and Table 2 below. We denote a composition $n = \ell_1 + \ldots + \ell_m$ by the m-tuple $(\ell_1, \ldots, \ell_m)$ and the associated parabolic subgroup by $Q_{\ell_1,\ldots,\ell_m}$ (cf. Example I.4.4).

$(\ell_1, \ldots, \ell_m)$	$P_{H^*(\mathfrak{g},K;\pi_{Q_{\ell_1,\ldots,\ell_m}})}(t) =$
$(1,1,1)$	$t^3 + 2t^4 + t^5$
$(1,2)$	$t^2 + t^3 + t^5 + t^6$
$(2,1)$	$t^2 + t^3 + t^5 + t^6$
(3)	$1 + t^3 + t^5 + t^8$

Table 1. Poincaré polynomials of the irreducible unitary representations of $SL_3(\mathbb{C})$ with nontrivial $(\mathfrak{g}, K)$-cohomology.

$(\ell_1, \ldots, \ell_m)$	$P_{H^*(\mathfrak{g},K;\pi_{Q_{\ell_1,\ldots,\ell_m}})}(t) =$
$(1,1,1,1)$	$t^6 + 3t^7 + 3t^8 + t^9$
$(1,1,2)$	$t^5 + 2t^6 + t^7 + t^8 + 2t^9 + t^{10}$
$(1,2,1)$	$t^5 + 2t^6 + t^7 + t^8 + 2t^9 + t^{10}$
$(2,1,1)$	$t^5 + 2t^6 + t^7 + t^8 + 2t^9 + t^{10}$
$(1,3)$	$t^3 + t^4 + t^6 + t^7 + t^8 + t^9 + t^{11} + t^{12}$
$(2,2)$	$t^4 + t^5 + 2t^7 + 2t^8 + t^{10} + t^{11}$
$(3,1)$	$t^3 + t^4 + t^6 + t^7 + t^8 + t^9 + t^{11} + t^{12}$
(4)	$1 + t^3 + t^5 + t^7 + t^8 + t^{10} + t^{12} + t^{15}$

Table 2. Poincaré polynomials of the irreducible unitary representations of $SL_4(\mathbb{C})$ with nontrivial $(\mathfrak{g}, K)$-cohomology.

Remark. One can see that already in case $n = 4$ there are many different representations that have cohomology in the same degree. As is also apparent from the formula in Theorem 3.2, the cohmology of the representation π_Q only depends on the number and the size of the blocks of Q. Different parabolic subgroups that have the same number and same size of blocks but in different order give rise to nonequivalent representations of G; however, their $(\mathfrak{g}, K)$-cohomology is the same.

3.5. Let us relate our findings to the results of Section III.6. Assume we are given a cocompact discrete subgroup $\Gamma \subset SL_n(\mathbb{C})$. The irreducible unitary representations with trivial infinitesimal character that we have classified in the previous sections are exactly the representations that can possibly contribute to the cohomology of X/Γ via Theorem 1.1. However, only those representations whose corresponding multiplicity does not vanish do actually contribute to the cohomology. In general, the question of whether or not for a given representation $\pi \in \widehat{G}$ one has $m(\Gamma, \pi) \neq 0$ is still open. However, as we have already discussed in Section 1.2, the nonvanishing results for the cohomology in Theorem III.6.4 imply the existence of (at least) one nontrivial automorphic representation for $SL_n(\mathbb{C})$ w. r. t. Γ. For small values of n, we can even identify explicit representations with nonvanishing multiplicity. The idea is simple: For a cocompact discrete subgroup $\Gamma \subset SL_n(\mathbb{C})$ obtained as in Theorem III.6.4, we have shown the nonvanishing of the cohomology of X/Γ in certain degrees. If for one of those degrees there is exactly one representation π with nontrivial $(\mathfrak{g}, K)$-cohomology that contributes in that degree, we can deduce that the corresponding multiplicity $m(\pi, \Gamma)$ is not zero.

To be able to compare the degrees in which we have cohomology coming from geometric cycles and the degrees to which our representations can possibly contribute, we summarize this information in Tables 3–5 for $n = 2, 3, 4$. In the right half of each table, we see the degrees in which the space X/Γ can have nontrivial cohomology, from 0 to $\dim X/\Gamma$. In the first two rows, one can read off the degrees of $H^*(X/\Gamma; \mathbb{C})$ in which there exist nontrivial cohomology classes, coming either from the cycles constructed in Section III.5 or from the trivial representation. Note that these classes do not coincide, as we have seen in Theorem III.6.4 that the classes induced by special cycles are not coming from the compact dual symmetric space, i. e. they are not related to the trivial representation. In the lower part of the tables, we indicate the degrees to which the representations π_Q may possibly contribute. As above, we denote a representation π_Q by the associated tuple $(\ell_1, \ldots, \ell_m)$.

	0	1	2	3
Cycles		$\times$	$\times$	
trivial rep.	$\times$			$\times$
$(1,1)$		$\times$	$\times$	

TABLE 3. Complex Case: Contribution to the cohomology of X/Γ, $n = 2$.

COROLLARY 3.6. *Let $n \in \{2, 3\}$ and let Q_0 denote a minimal parabolic subgroup of $SL_n(\mathbb{C})$. Then there exists a cocompact discrete subgroup $\Gamma \subset SL_n(\mathbb{C})$ such that the multiplicity $m(\Gamma, \pi_{Q_0}) \neq 0$.*

PROOF. We choose Γ as in Theorem III.6.4 for $n = 2$ or $n = 3$. Then the result can be read off from Table 3 and Table 4: In case $n = 2$, we have cycles contributing to the cohomology in degrees one and two, and π_{Q_0} is the only unitary representation that has

	0	1	2	3	4	5	6	7	8
Cycles				×	×	×			
trivial rep.	×			×		×			×
$(1,1,1)$				×	×	×			
$(2,1)$			×	×		×	×		
$(1,2)$			×	×		×	×		

TABLE 4. Complex Case: Contribution to the cohomology of X/Γ, $n = 3$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Cycles						×	×	×	×	×	×					
trivial rep.	×			×		×		×	×		×		×			×
$(1,1,1,1)$							×	×	×	×						
$(1,1,2)$						×	×	×	×	×	×					
$(1,2,1)$						×	×	×	×	×	×					
$(2,1,1)$						×	×	×	×	×	×					
$(1,3)$				×	×		×	×	×	×		×	×			
$(2,2)$					×	×		×	×		×	×				
$(3,1)$				×	×		×	×	×	×		×	×			

TABLE 5. Complex Case: Contribution to the cohomology of X/Γ, $n = 4$.

cohomology in these degrees. Therefore, we conclude $m(\Gamma, \pi_{Q_0}) \neq 0$. Similarly, for $n = 3$, we have cycles contributing to degree four, and π_{Q_0} is the only unitary representation of $SL_3(\mathbb{C})$ that has cohomology in degree four. □

REMARK. Already in case $n = 4$, this reasoning is not successful anymore. As one can see in Table 5, there is no degree in the cohomology of X/Γ in which we have a nontrivial class coming from a cycle and to which only one representation can contribute. Unfortunately, for bigger n the situation will get even more complicated, as there are more representations which have more cohomology and hence more possible overlaps in the degrees.

4. Some Remarks on the Case $G = SL_n(\mathbb{R})$

So far, we have only dealt with the representation theory of complex Lie groups, in particular with the group $SL_n(\mathbb{C})$. We have explicitly classified all irreducible unitary representations of $SL_n(\mathbb{C})$ with nontrivial $(\mathfrak{g}, K)$-cohomology and related this information to the results from Section III.6. Let us conclude this chapter by making some remarks on the case of $SL_n(\mathbb{R})$.

Using the results of Speh [**62**] or Vogan and Zuckerman [**67**], one can classify the irreducible unitary representations of $SL_n(\mathbb{R})$ that have non-trivial $(\mathfrak{g}, K)$-cohomology. We will not make this result explicit here, but concentrate on the contribution of the trivial representation to the right hand side of Matsushima's formula. In the remark at the end of Section I.6.3, we have seen that this contribution is given by the cohomology of the compact symmetric space $X_u = SO(n)\backslash SU(n)$.

4.1. The Cohomology of the Compact Dual of $SL_n(\mathbb{R})/SO(n)$. The cohomology of compact symmetric spaces can be computed with the methods in [**24**, Chapter XI]. For the space $X_u = SO(n)\backslash SU(n)$, this yields the following result:

PROPOSITION 4.2. *The Poincaré polynomial of the cohomology of the compact symmetric space $X_u = SO(n)\backslash SU(n)$ is given by*

$$P_{H^*(X_u;\mathbb{C})}(t) = \begin{cases} (1+t^{2m})\cdot\prod_{k=1}^{m-1}(1+t^{4k+1}) & \text{if } n=2m \text{ is even,}\\ \prod_{k=1}^{m}(1+t^{4k+1}) & \text{if } n=2m+1 \text{ is odd.}\end{cases}$$

SKETCH OF THE PROOF. This is an application of [**24**, Thm. XI.V]. Set $U := SU(n)$ and $K := SO(n)$. Since $K\backslash U$ is a symmetric space, the pair (U,K) is a *Cartan pair* in the sense of Thm. XI.IV *ibid.*, and we can apply Thm. XI.V *ibid.* to obtain the cohomology of $K\backslash U$. To do this, first we need to determine the Poincaré polynomials of the cohomology of the compact groups $SU(n)$ and $SO(n)$ and of the Samelson space $S_{(U,K)}$ of the pair (U,K), cf. *ibid.* §11.2. The Poincaré polynomials of $H^*(SU(n);\mathbb{C})$ and $H^*(SO(n);\mathbb{C})$ are given in §6.28 and §6.29 of [**24**] and are of the form

$$P_{H^*(SU(n);\mathbb{C})}(t) = \prod_{k=2}^{n}(1+t^{2k-1})$$

$$\text{and}\quad P_{H^*(SO(n);\mathbb{C})}(t) = \begin{cases} (1+t^{2m-1})\prod_{k=1}^{m-1}(1+t^{4k-1}) & \text{if } n=2m \text{ is even,}\\ \prod_{k=1}^{m}(1+t^{4k-1}) & \text{if } n=2m+1 \text{ is odd.}\end{cases}$$

Analogous to the Example of $SO(n)\backslash U(n)$, cf. *ibid.* Example 11.11.4, we see that the Poincaré polynomial of the Samelson space is given by

$$P_{S_{(U,K)}}(t) = \begin{cases} \prod_{k=1}^{m-1}(1+t^{4k+1}) & \text{if } n=2m \text{ is even,}\\ \prod_{k=1}^{m}(1+t^{4k+1}) & \text{if } n=2m+1 \text{ is odd.}\end{cases}$$

Now, applying the formula for $P_{H*(K\backslash U)}$ given in [**24**, Thm. XI.V] leads to

$$\begin{aligned} P_{H^*(K\backslash U;\mathbb{C})}(t) &= \frac{\prod_{k=1}^{m}(1-t^{4k})}{(1-t^{2m})\prod_{k=1}^{m-1}(1-t^{4k})}\prod_{k=1}^{m-1}(1+t^{4k+1})\\ &= \frac{(1-t^{4m})}{(1-t^{2m})}\prod_{k=1}^{m-1}(1+t^{4k+1}) = (1+t^{2m})\prod_{k=1}^{m-1}(1+t^{4k+1}),\end{aligned}$$

if n is even, and

$$P_{H^*(K\backslash U;\mathbb{C})}(t) = \frac{\prod_{k=1}^{m}(1-t^{4k})}{\prod_{k=1}^{m}(1-t^{4k})}\prod_{k=1}^{m}(1+t^{4k+1}) = \prod_{k=1}^{m}(1+t^{4k+1}),$$

if n is odd. □

EXAMPLE 4.3. Using the formula of the above proposition, we can easily determine the cohomology of the compact space $SO(n)\backslash SU(n)$ for some small values of n:

$$\begin{aligned} n = 2: &\quad P_{H^*(X_u;\mathbb{C})}(t) = 1 + t^2 \\ n = 3: &\quad P_{H^*(X_u;\mathbb{C})}(t) = 1 + t^5 \\ n = 4: &\quad P_{H^*(X_u;\mathbb{C})}(t) = (1+t^4)(1+t^5) = 1 + t^4 + t^5 + t^9. \end{aligned}$$

4.4. Finally, let us compare these results with those from Section III.6. Let n be even and let $\Gamma \subset SL_n(\mathbb{R})$ be a cocompact discrete subgroup of $SL_n(\mathbb{R})$ that is obtained as in Theorem III.6.2. As we have pointed out in Section 1.2, the nonvanishing result from Theorem III.6.2 implies the existence of a nontrivial automorphic representation for $SL_n(\mathbb{R})$ with respect to Γ. For $n \neq 2$ this cannot be a discrete series representation.

To make this result more precise, we look at the irreducible unitary representations with nontrivial $(\mathfrak{g}, K)$-cohomology for the cases $n = 2$ and $n = 4$. As in paragraph 3.5, we will compare the degrees in which X/Γ has nontrivial cohomology induced by geometric cycles with those degrees in which there is cohomology coming from such a representation.

For $n = 2$ the situation is simple: The symmetric space $SL_2(\mathbb{R})/SO(2)$ is of dimension two. There are two geometric cycles, both inducing nontrivial cohomology classes in degree 1. On the other hand, there are three irreducible unitary representations with nontrivial $(\mathfrak{g}, K)$-cohomology: The trivial representation that has cohomology in degrees 0 and 2 (cf. 4.3), and the holomorphic and antiholomorphic discrete series representations $D_{+,2}$ and $D_{-,2}$ of weight 2, each having cohomology in degree 1 only. These exhaust the irreducible unitary representations of $SL_2(\mathbb{R})$ with nontrivial $(\mathfrak{g}, K)$-cohomology up to equivalence, see e. g. [**56**, Satz 7.9].

	0	1	2
Cycles		$\times$	
trivial rep.	$\times$		$\times$
$D_{+,2}$		$\times$	
$D_{-,2}$		$\times$	

TABLE 6. Real Case: Contribution to the cohomology of X/Γ for $n = 2$.

COROLLARY 4.5. *The multiplicities $m(D_{+,2}, \Gamma)$ and $m(D_{-,2}, \Gamma)$ with which the discrete series representations $D_{+,2}$ and $D_{-,2}$ occur in the decomposition of $L^2(SL_2(\mathbb{R})/\Gamma)$ are nontrivial. Stated differently, this shows the existence of holomorphic and antiholomorphic modular forms of weight two for Γ.*

PROOF. The existence of a nontrivial cohomology class in degree one that is induced from a geometric cycle implies that at least one of the multiplicities $m(D_{+,2}, \Gamma)$ and $m(D_{-,2}, \Gamma)$ is nonzero. However, if there exists a holomorphic modular form of weight 2 for Γ, there is also an antiholomorphic one, and vice versa. Therefore, the two multiplicities can only be nontrivial simultaneously. $\square$

For $n = 4$, the symmetric space is of dimension 9. The irreducible unitary representations with nontrivial $(\mathfrak{g}, K)$-cohomology can be determined using the results of Vogan and Zuckerman [**67**] or Speh [**62**]. We quote the result from [**2**, p. 65]: Up to equivalence, there are six irreducible unitary representations with nontrivial $(\mathfrak{g}, K)$-cohomology, which we denote by $\pi_1, \ldots, \pi_6$. Here, π_1 stands for the trivial representation. We indicate the degrees in the cohomology of X/Γ to which the cycles and the representations contribute in Table 7.

	0	1	2	3	4	5	6	7	8	9
Cycles				$\times$	$\times$	$\times$	$\times$			
π_1 = trivial rep.	$\times$				$\times$	$\times$				$\times$
π_2					$\times$	$\times$				
π_3					$\times$	$\times$				
π_4				$\times$			$\times$			
π_5				$\times$			$\times$			
π_6				$\times$	$\times$	$\times$	$\times$			

TABLE 7. Real Case: Contribution to the cohomology of X/Γ for $n = 4$.

REMARK. In both cases, $n = 2$ and $n = 4$ there are nontrivial cohomology classes detected by cycles in degrees to which the trivial representation does not contribute. Therefore, the approach of constructing nontrivial cohomology classes via geometric cycles really gives us *new* degrees in which the cohomology of X/Γ does not vanish, in the sense that these degrees are not apparent from the cohomology of the compact dual symmetric space.

For $n = 4$, as in the complex case, there are at least two representations with nontrivial cohomology contributing to each degree in which we have cohomology coming from a geometric cycle. Thus, we cannot decide which one of the multiplicities is nontrivial in this case.

APPENDIX A

A Quaternion Division Algebra over $\mathbb{Q}(\sqrt[4]{2})$)

In Example III.3.3 we have claimed that the quaternion algebra $Q(5,73\,|\,\mathbb{Q}(\sqrt[4]{2}))$ is a division algebra that splits at all archimedean places. We will now give a proof of this claim.[1]

We start by stating a well-known criterion for the ramification of a quaternion algebra over an arbitrary field k:

Lemma A.1. *A quaternion algebra $Q = Q(a,b|k)$ ramifies over k if and only if the quadratic form*

$$ax^2 + by^2 - abz^2$$

is anisotropic, i. e. has no nontrivial zeroes in k.

Proof. See [**41**, Thm. 2.3.1 (d)]. □

Recall the notation from Example III.3.3: Let $F := Q(\sqrt{2})$ and $E := \mathbb{Q}(\sqrt[4]{2})$. We denote the real places of E by w_1 and w_2 and the complex place by w'. Consider the quaternion algebra $Q_0 := Q(5,73\,|\,\mathbb{Q})$. At the archimedean place, we have $(Q_0)_\infty = Q_0 \otimes_{\mathbb{Q}} \mathbb{R} = Q(5,73\,|\,\mathbb{R})$. Since 5 and 73 are positive, Q_0 splits at ∞.

Proposition A.2. *The quaternion algebra Q_0 ramifies at the finite place corresponding to the prime* 73.

Proof. We need to show that the quaternion algebra $Q(5,73\,|\,\mathbb{Q}_{73})$ is a division algebra. By Lemma A.1, it suffices to show that the equation

$$5x^2 + 73y^2 = 5 \cdot 73z^2 \tag{A.1}$$

does not have a solution in $\mathbb{Q}_{73}$. Indeed, we can even restrict to $\mathbb{Z}_{73}$ since $\mathbb{Q}_{73} = \mathrm{Quot}(\mathbb{Z}_{73})$ and hence from a solution in $\mathbb{Q}_{73}$ we obtain one in $\mathbb{Z}_{73}$ by multiplying with a common denominator.

Let us assume that there is a triple $(x,y,z) \in \mathbb{Z}_{73}^3$ that satisfies the above equation. Taking the 73-adic absolute value on both sides we get

$$1 > |73|_{73} \cdot |z|_{73}^2 = |5x^2 + 73y^2|_{73} = \max\{|5|_{73} \cdot |x|_{73}^2\,, |73|_{73} \cdot |y|_{73}^2\}, \tag{A.2}$$

where we have used the ultrametic inequality[2] in the last step. Let v_{73} denote the 73-adic valuation on $\mathbb{Q}_{73}$. Then the equation implies $v_{73}(x) \geq 1$ and $v_{73}(y) = v_{73}(z)$. We may thus assume that neither y nor z is a multiple of 73 (otherwise we divide x, y and z by 73 and apply the same argument again). Now we can divide equation (A.1) by 73 and get

[1]This example and the idea of the proof are inspired by a discussion on MathOverflow: `www.mathoverflow.net`.

[2]Here we can use the "strong" form of the ultrametic inequality (see [**22**, Prop. 2.3.3]), since the 73-adic valuations of $5x^2$ and $73y^2$ cannot be equal – one is even while the other is odd.

$5 \cdot 73(x/73)^2 + y^2 - 5z^2 = 0$, where y and z are not divisible by 73. Therefore, modulo 73, we get the identity

$$y^2 - 5z^2 = 0 \mod 73.$$

This is a contradiction, since 5 is not a quadratic residue modulo 73. □

Now we look at the algebra $Q := Q_0 \otimes_{\mathbb{Q}} E$. Let us assume that the prime 73 of $\mathbb{Q}$ is completely split in E. Then for any place w of E with $w|73$, we have $E_w \cong \mathbb{Q}_{73}$ and hence $Q \otimes_E E_w \cong (Q_0 \otimes_{\mathbb{Q}} E) \otimes_E E_w \cong Q_0 \otimes_{\mathbb{Q}} \mathbb{Q}_{73}$. Therefore, it follows that Q is ramified at all places of E lying over 73 if we can show that 73 is completely split in E. This is indeed the case:

Proposition A.3. *The prime* 73 *is completely split in* E.

For the proof of this proposition we use a nice criterion by Kummer about the factorization of prime ideals in finite extensions of number fields:

Theorem A.4 (Kummer's Criterion). *Let ℓ/k be a finite extension of number fields such that $\ell = k(\theta)$ for some $\theta \in \mathcal{O}_\ell$. Assume that $\mathcal{O}_\ell = \mathcal{O}_k[\theta]$. Let $m_{\theta,k}$ be the minimal polynomial of θ over k, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_k$ and denote by $\overline{m}_{\theta,k}$ the polynomial obtained by reducing the coefficients of $m_{\theta,k}$ modulo $\mathfrak{p}$. Suppose*

$$\overline{m}_{\theta,k}(X) = g_1(X)^{e_1} \cdot \ldots \cdot g_t(X)^{e_t}$$

is the factorization of $\overline{m}_{\theta,k}$ as a product of distinct irreducible polynomials $g_1 \ldots g_t \in (\mathcal{O}_k/\mathfrak{p})[X]$. Then $\mathfrak{p}\mathcal{O}_\ell$ has a factorization of the form

$$\mathfrak{p}\mathcal{O}_\ell = \mathfrak{P}_1{}^{e_1} \cdot \ldots \cdot \mathfrak{P}_t{}^{e_t}$$

for certain prime ideals $\mathfrak{P}_i$ of $\mathcal{O}_\ell$ corresponding one to one with the irreducible factors g_i and such that the inertia degree $f_i = [\mathcal{O}_\ell/\mathfrak{P}_i : \mathcal{O}_k/\mathfrak{p}]$ equals the degree of g_i.

Proof. A slightly more general version of this result can be found in [**30**, Thm.I.7.4]. □

Proof of Proposition A.3. First of all, note that we have $\mathcal{O}_E = \mathbb{Z}[\sqrt[4]{2}]$ (see [**19**, Thm. 1]), so we may apply Kummer's criterion. Thus, to see that 73 is completely split in $E = \mathbb{Q}(\sqrt[4]{2})$, we need to find a factorization modulo 73 of the minimal polynomial $m_{\sqrt[4]{2},\mathbb{Q}} = X^4 - 2$ of $\sqrt[4]{2}$ over $\mathbb{Q}$. Over $\mathbb{C}$, we have a factorization

$$X^4 - 2 = (X - \sqrt[4]{2})(X + \sqrt[4]{2})(X - i\sqrt[4]{2})(X + i\sqrt[4]{2}),$$

so we need to find a square root of -1 and a fourth root of 2 modulo 73. We have

$$25^4 = 225^2 \equiv 41^2 \equiv 2 \mod 73$$

and

$$27^2 = 729 \equiv -1 \mod 73.$$

Therefore, we get the identity

$$\begin{aligned} X^4 - 1 &\equiv (X - 25)(X + 25)(X - 27 \cdot 25)(X + 27 \cdot 25) && \mod 73 \\ &\equiv (X + 48)(X + 25)(X + 66)(X + 7) && \mod 73\,, \end{aligned}$$

which is a factorization into four distinct linear factors. By Theorem A.4, this implies that $73\mathcal{O}_E$ splits into 4 distinct prime ideals, i. e. is completely split. □

Now we can put everything together to obtain the following

COROLLARY A.5. *The quaternion algebra* $Q = Q(5, 73 \,|\, \mathbb{Q}(\sqrt[4]{2}))$ *is a division algebra that splits at all archimedean places.*

PROOF. It is clear that Q splits at the complex place of E and since Q_0 splits at ∞ we know that Q also splits at the real places of E. Using Propositions A.2 and A.3, we see that Q ramifies at all places $w|73$ and thus must be a division algebra. $\square$

APPENDIX B

Biquadratic Extensions of Number Fields

Let F denote an algebraic number field. A quartic extension E of F is called *biquadratic* if it is a Galois extension with $\mathrm{Gal}(E/F) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

PROPOSITION B.1. *A biquadratic extension E is of the form $E = F(\sqrt{D_1}, \sqrt{D_2})$ for two elements $D_1, D_2 \in F$ such that neither of D_1, D_2 and D_1D_2 is a square in F. The extension E/F has three intermediate fields corresponding to the nontrivial elements of $\mathcal{G}al(E/F)$. These are quadratic extensions of F of the form $L_1 = F(\sqrt{D_1})$, $L_2 = F(\sqrt{D_2})$ and $L_3 = F(\sqrt{D_1D_2})$.*

SKETCH OF THE PROOF. By the Fundamental Theorem of Galois Theory, there are three distinct intermediate fields of index 2 in E, corresponding to the subgroups of order 2 in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We denote these subfields by L_1, L_2 and L_3. Each field L_i is of the form $L_i = F(\sqrt{D_i})$ for some $D_i \in F$ that is not a square. Since $L_1 \neq L_2$, also D_1D_2 cannot be a square and we deduce $D_3 = D_1D_2$. Finally, $[E : F] = 4$ implies that E is the compositum of L_1 and L_2 and hence $E = L_1L_2 = F(\sqrt{D_1}, \sqrt{D_2})$. $\square$

REMARK. Let us denote by ι_i the nontrivial Galois automorphism of E/L_i for $i \in \{1, 2, 3\}$. Then ι_1 and ι_2 generate $\mathrm{Gal}(E/F)$ and we have $\iota_3 = \iota_1\iota_2 = \iota_2\iota_1$. Moreover, $\iota_i|_{L_j}$ is the nontrivial Galois automorphism of L_j/F for all $i, j \in \{1, 2, 3\}$ with $i \neq j$.

This gives the following diagram of the field extension E/F and its intermediate fields:

$$\begin{tikzcd}
 & E \arrow[dl, dash, "\iota_1"'] \arrow[d, dash, "\iota_2"] \arrow[dr, dash, "\iota_3"] & \\
L_1 \arrow[dr, dash, "\iota_2|_{L_1}=\iota_3|_{L_1}"'] & L_2 \arrow[d, dash, "\iota_1|_{L_2}=\iota_3|_{L_2}"] & L_3 \arrow[dl, dash, "\iota_1|_{L_3}=\iota_2|_{L_3}"] \\
 & F &
\end{tikzcd}$$

The labels at the edges specify the nontrivial Galois automorphism of the respective quadratic extension.

Now assume that F is totally real and E is totally complex. This implies that for any real place $v_0 \in F$, there are two complex places w_1 and w_2 of E with $w_1|v_0$ and $w_2|v_0$ and these are the only places of E lying above v_0.

PROPOSITION B.2. *For a real place v_0 of F there exists $i \in \{1, 2, 3\}$ such that v_0 is decomposed in L_i and v_0 is nondecomposed in L_j, $j \neq i$.*

PROOF. Let us denote by $(D_1)_{v_0}$ and $(D_2)_{v_0}$ the images of D_1 and D_2 in F_{v_0} under the embedding corresponding to the place v_0. If $(D_1)_{v_0}$ is positive, v_0 is decomposed in L_1, i. e. there are two real places v_1, v_2 of L_1 lying above v_0. This implies that $(D_2)_{v_0}$

(and therefore also $(D_1D_2)_{v_0}$) is negative, as otherwise we would not have two complex places w_1 and w_2 lying over v_1 and v_2, respectively, and E would not be totally complex. In particular, v_0 is nondecomposed in $L_2 = F(\sqrt{D_2})$ and $L_3 = F(\sqrt{D_1D_2})$. If $(D_1)_{v_0}$ is negative, v_0 is nondecomposed in L_1, i. e. there exists one complex place v of L_1 lying above L. In this case, one of D_2 and D_1D_2 is positive, while the other one is negative, and thus v_0 is decomposed in L_2 (resp. L_3) and nondecomposed in L_1 and L_3 (resp. L_2). $\square$

Corollary B.3. *Let E/F be a totally complex biquadratic extension of a totally real number field. If we assume that L_1 has one complex place lying above a place v_0 of F and only real archimedean places otherwise, then one of L_2 and L_3 is totally complex while the other one has two real places (lying above v_0) and only complex places otherwise.*

Remark. We have used the statement of the corollary in Section III.5.

APPENDIX C

Nonabelian Galois Cohomology

We summarize some definitions from the theory of nonabelian Galois cohomology. For a more detailed presentation, the reader is referred to [**61**] or [**36**, Chapter VII].

C.1. Let Θ be a group acting on a set A. We denote the action of an element $s \in \Theta$ on $a \in A$ by ${}^s a$. We define the set $H^0(\Theta, A) := \mathrm{Fix}(\Theta, A)$ of elements in A fixed by Θ.

From now on, we assume that A has a group structure and that Θ acts on A by automorphisms, i. e. such that ${}^s(aa') = {}^s a\, {}^s a'$. Then we can define a *1-cocycle of* Θ *in* A as a map

$$\gamma\colon \Theta \to A, \quad s \mapsto \gamma_s$$

such that for all $s, s' \in \Theta$ we have $\gamma_{ss'} = \gamma_s\, {}^s\gamma_{s'}$. We denote the set of 1-cocycles of Θ in A by $Z^1(\Theta, A)$. Clearly, the constant map $\Theta \to A$, $s \mapsto 1_A$ defines an element in $Z^1(\Theta, A)$, the so-called trivial 1-cocycle.

On the set of 1-cocycles, we define an equivalence relation by setting $\gamma \sim \delta$ if and only if there exists an element $a \in A$ satisfying $\delta_s = a\gamma_s\, {}^s a^{-1}$ for all $s \in \Theta$. The set $H^1(\Theta, A)$ of equivalence classes of 1-cocycles is a pointed set with base point the class of the trivial cocycle and is called the *first nonabelian Galois cohomology set of* Θ *in* A.

C.2. Restriction Maps. Let $\Xi \subset \Theta$ be a subgroup of Θ. By restricting a 1-cocycle of Θ in A to Ξ, we get a map $Z^1(\Theta, A) \to Z^1(\Xi, A)$ that induces the so-called *restriction map*

$$\mathrm{res}\colon H^1(\Theta, A) \to H^1(\Xi, A)$$

of pointed sets.

Remark. We note that the kernel of this map plays an important role in parametrizing the connected components of the intersection of two special cycles, cf. Section II.1.4.

C.3. Twisting. Assume we are given a set B that is endowed with an action of Θ and an action of A such that ${}^s(a \cdot b) = {}^s a\, {}^s b$ for all $a \in A$, $b \in B$ and $s \in \Theta$. Let $\gamma \in Z^1(\Theta, A)$ be a 1-cocycle of Θ in A. Then we can define a γ*-twisted action* of Θ on B by setting $s * b := \gamma_s \cdot {}^s b$. The fixed points of this γ-twisted Θ-action in B are denoted by $B(\gamma)$.

Example. This example explains the introduced notions in the context of algebraic groups, as used in Section II.1.

Let $\mathbf{G}$ be a reductive algebraic $\mathbb{Q}$-group and Θ a finite group of commuting $\mathbb{Q}$-rational automorphisms of $\mathbf{G}$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup that is Θ-stable. Then Θ acts on Γ by automorphisms and we have the first nonabelian Galois cohomology set $H^1(\Theta, \Gamma)$ as defined above.

Let $K \subset \mathbf{G}(\mathbb{R})$ be a Θ-stable maximal compact subgroup and set $X := K\backslash\mathbf{G}(\mathbb{R})$. Then Γ acts on $\mathbf{G}$ (more precisely, on all groups $\mathbf{G}(R)$ for any commutative $\mathbb{Q}$-algebra R)

by conjugation. In particular, an element $g \in \Gamma$ acts on $X = K\backslash\mathbf{G}(\mathbb{R})$ by $(g,x) \mapsto xg^{-1}$. Given a 1-cocycle $\gamma\colon \Theta \to \Gamma$, we obtain the following γ-twisted Θ-actions on $\mathbf{G}$ and X: For any commutative $\mathbb{Q}$-algebra R, the new action on $\mathbf{G}(R)$ is given by $s * g := \gamma_s\, {}^s g \gamma_s^{-1}$, and the one on X is given by $s * x := {}^s x \gamma_s^{-1}$.

APPENDIX D

Abstract

The main objective of this thesis is the geometric construction of nontrivial cohomology classes for cocompact discrete subgroups of the real and complex special linear group. The cocompact discrete subgroups in question are of arithmetic nature, i.e. they arise as arithmetic subgroups Γ of a suitable semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, whose group of real points is (up to compact factors) isomorphic to $SL_n(\mathbb{R})$ or $SL_n(\mathbb{C})$.

For such a group Γ that is in addition torsion-free, it is well-known that one has an isomorphism $H^*(\Gamma;\mathbb{C}) \cong H^*(X/\Gamma,\mathbb{C})$, where $X := K\backslash\mathbf{G}(\mathbb{R})$ denotes the symmetric space associated with the real Lie group $\mathbf{G}(\mathbb{R})$ for a maximal compact subgroup $K \subset \mathbf{G}(\mathbb{R})$. Therefore, it is sufficient to study the cohomology of the Riemannian symmetric space X/Γ, which can be done using geometric methods.

The method used in this thesis was developed by Rohlfs and Schwermer [**53**] based on the work of Millson [**44**] and Millson-Raghunathan [**45**]. It is is applicable under the assumption that the compact quotient X/Γ is orientable. The main idea is to find oriented submanifolds of X/Γ whose fundamental classes induce nontrivial cohomology classes in $H^*(X/\Gamma;\mathbb{C})$ via Poincaré duality. The submanifolds arise as arithmetic quotients attached to fixed point groups of certain $\mathbb{Q}$-rational automorphism of finite order on $\mathbf{G}$ and are called *special cycles*.

For an algebraic group $\mathbf{G}$ and an arithmetic subgroup Γ as above, we construct pairs of special cycles of complementary dimension and show that these contribute nontrivially to the cohomology of X/Γ. This yields the existence of nontrivial cohomology classes in many degrees.

The cohomology of the compact quotient X/Γ is also related to the $(\mathfrak{g},K)$-cohomology of irreducible unitary representations of the group $\mathbf{G}(\mathbb{R})$ via Matsushima's formula. In the second part of the thesis, we study the representation theory of the group $SL_n(\mathbb{C})$. By making explicit results of Delorme [**14**] and Enright [**17**], we classify all irreducible unitary representations of $SL_n(\mathbb{C})$ with nontrivial $(\mathfrak{g},K)$-cohomology up to unitary equivalence. Combining these results with those from the first part implies the existence of certain automorphic representations.

APPENDIX E

Zusammenfassung

Ziel dieser Arbeit ist die Konstruktion von nichttrivialen Kohomologieklassen für kokompakte, diskrete Untergruppen der reellen und komplexen speziellen linearen Gruppe mit Hilfe von geometrischen Methoden. Die betrachteten kokompakten, diskreten Untergruppen sind arithmetisch definiert, das heißt, sie sind arithmetische Untergruppen einer halbeinfachen algebraischen $\mathbb{Q}$-Gruppe $\mathbf{G}$, deren Gruppe der reellen Punkte bis auf kompakte Faktoren isomorph zu $SL_n(\mathbb{R})$ oder $SL_n(\mathbb{C})$ ist.

Ist Γ eine solche Gruppe, die außerdem torsionsfrei ist, so gibt es einen Isomorphismus $H^*(\Gamma;\mathbb{C}) \cong H^*(X/\Gamma, \mathbb{C})$. Dabei bezeichnet $X := K\backslash\mathbf{G}(\mathbb{R})$ den zu $\mathbf{G}(\mathbb{R})$ gehörigen symmetrischen Raum für eine beliebige maximal kompakte Untergruppe K. Es genügt daher, die Kohomologie des riemannschen symmetrischen Raumes X/Γ zu untersuchen. Dazu können geometrische Methoden verwendet werden.

Wir benutzen für unser zentrales Resultat eine Methode von Rohlfs und Schwermer [**53**], die auf Basis der Ideen von Millson [**44**] und Millson-Raghunathan [**45**] entwickelt wurde. Die Methode lässt sich auf den Fall kompakter, orientierbarer Quotienten X/Γ anwenden. Die zugrundliegende Idee ist, orientierte Teilmannigfaltigkeiten in X/Γ zu finden, deren Fundamentalklasse mittels Poincaré-Dualität nichttriviale Kohomologieklassen in der Kohomologie von X/Γ induziert. Geeignete Teilmannigfaltigkeiten sind arithmetische Quotienten von Fixpunktgruppen $\mathbb{Q}$-rationaler Automorphismen endlicher Ordnung auf $\mathbf{G}$. Solche Teilmannigfaltigkeiten werden *spezielle Zykel* genannt.

In dieser Arbeit konstruieren wir zunächst Paare von speziellen Zykeln für algebraische Gruppen $\mathbf{G}$ mit der Eigenschaft, dass $\mathbf{G}(\mathbb{R})$ bis auf kompakte Faktoren mit $SL_n(\mathbb{R})$ oder $SL_n(\mathbb{C})$ übereinstimmt und zeigen, dass diese nichttriviale Klassen in verschiedenen Graden von $H^*(X/\Gamma, \mathbb{C})$ induzieren. Im zweiten Teil der Arbeit beschäftigen wir uns mit dem Zusammenhang zur Darstellungstheorie mittels der Formel von Matsushima. Im Besonderen untersuchen wir die Darstellungen der Gruppe $SL_n(\mathbb{C})$ und klassifizieren alle unitären irreduziben Darstellungen mit nichttrivialer $(\mathfrak{g}, K)$-Kohomologie bis auf Äquivalenz. Dabei orientieren wir uns an den Resultaten von Delorme [**14**] und Enright [**17**]. In Kombination mit den konstruierten nichttrivialen Kohomologieklassen für X/Γ liefert dies die Existenz gewisser automorpher Darstellungen.

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