DIPLOMARBEIT

Titel der Diplomarbeit

Quantum Entanglement and Relativity

Verfasserin

Irena Bojic

angestrebter akademischer Grad

Magistra der Naturwissenschaften (Mag.rer.nat)

Wien, im Februar 2013

Studienkennzahl lt. Studienblatt:	A 411
Studienrichtung lt. Studienblatt:	Physik
Betreuer:	Univ. Doz. Dr. Beatrix C. Hiesmayr

Quantum Entanglement and Relativity

February 19, 2013

Contents

Abstract

The phenomenon of quantum entanglement lies at the heart of quantum physics. Mathematically, entanglement is defined by the inseparability of quantum states that are used to describe quantum objects. States, which are inseparable are called entangled. These entangled objects can be spatially separate objects or distinct features of the same object, which can be described only by a common mathematical state vector. An separate description for the individual objects is not possible.

Physically, entanglement manifests itself e.g. in nonlocal features, which can be tested experimentally. Classical local concepts like e.g. the Bell inequalities are violated by the predictions of quantum mechanics due to entanglement, which can be verified in experiment as well. So, the classical concepts of locality and reality were exposed to be misleading towards a better understanding of the concept of nature.

Also, various applications of entanglement theory opened up a broad field of research, like e.g. quantum information, quantum information processing and quantum cryptography.

The theory of (special) relativity is a classical theory describing the transformation properties between inertial frames of reference, wherein the laws of physics remain unchanged.

In this thesis we discuss the connection of entanglement and relativity, that is, an introduction into both fields and some of the recent research topics in relativistic entanglement theory is presented. Previous works in this field suggest that entanglement is not frame independent in general, that is, not Lorentz invariant or observer independent, if different partitions of the Hilbert space are considered. This is shown e.g. for the case of two entangled spin-1/2 and and also two entangled spin-1 particles for different quantum states.

Furthermore, this thesis presents a new example of three entangled spin-1 particles. Their relativistic transformation properties are examined in the last chapter.

The detection of multipartite entanglement is discussed for different reference frames as well.

Zusammenfassung

Das Phänomen der *Verschränkung* hat eine grundlegende Bedeutung in der Quantenphysik. Mathematisch is Verschränkung definiert durch die *Inseparabilität* von Quantenzuständen, die verwendet werden um Quantenobjekte zu beschreiben. Zustände, die nicht *separabel* sind, werden *verschränkt* genannt. Diese können räumlich getrennte Objekte oder verschiedene Eigenschaften desselben Objektes sein, welche nur durch einen gemeinsamen mathematischen Zustandvektor beschrieben werden können. Für die einzelnen Objekte gibt es keine separable Beschreibung.

Physikalisch äußert sich Verschränkung zum Beispiel in nichtlokalen Eigenschaften, die auch experimentell nachgewiesen werden können. Klassische, lokale Konzepte, wie z.B. die Bell-Ungleichungen, werden durch Voraussagen der Quantenmechanik und aufgrund von Verschränkung verletzt, was man auch experimentell nachweisen kann.

Man kann zeigen, dass klassische Konzepte der Lokalität und Realität hinderlich sind für ein besseres Verständnis der Natur.

Verschiedene Anwendungen der Verschränkungstheorie haben breite Forschungsfelder eröffnet, wie zum Beispiel Quanteninformations-Theorie und -Verarbeitung und Quantenkryptographie.

Die (spezielle) Relativitätstheorie ist eine klassische Theorie, die die Transformationseigenschaften zwischen Inertialsystemen beschreibt, in welchen die Gesetze der Physik unverändert bleiben.

Diese Diplomarbeit beschäftigt sich mit der Verbindung von Verschränkung und Relativitätstheorie, beziehungsweise, bietet eine Einleitung in beide Felder und die neueren Forschungsgegenstände der relativistischen Verschränkungstheorie. Frühere Arbeiten auf diesem Gebiet haben gezeigt, dass Verschränkung im Allgemeinen nicht Beobachter-unabhängig oder Lorentz-invariant ist, wenn verschiedene Partitionen des Hilbertraumes betrachtet werden.

Dies wurde anhand von zwei verschränkten Spin-1/2 oder zwei verschränkten Spin-1 Teilchen für verschiedene Quantenzustände gezeigt.

Weiters präsentiert diese Diplomarbeit auch ein neues Beispiel für drei verschränkte Spin-1 Teilchen. Ihre relativistischen Transformationseigenschaften werden im letzten Kapitel untersucht. Es wird ebenfalls die Detektion von Vielkörperverschränkung für verschiedene inertiale Beobachter diskutiert.

1 Introduction

The aim of this work is to give an introduction into the field of *relativistic quantum entanglement theory*, that is, *quantum entanglement* from a *special relativistic* point of view.

In order to understand this relatively young field of research we need to combine the tools of *quantum mechanics*, which are applicable to the problems of quantum information theory, and the tools of special relativity.

Chapter 2 and it's section 2.1 is concerned with the introduction and definition of basic mathematical tools and formalism concerning entanglement, which is needed throughout this work. Also, the *definition*, *detection*, *quantification* and *classification* of entanglement are discussed in section 2.2.

Section 2.3 treats the phenomenon of *quantum nonlocality*, which can be linked to entangled quantum systems and which has led to a lot of philosophical and conceptual problems in the past.

In chapter 3 we are going to review the basic terms and definitions of special relativity in order to understand the concept of inertial *frames of reference* and coordinate transformations between those, also known as *Lorentz* or *Poincaré transformations*.

Finally, chapter 4 is concerned with the relativistic transformation properties of entangled systems. The key ingredient within these relativistic transformation, the *Wigner Rotation*, which is resulting from the fact that we demand *unitary* transformations of quantum states, is examined. We are going to review the relativistic transformation properties of the amount of entanglement in *spin-1/2 systems*, which can be modeled as *qubit systems* within quantum information theory.

The conversion to entangled relativistic *spin-1 particles*, which can be modeled as *qutrit systems*, is introduced in section 4.2. There, the change of entanglement corresponding to a change of reference frame is discussed for a certain family of states, which in the maximally entangled case is also known as *Aharonov state*, and the symmetric version thereof. Also, the detection of *multipartite entanglement* in different frames is considered.

2 Quantum mechanics

The theory of quantum mechanics - like every other physical theory - describes the state and evolution of (certain) physical systems with the final cause being the prediction of the outcome of experiments. For quantum mechanical systems the appropriate mathematical formalism proved to be the formalism of vectors and operators in Hilbert space, density operators in Hilbert-Schmidt space, linear (hermitian) operators in Liouville space and theory of probabilities as quantum mechanics makes probabilistic predictions.

Each of the mentioned terms and definitions are going to be clarified briefly in the following sections. We are going to use conventional Dirac notation (see Ref.[68]) for state vectors and restrict ourselves to d-dimensional Hilbert spaces (d=2,3...) as for problems considered in this work a d-dimensional description suffices. For further reading consult e.g. [4, 5, 6].

2.1 Basics and mathematical description of quantum systems

2.1.1 Quantum mechanical system

Simply put, a quantum mechanical system (or QM-system) is every physical system that can be formulated and treated within qm-formalism. The spin component of a particle e.g. requires an explanation within QM whereas e.g. the mass of the same particle is treated classically. A possible interpretation (see e.g. Ref.[6]) of a QM-system is that the same is an abstraction of certain aspects of nature which occur under certain conditions created in the laboratory or a *Gedankenexperiment*. Interaction between a QM-system and the experimenter causes a certain disturbance of the system which leads to the *operational character* of quantum mechanics.

2.1.2 Quantum mechanical state

The state of a QM-system is characterized operationally, i.e. for a certain kind of preparations the experimental outcome show a certain kind of statistical behavior. After many experiments on equally prepared QM-Systems (statistical ensemble) we obtain certain probabilities for the experimental outcome defining the QM-state. Now, the QM-Formalism provides the mathematical framework to compute and predict other probabilities in a different set up or physical scenario.

Definition 2.1 A quantum mechanical state is formally described by a wave function $\psi(\vec{x})$ which is element of a complete function space called Hilbert space $\mathscr{H}$ with inner product in $\mathbb{C}$.

Definition 2.2 The inner product for $\psi, \varphi \in \mathscr{H}$ is defined by

$$\langle \psi \mid \varphi \rangle := \int\limits_{-\infty}^{\infty} d^3x \psi^*(\vec{x}) \varphi(\vec{x}). \tag{1}$$

Here we have also used the Dirac notation for quantum states where the state vector $\mid \varphi\rangle$, also called "ket", corresponds to the wave function $\varphi(\vec{x})$ and the dual vector $\langle \psi \mid \in \mathscr{H}*$(dual vector space) is called "bra", corresponds to the complex conjugate $\psi^*(\vec{x})$ of $\psi(\vec{x})$. Wave functions are elements of the infinite dimensional Hilbert space while state vectors are elements of a finite dimensional vector space used preferably in Quantum Information Theory and Quantum Computation. The two spaces correspond to each other except that in the infinite case certain restrictions need to be satisfied beyond those of the finite complex vector space.

In this work we are not concerned with infinite dimensions and therefore the term *Hilbert space* means always *finite dimensional complex vector space*. The dual vector can also be interpreted as a linear operator from the vector space $\mathscr{H}$ to the complex numbers $\mathbb{C}$.

With Def. 2.2 we are able to define the norm of a vector.

Definition 2.3 The *norm* of a vector is defined by

$$\parallel \langle \psi \mid \psi \rangle \parallel := \sqrt{\langle \psi \mid \psi \rangle}. \tag{2}$$

If the norm of a vector equals 1 the vector is called *unit vector* or *normalized*. Two vectors are orthonormal, if their norm equals 1 and if they are orthogonal, i.e. their inner product equals 0. State vectors associated with a quantum state need to be normalized vectors.

Actions on the elements of $\mathscr{H}$ are represented by the class of linear operators mapping the vector $\mid \psi\rangle$ to a different vector $\mid \varphi\rangle$.

Definition 2.4 A *linear operator* A is defined by its action on "ket"-vectors $\mid \psi\rangle$ by

$$A \mid \psi\rangle =: \mid A\psi\rangle = \mid \varphi\rangle. \tag{3}$$

Linear operators A can be exemplified by their matrix representation A_{ij} with entries ij.

For every linear operator A on $\mathscr{H}$ we can define the *adjoint operator* $A^\dagger$ by

Definition 2.5 A linear operator $A^\dagger$ is called *adjoint* to A and is defined by its action on a dual vector $\langle\psi \mid \in \mathscr{H}^*$ such that following relations are true

$$\langle A\psi \mid \varphi\rangle = \langle\psi \mid A^\dagger\varphi\rangle = \langle\psi \mid A^\dagger \mid \varphi\rangle, \tag{4}$$

$$\langle A^\dagger\psi \mid \varphi\rangle = \langle\psi \mid A\varphi\rangle = \langle\psi \mid A \mid \varphi\rangle.$$

A subclass of the class of linear operators are the *hermitian operators*. They have an important physical meaning. Their eigenvalues are elements of the real numbers and are therefore associated with physical quantities and referred to as *observables*. But first we have to define normal operators:

Definition 2.6 A linear operator N is *normal* if it is diagonalizable w.r.t. some $\mathscr{ONB}$ and vice versa. In other words, N is normal if there exists a *spectral decomposition* of N

$$N = \sum_i \lambda_i \mid i\rangle\langle i \mid \tag{5}$$

with eigenvectors$\mid i\rangle$ such that $\{\mid i\rangle\}$ is an $\mathscr{ONB}$ and associated with every $\mid i\rangle$ is an eigenvalue λ_i.

Definition 2.7 A linear operator A is called *hermitian* or *self-adjoint* if following relation is true

$$A^\dagger = A. \tag{6}$$

Hermitian operators possess the special spectral decomposition

$$A = \sum_i r_i \mid i\rangle\langle i \mid \tag{7}$$

with $r_i \in \mathbb{R}$ and $\mathscr{ONB}$ $\{\mid i\rangle\}$. Eigenvectors of A belonging to its different eigenvalues are orthogonal.

With help of Definition 2.6 it is easy to verify that the expectation value of some hermitian A, an important physical quantity (see Def. 2.7.), is real for arbitrary state vectors $\mid \psi\rangle$.

Definition 2.8. Measurement outcome for some observable A are represented by its expectation value

$$\langle\psi \mid A \mid \psi\rangle =: \langle A\rangle_\psi \tag{8}$$

for some arbitrary state $\mid \psi\rangle$.

Special representatives of hermitian operators are projection operators P.

Definition 2.9 An hermitian operator P , defined by

$$P = \sum_{i=1}^{k} | \, i\rangle\langle i \, |, \tag{9}$$

is a *projection* onto the $k - dimensional$ subspace of a $d - dimensional$ Hilbertspace $\mathscr{H}$ with $k \leq d$ and $\mathscr{O}\mathscr{N}\mathscr{B}$ $\{| \, i\rangle\}$. For P also the relation

$$P = P^2 \tag{10}$$

is true.

Another important subclass of the class of linear operators are *unitary operators.*

Definition 2.10 A linear operator U is called *unitary* if

$$U^\dagger = U^{-1} \tag{11}$$

holds and with it the relation

$$U^\dagger U = U^{-1}U = \mathbb{I} \tag{12}$$

as well.
Unitary operators possess the special spectral decomposition

$$U = \sum_i e^{i\varphi_i} \, | \, i\rangle\langle i \, | \tag{13}$$

with $\varphi_i \in \mathbb{R}$ and $\mathscr{O}\mathscr{N}\mathscr{B}$ $\{| \, i\rangle\}$. The eigenvalue here is called *phase factor*. An important feature and physical meaning of unitary operators is that they preserve inner products of state vectors, i.e. for some vectors $| \, \psi\rangle$, $| \, \varphi\rangle$ the relation

$$\langle\psi \mid \varphi\rangle = \langle\psi \mid U^\dagger U \mid \varphi\rangle \tag{14}$$

is true.

2.1.3 Quantum dynamics

According to the *Schrödinger picture,* the time evolution of closed quantum systems is described by the action of an unitary operator U on the state vector $\mid \psi\rangle$ representing the system. The unitary operator $U(t, t_0)$ relates the state vectors of the system at time t_0 (a fixed time) and a time t :

$$\mid \psi(t)\rangle = U(t, t_0) \mid \psi(t_0)\rangle. \tag{15}$$

The time evolution of $U(t, t_0)$ is given by the differential equation[1]

$$i\hbar\frac{\partial}{\partial t}U(t, t_0) = H(t)U(t, t_0), \tag{16}$$

where the Hamilton operator[2] $H(t)$ is, by postulate, the hermitian operator describing the total energy of the system.

With (15) and (16) we receive the well known *Schrödinger equation*

$$i\hbar\frac{\partial}{\partial t} \mid \psi(t)\rangle = H(t) \mid \psi(t)\rangle \tag{17}$$

where the time evolution of $\mid \psi(t)\rangle$ is governed by the particular Hamiltonian of the considered system.

At this point we should emphasize that quantum mechanics doesn't tell us the explicit form of the Hamiltonian of the system. The time evolution

[1] Except for this chapter, the *Planck constant* $\hbar = 1,054571726\times10^{-34} J \cdot s$ is going to be set equal to one throughout this work.

[2] $H\,(t)$ *can* be explicitly time dependent. If H is *not* time dependent, the Schrödinger equation reduces to the eigenvalue equation $H \mid \psi\rangle = E \mid \psi\rangle$ with energy eigenstates E.

is described by the Schrödinger equation and if the Hamiltonian is known (by experiment), we are able to determine the state vector $\mid \psi(t)\rangle$, which , itself, is not a physical quantity. By postulate, $\mid \psi(t)\rangle$ becomes physical when being considered as "probability amplitude" which is used to predict a certain behavior of the quantum system, i.e. the square of the norm

$$\| \mid \psi(t)\rangle \|^{2} \tag{18}$$

gives the probability of a certain measurement outcome.

2.1.4 State vectors in Hilbert space

In this thesis the quantum mechanical system of interest is described by a finite dimensional state vector which is element of a finite dimensional complex vector space equipped with an inner product called Hilbert space $\mathscr{H} = \mathbb{C}^d$. The vector is denoted in Dirac notation as "ket" $\mid \psi\rangle$ and contains in principle all accessible information about the considered physical quantities or degrees of freedom.

In two dimensional Hilbert space (d=2) we get the simplest QM-System and also the most important in Quantum Information Theory, the *qubit*, which is denoted by the superposition of two basis vectors $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$, i.e.

$$\mid \psi\rangle = \alpha \mid \psi_1\rangle + \beta \mid \psi_2\rangle, \tag{19}$$

where $\{\mid \psi_1\rangle, \mid \psi_2\rangle\}$ form an $\mathscr{ONB}$ (orthonormal basis) in $\mathscr{H}$ and the parameters α and β are complex numbers fulfilling the normalization condition
$\mid \alpha \mid^2 + \mid \beta \mid^2 = 1$. For d=2 the $\mathscr{ONB}$ is often denoted as $\{\mid 0\rangle, \mid 1\rangle\}$ and is also called "computational basis" and the QM-system is referred to as "*2-level-system*" with two *degrees of freedom*. These degrees of freedom are, in principle, measurable physical quantities, e.g. the spin of a particle or the polarization of a photon.

That this superposition of state vectors is possible originates from the Schrödinger equation (17), which is a linear differential equation allowing for the superposition of several solutions thereof.

The most important system in this work would be the "3-level-system" consisting of *"qutrits"*. A qutrit, formally looking like

$$| \psi\rangle = \alpha \mid \psi_1\rangle + \beta \mid \psi_2\rangle + \gamma \mid \psi_3\rangle, \tag{20}$$

has to fulfill the same normalization conditions for α, β, γ as in the qubit case, that is $| \alpha |^2 + | \beta |^2 + | \gamma |^2 = 1$. The basis states $\{| \psi_1\rangle, | \psi_2\rangle, | \psi_3\rangle\}$ are often denoted by $\{| 0\rangle, | 1\rangle, | 2\rangle\}$.

The most general d-dimensional state vector in d-dimensional Hilbert space $\mathscr{H}$, a so called *qudit*, describing a d-level QM-System is denoted by

$$| \psi\rangle = \sum_{i=0}^{d} \alpha_i \mid i\rangle. \tag{21}$$

2.1.5 Statistical mixtures - mixed states

Up to this point we were concerned with *pure states* which can be treated in the state vector formalism. A generalization of this formulation is the *density operator* formalism, where, up to a phase, every quantum state is determined by an operator, or the matrix representation thereof, w.r.t. a chosen basis.

More precisely, in case of maximal information about the considered degrees of freedom of some quantum system we can use the pure state description in terms of state vectors. In the more realistic case we may not have maximal information about a certain state, i.e. there is loss of information due to interaction with the environment, an uncertain preparation or when the considered system is a subsystem of an *entangled* bigger *pure* system (see chapter 2.2).

In these situations the state is described by a *statistical ensemble*, i.e. the density matrix is a *convex sum* of pure state *density matrices*.

The most general quantum state can be formulated by the density matrix, where pure states are contained, but represent an ideal special case.

Definition 2.11 A *general* state is represented by a statistical mixture of pure state vectors or pure state density matrices,

$$\rho = \sum_i p_i \mid \psi_i\rangle\langle\psi_i \mid = \sum_i p_i \rho_i, \tag{22}$$

with

$$p_i \geq 0, \tag{23}$$

and

$$\sum_i p_i = 1, \tag{24}$$

i.e. the pure state density matrices $\rho_i = \mid \psi_i\rangle\langle\psi_i \mid$ are weighted by their respective probabilities p_i, which they occur with (in experiment).

The probabilities p_i are elements of the positive real numbers and normalized according to (24) , i.e. $0 \leq p_i \leq 1$.

The general operator ρ has to fulfill the following properties:

- $\langle\varphi \mid \rho \mid \varphi\rangle \geq 0, \forall \mid \varphi\rangle \in \mathscr{H}$, positivity and following from that, more precisely from $\rho \geq 0$
- $\rho^\dagger = \rho$, hermicity.
- $Tr(\rho) = 1$, following from the normalization of $\mid \psi\rangle$.
- $Tr(\rho^2) \leq 1$,where the equality is valid for pure states.

For the special case of *pure states*, i.e. the state can be described by a *pure state density matrix*

$$\rho = \sum_{i=1}^{1} p_i \mid \psi_i\rangle\langle\psi_i \mid = \sum_{i=1}^{1} p_i \rho_i, \tag{25}$$

with *one* $p_i \neq 0$, and as $\sum_{i=1}^{1} p_i = 1 \Rightarrow p_1 = 1$. Also, following properties are fulfilled by a *pure* ρ:

- $\rho^2 = \rho$, and following from that

- $Tr\,(\rho) = Tr\,(\rho^2) = 1$.

Definition 2.12 The *trace* of a matrix A is defined by

$$Tr\,(A) := \sum_i \langle i \mid A \mid i \rangle, \tag{26}$$

for any $\{\mid i\rangle\}$ is an $\mathscr{O}\mathscr{N}\mathscr{B}$ of the Hilbert space.

With help of the property $Tr(\rho^2) \leq 1$ we are able to introduce the quantity

Definition 2.13 *mixedness* of a density matrix, which can be defined by

$$M(\rho) := (1 - Tr(\rho^2)), \tag{27}$$

$M(\rho)$ scales between 0 for *all* pure states and $1 - \frac{1}{d}$ for the maximally mixed state, $\frac{1}{d}\mathbb{I}$. This quantity can be normalized and is then called linear entropy.

Definition 2.14 We define the *linear entropy* of a quantum state, represented by a density matrix, according to

$$S_L\,(\rho) := \frac{d}{d-1}\left(1 - Tr(\rho^2)\right) = \frac{d}{d-1} M\,(\rho)\,, \tag{28}$$

which is a function ranging from 0 to 1 and represents a simple way, as we are going to see, to detect and measure entanglement contained in pure states.

There are also other quantum entropies like e.g. the *von Neumann entropy,* which is going to be defined in chapter 2.2.4.

Furthermore we can define the expectation value in terms of densitiy matrices, i.e.

Definition 2.15 The *expectation value* $\langle A \rangle_\rho$ of some observable A is defined by

$$\langle A \rangle_\rho = Tr\left(\rho A\right). \tag{29}$$

For reasons of completeness we should also mention the unitary dynamics (Schrödinger picture) adapted for ρ, i.e.

$$\rho(t) = U(t, t_0)\rho(t_0)U^{-1}(t, t_0) \tag{30}$$

and the *von-Neumann equation* describing the dynamics of ρ,i.e.

$$i\hbar\frac{\partial\rho(t)}{\partial t} = \left[H, \rho(t)\right]. \tag{31}$$

2.2 Quantum entanglement

This section is devoted to the most puzzling non-classical feature of quantum mechanics known as quantum *entanglement*. This feature of *composite quantum systems*, first discovered by Schrödinger in 1935 (see the famous paper [1]), implies that there is something "spooky" about quantum mechanics and that "the whole is greater than the sum of its parts[3]".

For years the debate around entanglement and its consequences was considered a purely philosophical one. In 1964 it was John Bell and his famous inequalities (see [3]) who pointed out the physical implications and opened up whole new research fields based on the phenomenon of entanglement, namely quantum information theory, quantum computation, quantum cryptography and dense coding.

Due to the phenomenon of entanglement the total system is in a clearly defined state, whereas the individual subsystems are indetermined[4]. Mathematically speaking, there does not exist a state vector description for the single subsystems of the total system in the presence of entanglement. Physically, entanglement manifests itself in *non-classical correlations* in measurement results of the entangled subsystems (or particles[5]), or of considered degrees of freedom of a single particle. In experiment, these correlations (also called *EPR-correlations*, named after the famous paper of Einstein, Podolsky and Rosen, see Ref. [2], where they employed the phenomenon of entanglement to show that quantum mechanics is incomplete under certain assumptions), can be maintained over large[6] distances. Today, this phenomenon is known as

[3]The phrase "The whole is greater than the sum of it's parts" is originally by Aristotle, but very well suited to describe entanglement qualitatively.

[4]Example: Given the maximally entangled Bell state $\mid \Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ (or equivalently$\mid \Psi\rangle = \frac{1}{\sqrt{2}}(\mid 01\rangle - \mid 10\rangle)$, see (42)), Alice is going to measure $\uparrow$ or $\downarrow$ with probability $p_i = \frac{1}{2}$ each.

Generally, in the limit of many measurements Alice's information about her system is proportional to $\frac{1}{d}\mathbb{I}$, i.e. the maximally mixed state, where d is the dimension of the system (d=2 in the discussed example).

[5]In this thesis the general term "subsystem" can mean a single particle, many particles, or certain degrees of freedom of a single particle. Mathematically, these are different partitions of the Hilbert space. From the context it is clear, which partition is considered.

[6]A distance of 144 km separating entangled photons, which "exceeds all previous tests by more than one order of magnitude" was achieved experimentally in 2007, see Refs. [84][85]. There the entangled photon was transmitted between the Canary Islands of La Palma and Tenerife. Also, a recent experiment (2012) extended the communication distance in quantum teleportation up to 143 km and can be seen as the basis for future

nonlocality[7], clearly interfering with a classical, local-realistic[2] description of the (macroscopic) world.

That these manifestations of entanglement are still present, despite a spatial separation of the subsystems, is maybe the most extraordinary feature of modern physics which caused a lot of controversy in the past ("spooky action at a distance", A. Einstein), while different features of entanglement are still not fully understood in the present (e.g. multipartite entanglement, bound entanglement etc.).

In the following chapters we are going to define the (different kinds of) entanglement formally and the various physical manifestations of it. Furthermore, we are going to discuss the detection, quantification and classification of entanglement and give a few examples for the different detection criteria and entanglement measures. For a general overview on the topic see e.g. references [4][5][7].

2.2.1 Composite quantum systems

The most important mathematical tool needed for the mathematical description of composite quantum systems is the *tensor product* or *Kronecker product* denoted by the symbol $\otimes$.

Two state vectors

$$\mid \psi^A\rangle = \sum_i a_i \mid i\rangle \in \mathscr{H}^A, \tag{32}$$

and

$$\mid \psi^B\rangle = \sum_j b_j \mid j\rangle \in \mathscr{H}^B \tag{33}$$

can formally be composed to a state vector

$$\mid \psi^A\rangle\otimes \mid \psi^B\rangle = \sum_{i,j} a_i b_j \mid i\rangle\otimes \mid j\rangle =: \mid \psi^A, \psi^B\rangle \in \mathscr{H}^A \otimes \mathscr{H}^B =: \mathscr{H}^{AB} \tag{34}$$

describing a composite total *bipartite* system, i.e. consisting of the *two*

satellite-based quantum teleportation, see Ref. [86].

[7] Pioneering experiments showing the nonlocal features of quantum mechanics were e.g. [45, 46, 47, 48, 49]

subsystems . This is an axiom of quantum mechanics. The tensor product of the two vectors, strictly speaking the product space spanned by the two respective $\mathscr{ONB}$s $\{\mid i\rangle, \mid j\rangle\}$, forms a complete $\mathscr{ONB}$ in $\mathscr{H}^{AB}$.

Every state vector $\mid \psi^{AB}\rangle \in \mathscr{H}^{AB}$ can be decomposed w.r.t. to this $\mathscr{ONB}$ in the following way:

$$\mid \psi^{AB}\rangle = \sum_{ij} c_{ij} \mid i\rangle \otimes \mid j\rangle, \tag{35}$$

with c_{ij} being complex coefficients and in general $c_{ij} \neq a_i b_j$.

In other words, there exist elements in $\mathscr{H}^{AB}$ which can not be described appropriately by the composition of vectors (32) and (33), i.e. the decomposition into vectors (32) and (33) does not exist for the state in (35) for arbitrary a_i and b_i. We call such states *entangled* w.r.t. the bipartition A and B. In case they can be described by the composition in (34), they are called *separable* $\neq$*entangled*.

Now, we can gather our results and define *separability* and *entanglement* properly:

Definition 2.16 *Separability* of pure quantum state vectors:

Two (or more, see chapter 2.2.3) quantum state vectors are said to be *separable,* iff[8] their composition according to (34) fully describes the total quantum state. These states are also called *product states*, i.e. they *factorize* w.r.t. the considered subsystems:

$$\mid \psi^{AB}\rangle = \mid \psi^{A}\rangle \otimes \mid \psi^{B}\rangle. \tag{36}$$

In this case the state is separable w.r.t. the *bi-partition* of the total Hilbert space, i.e. a decomposition or "cut" of $\mathscr{H}^{AB}$ into the subspaces $\mathscr{H}^{A}$ and $\mathscr{H}^{B}$.

[8]The term iff means if and only if!

Definition 2.17 *Separability* of general quantum states:

In order to describe general quantum states we need to use the density matrix formalism. A state ρ^{AB} is said to be separable w.r.t. to a partition into subspaces of the total Hilbert space, i.e. into the bipartition A and B, iff it can be written as a convex sum of pure product states of the respective subsystems (see Ref. [24]),

$$\rho^{AB} = \sum_{i=1}^{k} p_i \rho_i^A \otimes \rho_i^B, \tag{37}$$

where $\sum_i p_i = 1$ and $p_i \geq 0$.

That means, that there exists a decomposition into an ensemble of pure states weighted by their probabilities p_i.

There exist inifinitely many different decompositions (see (37)) w.r.t. which a state ρ^{AB} can be represented, meaning that ρ^{AB} can also be represented by a different decomposition

$$\rho^{AB} = \sum_{j=1}^{k'} \tilde{p}_j \tilde{\rho}_j^A \otimes \tilde{\rho}_j^B. \tag{38}$$

It was shown, however, that there exists an decomposition with an upper bound for k, namely $k \leq \left(d^{AB}\right)^2 = \left(d^A \cdot d^B\right)^2$, with $d^A = dim\left(\mathscr{H}^A\right)$ and $d^B = dim\left(\mathscr{H}^B\right)$ for every ρ^{AB} in principle, see [25][26]. But such a decomposition does not have to be known.

For the special case of separable pure quantum states, where there is *one* $p_i \geq 0$, i.e. $\sum_{i=1}^{1} p_i = 1 \Rightarrow p_1 =1$, equation (37)reduces to

$$\rho^{AB} = \rho^A \otimes \rho^B. \tag{39}$$

At this point we need to define the partial trace operation which we are going to use later on to detect and quantify pure state entanglement.

In order to receive the subsystem ρ^A or ρ^B out of a composite system ρ^{AB} , we have to calculate the reduced density matrices by "tracing out" the respective other subsystem in the following manner:

Definition 2.18 The *trace − operation* Tr is defined by

$$\rho^A = Tr_B\left(\rho^{AB}\right) = \sum_j \langle j^B \mid \rho^{AB} \mid j^B \rangle, \tag{40}$$

$$\rho^B = Tr_A\left(\rho^{AB}\right) = \sum_i \langle i^A \mid \rho^{AB} \mid i^A \rangle, \tag{41}$$

where $\left\{\mid i^A\rangle, \mid j^B\rangle\right\}$ are $\mathscr{ONB}$s of subspaces $\mathscr{H}^A$ and $\mathscr{H}^B$.

In case of *mixed states* it is rather difficult to check whether an arbitrary state is separable or not. There are just a few special cases (2×2 and 2×3 dimensions) where operational criteria are known. There the *PPT criterion* is a necessary and sufficient condition for separability, see Ref. [28]. We will come to this in section 2.2.2.

Definition 2.19 *Entanglement*:

A quantum state is said to be *entangled*, iff it is *not separable* w.r.t. to certain considered subsystems of the total system, i.e. Definitions 2.16 (for state vectors) and 2.17 (for general states) do not hold.

The most famous example for pure bipartite entangled states are the maximally entangled *Bell states*. They are of the form

$$\mid \Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(\mid 01\rangle \pm \mid 10\rangle\right), \tag{42}$$

$$\mid \Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(\mid 00\rangle \pm \mid 11\rangle\right). \tag{43}$$

One easy way to see that these states are entangled is by taking advantage of Def. 2.18 and Def. 2.13, i.e. calculating the reduced density matrices w.r.t. A and B and the mixedness thereof. For both of the subsystems we get the maximal value indicating maximal mixedness, i.e. maximal information loss when considering the subsystems separately and therefore *maximal entanglement*.

The mixedness of the reduced density matrix can also be used to define entanglement for *pure* bipartite states:

Definition 2.20 A pure state is (maximally) *entangled* w.r.t. a bipartition, if the reduced density matrices are calculated to be (maximally) mixed.

2.2.2 Bipartite separability/entanglement criteria

One of the central problems in entanglement theory is the question whether a given quantum state is separable or not, also known as the *separability problem*.

In accordance with the definitions of the former chapter a bipartite quantum state is entangled, iff is is *not* separable. Due to the fact that entanglement is an important resource in quantum information and applications thereof, it is of great interest to know, whether the state is entangled. This is followed by the question of 'how much' it is entangled and will be discussed in chapter 2.2.4 and 2.2.6.

As entanglement is defined via separability, that is, a state is entangled, iff it is not separable, there are several *separability criteria* some of which we are going to review in this section.

Schmidt decomposition/Schmidt rank

In the case of pure bipartite states the *Schmidt decomposition* proved to be one possible separability criterion. Based on a theorem of linear algebra it states the following [4][34]

Definition 2.21 Any pure bipartite state $\mid \psi^{AB}\rangle$ can be decomposed w.r.t. the $\mathscr{O}\mathscr{N}\mathscr{B}s$, $\left\{\mid \varphi_i^A\rangle\right\} \in \mathscr{H}^A$ and $\left\{\mid \varphi_i^B\right\} \in \mathscr{H}^B$, according to

$$\mid \psi^{AB}\rangle = \sum_{i=1}^{k} \sqrt{p_i} \mid \varphi_i^A\rangle \otimes \mid \varphi_i^B\rangle, \tag{44}$$

where $p_i > 0$ are real numbers and $\sum_i p_i = 1$.

The number $k \leq min\left(dim\left(\mathscr{H}^A\right), dim\left(\mathscr{H}^B\right)\right)$ is called the *Schmidt rank* of $\mid \psi^{AB}\rangle$.

The Schmidt rank also tells us, whether a state is separable or not. This is the case for $k = 1$. Any $k \geq 2$ indicates entanglement and also how many degrees of freedom are entangled.

If $k = min\left(dim\left(\mathscr{H}^A\right), dim\left(\mathscr{H}^B\right)\right)$ and if the Schmidt coefficients

$$\sqrt{p_i} = \frac{1}{\sqrt{min\left(dim\left(\left(\mathscr{H}^A\right), dim\left(\mathscr{H}^B\right)\right)\right)}} \tag{45}$$

assume the same values $\forall i$, the state is *maximally entangled* (see Bell states (42), (38)) and the probability of local measurement outcome is equally distributed over all possibilities.

Note that bases $\left\{\mid \varphi_i^A\rangle\right\}$ and $\left\{\mid \varphi_i^B\right\}$ also form the $\mathscr{O}\mathscr{N}\mathscr{B}s$ of the reduced density matrices $\rho^A = Tr_B(\rho^{AB})$ and $\rho^B = Tr_A(\rho^{AB})$ respectively with corresponding eigenvalues $\sqrt{p_i}^2$. So ρ^A and ρ^B possess the same eigenvalues, which also indicates that functions of the reduced density matrices, i.e. functions of their eigenvalues, coincide. This is helpful when calculating quantum entropies, e.g. the von Neumann entropy, in order to decide, whether a state is separable or not. It is sufficient to look at one reduced density matrix.

The Schmidt rank can also be generalized for general density matrices [35]:

Definition 2.22 The *Schmidt number* r of a bipartite density matrix ρ^{AB} corresponds to the Schmidt rank k of the pure state decomposition of ρ^{AB}, i.e. there exists a decomposition into pure states, where at least one of them has Schmidt rank $k = r$ or a decomposition into pure states where all of them have at most Schmidt rank $k = r$.

PPT criterion (Positive Partial Trace criterion)

A very strong separability criterion was first introduced by Asher Peres (see Ref. [27]) and was proven to be necessary and sufficient for separability in 2⊗2 and 2⊗3 dimensions by the Horodeckis (see Ref.[28]). It is also referred to as Peres-Horodecki-criterion.

The criterion is based on the notion of positive and completely positive maps, more precisely the partial transpose which is the standard matrix transpose in one subsystem.

Definition 2.23 The partial transposition $\mathbb{I} \otimes T$ of subsystem B of the bipartite density matrix

$$\rho_{AB} = \sum_{i,j,k,l} \langle ik \mid \rho_{AB} \mid jl \rangle \mid i \rangle \langle j \mid \otimes \mid k \rangle \langle l \mid, \tag{46}$$

where $\langle ik \mid \rho_{AB} \mid jl \rangle$ are the matrix elements and $\mid i \rangle \langle j \mid \otimes \mid k \rangle \langle l \mid$ the computational product basis, is defined as

$$\begin{aligned} \rho_{AB}^{T_B} := (\mathbb{I} \otimes T)(\varrho_{AB}) &= \textstyle\sum_{ijkl} \langle i,k \mid \rho_{AB} \mid j,l \rangle \mid i \rangle \langle j \mid \otimes (\mid k \rangle \langle l \mid)^T \\ &= \textstyle\sum_{ijlk} \langle i,k \mid \rho_{AB} \mid j,l \rangle \mid i \rangle \langle j \mid \otimes \mid l \rangle \langle k \mid \end{aligned} \tag{47}$$

It could have been also defined as a map $T \otimes \mathbb{I}$, i.e. a transposition in subsystem A, in an analogue manner.

Now, if an arbitrary bipartite state ρ_{AB} is separable it is positive under PT (partial transposition), i.e.

$$\rho_{AB} = \textit{separable} \Rightarrow (\mathbb{I} \otimes T)(\rho_{AB}) > 0. \tag{48}$$

States which are positive under PT are also called PPT states and are separable in 2x2 and 2x3 dimensions. The PPT criterion necessary and suffiencient condition for separability in 2x2 and 2x3 dimensions. In higher dimensions PPT states can also be detected as entangled by other, stronger criteria and are then called *bound entangled*. States which are negative under PT are called NPT states and are *always* entangled.

Entanglement witnesses

Entanglement witnesses are hermitian operators (observables) which can be used for entanglement detection in experiment and are therefore of great physical importance (see [28]).

Definition 2.24 An hermitian operator W is an *entanglement witness*, i.e. detects entanglement, iff it satisfies the following conditions:

For one entangled state ρ_{ent} there exists an operator W fulfilling

$$\langle W, \rho_{ent} \rangle = Tr(W \rho_{ent}) < 0, \tag{49}$$

and for the set of all separable states $\{\rho_{sep}\}$

$$\langle W, \rho_{sep} \rangle = Tr(W \rho_{sep}) \geq 0. \tag{50}$$

This is a consequence from the Hahn-Banach-theorem (functional analysis) which states that given a convex set of operators (in our context the set of separable states $\{\rho_{sep}\}$) and a point outside this set (entangled state ρ_{ent}), then there is at least one hyperplane separating them.

This hyperplane is the entanglement witness W and in the *optimal* case, there exists at least one ρ_{sep} for which

$$\langle W, \rho_{sep} \rangle = Tr(W \rho_{sep}) = 0 \tag{51}$$

is valid. W is then called *optimal entanglement witness* and forms a tangent plane to the set of separable states $\{\rho_{sep}\}$. The set of all optimal entanglement witnesses $\{W\}$ forms a border to the set of all separable states.

Matrix element inequalities

Another separability criterion, more precisely, a general framework allowing to construct separability criteria which are implementable in various situations, can be derived by means of certain inequalities composed of matrix elements $\langle \Phi \mid \rho \mid \Phi \rangle$ of a density matrix ρ. The state $\mid \Phi\rangle$ is separable (in the multipartite case *fully separable* (see chapter 2.2.3 and Def. 2.26) w.r.t. the number of subsystems it is composed of). More on the vector $\mid \Phi\rangle$ and its properties can be also found in chapter 2.2.5.

Definition 2.25 For *bipartite* states $\rho \in \mathscr{H}^A \otimes \mathscr{H}^B$, and all *separable* states $\mid \phi_{1_1}\rangle, \mid \phi_{1_2}\rangle \in \mathscr{H}^A$, $\mid \phi_{2_1}\rangle, \mid \phi_{2_2}\rangle \in \mathscr{H}^B$

the inequality

$$\begin{aligned} &\mid \langle \phi_{1_1} \mid \langle \phi_{2_1} \mid \rho \mid \phi_{1_2}\rangle \mid \phi_{2_2}\rangle \mid \\ &-\sqrt{\langle \phi_{1_1} \mid \langle \phi_{2_2} \mid \rho \mid \phi_{1_1}\rangle \mid \phi_{2_2}\rangle \langle \phi_{1_2} \mid \langle \phi_{2_1} \mid \rho \mid \phi_{1_2}\rangle \mid \phi_{2_1}\rangle} \leq 0 \end{aligned} \qquad (52)$$

was shown to hold for all *separable* states [43].

The general formulation in terms of permutation operators for arbitrary dimensions is given by Def. 2.39, see chapter about multipartite entanglement.

But first we need to define entanglement in the multipartite scenario.

2.2.3 Multipartite entanglement

For composite systems consisting of more than just 2 subsystems of the total system, i.e. multipartite systems described by a state vector $\mid \psi^{ABC....N}\rangle \in \mathscr{H}^{ABC....N}$ or density matrix $\rho^{ABC....N}$, we define separability in accordance to Def. 2.16.

Definition 2.26 A multipartite (N-partite) pure state is *fully separable*, iff the decomposition

$$\mid \psi^{ABC....N}\rangle = \mid \psi^{A}\rangle \otimes \mid \psi^{B}\rangle \otimes \mid \psi^{C}\rangle \otimes .... \otimes \mid \psi^{N}\rangle \tag{53}$$

exists and fully describes the total state. A simple way to check whether a given pure state is *fully separable* is via the *mixedness* when considering every possible bi-partition of the system. All reduced density matrices are pure, i.e. the mixedness is zero w.r.t. every bipartition. So, a state is *fully separable* when it is separable w.r.t. to every possible bipartition of the total system. (e.g. bipartitions $\{A \mid BC\}$, $\{AB \mid C\}$, $\{AC \mid B\}$ when the state is a composition of 3 subsystems).

Def. 2.18 for mixed bipartite states can be extended to multipartite states as well.

Definition 2.27 A multipartite (N-partite) mixed state is said to be *fully separable* w.r.t. N-partitions, iff it can be written as a convex sum over N pure product states corresponding to the N subsystems,

$$\rho^{ABC....N} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C \otimes .... \otimes \rho_i^N, \tag{54}$$

with probabilities p_i satisfying

$$p_i \geq 0 \tag{55}$$

and

$$\sum_i p_i = 1. \tag{56}$$

Definition 2.28 If definitions 2.26 and 2.27 do not hold, the multipartite state is at least *partially entangled*.

This is a novel feature inherent only to multipartite systems and allows for a more complex classification of entanglement as different subsystems can be entangled in different ways whilst being separable to other subsystems etc. A way of characterization is the following (see Ref. [7][29])

Definition 2.29 A pure N-partite state is called *k-separable* iff it is separable w.r.t. k partitions (k "cuts") of the total Hilbert space, i.e. the state vector

$$\mid \psi\rangle^k =\mid \psi_1\rangle\otimes \mid \psi_2\rangle \otimes ...\otimes \mid \psi_k\rangle \tag{57}$$

factorizes into $1 \leq k \leq N$ state vectors describing one or more subsystems.

A state with $k = N$ is *fully separable* (see Def. 2.27).

For $k = 2$ we get the bipartite case, i.e. the state is *bi-separable*.

Definition 2.30 A mixed state ρ^k is called $k-separable$ iff the decomposition w.r.t. to pure states,

$$\rho^k = \sum_i p_i \mid \psi_i\rangle^{(k)}\langle\psi_i \mid^{(k)}, \tag{58}$$

where each $\mid \psi_i\rangle^{(k)}$ is at least a $k-separable$ pure state. It is important to note that the pure states composing the mixed ρ^k are allowed to be not separable w.r.t. the same $k-partition$ in general, which makes it very difficult to detect $k-separability$ in mixed states. Otherwise, if the state is separable w.r.t. certain subsystems, we call it $\gamma_k - separable$, i.e.

Definition 2.31 A general multipartite state ρ is $\gamma_k - \textit{separable}$, iff it can be decomposed according to

$$\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i \otimes ... \otimes \rho_k^i, \tag{59}$$

with probabilities p_i satisfying

$$p_i \geq 0 \tag{60}$$

and

$$\sum_i p_i = 1. \tag{61}$$

Each of the ρ_j^i with $1 \leq j \leq k$ correspond to one or more subsystems.

Each state which is $k-\textit{separable}$ or $\gamma_k-\textit{separable}$ is also $k-l-\textit{separable}$ or $\gamma_{k-l} - \textit{separable}$ for $0 \leq l \leq k-1$. So, a state which is e.g. 3-separable is also 2-separable and 1-separable. So the Sets of k-separable states are convexly embedded with decreasing k.

Definition 2.32 A general state which is not 2-separable w.r.t. any bipartition is called *genuinely multipartite entangled.*

This kind of entanglement is the most favoured by all applications of entanglement theory using multipartite states, e.g. quantum information processing task such as quantum secret sharing, dense coding etc.

Examples:

Let us have a look on these exemplary states in order to understand better the notions of separability and the different kinds of multipartite entanglement defined above:

$$\mid \psi_1\rangle =\mid 011\rangle \tag{62}$$

$$\mid GHZ\rangle = \frac{1}{\sqrt{2}}\left(\mid 000\rangle + \mid 111\rangle\right) \tag{63}$$

$$\mid W\rangle = \frac{1}{\sqrt{3}}\left(\mid 100\rangle + \mid 010\rangle + \mid 001\rangle\right) \tag{64}$$

$$\mid \psi_{Ah}\rangle = \frac{1}{\sqrt{6}}\left(\mid 012\rangle + \mid 201\rangle + \mid 120\rangle - \mid 102\rangle - \mid 021\rangle - \mid 210\rangle\right) \tag{65}$$

$$\mid \psi_2^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(\mid 011\rangle \pm \mid 101\rangle\right) =\mid \Psi^{\pm}\rangle \otimes \mid 1\rangle \tag{66}$$

$$\rho_1^{\pm} = \alpha\left(\mid \Psi^{\pm}\rangle\langle\Psi^{\pm}\mid \otimes \mid 1\rangle\langle 1\mid\right) + \beta\left(\mid \Psi^{\pm}\rangle\langle\Psi^{\pm}\mid \otimes \mid 0\rangle\langle 0\mid\right) \tag{67}$$

$$\rho_2^{\pm} = \alpha\left(\mid \Psi^{\pm}\rangle\langle\Psi^{\pm}\mid \otimes \mid 0\rangle\langle 0\mid\right) + \beta\left(\mid 1\rangle\langle 1\mid \otimes \mid \Psi^{\pm}\rangle\langle\Psi^{\pm}\mid\right) \tag{68}$$

All these states are tripartite, i.e. a composition of the subsystems $\{ABC\}$. Only the first state, $\mid \psi_1\rangle$, is *fully separable* w.r.t. "cuts" $\{A \mid B \mid C\}$.

The states $\mid GHZ\rangle$,[9] $\mid W\rangle$ and $\mid \psi_{Ah}\rangle$ are non-separable w.r.t. any partition and therefore *genuinely multipartite entangled.*

[9]Named after Greenberger, Horne and Zeilinger. Originally constructed as *Gedanken-experiment* to show the non-local features of quantum mechanics. See also Ref. [33]

The states $| \psi_2^{\pm} \rangle$, $\rho_1^{\pm}$ and $\rho_2^{\pm}$ can be characterized as follows: Apart from $| \psi_2^{\pm} \rangle$ being pure and $\rho_1^{\pm}, \rho_2^{\pm}$ mixed, they all are 2-separable. The mixed states possess decompositions into pure states, which are 2-separable.

The states $| \psi_2^{\pm} \rangle, \rho_1^{\pm}$ are also $\gamma_2 -$ *separable*, that is, *globally* separable w.r.t. a certain partition. In this case that is the partition $\{AB \mid C\}$.

These examples also demonstrate that $k-$*separability* and γ_k-*separability* coincide in the pure state case when $k = N$. For $k < N$ and mixed states the $\gamma_k -$ *separability* represents a stronger condition including the $k -$ *separability*. The inverse does not hold.

2.2.4 Entanglement measures for bipartite quantum systems

Once the task is completed of deciding whether a state is separable or not (see sections 2.2.2 and 2.2.5), the latter meaning entangled, one would want to quantify the entanglement present in a system. For bipartite pure systems the linear entropy (see Def. 2.15) is a suitable measure. There are other quantum entropies reviewed in the following section.

Quantum entropies

- **Von Neumann entropy**

Definition 2.33 the *von Neumann entropy* of a density matrix ρ is defined as

$$S(\rho) = -Tr(\rho \log(\rho)) = -\sum_{i=1}^{d} p_i \log(p_i), \tag{69}$$

where p_i are the eigenvalues of ρ.

When the logarithm is chosen to the base of $d = dim(\rho)$ it scales between 0 and 1, i.e.

$$0 \leq S(\rho) \leq 1, \tag{70}$$

where 0 indicates a pure and 1 a maximally mixed state. Therefore, like the linear entropy, this is applicable only for pure state entanglement quantification. Under the condition that there is entanglement in a bipartite system, the von Neumann entropy of either of the two reduced density matrices is $S(\rho) > 0$.

The linear entropy is an approximation of the von Neumann entropy when the logarithm is chosen to be the natural logarithm, i.e.

$$-Tr(\rho \ln \rho) \approx -Tr(\rho(\rho - 1)) = Tr(\rho) - Tr\left(\rho^2\right) = 1 - Tr\left(\rho^2\right). \tag{71}$$

The von Neumann entropy is the quantum mechanical analogue to the *Shannon entropy* from classical information theory[36]. It is a measure either for the uncertainty about a random variable X or the gain of information when the variable is known.

Definition 2.34 The Shannon entropy is defined as

$$H(X) = H(\{p_i\}) = -\sum_i p_i \log(p_i), \tag{72}$$

where $\{p_i\}$ is a probability distribution for the variable X with $\sum_i p_i = 1$.

It is convention to take the log to the base 2 so that the entropies are measured in bits.

As stated before, any quantum entropy is a necessary and sufficient separability criterion only for pure states. For mixed states we can derive an upper bound.

Definition 2.35 For mixed states $\sum_i p_i \rho_i$ following inequality holds

$$\sum_i p_i S(\rho_i) \leq S\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i S(\rho_i) + H(p_i). \tag{73}$$

A proof of this inequality and more information on entropies can be found e.g. in Refs. [4][5].

Entanglement of formation

A possible way to detect and quantify entanglement in mixed states is represented by the *entanglement of formation* [38]. For pure states it reduces to the von Neumann entropy and for mixed states it is defined via a so called *convex roof* construction. Historically, it was the first entanglement measure aimed for mixed states:

Definition 2.36 The entanglement contained in a mixed state ρ is measured by

$$E_F(\rho) := \inf_{\{(p_i, \rho_{i,red})\}} \sum_i p_i S(\rho_{i,red}) . \tag{74}$$

Here, $S(\rho_{i,red})$ is the von Neumann entropy of the reduced density matrices representing one subsystem of the bipartite total system. Furthermore the infimum has to be taken over all possible pure state decompositions $\{(p_i, \rho_{i,red})\}$ of ρ.

In general it is a nontrivial task to compute the entanglement of formation, as the convex roof construction requires elaborate optimization. Only for the special case of two qubits is an analytical method known (see Ref.[37]) that we introduce now.

For two qubits the entanglement of formation can be computed and related to the *concurrence* C (see Def. 2.38):

Definition 2.37 The entanglement of formation for a state of two qubits is given by

$$E_F = h\left(\frac{1+\sqrt{1-C^2}}{2}\right), \tag{75}$$

where $h = h(x)$ is the binary entropy function

$$h(x) = -x \log x - (1-x) \log(1-x) \tag{76}$$

and C is given by Def. 2.38.

Concurrence

The concurrence was originally introduced in [38] but named *concurrence* by Hill and Wootters first in Ref. [39]. For pure states it takes the very simple form of

$$C\left(\rho_{pure}\right) = \sqrt{2\left(1 - Tr(\rho_{red}^2)\right)}, \tag{77}$$

where ρ_{red} is the reduced density matrix of ρ_{pure}, i.e. $Tr_A\left(\rho_{pure}\right) = \rho_{red}$.

For mixed bipartite states it can be obtained via the convex roof construction

$$C\left(\rho_{mix}\right) = \inf_{\{(p_i, \rho_{i,pure})\}} \sum_i p_i C\left(\rho_{i,pure}\right), \tag{78}$$

where we have to take again the infimum over all pure state decompositions of ρ_{mix}. For the bipartite qubit case the infimum is proven to be obtained by:

Definition 2.38 The *concurrence* of a certain state ρ is given by

$$C\left(\rho\right) = \max\left\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\right\}, \tag{79}$$

where λ_1 is the greatest value and λ_i are the square roots of the eigenvalues of $\rho\left(\sigma_y \otimes \sigma_y\right)\rho^*\left(\sigma_y \otimes \sigma_y\right)$ and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ are the Pauli-σ_y-matrices, i.e. "spin-flip"-matrices.

Computing the entanglement of formation as well as the concurrence is a non-trivial problem for general (mixed) states, i.e. bipartite qudits. There is no standard-procedure but for certain special cases, e.g. the isotropic states[10] (see e.g. Ref.[41]), explicit expressions were given.

The concurrence can be generalized to a multidimensional and multipartite entanglement measure, which also helps in classifying different kinds of entanglement (see Ref. [40]). But this will be discussed in the chapter about multipartite entanglement (see section 2.2.6).

[10] The class of isotropic states is of the form: $\rho_\alpha = \alpha \mid \phi^+\rangle\langle\phi^+ \mid + \frac{1-\alpha}{d^2}\mathbb{I}$, $0 \leq \alpha \leq 1$, $\mid \phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d \mid ii\rangle$. See Ref. [42]

2.2.5 Separability criteria for multipartite quantum systems

According to Def. 2.32 a multipartite pure state is genuinely multipartite entangled, if it is not biseparable w.r.t. any bipartition $\{A \mid BC..\}$. In the pure state case it is straightforward to check, whether it is biseparable or not by employing e.g. the PPT-criterion discussed in section 2.2.2. A mixed multipartite state is genuinely multipartite entangled, if the pure state decomposition is not biseparable, i.e. the pure states are not biseparable w.r.t. any bipartition $\{A \mid BC..\}$, $\{AB \mid C...\}$ ..

As mentioned before, the specific elements of the pure state decomposition do not need to be biseparable w.r.t. the *same* bipartition , i.e. one can think of the arbitrary decomposition of an N-partite state

$$\rho^{ABC....N} = \sum_i p_i \rho^A \otimes \rho^{BC...N} + \sum_i q_i \rho^{AB} \otimes \rho^{C...N} + \sum_i r_i \rho^{BC} \otimes \rho^{A...N} + ..., \quad (80)$$

with probabilities p_i, q_i, r_i satisfying

$$p_i, q_i, r_i \geq 0 \quad (81)$$

and

$$\sum_i p_i + \sum_i q_i + \sum_i r_i = 1, \quad (82)$$

which is biseparable but not globally separable w.r.t. a certain bipartition.

Therefore it is a nontrivial task to detect non-biseparability, i.e. genuine mutlipartite entanglement in mixed states, and new criteria are necessary.

Such criteria can be constructed with help of *matrix element inequalities*:

Matrix element inequalities

We have already introduced the notion of matrix element inequalities in chapter 2.2.2, which we are going to adapt now for use in the multipartite case.

A generalization of the bipartite separability criterion (see Def. 2.25) to the multipartite case is given by

Definition 2.39 The inequality

$$\sqrt{\langle\Phi \mid \rho^{\otimes 2}\Pi \mid \Phi\rangle} - \sum_i \sqrt{\langle\Phi \mid \mathcal{P}_i^\dagger \rho^{\otimes 2}\mathcal{P}_i \mid \Phi\rangle} \leq 0, \tag{83}$$

where $\rho^{\otimes 2} \in \mathscr{H}^{\otimes 2}$ is the 2-fold copy of the state ρ, the state vector $\mid \Phi\rangle$ is a fully separable vector. The permutation operators Π and $\mathcal{P}_i$ need to be defined yet:

Definition 2.40 The operator Π is the global permutation operator performing simultaneous permutations on the n subsystems of the two-fold copy, i.e.

$$\Pi \mid \Phi\rangle = \Pi \mid \phi_{1_1}\phi_{1_2}...\phi_{1_n}\rangle \otimes \mid \phi_{2_1}\phi_{2_2}...\phi_{2_n}\rangle =$$

$$= \mid \phi_{2_1}\phi_{2_2}...\phi_{2_n}\rangle \otimes \mid \phi_{1_1}\phi_{1_2}...\phi_{1_n}\rangle, \tag{84}$$

meaning that *all* (in this case n) subsystems are permuted.

Definition 2.41 The operator $\mathcal{P}_i$ is the permutation operator permuting the $i'th$ subsystem of the two-fold copy, i.e. an example would be

$$\mathcal{P}_i \mid \Phi\rangle = \mathcal{P}_i \mid \phi_{1_1}\phi_{1_2}...\phi_{1_i}...\phi_{1_n}\rangle \otimes \mid \phi_{2_1}\phi_{2_2}...\phi_{2_i}...\phi_{2_n}\rangle =$$

$$\mid \phi_{1_1}\phi_{1_2}...\phi_{2_i}...\phi_{1_n}\rangle \otimes \mid \phi_{2_1}\phi_{2_2}...\phi_{1_i}...\phi_{2_n}\rangle. \tag{85}$$

In general, $\mathcal{P}_i$ is an operator permuting arbitrary subsystems i, which can be also a set of subsystems. The operator $\mathcal{P}_{\{1,2\}}$, for instance, permutes subsystems 1 and 2.

The fully separable state vector $\mid \Phi\rangle$ is element on the twofold copy Hilbert space $\mathscr{H}^{\otimes 2}$, i.e. $\mid \Phi\rangle = \mid \phi_1\rangle \otimes \mid \phi_2\rangle$ consists of two biseparable parts where each part is fully separable as well.

For $\mid \Phi\rangle = \mid \phi_{1_1}\phi_{1_2}\rangle \otimes \mid \phi_{2_1}\phi_{2_2}\rangle$, that is the bipartite case (n=2), the inequality assumes the simple form of (52).

Although $\mid \Phi\rangle$ can be chosen freely[11], the detection of genuine multipartite entanglement, i.e. violation of inequality (83), might depend on the appropriate choice of $\mid \Phi\rangle$. One way to gain optimal results, i.e. maximal violation of (83), is by employing (numerical) optimization methods like the one presented in [56]. We will come to this in chapter 2.2.7. However, note that it is often simple to "guess" the optimal choice that makes these criteria applicable.

With help of the permutation operators defined above the separability problem of a state ρ is shifted to a separability problem in between the two copies of the state. This can be interpreted as follows: If by the permutation operator the subsystem A_i of the two copies of the total state is exchanged and this subsystem is separable w.r.t. the remaining subsystem(s) of the state, then the two copies keep exactly the same product form and remain separable w.r.t. each other.

On the contrary, if the chosen subsystems are not separable w.r.t. the remaining subsystem(s), then the two copies themselves are left non separable after the permutation. This non separability between the two copies of the total state is detected by inequality (83).

As these inequalities are convex and can be shown to hold for separable pure states, they are also hold for mixed states.

This inequality can be extended to the multipartite case and applied to detect *genuine multipartite entanglement* and also *k-in-separability*[44].

Also, inequality (83) can be used to construct separability criteria adapted to various situations and different states, like e.g. multipartite entanglement in n-qubit *Dicke-states*[12], see Ref. [50].

[11] The two parts of $\mid \Phi\rangle = \mid \phi_1\rangle \otimes \mid \phi_2\rangle$ need to be different in order for the permutation to make sense. Otherwise the permuted vector would assume exactly the same form. More details can be found in Ref. [53].

[12] Originally introduced in [51] they offer a lot of advantages in quantum communication tasks. See e.g. Ref. [50] and References therein.

They are genuine multipartite entangled and of the form

$\mid D^n_m\rangle = \begin{pmatrix} n \\ m \end{pmatrix}^{-\frac{1}{2}} \sum_{\{\alpha\}} \mid d_{\{\alpha\}}\rangle$,

n is the number of qubits, m is the number of excitations, i.e. denoted by $\mid 1\rangle$, $\mid d_{\{\alpha\}}\rangle = \bigotimes_{i\notin\{\alpha\}} \mid 0\rangle_i \bigotimes_{i\in\{\alpha\}} \mid 1\rangle_i$, $\{\alpha\}$ is a set of indices giving the number of subsystems in excitation, the sum runs over all in-equivalent $\{\alpha\}$ with $\mid \{\alpha\} \mid = m$.

$m = 1$ corresponds to the $\mid W\rangle$ state.

Inequality (83) can be used for the construction of criteria detecting *k-separability* as well, see Ref.[44][53].

Definition 2.42 The inequality

$$\sqrt{\langle \Phi \mid \rho^{\otimes 2}\mathcal{P}_{tot} \mid \Phi \rangle} - \sum_{\{\gamma\}} \left(\prod_{i=1}^{k} \langle \Phi \mid \mathcal{P}^{\dagger}_{\gamma_i} \rho^{\otimes 2} \mathcal{P}_{\gamma_i} \mid \Phi \rangle \right)^{\frac{1}{2k}} \leq 0, \tag{86}$$

holds for all *k-separable states* ρ, where$\mid \Phi\rangle$ is a fully separable state. Here the sum is taken over all *k-partitions* $\{\gamma\}$.

The permutation operator $\mathcal{P}_{tot}$ permutes all subsystems at once, i.e. is defined as Π in (84), but in this section denoted differently to avoid conflict in notation.

The permutation operator $\mathcal{P}_{\gamma_i}$ permutes all of the subsystems contained in the i'th subset of the partition γ.

This inequality can be used to detect *k-in-separability,* that is, (86) is violated for k-in-separable states and represents the first such criterion known. Clearly, if the inequality in (86) is violated for k=2 , the state is genuinely multipartite entangled. For the general detection of entanglement it is comparable and as good as the PPT-criterion in some cases, but the advantage of this criterion is clearly that they reveal a substructure.

Inequality (83) can also be used for the construction of criteria detecting *multilevel genuine multipartite entanglement*, see Ref. [52]. First we need to clarify the notion of *multilevel genuine multipartite entanglement:*

Apart from the detection and quantification of entanglement, we could ask in "how many degrees of freedom" a certain state is entangled. Again, this question is difficult to answere for mixed states.

When considering e.g. the tripartite (d=3) mixed state [52]

$$\rho_C = \frac{1}{2} \mid GHZ_3\rangle\langle GHZ_3 \mid + \frac{1}{6} \sum_{i=0}^{2} \mid iii\rangle\langle iii \mid, \tag{87}$$

Via LOCC operations they are transformable into e.g. $\mid GHZ\rangle$−and $\mid W\rangle - states$.

with

$$| \, GHZ_3 \rangle = \frac{1}{\sqrt{3}} \left(| \, 000 \rangle + | \, 111 \rangle + | \, 222 \rangle \right) \tag{88}$$

we would say that all degrees of freedom, i.e. $f = 3$, are involved. Another decomposition of the same state is given by

$$\rho_C =$$

$$\frac{1}{3} \left(| \, GHZ_{1,2} \rangle \langle GHZ_{1,2} \, | + | \, GHZ_{1,3} \rangle \langle GHZ_{1,3} \, | + | \, GHZ_{2,3} \rangle \langle GHZ_{2,3} \, | \right), \tag{89}$$

with

$$| \, GHZ_{i,j} \rangle = \frac{1}{\sqrt{2}} \left(| \, iii \rangle + | \, jjj \rangle \right), \tag{90}$$

we would say that only 2 degrees of freedom, i.e. $f = 2$, are entangled.

For pure states, one necessary (but not sufficient) criterion is to determine the *Schmidt rank* k of the reduced density matrices w.r.t. all bipartitions of the system. The state contains $f - level$ entanglement, when at least *one* bipartition respects $k \geq f$.

The state is $f - level$ multipartite entangled, iff $k \geq f$ is valid for *all* bipartitions.

This can be generalized to mixed states as well. A mixed state is *$f -$ level multipartite entangled,* iff at least one pure state from its pure state decomposition is *$f -$ level multipartite entangled*.

Inequality (83) can be generalized to detect *genuine $f -$ level multipartite entanglement* in arbitrary mixed states.

Definition 2.43 The quantities

$$Q_0 = \sum_{k \neq l}^{d-1} \left(| \langle k \mid \rho^{\otimes 2} \mid l \rangle | - \sum_{\{\gamma\}} \sqrt{\langle k, l \mid \mathcal{P}^{\dagger}_{\{\gamma_1\}} \rho^{\otimes 2} \mathcal{P}_{\{\gamma_1\}} \mid k, l \rangle} \right), \tag{91}$$

where $\mid k \rangle = \mid k \rangle^{\otimes 2}$, $\mid l \rangle = \mid l \rangle^{\otimes 2}$, $\mid k, l \rangle = \mid k \otimes l \rangle^{\otimes n}$ are elements of an $\mathscr{ONB}$ of dimension $dim(\mathscr{H}) = d$, and the sum is taken over all possible bipartitions $\{\gamma\}$, and

$$Q_m = \frac{1}{m} \left(\sum_{k,l=0}^{d-2} \sum_{\{\sigma\}} \left(| \langle \alpha^k \mid \rho \mid \beta^l \rangle | - \sum_{\{\delta\}} \sqrt{\langle \alpha^k \mid \langle \beta^l \mid \mathcal{P}^{\dagger}_{\{\delta\}} \rho^{\otimes 2} \mathcal{P}_{\{\delta\}} \mid \alpha^k \rangle \mid \beta^l \rangle} \right) \right.$$
$$\left. - N_D \sum_{l=0}^{d-2} \sum_{\{\alpha\}} \langle \alpha^l \mid \rho \mid \alpha^l \rangle \right), \tag{92}$$

with $1 \leq m \leq \frac{n}{2}$, $\mid \alpha^l \rangle$ are product vectors where m of the n subsystems out of the set $\{\alpha\}$ are in the state $\mid l+1 \rangle$ and the remaining ones, $n-m$, are in the state $\mid l \rangle$, i.e.

$$\mid \alpha^l \rangle = \bigotimes_{i \in \{\alpha\}} \mid l+1 \rangle_i \bigotimes_{i \notin \{\alpha\}} \mid l \rangle_i . \tag{93}$$

The same holds for $\mid \beta^l \rangle$.

$$N_D = (d-1)\, m\, (n-m-1) \tag{94}$$

and the set

$$\{\sigma\} = \{\{\alpha\}, \{\beta\} : \mid \{\alpha\} \cap \{\beta\} \mid = m-1\}. \tag{95}$$

The set

$$\{\delta\} = \left\{ \begin{array}{ll} \{\alpha\}, & for\, k = l \\ \left\{ \{\delta\} \mid \{\delta\} \subset \overline{\{\alpha\} \setminus \{\beta\}} \right\}, & for\, k < l \\ \left\{ \{\delta\} \mid \{\delta\} \subset \overline{\{\beta\} \setminus \{\alpha\}} \right\}, & for\, k > l \end{array} \right\}, \tag{96}$$

where the over-line denotes the complement w.r.t. the set $\{1, ..., n\}$.

Definition 2.44 If for a given state ρ any of the functions defined above yields a value greater than $f-2$ for any integer $f \geq 2$, i.e.

$$Q_i > f - 2, \tag{97}$$

for any i, $(i = 0, m)$, then this state is at least $f-$*dimensionally* or $f-$*level genuinely multipartite entangled.*

Due to construction, these criteria perform optimal for states close to GHZ- and Dicke-states. Q_0 yields the value 1 for a n-qubit-GHZ-state which corresponds to a maximal violation, as the upper bound of Q_0 is by construction $f-1$.

2.2.6 Entanglement measures for multipartite quantum systems

As mentioned before there are different kinds of multipartite entanglement, i.e. there are partially separable (partially entangled) and genuinely multipartite entangled states. For all these situations different measures can be derived best suited for certain kinds of entanglement.

Again, as in the bipartite case, once a measure is found for multipartite pure states it can be generalized for mixed states via the convex roof construction. The latter is in general very hard to compute, as it involves nontrivial optimization procedures.

We are going to discuss a measure for genuine multipartite entanglement which is based on the *concurrence* with computable lower bounds (see Ref. [57]).

The concurrence defined in chapter 2.2.4 can be generalized to an entanglement measure for multipartite pure states according to:

Definition 2.45 The *gme-concurrence* is defined as

$$C_{gme}\left(\rho_{pure}\right) := \min_{\gamma_i \in \{\gamma\}} \sqrt{2\left(1 - Tr\left(\rho^2_{A_{\gamma_i}}\right)\right)}, \tag{98}$$

where $\{\gamma\}$ is the set of all possible bipartitions $\{A_i \mid B_i\}$ and $\rho^2_{A_{\gamma_i}}$ is the reduced density matrix w.r.t. the part γ_i of $\{\gamma\}$.

The generalization to mixed states is straight forward via convex roof construction, i.e.

Definition 2.46 The *gme-concurrence* for mixed states ρ_{mix} is given by

$$C_{gme}\left(\rho_{mix}\right) := \inf_{\{(p_i,\rho_{i,pure})\}} \sum_i p_i C_{gme}\left(\rho_{i,pure}\right), \tag{99}$$

where the infimum is taken over all possible pure state decompositions of ρ_{mix}.

For all biseparable states $C_{gme} = 0$ and for all genuinely multipartite entangled states $C_{gme} > 0$.

We can derive a lower bound on (99) with help of the separability criterion defined in (83).

Definition 2.47 The concurrence defined for genuine multipartite entanglement, C_{gme}, is bounded by

$$C_{gme}\left(\rho_{mix}\right) \geq 2\left(\sqrt{\langle\Phi \mid \rho_{mix}^{\otimes 2}\Pi \mid \Phi\rangle} - \sum_i \sqrt{\langle\Phi \mid \mathcal{P}_i^{\dagger}\rho_{mix}^{\otimes 2}\mathcal{P}_i \mid \Phi\rangle}\right). \tag{100}$$

The proof of (100) and examples for the tripartite qubit case can be found in [57].

The bound can be optimized over all $\{\mid \Phi\rangle\}$ by the procedure described in the following chapter.

2.2.7 Optimization and composite parametrization

The inequalities based on matrix elements, which were discussed in the chapter 2.2.5, perform quite well without optimization, but in order to achieve the best possible results we can apply an optimization over all local unitary transformations of the used basis elements $\mid \Phi\rangle$ of the kind

$$\mid \Phi\rangle \rightarrow U \mid \Phi\rangle, \tag{101}$$

where

$$U = U_1 \otimes U_2 \otimes ... \otimes U_n. \tag{102}$$

This unitary transformation U needs to be parametrized in order to achieve an optimization, i.e. vary over the parameters that are used in the composition of U.

A possible and beneficial way to implement such a parametrization[13] was introduced in [55][56].

Here the unitary group $U(d)$, i.e. the set of all unitary transformations U on *d-dimensional* Hilbert space $\mathscr{H}^d$, is parametrized as follows:

Definition 2.48 The *composite parametrization* U_C is defined by

$$U_C = \left(\prod_{m=1}^{d-1} \left(\prod_{n=m+1}^{d} \exp\left(iP_n \lambda_{n,m}\right) \exp\left(iY_{m,n} \lambda_{m,n}\right) \right) \right) \cdot \left(\prod_{l=1}^{d} \exp\left(iP_l \lambda_{l,l}\right) \right). \tag{103}$$

Every unitary transformation $U \in U(d)$ on $\mathscr{H}^d$ can be represented by a composition, i.e. U_C, of d^2 real valued parameters $\lambda_{m,n} \in [0, 2\pi]$ for $m \geq n$ and $\lambda_{m,n} \in \left[0, \frac{\pi}{2}\right]$ for $m < n$ with $m, n \in \{1, ..., d\}$.

The operators P_l are one dimensional projection operators, i.e.

$$P_l = \mid l\rangle\langle l \mid . \tag{104}$$

The $Y_{m,n}$ are the generalized anti-symmetric σ_y Pauli matrices

$$Y_{m,n} = -i \mid m\rangle\langle n \mid +i \mid n\rangle\langle m \mid, \tag{105}$$

with $1 \leq m < n \leq d$.

The term $\exp\left(iY_{m,n}\lambda_{m,n}\right)$ corresponds to a rotation in two dimensional subspace spanned by $\{\mid m\rangle, \mid n\rangle\}$, parametrized by one parameter $\lambda_{m,n}$. The

[13] One possible parametrization and also the simplest one is the canonical parametrization. Here a unitary operator U is decomposed w.r.t. exponentials of hermitian operators H, i.e.

$$U = \exp(iH),$$

where H is a linear combination of $d^2 - 1$ generalized Gell-Mann matrices and the unity $\mathbb{I}_d$.

term $\exp(iP_n\lambda_{n,m})$ adds a relative phase in between these rotated vectors. All products of the last two terms give $2 \times d(d-1)/2 = d^2 - d$ parameters.

The last factor in (103) corresponds to a global phase attached to each vector of the $\mathscr{ONB}$ of $\mathscr{H}^d$, i.e. $\{\mid 0\rangle, ..., \mid d\rangle\} \to \left\{e^{i\lambda_{1,1}} \mid 0\rangle, ..., e^{i\lambda_{d,d}} \mid d\rangle\right\}$, which gives us another d parameters. In total we have $d^2 - d + d = d^2$parameters.

This can be illustrated by a $d \times d-$matrix representation, i.e

$$U_C = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,d} \\ \vdots & \ddots & \vdots \\ \lambda_{d,1} & \cdots & \lambda_{d,d} \end{pmatrix}, \tag{106}$$

where the diagonal elements correspond to the global phase, the upper right to rotations and the lower left to relative phase transformations.

A subgroup of the unitary group $U(d)$ forms the special unitary group $SU(d)$. Elements U of $SU(d)$ have to fulfill the constraint $\det U = 1$ and therefore less parameters, namely $d^2 - 1$, are required for parametrization of the $SU(d)$.

Definition 2.49 Every operator $U \in SU(d)$ can be decomposed according to

$$U_C = \left(\prod_{m=1}^{d-1}\left(\prod_{n=m+1}^{d} \exp(iZ_{m,n}\lambda_{n,m}) \exp(iY_{m,n}\lambda_{m,n})\right)\right) \cdot \left(\prod_{l=1}^{d-1} \exp(iZ_{l,d}\lambda_{l,l})\right), \tag{107}$$

with $d^2 - 1$ real parameters $\lambda_{m,n} \in [0, \pi]$ for $m > n$, $\lambda_{m,n} \in \left[0, \frac{\pi}{2}\right]$ for $m < n$ and $\lambda_{m,n} \in [0, 2\pi]$ for $m = n$.

The operators $Z_{m,n}$ are diagonal and traceless[14], i.e.

$$Z_{m,n} = \mid m\rangle\langle m \mid - \mid n\rangle\langle n \mid, \tag{108}$$

which are generalizations of the diagonal and traceless Pauli matrix σ_z with $1 \leq m < n \leq d$.

The term $\exp\left(iZ_{m,n}\lambda_{n,m}\right)$ in (107) has the same effect as $\exp\left(iP_n\lambda_{n,m}\right)$ in (103), that is inducing relative phase shifts between the vectors $\mid m\rangle$ and $\mid n\rangle$.

The last term in (107), i.e. $\prod_{l=1}^{d-1}\exp\left(iZ_{l,d}\lambda_{l,l}\right)$ which is responsible for global phase operations, can be explained as follows:

As there are only $d-1$ parameters required instead of d like in (103), there are only $d-1$ global phase shifts induced according to $\{\mid 0\rangle, ..., \mid d-1\rangle\} \rightarrow \left\{e^{i\lambda_{1,1}} \mid 0\rangle, ..., e^{i\lambda_{d-1,d-1}} \mid d-1\rangle\right\}$.

The last vector $\mid d\rangle$ gets phase shifted in the overall inverse direction according to $\mid d\rangle \rightarrow e^{-i\sum_{l=1}^{d-1}\lambda_{l,l}} \mid d\rangle$. There is no parameter $\lambda_{d,d}$ and in total we have d^2-1 parameters $\lambda_{m,n}$.

All properties of the parametrization of the unitary group $U(d)$ are also valid for the special unitary group $SU(d)$, except that there are d^2-1 parameters instead of d^2. Now, with help of the parametrization introduced in (103) and (107) we are able to construct every density matrix according to

$$\rho = \sum_{n=1}^{k} p_n U_C \mid n\rangle\langle n \mid U_C^\dagger, \tag{109}$$

i.e. any $\mathscr{ONB}$ representing ρ can be achieved by applying U_C.

[14]Decompositions of operators $U \in SU(d)$ w.r.t. exponentials of operators are of the form

$$U = \exp\left(iH\alpha\right),$$

where the hermitian operator H has to be traceless, $\alpha \in \mathbb{R}$. As the projectors we used in the decomposition (88) for operators $U \in U(d)$ are *not* traceless, we have to replace them with traceless ones described in (95). Generalizations of the Pauli matrices, i.e. $Y_{m,n}$ and $Z_{m,n}$ are already traceless.

Also, the coefficients p_n can be represented by $(k-1)$ real parameters, i.e.

$$\begin{aligned} & p_1 = \cos^2 \vartheta_1 \\ & p_n = \cos^2 \vartheta_n \prod_{i=1}^{n-1} \sin^2 \vartheta_i, \quad for\ 1 < n < k \\ & p_k = \prod_{i=1}^{k-1} \sin^2 \vartheta_i \end{aligned} \tag{110}$$

where parameters $\vartheta_i \in \left[0, \frac{\pi}{2}\right]$ and $p_i \geq 0$, $\sum_{i=1}^{k} p_i = 1$.

The advantage of this composite parametrization is that for certain density matrices ρ not all of the d^2 or d^2-1 parameters $\lambda_{m,n}$ are required. Parameters representing the global phase cancel out when building $U_C \mid n\rangle\langle n \mid U_C^\dagger$, while for matrices with rank $k < m, n$ the parameters with corresponding m, n are redundant and U_C defined in (103) reduces to

Definition 2.50

$$U_{CD} = \prod_{m=1}^{k} \left(\prod_{n=m+1}^{d} \exp\left(iP_n \lambda_{n,m}\right) \exp\left(iY_{m,n} \lambda_{m,n}\right) \right), \tag{111}$$

where a total of $k\,(2d-k-1)$ parameters $\lambda_{m,n}$ and $k-1$ parameters ϑ_i, i.e. $2dk - k^2 - 1$ are necessary in order to parametrize a density matrix ρ with rank $k < d$. Full rank matrices, i.e. $k = d$ require $d^2 - 1$ and pure state density matrices $2\,(d-1)$ parameters.

2.3 Quantum nonlocality

This chapter is devoted to the phenomenon of *quantum nonlocality,* in which entangled quantum states are responsible for *quantum correlations*[15] among measurement results obtained from (spatially separated) quantum objects.

In the early years of quantum mechanics, this phenomenon gave rise to a debate, whether there is some kind of influence or instantaneous action at a distance between these objects, contradicting special relativity.

Under these circumstances, especially Einstein, who was a proponent of *local realism*, was motivated to believe that quantum mechanics is an *incomplete* theory. Together with Podolsky and Rosen, he published a paper in 1935 (see Ref.[2]), which illustrates by a Gedankenexperiment that quantum mechanics can not fulfill certain requirements, which every physical theory should (intuitively) fulfill. We are going to review their arguments in the next section, which mainly follows lecture notes made during a course held by Reinhold Bertlmann at the University of Vienna and Ref. [6].

2.3.1 Quantum mechanics and the EPR-paradox

According to Einstein, Podolsky and Rosen (EPR), a physical theory must fulfill the following requirements:

1. Completeness

A physical theory is considered to be *complete*, if every element of the theory corresponds to an element of reality.

2. Realism

Is often also referred to as *Einstein-reality.* A physical quantity is element of physical reality, if this quantity can be predicted without disturbing the system, i.e. can be assigned a value with *certainty.*

[15] There are also *classical correlations*, which can arise from the preparation of mixed quantum states, i.e. a mixture of pure product states. They have nothing to do with *quantum correlations,* which originate from quantum entanglement.

3. Locality

According to special relativity, there is no transmission of information faster than the speed of light. Communication and action between spacelike separated objects (see chapter 3.3) contradict the principles of special relativity. Einstein called that "no action at a distance" between two spatially separated objects, which is faster than the speed of light.

Under the above assumptions 1., 2., and 3., quantum mechanics was considered to be *incomplete* by EPR, meaning that there have to be elements of reality, which are not described within the theory.

The original argument was formulated in terms of the complementary observables *position* and *momentum* (see e.g. also Ref. [6]). The *spin* version of the argument was proposed by Bohm and Aharonov (see Ref.[59]), which we are now going to present briefly.

Assume that a system of two initially interacting particles with total spin 0 dissociates in diametrical directions, then their spin is evaluated by two experimenters Alice and Bob. The two subsystems are in an entangled state w.r.t. their spins, that is, the joint spin state is described by the antisymmetric maximally entangled Bell state

$$| \Psi^- \rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) . \tag{112}$$

Let's say, Alice, without loss of generality, is performing a measurement in $x - direction$. She will obtain either $|\uparrow\rangle$ or $|\downarrow\rangle$ with probability $\frac{1}{2}$ for each possibility. Now, quantum mechanics predicts, that Bob with *certainty*, that is, with probability 1, is going to measure the opposite spin direction, which is $|\downarrow\rangle$ or $|\uparrow\rangle$ depending on Alice's result. The measurement results are always *anti-correlated,* when choosing the same measurement direction.

The EPR *locality* and *realism* arguments are implemented as follows: Supposing that there is no action at a distance, i.e. Bob's measurement is not influenced by Alice's measurement, Bob's result can be predicted with certainty and is therefore an element of physical reality. Following the same line of reasoning, we could have chosen any arbitrary direction, e.g. perpendicular to the $x - direction$, the obtained result would have been an element of a simultaneous physical reality. Since there is no interaction between those particles, we are able to predict Bob's result for the perpendicular direction. In that way all three components of spin on Bob's side, which are perpendicular w.r.t. each other, i.e. *complementary observables*[16], can be predicted *with certainty* and correspond to *elements of reality* as defined by EPR.

On the other hand, the *uncertainty principle* of quantum mechanics states that *complementary observables* can *not* be evaluated at the *same time*. This is clearly causing conceptual difficulties and EPR *"are forced to conclude that the quantum mechanical description of physical reality given by the wave functions is not complete*[2]" , leaving open the question, whether or not such a description exists at all.

This is the *EPR-paradox*.

According to EPR, quantum mechanics is incomplete, since the requirements of reality and/or locality are not met by the theory.

The EPR article was criticized by Bohr (see Ref. [87]) in his reply, where he disagreed with EPR's interpretation of the notion of locality. He agreed that "*there is no question of a mechanical disturbance of the system under investigation*" by a measurement performed on the other spatially separated system. But "*there is essentially the question of an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system.*" According to him, not all types of predictions are possible. He called that "principle of complementarity". There are incompatible experiments and each experimental setup has to be considered separately and no conclusions can be made by comparison of mutually incompatible experiments.

However, Einstein was not satisfied with Bohr's arguments. He believed that the paradox forces us to drop the requirement of locality or the assertion that the description by the wave function is complete. The former, in his opinion, was indisputable.

One could give up on the requirement of locality, in order to save reality (or the notion that the description by the wave function is complete), but

[16]Spin observables evaluating the spin in perpendicular directions are complementary observables, i.e. do not commute. The Pauli matrices satisfy the commutation relation

$$\left[\sigma^i, \sigma^j\right] = 2i\varepsilon^{ijk}\sigma^k.$$

that would interfere with the concept of causality and special relativity, stating that there can not be *instantaneous* action between spatially separated particles[17]. One example, where the requirement of locality is dropped is the Bohmian interpretation of quantum mechanics (see e.g. Ref. [62]) by David Bohm, which is a deterministic, but *nonlocal*[18] theory.

Conversely, we could give up on the concept of (Einstein-) reality, meaning that the obtained measurement results are not pre-determined but depend on the chosen measurement directions (are *contextual*, i.e. depending on the context of the measurement, see also Kochen and Specker theorem[61]).

The *Copenhagen interpretation* of quantum mechanics drops the concept of reality, e.g. assigns no physical reality to quantum mechanical quantities pre measurement. The measurment induces a "*collapse of the wave function*" and singles out one of the various possibilities encoded in the wave function describing the system.

The Copenhagen interpretation is widely accepted as the standard interpretation of quantum mechanics.

Either way, quantum mechanics can not be a *local-realistic* theory, i.e. a *classical* physical theory, in the sense of EPR, as, according to them, it does not fulfill the above discussed requirements. This also becomes apparent when we consider the attempt to complete the theory and "save" local realism within quantum mechanics as was done in terms of *Local Hidden Variable Theories* (LHVT). Therein *hidden parameters* were postulated, which assign *locally,* determined values to observables.

One such LHVT was derived by John Bell in 1964 where he showed by his famous *Bell inequality*[3] that all LHVTs and therefore all local realistic

[17]Nowadays, however, this problem is solved by the fact that there is no faster than the speed of light transmission of information between the particles, i.e. correlations have to obey the *no-signaling condition*, which roughly states that there is no communication of (classical) signals between spacelike separated parties. In that way quantum mechanics is compatible with special relativity despite its *nonlocal* features and EPR-non-classical correlations.

Nevertheless, alternative formulations of quantum mechanics were considered, which lead to the possibility of superluminal communication, see e.g. [63] and references therein.

There are also alternative models, wherein quantum correlations are explained in terms of "superluminal hidden communication", see e.g. Ref. [64].

These alternative models, however, lead to signaling in certain situations, which is in contradiction with special relativity.

[18]This is a *nonlocal* hidden variable theory. See in comparison *local* hidden variable theories, like e.g. Bell's inequality discussed later on in the chapter.

theories are *incompatible* with the predictions of quantum mechanics. This is also known as the *Bell Theorem*, which we are going to point out in Definition once again due to its importance:

Bell's Theorem:

All local realistic theories are incompatible with the predictions of quantum mechanics in certain experimental setups.

Mathematically, this can be formulated in terms of Bell inequalities. These inequalities exist in various formulations applicable to various situations. e.g. for systems of higher dimensions, like the *CGLMP inequality* for bipartite qudit-systems[65], or representatives applicable in particle physics, e.g. the entangled neutral kaon system and Bell inequalities sensitive to CP-violation, see e.g. Ref.[66] and references therein.

We are going to discuss the version of Clauser, Horne, Shimony and Holt (CHSH)[60], which is also easier implementable in experiment than Bell's original inequality.

2.3.2 Example for a Bell inequality: The CHSH inequality

Every *local realistic theory* has to satisfy a *Bell inequality.* We are going to discuss the *CHSH-version* of the Bell inequality in order to understand the concept of these kind of inequalities, which can be derived as follows:

Starting point for our consideration is an EPR-like situation, where a source produces entangled particles in the $\mid \Psi^- \rangle - spin-$singlett state, i.e. entangled in the spin degrees of freedom.

The experimenters Alice (A) and Bob (B) perform spin measurements in arbitrary directions $\vec{a}$ and $\vec{b}$, respectively. The Observables A and B then look like

$$A(\vec{a}) = \vec{\sigma} \cdot \vec{a}, \quad B(\vec{b}) = \vec{\sigma} \cdot \vec{b}. \tag{113}$$

and where $\vec{\sigma}$ is the vector of Pauli matrices.

The EPR arguments are implemented as follows: The argument of reality assigns values to observables A and B in dependence of a hidden parameter λ, i.e.

$$A(\vec{a},\lambda) = \vec{\sigma}\cdot\vec{a}, \quad B(\vec{b},\lambda) = \vec{\sigma}\cdot\vec{b}. \tag{114}$$

The argument of locality (also *Bell-locality*) assigns values to A and B in a way, so that A is independent of the choice of measurement direction in B and vice versa, i.e.

$$A(\vec{a},\lambda) \neq A(\vec{b},\lambda), \quad B(\vec{b},\lambda) \neq B(\vec{a},\lambda). \tag{115}$$

Now, we are considering *expectation values* E of the joint spin measurement in A and B, defined as

$$E(\vec{a},\vec{b}) = \int A(\vec{a},\lambda)B(\vec{b},\lambda)\varrho(\lambda)d\lambda, \tag{116}$$

where

$$\int \varrho(\lambda)d\lambda := \int d\varrho = 1 \tag{117}$$

is a normalized distribution function for the parameter λ.

Furthermore, we want to assume that there are three possible values the observables can have, i.e.

$$A(\vec{a},\lambda),\ B(\vec{b},\lambda) \in \{+1,-1,0\}, \tag{118}$$

which are assigned, theoretical values for the measurement outcome "spin up", spin down" and "no measurement outcome", respectively.

When considering certain combinations of expectation values defined in (116) , but in dependence of 4 different measurement directions $\vec{a},\vec{a}',\vec{b},\vec{b}'$, we obtain

$$\begin{aligned} E(\vec{a},\vec{b})-E(\vec{a},\vec{b}') &= \int \Big(A(\vec{a},\lambda)B(\vec{b},\lambda)-A(\vec{a},\lambda)B(\vec{b}',\lambda)\Big)\,d\varrho \\ &= \int A(\vec{a},\lambda)B(\vec{b},\lambda)\Big(1\pm A(\vec{a}',\lambda)B(\vec{b}',\lambda)\Big)\,d\varrho- \\ &\quad -\int A(\vec{a},\lambda)B(\vec{b}',\lambda)\Big(1\pm A(\vec{a}',\lambda)B(\vec{b},\lambda)\Big)\,d\varrho \end{aligned} \tag{119}$$

where the second line is obtained by adding and subtracting the same term.

The terms $\mid \int A(\vec{a},\lambda)B(\vec{b},\lambda)d\varrho \mid$ and $\mid \int A(\vec{a},\lambda)B(\vec{b}',\lambda)d\varrho|$ have to be per definitionem ≤ 1. So, when considering the absolute value of (119), we obtain the estimated values

$$\begin{aligned} \mid E(\vec{a},\vec{b})-E(\vec{a},\vec{b}')\mid &\leq \mid \int \Big(1\pm A(\vec{a}',\lambda)B(\vec{b}',\lambda)\Big)\,d\varrho\mid + \\ &\quad +\mid \int \Big(1\pm A(\vec{a}',\lambda)B(\vec{b},\lambda)\Big)\,d\varrho\mid \\ \mid E(\vec{a},\vec{b})-E(\vec{a},\vec{b}')\mid &\leq 2\pm \mid E(\vec{a}',\vec{b}')+E(\vec{a}',\vec{b})\mid \end{aligned} \tag{120}$$

and by rewriting (120) the CHSH-inequality

$$\mid E(\vec{a},\vec{b})-E(\vec{a},\vec{b}')\mid + \mid E(\vec{a}',\vec{b}')+E(\vec{a}',\vec{b})\mid \leq 2, \tag{121}$$

where we chose the plus sign in favour of a greater value.

We can reformulate the CHSH inequality by introducing the *Bell observable*

$$S(\vec{a},\vec{a}',\vec{b},\vec{b}') := E(\vec{a},\vec{b})-E(\vec{a},\vec{b}')+E(\vec{a}',\vec{b}')+E(\vec{a}',\vec{b}), \tag{122}$$

that is

$$\mid S(\vec{a},\vec{a}',\vec{b},\vec{b}')\mid \leq 2. \tag{123}$$

Now, we can check, whether quantum mechanics can be described by a LHVT, that is, satisfies the CHSH inequality defined in (123) .To do so, we need the quantum mechanical expectation values for the joint spin measurement performed by Alice and Bob, which is obtained via

$$E(\vec{a},\vec{b}) = \langle \psi^- \mid \vec{\sigma}\vec{a} \otimes \vec{\sigma}\vec{b} \mid \psi^- \rangle = -\vec{a}\cdot\vec{b}, \tag{124}$$

which can be shown by a straight forward calculation.

The right hand side of (124) can also be expressed as $-\cos(\theta)$, with θ being the angle between the measurement directions $\vec{a}$ and $\vec{b}$.

As we have 4 expectation values in (122) , we have 4 angles between the chosen measurement directions $\vec{a}, \vec{a}', \vec{b}, \vec{b}'$, which are chosen such, that they yield the greatest value, i.e. when inserting

$$E\left(\vec{a},\vec{b}\right) = -\cos(\theta_1) = -\cos\left(\frac{\pi}{4}\right), \tag{125}$$

$$E\left(\vec{a}',\vec{b}\right) = -\cos(\theta_2) = -\cos\left(\frac{\pi}{4}\right), \tag{126}$$

$$E\left(\vec{a}',\vec{b}'\right) = -\cos(\theta_3) = -\cos\left(\frac{\pi}{4}\right), \tag{127}$$

$$E\left(\vec{a},\vec{b}'\right) = -\cos(\theta_4) = -\cos\left(\frac{3\pi}{4}\right) \tag{128}$$

into (122), more exactly (123), we obtain a value

$$\mid S \mid = 2\sqrt{2} > 2, \tag{129}$$

which is clearly a violation of the CHSH inequality. The the theoretical value $2\sqrt{2}$ represents the maximal possible violation of the CHSH inequality reached by pure maximally entangled bipartite qubit states. To reach this value an optimization has to be performed over the possible measurement directions.

What can be concluded is that quantum mechanics can not be described by LHVTs due to its *nonlocal* behaviour, i.e. violation of Bell inequalities. This result was also confirmed by various experiments with entangled photons, see e.g. Refs.[45][46][47][48][49], first of which performed by Freedman and Clauser in 1972.

A state, which violates the Bell inequality is a *nonlocal* state. Pure entangled bipartite states are always nonlocal[67]. For mixed states, however, the situation is more complicated. Violation of a Bell inequality can be traced back to *entanglement*, but this does not necessarily mean, that all entangled mixed states are nonlocal. There are also entangled, but local states[24].

One representative is the family of Werner states, i.e. states of the form

$$\rho_W = p \mid \Psi^- \rangle \langle \Psi^- \mid + \frac{1-p}{4}\mathbb{I}. \tag{130}$$

The parameter $\frac{1}{3} < p \leq 1$ indicates an entangled state, while nonlocality is only obtained for $\frac{1}{\sqrt{2}} < p \leq 1$.

This example also shows that Bell inequalities can not be good *entanglement criteria*, especially for mixed states. There are also far better ones, because easier to compute, for pure states (see chapter 2.2. for some representatives of entanglement criteria).

3 Relativity

This section is devoted to the study of relativistic behavior of (classical) physical systems. As the main focus of this work lies on the relativistic behavior of entangled quantum systems, we can only give a short introduction into the topic of special relativity and point out the central aspects needed for a proper understanding of our considerations in chapter 4.

More on the topic can be found e.g. in [32][30], whereas [31] is focusing on the applications in quantum theory and particle physics.

3.1 Galilean relativity

The laws of nature, technically, our mathematical framework we developed to describe certain phenomena, are not supposed to change when viewed by different observers who are in relative motion w.r.t. each other. Especially, these observers should be able to compare their observation of one and the same phenomenon independently of their state of motion.

Every physical observation we make happens in front of *space* and *time.* In *Newtonian physics,* the right mathematical concepts to describe our spatial world and changes therein turned out to be the *Galilean space*, which is an *affine space*[19] over a three-dimensional *Euclidean vector space*[20]. According to Newton, space and time are two separate and absolute aspects of reality and exist independently of the observer.

Newton on space:

"Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies: and which is vulgarly taken for immovable space ... Absolute motion is the translation of a body from one absolute place into another: and relative motion, the translation from one relative place into another[21]*."*

[19] In principle a continuum of points without a distinguished origin.

[20] Determines the geometry of space, i.e. the euclidean metric is the appropriate metric determining the lengths and angles of our spatial world.

[21] Sir Isaac Newton, *Philosophiæ Naturalis Principia Mathematica*, "Mathematical Principles of Natural Philosophy" (1687)

Newton on time:

"Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration: relative, apparent and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time ...[3]*"*

So, everything we observe has an absolute state of motion or is in absolute rest w.r.t. absolute space and time, but is in relative motion w.r.t. our state of motion. This implies that there are various *frames of reference* (mathematically speaking coordinate systems) depending on the state of motion relative to absolute space and time. Within these frames of reference there exists a certain class, the class of *inertial frames,* and within these the laws of physics assume one and the same form.

Definition 3.1 *Special principle of relativity*

There is a certain class of frames, the *inertial frames of reference*, w.r.t. which the laws of physics assume the same (mathematical) form. Observers within these reference frames, the *inertial observers*, are able to formulate the laws of physics *form-invariant.*

In the Galilean context, these inertial frames are coordinate systems in Galilean space.

There are certain coordinate transformations, the Galilean transformations, which guarantee the (Newtonian) laws of physics to be form-invariant.

Definition 3.2 Under *Galilean coordinate transformations*, i.e. under their most general formulation

$$t' = t + t'_0, \tag{131}$$

$$x^{i\prime} = R^i_k x^k - v^i t + x'_0, \tag{132}$$

where $t'$and x'_0 are temporal and spatial translations, $R^i_k \in O\,(3)$ is either a reflection or a rotation in which case $R^i_k \in SO\,(3)$ and v^i represents a velocity "boost".

A special class within Galilean transformations are *Galilean boosts*. They assume the form

$$t' = t, \tag{133}$$

$$x^{i\prime} = x^i - v^i t, \tag{134}$$

and are a coordinate transformation between inertial observers in relative uniform motion w.r.t. each other.

It is important to emphasize that Galilean transformations only leave Newtonian classical physics invariant.

3.2 Lorentzian relativity

According to the principle of relativity, the fundamental laws of physics have to be Galilei-invariant. It turned out, however, that the *wave equation*[22]

$$\Box\phi\,(t,x) = \frac{1}{c^2}\partial_t^2\phi\,(t,x) + \delta^{ij}\partial_i\partial_j\phi\,(t,x) = 0 \tag{135}$$

and as a consequence also the *Maxwell equations*

$$\begin{aligned} \vec{\nabla}\vec{E} &= 0, & \vec{\nabla}\vec{B} &= 0, \\ \vec{\nabla}\times\vec{E} &= -\partial_t\vec{B}, & \vec{\nabla}\times\vec{B} &= \tfrac{1}{c^2}\partial_t\vec{E}, \end{aligned} \tag{136}$$

are *not* Galilei-invariant. This was considered irrelevant due to the fact that the wave equation and the Maxwell equations are not fundamental but derived equations. They are based on a “background solution”, i.e. a medium which is a distinguished frame of reference. So the main idea was that all kind

[22]quantity c describes the velocity of the propagation of waves in general. It could be the speed of sound c_s, the speed of light in vacuum c_0 etc.

of waves, i.e. also electromagnetic waves, require a propagation medium. In case of light that medium is called *aether*.

So, in the 19th century a lot of experiments were carried out in order to describe the properties of the aether. When there is aether, what is the motion of the earth relative to the aether etc. The most famous experiment is the Michelson and Morley experiment. They used a light beam interferometer in order to show that relative motion of the earth through the aether results in interference effects of the light beams traveling relative and in different directions w.r.t. aether and the earth.

The result was *null*, several times[23].

Long story short, there is no aether (A. Einstein 1905).

So, when there is no aether, the Maxwell equations are not derived but rather fundamental equations. Also, when taking the principle of relativity serious, there have to be some kind of transformation, which leaves the Maxwell equations (and the wave equation as well) invariant under a change of reference frame.

These coordinate transformations are the *Lorentz/Poincaré transformations.*

Definition 3.3 The Lorentz transformations is the appropriate coordinate transformation between inertial frames leaving invariant the laws of physics, e.g. the Maxwell and the wave equations. Their general form reads

$$x^{\mu\prime} = T\left(\Lambda, b\right) x^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + b^{\mu\prime}, \tag{137}$$

where Λ^{μ}_{ν} is a Lorentz transformation matrix yet needed to be defined and b^{μ} a translation vector. The latter defines the Poincaré transformation.

Resulting from the requirement that the Maxwell equations stay invariant under changes of the inertial reference frame[24], there is no absolute time and

[23] There were many rather desperate attempts to save the aether-theory. The Michelson and Morley interferometry experiment was also conducted on mountains and even in balloons because the original laboratory was situated in a basement and that could have had effects on the aether...

[24] Also this requirement demands that the speed of light is constant. In vacuum it is defined as $c = 299792458 m/s$ in SI units. In all our considerations $c = 1$.

space. Time and space are combined to a spacetime, where they are treated equally and in dependence of the inertial observer.

In the special relativistic description we formulate vectors in terms of 4-component vectors, i.e. the vector reads $x^\mu = (x^0, x^1, x^2, x^3)$. The x^0 component includes the time component, i.e. $x^0 = ct$.[25]But more on this formalism can be found in the following section.

3.3 Minkowski spacetime

The Lorentz transformation can be defined as the transformation that leaves the *Minkowski spacetime metric* invariant. First let us define the Minkowski space:

Definition 3.4 The *Minkowski spacetime* is a four dimensional affine space over a 4-dimensional *Lorentzian vector space.*

Definition 3.5 A *Lorentzian vector space* is a 4-dimensional vector space with an inherent special geometric structure, namely the *Minkowski metric* or pseudo-scalar product.

Definition 3.6 The *Minkowski metric* determining the geometric structure of our spacetime is defined as

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{138}$$

i.e. possesses the signature $\begin{pmatrix} - & + & + & + \end{pmatrix}$.[26] It can be interpreted as pseudo-orthonormal pseudo-scalar product $\eta(\cdot, \cdot)$.

The basis vectors $\{e_1, e_2, e_3\}$ form the orthonormal complement to $\{e_0\}$.

[25]time is then measured in light meters, i.e. the time it takes light to travel trough one meter.

[26]There exists a different convention especially used in particle physics. There the metric is defined as $\eta_{\mu\nu} = diag\,(+1, -1, -1, -1)$.

The pseudo-scalar product $\eta(v, w)$ of two four-vectors reads in components

$$\eta(v, w) = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3, \tag{139}$$

or can be denoted in index notation by

$$\eta(v, w) = \eta_{\mu\nu} v^\mu w^\nu, \tag{140}$$

and in matrix notation by

$$\eta(v, w) = v^T \eta w. \tag{141}$$

Points in the Minkowski space are called *events.* When choosing an origin in Minkowski spacetime this *event* can be described in terms of *four-vectors* w.r.t. this inertial observer.

The Lorentz Transformation can now be defined in terms of the Minkowski metric.

Definition 3.7 The Lorentz transformations are all kind of transformations which leave the Minkowski metric $\eta_{\mu\nu}$ and thereby the pseudo-scalar product $\eta(\cdot, \cdot)$ invariant, i.e.

$$\eta(\Lambda v, \Lambda w) = \eta(v, w). \tag{142}$$

This also means that

$$\Lambda^T \eta \Lambda = \eta \tag{143}$$

in matrix notation.

The Minkowski metric η defines the geometric structure of spacetime, i.e. is the same for every inertial observer. When considering the spacetime geometry, we are able to formulate the laws of physics in a *coordinate indepentent*, i.e. *observer independent* manner.

Due to the nature of the pseudo-scalar product $\eta(\cdot, \cdot)$, the square of a four-vector v , that is $\eta(v, v) = \eta_{\mu\nu} v^\mu v^\nu = v^\mu v_\mu$, can have any sign.

Definition 3.8 Four-vectors v fulfilling

$$\eta(v,v) > 0 \tag{144}$$

are called *spacelike*. In contrast, four-vectors v fulfilling

$$\eta(v,v) < 0 \tag{145}$$

are called *timelike* and in case

$$\eta(v,v) = 0 \tag{146}$$

four-vectors v are called *null*.

The basis underlying the Minkowski space is the pseudo-orthonormal basis $\{e_0, e_1, e_2, e_3\}$ with

$$\eta(e_\mu, e_\nu) = \eta_{\mu\nu} \tag{147}$$

or

$$\eta(e_0, e_0) = -1, \quad \eta(e_0, e_i) = 0, \quad \eta(e_i, e_j) = \delta_{ij} \ . \tag{148}$$

So e_0 is clearly distinguished from the other basis vectors, i.e. timelike, and therefore often given a different symbol like e.g. u. It is also called *four-velocity* of an observer and reads

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{149}$$

w.r.t. the basis $\{e_0, e_1, e_2, e_3\}$. It gives the *timeline* of an observer and the orthogonal complement w.r.t. u are the *planes of simultaneity*, i.e. planes of constant time whose events happen *simultaneously*. The orthogonal complement of u always consists of spacelike vectors.

When choosing a second inertial observer, the four velocity defined in (149) of the first observer w.r.t. the second frame reads

$$u' = \gamma \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}, \tag{150}$$

where $\vec{v}$ is the relative *three velocity* of the first observer as seen by the second one.

Particles (more precisely, point particles) are represented by *world lines* in Minkowski space, i.e. a curve

$$\lambda \mapsto x(\lambda), \tag{151}$$

with $\lambda \in \mathbb{R}$ being an arbitrary parameter.

When forming the field of tangent vectors to the curve $x(\lambda)$ according to

$$\frac{d}{d\lambda} x(\lambda) = w(\lambda), \tag{152}$$

we can decide whether the curve $x(\lambda)$ is timelike, null, or spacelike for all parameters λ by deciding whether the four vector $w(\lambda)$ is timelike, null, or spacelike. Particles are *only* represented by timelike or spacelike curves. Hypothetical particles, called *Tachyons*, that are represented by spacelike curves, i.e. moving with superluminous velocities, would mean a violation of causality.

Often the parameter λ is chosen to be the *proper time* along the world line, especially when we want to measure the time that elapses for the particle between two events on the world line, e.g. $x(\lambda)$ and $x(\lambda + d\lambda)$.

Definition 3.9 The *momentary rest frame* is an observer who is momentarily at rest w.r.t. the particle at fixed λ. Those observers four velocity u is parallel to the tangent vector $w(\lambda)$ and given by

$$u = \frac{w(\lambda)}{\sqrt{-\eta(w(\lambda), w(\lambda))}}. \tag{153}$$

In general, this particle has no global rest frame, i.e. one rest frame along the total length of the curve $x(\lambda)$, but rather local rest frames which change with the parameter λ. When choosing one distinctive rest frame w.r.t. the particle, we can choose a parametrization[27] w.r.t. the coordinate time t of one observers particular frame.

Definition 3.10 The time that elapses for the particle along its worldline $x(\lambda)$ between the events $x(\lambda)$ and $x(\lambda + d\lambda)$ is denoted by ds. It is given by[28]

$$ds = \sqrt{-\eta\left(w(\lambda), w(\lambda)\right)}d\lambda. \tag{154}$$

The proper time s is obtained by integrating (154) over the particles world line, i.e.

$$s = \int \sqrt{-\eta\left(w(\lambda), w(\lambda)\right)}d\lambda. \tag{155}$$

Definition 3.11 The *four momentum* of a massive particle represented by a timelike wordline $s \mapsto x(s)$ is given by

$$p = mu, \tag{156}$$

where m is the mass and u the four velocity of the particle.

[27] As mentioned before, the parametrization is completely arbitrary. We can parametrize the world line w.r.t. any observers coordinate time t.

[28] This can be seen by expressing $w(\lambda)$ in (153) w.r.t. the coordinates of the observer whose four velocity is $u = \left(1, \vec{0}\right)$. $w(\lambda)$ then reads

$$w(\lambda) = \begin{pmatrix} w^0(\lambda) \\ \vec{w}(\lambda) \end{pmatrix} = \begin{pmatrix} \sqrt{-\eta\left(w(\lambda), w(\lambda)\right)} \\ \vec{0} \end{pmatrix}.$$

Furthermore, $x(\lambda + d\lambda) = x(\lambda) + w(\lambda)\,d\lambda$. Inserting the above defined $w(\lambda)$ defined w.r.t. u and calculating $x(\lambda + d\lambda) - x(\lambda)$, yields the connection of the events $x(\lambda)$ and $x(\lambda + d\lambda)$, whose time component is the proper time ds.

Considering the four momentum w.r.t. an arbitrary inertial observer gives

$$p = \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix} = mu = m\gamma \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}, \tag{157}$$

where $\vec{p} = m\gamma\vec{v}$ is the *three momentum*[29] w.r.t. this observer.

3.4 Lorentz transformations

In the former sections we have established the Lorentz transformations as the kind of transformations, which leave the laws of physics unchanged and as the proper coordinate transformation between inertial frames.

In this section we are going to discuss the possible representations.

A possible Lorentz transformation is spatial rotation, i.e.

$$\Lambda = \begin{pmatrix} 1 & \\ & \boxed{R_{3x3}} \end{pmatrix}, \tag{158}$$

where $R = (R)_{ij}$ is a $(3x3)$ rotation matrix, i.e. $R \in SO(3)$. Spatial rotations map inertial frames into inertial frames.

An important class of Lorentz transformations are the so called *Lorentz boosts*. This coordinate transformation involves the time coordinate as well and relates two inertial observers who are in constant uniform motion w.r.t. each other. A boost in x^1direction is a coordinate transformation from a reference frame S to another frame S' which is moving with constant relative velocity $\vec{v} = (v_x, 0, 0)^T$ in x-direction as seen from frame S. It is given by

$$\Lambda_x(\vec{v}) = \begin{pmatrix} \gamma & -\gamma v_x & 0 & 0 \\ -\gamma v_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{159}$$

[29]It differs from the classical momentum by the $\gamma-$ factor, i.e. is *not* proportional to the three velocity. However, for small velocities we obtain the classical momentum $\vec{p} = m\vec{v}$. The mass m remains a constant, i.e. there is no such thing as 'relativistic mass', in contrast to the claims of many textbooks.

Definition 3.12 The $\gamma - factor$ (or Lorentz factor) is defined as

$$\gamma = \gamma(\vec{v}) = \frac{1}{\sqrt{1-\vec{v}^2}}. \tag{160}$$

The Lorentz boosts associated with the other axes are given by

$$\Lambda_y(\vec{v}) = \begin{pmatrix} \gamma & 0 & -\gamma v_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma v_y & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{161}$$

$$\Lambda_z(\vec{v}) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v_z & 0 & 0 & \gamma \end{pmatrix}, \tag{162}$$

where Λ_y and Λ_z are boosts in the x^2 and x^3 direction respectively[30].

A general Lorentz boost in an arbitrary direction $\vec{v} = (v_x, v_y, v_z)^T$ is given by

$$\Lambda(\vec{v}) = \begin{pmatrix} \gamma & -\gamma\vec{v}^T \\ -\gamma\vec{v} & \boxed{\mathbb{I} + \frac{\gamma-1}{|\,\vec{v}\,|^2}\vec{v}\vec{v}^T} \end{pmatrix}. \tag{163}$$

A general Lorentz boost can also be expressed by a boost in a chosen direction (e.g. the x-direction) and suitable rotations, as

$$\Lambda(\vec{v}) = \begin{pmatrix} 1 & \\ & \boxed{R^T_{3x3}} \end{pmatrix} \begin{pmatrix} \gamma & -\gamma v_x & 0 & 0 \\ -\gamma v_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \boxed{R_{3x3}} \end{pmatrix}, \tag{164}$$

[30] We denote spatial components by x,y and z. Instead we could stick to the relativistic convention and name them x^1, x^2 and x^3 as well, but the former description is more illustrative.

where the rotation $R = R(\vec{v})$ takes the x axis into the direction of $\vec{v}$.

A Poincaré transformation adds a spacetime translation $b^{\mu\prime}$, i.e. maps coordinates x^{μ} into $x^{\mu'}$ according to

$$x^{\mu'} = T(\Lambda, b)\, x^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + b^{\mu\prime}. \tag{165}$$

The Lorentz transformations can also be expressed in terms of a parameter ξ, also called *rapidity,* instead of the velocity $\mid \vec{v} \mid$. It is defined as

$$\mid \vec{v} \mid = \tanh \xi \tag{166}$$

and thereby

$$\gamma(\vec{v}) = \cosh \xi \tag{167}$$

and

$$\gamma(\vec{v}) \mid \vec{v} \mid = \sinh \xi. \tag{168}$$

Inserting (166), (167) and (168) into e.g. (145), that is the Lorentz boost in $\mid \vec{v} \mid = v_x - direction$, yields

$$\Lambda_x(\xi) = \begin{pmatrix} \cosh \xi & -\sinh \xi & 0 & 0 \\ -\sinh \xi & \cosh \xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{169}$$

4 Relativity in entanglement

In the previous chapters we have given an introduction to the topic of quantum entanglement, i.e. we have discussed its definition, detection, quantification and its nonlocal manifestation in terms of Bell inequalities.

Further, we have given an introduction into the theory of special relativity. Especially, we have established the Lorentz transformations as the appropriate transformations between different inertial observers.

Focus of the following chapter is the connection of these two theories, that is, the relativistic description of entangled systems and its implications.

We are going to see, if and how entanglement properties change under Lorentz boosts and actually, whether entanglement is Lorentz invariant or not.

Entanglement in a relativistic setting was studied by various authors in recent years, see e.g. References [10]-[21] and references therein. The discussion ranges from finding the appropriate relativistic spin observable in a relativistic EPR-setting [10] to analyzing various (bipartite) quantum states and their entanglement properties under a change of reference frame.

Controversial results were obtained ranging from 'the entanglement fidelity[31] is preserved under Lorentz transformations' in [13] to 'no sums of entanglement are found to be unchanged' in [16] for certain types of two-particle states.

A thorough investigation for a broad class of two-particle momentum and spin states in [19], where the momentum and spin degrees of freedom of two massive spin-$\frac{1}{2}$ particles are treated in the qubit formalism, has shown that the change in spin and momentum entanglement depend on the chosen initial state and the partition of the total Hilbert space. While the overall entanglement in the particle-particle partition indeed is Lorentz invariant, i.e. the same for all inertial observers, the overall entanglement in different partitions is frame dependent and state dependent. The Bell states $\mid \Psi^+\rangle$ and $\mid \Psi^-\rangle$ e.g. behave differently, i.e. the overall entanglement distributed between the spin and momentum degrees of freedom is Lorentz invariant for the $\mid \Psi^+\rangle$-spin- state, while it changes for a $\mid \Psi^-\rangle$-spin state.

Also, the maximal violation of the Bell inequality was stated to be frame-dependent in [17] while entanglement is claimed to be frame-independent. The former statement is only true for not appropriately chosen measurement

[31] The quantum *fidelity* F of two quantum states ρ and σ is defined as

$$F\left(\rho,\sigma\right) = Tr\left(\sqrt{\rho}\sigma\sqrt{\rho}\right)$$

and can be interpreted as a distance measure between the two states ρ and σ (see Refs. [58][5]). Strictly speaking, the fidelity defines the *closeness* of two states ρ and σ, but can be related to the *Bures distance*

$$D_B\left(\rho,\sigma\right) = 2\left(1-\sqrt{F\left(\rho,\sigma\right)}\right),$$

which is a distance measure.

For pure states $\rho = \mid \phi\rangle\langle\phi\mid$ and $\sigma = \mid\psi\rangle\langle\psi\mid$ the fidelity reduces to the simple expression of

$$F\left(\rho,\sigma\right) = \mid\langle\phi\mid\psi\rangle\mid .$$

This is also called the *overlap* between two states $\mid\phi\rangle$ and $\mid\psi\rangle$. Furthermore, $F\left(\rho,\sigma\right)^2$ is the probability of the system, which was prepared in $\mid\psi\rangle$, being in the state $\mid\phi\rangle$ after a measurement.

The fidelity is invariant under unitary transformations, i.e.

$$F\left(U\rho U^\dagger, U\sigma U^\dagger\right) = F\left(\rho,\sigma\right).$$

directions. The maximal violation can be recovered for all inertial observers by appropriately transforming the measurement directions, as was shown in [19]. Strictly speaking, by relativistically transforming the state and the obervable we can guarantee the Lorentz invariance of the expectation values w.r.t. all inertial observers, i.e. every inertial observer will obtain the same measurement results. These Lorentz invariant expectation values are then used to evaluate the Bell inequality (see chapter. 4.2 for more details).

In order to study the relativistic behavior of entanglement, first of all we need to establish the appropriate relativistic transformation of quantum states. A Lorentz transformation works only for the momenta, while it induces a *Wigner rotation* on the spins.

The following section reviews the role of *Wigner rotations* and its implications.

4.1 Relativistic transformation of quantum mechanical states

According to the principle of special relativity, physics is not supposed to change when viewed by different observers moving with constant velocity w.r.t. each other, i.e. under the change of coordinate system the quantifiable physical properties of a system remain the same. The appropriate transformation between the reference frames is the Lorentz transformation (or Poincaré transformation, when considering translations b of the origin of a coordinate system as well):

$$x^{\mu'} = T\left(\Lambda, b\right) x^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + b^{\mu} \tag{170}$$

In the case of quantum states these transformations need to be *unitary* in order to conserve the length, i.e. the normalization of states, and therefore the physical quantities like expectation values. As was demonstrated by Wigner [8], finite dimensional representations of Lorentz boosts are *non-unitary*. How unitarity can be reestablished can be seen as follows (see e.g. Ref.[13] and [23] for further information) :

A quantum state vector

$$\mid p, s\rangle = \mid p\rangle \otimes \mid s\rangle, \tag{171}$$

where $\mid p\rangle$ is the momentum and $\mid s\rangle$ the spin state vector, transforms *unitarily* under the homogenous, orthochronous Lorentz group (homogenous Lorentz transformations Λ) to a linear combination of the state vectors $\mid \Lambda p, s'\rangle$, i.e[32].

$$U\left(\Lambda\right) \mid p, s\rangle = \sum_{s'} C_{s's}\left(\Lambda, p\right) \mid \Lambda p, s'\rangle \tag{172}$$

[32]Weinberg's treatment includes a normalization factor $\sqrt{\frac{(\Lambda p)^0}{p^0}}$, i.e. $U\left(\Lambda\right) \mid p, s\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} C_{s's}\left(\Lambda, p\right) \mid \Lambda p, s'\rangle$.

where the matrix $C_{s's}$ is a suitable representation, depending on the spin of the particle (for further information, see e.g. Ref. [31]).

We take the momentum (of a massive) particle to be the *standard momentum* $k^\mu = (1, 0, 0, 0)$, which is the *rest frame momentum.* Every other momentum p^μ can be expressed by Lorentz-transforming k^μ:

$$p^\mu = L^\mu_\nu\,(p)\,k^\nu \tag{173}$$

A state vector $\mid p, s\rangle$ can then be expressed as

$$\mid p, s\rangle \equiv U\,(L\,(p)) \mid k, s'\rangle. \tag{174}$$

It has to be emphasized that only momentum eigenstates are used, as all our investigations are concerned with discrete 3-level systems (see chapter 4.3).

The four momenta p^μ are eigenstates of the four momentum operator P^μ, i.e.

$$P^\mu \mid p, s\rangle = p^\mu \mid p, s\rangle, \tag{175}$$

satisfying the normalization condition

$$\int d\mu\,(p)\,\langle p' \mid p\rangle = 1, \tag{176}$$

with a Lorentz-invariant integration measure $d\mu\,(p)$. We are going to use the shorthand notation

$$\langle p' \mid p\rangle = 1, \tag{177}$$

where the integration is implied, whenever skalar products of momenta are calculated.

By using this "sharp" momenta, we guarantee that there are single resulting *Wigner Rotations* (see (180)), which depend on the momentum of the particle and act on the spin within the relativistic transformation of the state $\mid p, s\rangle$. It is a frequent approximation[33] used in all comparable works, see e.g. Refs. [16][19][20][21].

The transformation $U(\Lambda)$ can be derived as follows:

For an arbitrary observer's frame, achieved by a Lorentz transformation Λ, the state in (174) is

$$U(\Lambda) \mid p, s\rangle = U(\Lambda L(p)) \mid k, s'\rangle. \tag{178}$$

[33]This approximation is usefull, as shown in Ref. [14] (see Ref. [18] for further details as well) out of the following reasons: when considering *localized* particles instead of plane waves then a two particle ket state is of the form

$$\mid \psi\rangle_{2-particle} = \sum_{\sigma_1,\sigma_2} \int d\mu(p_1, p_2)\, f_{\sigma_1,\sigma_2}(p_1, p_2) \mid p_1, \sigma_1\rangle \otimes \mid p_2, \sigma_2\rangle,$$

where $f_{\sigma_1,\sigma_2}(p_1, p_2)$ is a distribution function satisfying the normalization condition

$$\sum_{\sigma_1,\sigma_2} \int d\mu(p_1, p_2) \mid f_{\sigma_1,\sigma_2}(p_1, p_2) \mid^2 = 1.$$

The integration measure $d\mu(p_1, p_2) = d\mu(p_1)\, d\mu(p_2)$ has to be Lorentz invariant. A possible choice is $d\mu(p) = \frac{d^3p}{2E_p}$, $E_p = p^0 > 0$, i.e. $E_p = \sqrt{m^2 + \vec{p}^2}$, i.e. the domain of integration is the *unit mass shell*, leaving the same invariant under Lorentz transformations. For further details see e.g. Ref. [23].

The distribution function $f_{\sigma_1,\sigma_2}(p_1, p_2)$ can be chosen to be of the form $f_{\sigma_1,\sigma_2}(p_1, p_2) = \frac{1}{\sqrt{2}}\delta_{\sigma_1\sigma_2} g(p_1)\, g(p_2)$, where $g(p_1)$ *and* $g(p_2)$ are *gaussian distribution* functions of width w centered around fixed values p_1 and p_2 and δ is the *delta distribution,* giving sharp spins σ_1 *and* σ_2. The authors of Ref. [14] showed that, due to the partial trace operation, the information loss depends on the parameter w (strictly speaking $\frac{w}{m}$, m is the mass of the particle), i.e. becomes bigger with increasing width w of the distribution. They considered the momentum-spin partition of the Hilbert space after the Lorentz boost, which entangles the momenta and the spins (see also chapter 4.3 of this thesis), then traced out the momentum subsystem and evaluated the entanglement in the spin partition. The entanglement (measured by the concurrence) decreases with increasing width w. For sufficiently narrow distributions ($\frac{w}{m} \underset{\approx}{<} 3.377$) in momentum space, they find that the value obtained by the concurrence saturates at nonzero values for the rapidity $\xi \to \infty$.

Therefore it is usefull to consider sharp instead of a continuous momentum distribution, i.e. momentum eigenstates, beneath the fact that sharp momenta result in single Wigner rotations, since the Wigner rotation depends on the momentum p.

For (non-relativistic) entanglement theory in continuous variable (infinite dimensional) systems see e.g. Refs. [89][90] and References therein.

Using a "trick" by inserting the identity $\mathbb{I}$ and applying group composition rules we get

$$\begin{aligned} U(\Lambda) \mid p, s\rangle &= \underbrace{U(L(\Lambda p))\, U\left(L^{-1}(\Lambda p)\right)}_{U(\mathbb{I})} U(\Lambda L(p)) \mid k, s'\rangle \\ &= U(L(\Lambda p))\left[U\left(L^{-1}(\Lambda p)\, \Lambda L(p)\right)\right] \mid k, s'\rangle \\ &= U(L(\Lambda p))\, U(W(\Lambda, p)) \mid k, s'\rangle \end{aligned} \tag{179}$$

where

$$W(\Lambda, p) := L^{-1}(\Lambda p)\, \Lambda L(p) \tag{180}$$

is a product of Lorentz transformations, the so called *Wigner rotation,* which is a subgroup ((Wigner's) little group) of the homogenous Lorentz group, leaving the standard momentum k^μ invariant. $W(\Lambda, p)$ is a rotation because $L(p)$ takes k to p, Λ takes p to Λp and $L^{-1}(\Lambda p)$ takes Λp back to k.

The total transformed state can then be written as as linear combination of Lorentz-transformed momenta and Wigner-rotated spins, i.e.

$$U(\Lambda) \mid p, s\rangle = \sum_{s'} D_{s's}(W(\Lambda, p)) \mid \Lambda p, s'\rangle, \tag{181}$$

where $D_{s's}(W(\Lambda, p))$ is a suitable representation of the *little group.*
In case of spin-$\frac{1}{2}$ particles a suitable representation is the $SU(2)$ group. In our example discussed in chapter 4.2, spin-1 particles, this would be the unitary group $SO(3)$.

A construction of W[34], where $L(p)$ represents e.g. a boost rotating k^μ into p^μ towards the x-direction (e.g. for a particle with momentum $\vec{p}_0$ towards the

[34]See also example for spin-1/2-particles, [13], where the Wigner rotation was calculated explicitly from the Lorentz boost matrices, i.e. $W(\Lambda, p) := L^{-1}(\Lambda p)\, \Lambda L(p)$.

direction x) and a boost Λ in z-direction, representing the observer, reveals an axis of rotation $\vec{n}$ anti-parallel to $\vec{u} \times \vec{v}$, where $\vec{u}$ and $\vec{v}$ are the velocities of particle and observer, respectively, that is

$$\vec{n}_i \propto -\frac{\vec{u}_i \times \vec{v}}{\mid \vec{u}_i \times \vec{v} \mid}, \tag{182}$$

where $\vec{u}_i$ is the velocity of the i'th particle. The two Lorentz transformations are taken to be in perpendicular directions, since for perpendicular directions the Wigner rotation angle δ can be calculated (see Ref. [13]) according to

$$\tan \delta = \frac{\sinh \eta \sinh \xi}{\cosh \eta + \cosh \xi}, \tag{183}$$

with the rapidities η and ξ, as defined in (166), (167) and (168).

The Wigner rotation angle is greatest for perpendicular directions, see e.g. Ref. [18], and increasing with increasing velocities of particle and observer. the greatest value, $\delta = \frac{\pi}{2}$, is achieved for both particle and observer approaching the speed of light.

4.2 Relativistic Bell inequality

When studying the behaviour of entangled quantum states in a relativistic setting, then quite naturally the question occurs, if and how that might affect nonlocal effects, i.e. the violation of a Bell inequality (see chapter 2.3 for more details on nonlocality and the CHSH version of the Bell inequality).

There are results, which show diminished correlations or no correlations at all (see Ref. [12]) in the ultra relativistic limit for the joint spin measurement performed by Alice and Bob. This result suggests that there is no non-locality in the ultra relativistic limit.

The authors studied Lorentz-transformed Bell states and the *relativistic spin observable* related to the *Pauli-Ljubanski pseudo vector*[35] (see also Ref. [10] for more information on the relativistic spin observable), but kept measurement directions, which in the non-relativistic case would have lead to a maximum violation of the Bell inequality. When thinking about the effect the Wigner rotations have on the spin orientation (see chapter 4.1), the original measurement directions will lead to different expectation values and therefore to different results for the Bell inequality.

It was shown that the maximum violation of the Bell inequality (in the CHSH version) can be restored by a proper adjustment, i.e. a rotation, of the spin measurement directions (see Refs. [18][19]).

In this chapter, the main points are reviewed briefly.

4.2.1 The relativistic spin observable

The observable $\hat{a}$ for a spin measurement in the non-relativistic case along a direction $\vec{a}$ is

$$\hat{a} = \frac{\vec{a} \cdot \vec{\sigma}}{|\vec{a}|} \tag{184}$$

The author of Ref. [10] suggested that the correct spin observable in the relativistic case is related to

[35] Originally, it was shown by Fleming in 1965 (see Ref. [9]) that the covariant (relativistically invariant) spin operator can be derived from the Pauli Ljubanski pseudo vector. It reduces to the ordinary spin operator in the non-relativistic limit.

$$\vec{\sigma}_p = \frac{\vec{W}}{p^0} = \sqrt{1-\vec{v}^2}\vec{\sigma}_\perp + \vec{\sigma}_\parallel \tag{185}$$

where $\vec{W}$ is the spatial part of the *Pauli-Ljubanski vector*

$$W^\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} p^\sigma. \tag{186}$$

The square $W_\mu W^\mu$ is one of the *Casimir operators* of the *Poincaré group*, and therefore a *relativistic invariant*. The total angular momentum or spin is contained in $J^{\nu\rho}$, but see Ref. [18] and references therein for further details. The vector $\vec{\sigma}_p$ is the vector of Pauli matrices in a reference frame where the particle has momentum $p = (p^0, \vec{p})$ and $\vec{\sigma}_\perp$ and $\vec{\sigma}_\parallel$are components perpendicular and parallel, respectively, to the spatial part of p, i.e. $\vec{p}$, corresponding to the particle's moving direction.
When evaluating the spin along a direction $\vec{a}$, one has to use the normalized binary observable

$$\hat{a}(p) = \frac{\vec{a}\cdot\vec{\sigma}_p}{|\lambda(\vec{a}\cdot\vec{\sigma}_p)|}, \tag{187}$$

with λ being the eigenvalue of $\vec{a}\cdot\vec{\sigma}$.
Equation (187) can be re-expressed as

$$\hat{a}(p) = \vec{a}_p\cdot\vec{\sigma} = \frac{\left(L^{-1}(p)\,a\right)^i \sigma_i}{\left|\left(L^{-1}(p)\,a\right)^j\right|} \tag{188}$$

with $\vec{a}_p = \frac{\sqrt{1-\vec{v}^2}\vec{a}_\perp + \vec{a}_\parallel}{\sqrt{1+\vec{v}^2\left(\vec{a_\parallel}^2-1\right)}}$ being the direction $\vec{a}$ viewed from the rest frame S of the particle. $L^{-1}(p)\,a$ is the Lorentz transformed orientation four- vector a. $L(p)$ is the Lorentz boost taking the standard momentum k to p.

In the relativistic case we have to keep in mind that due to Lorentz contraction the different observers, e.g. observer S in the rest frame of the particle and a boosted observer in S', won't agree on directions.

From the perspective of frame S' the observable takes the form

$$\hat{a}' = \frac{\left(L^{-1}(p)\, a'\right)^{i} \sigma^{i}}{\left|\left(L^{-1}(p)\, a'\right)^{j}\right|}, \tag{189}$$

with $\hat{a}'$and a' being the corresponding observable and direction. For $a' = L(p)\, a$ we get the same observable as in (184), and therefore both observers will get the same results, i.e. inserting $a' = L(p)\, a$ into equation (189) yields

$$\frac{a^{i}\sigma_{i}}{|a^{j}|} = \hat{a}. \tag{190}$$

An observer S" moving in perpendicular direction to the particle momenta in S' will, with the corresponding Lorentz transformation Λ, obtain the momentum

$$\Lambda p = \Lambda L(p)\, k \tag{191}$$

and the direction

$$a'' = \Lambda a' = \Lambda L(p)\, a. \tag{192}$$

The correct observable giving the same results corresponding to the ones obtained in frames S and S' then is

$$\hat{a}'' = \frac{\left(L^{-1}(\Lambda p)\, a''\right)^{i} \sigma_{i}}{\left|\left(L^{-1}(\Lambda p)\, a''\right)^{j}\right|}. \tag{193}$$

By inserting (192) into (193) we recognize the familiar expression of the Wigner rotation,

$$\hat{a}'' = \frac{\left(W\left(\Lambda,p\right)a\right)^{i}\sigma_{i}}{\left|\left(W\left(\Lambda,p\right)a\right)^{j}\right|} = \left(R\left(W\left(\Lambda,p\right)\right)\frac{\vec{a}}{|\vec{a}|}\right)\cdot\vec{\sigma} = U\left(W\left(\Lambda,p\right)\right)\left[\frac{\vec{a}\cdot\vec{\sigma}}{|\vec{a}|}\right]U^{\dagger}\left(W\left(\Lambda,p\right)\right). \tag{194}$$

The equality $\left|\left(W\left(\Lambda,p\right)a\right)^{j}\right| = |\vec{a}|$ arises from the fact that W is an unitary transformation, i.e. rotation, leaving the norm of the spatial part of a invariant.

That $\hat{a}''$ is the same observable as $\hat{a}$ can be seen by comparing the expectation values in frames S" and S.

For the particle state $\mid k,\sigma\rangle$ in S the expectation value for observable (184) is given by

$$\langle k,\sigma \mid \frac{\vec{a}\cdot\vec{\sigma}}{|\vec{a}|} \mid k,\sigma\rangle = \langle\sigma \mid \frac{\vec{a}\cdot\vec{\sigma}}{|\vec{a}|} \mid \sigma\rangle. \tag{195}$$

As we know (see chapter 4.1), the state in S" is given by $U\left(W\left(\Lambda,p\right)\right)\mid\Lambda p,\sigma\rangle$. The expectation value in S" then is

$$\langle\Lambda p,\sigma \mid U^{\dagger}\left(W\left(\Lambda,p\right)\right)\hat{a}''U\left(W\left(\Lambda,p\right)\right)\mid\Lambda p,\sigma\rangle = \langle\sigma \mid \frac{\vec{a}\cdot\vec{\sigma}}{|\vec{a}|} \mid \sigma\rangle, \tag{196}$$

where we recover the same result as in S on the right hand side after inserting the r.h.s. of (194) into the l.h.s. of (196). The unitary transformations therein compensate each other.

We have shown that it is possible to obtain the same spin expectation values in every reference frame after choosing a measurement direction in the rest frame of the particle and appropriately transforming the measurement direction w.r.t. the considered observer.

4.2.2 Testing the CHSH inequality for two relativistic, massive spin-$\frac{1}{2}$ particles

Now, we are in a position where these results of chapters 4.1 and 4.2.1 can be applied to a set up of 2 spin-$\frac{1}{2}$ particles and Alice and Bob performing a joint spin measurement in order to test a Bell inequality.
Strictly speaking, we are evaluating the *Bell observable* (see equation (122)) in the CHSH inequality for 2 particles moving in the $\pm z$-direction w.r.t the rest frame S' of the source along the measurement directions $\vec{a}$,$\vec{\alpha}$,$\vec{b}$,$\vec{\beta}$:

$$S\left(\vec{a},\vec{\alpha},\vec{b},\vec{\beta}\right) = \left|E\left(\vec{a},\vec{b}\right) - E\left(\vec{a},\vec{\beta}\right)\right| + \left|E\left(\vec{\alpha},\vec{\beta}\right) + E\left(\vec{\alpha},\vec{b}\right)\right| \leq 2. \quad (197)$$

Alice and Bob are choosing their measurement directions in the x-y plane, which is perpendicular to the particle moving direction. The relativistic spin observable reduces to the non-relativistic spin observable (184), and there is no change in the directions[36], i.e.

$$a' = a = \begin{pmatrix} 0 \\ a_x \\ a_y \\ 0 \end{pmatrix}, \; b' = b = \begin{pmatrix} 0 \\ b_x \\ b_y \\ 0 \end{pmatrix}, \quad (198)$$

which corresponds to the "normal" situation and we get the familiar expectation value when assuming the particles to be in the state $\mid \Psi^-\rangle$, i.e

$$E\left(\vec{a},\vec{b}\right) = \langle \psi^- \mid \langle k \mid \hat{a} \otimes \hat{b} \mid k\rangle \mid \psi^-\rangle = -\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\left|\vec{b}\right|}. \quad (199)$$

In that way we obtain all expectation values necessary (see equation (197))in order to achieve
the maximal violation of (197) by the value $2\sqrt{2}$.

[36] The same is valid for measurement directions $\vec{\alpha}$ and $\vec{\beta}$, i.e. the four vectors α and β are equal to α' and β', respectively.

As has been demonstrated, it is possible to achieve maximal violation for a suitable choice of measurement directions in the x-y- plane.

Now, when considering the situation for the boosted frame S”, where Alice and Bob are moving in the x-direction w.r.t. the source, the state gets Wigner rotated. With the appropriate spin observable for this frame, see equation (194), it is straightforward to show that we get the same expectation value as in (199). That is, because the unitary transformations in state and observable compensate each other, see equation (196).
The maximal violation can be recovered in all frames once there is one frame where maximal violation can be achieved.

4.3 Relativistic transformation properties of three entangled spin-1-particles

Aim of the following chapter is the investigation of the relativistic transformation properties of a *massive spin-1 three-particle state*, where the momentum and spin degrees of freedom are being considered and modeled as *qutrit-system* consisting of 3 momentum and 3 spin qutrits. That means that 3 parties, i.e. Alice, Bob and Charlie, are evaluating the momenta and spins of the particles with 3 possible values for momentum and spin for each particle. Also, the authors of Ref. [21] discuss the situation for spin-1 particles, but for *two* particles and different states.

Our total Hilbert space is a composition of 6 subsystems, i.e.

$$\mathscr{H} = \mathscr{H}_p^A \otimes \mathscr{H}_s^A \otimes \mathscr{H}_p^B \otimes \mathscr{H}_s^B \otimes \mathscr{H}_p^C \otimes \mathscr{H}_s^C, \tag{200}$$

where p and s denote momentum and spin in Alice's, Bob's and Charlie's subsystem, respectively.

The spins of the particles are assigned values element of $\{0, 1, 2\}$, which correspond to three possible measurement outcomes, and it is assumed that the spin quantization axis is the $z - axis$.

Without loss of generality and in analogy to former works (see e.g. [19, 21]), we choose the initial tripartite spin-1 particle state $|\psi\rangle$ to be separable between momentum, $|\psi_p\rangle$, and spin, $|\psi_s\rangle$, degrees of freedom, i.e.

$$| \psi\rangle_{total} =| \psi_p\rangle\otimes | \psi_s\rangle. \tag{201}$$

Furthermore, we choose the momentum state to be *maximally entangled in three degrees of freedom,* as the greatest effect due to the boost is to be expected for maximally entangled momentum states[37] (see also Ref. [19]).

We choose the momentum degrees of freedom to be a *totally antisymmetric* state, which is also known as *Aharonov state*[38] (see e.g. Refs. [81][82][83]), i.e.

$$| \psi_p\rangle =$$

$$\frac{1}{\sqrt{6}}\left(| p_0p_1p_2\rangle- | p_1p_0p_2\rangle+ | p_2p_0p_1\rangle- | p_0p_2p_1\rangle+ | p_1p_2p_0\rangle- | p_2p_1p_0\rangle\right) \tag{202}$$

where

[37]In general, there has to be some entanglement in the initial momentum state to observe an increase in entanglement in the spin state after the boost. This becomes apparent when considering the form of the transformation, see equation (206). The effect becomes greater the more entanglement there is in the unboosted momentum state. Also, there can not be a further increase of entanglement of already maximally entangled initial spin states after the boost, due to saturation. In general, the greatest effect after the boost is to be expected for maximally entangled initial momentum states and separable initial spin states.

[38]The Aharonov state is *symmetric,* i.e. *invariant* under *local unitary operations*, i.e. the n-partite (pure) Aharonov state satisfies

$$| \psi\rangle_{Ah}^{\otimes n} = U^1\otimes U^2\otimes ...\otimes U^n \mid \psi\rangle_{Ah}^{\otimes n},$$

i.e. is invariant under *local unitaries* $U^1\otimes U^2\otimes ...\otimes U^n$, where each U^i is unitary and acts *locally* on the subspace i.

Other examples for (bipartite mixed) states, which are invariant under local unitaries are the isotropic state $\rho_\alpha = \alpha \mid \phi^+\rangle\langle\phi^+ \mid + \frac{1-\alpha}{d^2}\mathbb{I}$, $0\leq\alpha\leq 1$, $| \phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} | ii\rangle$,and the Werner state $\rho_W = \frac{\beta\left(\sum_{i,j=0}^{d-1}|ij\rangle\langle ji|\right)+\mathbb{I}}{d(d+\beta)}$, $-1\leq\beta\leq 1$. Same same can be extended for the multipartite case (for further information see e.g. Ref. [54]).

$$p_{0,1,2} = \begin{pmatrix} p^0 \\ \vec{p}_{0,1,2} \end{pmatrix} \qquad (203)$$

are the the momentum four-vectors of the three particles.

We are going to treat the momenta in the qutrit formalism (i.e. the momenta are sharp distributions in momentum space, so that they assume *discrete* values) to guarantee *single* resulting Wigner rotations acting on the spins, as was mentioned before. Only the direction of the momentum is taken into account and $\mid \vec{p}_0 \mid = \mid \vec{p}_1 \mid = \mid \vec{p}_2 \mid$ should be satisfied, whereas the velocity affects the Wigner angle δ in the resulting Wigner rotations (depending on $\vec{p}$).

The momentum state can then be written as

$$\mid \psi_p \rangle = \frac{1}{\sqrt{6}} \left(\mid 012 \rangle - \mid 102 \rangle + \mid 201 \rangle - \mid 021 \rangle + \mid 120 \rangle - \mid 210 \rangle \right), \qquad (204)$$

with the qutrit basis in $\mathscr{H}_p$ given in the standard basis

$$\mid 0 \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mid 1 \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mid 2 \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \qquad (205)$$

The relativistic transformation then entangles the spin and momentum degrees of freedom, i.e. the total state after the boost is given by

$$
\begin{aligned}
|\,\psi^{\Lambda}\rangle_{total} \;=\; \frac{1}{\sqrt{6}} \quad \Big(& \,|\,\Lambda p_0, \Lambda p_1, \Lambda p_2\rangle\,(U_0 \otimes U_1 \otimes U_2)\,|\,\psi_s\rangle \\
& + |\,\Lambda p_2, \Lambda p_0, \Lambda p_1\rangle\,(U_2 \otimes U_0 \otimes U_1)\,|\,\psi_s\rangle \\
& + |\,\Lambda p_1, \Lambda p_2, \Lambda p_0\rangle\,(U_1 \otimes U_2 \otimes U_0)\,|\,\psi_s\rangle \\
& - |\,\Lambda p_1, \Lambda p_0, \Lambda p_2\rangle\,(U_1 \otimes U_0 \otimes U_2)\,|\,\psi_s\rangle \\
& - |\,\Lambda p_0, \Lambda p_2, \Lambda p_1\rangle\,(U_0 \otimes U_2 \otimes U_1)\,|\,\psi_s\rangle \\
& - |\,\Lambda p_2, \Lambda p_1, \Lambda p_0\rangle\,(U_2 \otimes U_1 \otimes U_0)\,|\,\psi_s\rangle \Big) \quad ,
\end{aligned} \tag{206}
$$

where Λ represents the Lorentz transformation and $U_{0,1,2}$ are the Wigner rotations depending on the momenta $p_{0,1,2}$. That the transformed state in (206) is going to be entangled w.r.t. momentum and spin degrees of freedom is apparent, as, in general, (206) does not factorize w.r.t. momentum and spin. Also, since the operation acting on the spin state $|\,\psi_s\rangle$ can not be written as a single local unitary transformation acting on the subspaces (a local unitary transformation is of the form $U^1 \otimes U^2 \otimes U^3\,|\,\psi_s\rangle$ for tripartite spin states), the spin state is affected by the transformation.

We want to assume, that the particle motion (Alice, Bob, Charlie) is in the $x-y-plane$, that is, the angles between the particles' velocities $\vec{u}_{0,1,2}$ are equally amounting 120 degrees. Alice's particle is moving in $-x-direction$, Bob's and Chalie's particle's direction of motion is lying in the $x-y-plane$ separated by an angle of 120 degrees from the former, respectively (see Fig. 1).

A boosted observer, let's call him Rob, is moving perpendicular to the particles' plane of motion in the $z-direction$ with velocity $\vec{v}$. The Wigner rotation axes are then given by (182) . The axis of rotation e.g. for the spin of Alice's particle is the $y-direction$.

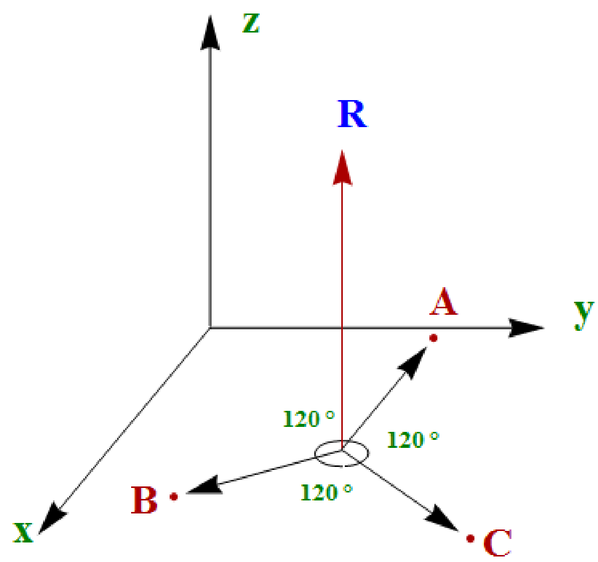

Figure 1: *Scheme of particle motion in Alice's, Bob's and Charlie's frame of reference and motion of the boosted observer Robert.*

The Wigner Rotations are given by

$$U_0 = \begin{pmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{pmatrix}, \tag{207}$$

$$U_1 = \begin{pmatrix} \frac{3}{4} + \frac{\cos\delta}{4} & -\frac{\sqrt{3}}{4} + \frac{\cos\delta}{4} & \frac{\sin\delta}{2} \\ -\frac{\sqrt{3}}{4} + \frac{\cos\delta}{4} & \frac{1}{4} + \frac{3\cos\delta}{4} & \frac{\sqrt{3}}{2}\sin\delta \\ -\frac{\sin\delta}{2} & -\frac{\sqrt{3}}{2}\sin\delta & \cos\delta \end{pmatrix}, \tag{208}$$

$$U_2 = \begin{pmatrix} \frac{3}{4} + \frac{\cos\delta}{4} & \frac{\sqrt{3}}{4} - \frac{\cos\delta}{4} & \frac{\sin\delta}{2} \\ \frac{\sqrt{3}}{4} - \frac{\cos\delta}{4} & \frac{1}{4} + \frac{3\cos\delta}{4} & -\frac{\sqrt{3}}{2}\sin\delta \\ -\frac{\sin\delta}{2} & \frac{\sqrt{3}}{2}\sin\delta & \cos\delta \end{pmatrix}. \tag{209}$$

That they have to be of the form (207) , (208) and (209) depending on the momenta p_0, p_1 and p_2, respectively, arises from the following considerations:

The three dimensional rotation matrix $R(\vec{n},\delta)$ w.r.t. an arbitrary axis of rotation in direction $\vec{n} = (n_1, n_2, n_3)$, fulfilling the normalization condition $|\vec{n}| = 1$, is given by

$$R(\vec{n},\delta) = \tag{210}$$

$$\begin{pmatrix} n_1^2(1-\cos\delta)+\cos\delta & n_1 n_2(1-\cos\delta) - n_3\sin\delta & n_1 n_3(1-\cos\delta) + n_2\sin\delta \\ n_2 n_1(1-\cos\delta) + n_3\sin\delta & n_2^2(1-\cos\delta)+\cos\delta & n_3 n_2(1-\cos\delta) - n_1\sin\delta \\ n_1 n_3(1-\cos\delta) - n_2\sin\delta & n_3 n_2(1-\cos\delta) + n_1\sin\delta & n_3^2(1-\cos\delta)+\cos\delta \end{pmatrix}.$$

The axes of rotation $\vec{n_i}$, $i = 0, 1, 2$, corresponding to the three particle's motion and the observer motion, is given by equation (182). Their explicit form is given by

$$\vec{n}_0 = (0, 1, 0)\,, \tag{211}$$

$$\vec{n}_1 = \left(\sin\left(-\frac{\pi}{3}\right), \cos\left(-\frac{\pi}{3}\right), 0\right), \tag{212}$$

$$\vec{n}_2 = \left(\sin\left(\frac{\pi}{3}\right), \cos\left(\frac{\pi}{3}\right), 0\right). \tag{213}$$

Inserting (211), (212) and (213) into (210) yields the Wigner rotation matrices (207) ,(208) and (209) , respectively.

The spin state is chosen such that with a 3-dimensional parametrization with spherical coordinates a broad class of states can be reached, up to the maximally entangled states. We want to assume that the spin state is in the totally antisymmetric state

$$\mid \psi_{s1}\rangle = \frac{\alpha}{\sqrt{2}}(\mid 012\rangle - \mid 102\rangle) + \frac{\beta}{\sqrt{2}}(\mid 201\rangle - \mid 021\rangle) + \frac{\gamma}{\sqrt{2}}(\mid 120\rangle - \mid 210\rangle)\,, \tag{214}$$

and the symmetric version thereof, i.e.

$$\mid \psi_{s2}\rangle = \frac{\alpha}{\sqrt{2}}(\mid 012\rangle + \mid 102\rangle) + \frac{\beta}{\sqrt{2}}(\mid 201\rangle + \mid 021\rangle) + \frac{\gamma}{\sqrt{2}}(\mid 120\rangle + \mid 210\rangle)\,, \tag{215}$$

where parameters α, β and γ are given as

$$\alpha = \cos\theta\cos\phi, \tag{216}$$

$$\beta = \cos\theta\sin\phi, \tag{217}$$

$$\gamma = \sin\theta. \tag{218}$$

With this parametrization we can visualize the amount of entanglement changes in entanglement after the boost in 3-dimensional space, as illustrated by graphics later in the chapter.

The density matrices of the total states are given by

$$\rho = \mid \psi\rangle\langle\psi \mid, \tag{219}$$

where the state $\mid \psi\rangle$ is the state in (201), i.e. the momentum state in (204) and either the antisymmetric spin state in (214) or the symmetric spin state in (215).

The density matrices of the *boosted* total states are given by

$$\rho^{\Lambda} = \mid \psi^{\Lambda}\rangle\langle\psi^{\Lambda} \mid, \tag{220}$$

where $\mid \psi^{\Lambda}\rangle$ is the boosted state vector defined in (206) with spin states either antisymmetric (214) or symmetric(215) , again.

4.3.1 Amount of entanglement in different partitions

The first part of our considerations is the evaluation of the total amount of entanglement contained the system w.r.t. certain partitions of the total Hilbert space, as has been done in Refs. [19][21], that is the partition w.r.t.

I) one subsystem versus the remaining 5 subsystems (i.e. Alice's momentum versus the other subsystems, and then Alice's spin versus the other remaining subsystems and so forth.)

II) the momentum versus the spin partition.

III) the particle partitions, i.e. the bipartitions A versus B and C, B versus A and C, and C versus A and B.

All this partitions are obtained by tracing out the respective other degrees of freedom (see also equation (26)).

A suitable entanglement measure E is the linear entropy (see (28)), as we are dealing with pure states, i.e.

$$E = \frac{d}{d-1} \sum_i 1 - Tr\left(\rho_i^2\right), \tag{221}$$

with ρ_i being the reduced density matrix obtained from the initial state (219) or boosted state (220) by tracing over all subsystems except the i'th.

The 1 vs. 5 qutrit-partition

Figure 2 shows the total amount of entanglement present in the unboosted state,

$$E(\rho). \tag{222}$$

Thereby, it makes no difference, whether the antisymmetric or symmetric momentum and spin states (or different combinations of symmetric or antisymmetric for spin and momentum) are chosen, the amount of entanglement is distributed in dependence of parameters θ and ϕ according to Figure 2. The maximal entanglement scales to 6 due to the chosen parameter $\frac{d}{d-1}$ defined in (28). Also, the amount of entanglement is never zero due to the chosen maximally entangled momentum state. The global maxima correspond to the maximally entangled spin state, whereas the global minima do not correspond to the fully separable spin state, as, due to the chosen superposition and parametrization (see (214)and (215)) we are not able to reach full separability.

As can be seen in Figure 2, the entanglement maxima correspond to the maximally entangled initial spin state where the parameters are equal to $\phi \in \left\{\frac{\pi}{4} + n\frac{\pi}{2}\right\}$and $\theta \in \left\{\frac{1}{5}\pi, \frac{4}{5}\pi, \frac{6}{5}\pi, \frac{9}{5}\pi\right\}$. The entanglement minima correspond to parameters equal to $\phi \in [0, 2\pi]$and $\theta \in \left\{\frac{\pi}{2} + n\pi\right\}$ and also for parameters $\phi \in \left\{\frac{n\pi}{2}\right\}$ and $\theta = n\pi$.

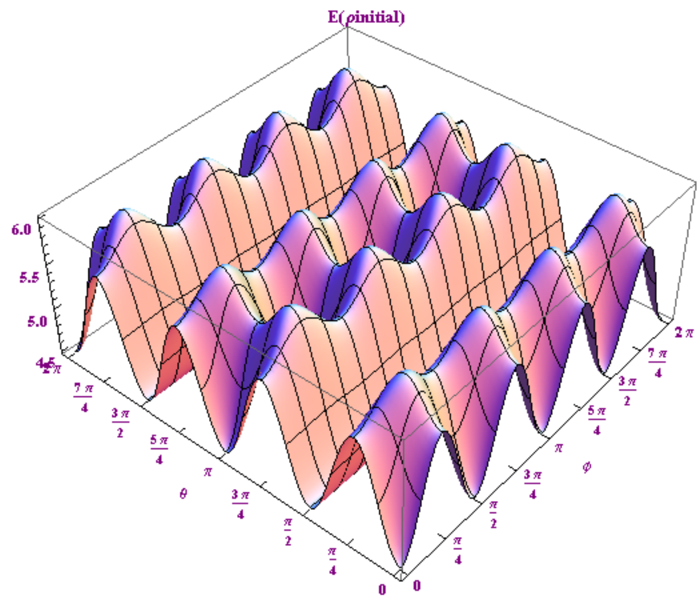

Figure 2: *Entanglement of the unboosted (antisymmetric and symmetric) state in the 1 vs. 5 partition.*

When considering the boosted reference frame of the relativistic observer Rob, we have to calculate the linear entropy for the boosted state, which is

$$E\left(\rho^{\Lambda}\right). \tag{223}$$

There the amount of entanglement changes in dependence of the Wigner angle $\delta \in \left[0, \frac{\pi}{2}\right]$, where $\frac{\pi}{2}$ corresponds to the limit of the speed of light.

Now, we can calculate the difference in entanglement, i.e.

$$E\left(\rho^{\Lambda}\right) - E\left(\rho\right), \tag{224}$$

which is illustrated for the Wigner angles $\delta = \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}$ in the plot in Figure 3. Also here, it makes no difference, whether the symmetric or antisymmetric states are chosen for momentum and spin.

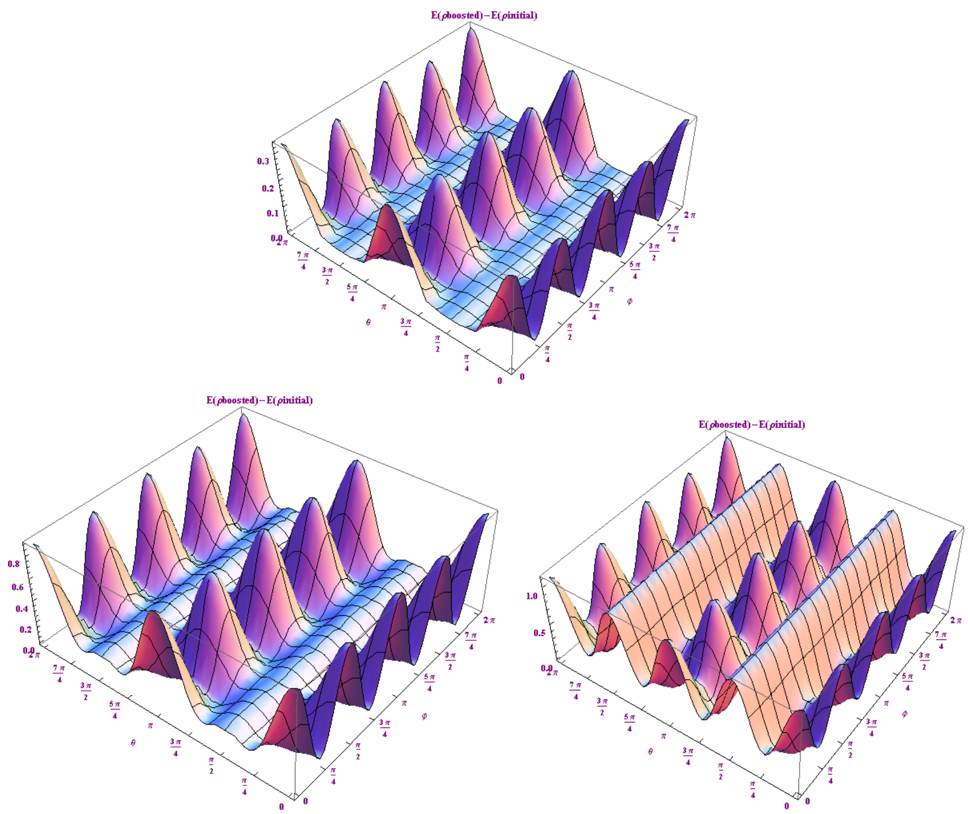

Figure 3: *Change in Entanglement for Wigner angles* $\delta = \frac{\pi}{8}$ *(above),* $\delta = \frac{\pi}{4}$*(below,left),* $\delta = \frac{\pi}{2}$ *(below, right) for the antisymmetric and symmetric states.*

For the formerly maximally entangled spin states we observe no change in entanglement, which was to be expected and shown in previous works. That is, because the amount of entanglement can not be further increased for states, which are allready maximally entangled. The form of the states changes after the boost, but the total amount of entanglement is saturated when initially chosen to be maximally entangled (see also Refs. [19][21]).

The former minima (of the unboosted state) show a different behaviour in dependence of the Wigner angle δ, i.e. the minimally entangled states corresponding to parameters $\phi \in \left\{\frac{n\pi}{2}\right\}$ and $\theta = \{n\pi\}$, i.e. the states

$$|\,\psi_{s1}\rangle = \pm\frac{1}{\sqrt{2}}\left(|\,201\rangle - |\,021\rangle\right), \tag{225}$$

$$|\,\psi_{s2}\rangle = \pm\frac{1}{\sqrt{2}}\left(|\,201\rangle + |\,021\rangle\right), \tag{226}$$

$$|\,\psi_{s1}\rangle = \pm\frac{1}{\sqrt{2}}\left(|\,012\rangle - |\,102\rangle\right), \tag{227}$$

$$|\,\psi_{s2}\rangle = \pm\frac{1}{\sqrt{2}}\left(|\,012\rangle + |\,102\rangle\right), \tag{228}$$

show the maximal change in entanglement, whereas the entanglement change of states corresponding to parameters $\phi \in [0, 2\pi]$ and $\theta \in \left\{\frac{\pi}{2} + n\pi\right\}$, i.e. the states

$$|\,\psi_{s1}\rangle = \pm\frac{1}{\sqrt{2}}\left(|\,120\rangle - |\,210\rangle\right), \tag{229}$$

$$|\,\psi_{s2}\rangle = \pm\frac{1}{\sqrt{2}}\left(|\,120\rangle + |\,210\rangle\right), \tag{230}$$

show slower increase with increasing δ and coincide with the behaviour of (225) and (226) only in the limit $\delta \to \frac{\pi}{2}$.

The momentum-spin partition

In the *momentum-spin partition*, which we obtain by tracing over the spin or momentum degrees of freedom, respectively, i.e.

$$\begin{aligned}
\rho_{mom}^{ABC} &= Tr_{spin}^{ABC}(\rho)\,, \quad \rho_{mom}^{\Lambda,\,ABC} = Tr_{spin}^{\Lambda,\,ABC}\left(\rho^{\Lambda}\right), \\
\rho_{spin}^{ABC} &= Tr_{mom}^{ABC}(\rho)\,, \quad \rho_{spin}^{\Lambda,\,ABC} = Tr_{mom}^{\Lambda,\,ABC}\left(\rho^{\Lambda}\right),
\end{aligned} \tag{231}$$

in the initial and boosted density matrix of the total state, the results are the following:

The entanglement w.r.t. the momentum-spin partition is in the unboosted frame by construction (see (201)) zero, as we chose the momentum and spin state to be separable w.r.t. each other. The relativistic transformation entangles the momentum and spin degrees of freedom. The entanglement after the boost, i.e.

$$E\left(\rho_{boosted}\right) - E\left(\rho_{initial}\right) = E\left(\rho_{boosted}\right) - 0 = E\left(\rho_{boosted}\right) \tag{232}$$

is illustrated in Figures 4, 5 and 6 for Wigner angles $\delta = \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}$, respectively, and the symmetric and antisymmetric states.

As can be seen, this partition is not Lorentz invariant, as the amount of entanglement, which is zero in the unboosted frame, depends on the parameters θ, ϕ and the Wigner angle δ in the boosted frame, as has been shown in Refs. [19][21].

Also, the symmetric and antisymmetric states behave differently w.r.t. this partition, as is also the case for spin-$\frac{1}{2}$-particles, i.e. for the symmetric and antisymmetric Bell states in Ref.[19]. Also, it does not make any difference, whether the antisymmetric or symmetric momentum state is used, the effects on the symmetric spin state are shown in the left pictures of Figures 4, 5 and 6. The same is true for the antisymmetric spin state, whose behaviour is independent of the chosen momentum state, whether be it symmetric or antisymmetric.

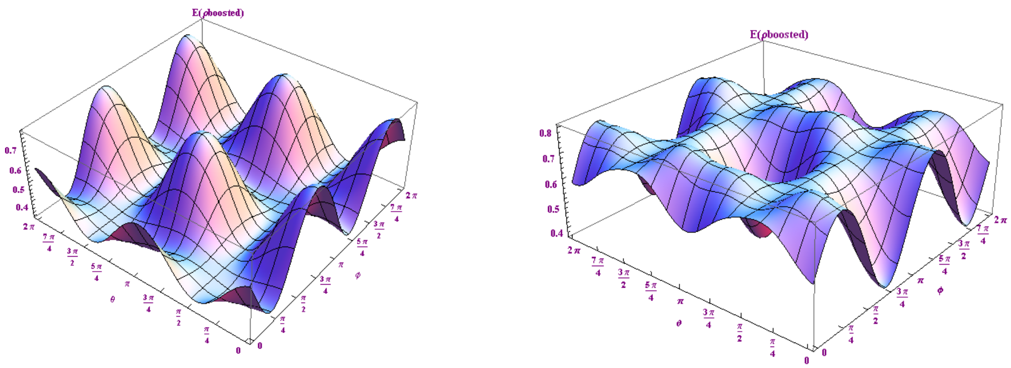

Figure 4: *Entanglement in boosted state w.r.t. momentum-spin partition and for Wigner angle* $\delta = \frac{\pi}{8}$. *Left: Symmetric state. Right: Antisymmetric state.*

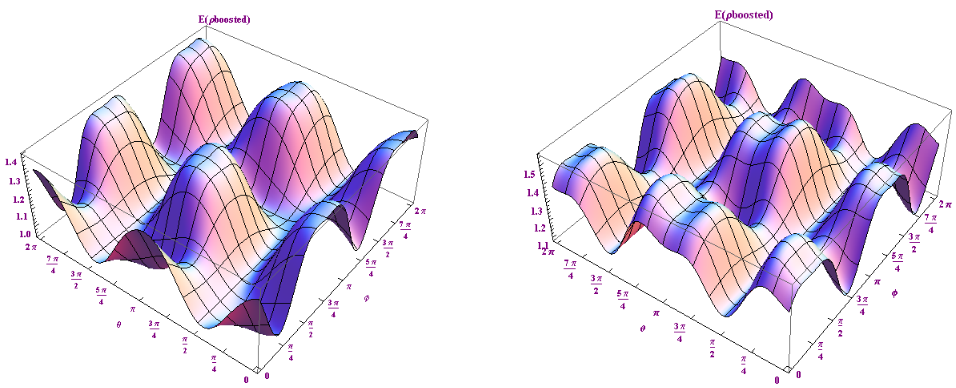

Figure 5: *Entanglement in boosted state w.r.t. momentum-spin partition and for Wigner angle* $\delta = \frac{\pi}{4}$. *Left: Symmetric state. Right: Antisymmetric state.*

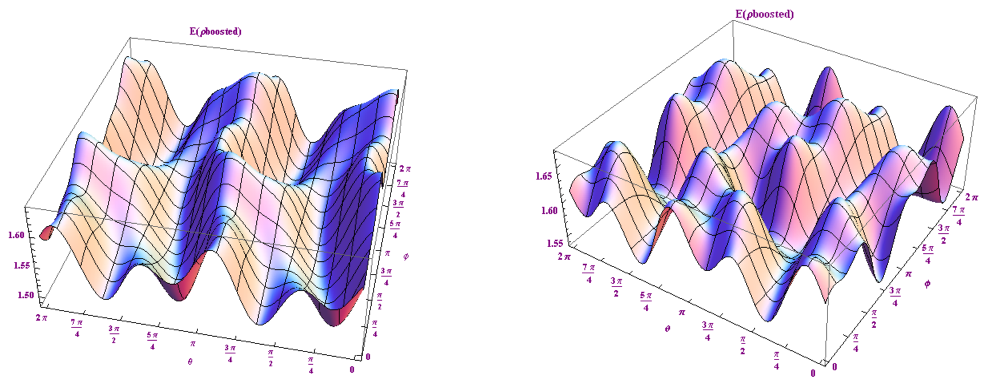

Figure 6: *Entanglement in boosted state w.r.t. momentum-spin partition and for Wigner angle* $\delta = \frac{\pi}{2}$*. Left: Symmetric state. Right: Antisymmetric state.*

The particle partition

The last partition we are considering is the *particle-partition*. Here the reduced density matrices are obtained as before by tracing over the respective other degrees of freedom, i.e. Alice's particle density matrix is obtained by

$$\rho^{A}_{mom-spin} = Tr^{BC}_{mom-spin}\left(\rho\right), \tag{233}$$

and Bob's and Charlie's particle density matrix is obtained analogically by tracing over the degrees of freedom of the respective other particles.

In this partition entanglement is *Lorentz invariant*, in accordance to the results of previous works, see e.g. Ref. [13], i.e. the entanglement in the unboosted frame equals the entanglement in the boosted frame,

$$E\left(\rho\right) = E\left(\rho^{\Lambda}\right), \tag{234}$$

regardless of the state and Wigner angle δ.

That is, because the relativistic transformation $U\left(\Lambda\right)$ is a unitary transformation, strictly speaking local unitary w.r.t. to the particle subsystems of the total Hilbert space. The entanglement measure[39], i.e. the linear entropy, is invariant under local unitaries, that is

$$E\left(\rho^{ABC}\right) = E\left(U^{A}\left(\Lambda\right) \otimes U^{B}\left(\Lambda\right) \otimes U^{C}\left(\Lambda\right) \rho^{ABC} U^{A\dagger}\left(\Lambda\right) \otimes U^{B\dagger}\left(\Lambda\right) \otimes U^{C\dagger}\left(\Lambda\right)\right), \tag{235}$$

when e.g. considering a tripartite state ρ^{ABC}.

Physically, it means that $U\left(\Lambda\right)$ has to be a local unitary transformation in order to satisfy the principle of special relativity, i.e. local measurement on ρ^{A}, ρ^{B} and ρ^{C} are independent of the observer.

[39] An proper entanglement measure has to fulfill certain requirements, one of which is that it has to be invariant under local unitaries, or more strictly, does not increase under LOCC (local operations and classical communication). The linear entropy is a proper entanglement measure. For more information consult e.g. Ref. [88].

4.3.2 Detection of multipartite entanglement in different frames

In this section we are going to test the detection quality of quantity Q_0 defined in chapter 2.2.5 (see Def. 2.43) for different observers, i.e. different reference frames.

We are going to use the states defined in chapter 4.3 (see (204) ,(214) and (215)) and the same relativistic transformation (see (206) ,(207) ,(208) and (209)). We are investigating the effects the relativistic transformation has on the spin states, i.e. Q_0 is applied to the unboosted and boosted spin density matrix, after tracing out the momentum degrees of freedom.

In order to obtain optimal results, i.e. a maximal violation of inequality $Q_0 \leq 0$, which indicates genuine multipartite entanglement, we can perform an optimization presented in chapter 2.2.7 for the used states. We are going to employ the parametrization introduced for the special unitary group $SU(d)$, in our case $SU(3)$, and set the parameters corresponding to the global phases, i.e. diagonal elements of (106), to zero. The global phases are physically irrelevant and cancel out when density matrices instead of state vectors are considered. The parameter $\lambda_{3,3}$ is zero, by construction, for the parametrization of the special unitary group.

Also, the parameters corresponding to relative phase shifts, i.e. $\lambda_{2,1}$, $\lambda_{3,1}$, $\lambda_{3,2}$ are going to be set to zero, which can be done, because they are redundant in this case (without proof). This is an advantage of the composite parametrization.

We are left with 3 parameters

$$\lambda_{1,2}, \lambda_{1,3}, \lambda_{2,3} \tag{236}$$

right of the diagonal, corresponding to rotations, which are going to be used for the optimization of Q_0.

When optimizing e.g. the initial spin state introduced in (214), the total transformation U_C for the state is of the form

$$U_C = U_C^1\left(\lambda_{i,j}^1\right) \otimes U_C^2\left(\lambda_{i,j}^2\right) \otimes U_C^3\left(\lambda_{i,j}^3\right), \tag{237}$$

corresponding to 3 involved parties. In matrix notation introduced in equation (106), U_C reads

$$U_C = \begin{pmatrix} 0 & \lambda^1_{1,2} & \lambda^1_{1,3} \\ 0 & 0 & \lambda^1_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \lambda^2_{1,2} & \lambda^2_{1,3} \\ 0 & 0 & \lambda^2_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \lambda^3_{1,2} & \lambda^3_{1,3} \\ 0 & 0 & \lambda^3_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \quad (238)$$

The parameters $\lambda^{1,2,3}_{i,j}$ are the parameters, which induce a rotation in the subspace $\mid i\rangle$ and $\mid j\rangle$.

The unboosted spin state density matrices, i.e.

$$\rho_{s1} =\mid \psi_{s1}\rangle\langle\psi_{s1} \mid, \quad (239)$$

and

$$\rho_{s2} =\mid \psi_{s2}\rangle\langle\psi_{s2} \mid \quad (240)$$

are transformed according to

$$U_C \rho_{s1} U_C^\dagger \quad (241)$$

and

$$U_C \rho_{s2} U_C^\dagger, \quad (242)$$

which are used for evaluation of Q_0, after numerically optimizing the parameters $\lambda^{1,2,3}_{i,j}$.

The boosted spin state density matrices are obtained according to equation (231) , then Q_0 is optimized with help of U_C.

Figure 7 illustrates the detection quality of genuine multipartite entanglement by Q_0 in the initial symmetric (240) and antisymmetric (239) spin states. A value $Q_0 > 0$ is an *necessary* criterion for the detection of genuine multipartite entanglement (GME) and is obtained for all parameters $\{\phi, \theta\}$ when optimized for all $\{\phi, \theta\}$, since *all* regions $\{\phi, \theta\}$ are indicated to be *greater or equal to zero*, i.e. are *blue* in Figure 7. This means, that every state parametrized by $\{\phi, \theta\}$ can be detected to be GME-entangled when optimized over U_C. The right hand side of Figure 7 shows values $Q_0 > 0$ for the same states as before, but in comparison, without an optimization. The inequality is satisfied, i.e. $Q_0 \leq 0$ is valid for all parameters $\{\phi, \theta\}$, i.e. GME detection fails. This also shows that in this case an optimization is necessary in order to detect GME successfully.

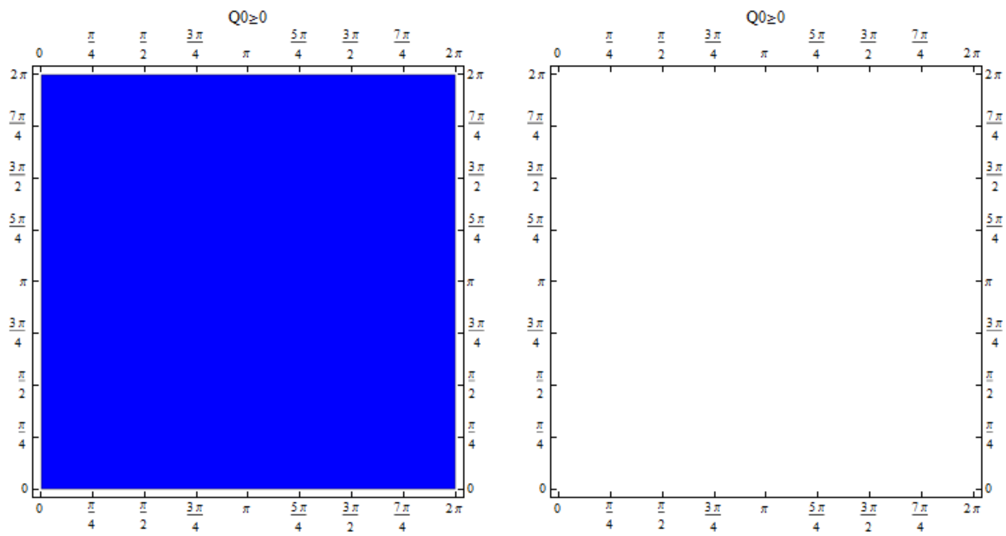

Figure 7: *Detection of genuine multipartite entanglement, i.e.* $Q_0 \geq 0$. ***Left:*** *Symmetric and antisymmetric* ***unboosted*** *spin state,* ***optimized;*** ***Right:*** *Symmetric and antisymmetric* ***unboosted*** *spin state,* ***nonoptimized****.*

When evaluating Q_0 in the boosted frame, we have to trace out the momentum degrees of freedom from the density matrix describing the total state (see (231)) and proceed as described for the unboosted spin state above.

Figure 8 (left hand side) shows the violation of inequality Q_0, i.e. the values $Q_0 > 0$, which is achieved for all parameters $\{\phi, \theta\}$ and for all Wigner angles δ, when optimized over U_C. Without optimization, which is illustrated by the right hand side of Figure 8, the criterion fails for all Wigner angles δ for all $\{\phi, \theta\}$, i.e. the values Q_0 are below zero, i.e. not indicated by the plot. Without an optimization of the measurement directions, the detection of GME-entanglement is not possible in boosted frames for all velocities, i.e. all Wigner angles δ.

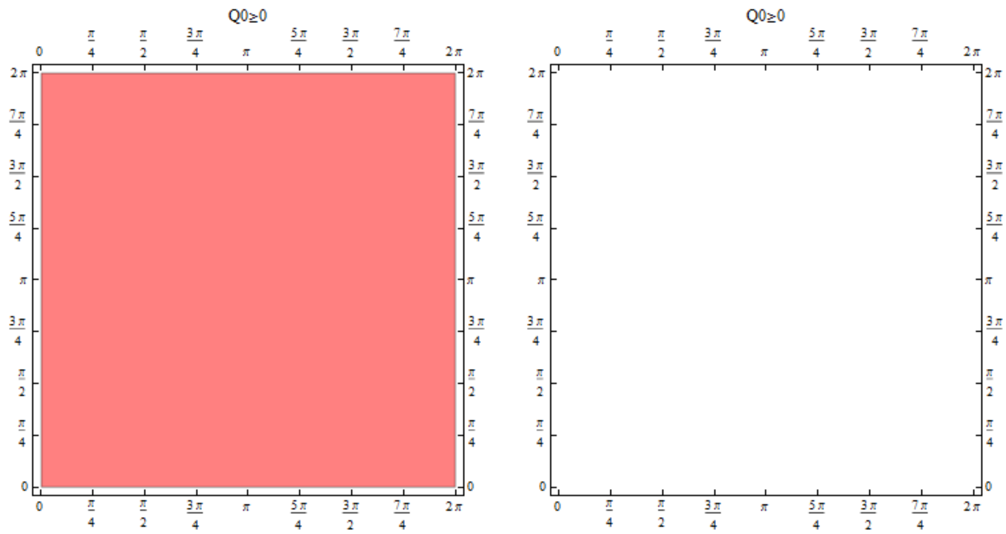

Figure 8: *Detection of genuine multipartite entanglement in* ***boosted*** *reference frame, i.e.* $Q_0 \geq 0$. ***Left****: Symmetric and antisymmetric boosted spin state,* ***optimized****;* ***Right****: Symmetric and antisymmetric boosted spin state,* ***nonoptimized****. (Valid for all Wigner angles* δ*!)*

All boosted states parametrized by $\{\phi, \theta\}$ are detected to be genuinely multipartite entangled, i.e. $Q_0 > 0$, when optimized accordingly. That means that the expectation values, or matrix elements used for the construction of Q_0 are invariant quantities for all inertial observers, when the basis elements or physically speaking, measurements directions, obtained and optimized by U_C are chosen appropriately.

This is in accordance with the result for the relativistic Bell inequalities discussed in chapter 4.2, that is, invariant quantities, i.e. expectation values, were obtained by appropriately Wigner-rotating the spin measurement directions.

The parameters $\lambda_{i,j}$ chosen in U_C correspond to rotations in subspace $\mid i\rangle$ and $\mid j\rangle$ and leave invariant, when optimized, the expectation values required to evaluate Q_0 for all Wigner angles δ, i.e. for all velocities and inertial observers.

Conclusion

This thesis is concerned with relativistic entanglement theory in *inertial frames*, where the number of particles is constant and can be treated *without* all the machinery within *quantum field theory*.

We have studied the behaviour of relativistic entangled particles, especially spin-1 particles, in inertial frames of reference within the *Wigner rotation* formalism.

To do so, we first presented and defined non-relativistic entanglement theory, second the basics of special relativity and the relativistic transformation properties of entangled states.

The main results for 3 relativistic spin-1 particles recover those of spin-1/2 particles studied before (see. e.g. [19]), meaning that their qualitative behaviour under a change of reference frame reflects those of entangled relativistic spin-1/2 particles.

Entanglement is partition dependent, i.e. only Lorentz invariant in the particle-partition of Hilbert space, whereas the momentum-spin-partition bears some surprises. There the amount of entanglement is clearly depending on the initial state and the velocity of the particles, i.e. the Wigner rotation angle. Also, symmetric and antisymmetric initial spin states behave differently in boosted frames.

When considering the relativistic behaviour of multipartite entanglement, i.e. the detection quality of certain entanglement criteria, we find that the latter, without optimization of the measurement directions, is velocity dependent, i.e. also depends on the Wigner rotation angle. When optimizing the measurement directions, the criterion works for all velocities and all states (considered in this thesis), i.e. detects genuine multipartite entanglement. This result is in accordance with the one obtained for the relativistic Bell inequalities, where (maximal) violation can be reestablished when appropriately rotating the measurement directions.

Outlook

In recent years also *non-inertial frames*, i.e. *accelerated frames*, *curved space-times* and *gravitational effects* were considered (see Refs. [69]-[80] and References therein) as well, expanding the field tremendously.

For the sake of completeness it should be mentioned that there is an ongoing debate about the physical interpretation of the Wigner rotation formalism used in this work and many other similar works (see Ref.[22] and References therein for further information). The authors point out that there might be conceptual problems resulting from the formalism, when considering spin measurements in measuring devices, as the e.g. the Stern-Gerlach-apparatus, when the spin couples to the electromagnetic field. There the conceptual problems arise from the fact that the mostly used relativistic spin observable related to the Pauli-Lubanski vector (see Ref. [10] for further information on the relativistic spin observable) transforms differently than the quantity the spin couples to. This quantity is part of a four-vector.

It should be mentioned as well, that it is not clear, whether and how the spin couples to the electromagnetic field.

Also, they point out that, as a consequence of the Wigner rotation formalism, the spin quantization axes depend on the particle momentum. The spin of relativistic particles *is* momentum dependent (see chapter 4 of this thesis). So, it might lead to paradoxes to consider momentum-spin partitions of the Hilbert space in the relativistic case.

This also shows that the relatively young field of relativistic quantum information theory is far from being complete and that there is a lot of potential for further investigation and discussion.

Acknowledgments

I would like to thank my colleagues from the University of Vienna I studied with during the last years, especially the (former) members of the quantum particle group, who supported me during the writing process: Christoph Spengler, Andreas Gabriel, Hansi Schimpf, Gerd Krizek, Marcus Huber, Theodor Adaktylos, Paul Erker, Philipp Köhler, Karoline Mühlbacher and Heidi Waldner.

Thank you for interesting and the fun times at conferences as well!

Thanks to Nicolai Friis for his extraordinary diploma thesis and valuable remarks.

Thanks to my supervisor Beatrix C. Hiesmayr, for the interesting topic, for her patience, support and for the opportunity to gather a little bit of experience in the world of research.

I am grateful to my mother, Ljiljana Medic, who supported me in every possible way during my studies and never stopped believing in me during difficult times.

Last, but not least, I would like to thank Michael Usleber, for proofreading, the emotional support and encouragement, and for being the person who you are.

List of Figures

Curriculum vitae

Name: Irena Bojic

Geburtsdatum: 3.4.1985

Geburtsort: Brcko, Bosnien u. Herzegowina

Wohnort: Wien

Staatsbürgerschaft: Österreich

Ausbildung

seit WS 2004 Studium der Physik, Universität Wien

2001-2004 Bundesgymnasium Wien 1230, Anton Baumgartnerstraße 123

1996-2001 Bundesgymnasium Nonntal, Salzburg

1992-1996 Volksschule Parsch, Salzburg

Konferenzen und Workshops

Juni 2011 Teilnahme: 8th CEQIP 2011, Central European Quantum Information Processing Workshop, Znojmo, Tschechische Republik

September 2011 Teilnahme: Relativity meets Entanglement and High Energy Workshop, London, GB

Juni 2012 Teilnahme: 9th CEQIP 2012, Central European Quantum Information Processing Workshop, Smolenice, Slowakische Republik

November 2012 Teilnahme und Mithilfe: Theory Meets Experiment: Finding Proposals Towards Testing Foundational Issues in Particle Physics, Workshop, Wien

Besondere Kenntnisse

Sprachen Serbokroatisch (Muttersprache), Deutsch (fließend in Wort und Schrift), Englisch (fließend in Wort und Schrift), Spanisch (Grundkenntnisse)

EDV Erfahrung mit LaTeX, LyX, Mathematica

References

[1] E. Schrödinger, *Die gegenwärtige Situation in der Quantenmechanik, Naturwissenschaften*, **23**, 807, 823, 844, (1935)

[2] A. Einstein, B. Podolsky, N. Rosen, *Can quantum mechanical description of physical reality be considered complete?,* Phys. Rev., **47**, 777 (1935)

[3] J. S. Bell, *On the Einstein Podolsky Rosen Paradox*, Physics Vol. 1, **No. 3**, 195-200, (1964)

[4] J. Audretsch, *Entangled Systems: New Directions in Quantum Physics*, textbook, WILEY-VCH (2007)

[5] M. A. Nielsen, I. L. Chuang, *Quantum Computation and Quantum Information*, textbook, Cambridge University Press (2000)

[6] A. Peres, *Quantum Theory: Concepts and Methods*, textbook, Kluwer Academic Publishers (1995)

[7] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, *Quantum Entanglement*, Rev. Mod. Phys. **81**, 865-942, (2009)

[8] E. P. Wigner, *On Unitary Representations of the Inhomogeneous Lorentz Group,* Ann. Math. **40**, No 1, 149-204, (1939)

[9] G. N. Fleming, Phys. Rev. **137**, B188, (1965)

[10] M. Czachor, *Einstein-Podolsky-Rosen-Bohm experiment with relativistic massive particles*, Phys. Rev. A **55**, 72, (1997)

[11] A. Peres, P. F. Scudo, D. R. Terno, *Quantum Entropy and Special Relativity*, Phys. Rev. Lett. **88**, 230402, (2002)

[12] D. Ahn, H. Lee, S. W. Hwang, *Relativistic entanglement of quantum states and nonlocality of Einstein-Podolsky-Rosen (EPR) paradox,* arXiv:quant-ph/0207018v2 (Aug 2002)

[13] P.M. Alsing, G.J. Milburn, *On Entanglement and Lorentz Transformations*, Quant. Inf. Comp. **2**, No. 6 (2002) 487-512; same paper under the different title *Lorentz Invariance of Entanglement*, arXiv:quant-ph/0203051v1 (2002)

[14] R.M. Gingrich, Ch. Adami, *Quantum Entanglement of Moving Bodies*, Phys. Rev. Lett. **89**, 2704002, (2002)

[15] A. Peres, D. R. Terno, *Quantum information and relativity theory*, Rev. Mod. Phys. **76**, No. 1, 93-123, (2004)

[16] F. F. Jordan, A. Shaji, E. C. G. Sudarshan, *Lorentz transformations that entangle spins and entangle momenta*, Phys. Rev. A **75**, 022101, (2007)

[17] A. Chakrabarti, *Entangled states, Lorentz transformations and spin precession in magnetic fields*, J. Phys. A: Math. Theor. **42**, 245205, (2009)

[18] Nicolai Friis, *Relativistic Effects in Quantum Entanglement,* Diploma thesis (Feb 2010), arXiv:1003.1874v1 [quant-ph]

[19] N. Friis, R. A. Bertlmann, M. Huber, B. C. Hiesmayr, *Relativistic entanglement of two massive particles*, Phys. Rev. A **81**, 042114, (2010)

[20] M. Huber, N. Friis, A. Gabriel, C. Spengler, B. C. Hiesmayr, *Lorentz invariance of entanglement classes in multipartite systems*, EPL 95, Number 2, 20002, (2011)

[21] E. Castro-Ruiz, E. Nahmad-Achar, *Entanglement properties of a two spin-one particle system under a Lorentz transformation*, arXiv:1210.1889v1 [quant-ph] , (2012)

[22] P. L. Saldanha, V. Vedral, *Physical interpretation of the Wigner rotations and its implications for relativistic quantum information*, New J. Phys. **14**, 023041, (2012)

[23] S. Weinberg, *The Quantum Theory of Fields* Vol. 1, Cambridge University Press, (1995)

[24] R. Werner, *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys. Rev. A, **40**, 4277, (1989)

[25] M. Horodecki, P. Horodecki, R. Horodecki, *Inseparable two spin-1/2 density matrices can be distilled to a singlet form*, Phys. Rev. Lett., **78**, 574, (1997)

[26] V. Vedral, M. B. Plenio, *Entanglement measures and purification procedures*, Phys. Rev. A, **57**, 1619, (1998)

[27] A. Peres, *Separability criterion for density matrices*, Phys. Rev. Lett. **77**, 1413, (1996)

[28] M. Horodecki, P. Horodecki, R. Horodecki, *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett. A, **223**, 1, (1996)

[29] A. Gabriel, B. C. Hiesmayr, M. Huber, *Criterion for k-separability in mixed multipartite states*, Quant. Inf. Comp., **10**, 829, (2010)

[30] N. M. J. Woodhouse, *Special Relativity*, Springer (2008)

[31] H. K. Urbantke, R. U. Sexl, *Relativity, Groups, Particles: Special Relativity and Relativistic Symmetry in Field and Particle Physics,* Springer (2008)

[32] J. M. Heinzle, Introduction to Relativity and Cosmology I, lecture notes, http://gravity.univie.ac.at/fileadmin/GP/webdesign/Neuer_Ordner/RThI_Version8.pdf

[33] D. M. Greenberger, M. A. Horne, A. Shimony, A. Zeilinger, *Bell's theorem without inequalities*, Am. J. Phys. **58**, 1131–1143, (1990)

[34] E. Schmidt, *Zur Theorie der linearen und nichtlinearen Integralgleichungen*, Math. Ann. 63, 433. (1906)

[35] B. M. Terhal, P. Horodecki, *A Schmidt number for density matrices*, Phys. Rev. A Rapid Communications **61**, 040301 (2000)

[36] C. E. Shannon, *A Mathematical Theory of Communication*, Bell Systems Technical Journal **27** , 379–423, 623-656, (1948)

[37] W. K. Wootters, *Entanglement of Formation of an Arbitrary State of Two Qubits*, Phys. Rev. Lett. **80**, No. 10, 2245-2248, (1998)

[38] C. H. Bennett, D. P. Vincenzo, J. A. Smolin, W. K. Wootters, *Mixed-state entanglement and quantum error correction*, Phys. Rev. A **54**, No. 5, 3824-3851, (1996)

[39] S. Hill, W. K. Wootters, *Entanglement of a Pair of Quantum Bits*, Phys. Rev. Lett. **78**, 5022-5025, (1997)

[40] B. C. Hiesmayr, M. Huber, *Multipartite entanglement measure for all discrete systems*, Phys. Lett. A **372**, 3608 (2008)

[41] B. M. Terhal, K. G. H. Vollbrecht, *Entanglement of formation for isotropic states*, Phys. Rev. Lett. **85**, 2625, (2000)

[42] M. Horodecki, P. Horodecki, ,Phys. Rev. A, 4206-4216, (1999)

[43] M. Huber, F. Mintert, A. Gabriel and B. C. Hiesmayr, *Detection of high-dimensional genuine multi-partite entanglement of mixed states*, Phys. Rev. Lett. **104**, 210501, (2010)

[44] A. Gabriel, B. C. Hiesmayr, M. Huber, *Criterion for k-separability in mixed multipartite systems*, Quant. Inf. Comp. **10**, No. 9 & 10, 0829-0836, (2010)

[45] S. J. Freedman, J. F. Clauser, *Experimental Test of Local Hidden-Variable Theories*, Phys. Rev. Lett. **28,** 14, 938–941, (1972)

[46] A. Aspect, P. Grangier, G. Roger, *Experimental Tests of Realistic Local Theories via Bell's Theorem*, Phys. Rev. Lett. **47**, 7, 460-463, (1981)

[47] A. Aspect, P. Grangier, G. Roger, *Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities*, Phys. Rev. Lett. **49** , 2 , 91–94, (1982)

[48] A. Aspect, J. Dalibard, G. Roger, *Experimental Test of Bell's Inequalities Using Time-Varying Analyzers*, Phys. Rev. Lett. **49**, 25, 1804-1807, (1982)

[49] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, A. Zeilinger, *Violation of Bell's inequality under strict Einstein locality conditions*, Phys. Rev. Lett. **81**, 5039-5043, (1998)

[50] M. Huber, P. Erker, H. Schimpf, A. Gabriel, B. C. Hiesmayr, *Experimentally feasible set of criteria detecting genuine multipartite entanglement in n-qubit Dicke states and in higher dimensional systems*, Phys. Rev. A **83**, 040301(R), (2011); Phys. Rev. A **84**, 039906(E), (2011)

[51] R. H. Dicke, *Coherence in spontaneous radiation processes*, Phys. Rev. A, **93**, 99, (1954)

[52] C. Spengler, M. Huber, A. Gabriel, B. C. Hiesmayr, *Examining the dimensionality of genuine multipartite entanglement*, Quantum Inf. Process. **12**, 269, (2013)

[53] A. Gabriel, *Detection and Characterisation of Multipartite Entanglement*, Dissertation, University of Vienna, (2013)

[54] C. Spengler, *Entanglement and correlations in composite quantum systems*, Dissertation, University of Vienna (2012)

[55] C. Spengler, M. Huber, B. C. Hiesmayr, *A composite parameterization of unitary groups, density matrices and subspaces*, J. Phys. A: Math. Theor. **43**, 385306, (2010)

[56] C. Spengler, M. Huber, B. C. Hiesmayr, *Composite parameterization and Haar measure for all unitary and special unitary groups*, J. Math. Phys. **53**, 013501, (2012)

[57] Zhi-Hao Ma, Zhi-Hua Chen, Jing-Ling Chen, C. Spengler, A. Gabriel, M. Huber, *Measure of genuine multipartite entanglement with computable lower bounds*, Phys. Rev. A **83**, 062325, (2011)

[58] R. Jozsa, *Fidelity for mixed quantum states*, Journal of Modern Optics, **41**, 2315-2323, (1994)

[59] D. Bohm, Y. Aharonov, *Discussion of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky*, Phys. Rev., **108**, no. 4, 1070–1076, (1957)

[60] J. F. Clauser, M. A. Horne, A. Shimony, R. A. Holt, *Proposed Experiment to Test Local Hidden-Variable Theories*, Phys. Rev. Lett. **23**, no. 15, 880-884, (1969)

[61] S. Kochen, E. P. Specker, *The Problem of Hidden Variables in Quantum Mechanics*, J. Math. Mech. **17**, no. 1, 59-87, (1967)

[62] P. Holland, *Quantum Theory of Motion*, Cambridge University Press, (1993)

[63] N. Gisin, *Weinberg's non-linear quantum mechanics and supra-luminal communications*, Phys. Lett. A, **143**, 1, (1990)

[64] V. Scarani , N. Gisin, *Superluminal influences, hidden variables, and signaling*, Phys. Lett. A, **295**, 167, (2002)

[65] D. Collins, N. Gisin, N. Linden, S. Massar, S. Popescu, *Bell Inequalities for Arbitrarily High-Dimensional Systems*, Phys. Rev. Lett. **88**, 040404, (2002)

[66] B. C. Hiesmayr, A. Di Domenico, C. Curceanu, A. Gabriel, M. Huber, J.-Å. Larsson, P. Moskal, *Revealing Bell's nonlocality for unstable systems in high energy physics*, Eur. Phys. J. C **72**, 1856, (2012)

[67] N. Gisin, *Bell's inequality holds for all non-product states*, Phys. Lett. A, **154**, 201, (1991)

[68] P. A. M. Dirac, *A new notation for quantum mechanics*, Mathematical Proceedings of the Cambridge Philosophical Society, Vol. **35** , Issue 03, 416-418, (1939)

[69] P. M. Alsing, D. McMahon, G.J. Milburn, *Teleportation in a non-inertial frame*, arXiv:quant-ph/0311096, submitted to J. Opt. B: Q. & Semi. Class. Opt, (2003)

[70] I. Fuentes-Schuller, R. B. Mann, *Alice falls into a Black Hole: Entanglement in Noninertial Frames*, Phys. Rev. Lett. **95**., 120404, (2005)

[71] P. M. Alsing, I. Fuentes-Schuller, R. B. Mann, T. E. Tessier, *Entanglement of Dirac fields in noninertial frames*, Phys. Rev. A **74**, 032326, (2006)

[72] G. Adesso, I. Fuentes Schuller, *Correlation loss and multipartite entanglement across a black hole horizon*, Quant. Inf. Comp. **9**, No. 7&8. 0657-0665, (2009)

[73] I. Fuentes, R. B. Mann, E. Martin-Martinez, S. Moradi, *Entanglement of Dirac fields in an expanding spacetime*, Rev. D **82**, 045030, (2010)

[74] N. Friis, P. Köhler, E. Martin-Martinez, R. A. Bertlmann, *Residual entanglement of accelerated fermions is not nonlocal*, Phys. Rev. A **84**, 062111, (2011)

[75] E. Martin-Martinez, *Relativistic Quantum Information: developments in Quantum Information in general relativistic scenarios*, Dissertation, arXiv:1106.0280v1, (2011)

[76] N. Friis, I. Fuentes, *Entanglement generation in relativistic quantum fields*, Journal of Modern Optics, (online publication), (2012)

[77] N. Friis, M. Huber, I. Fuentes, D. E. Bruschi, *Quantum gates and multipartite entanglement resonances realized by non-uniform cavity motion*, Phys. Rev. D **86**, 105003, (2012)

[78] D. E. Bruschi, A. Dragan, I. Fuentes, J. Louko, *Particle and anti-particle bosonic entanglement in non-inertial frames*, Phys. Rev. D **86**, 025026, (2012)

[79] E. Martin-Martinez, *Relativistic Quantum Information: developments in Quantum Information in general relativistic scenarios*, Dissertation, arXiv:1106.0280v1, (2011)

[80] M. C. Palmer, M. Takahashi, H. F. Westman, *Localized qubits in curved spacetimes*, Annals of Physics **327**, 1078-1131, (2012)

[81] A. Cabello, *N-Particle N-Level Singlet States: Some Properties and Applications*, Phys. Rev. Lett. **89**,100402, (2002)

[82] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen (De), U. Sen, B. Synak-Radtke, *Local versus nonlocal information in quantum-information theory: Formalism and phenomena* , Phys. Rev. A **71**, 062307, (2005)

[83] L. Yuan, Z. Gui-Hua, Commun. Theor. Phys. **50**, 307, (2008)

[84] T. Schmitt-Manderbach, H. Weier, M. Fürst, R. Ursin, F. Tiefenbacher, Th. Scheidl, J. Perdigues, Z. Sodnik, J. G. Rarity, A. Zeilinger, H. Weinfurter, *Experimental Demonstration of Free-Space Decoy-State Quantum Key Distribution over 144 km*, Phys. Rev. Lett. **98**, 010504, (2007)

[85] R. Ursin, F. Tiefenbacher, T. Schmitt-Manderbach, H. Weier, T. Scheidl, M. Lindenthal, B. Blauensteiner, T. Jennewein, J. Perdigues, P. Trojek, B. Ömer, M. Fürst, M. Meyenburg, J. Rarity, Z. Sodnik, C. Barbieri, H. Weinfurter, A. Zeilinger, *Entanglement based quantum communication over 144 km*, Nature Physics **3**, 481 - 486, (2007)

[86] Xiao-song Ma, Th. Herbst, Th. Scheidl, D. Wang, S. Kropatschek, W. Naylor, A. Mech, B. Wittmann, J. Kofler, E. Anisimova, V. Makarov, Th. Jennewein, R. Ursin, A. Zeilinger, *Quantum teleportation using active feed-forward between two Canary Islands*, Nature **489**, 269, (2012)

[87] N. Bohr, *Can Quantum-Mechanical Description of Physical Reality be Considered Complete?*, Phys. Rev. **48**, 696, (1935)

[88] D. Bruß, *Characterizing Entanglement*, J. Math. Phys. **43**, 4237, (2002)

[89] S. Parker, S. Bose, M. B. Plenio, *Entanglement quantification and purification in continuous-variable systems*, Phys. Rev. A **61**, 032305, (2000)

[90] A. Gabriel, M. Huber, S. Radic and B. C. Hiesmayr, *A Computable Criterion for Partial Entanglement in Continuous Variable Quantum Systems*, Phys. Rev. A **83**, 052318, (2011)