universität wien

DIPLOMARBEIT

Titel der Diplomarbeit

On a Cosmological Model with Torsion

verfasst von

Hedda Alice Gressel

angestrebter akademischer Grad

Magistra der Naturwissenschaften (Mag. rer. nat.)

Wien, 2013

Studienkennzahl lt. Studienblatt:	A411
Studienrichtung lt. Studienblatt:	Diplomstudium Physik
Betreut von:	ao. Univ.-Prof. Dr. Helmut Rumpf

Contents

1 Introduction

The phenomenon of dark matter remains one of the challenging mysteries of cosmology. Discrepancies between astronomical observations and calculated predictions gave rise to the idea of a new kind of matter that neither absorbs nor emits electromagnetic radiation up to a measurable level and, therefore, hasn't been detected yet.

One of the above mentioned discrepancies is the fact that the sum of all matter observed within clusters of galaxies would not be sufficient to let them stay bound. Another discrepancy lies within the behaviour of the rotation curves of spiral galaxies which is the discrepancy we will focus on in this work.

Calculations predicted that the speed of rotation in the inner regions of the spiral galaxy would rise linearly with the radius and eventually would reach a maximum. At that point the rotational velocity would start decreasing with increasing value of the radius. Observations could not confirm the latter prediction. On the contrary, the observations showed that after having reached the maximum the rotational velocity remained remarkably constant up to very large radii.[2]

A possible explanation for clusters of galaxies staying bound as well as for the anomalous rotation curves could be the existence of a yet undetected dark matter.

Another example that suggests the existence of dark matter is given by the arrangement of the peaks in the power spectrum of the cosmic microwave background anisotropies. The peaks origin in acoustic oscillations in the photon-baryon plasma in the early universe, resulting from an interplay of gravitation and radiation pressure. Naturally, dark matter does not interact with any kind of radiation but it interacts through gravity and, thereby, has a measurable effect on acoustic oscillations.

The first peak of the power spectrum gives informations on the curvature of the universe, i.e. that the universe is approximately spatially flat. From the peak-hight relationship between the second and the third peak we can derive the matter - dark matter ratio.

The Friedmann universe with the Friedmann-Lematre-Robertson-Walker metric:

$$ds^2 = g_{ik}dx^i dx^k = -dt^2 + R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{1}$$

where $R(t)$ is the cosmological scale factor and $k = (-1, 0, 1)$ a constant representing the curvature, is regarded as the standard model of cosmology. Note that we set the speed of light c equal to one.

(1) is in accordance with the cosmological principle requiring that the universe is homogeneous and isotropic.

The Friedmann model is based on the Friedmann equations and the equation of state with one of the two Friedmann equations reading

$$3\frac{\dot{R}^2(t)+k}{R^2(t)} = \kappa\mu + \Lambda \tag{2}$$

with $\kappa = 8\pi G$, G being the gravitational constant and Λ the cosmological constant.
If $k > 0$ it follows that [7]

$$\mu_{\text{tot}} \equiv 3\frac{\dot{R}^2(t)+k}{R^2(t)\kappa} = \mu + \frac{\Lambda}{\kappa} > 3\frac{\dot{R}^2(t)}{R^2(t)\kappa} \equiv \mu_{\text{crit}} \tag{3}$$

with μ_{crit} constituting the critical value between an open and a closed universe and μ_{tot} being the total energy density. Introducing the Hubble constant $H = \frac{\dot{R}(t)}{R(t)}$ the critical energy density reads

$$\mu_{\text{crit}} = \frac{3H^2}{8\pi G}. \tag{4}$$

According the power spectrum mentioned above the energy density μ_{tot} approximately equals the critical energy density μ_{crit} and, therefore, we conclude that the universe is approximately spatially flat.
The density parameter $\Omega = \frac{\mu_{\text{tot}}}{\mu_{\text{crit}}}$ is composed of four density contributions: [7]

$$\Omega = \Omega_m + \Omega_\Lambda + \Omega_\nu + \Omega_\gamma$$

where Ω_Λ is the dark energy density parameter, Ω_ν and Ω_γ the neutrino and photon density parameter and $\Omega_m = \Omega_b + \Omega_{DM}$ the matter density parameter consisting of the baryonic and dark matter density parameters Ω_b and Ω_{DM}, respectively. Dark energy and dark matter combined provide the main contribution to the energy density parameter with $\Omega_\Lambda = 0.75$ and $\Omega_{DM} = 0.2$, respectively. The baryonic contribution only amounts to 0.04, while the photon and neutrino contributions are negligible.
There are several theories giving explanations on the origin of dark matter which can be categorised into three groups: [9], [8]

- hot dark matter (HDM)
- cold dark matter (CDM)
- warm dark matter (WDM)

HDM: hot dark matter is considered to be relativistic at their decoupling from plasma. The theory predicts a "top-bottom" structure formation of the universe in which large superclusters were formed, first resembling a flat pancake, which subsequently collapsed. However, this scenario is not supported by observations.
CDM: in this theory the particles were non-relativistic when decoupled from hot plasma.

The structure formation predicted by this theory is the so-called "bottom-up" scenario. In this theory, the structure formation starts with small objects, which collapse due to gravity, merging to more massive objects. Unlike the scenario predicted by HDM the "bottom-up" scenario is in accordance with observations. [8]
There are several candidates for a dark matter particle with the most prominent one being the WIMP (weakly interacting massive particle) theory predicting that the dark matter consists of particles only interacting through gravity and the weak force.
Other possible candidates for DM are axions being light pseudoscalar bosons or MACHOs (massive compact halo object), which are objects consisting of baryonic matter such as brown dwarfs. MACHOs emit little or no radiation and are therefore hard to detect. However, MACHOs act as gravitational microlenses when being located exactly in front of a star [9]. Nevertheless, a purely baryonic dark matter can be excluded for reasons given by primordial nucleosynthesis.
There are, as well, supersymmetric particles such as the neutralino, which meet the requirements for a dark matter particle.
WDM: warm dark matter is a mixture of cold dark matter and hot dark matter. Depending on its composition of hot and cold dark matter the theory of warm dark matter provides either the "bottom-up" or the "top-bottom" scenario.
In this work we shall focus on a less known theory describing a gravitational field similar to the field attributed to dark matter being generated by a torsion field.
In the last decades there has been an increasing interest of a generalisation of General Relativity involving torsion. Especially in the 1970s the concept of torsion in a theory of gravitation gained a great deal of attention (see [3] - [6] for a representative sample) and has, thus, been extensively elaborated.
The current work builds up on a recently published paper by Hubert Bray that explains the phenomenon of dark matter by introducing torsion to a 4 dimensional space-time. The simulations described in the work of Hubert Bray provide rotational curves which resemble the actually observed ones and, thus, seem to be an interesting approach for further investigations into the field of dark matter.
This work aims to provide a potential approach to detect the torsion field proposed in the work [1] of Hubert Bray.

2 A Dark Matter Model Generated by Introducing Torsion into the 4 Dimensional Space-Time

The model adapted in this section is based on the paper "On Dark Matter, Spiral Galaxies, and the Axioms of General Relativity" from Hubert L. Bray [1]. In the next section potential interactions between the dark matter model - in particular its field originated from the added torsion - and the polarization vector will be analysed.
But first we may start with a brief introduction to the dark matter model based on [1].

2.1 The axioms

In [1] the author approaches General Relativity in an axiomatic way introducing two axioms in relation to the well known GR.

Axiom 1 *Let N be a manifold with metric g of signature $(-+++)$ and connection Γ , smooth, which is both Hausdorff and second countable. The metric g and the connection Γ are assumed smooth as well.*

Definition 1 *Let $\Phi : \Omega \subset N \to R^4$ be a coordinate chart, let $\{\partial_i\}$, $0 \leq i \leq 3$ be the associated basis of tangent vector fields and $g_{ij} = g(\partial_i, \partial_j)$ and $\Gamma_{ijk} = g\left(\nabla_{\partial_i}\partial_j, \partial_k\right)$ the coefficients of the metric g and the connection Γ.*

From now on all calculations will be done in the coordinate chart Φ.

Definition 2 *Let $M = \{g_{ij}\}$ and $C = \{\Gamma_{ijk}\}$ and $M' = \{g_{ij,k}\}$ and $C' = \{\Gamma_{ijk,l}\}$ be the components of the metric and the connection and all the first derivatives.*

Definition 3 *Let $Quad_Y(\{x_\alpha\}) = \sum_{\alpha,\beta} F^{\alpha\beta}(Y)x_\alpha x_\beta$ be the quadratic expression of the $\{x_\alpha\}$ with coefficients in Y for some functions $\{F^{\alpha\beta}\}$.*

Axiom 2 *For all coordinate charts $\Phi : \Omega \to R^4$ and open sets U, whose closure is compact and within the interior of Ω, (g_{ij}, Γ_{ijk}) is a critical point of the functional*

$$F_{\Phi(U)} = \int_{\Phi(U)} Quad_M(M' \cup M \cup C' \cup C)dV_{R^4} \tag{5}$$

with respect to smooth variations of the metric and connection compactly supported in U, for some fixed quadratic functional $Quad_M$ with coefficients in M. The functional $F_{\Phi(U)}$ is invariant under diffeomorphism transformations in order to meet the requirement of the principle of general covariance.

The first axiom determines the fundamental conditions in which the theory is embedded, whereas the second axiom restricts the dynamics, which is more general than in GR. In particular, the connection is not necessarily taken to be the standard Levi-Civita connection but, in general, a connection on the tangent bundle of space-time independent of the metric g_{ij}.
As this is a very general approach, we may find suitable restrictions for the general connection, so that, as a result, the connection can be expressed as a superposition of a metric-independent and a metric-dependent part. In particular, in (5) the standard Levi-Civita-connection is a linear expression in elements of the set M' whereas C and C' are considered to be metric-independent.

2.2 Does the second axiom include GR?

If the quadratic expression in (5) is reduced to $Quad_M(M')$, it yields the vacuum Einstein equations. The variation of $Quad_M(M' \cup M)$, however, yields the vacuum Einstein equations with cosmological constant.
Note that we set the speed of light c equal to one, unless noted otherwise.

2.2.1 Example $Quad_M(M')$

We may verify the examples given:
It needs to be shown that in the first case

$$F_{\Phi(U)} = \int_{\Phi(U)} Quad_M(M')\, dV_{R^4} \tag{6}$$

reproduces the Einstein-Hilbert action

$$H_{\Phi(U)}(g_{ij}) = \int_{\Phi(U)} R\,|g|^{\frac{1}{2}}\, dV_{R^4} \tag{7}$$

yielding the Euler-Lagrange equations $G_{ij} = 0$ better known as the vacuum Einstein equations, with $G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$ and $|g| = |\det([g_{ij}])|$.
For our calculations we may use the curvature tensor $R^r{}_{ikj}$, the Ricci tensor R_{ik} and the scalar curvature R defined as follows

$$R^r{}_{ikj} := \Gamma^r{}_{ij,k} - \Gamma^r{}_{ik,j} + \Gamma^r{}_{mk}\Gamma^m{}_{ij} - \Gamma^r{}_{mj}\Gamma^m{}_{ik}, \tag{8}$$

$$R_{ik} := R^r{}_{ikr} \quad \text{and} \tag{9}$$

$$R := g^{ik} R_{ik}. \tag{10}$$

Let us now express the scalar curvature R in terms of the metric g_{ij}:

$$\begin{aligned} R =&(g^{ij}g^{kl} - g^{ik}g^{jl})g_{ij,kl} + g_{ij,k}g_{ab,c} \\ &(\frac{1}{2}g^{ia}g^{jc}g^{kb} - \frac{3}{4}g^{ia}g^{jb}g^{kc} + g^{ia}g^{jk}g^{bc} + \frac{1}{4}g^{ij}g^{ab}g^{kc} - g^{ij}g^{ac}g^{kb}) \end{aligned} \tag{11}$$

Inserting (11) into (7) and using the metric volume form $dV = |g|^{\frac{1}{2}}dV_{R^4}$, the identities ${g^{ij}}_{,k} = -g^{ia}g^{jb}g_{ab,k}$ and $|g|_{,l} = |g|g^{mn}g_{mn,l}$ give:

$$\begin{aligned} \int_U R\,dV =& \int_{\Phi(U)} \left((g^{ij}g^{kl} - g^{ik}g^{jl})g_{ij,k}|g|^{\frac{1}{2}}\right)_{,l} dV_{R^4} \\ &- \int_{\Phi(U)} \Bigg[g_{ij,k}\left({g^{ij}}_{,l}g^{kl} + g^{ij}{g^{kl}}_{,l} - {g^{ik}}_{,l}g^{jl} - g^{ik}{g^{jl}}_{,l} + \frac{1}{2}g^{ab}g_{ab,l}\right) \\ &- \frac{3}{4}g^{ia}g^{jb}g^{kc} + \frac{1}{2}g^{ia}g^{jc}g^{kb} + g^{ia}g^{jk}g^{bc} + \frac{1}{4}g^{ij}g^{ab}g^{kc} - g^{ij}g^{ac}g^{kb}\Bigg] |g|^{\frac{1}{2}}dV_{R^4} \end{aligned} \tag{12}$$

and via the divergence theorem the functional can be expressed as

$$\begin{aligned} \int_U R\,dV =& \int_{\partial(\Phi(U))} (g^{ij}g^{kl} - g^{ik}g^{jl})g_{ij,k}\nu_l|g|^{\frac{1}{2}}dA_{\partial(\Phi(U))} + \int_{\Phi(U)} g_{ij,k}g_{ab,c} \\ &\Bigg(\frac{1}{4}g^{ia}g^{jb}g^{kc} + g^{ij}g^{ka}g^{cb} - \frac{1}{4}g^{ij}g^{ab}g^{kc} + \frac{1}{2}g^{ij}g^{ac}g^{kb} + g^{ia}g^{jk}g^{bc} \\ &-g^{ik}g^{ja}g^{cb} - \frac{1}{2}g^{ia}g^{jc}g^{kb}\Bigg) |g|^{\frac{1}{2}}dV_{R^4} \end{aligned} \tag{13}$$

where ν_l is the outward pointing unit normal field of the boundary $\partial\Phi(U)$ in the R^4 coordinate chart.

Equation (13) displays the Einstein-Hilbert action in terms of the metric g_{ij}. Recalling Axiom 2 it is obvious that the Einstein-Hilbert action takes the form of

$$\int_U R\,dV = \int_{\Phi(U)} Quad_M(M')dV_{R^4} + \text{boundary term}, \tag{14}$$

where

$$M = \{g_{ij}\} \quad \text{and} \quad M' = \{g_{ij,k}\} \quad .$$

2.2.2 Example $Quad_M(M \cup M')$

Since the expression $Quad_M(M) = g_{ij}g_{kl}\left(g^{ij}g^{kl}\right) = 16$ gives a constant, it can be absorbed into any other constant, in our case, into the cosmological constant Λ. Thus, $Quad_M(M \cup M')$ yields the Lagrangian $R - 2\Lambda$ from which we obtain the vacuum Einstein equations with cosmological constant.

2.3 Derivation of the Einstein-Klein-Gordon Equations with Cosmological Constant from Axiom 2

We may derive the Einstein-Klein-Gordon equations from the generalised connection Γ_{ijk} introduced by Axiom 1.

2.3.1 The general connection Γ_{ijk} and its $Quad_M(C \cup C')$ part

Summarising the author's approach [1], he uses a general connection in order to derive new features which possibly explain the phenomenon of dark matter. We begin our calculation by analysing the difference tensor D_{ijk} denoted as

$$D_{ijk} = \Gamma_{ijk} - \bar{\Gamma}_{ijk}, \tag{15}$$

being the difference between the general connection Γ_{ijk} and the Levi-Civita connection $\bar{\Gamma}_{ijk}$ defined as

$$\bar{\Gamma}_{ijk} = \frac{1}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right). \tag{16}$$

Recall that Axiom 2 restricts the functional in the following way:

$$F_{\Phi(U)} = \int_{\Phi(U)} Quad_M(M' \cup M \cup C' \cup C) dV_{R^4} \tag{17}$$

with $M = \{g_{ij}\}$ and $C = \{\Gamma_{ijk}\}$ and $M' = \{g_{ij,k}\}$ and $C' = \{\Gamma_{ijk,l}\}$.
In the previous section we assigned $(R - 2\Lambda)\,|g|^{\frac{1}{2}}$ to $Quad_M(M' \cup M)$ at least up to boundary terms.

We now turn to $Quad_M(C' \cup C)$ starting with $Quad_M(C)$:
We split the general connection into $\Gamma_{ijk} = D_{ijk} + \bar{\Gamma}_{ijk}$. The Levi-Civita connection $\bar{\Gamma}_{ijk}$, consisting of derivatives of the metric, is to be understood as an element of the set $\{M'\}$. As $\bar{\Gamma}_{ijk}$ is not a tensor, any crossterm, that we obtain squaring Γ_{ijk}, is excluded due to the requirement of diffeomorphism invariance for the functional $F_{\Phi(U)}$.
But before we proceed to determine the quadratic form of D_{ijk}, we first need to analyse $Quad_M\,(C')$:
The expression $Quad_{g_{ij}}(D_{ijk,l})$ or $Quad_{g_{ij}}(D_{ijk;l})$, whereby the comma denotes the derivative with respect to x^l and the semicolon denotes the covariant derivative with respect to x^l, is not allowed by Axiom 2. Taking the derivative with respect to x^l of D_{ijk} and squaring this expression will result in quadratic terms of second derivatives of the metric and therefore, it is not allowed by Axiom 2. Taking the covariant derivative with the Levi-Civita connection violates Axiom 2 as $\bar{\Gamma}_{ijk}$ already contains derivatives of the

metric g_{ij}. Even the covariant derivative with the general connection Γ_{ijk} is not allowed by the Axiom 2 as it would result in terms like ${\Gamma^a}_{jk} D_{akl}$, which, being squared, would violate the restrictions of the Axiom 2.
To satisfy constraints like those mentioned above, we may consider the torsion part of D_{ijk}:

$$T_{ijk} = D_{ijk} - D_{ikj} \tag{18}$$

which is antisymmetric with respect to the last two indices. Thus, all parts of the equation (18) containing $\bar{\Gamma}_{ijk}$ cancel:

$$\begin{aligned} T_{ijk} &= D_{ijk} - D_{ikj} && (19) \\ &= (\Gamma_{ijk} - \bar{\Gamma}_{ijk}) - (\Gamma_{ikj} - \bar{\Gamma}_{ikj}) && (20) \\ &= \Gamma_{ijk} - \Gamma_{ikj} && (21) \end{aligned}$$

Keeping in mind that the exterior derivative of a 3-form meets the requirements of Axiom 2, we may define the totally antisymmetric part of Γ_{ijk}:

$$\begin{aligned} \gamma_{ijk} &= \frac{1}{6}(T_{ijk} + T_{jki} + T_{kij}) && (22) \\ &= \frac{1}{6}(\Gamma_{ijk} - \Gamma_{ikj} + \Gamma_{jki} - \Gamma_{jik} + \Gamma_{kij} - \Gamma_{kji}) && (23) \end{aligned}$$

We now consider the exterior derivative of γ_{ijk}

$$(d\gamma)_{ijkl} = \gamma_{jkl,i} - \gamma_{kli,j} + \gamma_{lij,k} - \gamma_{ijk,l} \tag{24}$$

which does not contain any derivative of the metric and, thereby, satisfies the requirements of Axiom 2.
Hence, the functional yields

$$F_{\Phi,U}(g_{ij}, \Gamma_{ijk}) = \int_U \Big(R - 2\Lambda - \frac{c_3}{24}|d\gamma|^2 - Quad_{g_{ij}}(D_{ijk})\Big)\, dV \tag{25}$$

with the constant c_3.
[1] contains an important conjecture, which, however, remains unproven:

Conjecture 1 *All functionals for which there exists a smooth metric and smooth connection which satisfy Axiom 2 with that functional are of the form of equation* (25).

We now determine the quadratic form $Quad_M(C)$. Since (25) only contains a kinetic term for γ_{ijk}, only the variation of the totally antisymmetric part of D_{ijk}, which is γ_{ijk}, gives non-trivial field equations. In other words, the symmetric part of D_{ijk} turns out

to be zero. Therefore, we may only consider the quadratic form of γ_{ijk}.
Based on our last considerations, the action functional can be expressed as

$$F_{\Phi,U}(g_{ij},\Gamma_{ijk}) = \int_U \left(R - 2\Lambda - \frac{c_3}{24}|d\gamma|^2 - \frac{c_4}{6}|\gamma|^2\right) dV. \tag{26}$$

$|\gamma|^2$ can be expressed more conveniently using

$$\gamma_{ijk} = \epsilon_{ijk}{}^{l} v_l, \tag{27}$$

thereby replacing the 3-form γ_{ijk} by a 1-form v_l.
Thus, the squared norms $|\gamma|^2$ and $|d\gamma|^2$ can be expressed as

$$|\gamma|^2 = -6|v|^2 \quad \text{and} \quad |d\gamma|^2 = -24(\nabla_i v^i)^2 \tag{28}$$

where the minus sign is due to the Lorentzian signature of the metric.
Inserting (28) into (26) yields

$$F_{\Phi,U}(g^{ij},v^i) = \int_U \left(R - 2\Lambda + c_3(\nabla_i v^i)^2 + c_4|v|^2\right) dV \tag{29}$$

$$\equiv \int_{\Phi(U)} W(g^{ij}, g^{ij}{}_{,k}, v^i, v^i{}_{,j})\sqrt{|g|}dV_{R^4}. \tag{30}$$

2.3.2 Variation of the action functional and the Einstein-Klein-Gordon equation

Unlike most of the authors dealing with the subject of General Relativity the author of [1] decides to vary δg_{ij} instead of δg^{ij}. For a couple of reasons I decided to use the familiar variation δg^{ij}, mostly because the variation results in the covariant Einstein-Klein-Gordon equations and additionally turns out to be easier to calculate.
The variation of the Lagrangian $W\left(g^{ij}, g^{ij}{}_{,k}, v^i, v^i{}_{,j}\right)$ takes the form:

$$\delta W\left(g^{ij}, g^{ij}{}_{,k}, v^i, v^i{}_{,j}\right) = \delta\left((R-2\Lambda)\sqrt{|g|}\right) + \delta\left(\left(c_3(v^i{}_{;i})^2 + c_4 v^i v_i\right)\sqrt{|g|}\right) \tag{31}$$

where $\delta\bar{W}(g^{ij}, g^{ij}{}_{,k}) = \delta\left((R-2\Lambda)\sqrt{|g|}\right)$ leads, up to boundary terms, to the familiar Einstein equations with cosmological constant Λ [11]:

$$G_{ij} + g_{ij}\Lambda = 0$$

We will now turn to the additional terms we obtain from varying the second term on the right hand side of (31):

$$\delta\left(\left(c_3(v^i{}_{;i})^2+c_4v^iv_i\right)\sqrt{|g|}\right)=$$
$$c_3 2\,v^i{}_{;i}\left(\delta\left(v^i{}_{,i}\right)+\frac{1}{2}\,\delta\left(g^{im}g_{im,l}\right)v^l+\frac{1}{2}\,g^{im}g_{im,l}\delta v^l\right)\sqrt{|g|}+c_4\,\delta g_{ij}\,v^iv^j\sqrt{|g|}$$
$$+c_4 2\,v_i\,\delta v^i\sqrt{|g|}-\frac{1}{2}\sqrt{|g|}\,g_{im}\,\delta g^{im}\left(c_3(v^i{}_{;i})^2+c_4v^iv_i\right) \tag{32}$$

The variation with respect to v^i yields

$$2\,c_3\sqrt{|g|}\,v^l{}_{;l}\,\delta v^i{}_{;i}+2\,c_4\sqrt{|g|}\,v_i\,\delta v^i=0$$
$$\Leftrightarrow\ \left(-c_3v^l{}_{;l;i}+c_4v_i\right)\delta v^i=0$$
$$\Rightarrow\ v^l{}_{;l;i}=\frac{c_4}{c_3}v_i \tag{33}$$

implying

$$\delta(v^i{}_{,i})+\frac{1}{2}g^{im}g_{im,l}\delta v^l=(\delta v^i)_{;i} \tag{34}$$

Considering the variation with respect to the metric g_{im}, we see that

$$\delta\left(g^{im}g_{im,l}\right)=\delta g^{im}g_{im,l}+g^{im}\delta g_{im,l}=\delta g^{im}g_{im,l}-\left(g_{ia}g_{mb}\delta g^{ab}\right)_{,l}$$
$$=\delta g^{im}g_{im,l}-\delta g^{ai}g_{ai,l}-\delta g^{ab}g_{ab,l}-g_{ab}\delta g^{ab}{}_{,l}$$
$$=-\delta g^{im}\,g_{im,l}-\delta g^{im}{}_{,l}\,g_{im}=-\left(\delta g^{im}\,g_{im}\right)_{,l}=-\left(\delta g^{im}\,g_{im}\right)_{;l} \tag{35}$$

using $\delta g_{im}=-g_{ia}\,g_{mb}\,\delta g^{ab}$.

Further on, rearranging the term of equation (32) containing (35) gives

$$-\,c_3\,v^j{}_{;j}\left(\delta g^{im}\,g_{im}\right)_{;l}v^l\sqrt{|g|}=-\left(c_3\,v^j{}_{;j}\,\delta g^{im}\,g_{im}\,v^l\sqrt{|g|}\right)_{;l}$$
$$+\,c_3v^j{}_{;j}\,v^l{}_{;l}\,\delta g^{im}\,g_{im}\sqrt{|g|}+c_3v^j{}_{;j\,;l}\,v^l\,\delta g^{im}\,g_{im}\sqrt{|g|}$$
$$=+c_3\left(v^j{}_{;j}\right)^2\delta g^{im}\,g_{im}\sqrt{|g|}+c_4\,v^lv_l\,\delta g^{im}\,g_{im}\sqrt{|g|} \tag{36}$$

using equation (33).

Substituting the respective results into equation (32) yields

$$\sqrt{|g|}\,c_3\left(v^j{}_{;j}\right)^2g_{im}\,\delta g^{im}+\sqrt{|g|}c_4\,v^lv_l\,g_{im}\,\delta g^{im}-\sqrt{|g|}\,v_iv_m\,\delta g^{im}$$
$$-\sqrt{|g|}\,\frac{1}{2}\,g_{im}\left(c_3\left(v^j{}_{;j}\right)^2+c_4v^lv_l\right)\delta g^{im} \tag{37}$$
$$=\sqrt{|g|}\left(-c_4v_iv_m+\frac{1}{2}\,g_{im}\left(c_3\left(v^j{}_{;j}\right)^2+c_4v^lv_l\right)\right)\delta g^{im} \tag{38}$$

Consequently, the variation (31) results in

$$-G_{im} - g_{im}\Lambda - c_4 v_i v_m + \frac{1}{2}\left(c_3\left(v^j{}_{;j}\right)^2 + c_4 v^l v_l\right) g_{im} = 0 \tag{39}$$

$$\Rightarrow G_{im} + g_{im}\Lambda = -c_4 v_i v_m + \frac{1}{2}\left(c_3\left(v^j{}_{;j}\right)^2 + c_4 v^l v_l\right) g_{im} \tag{40}$$

Lest the dominant energy condition, which is defined as $T^{ab}u_b$ being future-directed or null for all future-directed timelike u_b, where T^{ab} is the stress-energy tensor, is violated [13], it is essential that $c_3, c_4 \geq 0$. Moreover, considering (33) we define $c_3 \neq 0$ in order to obtain a nontrivial equation for v_i. Based on physical considerations $c_4 \neq 0$ is required in order to reduce the degrees of freedom of the vector field v^i.
Let us return to the dominant energy condition:
Usually the stress-energy tensor describes the density and current density of energy and momentum in space-time. It is contained in the Einstein field equations

$$G_{ik} = -\kappa T_{ik} \tag{41}$$

with $\kappa = \frac{8\pi G}{c^4}$ and G being the gravitational constant, and is identified with the source of the gravitational field.
In the present calculation we notice the absence of matter but, nevertheless, we may define an effective stress-energy tensor by the right hand side of equation (40):

$$T^{ab} = \frac{1}{\kappa}\left[c_4 v^a v^b - \frac{1}{2}\left(c_3(v^a{}_{;a})^2 + c_4 v^l v_l\right) g^{ab}\right] \tag{42}$$

Without loss of generality, we define the future-directed timelike vector u_b as $u_b = (-1, 0, 0, 0)$. Hence, we need to show that

$$T^{ab}u_b = -T^{a0} = -\frac{c_4}{\kappa} v^a v^0 + \frac{1}{2\kappa}\left(c_3(v^a{}_{;a})^2 + c_4 v^l v_l\right) g^{a0}$$

is a future-directed timelike vector:

$$\begin{aligned} g_{ac}T^{a0}T^{c0} &= \frac{1}{\kappa^2}\left(c_4 v_c v^0 - \frac{1}{2}\left(c_3(v^a{}_{;a})^2 + c_4 v^l v_l\right)\delta^0_c\right)\left(c_4 v^c v^0 - \frac{1}{2}\left(c_3(v^a{}_{;a})^2 + c_4 v^l v_l\right) g^{c0}\right) \\ &= \frac{1}{\kappa^2}\left[c_4^2 v_c v^c v^0 v^0 - c_4 v^0 v^0\left(c_3\left(v^a{}_{;a}\right)^2 + c_4 v_a v^a\right) + \frac{1}{4}\left(c_3\left(v^a{}_{;a}\right)^2 + c_4 v_a v^a\right)^2 g^{00}\right] \\ &= -\frac{c_4 c_3}{\kappa^2} v^0 v^0\left(v^a{}_{;a}\right)^2 + \frac{1}{4\kappa^2}\left(c_3\left(v^a{}_{;a}\right)^2 + c_4 v_a v^a\right)^2 g^{00} < 0 \end{aligned} \tag{43}$$

Note that $g^{00} < 0$.
Having proven our claim that the dominant energy condition is not violated, it follows that the right hand side of (40) may as well be identified with the source of the gravi-

tational field of GR. Thus, we relate our results to the behaviour commonly associated with dark matter.

Recall that we set the speed of light $c = 1$. Furthermore, we shall set the gravitational constant G equal to one and, thus, we obtain $\kappa = 8\pi$.

With the intention to eventually derive the Klein-Gordon equation from (33), we define a new function f:

$$f = \left(\frac{c_3}{c_4}\right)^{\frac{1}{2}} v^i{}_{;i}. \tag{44}$$

By means of (33), v_i can be expressed as follows:

$$v_i = \left(\frac{c_3}{c_4}\right)^{\frac{1}{2}} f_{;i} \tag{45}$$

Substituting (44) and (45) into (40) yields:

$$G_{ik} + \Lambda g_{ik} = -c_3 \left(f_{,i}\, f_{,k} - \frac{1}{2}\left(g^{lm} f_{,l} f_{,m} + \frac{c_4}{c_3} f^2 \right) g_{ik} \right) \tag{46}$$

$$\Box_g f = \frac{c_4}{c_3} f \tag{47}$$

where $\Box_g$ is the d'Alembert operator with respect to the Lorentzian metric g_{ik}. Note, that we carried out an integration in order to obtain the Klein-Gordon equation and that although (47) is not implied by (33), (47) implies (33).

As a result of the equations (27) and (44), the connection takes the form

$$\Gamma_{ijk} = \left(\frac{c_3}{c_4}\right)^{1/2} (\star f_{,l})_{ijk} + \bar{\Gamma}_{ijk} \tag{48}$$

where $\star$ is the Hodge-operator $(\star f_{,l})_{ijk} = \epsilon_{ijk}{}^{l} f_{,l}$.

It is noteworthy with regard to (48) and (46) that if the scalar field f is zero the connection is reduced to the Levi-Civita connection and the Einstein equations turn into the known vacuum field equations.

Furthermore, we may introduce the new constants Υ and $\bar{c}_4$ defined as

$$\frac{c_4}{c_3} = \Upsilon^2 \quad \text{and} \quad c_4 = 16\pi \bar{c}_4 \tag{49}$$

so that (46) and (47) can be written as

$$G_{ik} + \Lambda g_{ik} = -8\pi \left(2\frac{f_{,i}\, f_{,k}}{\Upsilon^2} - \left(\frac{g^{lm} f_{,l} f_{,m}}{\Upsilon^2} + f^2 \right) g_{ik} \right) \tag{50}$$

$$\Box_g f = \Upsilon^2 f \tag{51}$$

where f is a scalar field representing the dark matter and Υ is a new fundamental constant whose value will be estimated later on. The constant $\bar{c}_4$ has been absorbed into the scalar field f.

Checking the dimensions of f, Υ and $\bar{c}_4$ we have (L denoting the length)

$$[f] = \frac{1}{L}, \quad [\Upsilon] = \frac{1}{L}, \quad [\bar{c}_4] = \text{dimensionless}\,. \tag{52}$$

Thus, the dimension $\left[\frac{1}{\mathrm{L}}\right]$ of (48) meets the requirement for a connection.

Equation (51) is, in fact, not a Klein-Gordon equation in the quantum mechanical sense, because neither the constant Υ is a mass, nor does it contain Planck's constant. Paper [1] contains quantum mechanical reflections on the boson star case where the scalar field f is the wave function for very light bosons with masses of the order 10^{-23}eV [12]. The constant Υ that replaces the mass term in (51) is related to the Compton wavelength λ in the following way:

$$\lambda = \frac{2\pi}{\Upsilon} = \frac{h}{m} \approx 13\,\text{lightyears} \tag{53}$$

(if $m = 10^{-23}$eV, then $\Upsilon \approx \frac{1}{2\,\text{lightyears}}$). However, our considerations are purely geometrically motivated exemplified by f being a real-valued scalar field and not a probability density, therefore, in this section we will not deal with the subject of quantum mechanics any further.

2.4 The Scalar Field f and the Correlation to Spiral Galaxies

Next, we try to identify a potential link between the scalar field f and the structure of spiral galaxies. We simplify (46) by approximating the cosmological constant with zero due to its sufficiently small value on the galactic scale, for example when compared to Υ. Consequently, the Einstein-Klein-Gordon equations read

$$G_{ik} = -8\pi\left(2\frac{f_{,i}\,f_{,k}}{\Upsilon^2} - \left(\frac{g^{lm}f_{,l}f_{,m}}{\Upsilon^2} + f^2\right)g_{ik}\right) \tag{54}$$

$$\Box_g f = \Upsilon^2 f \tag{55}$$

with $\bar{c}_4$ being absorbed by f.

The equations yield, as a trivial solution, the Minkowski space-time provided that the scalar field is zero. Furthermore, every vacuum solution of the Einstein equations is a solution of (54) with the trivial solution of (55). Unfortunately, most of the solutions, both complex and real, are unstable [1]. There are, however, static solutions for complex scalar fields [1], yet in this classical context we only consider real scalar fields. Therefore, we combine a stable "ground state" solution with real higher order solutions in order to, eventually, obtain dynamic spherically symmetric solutions which are stable [1]. We may note that the galaxies discussed here have a higher probability of being stable if dark matter halos have angular momentum, yet this remains an assumption [1].

2.4.1 Solutions to the Klein-Gordon equation in Minkowski space-time

In Minkowski space-time the Klein-Gordon equation (51) yields

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta_x\right) f = \Upsilon^2 f, \tag{56}$$

where Δ_x is the standard Laplacian on R^3. An appropriate ansatz for the solution is factorising it into functions solely dependent on r, ϕ, θ and t. The angular part has to be a spherical harmonic. We opt for the ansatz:

$$f = \sum_n A_n \cos\omega t \cdot Y_n(\theta, \phi) \cdot r^n \cdot f_{\omega,n}(r). \tag{57}$$

Substituting (57) into (56), we obtain

$$f''_{\omega,n}(r) + \frac{2(n+1)}{r} f'_{\omega,n}(r) = (\Upsilon^2 - \omega^2) f_{\omega,n}(r). \tag{58}$$

We impose $f'_{\omega,n}(0) = 0$ as boundary condition. Note that an overall normalisation has not yet been specified. The boundary condition excludes a source at the origin. In equation (58) we neglect the term $\frac{n(n-1)+2n}{r^2} f_{\omega,n}(r)$. Since it decreases by the factor $\frac{1}{r^2}$ its contribution to the solution is negligible.

The choice of $\cos\omega t$ is arbitrary as using $\sin\omega t$ instead also yields solutions to (56). In fact both $\cos\omega t$ and $\sin\omega t$ are necessary in order to get a complete basis of solutions.

2.4.2 Solutions to the Klein-Gordon equation in a spherically symmetric static space-time

We may approximate the spherically symmetric static space-time metric as

$$ds^2 = -V(r)^2 dt^2 + V(r)^{-2}(dx^2 + dy^2 + dz^2) \tag{59}$$

where $V(r)^2 = 1 + 2\bar{V}(r)$ in the first post-Newtonian approximation with $\bar{V}(r)$ the Newtonian potential and $V(r)$ is to be understood as "potential well" of the galaxy's mass distribution.

Consequently, (58) reads

$$V(r)^2\left(f''_{\omega,n}(r)+\frac{2(n+1)}{r}f'_{\omega,n}(r)\right)=\left(\Upsilon^2-\frac{\omega^2}{V(r)^2}\right)f_{\omega,n}(r). \tag{60}$$

Since the dark matter distribution is known to be approximately spherically symmetric, we approximate the space-time metric as spherically symmetric and static. The general spherically symmetric case differs in some respects from the Minkowskian one:
In Minkowski space-time $r^n f_{\omega,n}(r)$ always decays as a sine or cosine divided by r. The energy density decays with the square of $r^n f_{\omega,n}(r)$. Consequently, the solutions in this case have no finite total energy. However, in the general spherically symmetric case, there are solutions, which, for ω small enough, display an exponential decay [1]. It is also possible to find solutions similar to those in Minkowski space-time, but we are only interested in those solutions that decay exponentially for small ω. Although for most values of ω the solutions are exponentially increasing, there is also a discrete set of values of ω that displays an exponentially decreasing behaviour. Consequently, the energy density is exponentially decaying, thus resulting in a finite total energy.
The exponential behaviour is determined by the following term:

$$\left(\Upsilon^2-\frac{\omega^2}{V(r)^2}\right). \tag{61}$$

If this term is negative, (60) becomes an equation of the form $A\,f''_{\omega,n}(r)+B\,\frac{f'_{\omega,n}(r)}{r}=-C\,f_{\omega,n}(r)$, where $C>0$. Solutions to this differential equation are characterised by oscillation. On the other hand, if (61) is positive, the equation takes the form $A\,f''_{\omega,n}(r)+B\,\frac{f'_{\omega,n}(r)}{r}=+C\,f_{\omega,n}(r)$ and its solutions display an exponential behaviour.
At this point we may compare our results to those corresponding to bound states in one-dimensional potential wells in quantum mechanics. Though we are looking at a different branch of physics the mathematical and, to a certain extent, the physical background is quite similar. We consider the time-independent Schrödinger equation with a piecewise constant potential energy function $V_{pw}(x)$.
Consider a symmetric square potential well of depth V_0 and width $2a$:

$$V_{pw}(x)=\begin{cases}0, & \text{for } -a<x<a\\ -V_0, & \text{for } \quad |x|\geq a\end{cases}$$

Then, the corresponding Schrödinger equation reads

$$\left(-\frac{\hbar^2}{2m}\frac{d}{dx^2}+V_{pw}(x)\right)\phi(x)=E\phi(x) \tag{62}$$

where $\hbar$ is the reduced Planck constant, m the particle mass, $\phi(x)$ the Schrödinger wave function and E the energy of the particle moving along the x-axis, where $E < 0$ is required in order to obtain a bound state. Recall that $-V_0 < E$.

Both equations (62) and (60) give oscillating as well as exponentially decaying solutions by virtue of the change of signs on the right side of the equations (62) and (60), more precisely the terms $(E - V_{pw}(x))$ and $\left(\Upsilon^2 - \frac{\omega^2}{V(r)^2}\right)$, respectively. It is worth mentioning, that in any case $V(r)$ and $V_{pw}(r)$ are responsible for this change of signs, though both cases are physically distinct. The finite square potential well is a quantum mechanical concept usually applied to atomic scales whereas the solutions of (60) apply to galactic scales.

We may return to the dark matter model:

The values for Υ and ω have not yet been determined. However, as we try to find a solution which oscillates for small r and decays for large r, the behaviour of the potential $V(r)$, particularly $\frac{\omega^2}{V(r)^2}$, will be of importance.

Hence, in order to obtain solutions with the properties just mentioned, we must choose ω within the range

$$\Upsilon\, V(0) < \omega < \Upsilon \lim_{r\to\infty} V(r) \tag{63}$$

There will be a discrete set of admissible values of ω in the interval defined by (63). The value of ω determines the size of a galaxy, if the energy-momentum distribution is interpreted in this way. We may explain this further:

To begin with, we introduce a new quantity $Rad(f_{\omega,n}(r))$, which denotes the value of r at which the right hand side of equation (60) equals zero, that is $\left(\Upsilon^2 - \frac{\omega^2}{V(r)^2}\right) = 0$, (tacitly, we have excluded the trivial case where $f_{\omega,n}(r) = 0$). We assume that $V(r)$ is an increasing function of r derived from a positive matter density. Thus, $\forall r > Rad(f_{\omega,n}(r))$, the bound solutions of equation (60) display a rapid exponential decay whereas $\forall r < Rad(f_{\omega,n}(r))$ the solutions oscillate. The quantity $Rad(f_{\omega,n}(r))$ gives an idea of the size of the galaxy since $Rad(f_{\omega,n}(r))$ is the crucial value which marks the beginning of the exponential decay. We may identify $Rad(f_{\omega,n}(r))$ with the radius of the galaxy, (though the galaxy does not abruptly end at $r \geq Rad(f_{\omega,n}(r))$ but the energy-momentum tensor displays a fast exponential decay, as described above).

Let us return to our previous statement and analyse the correlation between ω and the size of the galaxy.

Again, we take a look at the term $\left(\Upsilon^2 - \frac{\omega^2}{V(r)^2}\right)$ taken from the right hand side of (60). If $\omega \approx \Upsilon \lim_{r\to\infty} V(r)$, we obtain $Rad(f_{\omega,n}(r)) \to \infty$. If, however, we choose a smaller value of ω, $Rad(f_{\omega,n}(r))$ takes a smaller value which in turn leads to a smaller galaxy.

Moreover, provided that $V(r)$ is sufficiently asymptotic to $A - \frac{B}{r}$ at infinity with A and

B being two constants, there is an infinite number of discrete values of ω for $V(r)$ and for each n which give finite energy.
The phenomenon is essentially identical with quantum mechanical energy quantization in a potential with similar behaviour for $r \to \infty$.

Without loss of generality we may choose $V(0) = 1$. In addition we set $V(r) = 1 + \epsilon$, where $\epsilon > 0$, since we operate within the weak field limit of general relativity. We already argued that the right hand side of (40) can be identified with a source of a gravitational field of General Relativity. In consequence, the scalar field f contributes to the potential $V(r)$. Choosing $V(r)$ in this way is therefore only the first approximation to a self-consistent solution. Note that for now we do not determine the potential energy function $V(r)$ any further.
Now, we may compare the solutions of equation (58) and (60) by choosing solutions with the same value of ω. For small r the solutions of (58) and (60) will be very similar because $V(0) = 1$. For increasing values of r the solution of (60) decreases faster than the Minkowski space-time solution. Moreover, at $r = Rad(f_{\omega,n}(r))$ the solution of (60) stops oscillating and begins to decay exponentially.
Having explored the properties of the solutions of (58) and (60) we shall now formulate a convenient approximation for these solutions in a spherically symmetric potential well: For $r < Rad(f_{\omega,n}(r))$, we may take the Minkowski-spacetime solution, while at $r = Rad(f_{\omega,n}(r))$ the solutions are cut off and for $r > Rad(f_{\omega,n}(r))$ the solutions are set equal to zero.
This approximation retains all crucial features of the spherically symmetric solutions and displays the solutions of (60) in a qualitatively correct way.
Note that because of the lack of further knowledge of $V(r)$, we are not able to determine the value of $Rad(f_{\omega,n}(r))$ via (60). Thus, we are left to choose $Rad(f_{\omega,n}(r))$ and ω.

Three approximations. The author of [1] emphasises three important approximations:

- We approximate the solutions in a spherically symmetric potential well with solutions in the Minkowski space-time which we arbitrarily cut off at some radius.
- The exact solutions in a spherically symmetric potential well are still approximations, since the dark matter density in the rotating dark matter model we are studying here is not spherically symmetric distributed, although it can be approximated as such.
- It has not yet been shown that there even exist solutions to the full Einstein-Klein-Gordon equations which qualitatively accord with these assumptions.

We proceed to find a solution of equation (55) using the ansatz (57):

$$f(t, r, \theta, \phi) = A_n \cos(\omega t) Y_{n,m}(\theta, \phi) r^n f_{\omega_n, n}(r)$$

where $n \in \mathbb{N}$, $|m| \leq n$ and $Y_{n,m}(\theta, \phi)$ is the spherical harmonic of degree (n, m).
In the previous section we analysed the differential equation, which determines $f_{\omega,n}(r)$. Now we shall turn to the spherical harmonics of the ansatz (57). It is obvious that any $Y_{n,m}(\theta, \phi)$ solves equation (55). It is up to us to determine those $Y_{n,m}(\theta, \phi)$, which can be excluded due to physical considerations and finally come up with a solution of (55) displaying a reasonable dark matter distribution.
To begin with, we may list all spherical harmonics up to degree 2. We shall later explain why we do not consider any $Y_{n,m}$ with $n > 2$.

- $Y_{0,0}(\theta, \phi) = \sqrt{\frac{1}{4\pi}}$
- $Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad Y_{1,\pm 1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$
- $Y_{2,0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right), \quad Y_{2,\pm 1}(\theta, \phi) = \mp\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{\pm i\phi}$,
  $Y_{2,\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{\pm 2i\phi}$

where $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi]$.
It turns out that any $Y_{n,m}(\theta, \phi)$ with the exception of $Y_{0,0}(\theta, \phi)$ and $Y_{2,\pm 2}(\theta, \phi)$, have to be excluded as they do not preserve the center of mass at the origin or must be excluded due to reasons of stability. To elaborate the argument given above, we analyse the functions $Y_{n,m}(\theta, \phi) r^n f_{\omega_n, n}(r)$ appearing in (57):
for simplicity, we denote $g_{n,m}(r, \theta, \phi) = Y_{n,m}(\theta, \phi) r^n f_{\omega_n, n}(r)$ so that the ansatz (57) can be expressed as $f(t, r, \theta, \phi) = A_n \cos(\omega_n t) g(r, \theta, \phi)$.
Recall the common representation of spherical harmonics displaying the positive and negative regions of the real part of $Y_{n,m}(\theta, \phi)$ mapped on the unit sphere. We shall denote the positive region by A and the negative region by B. Any spherical harmonic $\propto \cos\theta$ (namely $Y_{1,0}(\phi, \theta)$ and $Y_{2,\pm 1}(\phi, \theta)$) equals zero along the equator and can therefore be excluded due to reasons of stability. A similar argument holds for $Y_{2,0}(\phi, \theta)$, which can be excluded due to the two zeros of $3\cos^2\theta - 1$ for $\theta \in [0, \pi]$.
Next, we consider $Y_{1,\pm 1}(\theta, \phi)$. Without loss of generality we may focus only on $Y_{1,1}(\theta, \phi)$. We consider a straight line $\mathscr{C}$ embedded in the $\theta = \pi/2$-plane and containing the origin. The line $\mathscr{C}$ intersects the unit sphere either in the negatively valued region B and the positively valued region A or intersects at the points where $Y_{1,1} = 0$, but the latter case is not of interest here. Without loss of generality we choose the line $\mathscr{C}$ to intersect the unit sphere at $\phi = 0$ in region A and at $\phi = \pi$ in region B. Next we consider $g_{1,1}(r, \theta, \phi)$

along the line $\mathscr{C}$. In region A, $g_{1,1}(r,\theta=\frac{\pi}{2},\phi=0)$ gives $r\,f_{\omega_1,1}(r)$, but in region B $g_{1,1}(r,\theta=\frac{\pi}{2},\phi=\pi)$ gives: $-r\,f_{\omega_1,1}(r)$.

In order to prove that $Y_{1,1}(\theta,\phi)$ has to be excluded we refer to the two spherical harmonics $Y_{0,0}(\theta,\phi)$ and $Y_{2,\pm2}(\theta,\phi)$, we have mentioned before.

According to the ansatz (57) and the argument given above, $g_{n,m}(r,\theta,\phi)$ equals

$$g_{n\le 2,|m|\le n}(r,\theta,\phi)=f_{\omega_0,0}(r)+r\,f_{\omega_1,1}(r)Y_{1,1}(\theta,\phi)+r^2 f_{\omega_2,2}(r)Y_{2,\pm2}(\theta,\phi) \tag{64}$$

It suffices to only take the first two terms of (64) into account. Again, we observe the expression $g_{n\le 1,|m|\le n}(r,\theta,\phi)=f_{\omega_0,0}(r)+r\,f_{\omega_1,1}(r)Y_{1,1}(\theta,\phi)$ along the line $\mathscr{C}$. In region A,

$$g_{n\le 1,|m|\le n}(r,\theta=\frac{\pi}{2},\phi=0)=f_{\omega_0,0}(r)+r\,f_{\omega_1,1}(r), \tag{65}$$

yet in region B the expression gives

$$g_{n\le 1,|m|\le n}(r,\theta=\frac{\pi}{2},\phi=\pi)=f_{\omega_0,0}(r)-r\,f_{\omega_1,1}(r). \tag{66}$$

In order to get the energy density we have to square

$$\sum_n\sum_{|m|<n}A_n r^n f_{\omega_n,n}(r)Y_{n,m}(\theta,\phi).$$

Due to the mixed terms we obtain when squaring (65) and (66), the centre of mass is not preserved at the origin if solutions of degree $n=1$ are taken into account.

Thus, the ansatz (57) yields

$$f=A_0\cos(\omega_0 t)f_{\omega_0,0}(r)+A_2\cos(\omega_2 t-2\phi)\sin^2(\theta)r^2 f_{\omega_2,2}(r) \tag{67}$$

In (67) higher degree solutions, i.e. solutions of order 3 and higher, are not included. Any higher degree solutions are assumed to be small enough to be approximated by zero. Consequently, in the present case we consider a 180° rotational symmetry with rigidly rotating densities (somewhat resembling a spoke). If the two density peaks caused by the second degree solution (particularly by $\cos(-2\alpha)$) are applied on a number of test particles the outcome resembles the appearance of a two armed spiral galaxy. If we take higher degree solutions into account, this will result in different rotational symmetries, for example 120°, pertaining to three or more spiral arms.

In [1] the Einstein-Klein-Gordon equations (50) (51) are solved numerically in a fixed spherically symmetric potential well and compared to the approximation in (67). The data of the numerically solved solution is based on the Milky Way Galaxy. We obtain

an oscillating and decaying scalar field which rotates with a certain angular velocity and, compared to the solution derived in the previous section, the approximation (67) is sufficient for our purpose. For further information see [1].
We may define the angle α as

$$\alpha = \phi - \left(\frac{\omega_2 - \omega_0}{2}\right) t \tag{68}$$

to clearly identify the rotation of the dark matter distribution. Hence, equation (67) reads

$$f = A_0 \cos(\omega_0 t) f_{\omega_0,0}(r) + A_2 \cos(\omega_0 t - 2\alpha) \sin^2(\theta)\, r^2 f_{\omega_2,2}(r). \tag{69}$$

Via inequality (63) we conclude that $|\omega_2 - \omega_0|$ has to be very small, therefore, $\alpha(t)$ describes a very slow rotation. For a constant α the solution (69) gives a fixed interference pattern since both the terms are oscillating in time with the same frequency ω_0. But in fact the interference pattern is rotating according to the formula

$$\phi_{\rm rot}(t) = \left(\frac{\omega_2 - \omega_0}{2}\right) t \tag{70}$$

with period

$$T_{DM} = \frac{4\pi}{\omega_2 - \omega_0}. \tag{71}$$

2.4.3 Energy density μ_{DM}

The energy density μ_{DM} is obtained from equation (50):

$$\mu_{DM} = -\frac{1}{8\pi} G_{00} = 2\frac{f_{,0}\, f_{,0}}{\Upsilon^2} - \left(\frac{g^{lm} f_{,l} f_{,m}}{\Upsilon^2} + f^2\right) g_{00} \tag{72}$$

$$\approx 2\frac{f_{,0}\, f_{,0}}{\Upsilon^2} - \left(\frac{-f_{,0}\, f_{,0} + g^{\mu\nu} f_{,\mu} f_{,\nu}}{\Upsilon^2} + f^2\right)(-1) \tag{73}$$

$$= \frac{1}{\Upsilon^2}\left(f_{,0}\, f_{,0} + g^{\mu\nu} f_{,\mu} f_{,\nu}\right) + f^2 \tag{74}$$

$$\approx \left(\frac{f_{,0}}{\Upsilon}\right)^2 + f^2 \tag{75}$$

where i runs over values $(0, 1, 2, 3)$ and μ, ν over $(1, 2, 3)$. In (75) $g^{\mu\nu} f_{,\mu} f_{,\nu}$ is neglected due to the long wavelength approximation we assume.
Inserting $\cos(\omega_0 t - 2\alpha) = \cos\omega_0 t \cos 2\alpha + \sin\omega_0 t \sin 2\alpha$ into (69) yields

$$f = \cos(\omega_0 t)\left[A_0 f_{\omega_0,0}(r) + A_2 \cos(2\alpha) \sin^2(\theta) r^2 f_{\omega_2,2}(r)\right] + \sin(\omega_0 t) A_2 \sin(2\alpha) \sin^2(\theta) r^2 f_{\omega_2,2} \tag{76}$$

Now substituting (76) into (75) we obtain the following equations:

- $\left(\frac{f_{,0}}{\Upsilon}\right)^2$:

$$\left(\frac{f_{,0}}{\Upsilon}\right)^2 = \frac{1}{\Upsilon^2}\left(-\omega_0 \sin(\omega_0 t)[...] + \omega_0 \cos(\omega_0 t)[...]\right)^2 \tag{77}$$

$$= \frac{1}{\Upsilon^2}(\omega_0^2 \sin^2(\omega_0 t)[...]^2 - 2\omega_0^2 \sin(\omega_0 t)\cos(\omega_0 t)[...][...] + \omega_0^2 \cos^2(\omega_0 t)[...]) \tag{78}$$

  We have assumed $V(r) \approx 1$, so that the inequality (63) simplifies to $\omega_0 \approx \Upsilon \approx \omega_2$ and equation (78) to

$$\left(\frac{f_{,0}}{\Upsilon}\right)^2 = \sin^2(\omega_0 t)[...]^2 - 2\sin(\omega_0 t)\cos(\omega_0 t)[...][...] + \cos^2(\omega_0 t)[...]^2 \tag{79}$$

- f^2:

$$f^2 = \cos^2(\omega_0 t)\left[A_0 f_{\omega_0,0}(r) + A_2\cos(2\alpha)\sin^2(\theta) r^2 f_{\omega_2,2}(r)\right]^2 + 2\sin(\omega_0 t)\cos(\omega_0 t)\left[A_0 f_{\omega_0,0}(r) + A_2\cos(2\alpha)\sin^2(\theta) r^2 f_{\omega_2,2}(r)\right] \left[A_2\sin(2\alpha)\sin^2(\theta) r^2 f_{\omega_2,2}\right] + \sin^2(\omega_0 t)\left[A_2\sin(2\alpha)\sin^2(\theta) r^2 f_{\omega_2,2}\right]^2 \tag{80}$$

- $\mu_{DM} = \left(\frac{f_{,0}}{\Upsilon}\right)^2 + f^2$:

$$\mu_{DM} = (\sin^2(\omega_0 t) + \cos^2(\omega_0 t))\left[A_0 f_{\omega_0,0}(r) + A_2\cos(2\alpha)\sin^2(\theta) r^2 f_{\omega_2,2}(r)\right]^2 + (\cos^2(\omega_0 t) + \sin^2(\omega_0 t))\left[A_2\sin(2\alpha)\sin^2(\theta) r^2 f_{\omega_2,2}\right]^2 \tag{81}$$

$$= \left[A_0 f_{\omega_0,0}(r) + A_2\cos(2\alpha)\sin^2(\theta) r^2 f_{\omega_2,2}(r)\right]^2 + \left[A_2\sin(2\alpha)\sin^2(\theta) r^2 f_{\omega_2,2}\right]^2 \tag{82}$$

$$= A_0^2 f_{\omega_0,0}^2(r) + A_2^2 \sin^4(\theta) r^4 f_{\omega_2,2}^2(r) + 2A_0 A_2\cos(2\alpha)\sin^2(\theta) r^2 f_{\omega_0,0} f_{\omega_2,2}(r) \tag{83}$$

Because of the inequality (63) and assuming $V(r) \approx 1$ we can consider α to be approximately fixed in time.

Note that the energy density is not oscillating in time, but the components of the pressure p_α being the diagonal elements of the effective stress-energy tensor are. Nevertheless, on average the pressure p_α yields zero and, therefore, (42) can be considered as the stress-energy tensor of dust.

Since spherical harmonics defined on the unit sphere are restrictions of harmonic polynomials of the same degree, we may substitute them.

Decomposing (83) into spherical harmonics yields:

- $r^2 \cos 2\alpha \sin^2\theta = \left(x^2 - y^2\right)\cos 2\alpha$
- $r^4 \sin^4\theta = (r^2 - z^2)^2 = z^4 - 2r^2z^2 + r^4$
  $= \frac{3}{105}(35z^4 - 30r^2z^2 + 3r^4) - \frac{40}{105}r^2(3z^2 - r^2) + \frac{56}{105}r^4 \cdot 1$

Note that $\cos(2\alpha)\sin^2(\theta) = \cos(2\phi - 2\phi_{\mathrm{rot}})\sin^2(\theta)$ is a second degree rotated spherical harmonic.

Hence, (83) can be written as follows:

$$\begin{aligned}\mu_{DM} \quad \approx \quad & U_0(r) + U_2(r)(3z^2 - r^2) + U_4(r)(35z^4 - 30z^2r^2 + 3r^4) \\ & + \tilde{U}_2(r)(\cos(2\alpha)(r^2 - z^2))\end{aligned} \tag{84}$$

where

$$U_0(r) = A_0^2 f_{\omega_0,0}(r)^2 + \frac{56}{105}A_2^2 r^4 f_{\omega_2,2}(r)^2 \tag{85}$$

$$U_2(r) = -\frac{40}{105}A_2^2 r^2 f_{\omega_2,2}(r)^2 \tag{86}$$

$$U_4(r) = \frac{3}{105}A_2^2 f_{\omega_2,2}(r)^2 \tag{87}$$

$$\tilde{U}_2(r) = 2A_0A_2 f_{\omega_0,0}(r) f_{\omega_2,2}(r) \quad . \tag{88}$$

2.4.4 Gravitational potential of μ_{DM}

Solving the Poisson equation

$$\Delta_x V = 4\pi\mu_{DM} \tag{89}$$

results in a Newtonian potential for the dark matter density μ_{DM} for each value of t. Since the dark matter scalar field's rotation is very small, the group velocity is much less than the speed of light and the Newtonian approximation seems adequate for our purpose. For this calculation we may choose a similar ansatz as we did in (57):

$$\Delta_x\left(\sum_n W_n(r)\cdot r^n \cdot Y_n(\theta,\phi)\right) = \sum_n \left(W_n''(r) + \frac{2(n+1)}{r}W_n'(r)\right) r^n Y_n(\theta,\phi) \tag{90}$$

where Y_n is a spherical harmonic of degree n.

Note that we neglect the term $\propto \frac{W_n}{r^2}$ since its contribution to the solution is negligible. (cf. (58))

Obviously, the solution takes the form

$$\frac{V}{4\pi} \approx W_0(r) + W_2(r)(3z^2 - r^2) + W_4(r)(35z^4 - 30z^2r^2 + 3r^4) + \tilde{W}_2(r)(\cos(2\alpha)(r^2 - z^2)) \tag{91}$$

where

$$W_n''(r) + \frac{2(n+1)}{r}W_n'(r) = U_n(r) \tag{92}$$

with boundary conditions $W_n'(0) = 0$ and $\lim_{r\to\infty} W_n$ being finite and with $U_n(r)$ given by (85) - (88). Note that we can still multiply any solution $W_n(r)$ by a constant without violating the boundary conditions.
We may now focus on the range

$$Rad(f_{\omega,n}(r)) < r < \infty.$$

To approximate the spherically symmetric solutions in a potential well, we decided to cut off the Minkowski space solutions at $r = Rad(f_{\omega,n}(r))$. Therefore, we get the following differential equation for $r > Rad(f_{\omega,n}(r))$:

$$W_n''(r) + \frac{2(n+1)}{r}W_n'(r) = 0 \tag{93}$$

with the admissible solution

$$W_n(r) = \frac{k}{r^{2n+1}} \tag{94}$$

with k being a constant.
The solution is, as expected, decreasing and converges to zero for $r \to \infty$.

2.5 Simulations

In [1] several simulations were performed, which apply the potential function V derived from the dark matter distribution to a number of test particles. The computation of V is based on the approach that the dark matter distribution can be interpreted as a source of a gravitational field via equation (40), as already shown. The outcome of these simulations resemble the appearance of spiral galaxies and elliptical galaxies, respectively. In addition, we are able to compute rotation curves, which look similar to those of the spiral galaxies.
We may focus on "Spiral Galaxy Simulation # 1". Its data is, as far as possible, based on NGC1300, a barred spiral galaxy of type SBbc.

In the beginning of the simulation, the test particles are in circular motion distributed according to the formula $N(r) = N_0 e^{-r/r_0}$ where r_d is called the disc scale length of the galaxy and N_0 the number of particles at $r = 0$. As 25 million years elapse, the test particles form a two armed spiral galaxy due to the not exactly spherically symmetric potential V.

Comparison with realistic rotation curves suggests the following choice of constants:

- $Rad(f_{\omega,n}(r)) = 75000\text{ly}$
- $A_0 = 1272 \cdot \sqrt{\bar{c}_4 \pi}$ and $A_2 = -1648 \cdot \sqrt{\bar{c}_4 \pi}$[1]
- $\bar{c}_4 = 8.6 \cdot 10^{-13}$
- the spatial wavelengths

$$\lambda_0 = 2000\text{ly} \;\text{ and }\; \lambda_2 = 1990\text{ly} \tag{95}$$

  defined as

$$\lambda_k = \frac{2\pi}{\sqrt{\omega_k^2/c^2 - \Upsilon^2}} \tag{96}$$

- $T_{DM} = 25000000\text{y}$ and $T_{\text{total}} = 50000000$
- $n_{\text{particle}} = 5000$
- $dr = 1\text{ly}$ and $dt = 1000\text{y}$

where dr is the spatial stepsize, dt the stepsize in time, n_{particle} the number of test particles, $Rad(f_{\omega,n}(r))$ the radius of the galaxy, A_n the amplitudes of the solutions $f_{\omega,n}(r)$, $\bar{c}_4$ a constant (49), T_{DM} the period of rotation of the dark matter distribution, and T_{total} the total time of the simulation.

Note that the speed of light c and other constants are no longer set equal to one. Nevertheless, since we choose the units lightyears ly and years y, the speed of light becomes $c = 1\frac{\text{ly}}{\text{y}}$. Thus, in further calculations we omit c.

Using the formula for λ_0 and λ_2 (96) and for T_{DM} (71), we derive the needed values of ω_0, ω_2, and Υ as follows:

$$\omega_0 = \frac{\pi}{2}\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_0^2}\right) c^2\, T_{DM} - \frac{2\pi}{T_{DM}} \tag{97}$$

[1] A_0 and A_2 differ from the originally given values of the paper [1] ($A0_{[1]} = -A2_{[1]} = 1$). We absorbed the constant c_4 (49) into the amplitude A_n of $f_{\omega,n}(r)$ (cf. (50)), while the author of [1], whose interest is in the computation of the Newtonian potential V, absorbs c_4 later when computing V.

$$\omega_2 = \frac{\pi}{2}\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_0^2}\right) c^2\, T_{DM} + \frac{2\pi}{T_{DM}} \tag{98}$$

$$\Upsilon^2 = \frac{1}{c^2}\left(\frac{2\pi}{T_{DM}}\right)^2 + c^2\left(\frac{\pi T_{DM}}{2}\right)^2\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_0^2}\right)^2 - 2\pi^2\left(\frac{1}{\lambda_2^2} + \frac{1}{\lambda_0^2}\right) \tag{99}$$

The resulting values are

- $\Upsilon = 0.0988659\frac{1}{\mathrm{ly}}$
- $\omega_0 = 0.0989158\frac{1}{\mathrm{y}}$
- $\omega_2 = 0.0989163\frac{1}{\mathrm{y}}$.

2.5.1 Solutions $f_{\omega_n,n}(r)$

We solve equation (58) for $n = 0, 2$ with the boundary condition that $f_{\omega,n}(0)$ is finite. For simplicity, we define $a := |\Upsilon^2 - \omega_0^2| = 9.87 \cdot 10^{-6}\frac{1}{(\mathrm{ly})^2}$ and $b := |\Upsilon^2 - \omega_2^2| = 9.97 \cdot 10^{-6}\frac{1}{(\mathrm{ly})^2}$. The differential equation (58) choosing $n = 0$ yields

$$f''_{\omega,0}(r) + \frac{2}{r} f'_{\omega,0}(r) = -a f_{\omega,0}(r) \tag{100}$$

with the solution

$$f_{\omega,0}(r) = c_1 \frac{e^{-i\sqrt{a}r}}{r} + c_2 \frac{e^{i\sqrt{a}r}}{r}. \tag{101}$$

We split the solution in real and imaginary parts as follows

$$c_1 \frac{e^{-i\sqrt{a}r}}{r} + c_2 \frac{e^{i\sqrt{a}r}}{r} = c_1 \frac{\cos(\sqrt{a}r)}{r} - c_1 \frac{i\sin(\sqrt{a}r)}{r} + c_2 \frac{\cos(\sqrt{a}r)}{r} + c_2 \frac{i\sin(\sqrt{a}r)}{r} \tag{102}$$

Given the boundary condition, it follows that $c_1 + c_2 = 0$. As we only take real solutions into account, we modify the solution to

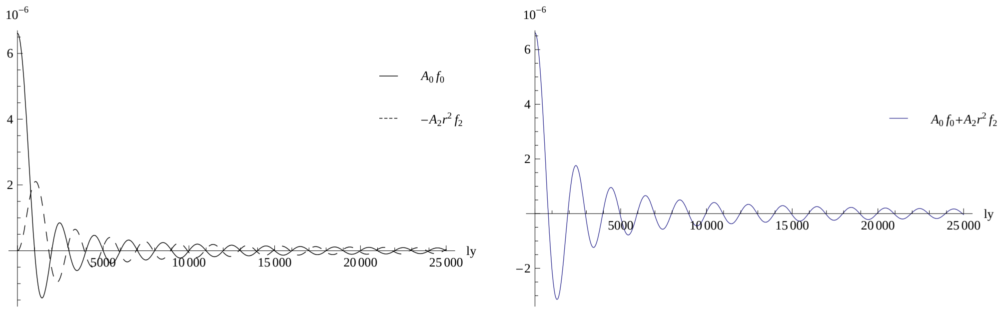
$$f_{\omega,0}(r) = \frac{\sin\sqrt{a}r}{r}. \tag{103}$$

Next, we solve the differential equation (58) for $n = 2$:

$$f''_{\omega,0}(r) + \frac{6}{r} f'_{\omega,0}(r) = -b f_{\omega,0}(r). \tag{104}$$

The solution is

$$\begin{aligned} f_{\omega,2}(r) = {} & c_1 \sqrt{\frac{2}{\pi}} \frac{\left(-\frac{3\cos(\sqrt{b}r)}{\sqrt{b}r} - \sin(\sqrt{b}r) + \frac{3\sin(\sqrt{b}r)}{br^2}\right)}{r^3} \\ & + c_2 \sqrt{\frac{2}{\pi}} \frac{\left(-\frac{3\sin(\sqrt{b}r)}{\sqrt{b}r} - \cos(\sqrt{b}r) + \frac{3\cos(\sqrt{b}r)}{br^2}\right)}{r^3} \end{aligned} \tag{105}$$



(a) $A_0 f_{\omega,0}(r)$ and $-A_2 r^2 f_{\omega,2}(r)$

(b) $f(t = 0, r, \alpha = 0, \theta = \pi/2) = A_0 f_{\omega,0}(r) + A_2 r^2 f_{\omega,2}(r)$

Figure 1: Figure (a) shows the solution $A_0 f_{\omega,0}(r)$ and $-A_2 r^2 f_{\omega,2}(r)$. In figure (b) we see the r-dependent part of the scalar field f.

The first term of the r.h.s. of (105) fulfills the boundary condition, whereas with respect to the second term we find that $c_2 = 0$.

Thus, the solution to (58) with $n = 2$ reads

$$f_{\omega,2}(r) = \sqrt{\frac{2}{\pi}} \frac{\left(-\frac{3\cos(\sqrt{b}r)}{\sqrt{b}r} - \sin(\sqrt{b}r) + \frac{3\sin(\sqrt{b}r)}{br^2}\right)}{r^3}. \tag{106}$$

2.5.2 Energy density μ_{DM}

According to the density parameter

$$\Omega = \frac{\mu_{\text{tot}}}{\mu_{\text{crit}}} \approx 1, \tag{107}$$

where μ_{tot} is the total energy density and $\frac{\mu_{\text{crit}}}{c^2} \approx 10^{-29} \frac{\text{g}}{\text{cm}^3}$ constitutes the crucial value between an open and a closed universe, the dark matter contribution Ω_{DM} to the density parameter amounts to 0.2.[7]

Recall that $\mu_{DM} = -\frac{G_{00}}{\kappa}$ with $\kappa = \frac{8\pi G}{c^4}$ and $\kappa c^2 = 1.87 \cdot 10^{-27} \frac{\text{cm}}{\text{g}}$ [11].

In our case the mass density μ_{DM} oscillates in space and decreases for increasing values of r. Therefore, we compute the mean value

$$\kappa\bar{\mu}_{DM} = \frac{1}{V_s} \int_0^R \int_0^{2\pi} \int_0^{\pi} \kappa\mu_{DM}(r, \alpha, \theta)\, r^2 \sin(\theta) dr d\alpha d\theta \tag{108}$$

$$= 9.25 \cdot 10^{-16} \frac{1}{\text{ly}^2} = 1.03 \cdot 10^{-51} \frac{1}{\text{cm}^2} \tag{109}$$

with V_s being the volume of a sphere with radius $R = 75000$ly, where R is the radius of the galaxy.

Thus, for the average energy density divided by c^2 $\frac{\bar{\mu}_{DM}}{c^2}$ we obtain

$$\frac{\bar{\mu}_{DM}}{c^2} = 5.49 \cdot 10^{-25} \frac{\mathrm{g}}{\mathrm{cm}^3}. \tag{110}$$

Note that Ω_{DM} denotes the dark matter density parameter of the universe. Thus, we expect the dark matter density parameter to be increased within a galaxy and conclude that the value we obtained for $\frac{\bar{\mu}_{DM}}{c^2}$ is consistent with the expected value according to (107).

2.5.3 $W_0(r)$, $W_2(r)$, $W_4(r)$ and $\tilde{W}_2(r)$

In a previous section we derived the potential function V using the energy density of the dark matter distribution and formulated the corresponding Poisson equation. The potential V derived is (cf.(91))

$$\begin{aligned} V(r,\alpha,z) \approx\, & W_0(r) + W_2(r)(3z^2 - r^2) + W_4(r)(35z^4 - 30z^2r^2 + 3r^4) \\ & + \tilde{W}_2(r)(\cos(2\alpha)(r^2 - z^2). \end{aligned} \tag{111}$$

In order to compute the potential V, we solve the respective differential equations (89) for $W_n(r)$ starting with $n = 0$: (cf. (85))

$$W_0''(r) + \frac{2}{r}W_0'(r) = A_0^2 f_{\omega,0}(r)^2 + \frac{56}{105}A_2^2 r^4 f_{\omega,2}(r)^2 \tag{112}$$

with the solution

$$W_0(r) = \frac{8A_2^2}{15\pi}\ln\left(\frac{r}{r_0}\right) + \frac{A_0^2}{2}\ln\left(\frac{r}{r_0}\right) - \frac{A_0^2}{2}\mathrm{Ci}\left(2\sqrt{a}r\right) - \frac{8A_2^2}{15\pi}\mathrm{Ci}\left(2\sqrt{b}r\right) \tag{113}$$

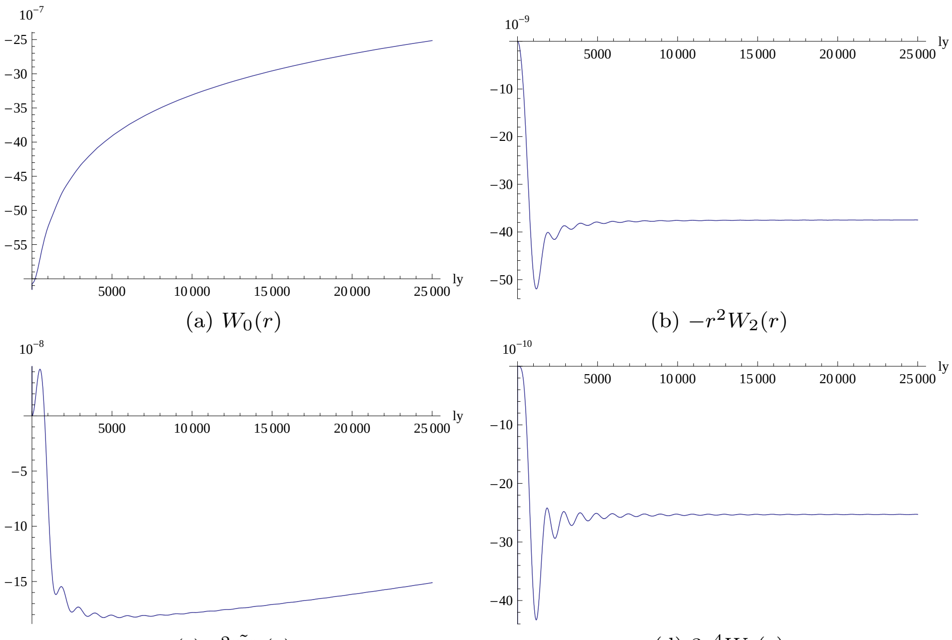
$$+ \frac{1}{r}\left(\frac{A_0^2}{4\sqrt{a}}\sin\left(2\sqrt{a}r\right) + \frac{4A_2 2}{15\sqrt{b}\pi}\sin\left(2\sqrt{b}r\right)\right) + \frac{1}{r^2}\left(\frac{4A_2^2}{5b\pi}\right) \tag{114}$$

$$+ \frac{1}{r^3}\left(-\frac{4A_2^2}{5\sqrt{b}^3\pi}\sin\left(2\sqrt{b}r\right)\right) + \frac{1}{r^4}\left(-\frac{2A_2^2}{5b^2\pi}\cos\left(2\sqrt{b}r\right) + \frac{2A_2^2}{5b^2\pi}\right), \tag{115}$$

where $\mathrm{Ci}(r)$ is the cosine integral defined as $\mathrm{Ci}(r) = -\int_r^\infty \frac{\cos(r')dr'}{r'}$.
The boundary condition $W_n(Rad(f_{\omega,n}(r))) = 0$ determines the constant r_0.
For $n = 2$ there are two differential equations: (cf. (86), (88))

$$W_2''(r) + \frac{6}{r}W_2'(r) = -\frac{40}{105}A_2^2 r^2 f_{\omega,2}(r)^2 \tag{116}$$

$$\tilde{W}_2''(r) + \frac{6}{r}\tilde{W}_2'(r) = 2A_0A_2 f_{\omega,0}(r) f_{\omega,2}(r) \tag{117}$$



(a) $W_0(r)$ (b) $-r^2W_2(r)$

(c) $r^2\tilde{W}_2(r)$ (d) $3r^4W_4(r)$

Figure 2: The solutions which give the radial functions of the potential.

with the solution

$$W_2(r) = -\frac{4A_2^2}{r^2 63\pi} + \frac{A_2^2}{r^4}\left(\frac{2\cos\left(2\sqrt{b}r\right)}{21b\pi} - \frac{2}{7b^2\pi}\right) - \frac{10A_2^2\sin\left(2\sqrt{b}r\right)}{r^5 21\sqrt{b}^3\pi} \tag{118}$$

$$+\frac{A_2^2}{r^6}\left(\frac{4}{7b^2\pi} - \frac{4\cos\left(2\sqrt{b}r\right)}{7b^2\pi}\right) \tag{119}$$

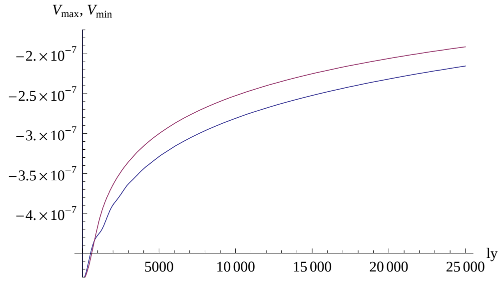
and

$$\begin{aligned}
\tilde{W}_2(r) = \quad & -2\cdot 10^{-17}\Gamma\left[-5,-1.58\cdot 10^{-5}ir\right] - 2\cdot 10^{-17}\Gamma\left[-5,\, 1.58\cdot 10^{-5}ir\right] \\
& -3.59613\cdot 10^{-11}\Gamma\left[-5,-6.3\cdot 10^{-3}ir\right] + 3.59613\cdot 10^{-11}\Gamma\left[-5,\, 6.3\cdot 10^{-3}ir\right] \\
& -2\cdot 10^{-17}\Gamma\left[-4,-1.58\cdot 10^{-5}ir\right] - 2\cdot 10^{-17}\Gamma\left[-4,\, 1.58\cdot 10^{-5}ir\right] \\
& +1.44115\cdot 10^{-11}\Gamma\left[-4,-6.3\cdot 10^{-3}ir\right] + 1.44115\cdot 10^{-11}\Gamma\left[-4,\, 6.3\cdot 10^{-3}ir\right] \\
& -1\cdot 10^{-17}\Gamma\left[-3,-1.58\cdot 10^{-5}ir\right] + 1\cdot 10^{-17}\Gamma\left[-3,\, 1.58\cdot 10^{-5}ir\right] \\
& -1.8214\cdot 10^{-12}\Gamma\left[-3,-6.3\cdot 10^{-3}ir\right] + 1.8214\cdot 10^{-12}\Gamma\left[-3,\, 6.3\cdot 10^{-3}ir\right],
\end{aligned} \tag{120}$$

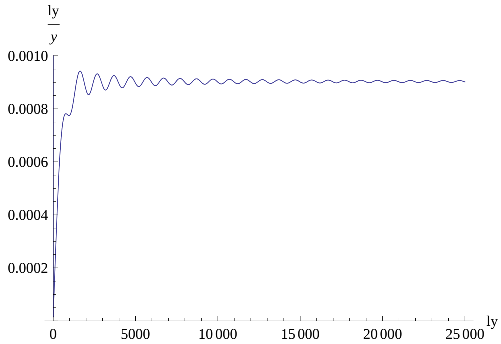
where $\Gamma\left[a,z\right] = \int_z^\infty dt\, \mathrm{e}^{-t}t^{a-1}$ is the upper incomplete Gamma function. For simplicity we already substituted the values of ω_0, ω_2, A_0, and A_2.

For $n = 4$ the differential equation reads: (cf. (87))

$$W_4''(r) + \frac{10}{r}W_4'(r) = \frac{3}{105}A_2^2 f_{\omega,2}(r)^2 \tag{121}$$



(a) $V(r, \alpha, \theta)$ with $V_{\max} = V(r, 0, \pi/2)$ and $V_{\min} = V(r, \pi/4, \pi/2)$



(b) The rotation curve

Figure 3: Fig. (a) shows the potential $V(r, \alpha, \theta)$, where $V_{\max} = V(r, \alpha = 0, \theta = \pi/2)$ and $V_{\min} = V(r, \alpha = \pi/4, \theta = \pi/2)$ and Fig. (b) the rotation curve based on the data of "Simulation # 1". [1]

with solution

$$W_4(r) = -\frac{A_2^2}{r^4\pi 700} - \frac{A_2^2}{r^6 210\pi\sqrt{b}} + \frac{A_2^2}{r^7}\left(\frac{\cos(2\sqrt{b}r)}{140\pi\sqrt{b}^3} - \frac{\sin(2\sqrt{b}r)}{28\sqrt{b}^3\pi}\right) \tag{122}$$

$$-\frac{A_2^2}{r^8}\left(\frac{9}{280b^2\pi} + \frac{19\cos(2\sqrt{b}r)}{280b^2}\pi\right) + \frac{A_2^2\sin(2\sqrt{b}r)}{r^9 20\pi\sqrt{b}^5} \quad . \tag{123}$$

Figure 2 shows the graphs of the different $W_n(r)$. Clearly, $W_0(r)$ dominates the potential function $V(r)$.

Figure 3a shows the potential (111) for $\alpha = 0$ and also for $\alpha = \frac{\pi}{4}$. The potential V behaves approximately like $\ln r$ caused by $W_0(r)$. For simplicity, the rotation curve is not obtained by computing the actual velocities of the test particles, but by taking $v(r) = \sqrt{r|V_{,i}|}$ instead, which gives the velocity of the test particles in circular motion in the spherically symmetric case. Unfortunately, the data of the rotation curve of NGC1300 is not available. But in comparison to other rotation curves of galaxies similar to NGC1300, for which the data is known, the rotation curve we derive displays a reasonable behaviour, namely that the rotation curve is flat from near the centre to the end of the galaxy. Note that by only using V to compute the rotation curve, it only accounts for the dark matter distribution.

3 The Effect of Torsion on Spinning Particles

We now turn to the measurable effects of the torsion field with one being the effect the torsion field has on a spinning particle. We compute the magnitude of the effect and thereby maybe offer a possibility to experimentally detect the torsion field derived in the previous section.

In order to analyse the effect torsion has on a spinning particle we may use two different approaches.

First, we consider a pseudo-classical approach based on [10]. The second approach enters the field of quantum mechanics where we solve the Dirac equation minimally coupled to the torsion field.

3.1 Pseudo-classical Approach

We operate within a supersymmetric model where classical spin particles are non-trivially coupled to the torsion field [10]. Our interest is in the effect torsion has on the polarization vector

$$S_i = -\frac{1}{2}\epsilon_{ijk\,l}S^{jk}u^l \tag{124}$$

where S_{jk} is the spin tensor and u^l the 4-velocity of the particle. In the supersymmetric approach the spin tensor S_{jk} is defined as $S^{jk} = im\xi^j\xi^k$ where the "spin variable" ξ^i is an odd Grassmann variable. Note that in the definition of the spin tensor the factor i is required since the product $\xi^j\xi^k$ is anti-Hermitian. Thus, solving the equations of motion for ξ^i derived below and inserting the respective results into (124) we are able to analyse the effect on spinning particles in a torsion field.

3.1.1 Derivation of the pseudo-classical equations of motion for ξ^i

We briefly summarise the derivation of the equations of motion for ξ^i based on [10]:

Free spinning particle:

The supersymmetric Lagrangian for a free spinning particle reads

$$\mathcal{L} = \frac{m}{2}\eta_{ab}\left(\dot{x}^a\dot{x}^b - i\xi^a\dot{\xi}^b\right) \tag{125}$$

with the dot denoting the differentiation w.r.t. a proper time s and m being the mass of the particle. The x^a and the ξ^a are even and odd Grassmann variables, respectively. We assume the Grassmann algebra to be generated by the four ξ^i. The "spin variable" ξ^i itself has no physical meaning but the product $S^{ij} = im\xi^i\xi^j$ being the spin tensor has.

The Lagrangian is invariant under the supersymmetry transformations

$$\delta x^a = i\epsilon\xi^a \text{ and } \delta\xi^a = \epsilon\dot{x}^a \tag{126}$$

with ϵ being a finite anticommuting parameter.
We introduce an anticommuting evolution parameter θ and a "supercoordinate" $X^a(s,\theta)$. The Taylor expansion of $X^a(s,\theta)$ about $\theta = 0$ gives:

$$X^a(s,\theta) = x^a(s) + i\theta\xi^a(s) \tag{127}$$

with the supersymmetry transformations (126) now expressed as

$$\theta \to \theta + \epsilon \text{ and } s \to s - i\epsilon\theta. \tag{128}$$

Furthermore we introduce a supercovariant derivative

$$D = \frac{\vec{\partial}}{\partial\theta} + i\theta\frac{\partial}{\partial s} \tag{129}$$

being form-invariant under the supersymmetry transformations (128).
Using

$$D^2X^a = \left(\frac{\vec{\partial}}{\partial\theta} + i\theta\frac{\partial}{\partial s}\right)(i\xi^a + i\theta\dot{x}^a) = \left(i\dot{x}^a - \theta\dot{\xi}^a\right) \tag{130}$$

$$= i\left(\dot{x}^a + i\theta\dot{\xi}^a\right) = i\frac{\partial X^a}{\partial s} \tag{131}$$

and

$$\int d\theta = 0 \text{ and } \int d\theta\ \theta = l, \tag{132}$$

where $l \in \mathbb{R}$, we can rearrange the Lagrangian (125) as follows:

$$\mathcal{L} = \frac{m}{2}\eta_{ab}\left(\dot{x}^a\dot{x}^b - i\xi^a\dot{\xi}^b\right) = -\frac{m}{2l}\int d\theta\eta_{ab}\left(-\dot{x}^a\dot{x}^b + i\xi^a\dot{\xi}^b\right)\theta \tag{133}$$

$$= -\frac{m}{2}\int d\theta\eta_{ab}\left(-\dot{x}^a\xi^b - \dot{x}^a\dot{x}^b\theta - i\dot{\xi}^a\xi^b\theta - i\dot{\xi}^a\dot{x}^b\theta\theta\right) \tag{134}$$

$$= -\eta_{ab}\frac{m}{2l}\int d\theta\left(i\dot{x}^a - \theta\dot{\xi}^a\right)\left(i\xi^b + i\theta\dot{x}^b\right) = -\frac{m}{2l}\int d\theta i\frac{\partial X^a}{\partial s}DX_a \tag{135}$$

$$= -\frac{m}{2l}\int d\theta\left(D^2X^a\right)DX_a. \tag{136}$$

Particle coupled to the torsion field:

The Lagrangian in a Riemann-Cartan space-time takes the following form:

$$\mathcal{L} = -\frac{m}{2l}\int d\theta \left(g_{ij}\left(X\right) D^2X^i DX^j + S_{ijk}\left(X\right) DX^i DX^j DX^k\right) \tag{137}$$

$$= \frac{m}{2}\left(g_{ij}\dot{x}^i\dot{x}^j + ig_{ij}\left(\dot{\xi}^i + \hat{\Gamma}^i{}_{mn}\dot{x}^m\xi^n\right)\xi^j + S_{ijk,l}\xi^i\xi^j\xi^k\xi^l\right) \tag{138}$$

with S_{ijk} being a totally antisymmetric torsion tensor, $\hat{\Gamma}^i{}_{mn} = \bar{\Gamma}^i{}_{mn} + 3S_{mn}{}^i$, where $\bar{\Gamma}^i{}_{mn}$ is the Christoffel symbol, and the indices i, j, k,... denote Riemann-Cartan space-time indices.

From (137) to (138) we used

$$\begin{aligned}
-\int d\theta g_{ij}(X)D^2X^iDX^j =& -\int d\theta\left(g_{ij}(x) + i\theta g_{ij,l}\xi^l\right) i\frac{\partial X^i}{\partial s}\left(i\xi^j + i\theta\dot{x}^j\right)\\
=& -\int d\theta\left(g_{ij}(x) + i\theta g_{ij,l}\xi^l\right)\left(i\dot{x}^i - \theta\dot{\xi}^i\right)\left(i\xi^j + i\theta\dot{x}^j\right)\\
=& -\int d\theta\left(g_{ij}(x) + i\theta g_{ij,l}\xi^l\right)\left(-\dot{x}^i\xi^j - \dot{x}^i\dot{x}^j\theta - i\theta\dot{\xi}^i\xi^j\right)\\
=& g_{ij}(x)\left(\dot{x}^i\dot{x}^j + i\dot{\xi}^i\xi^j\right) + ig_{ij,l}\xi^l\dot{x}^i\xi^j\\
=& g_{ij}(x)\left(\dot{x}^i\dot{x}^j + i\dot{\xi}^i\xi^j\right) + i\frac{1}{2}\left(g_{ij,l} + g_{jl,i} - g_{il,j}\right)\xi^l\dot{x}^i\xi^j \qquad (139)\\
=& g_{ij}(x)\left(\dot{x}^i\dot{x}^j + i\dot{\xi}^i\xi^j\right) + ig_{mj}\bar{\Gamma}^m{}_{il}\dot{x}^i\xi^l\xi^j \qquad (140)
\end{aligned}$$

and

$$\begin{aligned}
S_{ijk}\left(X\right)DX^iDX^jDX^k =& \left(S_{ijk} + i\theta S_{ijk,l}\xi^l\right)\left(i\xi^i + i\theta\dot{x}^i\right)\left(i\xi^j + i\theta\dot{x}^j\right)\left(i\xi^k + i\theta\dot{x}^k\right)\\
=& -i\left(S_{ijk}\left(\xi^i\xi^j\xi^k - \xi^i\dot{x}^j\xi^k\theta + \xi^i\xi^j\dot{x}^k\theta + \dot{x}^i\xi^j\xi^k\theta\right) - iS_{ijk,l}\xi^i\xi^j\xi^k\xi^l\theta\right)\\
\Rightarrow\quad -\int d\theta S_{ijk}\left(X\right)DX^iDX^jDX^k =& i3S_{ijk}\dot{x}^i\xi^j\xi^k + S_{ijk,l}\xi^i\xi^j\xi^k\xi^l. \qquad (141)
\end{aligned}$$

Since scalar particles do not couple to torsion, the scalar part ($\xi = 0$) alone cannot determine the supersymmetric Lagrangian in a Riemann-Cartan space-time unlike in cases when coupling a scalar particle to the electromagnetic field. A factor 3 appears in the first term of the r.h.s. of equation (141) which comes from the coupling term involving the third-rank torsion tensor in the supersymmetric Lagrangian (137). Later we will see that in the quantum mechanical approach the same factor 3 appears due to the torsion tensor being multiplied by the product of three gamma matrices.

Since the determinant g of the metric is assumed to be an element of the multiplicative group of invertible even Grassmann numbers we can introduce the inverse g^{ij} of the metric tensor g_{ij} used from (139) to (140). Furthermore we note that due to the supersymmetry transformations (126) the odd Grassmann variables ξ^j transform as vectors and, thus,

the Lagrangian is a scalar under general coordinate transformation.
The variation of the Lagrangian (138) with respect to the metric g_{ij} and the "spin variable" ξ^j yields

$$\ddot{x}^i = -\bar{\Gamma}^i{}_{jk}\dot{x}^j\dot{x}^k + \frac{i}{2}\hat{R}^i{}_{klm}\dot{x}^k\xi^l\xi^k - \frac{1}{2}g^{il}\bar{\nabla}_l S_{[opq,r]}\xi^o\xi^p\xi^q\xi^r \tag{142}$$

$$\dot{\xi}^i = -\hat{\Gamma}^i{}_{jk}\dot{x}^j\xi^k - 2ig^{il}S_{[mno,l]}\xi^m\xi^n\xi^o \tag{143}$$

with $\hat{R}^i{}_{klm}$ being the curvature tensor with respect to the connection $\hat{\Gamma}^i{}_{kl}$ and $\bar{\nabla}$ denoting the Christoffel covariant derivative.
We may focus on the latter equation:
the Euler-Lagrange equation reads

$$\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\xi^u} - \frac{d}{ds}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\dot{\xi}^u} = 0 \tag{144}$$

$$\begin{aligned}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\xi^u} =& ig_{ij}\dot{\xi}^i\delta^j_u + ig_{ij}\hat{\Gamma}^i{}_{mn}\dot{x}^m\left(\xi^n\delta^j_u - \xi^j\delta^n_u\right) + S_{ijk,l}\xi^i\xi^j\xi^k\delta^l_u \\ &- S_{ijk,l}\xi^i\xi^j\xi^l\delta^k_u + S_{ijk,l}\xi^i\xi^k\xi^l\delta^j_u - S_{ijk,l}\xi^j\xi^k\xi^l\delta^i_u \\ =& ig_{iu}\dot{\xi}^i + ig_{iu}2\hat{\Gamma}^i{}_{mn}\dot{x}^m\xi^n - ig_{un,m}\dot{x}^m\xi^n + 4S_{[ijk,u]}\xi^i\xi^j\xi^k\end{aligned} \tag{145}$$

because

$$\begin{aligned}ig_{ij}\bar{\Gamma}^i{}_{mn}\dot{x}^m\left(\xi^n\delta^j_u - \xi^j\delta^n_u\right) =& \frac{i}{2}g_{iu}g^{ik}\left(g_{km,n} + g_{kn,m} - g_{mn,k}\right)\dot{x}^m\xi^n \\ &- \frac{i}{2}g_{ij}g^{il}\left(g_{lm,u} + g_{lu,m} - g_{um,l}\right)\dot{x}^m\xi^j \\ =& \frac{i}{2}\left(g_{um,n} + g_{un,m} - g_{mn,u} - g_{nm,u} - g_{nu,m} + g_{um,n}\right)\dot{x}^m\xi^n \\ =& i\left(g_{um,n} - g_{mn,u}\right)\dot{x}^m\xi^n = i\left(g_{um,n} + g_{un,m} - g_{mn,u}\right)\dot{x}^m\xi^n - ig_{un,m}\dot{x}^m\xi^n \\ =& 2i\bar{\Gamma}_{umn}\dot{x}^m\xi^n - ig_{un,m}\dot{x}^m\xi^n = 2ig_{iu}\bar{\Gamma}^u{}_{mn}\dot{x}^m\xi^n - ig_{un,m}\dot{x}^m\xi^n\end{aligned} \tag{146}$$

and

$$ig_{ij}S_{mn}{}^i\dot{x}^m\left(\xi^n\delta^j_u - \xi^j\delta^n_u\right) = i\left(S_{mnu}\xi^n - S_{muj}\xi^j\right)\dot{x}^m = 2iS_{mnu}\xi^n. \tag{147}$$

Moreover we obtain

$$-\frac{d}{ds}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial\dot{\xi}^u} = -\frac{d}{ds}\left(-ig_{ij}\xi^j\delta^i_u\right) = ig_{uj,m}\dot{x}^m\xi^j + ig_{uj}\dot{\xi}^j. \tag{148}$$

Thus,

$$\mathcal{L}\frac{\overleftarrow{\partial}}{\partial \xi^u} - \frac{d}{ds}\mathcal{L}\frac{\overleftarrow{\partial}}{\partial \dot{\xi}^u} = ig_{iu}\dot{\xi}^i + ig_{iu}2\hat{\Gamma}^i{}_{mn}\dot{x}^m\xi^n - ig_{un,m}\dot{x}^m\xi^n \tag{149}$$

$$+ 4S_{[ijk,u]}\xi^i\xi^j\xi^k + ig_{uj,m}\dot{x}^m\xi^j + ig_{uj}\dot{\xi}^j \tag{150}$$

$$= i2g_{iu}\dot{\xi}^i + ig_{iu}2\hat{\Gamma}^i{}_{mn}\dot{x}^m\xi^n + 4S_{[ijk,u]}\xi^i\xi^j\xi^k = 0 \tag{151}$$

$$\Rightarrow \dot{\xi}^i = -\hat{\Gamma}^i{}_{mn}\dot{x}^m\xi^n - ig^{ui}2S_{[ljk,u]}\xi^l\xi^j\xi^k \tag{152}$$

Returning to our case we have

$$S_{ij}{}^k = \gamma^k{}_{ij}. \tag{153}$$

Thus, the equations of motion (152) read

$$\dot{\xi}^i = -\hat{\Gamma}^i{}_{jk}\dot{x}^j\xi^k - 2ig^{in}\gamma_{[jk\,l,n]}\xi^j\xi^k\xi^l \tag{154}$$

with $\hat{\Gamma}^i{}_{jk} = \bar{\Gamma}^i{}_{jk} + 3\gamma^i{}_{jk}$.

For simplicity we assume the space-time to be flat and hence $\bar{\Gamma}^i{}_{jk} = 0$ since we are only interested in the effect caused by torsion.

Recall that in the previous section we related the torsion tensor to the derivative of the scalar field f in the following way (48):

$$\gamma_{ijk} = \frac{1}{\Upsilon}\epsilon_{ijk}{}^l f_{,l}. \tag{155}$$

The scalar field f is defined as

$$f = A_0\cos(\omega_0 t)f_0(r) - A_2\cos(\omega_0 t - 2\alpha)r^2 f_2(r)\sin^2\theta, \tag{156}$$

where $\alpha \in [0, 2\pi)$, $\theta \in [0, \pi]$, $r \in \mathbb{R}^+$ and $f_0(r)$ as well as $f_2(r)$ are oscillating functions of r ((103), (106)):

$$f_0(r) = \frac{\sin(\sqrt{a}r)}{r} \quad \text{and} \quad f_2(r) = \sqrt{\frac{2}{\pi}\frac{\frac{-3\cos(\sqrt{b}r)}{\sqrt{b}r} - \sin(\sqrt{b}r) + \frac{3\sin(\sqrt{b}r)}{br^2}}{r^3}}\ . \tag{157}$$

Since the trigonometrical functions appearing in the scalar field f combine into spherical harmonics, we can easily change f from spherical to Cartesian coordinates, which simplifies future calculations.

Thus we may start with rearranging the cosine of the second part of the right hand side of equation (156):

$$\begin{aligned}\cos(\omega_0 t - 2\alpha)\, r^2 \sin^2\theta &= [\cos(\omega_0 t)\cos(-2\alpha) - \sin(\omega_0 t)\sin(-2\alpha)]\, r^2 sin^2\theta \\ &= \left[\cos(\omega_0 t)\left(\cos^2\alpha - \sin^2\alpha\right) + \sin(\omega_0 t)\left(2\cos\alpha\sin\alpha\right)\right] r^2\sin^2\theta \\ &= \cos(\omega_0 t)\left(x^2 - y^2\right) + \sin(\omega_0 t)\,(2xy) \end{aligned} \tag{158}$$

with

$$\begin{aligned} x &= r\cos\alpha\sin\theta \\ y &= r\sin\alpha\sin\theta \\ z &= r\cos\theta, \end{aligned} \tag{159}$$

where $x, y, z \in \mathbb{R}_0$.

Hence, the scalar field f expressed in Cartesian coordinates is

$$f(t,x,y,z) = A_0\cos(\omega_0 t) f_0(x,y,z) - A_2\left[\cos(\omega_0 t)\left(x^2-y^2\right) + \sin(\omega_0 t) 2xy\right] f_2(x,y,z). \tag{160}$$

We have denoted $\tilde{f}_i(x,y,z) = f_i(r(x,y,z))$ by the same symbol f_i.

The first and second derivatives of the scalar field f are:

$$\begin{aligned} f_{,0} =& -A_0\sin(\omega_0 t)\omega_0 f_0(x,y,z) - A_2\left[-\sin(\omega_0 t)\omega_0\left(x^2-y^2\right)\right. \\ &\left. + \cos(\omega_0 t)\omega_0 2xy\right] f_2(x,y,z) \end{aligned} \tag{161}$$

$$\begin{aligned} f_{,1} =& A_0\cos(\omega_0 t) f_{0,1}(x,y,z) - A_2\Big[\cos(\omega_0 t)\left(\left(x^2-y^2\right) f_2(x,y,z)\right)_{,1} \\ & + \sin(\omega_0 t)\left(2xy f_2(x,y,z)\right)_{,1}\Big] \end{aligned} \tag{162}$$

$$\begin{aligned} f_{,2} =& A_0\cos(\omega_0 t) f_{0,2}(x,y,z) - A_2\Big[\cos(\omega_0 t)\left(\left(x^2-y^2\right) f_2(x,y,z)\right)_{,2} \\ & + \sin(\omega_0 t)\left(2xy f_2(x,y,z)\right)_{,2}\Big] \end{aligned} \tag{163}$$

$$\begin{aligned} f_{,3} =& A_0\cos(\omega_0 t) f_{0,3}(x,y,z) - A_2\left[\cos(\omega_0 t)\left(x^2-y^2\right)\right. \\ & \left. + \sin(\omega_0 t) 2xy\right] f_{2,3}(x,y,z) \end{aligned} \tag{164}$$

$$\begin{aligned} f_{,00} =& -A_0\cos(\omega_0 t)\omega_0^2 f_0(x,y,z) + A_2\left[\cos(\omega_0 t)\omega_0^2(x^2-y^2)\right. \\ & \left. + \sin(\omega_0 t)\omega_0^2 2xy\right] f_2(x,y,z) \end{aligned} \tag{165}$$

$$\begin{aligned} f_{,11} =& A_0\cos(\omega_0 t) f_{0,11}(x,y,z) - A_2\left[\cos(\omega_0 t)\left[2f_2(x,y,z) + 4x f_{2,1}(x,y,z)\right.\right. \\ & \left. + (x^2-y^2) f_{2,11}(x,y,z)\right] + \sin(\omega_0 t)\left[4y f_{2,1}(x,y,z) + 2xy f_{2,11}(x,y,z)\right]\Big] \end{aligned} \tag{166}$$

$$\begin{aligned} f_{,22} =& A_0 \cos(\omega_0 t) f_{0,22}(x,y,z) - A_2 \left[\cos(\omega_0 t) \left[-2 f_2(x,y,z) - 4y f_{2,2} \right.\right. \\ &\left.\left. + (x^2 - y^2) f_{2,22} \right] + \sin(\omega_0 t) \left[4x f_{2,2}(x,y,z) + 2xy f_{2,22}(x,y,z) \right] \right] \end{aligned} \tag{167}$$

$$\begin{aligned} f_{,33} =& A_0 \cos(\omega_0 t) f_{0,33}(x,y,z) - A_2 \left[\cos(\omega_0 t) \left(x^2 - y^2 \right) \right. \\ &\left. + \sin(\omega_0 t) 2xy \right] f_{2,33}(x,y,z). \end{aligned} \tag{168}$$

3.1.2 Co-moving Particle

We now consider a particle co-moving with the spoke of the torsion field of the galaxy, which we have derived in the last section. Instead of assuming the particle and the spoke to be in motion we take both of them to be at rest. Recall that we already set $\alpha(t) \approx \alpha$ since the rotation of the spoke is very small.

Thus, w.l.o.g. we choose $\dot{x}^j = (1,0,0,0)$. As ${\gamma^i}_{jk}$ is totally antisymmetric, equation (154) yields

$$\begin{aligned} \dot{\xi}^0 =& -2i\eta^{00} \gamma_{[\alpha\beta\gamma,0]} \xi^\alpha \xi^\beta \xi^\gamma \\ =& 2i \frac{1}{4} \left(\gamma_{[\alpha\beta\gamma],0} - \gamma_{[\alpha\beta 0],\gamma} + \gamma_{[\alpha\gamma 0],\beta} - \gamma_{[\beta\gamma 0],\alpha} \right) \xi^\alpha \xi^\beta \xi^\gamma \\ =& \frac{i}{2} \left(\gamma_{\alpha\beta\gamma,0} - \gamma_{\alpha\beta 0,\gamma} + \gamma_{\alpha\gamma 0,\beta} - \gamma_{\beta\gamma 0,\alpha} \right) \xi^\alpha \xi^\beta \xi^\gamma \\ =& \frac{i}{2\Upsilon} \left({\epsilon_{\alpha\beta\gamma}}^0 f_{,00} - {\epsilon_{\alpha\beta 0}}^\gamma f_{,\gamma\gamma} + {\epsilon_{\alpha\gamma 0}}^\beta f_{,\beta\beta} - {\epsilon_{\beta\gamma 0}}^\alpha f_{,\alpha\alpha} \right) \xi^\alpha \xi^\beta \xi^\gamma \\ =& -\frac{3i}{\Upsilon} \left(-f_{,00} + f_{,11} + f_{,22} + f_{,33} \right) \xi^1 \xi^2 \xi^3 \end{aligned} \tag{169}$$

$$\begin{aligned} \dot{\xi}^\alpha =& -3{\gamma^\alpha}_{0\beta} \xi^\beta - 2i\eta^{\alpha\kappa} \gamma_{[ijk,\kappa]} \xi^i \xi^j \xi^k \\ =& -3{\gamma^\alpha}_{0\beta} \xi^\beta - \frac{2i}{\Upsilon} \eta^{\alpha\kappa} {\epsilon_{[ijk|}}^l f_{,l|\kappa]} \xi^i \xi^j \xi^k, \end{aligned} \tag{170}$$

where the Greek letters run over the three spatial coordinates $(1,2,3)$.

Note that since $\xi^i \xi^j = -\xi^j \xi^i$ we obtain

$$\begin{aligned} {\epsilon_{\alpha\beta\gamma}}^0 f_{,00} \xi^\alpha \xi^\beta \xi^\gamma =& {\epsilon_{123}}^0 f_{,00} \xi^1 \xi^2 \xi^3 + {\epsilon_{132}}^0 f_{,00} \xi^3 \xi^1 \xi^3 + {\epsilon_{213}}^0 f_{,00} \xi^2 \xi^1 \xi^3 \\ &+ {\epsilon_{231}}^0 f_{,00} \xi^2 \xi^3 \xi^1 + {\epsilon_{312}}^0 f_{,00} \xi^3 \xi^1 \xi^2 + {\epsilon_{321}}^0 f_{,00} \xi^3 \xi^2 \xi^1 \\ =& -6 g^{00} f_{,00} \xi^1 \xi^2 \xi^3 = 6 f_{,00} \xi^1 \xi^2 \xi^3 \end{aligned} \tag{171}$$

We position the particle on the spoke of the torsion field: $\vec{\bar{x}} = (\bar{x}, 0, 0)$.

Therefore, we find that $f_{0,2} = f_{0,3} = f_{2,2} = f_{2,3} = 0$.

Thus, the components of the torsion tensor $\gamma_{\alpha\, 0\beta}$ and the second derivatives of f are

- $\gamma_{302} = -\gamma_{203} = \frac{1}{\Upsilon} f_{,1} = \frac{1}{\Upsilon} \cos(\omega_0 t) \left(A_0 f_{0,1}(\vec{x}) \Big|_{\vec{x}=\vec{\bar{x}}} - A_2 \left[\bar{x}^2 f_{2,1}(\vec{x}) \Big|_{\vec{x}=\vec{\bar{x}}} + 2\bar{x} f_2(\vec{\bar{x}}) \right] \right)$
  $= \cos(\omega_0 t) \frac{\bar{h}}{3}$

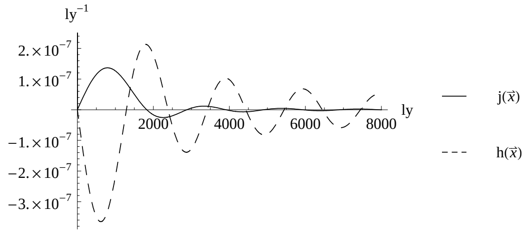


Figure 4: The figure displays the functions $h(\vec{x})$ and $j(\vec{x})$ along the spoke of the torsion field.

- $\gamma_{103} = -\gamma_{301} = \frac{1}{\Upsilon} f_{,2} = -\frac{1}{\Upsilon} A_2 \sin(\omega_0 t) 2\bar{x} f_2(\vec{\bar{x}}) = -\sin(\omega_0 t)\frac{\bar{j}}{3}$
- $\gamma_{201} = -\gamma_{102} = \frac{1}{\Upsilon} f_{,3} = 0$
- $f_{,00} = -A_0 \cos(\omega_0 t)\omega_0^2 f_0(\vec{\bar{x}}) + A_2 \cos(\omega_0 t)\omega_0^2 \bar{x}^2 f_2(\vec{\bar{x}}) = -\cos(\omega_0 t)\omega_0^2 \bar{d}\frac{\Upsilon}{3}$
- $f_{,11} = A_0 \cos(\omega_0 t) f_{0,11}(\vec{x})\Big|_{\vec{x}=\vec{\bar{x}}} - A_2 \cos(\omega_0 t)\left[4\bar{x} f_{2,1}(\vec{x})\Big|_{\vec{x}=\vec{\bar{x}}} + \bar{x}^2 f_{2,11}(\vec{x})\Big|_{\vec{x}=\vec{\bar{x}}} + 2 f_2(\vec{\bar{x}})\right]$
  $= \cos(\omega_0 t)\frac{\Upsilon}{3} h_{,1}(\vec{x})\Big|_{\vec{x}=\vec{\bar{x}}}$
- $f_{,22} = A_0 \cos(\omega_0 t) f_{0,22}(\vec{x})\Big|_{\vec{x}=\vec{\bar{x}}} - A_2 \cos(\omega_0 t)\left[-2 f_2(\vec{\bar{x}}) + \bar{x}^2 f_{2,22}(\vec{x})\Big|_{\vec{x}=\vec{\bar{x}}}\right]$
- $f_{,33} = A_0 \cos(\omega_0 t) f_{0,33}(\vec{x})\Big|_{\vec{x}=\vec{\bar{x}}} - A_2 \cos(\omega_0 t)\bar{x}^2 f_{2,33}(\vec{x})\Big|_{\vec{x}=\vec{\bar{x}}}$

where

$$\bar{h} \equiv h(\vec{\bar{x}}),\ \ h(\vec{x}) \equiv \frac{3}{\Upsilon}\left(A_0 f_{0,1}(\vec{x}) - A_2\left[x^2 f_{2,1}(\vec{x}) + 2x f_2(\vec{x})\right]\right), \tag{172}$$

$$\bar{j} \equiv j(\vec{\bar{x}}),\ \ j(\vec{x}) \equiv \frac{3}{\Upsilon} A_2 2x f_2(\vec{x}) \text{ and} \tag{173}$$

$$\bar{d} \equiv d(\vec{\bar{x}}),\ \ d(\vec{x}) \equiv \frac{3}{\Upsilon}\left(A_0 f_0(\vec{x}) - A_2 x^2 f_2(\vec{x})\right). \tag{174}$$

In order to choose the most convenient value for $\bar{x}$, we take a closer look at the scalar field f along its spoke. Figure 4 shows that on average $|h| \approx 10^2|j|$ along the torsion field's spoke starting from the value $r \approx 4000$ly with $r = \sqrt{x^2+y^2+z^2}$. Since the radius of the galaxy we are looking at is 75000ly, we do not take values of $r < 4000$ly into consideration and neglect j for $r > 4000$ly.

To maximise the effect we take $\bar{x} = 4915$ly as that is a maximum of the function $h(\vec{x})$. Eventually, equations (169) and (170) give:

$$\begin{aligned}\dot{\xi}^0 &= -\frac{3i}{\Upsilon}\left(-f_{,00} + f_{,11} + f_{,22} + f_{,33}\right)\xi^1\xi^2\xi^3 \\ &= -i\cos(\omega_0 t)\bar{k}\,\xi^1\xi^2\xi^3\end{aligned} \tag{175}$$

$$\begin{aligned}
\dot{\xi}^1 =& -3\left(\gamma^1{}_{02}\xi^2+\gamma^1{}_{03}\xi^3\right)-\frac{2i}{\Upsilon}\eta^{11}\epsilon_{[ijk|}{}^{l}f_{,l|1]}\xi^i\xi^j\xi^k\\
=& -\frac{3i}{\Upsilon}\left(-f_{,00}+f_{,11}+f_{,22}+f_{,33}\right)\xi^0\xi^2\xi^3\\
=& -i\cos(\omega_0 t)\bar{k}\,\xi^0\xi^2\xi^3 \qquad (176)\\
\dot{\xi}^2 =& -\left(\gamma^2{}_{01}\xi^1+\gamma^2{}_{03}\xi^3\right)-\frac{2i}{\Upsilon}\eta^{22}\epsilon_{[ijk|}{}^{l}f_{,l|2]}\xi^i\xi^j\xi^k\\
=& \frac{3}{\Upsilon}f_{,1}\xi^3+\frac{3i}{\Upsilon}\left(-f_{,00}+f_{,11}+f_{,22}+f_{,33}\right)\xi^0\xi^1\xi^3\\
=& \cos(\omega_0 t)\left(\bar{h}+i\bar{k}\,\xi^0\xi^1\right)\xi^3 \qquad (177)\\
\dot{\xi}^3 =& -3\left(\gamma^3{}_{01}\xi^1+\gamma^3{}_{02}\xi^2\right)-\frac{2i}{\Upsilon}\eta^{33}\epsilon_{[ijk|}{}^{l}f_{,l|3]}\xi^i\xi^j\xi^k\\
=& -\frac{3}{\Upsilon}f_{,1}\xi^2-\frac{3i}{\Upsilon}\left(-f_{,00}+f_{,11}+f_{,22}+f_{,33}\right)\xi^0\xi^1\xi^2\\
=& -\cos(\omega_0 t)\left(\bar{h}+i\bar{k}\,\xi^0\xi^1\right)\xi^2, \qquad (178)
\end{aligned}$$

where

$$\begin{aligned}
\frac{3}{\Upsilon}\left(-f_{,00}+f_{,11}+f_{,22}+f_{,33}\right) =& \cos(\omega_0 t)\left(\omega_0^2\bar{d}+a_{,1}(\vec{x})\Big|_{\vec{x}=\bar{\vec{x}}}\right.\\
&+\frac{3}{\Upsilon}\left(A_0 f_{0,22}(\vec{x})\Big|_{\vec{x}=\bar{\vec{x}}}-A_2\left(-2f_2(\bar{\vec{x}})+\bar{x}^2 f_{2,22}(\vec{x})\Big|_{\vec{x}=\bar{\vec{x}}}\right)\right)\\
&\left.+\frac{3}{\Upsilon}\left(A_0 f_{0,33}(\vec{x})\Big|_{\vec{x}=\bar{\vec{x}}}-A_2\bar{x}^2 f_{2,33}(\vec{x})\Big|_{\vec{x}=\bar{\vec{x}}}\right)\right)\equiv\cos(\omega_0 t)\bar{k}. \qquad (179)
\end{aligned}$$

We choose the ansatz

$$\xi^i = p^i{}_m(t)\xi_0^m + iq^i{}_{mno}(t)\xi_0^m\xi_0^n\xi_0^o, \qquad (180)$$

where $\xi_0^m = \xi^m(0)$ and $p^i{}_m(t)$ as well as $q^i{}_{mno}(t)$ are arbitrary tensors with $p^i{}_m(0) = \delta^i{}_m$ and $q^i{}_{mno}(0) = 0$, respectively.
We differentiate the ansatz (180) with respect to the time t and compare its coefficients with those of the differential equation (175):

$$\dot{\xi}^0 = \dot{p}^0{}_m\xi_0^m + i\dot{q}^0{}_{mno}\xi_0^m\xi_0^n\xi_0^o \qquad (181)$$

$$\Rightarrow \dot{p}^0{}_m = 0 \ \ \Rightarrow p^0{}_m\xi_0^m = const. \qquad (182)$$

Note that the implication does not hold if e.g. $\xi_0^0 \propto \xi_0^1\xi_0^2\xi_0^3$. This and similar relations will henceforth be excluded.
Thus we claim that

$$\xi^0 = \xi_0^0 + iq^0{}_{mno}\xi_0^m\xi_0^n\xi_0^o. \qquad (183)$$

Analogously we assume

$$\xi^1 = \xi_0^1 + iq^1{}_{abc}\xi_0^a\xi_0^b\xi_0^c. \qquad (184)$$

Next we compute the product $\xi^0\xi^1$:

$$\begin{aligned}\xi^0\xi^1 =& \xi_0^0\xi_0^1 + \xi_0^0 i{q^1}_{abc}\xi_0^a\xi_0^b\xi_0^c + \xi_0^1 i{q^0}_{mno}\xi_0^m\xi_0^n\xi_0^o \\ =& \xi_0^0\xi_0^1 + \xi_0^0 iQ_1\xi_0^1\xi_0^2\xi_0^3 + \xi_0^1 iQ_2\xi_0^0\xi_0^2\xi_0^3 = \xi_0^0\xi_0^1 + i\left(Q_1 - Q_2\right)\xi_0^0\xi_0^1\xi_0^2\xi_0^3,\end{aligned} \tag{185}$$

with Q_i being scalar functions of the time t and $\xi_0^0 {q^1}_{abc}\xi_0^a\xi_0^b\xi_0^c = Q_1\xi_0^0\xi_0^1\xi_0^2\xi_0^3$ due to the identity $\xi_0^i\xi_0^j = -\xi_0^j\xi_0^i$.

Substituting (185) into the differential equations (177) and (178) yields

$$\begin{aligned}\dot{\xi}^2 =& \cos(\omega_0 t)\left(\bar{h} + i\bar{k}\left[\xi_0^0\xi_0^1 + \left(Q_1 - Q_2\right)\xi_0^0\xi_0^1\xi_0^2\xi_0^3\right]\right)\xi^3 \\ =& \cos(\omega_0 t)\left(\bar{h} + i\bar{k}\xi_0^0\xi_0^1\right)\xi^3\end{aligned} \tag{186}$$

$$\begin{aligned}\dot{\xi}^3 =& -\cos(\omega_0 t)\left(\bar{h} + i\bar{k}\left[\xi_0^0\xi_0^1 + \left(Q_1 - Q_2\right)\xi_0^0\xi_0^1\xi_0^2\xi_0^3\right]\right)\xi^2 \\ =& -\cos(\omega_0 t)\left(\bar{h} + i\bar{k}\xi_0^0\xi_0^1\right)\xi^2\end{aligned} \tag{187}$$

or

$$\begin{pmatrix}\dot{\xi}^2 \\ \dot{\xi}^3\end{pmatrix} = \cos(\omega_0 t)\left(\bar{h} + i\bar{k}\xi_0^0\xi_0^1\right)\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}\xi^2 \\ \xi^3\end{pmatrix}. \tag{188}$$

The solution is

$$\begin{pmatrix}\xi^2 \\ \xi^3\end{pmatrix} = \exp\left(\int_0^t \cos(\omega_0 t')dt'\left(\bar{h} + i\bar{k}\xi_0^0\xi_0^1\right)\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\right)\begin{pmatrix}\xi_0^2 \\ \xi_0^3\end{pmatrix}. \tag{189}$$

Carrying out the integration

$$\int_0^t \cos(\omega_0 t')dt' = \frac{\sin(\omega_0 t)}{\omega_0}, \tag{190}$$

the exponential function in (189) gives:

$$\exp\left(\frac{\sin(\omega_0 t)}{\omega_0}\left(\bar{h} + i\bar{k}\xi_0^0\xi_0^1\right)\mathbb{J}\right) \tag{191}$$

with $\mathbb{J} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}$.

Using the identity

$$e^{\nu\mathbb{J}} = \sum_{n=0}^{\infty}\frac{(\nu\mathbb{J})^n}{n!} = \mathbb{1}\sum_{n=0}^{\infty}(-1)^n\frac{\nu^{2n}}{(2n)!} + \mathbb{J}\sum_{n=0}^{\infty}(-1)^n\frac{\nu^{2n+1}}{(2n+1)!} \tag{192}$$

with $\mathbb{J}^2 = -\mathbb{1}$ and $\nu = \frac{\sin(\omega_0 t)}{\omega_0}\left(\bar{h} + i\bar{k}\xi_0^0\xi_0^1\right)$, we expand (191) into a series.
Due to $\xi_0^0\xi_0^1 = -\xi_0^1\xi_0^0$ the exponentiation of the scalar ν yields:

$$\nu^n = \left(\frac{\sin(\omega_0 t)}{\omega_0}\left(\bar{h} + i\bar{k}\xi_0^0\xi_0^1\right)\right)^n = \frac{\sin^n(\omega_0 t)}{\omega_0^n}\left(\bar{h}^n + in\bar{h}^{n-1}\bar{k}\xi_0^0\xi_0^1\right). \tag{193}$$

Thus, defining $\beta(t) = \frac{\sin(\omega_0 t)}{\omega_0}$, the series expansion of the exponential function yields

$$\begin{aligned}
e^{\nu\mathbb{J}} =& \mathbb{1}\sum_{n=0}^{\infty}(-1)^n \frac{\beta^{2n}(t)\left(\bar{h}^{2n} + i2n\bar{h}^{2n-1}\bar{k}\xi_0^0\xi_0^1\right)}{(2n)!} \\
&+ \mathbb{J}\sum_{n=0}^{\infty}(-1)^n \frac{\beta^{2n+1}(t)\left(\bar{h}^{2n+1} + i(2n+1)\bar{h}^{2n}\bar{k}\xi_0^0\xi_0^1\right)}{(2n+1)!} \\
=& \mathbb{1}\sum_{n=0}^{\infty}(-1)^n \left[\frac{\left(\beta(t)\bar{h}\right)^{2n}}{(2n)!} - i\frac{\beta^{2(n+1)}\bar{h}^{2n+1}\bar{k}\xi_0^0\xi_0^1}{(2n+1)!}\right] \\
&+ \mathbb{J}\sum_{n=0}^{\infty}(-1)^n \left[\frac{\left(\beta(t)\bar{h}\right)^{2n+1}}{(2n+1)!} + i\frac{\beta^{2n+1}(t)\bar{h}^{2n}\bar{k}\xi_0^0\xi_0^1}{(2n)!}\right] \\
=& \mathbb{1}\left[\cos(\beta(t)\bar{h}) - i\sin(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right] \\
&+ \mathbb{J}\left[\sin(\beta(t)\bar{h}) + i\cos(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right]
\end{aligned} \tag{194}$$

and, eventually, the solutions to (186) and (187) read

$$\begin{aligned}
\xi^2 =& \left[\cos(\beta(t)\bar{h}) - i\sin(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right]\xi_0^2 \\
&+ \left[\sin(\beta(t)\bar{h}) + i\cos(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right]\xi_0^3,
\end{aligned} \tag{195}$$

$$\begin{aligned}
\xi^3 =& -\left[\sin(\beta(t)\bar{h}) + i\cos(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right]\xi_0^2 \\
&+ \left[\cos(\beta(t)\bar{h}) - i\sin(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right]\xi_0^3.
\end{aligned} \tag{196}$$

Next, we turn to the differential equations (175) and (176):
using

$$\begin{aligned}
\xi^2\xi^3 =& \left[\cos(\beta(t)\bar{h}) - i\sin(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right]\xi_0^2 \\
&\cdot\left[\cos(\beta(t)\bar{h}) - i\sin(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right]\xi_0^3 \\
&- \left[\sin(\beta(t)\bar{h}) + i\cos(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right]\xi_0^3 \\
&\cdot\left[\sin(\beta(t)\bar{h}) + i\cos(\beta(t)\bar{h})\beta(t)\bar{k}\xi_0^0\xi_0^1\right]\xi_0^2 \\
=& \left(\cos^2(\beta(t)\bar{h}) + \sin^2(\beta(t)\bar{h})\right)\xi_0^2\xi_0^3 \\
&- i\cos(\beta(t)\bar{h})\sin(\beta(t)\bar{h})\beta(t)\bar{k}\left[\xi_0^0\xi_0^1\xi_0^2\xi_0^3 + \xi_0^2\xi_0^0\xi_0^1\xi_0^3\right. \\
&\left.+\xi_0^3\xi_0^0\xi_0^1\xi_0^2 + \xi_0^0\xi_0^1\xi_0^3\xi_0^2\right] \\
=& \xi_0^2\xi_0^3
\end{aligned} \tag{197}$$

the equations read

$$\dot{\xi}^0 = -\, i\cos(\omega_0 t)\bar{k}\xi^1\xi_0^2\xi_0^3 \tag{198}$$

$$\dot{\xi}^1 = -\, i\cos(\omega_0 t)\bar{k}\xi^0\xi_0^2\xi_0^3. \tag{199}$$

or, in matrix representation,

$$\begin{pmatrix}\dot{\xi}^0\\ \dot{\xi}^1\end{pmatrix} = -i\cos(\omega_0 t)\bar{k}\xi_0^2\xi_0^3\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\xi^0\\ \xi^1\end{pmatrix} \tag{200}$$

with the solution being

$$\begin{pmatrix}\xi^0\\ \xi^1\end{pmatrix} = \exp\left(-i\int_0^t \cos(\omega_0 t')dt'\bar{k}\xi_0^2\xi_0^3\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)\begin{pmatrix}\xi_0^0\\ \xi_0^1\end{pmatrix}. \tag{201}$$

Due to the nilpotency of the product $\xi_0^2\xi_0^3$ the series expansion of the exponential function appearing in (201) yields

$$\exp\left(-i\frac{\sin(\omega_0 t)}{\omega_0}\bar{k}\xi_0^2\xi_0^3\mathbb{B}\right) = \mathbb{1} - i\frac{\sin(\omega_0 t)}{\omega_0}\bar{k}\xi_0^2\xi_0^3\mathbb{B} \tag{202}$$

with $\mathbb{B} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

Thus, the solution (201) reads

$$\xi^0 = \xi_0^0 - i\frac{\sin(\omega_0 t)}{\omega_0}\bar{k}\xi_0^2\xi_0^3\xi_0^1 = \xi_0^0 - i\beta(t)\bar{k}\xi_0^2\xi_0^3\xi_0^1 \tag{203}$$

$$\xi^1 = \xi_0^1 - i\frac{\sin(\omega_0 t)}{\omega_0}\bar{k}\xi_0^2\xi_0^3\xi_0^0 = \xi_0^1 - i\beta(t)\bar{k}\xi_0^2\xi_0^3\xi_0^0 \tag{204}$$

The results above enable us to compute the spin tensor $S^{ij} = im\xi^i\xi^j$. The respective components read

$$\begin{aligned}S^{01} =& im\xi^0\xi^1 = im\left[\xi_0^0 - i\beta(t)\bar{k}\xi_0^1\xi_0^2\xi_0^3\right]\left[\xi_0^1 - i\beta(t)\xi_0^0\xi_0^2\xi_0^3\right]\\ =& im\xi_0^0\xi_0^1 = S_0^{01}\end{aligned} \tag{205}$$

$$\begin{aligned}S^{02} =& im\xi^0\xi^2 = im\left[\xi_0^0 - i\beta(t)\bar{k}\xi_0^1\xi_0^2\xi_0^3\right]\left[\left(\cos(\beta(t)\bar{h}) - i\sin(\beta(t)\bar{h})\bar{k}\xi_0^0\xi_0^1\right)\xi_0^2\right.\\ &\left. + \left(\sin(\beta(t)\bar{h}) + i\cos(\beta(t)\bar{h})\bar{k}\xi_0^0\xi_0^1\right)\xi_0^3\right]\\ =& im\left[\cos(\beta(t)\bar{h})\xi_0^0\xi_0^2 + \sin(\beta(t)\bar{h})\xi_0^0\xi_0^3\right]\\ =& \cos(\beta(t)\bar{h})S_0^{02} + \sin(\beta(t)\bar{h})S_0^{03}\end{aligned} \tag{206}$$

$$
\begin{aligned}
S^{03} =& im\xi^0\xi^3 = im\left[\xi_0^0 - i\beta(t)\bar{k}\xi_0^1\xi_0^2\xi_0^3\right]\left[\left(\cos(\beta(t)\bar{h}) - i\sin(\beta(t)\bar{h})\bar{k}\xi_0^0\xi_0^1\right)\xi_0^3\right.\\
&\left. - \left(\sin(\beta(t)\bar{h}) + i\cos(\beta(t)\bar{h})\bar{k}\xi_0^0\xi_0^1\right)\xi_0^2\right]\\
=& im\left[\cos(\beta(t)\bar{h})\xi_0^0\xi_0^3 - \sin(\beta(t)\bar{h})\xi_0^0\xi_0^2\right]\\
=& \cos(\beta(t)\bar{h})S_0^{03} - \sin(\beta(t)\bar{h})S_0^{02} \qquad (207)\\
S^{12} =& im\xi^1\xi^2 = im\left[\xi_0^1 - i\beta(t)\bar{k}\xi_0^0\xi_0^2\xi_0^3\right]\left[\left(\cos(\beta(t)\bar{h}) - i\sin(\beta(t)\bar{h})\bar{k}\xi_0^0\xi_0^1\right)\xi_0^2\right.\\
&\left. + \left(\sin(\beta(t)\bar{h}) + i\cos(\beta(t)\bar{h})\bar{k}\xi_0^0\xi_0^1\right)\xi_0^3\right]\\
=& im\left[\cos(\beta(t)\bar{h})\xi_0^1\xi_0^2 + \sin(\beta(t)\bar{h})\xi_0^1\xi_0^3\right]\\
=& \cos(\beta(t)\bar{h})S_0^{12} + \sin(\beta(t)\bar{h})S_0^{13} \qquad (208)\\
S^{13} =& im\xi^1\xi^3 = im\left[\xi_0^1 - i\beta(t)\bar{k}\xi_0^0\xi_0^2\xi_0^3\right]\left[\left(\cos(\beta(t)\bar{h}) - i\sin(\beta(t)\bar{h})\bar{k}\xi_0^0\xi_0^1\right)\xi_0^3\right.\\
&\left. - \left(\sin(\beta(t)\bar{h}) + i\cos(\beta(t)\bar{h})\bar{k}\xi_0^0\xi_0^1\right)\xi_0^2\right]\\
=& im\left[\cos(\beta(t)\bar{h})\xi_0^1\xi_0^3 - \sin(\beta(t)\bar{h})\xi_0^1\xi_0^2\right]\\
=& \cos(\beta(t)\bar{h})S_0^{13} - \sin(\beta(t)\bar{h})S_0^{12} \qquad (209)\\
S^{23} =& im\xi^2\xi^3 = im\xi_0^2\xi_0^3 = S_0^{23}. \qquad (210)
\end{aligned}
$$

Note that the same solution for S^{ij} would have been obtained if we have neglected the cubic terms in the equations of motion (169) and (170) from the outset.
Therefore, the polarization vector $S_i = -\frac{1}{2}\epsilon_{ijk\,l}S^{jk}u^l$ (124) for a particle at rest reads:

$$
\begin{aligned}
S_0 =& -\frac{1}{2}\epsilon_{0jk0}S^{jk}u^0 = 0 \qquad (211)\\
S_1 =& -\frac{1}{2}\epsilon_{1jk0}S^{jk}u^0 = -\frac{1}{2}\left(\epsilon_{1230}S^{23} + \epsilon_{1320}S^{32}\right) = S^{23}\\
=& im\xi_0^2\xi_0^3 = \left(S_1\right)_0 \qquad (212)\\
S_2 =& -\frac{1}{2}\epsilon_{2jk0}S^{jk}u^0 = -\frac{1}{2}\left(\epsilon_{2130}S^{13} + \epsilon_{2310}S^{31}\right) = S^{31}\\
=& im\left[\cos(\beta(t)\bar{h})\xi_0^3\xi_0^1 - \sin(\beta(t)\bar{h})\xi_0^2\xi_0^1\right]\\
=& \cos(\beta(t)\bar{h})\left(S_2\right)_0 + \sin(\beta(t)\bar{h})\left(S_3\right)_0 \qquad (213)\\
S_3 =& -\frac{1}{2}\epsilon_{3jk0}S^{jk}u^0 = -\frac{1}{2}\left(\epsilon_{3120}S^{12} + \epsilon_{3210}S^{21}\right) = S^{12}\\
=& im\left[\cos(\beta(t)\bar{h})\xi_0^1\xi_0^2 + \sin(\beta(t)\bar{h})\xi_0^1\xi_0^3\right]\\
=& \cos(\beta(t)\bar{h})\left(S_3\right)_0 - \sin(\beta(t)\bar{h})\left(S_2\right)_0. \qquad (214)
\end{aligned}
$$

Evidently the torsion field causes the spatial polarization vector S_α to rotate about the x-axis. The rotation axis points in the same direction as the spatial gradient of the scalar field f. Since the argument of the trigonometric function $\beta(t)\bar{h}$ is an oscillating function of t, the angular velocity changes over time. $\beta(t)\bar{h}$ takes values from $\bar{h}\beta\left(\frac{\pi}{2\omega_0}\right) = -8.3\cdot 10^{-7}$

to $\bar{h}\beta\left(\frac{3\pi}{2\omega_0}\right) = 8.3 \cdot 10^{-7}$ with periodicity $T_0 = 2\pi/\omega_0 = 63.5$y. We notice that whenever $\beta(t)\bar{h}$ changes its sign the rotation axis gets reversed resulting in a change of the sense of rotation every 31.75 years.

As already noted the nonlinear terms appearing in the solutions (195), (196), (203) and (204) do not contribute to the spin tensor and, consequently, they do not contribute to the polarization vector.

Next, we compute the amplitude of the rotation of the polarization vector:
previously we saw that the argument $\bar{h}\beta(t)$ of the trigonometrical functions appearing in (213) as well as in (214) takes values only from $-8.3 \cdot 10^{-7}$ to $8.3 \cdot 10^{-7}$. Thus, the resulting rotation about the x-axis is very small and we can approximate the cosine and sine as follows:

$$\cos\left(\bar{h}\frac{\sin(\omega_0 t)}{\omega_0}\right) \approx 1 - \frac{1}{2}\left(\bar{h}\frac{\sin(\omega_0 t)}{\omega_0}\right)^2 \tag{215}$$

$$\sin\left(\bar{h}\frac{\sin(\omega_0 t)}{\omega_0}\right) \approx \bar{h}\frac{\sin(\omega_0 t)}{\omega_0} \tag{216}$$

The maximal amplitude of the rotation is reached at $t = \pi/2\omega_0$:

$$S_2 = \left(1 - 10^{-13}\right)(S_2)_0 - 8.3 \cdot 10^{-7}(S_3)_0 \tag{217}$$

$$S_3 = \left(1 - 10^{-13}\right)(S_3)_0 + 8.3 \cdot 10^{-7}(S_2)_0 \tag{218}$$

3.1.3 Radially moving particle

Recall that on solving equation (89) we derived a Newtonian potential for the dark matter density μ_{DM}. Operating within the Newtonian limit it is consistent to approximate a geodesic by a radial trajectory along the spoke of the torsion field.

We choose the 4-velocity of the particle to be $\dot{x}^i = \frac{dx^i}{ds} = \frac{dx^i}{dt}\frac{dt}{ds} = \gamma_v(1, v, 0, 0)$, where s denotes the proper time of the moving particle and $\gamma_v = 1/\sqrt{1-v^2}$.

As previously noted, the torsion field is related to the derivative of the scalar field f ($\gamma_{ijk} = \frac{1}{\Upsilon}\epsilon_{ijk}{}^{l} f_{,l}$ (48)), which, expressed in Cartesian coordinates, is given by (161) - (164). Since the particle is moving along the spoke, we have $y = z = 0$.

Recall that, due to its time oscillation, the torsion field changes its sign every 31.75y. Therefore, we may focus on this particular period.

The wavelength $\lambda_{f,i}$ (95) of the spatial oscillation of the gradient $f_{,i}$, which is given by the functions $h(\vec{x})$ (172), $j(\vec{x})$ (173) and $d(\vec{x})$ (174), is $\lambda_{f,i} \approx 2000$ly. Hence, even for the case of a particle with velocity $v = 1$ it is consistent to approximate the spatial oscillations $h(\vec{x})$, $j(\vec{x})$ and $d(\vec{x})$ by setting $h(\vec{x}) = h(\vec{\bar{x}}) \equiv \bar{h}$, $j(\vec{x}) = j(\vec{\bar{x}}) \equiv \bar{j}$ and $d(\vec{x}) = d(\vec{\bar{x}}) \equiv \bar{d}$ with $\vec{\bar{x}} = (\bar{x}, 0, 0)$.

Thus, using (172), (173) and (174) the gradient of the scalar field f (164) reads

$$f_{,0} = -\sin(\omega_0 t)\omega_0 \frac{\bar{d}\Upsilon}{3} \tag{219}$$

$$f_{,1} = \cos(\omega_0 t)\frac{\bar{h}\Upsilon}{3} \tag{220}$$

$$f_{,2} = -\sin(\omega_0 t)\frac{\bar{j}\Upsilon}{3} \approx 0 \text{ for } x > 4000\text{ly} \tag{221}$$

$$f_{,3} = 0 \tag{222}$$

Hence, using (169) and (170) the equations of motion (154) read

$$\begin{aligned}
\dot{\xi}^0 &= -3\gamma_v\left({\gamma^0}_{12}v\xi^2 + {\gamma^0}_{13}v\xi^3\right) - 2i\eta^{00}\gamma_{[jkl,0]}\xi^j\xi^k\xi^l \\
&= \frac{3\gamma_v v}{\Upsilon}\left(f_{,3}\xi^2 - f_{,2}\xi^3\right) - \frac{3i}{\Upsilon}\left(-f_{,00} + f_{,11} + f_{,22} + f_{,33}\right)\xi^1\xi^2\xi^3 \\
&= -\frac{3i}{\Upsilon}\left(-f_{,00} + f_{,11} + f_{,22} + f_{,33}\right)\xi^1\xi^2\xi^3
\end{aligned} \tag{223}$$

$$\begin{aligned}
\dot{\xi}^1 &= -3\gamma_v\left({\gamma^1}_{02}\xi^2 + {\gamma^1}_{03}\xi^3\right) - 2i\eta^{11}\gamma_{[jkl,1]}\xi^j\xi^k\xi^l \\
&= -\frac{3\gamma_v}{\Upsilon}\left(-f_{,3}\xi^2 + f_{,2}\xi^3\right) - \frac{3i}{\Upsilon}\left(-f_{,00} + f_{,11} + f_{,22} + f_{,33}\right)\xi^0\xi^2\xi^3 \\
&= -\frac{3i}{\Upsilon}\left(-f_{,00} + f_{,11} + f_{,22} + f_{,33}\right)\xi^0\xi^2\xi^3
\end{aligned} \tag{224}$$

$$\begin{aligned}
\dot{\xi}^2 &= -3\gamma_v\left({\gamma^2}_{01}\xi^1 + {\gamma^2}_{03}\xi^3 + {\gamma^2}_{10}v\xi^0 + {\gamma^2}_{13}v\xi^3\right) - 2i\eta^{22}\gamma_{[jkl,2]}\xi^j\xi^k\xi^l \\
&= -\frac{3\gamma_v}{\Upsilon}\left(f_{,3}\xi^1 - f_{,1}\xi^3 - f_{,3}v\xi^0 - f_{,0}v\xi^3\right) + \frac{3i}{\Upsilon}\left(-f_{,00} + f_{,11} + f_{,22} + f_{,33}\right)\xi^0\xi^1\xi^3 \\
&= \frac{3\gamma_v}{\Upsilon}\left(f_{,1} + f_{,0}v\right)\xi^3 + \frac{3i}{\Upsilon}\left(-f_{,00} + f_{,11} + f_{,22} + f_{,33}\right)\xi^0\xi^1\xi^3
\end{aligned} \tag{225}$$

$$\begin{aligned}
\dot{\xi}^3 &= -3\gamma_v\left({\gamma^3}_{01}\xi^1 + {\gamma^3}_{02}\xi^2 + {\gamma^3}_{12}v\xi^2 + {\gamma^3}_{10}v\xi^0\right) - 2ig^{33}\gamma_{[jkl,3]}\xi^j\xi^k\xi^l \\
&= -\frac{3\gamma_v}{\Upsilon}\left(-f_{,2}\xi^1 + f_{,1}\xi^2 + f_{,0}v\xi^2 + f_{,2}v\xi^0\right) - \frac{3i}{\Upsilon}\left(-f_{,00} + f_{,11} + f_{,22} + f_{,33}\right)\xi^0\xi^1\xi^2 \\
&= -\frac{3\gamma_v}{\Upsilon}\left(f_{,1} + f_{,0}v\right)\xi^2 - \frac{3i}{\Upsilon}\left(-f_{,00} + f_{,11} + f_{,22} + f_{,33}\right)\xi^0\xi^1\xi^2,
\end{aligned} \tag{226}$$

where $\dot{\xi}^i = d\xi^i/ds$ with s being approximately the proper time. Thus, $\frac{d\xi^i(s(t))}{ds} = \frac{d\xi^i}{dt}\frac{dt}{ds} \equiv \gamma_v \xi'^i$.

Substituting the constants $\bar{h}$, $\bar{d}$, and $\bar{k}$ defined in (172), (174) and (179) in equations (223) - (226) gives

$$\dot{\xi}^0 = \gamma_v\xi'^0 = -i\cos(\omega_0 t)\bar{k}\xi^1\xi^2\xi^3 \tag{227}$$

$$\dot{\xi}^1 = \gamma_v\xi'^1 = -i\cos(\omega_0 t)\bar{k}\xi^0\xi^2\xi^3 \tag{228}$$

$$\dot{\xi}^2 = \gamma_v\xi'^2 = \gamma_v\left[\cos(\omega_0 t)\bar{h} - \sin(\omega_0 t)\omega_0\bar{d}v\right]\xi^3 + i\cos(\omega_0 t)\bar{k}\xi^0\xi^1\xi^3 \tag{229}$$

$$\dot{\xi}^3 = \gamma_v\xi'^3 = -\gamma_v\left[\cos(\omega_0 t)\bar{h} - \sin(\omega_0 t)\omega_0\bar{d}v\right]\xi^2 - i\cos(\omega_0 t)\bar{k}\xi^0\xi^1\xi^2. \tag{230}$$

The equations of motion take the same form as they do in the case of a particle at rest except for some additional terms involving the velocity v.
The fact that the Grassmann variables ξ^i appear in the same order in (227) - (230) as they do in (175) - (178) enables us to solve the equations (227) - (230) using the ansatz (180). Therefore, when computing the spin tensor $S^{ij} = im\xi^i\xi^j$ all nonlinear terms appearing in the solutions ξ^i cancel.
So, being only interested in the effect torsion has on the polarization vector $S_i = -\frac{1}{2}\epsilon_{ijkl}S^{jk}u^l$, with S^{jk} being the spin tensor and u^l being the 4-velocity of the particle, we may neglect any nonlinear term in the equations of motion (227) - (230).
Thus, equations (227) - (230) read

$$\xi'^0 = 0 \tag{231}$$

$$\xi'^1 = 0 \tag{232}$$

$$\xi'^2 = \left[\cos(\omega_0 t)\bar{h} - \sin(\omega_0 t)\omega_0 \bar{d}v\right]\xi^3 \tag{233}$$

$$\xi'^3 = -\left[\cos(\omega_0 t)\bar{h} - \sin(\omega_0 t)\omega_0 \bar{d}v\right]\xi^2. \tag{234}$$

In matrix representation the equations (233) and (234) read:

$$\begin{pmatrix} \xi'^2 \\ \xi'^3 \end{pmatrix} = \left[\cos(\omega_0 t)\bar{h} - \sin(\omega_0 t)\omega_0 \bar{d}v\right] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \xi^2 \\ \xi^3 \end{pmatrix} \tag{235}$$

with the solution being

$$\begin{pmatrix} \xi^2 \\ \xi^3 \end{pmatrix} = \exp\left(\int_0^t \left[\cos(\omega_0 t')\bar{h} - \sin(\omega_0 t')\omega_0 \bar{d}v\right] dt' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \begin{pmatrix} \xi_0^2 \\ \xi_0^3 \end{pmatrix}. \tag{236}$$

Analogously to the case of a particle at rest we expand the exponential function in equation (236) into a series using the identity

$$e^{\phi\mathbb{J}} = \sum_{n=0}^{\infty} \frac{(\phi\mathbb{J})^n}{n!} = \mathbb{1}\sum_{n=0}^{\infty}(-1)^n \frac{\phi^{2n}}{(2n)!} + \mathbb{J}\sum_{n=0}^{\infty}(-1)^n \frac{\phi^{2n+1}}{(2n+1)!} \tag{237}$$

$$= \mathbb{1}\cos(\phi) + \mathbb{J}\sin(\phi) \tag{238}$$

with $\mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\phi = \int_0^t \left[\cos(\omega_0 t')\bar{h} - \sin(\omega_0 t')\omega_0 \bar{d}v\right] dt'$ and obtain:

$$\begin{pmatrix} \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{pmatrix} \begin{pmatrix} \xi_0^2 \\ \xi_0^3 \end{pmatrix}. \tag{239}$$

The integration gives:

$$\phi(t) = \int_0^t \left[\cos(\omega_0 t')\bar{h} - \sin(\omega_0 t')\omega_0 \bar{d} v\right] dt' \tag{240}$$

$$= \bar{h}\frac{\sin(\omega_0 t)}{\omega_0} + v\bar{d}(\cos(\omega_0 t) - 1) \tag{241}$$

In comparison with the case of a particle at rest we see that ξ^i undergoes an additional rotation caused by $f_{,0}$. Let us consider a simplified case where $\bar{h} = 0$ and, thus, $f_{,i} = (f_{,0}, 0, 0, 0)$:
The equations of motion (154) read

$$\dot{\xi}^i = -3\gamma_v \left(\gamma^i{}_{0k}\xi^k + v\gamma^i{}_{1l}\xi^l\right), \tag{242}$$

where we neglected the nonlinear terms. Since the torsion tensor is related to the scalar field via $\gamma_{ijk} \propto \epsilon_{ijk}{}^l f_{,l}$, all but the components $\gamma_{\alpha\beta\gamma}$ equal zero where α, β, γ run over the three spatial components $(1, 2, 3)$.
Thus, (242) reduces to

$$\dot{\xi}^\alpha = \gamma_v \xi'^\alpha = -3v\gamma_v\gamma^\alpha{}_{1\beta}\xi^\beta. \tag{243}$$

Due to the totally antisymmetric character of $\gamma_{\alpha\beta\gamma}$ we obtain:

$$\xi'^1 = -3v\gamma^1{}_{1\beta}\xi^\beta = 0 \tag{244}$$

$$\xi'^2 = -3v\gamma^2{}_{13}\xi^3 = \frac{3v}{\Upsilon} f_{,0}\xi^3 \tag{245}$$

$$\xi'^3 = -3v\gamma^3{}_{12}\xi^2 = -\frac{3v}{\Upsilon} f_{,0}\xi^2, \tag{246}$$

with the solutions

$$\xi^1 = const. \tag{247}$$

$$\xi^2 = \cos\left(v\bar{d}\left(\cos(\omega_0 t) - 1\right)\right)\xi_0^2 + \sin\left(v\bar{d}\left(\cos(\omega_0 t) - 1\right)\right)\xi_0^3 \tag{248}$$

$$\xi^3 = -\sin\left(v\bar{d}\left(\cos(\omega_0 t) - 1\right)\right)\xi_0^2 + \cos\left(v\bar{d}\left(\cos(\omega_0 t) - 1\right)\right)\xi_0^3. \tag{249}$$

Thus, the rotation axis is identical to the direction of motion of the particle. If $v = 0$ this effect vanishes. Hence we conclude that the rotation in (239) is a superposition of two rotations with one being caused by the spatial component and the other one by the time component of the gradient of the scalar field f. Since in (239) the rotation axes of both rotations, namely, the direction of motion and the spatial gradient of the scalar field f, coincide, we obtain two rotations about the same axis. As the plane of rotation is perpendicular to the direction of motion we do not expect to find special relativistic effects like Lorentz contractions. For a particle that is moving along the y-axis the rotation axes point in different directions. The resulting plane of rotation is no longer perpendicular to the direction of motion and, thus, due to special relativistic length-contraction, the rotation appears elliptic.

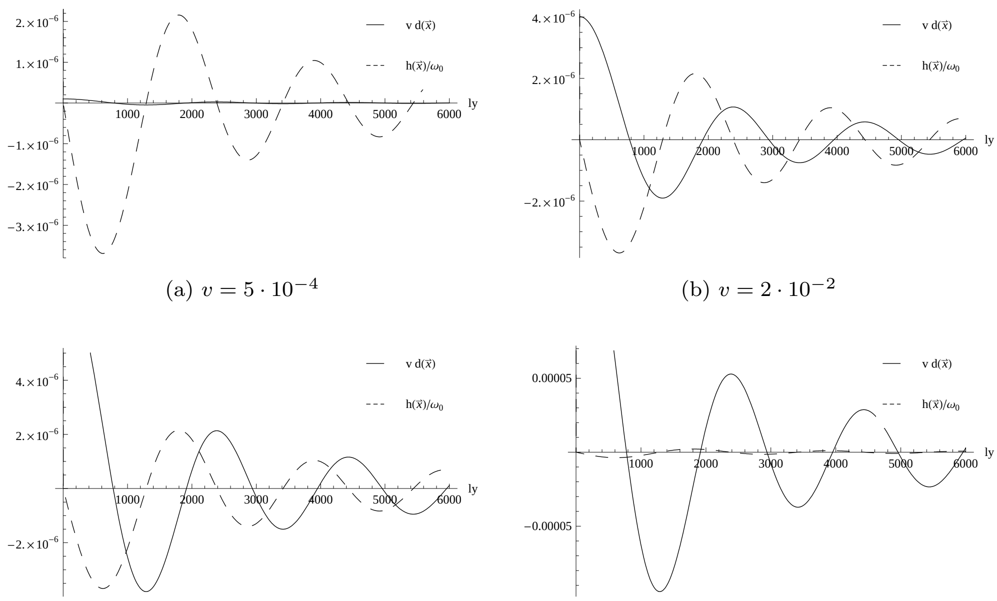
The spin tensor $S^{ij} = im\xi^i\xi^j$ is obtained analogously to the case of the particle at rest ((205) - (210)):

$$S^{01} =im\xi_0^0\xi_0^1 = S_0^{01} \tag{250}$$
$$S^{02} =im\left[\cos\phi(t)\xi_0^0\xi_0^2 + \sin\phi(t)\xi_0^0\xi_0^3\right] = \cos\phi(t)S_0^{02} + \sin\phi(t)S_0^{03} \tag{251}$$
$$S^{03} =im\left[\cos\phi(t)\xi_0^0\xi_0^3 - \sin\phi(t)\xi_0^0\xi_0^2\right] = \cos\phi(t)S_0^{03} - \sin\phi(t)S_0^{02} \tag{252}$$
$$S^{12} =im\left[\cos\phi(t)\xi_0^1\xi_0^2 + \sin\phi(t)\xi_0^1\xi_0^3\right] = \cos\phi(t)S_0^{12} + \sin\phi(t)S_0^{13} \tag{253}$$
$$S^{13} =im\left[\cos\phi(t)\xi_0^1\xi_0^3 - \sin\phi(t)\xi_0^1\xi_0^2\right] = \cos\phi(t)S_0^{13} - \sin\phi(t)S_0^{12} \tag{254}$$
$$S^{23} =im\xi_0^2\xi_0^3 = S_0^{23}. \tag{255}$$

Hence the polarization vector $S_i = -\epsilon_{ijkl}S^{jk}u^l$ reads

$$S_0 =-\frac{1}{2}\epsilon_{0jk1}S^{jk}u^1 = -\frac{1}{2}\epsilon_{0231}S^{23}v\gamma_v - \frac{1}{2}\epsilon_{0321}S^{32}v\gamma_v = -v\gamma_v S_0^{23} = (S_0)_0 \tag{256}$$
$$S_1 =-\frac{1}{2}\epsilon_{1jk0}S^{jk}u^0 = -\frac{1}{2}\epsilon_{1230}S^{23}\gamma_v - \frac{1}{2}\epsilon_{1320}S^{32}\gamma_v = \gamma_v S_0^{23} = (S_1)_0 \tag{257}$$
$$\begin{aligned} S_2 =& -\frac{1}{2}\epsilon_{2jk0}S^{jk}u^0 - \frac{1}{2}\epsilon_{2jk1}S^{jk}u^1 \\ =& -\frac{1}{2}\left(\epsilon_{2130}S^{13} + \epsilon_{2310}S^{31}\right)\gamma_v - \frac{1}{2}\left(\epsilon_{2031}S^{03} + \epsilon_{2301}S^{30}\right)v\gamma_v \\ =& -\gamma_v\left[\cos\phi(t)S_0^{13} - \sin\phi(t)S_0^{12} - v\left(\cos\phi(t)S_0^{03} - \sin\phi(t)S_0^{02}\right)\right] \\ =&\gamma_v\left[\cos\phi(t)\left(-S_0^{13} + vS_0^{03}\right) + \sin\phi(t)\left(S_0^{12} - vS_0^{02}\right)\right] \\ =& \cos\phi(t)\left(S_2\right)_0 + \sin\phi(t)\left(S_3\right)_0 \end{aligned} \tag{258}$$
$$\begin{aligned} S_3 =& -\frac{1}{2}\epsilon_{3jk0}S^{jk}u^0 - \frac{1}{2}\epsilon_{3jk1}S^{jk}u^1 \\ =& -\frac{1}{2}\left(\epsilon_{3120}S^{12} + \epsilon_{3210}S^{21}\right)\gamma_v - \frac{1}{2}\left(\epsilon_{3021}S^{02} + \epsilon_{3201}S^{20}\right)v\gamma_v \\ =&\gamma_v\left[\cos\phi(t)S_0^{12} + \sin\phi(t)S_0^{13} - v\left(\cos\phi(t)S_0^{02} + \sin\phi(t)S_0^{03}\right)\right] \\ =&\gamma_v\left[\cos\phi(t)\left(S_0^{12} - vS_0^{02}\right) - \sin\phi(t)\left(-S_0^{13} + vS_0^{03}\right)\right] \\ =& \cos\phi(t)\left(S_3\right)_0 - \sin\phi(t)\left(S_2\right)_0, \end{aligned} \tag{259}$$

with $u^l = \gamma_v(1, v, 0, 0)$ being the 4-velocity of the particle.
$(S_2)_0 = \gamma_v\left(-S_0^{13} + vS_0^{03}\right)$ and $(S_3)_0 = \gamma_v\left(S_0^{12} - vS_0^{02}\right)$ consist of two terms with one being proportional to the velocity v. This is due to the fact that a moving particle is subject to two rotations, where one corresponds to the rotation a particle at rest under-



(a) $v = 5 \cdot 10^{-4}$

(b) $v = 2 \cdot 10^{-2}$

(c) $v = 4 \cdot 10^{-2}$

(d) $v = 0.99$

Figure 5: The functions $h(\vec{x})/\omega_0$ and $vd(\vec{x})$ displayed in x-direction for different velocities. Note that the oscillations in time have been neglected.

goes and the other one is the additional rotation a moving particle experiences.

As we expected, the polarization vector undergoes a rotation about the x-axis. The rotation is very small and changes the direction whenever ϕ changes its sign. Note that in comparison with the case of the particle at rest, the argument of the trigonometrical functions $\phi(t)$ in (258) and (259) is extended to $\phi(t) = \bar{h}\beta + v\bar{d}(\cos(\omega_0 t) - 1)$, where $\bar{h}\beta(t)$ is the argument of the sine and cosine in (213) and (214).

Before substituting numbers in $\phi(t)$ we have to reconsider our choice of $\bar{x}$ with the aim of maximising the amplitude of the rotation. Recall that we neglected the spatial oscillations of $h(\vec{x})$ and $d(\vec{x})$ by setting $h(\vec{x}) = h(\bar{\vec{x}}) \equiv \bar{h}$ and $d(\vec{x}) = d(\bar{\vec{x}}) \equiv \bar{d}$, respectively. Since both the time and the spatial oscillations, the latter ones being reduced to the constants $\bar{h}$ and $\bar{d}$, appear in the argument $\phi(t)$ of the trigonometrical functions in the solution (239), we need to find a $\bar{x}$ so that $|\Delta\phi| = |\phi(t_2) - \phi(t_1)|$ is maximised, where t_1 and t_2 are values of the time t yielding two consecutive extrema of $\phi(t)$, one of them being a maximum and one a minimum.

We first consider the spatial oscillations $h(\vec{x})$ and $d(\vec{x})$:

figure 5 shows that the spatial oscillations of $h(\vec{x})$ and $d(\vec{x})$ do not run in parallel. Furthermore, we see that for velocities $v > 4 \cdot 10^{-2}$ on average $|vd| > |h/\omega_0|$ and for $v < 4 \cdot 10^{-2}$ on average $|vd| < |h/\omega_0|$. Thus, the optimal choice of $\bar{x}$ depends on the respective velocity v.

For velocities $v < 4 \cdot 10^{-2}$ we take the same value for $\bar{x}$ as we did in the case of a particle at

rest. For $v = 4 \cdot 10^{-2}$ we must not choose a maximum of one of the two functions as they are phase-shifted by nearly $\pi/2$. Instead we choose $\bar{x} = \frac{1}{2}(x_{h(\vec{x})_{max}} + x_{d(\vec{x})_{max}}) = 5177$ly where $h(\vec{x})_{max}$ and $d(\vec{x})_{max}$ are maxima of $h(\vec{x})$ and $d(\vec{x})$, respectively.
For velocities $v > 4 \cdot 10^{-2}$ we choose $\bar{x} = 4431$ly being a maximum of the function $h(\vec{x})$.
Next we choose the values t_1 and t_2 of the time t yielding consecutive extrema of the function $\phi(t)$.
Since $\bar{h}\frac{\sin(\omega_0 t)}{\omega_0}$ ranges from $-\frac{\bar{h}}{\omega_0}$ to $\frac{\bar{h}}{\omega_0}$ and $v\bar{d}\,(\cos(\omega_0 t) - 1)$ from $-2v\bar{d}$ to 0 the choice of t_1 and t_2 also depends on the respective velocity of the particle.
For $v < 4 \cdot 10^{-2}$ we choose $t_1 = \frac{\pi}{2\omega_0}$ and $t_2 = \frac{3\pi}{2\omega_0}$. For $v = 4 \cdot 10^{-2}$ we see that $|vd| \approx |h/\omega_0|$, thus, we choose $t_1 = \frac{\pi}{4\omega_0}$ and $t_2 = \frac{1}{2}\left(\frac{3\pi}{2\omega_0} + \frac{\pi}{\omega_0}\right) = \frac{5\pi}{4\omega_0}$ giving a maximum and a minimum, respectively, of the function $\phi(t)$.
For $v > 4 \cdot 10^{-2}$ we choose $t_1 = 0$ and $t_2 = \frac{\pi}{\omega_0}$ giving a maximum and a minimum, respectively, of the function $\cos(\omega_0 t) - 1$.

- $v = 10^{-4}$:
  $\bar{x} = 4915\,$ly, $t_1 = \frac{\pi}{2\omega_0}$ and $t_2 = \frac{3\pi}{2\omega_0}$:

$$\begin{aligned}\phi(t_1) &= -8.3053 \cdot 10^{-7} \sin(\omega_0 t_1) + 4 \cdot 10^{-11}(\cos(\omega_0 t_1) - 1) \\ &= -8.3053 \cdot 10^{-7} = -\phi(t_2) \qquad (260)\end{aligned}$$

$$|\Delta\phi| = 1.6617 \cdot 10^{-6} \qquad (261)$$

- $v = 4 \cdot 10^{-2}$:
  $\bar{x} = 5177\,$ly, $t_1 = \frac{\pi}{4\omega_0}$ and $t_2 = \frac{5\pi}{4\omega_0}$:

$$\begin{aligned}\phi(t_1) &= -5.7693 \cdot 10^{-7} \sin(\omega_0 t_1) - 6.3547 \cdot 10^{-7}(\cos(\omega_0 t_1) - 1) \\ &= -2.2183 \cdot 10^{-7} \qquad (262)\end{aligned}$$

$$\phi(t_2) = 1.49277 \cdot 10^{-6} \qquad (263)$$

$$|\Delta\phi| = 1.71459 \cdot 10^{-6} \qquad (264)$$

- $v = 0.99$:

$\bar{x} = 4431\,\text{ly}$, $t_1 = 0$ and $t_2 = \frac{\pi}{\omega_0}$:

$$\phi(t_1) \approx 0 \tag{265}$$

$$\begin{aligned}\phi(t_2) &= -1.4 \cdot 10^{-10} \sin(\omega_0 t_2) + 2.874576 \cdot 10^{-5}(\cos(\omega_0 t_2) - 1) \\ &= -5.749153 \cdot 10^{-5}\end{aligned} \tag{266}$$

$$|\Delta\phi| = 5.749153 \cdot 10^{-5} \tag{267}$$

3.2 Dirac Equation Minimally Coupled to the Torsion Field

In this section we turn to the quantum mechanical approach where we solve the Dirac equation minimally coupled to the torsion field γ_{ijk}.
For simplicity, we change the signature of the metric from $(-,+,+,+)$ to $(+,-,-,-)$.
In Minkowski space-time the Dirac equation reads

$$i\hbar\gamma^i\partial_i\psi = m\psi, \tag{268}$$

where ψ is a Dirac spinor, $\hbar$ is the reduced Planck constant and γ^i are the gamma matrices satisfying the relation

$$\left\{\gamma^k, \gamma^l\right\} = 2\eta^{kl}, \tag{269}$$

where the curly brackets denote the anticommutator.
We choose the following representation for the gamma matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \ \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix} \tag{270}$$

with $\mathbb{1}$ being the 2×2 unit matrix, σ^α being the Pauli matrices and with the index α running over the three spatial coordinates $(1,2,3)$.
Recall that since we choose the units lightyears ly and years y the speed of light c equals one.
In order to minimally couple the Dirac equation to the torsion field we need to find the covariant derivative for spinors.
It is necessary to introduce an anholonomic tangent space basis called tetrad field [3],[14]:
Due to the Equivalence Principle we can erect a local orthonormal frame at any point of space-time. The orthonormal frame comprises four mutually orthonormal tangent vectors $h_a^{\ i}$ with the upper index denoting the component of the vector w.r.t. a coordinate basis and the lower index denoted by letters from the beginning of the alphabet indicating the respective vector. The tetrads $h_a^{\ i}$ fulfill the following relations:

$$g_{ij}\, h_a^{\ i}\, h_b^{\ j} = \eta_{ab} \tag{271}$$

$$\eta_{ab}\, h_i^{\ a}\, h_b^{\ j} = g_{ij}, \tag{272}$$

where g_{ij} denotes the metric tensor of the Riemann-Cartan space-time and η_{ab} the metric tensor of the Minkowski space-time.
The $h_i^{\ a}$ are the components of the dual basis obtained from $h_a^{\ i}$ by raising the tangent space indices with η^{ab} and lower the indices $i, j, k\ldots$ by g_{ij}.

Under a general coordinate transformation the Dirac spinor ψ transforms like a scalar

$$\psi\left(x^k\right) \to \psi'\left(\bar{x}^l\right) = \psi\left(x^k\left(\bar{x}^l\right)\right). \tag{273}$$

However, under a Lorentz transformation of the tetrads ψ transforms according to the Dirac spinor representation of the Lorentz group

$$\psi\left(x^d\right) \to \psi'\left(x^d\right) = e^{-\frac{i}{2}\omega^{ab}\left(x^d\right)\Sigma_{ab}}\psi\left(x^d\right) \equiv S\left(\Lambda\right)\psi\left(x^d\right), \tag{274}$$

where $\Lambda = e^{-i\omega^{ab}\left(x^d\right)M_{ab}}$ is a standard Lorentz transformation and $M_{ab} = -M_{ba}$ are the six generators of the Lorentz group. In the Dirac representation $S\left(M_{ab}\right) = \Sigma_{ab} = \frac{i}{4}\left[\gamma_a, \gamma_b\right]$, where the square brackets denote the commutator of the gamma matrices.
We may focus on the latter equation.
In analogy to gauge theories we seek a covariant derivative that fulfills

$$\left(S\left(\Lambda\left(x^d\right)\right)\psi\right)_{;a} = S\left(\Lambda\left(x^d\right)\right)\psi_{;a} \tag{275}$$

under a local Lorentz transformation $\Lambda\left(x^d\right)$.
The covariant derivative that meets the requirement (275) reads [11]

$$D_i = \partial_i + \Gamma_i = \partial_i + \frac{i}{2}\Sigma^{ab}\Gamma_{iab} = \partial_i - \frac{i}{2}\Sigma^{ab}h^j{}_a h_{bj;i} \tag{276}$$

$$= \partial_i + \frac{1}{8}\left[\gamma^a, \gamma^b\right] h^j{}_a\left(h_{bj,i} - \Gamma^m{}_{ij}h_{bm}\right), \tag{277}$$

where $\Gamma^m{}_{ij}$ denotes the holonomic Riemann-Cartan connection.

We may now return to the Minkowski space-time with torsion:
note that in this case the tetrad may be chosen as holonomic: $h_a{}^i = \delta_a{}^i$.
Hence the covariant derivative (277) of a spinor ψ is

$$\psi_{;i} = \psi_{,i} - \frac{1}{8}\gamma_{ijk}\left[\gamma^j, \gamma^k\right]\psi, \tag{278}$$

where γ_{ijk} is the torsion tensor.
Thus, replacing $\psi_{,i}$ by $\psi_{;i}$ the Dirac equation (268) reads

$$i\hbar\gamma^i\left(\partial_i - \frac{1}{8}\gamma_{ijk}\left[\gamma^j, \gamma^k\right]\right)\psi = m\psi. \tag{279}$$

Schrödinger form of the Dirac equation:

Recall that the torsion field γ_{ijk} is related to the derivative of the scalar field f (48):

$$\gamma_{ijk} = -\frac{1}{\Upsilon}\epsilon_{ijk}{}^{l} f_{,l}. \tag{280}$$

A minus sign has been added due to the change of signature.

In order to obtain the Schrödinger form of the Dirac equation we may rearrange (279) as follows:

$$\left[i\hbar\gamma^i\left(\partial_i - \frac{1}{8}\gamma_{ijk}\left[\gamma^j,\gamma^k\right]\right) - m\right]\psi = 0 \tag{281}$$

$$\Leftrightarrow \left(i\hbar\gamma^i\partial_i - \frac{i\hbar}{4}\gamma_{ijk}\gamma^i\gamma^j\gamma^k - m\right)\psi = 0 \tag{282}$$

$$\Leftrightarrow \left(i\hbar\gamma^i\partial_i + \frac{i\hbar}{4\Upsilon}\epsilon_{ijkl}\eta^{lm}f_{,m}\gamma^i\gamma^j\gamma^k - m\right)\psi = 0 \tag{283}$$

$$\Leftrightarrow \left(i\hbar\gamma^i\partial_i - \frac{3\hbar}{2\Upsilon}f_{,l}\gamma^l\gamma^5 - m\right)\psi = 0 \tag{284}$$

$$\Leftrightarrow i\hbar\gamma^0\partial_0\psi = \left(-i\hbar\gamma^\alpha\partial_\alpha + \frac{3\hbar}{2\Upsilon}f_{,l}\gamma^l\gamma^5 + m\right)\psi \tag{285}$$

$$\Leftrightarrow \quad i\hbar\partial_0\psi = \left(-i\hbar\gamma^0\gamma^\alpha\partial_\alpha + \frac{3\hbar}{2\Upsilon}f_{,l}\gamma^0\gamma^l\gamma^5 + m\gamma^0\right)\psi, \tag{286}$$

because

$$\epsilon_{ijkl}\eta^{lm}f_{,m}\gamma^i\gamma^j\gamma^k = 6\left(\eta^{00}f_{,0}\epsilon_{1230}\gamma^1\gamma^2\gamma^3 + \eta^{11}f_{,1}\epsilon_{0231}\gamma^0\gamma^2\gamma^3\right. \tag{287}$$

$$\left.+\eta^{22}f_{,2}\epsilon_{0132}\gamma^0\gamma^1\gamma^3 + \eta^{33}f_{,3}\epsilon_{0123}\gamma^0\gamma^1\gamma^2\right) = 6\left(-f_{,0}\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\right. \tag{288}$$

$$\left.+f_{,1}\gamma^1\gamma^1\gamma^0\gamma^2\gamma^3 - f_{,2}\gamma^2\gamma^2\gamma^0\gamma^1\gamma^3 + f_{,3}\gamma^3\gamma^3\gamma^0\gamma^1\gamma^2\right) \tag{289}$$

$$= i6f_{,l}\gamma^l\gamma^5, \tag{290}$$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, where $\mathbb{1}$ is the 2×2 unit matrix, and the Greek letters denote the three spatial Cartesian coordinates.

3.2.1 Co-moving Particle

We consider a particle located on the spoke of the torsion field as we did in the quasi-classical approach. Since the spoke is rotating very slowly, we assume the spoke to be at rest and the 3-momentum of the particle to equal zero.

Recall that we neglected $f_{,2}$ for $x > 4000$ly as in this region its contribution to the torsion field is very small compared to $f_{,0}$ and $f_{,1}$.

Note that $f_{,3}(\vec{x})\Big|_{\vec{x}=\vec{\bar{x}}} = 0$.

As in the pseudo-classical approach we neglect the spatial oscillations given by the functions $h(\vec{x})$ (172) and $d(\vec{x})$ (174) by setting $h(\vec{x}) = h(\vec{\bar{x}}) \equiv \bar{h}$ and $d(\vec{x}) = d(\vec{\bar{x}}) \equiv \bar{d}$, respectively.

Due to the appearance of Pauli matrices in the gamma matrices we may, for simplicity, rotate our coordinate system so that the spoke of the torsion field coincides with the z-axis:

x-axis $\to$ $\check{z}$-axis, y-axis $\to$ $\check{x}$-axis and the z-axis $\to$ $\check{y}$-axis, where $\check{x}$, $\check{y}$ and $\check{z}$ denote the rotated coordinates.

Thus, $f_{,1}(t,x,y,z) = f_{,3}(t,\check{x},\check{y},\check{z}) \equiv f_{,\check{3}}$.

Therefore the Dirac equation in Schrödinger form resulting from (286) reads

$$i\hbar\partial_0\psi = \left(-i\hbar\gamma^0\gamma^\alpha\partial_\alpha + \frac{3\hbar}{2\Upsilon}\left(f_{,0}\gamma^5 + f_{,\check{3}}\gamma^0\gamma^{\check{3}}\gamma^5\right) + m\gamma^0\right)\psi \tag{291}$$

$$\text{or } i\hbar\partial_0\psi = \left(-i\hbar\gamma^0\gamma^\alpha\partial_\alpha + \frac{3\hbar}{2\Upsilon}\left(f_{,0}\gamma^5 + f_{,\check{3}}i\gamma^{\check{1}}\gamma^{\check{2}}\right) + m\gamma^0\right)\psi \tag{292}$$

In the case of a particle with vanishing 3-momentum equation (292) reduces to

$$i\hbar\partial_0\psi = \left(\frac{3\hbar}{2\Upsilon}\left(f_{,0}\gamma^5 + if_{,\check{3}}\gamma^{\check{1}}\gamma^{\check{2}}\right) + m\gamma^0\right)\psi. \tag{293}$$

Since $f_{,\check{3}}$ depends on the time t we have to modify the free solution given by

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} e^{-\frac{i}{\hbar}E\,t}, \tag{294}$$

where u and v are both normalised 2-components vectors and E denotes the energy, by replacing the product $E\,t$ by $\mathcal{E}(t) = \int_{t_0}^{t} E(t')dt'$. We call $E(t)$ the local energy.

Thus, the solution (294) reads

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} e^{-\frac{i}{\hbar}\mathcal{E}(t)}. \tag{295}$$

Substituting (295) in (293) yields

$$E(t)\psi = \frac{3\hbar}{2\Upsilon}\left(f_{,0}\gamma^5 + if_{,\check{3}}\gamma^{\check{1}}\gamma^{\check{2}}\right)\psi + \gamma^0 m\psi \tag{296}$$

$$\left(-E(t) + \frac{3\hbar}{2\Upsilon}\left(f_{,0}\gamma^5 + if_{,\check{3}}\gamma^{\check{1}}\gamma^{\check{2}}\right) + \gamma^0 m\right)\psi = 0 \tag{297}$$

and in matrix representation

$$\begin{pmatrix} m + cf_{,\breve{3}} - E & 0 & cf_{,0} & 0 \\ 0 & m - cf_{,\breve{3}} - E & 0 & cf_{,0} \\ cf_{,0} & 0 & -m + cf_{,\breve{3}} - E & 0 \\ 0 & cf_{,0} & 0 & -m - cf_{,\breve{3}} - E \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = 0, \quad (298)$$

with the constant $c = \frac{3\hbar}{2\Upsilon}$.

Thus, we obtain

$$\left(m + cf_{,\breve{3}} - E\right) u_1 + cf_{,0} v_1 = 0 \quad (299)$$

$$\left(m - cf_{,\breve{3}} - E\right) u_2 + cf_{,0} v_2 = 0 \quad (300)$$

$$cf_{,0} u_1 + \left(-m + cf_{,\breve{3}} - E\right) v_1 = 0 \quad (301)$$

$$cf_{,0} u_2 + \left(-m - cf_{,\breve{3}} - E\right) v_2 = 0. \quad (302)$$

We rearrange (299)

$$v_1 = -\frac{m + cf_{,\breve{3}} - E}{cf_{,0}} u_1 \quad (303)$$

and substitute the equation above into (301):

$$cf_{,0} u_1 - \left(-m + cf_{,\breve{3}} - E\right) \frac{m + cf_{,\breve{3}} - E}{cf_{,0}} u_1 = 0 \quad (304)$$

$$\Leftrightarrow c^2 f_{,0}^2 - \left(-m + cf_{,\breve{3}} - E\right)\left(m + cf_{,\breve{3}} - E\right) = 0 \quad (305)$$

$$\Leftrightarrow c^2 f_{,0}^2 - \left(cf_{,\breve{3}} - E\right)^2 + m^2 = 0 \quad (306)$$

$$\Rightarrow E_1 = cf_{,\breve{3}} + \sqrt{c^2 f_{,0}^2 + m^2} \quad (307)$$

$$= \frac{\hbar}{2}\cos(\omega_0 t)\bar{h} + \sqrt{\left(\frac{\hbar}{2}\sin(\omega_0 t)\bar{d}\right)^2 + m^2} \quad (308)$$

$$E_3 = cf_{,\breve{3}} - \sqrt{c^2 f_{,0}^2 + m^2} \quad (309)$$

$$= \frac{\hbar}{2}\cos(\omega_0 t)\bar{h} - \sqrt{\left(\frac{\hbar}{2}\sin(\omega_0 t)\bar{d}\right)^2 + m^2}. \quad (310)$$

The vectors u and v corresponding to the local energies E_1 and E_3 read:

$$u_{(1)} = N_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_{(1)} = N_1 \begin{pmatrix} \frac{cf_{,0}}{m + \sqrt{c^2 f_{,0}^2 + m^2}} \\ 0 \end{pmatrix}, \quad (311)$$

$$u_{(3)} = N_3 \begin{pmatrix} -\frac{cf_{,0}}{m + \sqrt{c^2 f_{,0}^2 + m^2}} \\ 0 \end{pmatrix}, \quad v_{(3)} = N_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (312)$$

with

$$N_{1,3} = \frac{1}{\sqrt{2}} \frac{m + \sqrt{m^2 + c^2 f_{,0}^2}}{\sqrt{m^2 + m\sqrt{m^2 + c^2 f_{,0}^2} + c^2 f_{,0}^2}}. \tag{313}$$

and the normalization condition $\bar{\psi}\psi = 1$ with $\bar{\psi} = \psi^\dagger \gamma^0$ and $\psi^\dagger$ being the adjoint of ψ. Analogously we rearrange equation (300) and substitute it into (302) obtaining

$$E_2 = -cf_{,\breve{3}} + \sqrt{c^2 f_{,0}^2 + m^2} \tag{314}$$

$$= -\frac{\hbar}{2}\cos(\omega_0 t)\bar{h} + \sqrt{\left(\frac{\hbar}{2}\sin(\omega_0 t)\bar{d}\right)^2 + m^2} \tag{315}$$

$$E_4 = -cf_{,\breve{3}} - \sqrt{c^2 f_{,0}^2 + m^2} \tag{316}$$

$$= -\frac{\hbar}{2}\cos(\omega_0 t)\bar{h} - \sqrt{\left(\frac{\hbar}{2}\sin(\omega_0 t)\bar{d}\right)^2 + m^2} \tag{317}$$

with

$$u_{(2)} = N_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_{(2)} = N_2 \begin{pmatrix} 0 \\ \frac{cf_{,0}}{m+\sqrt{c^2 f_{,0}^2 + m^2}} \end{pmatrix}, \tag{318}$$

$$u_{(4)} = N_4 \begin{pmatrix} 0 \\ -\frac{cf_{,0}}{m+\sqrt{c^2 f_{,0}^2 + m^2}} \end{pmatrix}, \quad v_{(4)} = N_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{319}$$

and

$$N_{2,4} = \frac{1}{\sqrt{2}} \frac{m + \sqrt{m^2 + c^2 f_{,0}^2}}{\sqrt{m^2 + m\sqrt{m^2 + c^2 f_{,0}^2} + c^2 f_{,0}^2}}. \tag{320}$$

The solutions

$$\psi_k = \begin{pmatrix} u_{(k)} \\ v_{(k)} \end{pmatrix} e^{-\frac{i}{\hbar}\mathcal{E}_k(t)}, \tag{321}$$

$k = 1, 2, 3, 4$, form an orthonormal basis. The spinors ψ_1 and ψ_2 with the positive local energies E_1 and E_2, respectively, are particle solutions whereas ψ_3 and ψ_4 with the negative local energies E_3 and E_4, respectively, are antiparticle solutions.

The solutions $\psi_{1,3}$ and $\psi_{2,4}$ are polarised in the z and the $-z$ direction, respectively.

We may point out the analogy to a spinning particle in a static magnetic field:

We consider the time independent Schrödinger equation for a spin$-\frac{1}{2}$ particle at rest in a constant magnetic field $\vec{B} = (0, 0, B_0)$ [16]:

$$E\phi = \hat{H}\phi, \tag{322}$$

where $\hat{H} = -\vec{\hat{\mu}}\vec{B} = -\frac{\hbar}{2}\gamma B_0\sigma_3$ denotes the Hamilton operator with the spin magnetic moment $\vec{\hat{\mu}} = \gamma\frac{\hbar}{2}\vec{\sigma}$ and γ denoting the gyromagnetic coefficient.
A basis of solutions is

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar}E_1 t} \text{ and } \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar}E_2 t} \tag{323}$$

with $E_1 = -\frac{1}{2}\gamma\hbar B_0$ and $E_2 = \frac{1}{2}\gamma\hbar B_0$.
We consider an arbitrary state $\phi(t) = \alpha\phi_1 + \beta\phi_2$ with $|\alpha|^2 + |\beta|^2 = 1$. The time evolution of ϕ gives

$$\phi(t) = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\frac{i}{2}\gamma B_0 t} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{i}{2}\gamma B_0 t} \tag{324}$$

$$= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\frac{i}{2}\omega_L t} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{i}{2}\omega_L t} \tag{325}$$

with

$$\omega_L = \frac{1}{\hbar}(E_1 - E_2) = \frac{1}{\hbar}\Delta E \tag{326}$$

$$= -\gamma B_0 \tag{327}$$

denoting the Larmor frequency.
The expectation value of $\hat{\mu}$ reads

$$\langle\hat{\mu}_x\rangle = 2\hbar\gamma\alpha\beta\cos(\omega_L t) \tag{328}$$

$$\langle\hat{\mu}_y\rangle = 2\hbar\gamma\alpha\beta\sin(\omega_L t) \tag{329}$$

$$\langle\hat{\mu}_z\rangle = 0 \tag{330}$$

with $\langle\hat{\mu}_k\rangle = \bar{\phi}\hat{\mu}_k\phi$.
Thus, the spin magnetic moment precesses about the z-axis with the angular velocity ω_L. If $\Delta E > 0$ the spin magnetic moment precesses with a mathematically positive sense of rotation and if $\Delta E < 0$ with a mathematically negative sense of rotation.
Classically, a spinning particle in a constant magnetic field is subject to a torque $\vec{M} = \vec{\mu}_s \times \vec{B}$, where $\vec{\mu}_s$ denotes the spin magnetic moment and $\vec{B}$ the magnetic field. The

potential energy corresponding to the torque reads $E_{pot} = -\vec{\mu}_s \vec{B}$. Hence, the particle tends to orient its spin parallel to the magnetic field resulting in a precession about the axis parallel to the direction of the magnetic field. [15]

In order to obtain the precession frequency in the case of a spinning particle in the torsion field we compute the expectations value of the polarization vector in the Dirac representation given by

$$S_D^i = \gamma^i \gamma^5. \tag{331}$$

Let us consider the state

$$\psi = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2 = \frac{N_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ v_1 \\ 0 \end{pmatrix} e^{-\frac{1}{\hbar}\mathcal{E}_1} + \frac{N_2}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ v_2 \end{pmatrix} e^{-\frac{1}{\hbar}\mathcal{E}_2}. \tag{332}$$

Note that $v_1 = v_2 \equiv v$ and, therefore, $N_1 = N_2 \equiv N$.

The expectation value $\langle S_D^i \rangle = \bar{\psi} S_D^i \psi$ with $\bar{\psi} = \psi^\dagger \gamma^0$ yields:

$$\langle S_D^0 \rangle = \frac{N^2}{2} \begin{pmatrix} 1 \\ 0 \\ -v \\ 0 \end{pmatrix}^T \begin{pmatrix} v \\ 0 \\ -1 \\ 0 \end{pmatrix} + \frac{N^2}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -v \end{pmatrix}^T \begin{pmatrix} 0 \\ v \\ 0 \\ -1 \end{pmatrix} = \frac{2v}{1+v^2} \tag{333}$$

$$\langle S_D^1 \rangle = \frac{N^2}{2} \begin{pmatrix} 1 \\ 0 \\ -v \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ -v \\ 0 \end{pmatrix} e^{\frac{i}{\hbar}(\mathcal{E}_1 - \mathcal{E}_2)} + \frac{N^2}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -v \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \\ 0 \\ -v \end{pmatrix} e^{\frac{i}{\hbar}(\mathcal{E}_2 - \mathcal{E}_1)} \tag{334}$$

$$= \frac{N^2}{2} \left[\left(1+v^2\right) \left(e^{\frac{1}{\hbar}(\mathcal{E}_1 - \mathcal{E}_2)} + e^{-\frac{1}{\hbar}(\mathcal{E}_1 - \mathcal{E}_2)} \right) \right] = \cos\left(\frac{\mathcal{E}_1 - \mathcal{E}_2}{\hbar} \right) \tag{335}$$

$$= \cos(\alpha_L), \tag{336}$$

$$\langle S_D^2 \rangle = \frac{iN^2}{2} \begin{pmatrix} 1 \\ 0 \\ -v \\ 0 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ v \\ 0 \end{pmatrix} e^{\frac{i}{\hbar}(\mathcal{E}_1 - \mathcal{E}_2)} + \frac{iN^2}{\hbar} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -v \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \\ 0 \\ -v \end{pmatrix} e^{\frac{i}{\hbar}(\mathcal{E}_2 - \mathcal{E}_1)} \tag{337}$$

$$= -\frac{i}{2} \left(e^{\frac{i}{\hbar}(\mathcal{E}_1 - \mathcal{E}_2)} - e^{-\frac{i}{\hbar}(\mathcal{E}_1 - \mathcal{E}_2)} \right) = \sin\left(\frac{\mathcal{E}_1 - \mathcal{E}_2}{\hbar} \right) \tag{338}$$

$$= \sin(\alpha_L) \text{ and} \tag{339}$$

$$\langle S_D^3 \rangle = 0 \tag{340}$$

with $\alpha_L = \frac{\mathcal{E}_1 - \mathcal{E}_2}{\hbar}$.
The expectation value (333) of S_D^0 differs from the result (211) in the pseudo-classical approach. This is due to the fact that the polarization vector in the Dirac representation is not an exact quantum mechanical analogue to the polarization vector in the pseudo-classical approach.
The Larmor frequency reads

$$\omega_L = \frac{d\alpha_L}{dt} = \frac{E_1 - E_2}{\hbar} = \bar{h}\cos(\omega_0 t) \text{ and} \tag{341}$$

$$\alpha_L = \bar{h}\frac{\sin(\omega_0 t)}{\omega_0}. \tag{342}$$

The phase α_L being the negative phase difference between the solutions ψ_1 and ψ_2 is identical to the negative classical rotation angle $\bar{h}\beta(t)$ of the polarization vector in the pseudo-classical approach ((213), (214)).
The polarization vector S_D^i precesses with mathematically positive sense of rotation and therefore with the opposite sense of rotation than the polarization vector does in the quasi-classical approach. The reason for the sign difference is still an open question to us.

In the Dirac representation the gamma matrices γ^i represent the analogue to "spin variable" ξ^i in the pseudo-classical approach. We may compare the expectation value $\langle\gamma^i\rangle$ to the solution of the equations of motion of ξ^i (152) in the pseudo-classical approach: we consider the state $\psi = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2$.
The expectation values read

$$\langle\gamma^0\rangle = \frac{N^2}{2}\begin{pmatrix}1\\0\\-v\\0\end{pmatrix}^T\begin{pmatrix}1\\0\\-v\\0\end{pmatrix} + \frac{N^2}{2}\begin{pmatrix}0\\1\\0\\-v\end{pmatrix}^T\begin{pmatrix}0\\1\\0\\-v\end{pmatrix} = 1 \tag{343}$$

$$\langle\gamma^1\rangle = \frac{N^2}{2}\begin{pmatrix}1\\0\\-v\\0\end{pmatrix}^T\begin{pmatrix}v\\0\\-1\\0\end{pmatrix} e^{\frac{i}{\hbar}(\mathcal{E}_1-\mathcal{E}_2)} + \frac{N^2}{2}\begin{pmatrix}0\\1\\0\\-v\end{pmatrix}^T\begin{pmatrix}0\\v\\0\\-1\end{pmatrix} e^{\frac{i}{\hbar}(\mathcal{E}_2-\mathcal{E}_1)} \tag{344}$$

$$= \frac{N^2}{2}2v\left(e^{\frac{i}{\hbar}(\mathcal{E}_1-\mathcal{E}_2)} + e^{-\frac{i}{\hbar}(\mathcal{E}_1-\mathcal{E}_2)}\right) = \frac{2v}{1+v^2}\cos\left(\frac{\mathcal{E}_1 - \mathcal{E}_2}{\hbar}\right) \tag{345}$$

$$= \frac{2v}{1+v^2}\cos\alpha_L, \tag{346}$$

$$\langle\gamma^2\rangle = \frac{iN^2}{2}\begin{pmatrix}1\\0\\-v\\0\end{pmatrix}^T\begin{pmatrix}-v\\0\\1\\0\end{pmatrix}e^{\frac{i}{\hbar}(\mathcal{E}_1-\mathcal{E}_2)} + \frac{iN^2}{2}\begin{pmatrix}0\\1\\0\\-v\end{pmatrix}^T\begin{pmatrix}0\\v\\0\\-1\end{pmatrix}e^{\frac{i}{\hbar}(\mathcal{E}_2-\mathcal{E}_1)} \tag{347}$$

$$= -\frac{iN^2}{2}2v\left(e^{\frac{i}{\hbar}(\mathcal{E}_1-\mathcal{E}_2)} - e^{-\frac{i}{\hbar}(\mathcal{E}_1-\mathcal{E}_2)}\right) = \frac{2v}{1+v^2}\sin\left(\frac{\mathcal{E}_1-\mathcal{E}_2}{\hbar}\right) \tag{348}$$

$$= \frac{2v}{1+v^2}\sin\alpha_L \text{ and} \tag{349}$$

$$\langle\gamma^3\rangle = 0. \tag{350}$$

The expectation values of the γ^i display a rotation in a mathematically positive sense, whereas the results in the pseudo-classical approach display a rotation in a mathematically negative sense.

3.2.2 Radially moving particle

We adopt the initial conditions used in the pseudo-classical approach. There, the particle moved along the spoke of the torsion field with a velocity that was approximated to be constant.

Furthermore we approximate the spatial oscillations of the torsion field given by the functions $h(\vec{x})$ (172) and $d(\vec{x})$ (174) by the constants $h(\vec{x}) = h(\vec{\bar{x}}) \equiv \bar{h}$ and $d(\vec{x}) = d(\vec{\bar{x}}) \equiv \bar{d}$, respectively.

As in the case of a particle at rest we rotate our coordinate system so that the spoke of the torsion field coincides with the $\check{z}$-axis. Thus, $\vec{\bar{x}} \mapsto \vec{\bar{\check{x}}} = (0, 0, \bar{\check{z}})$.

Since the components of the torsion field are functions of the time t we need to modify the classical plane-wave solution by replacing the product $E\,t$ by a time-dependent function $\mathcal{E}(t)$.

The solution to the Dirac equation (292) with constant momentum $p \equiv p^{\check{3}} = -p_{\check{3}}$ reads

$$\psi = \begin{pmatrix}u\\v\end{pmatrix}e^{-\frac{i}{\hbar}(\mathcal{E}(t)-p\check{z})} \tag{351}$$

with $E(t) = d\mathcal{E}(t)/dt$ denoting the local energy and u and v being both normalised 2-components vectors.

Substituting (351) into the Dirac equation (292) gives

$$\left(-E + \gamma^0\gamma^{\check{3}}p + \frac{3\hbar}{2\Upsilon}\left(f_{,0}\gamma^5 + if_{,\check{3}}\gamma^{\check{1}}\gamma^{\check{2}}\right) + \gamma^0 m\right)\psi = 0 \tag{352}$$

or, in matrix representation,

$$\begin{pmatrix} m + cf_{,\breve{3}} - E & 0 & p + cf_{,0} & 0 \\ 0 & m - cf_{,\breve{3}} - E & 0 & -p + cf_{,0} \\ p + cf_{,0} & 0 & -m + cf_{,\breve{3}} - E & 0 \\ 0 & -p + cf_{,0} & 0 & -m - cf_{,0} - E \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = 0 \quad (353)$$

with $c = \frac{3\hbar}{2\Upsilon}$.
Thus, we obtain

$$\left(m + cf_{,\breve{3}} - E\right) u_1 + (p + cf_{,0})\, v_1 = 0 \quad (354)$$

$$\left(m - cf_{,\breve{3}} - E\right) u_2 + (-p + cf_{,0})\, v_2 = 0 \quad (355)$$

$$(p + cf_{,0})\, u_1 + \left(-m + cf_{,\breve{3}} - E\right) v_1 = 0 \quad (356)$$

$$(-p + cf_{,0})\, u_2 + (-m - cf_{,0} - E)\, v_2 = 0 \quad (357)$$

We rearrange equation (354)

$$v_1 = -\frac{m + cf_{,\breve{3}} - E}{p + cf_{,0}} u_1 \quad (358)$$

and substitute the result into equation (356):

$$(p + cf_0)\, u_1 - \left(-m + cf_{,\breve{3}} - E\right) \frac{m + cf_{,\breve{3}} - E}{p + cf_{,0}} u_1 = 0 \quad (359)$$

$$\Leftrightarrow (p + cf_0)^2 + m^2 - \left(-E + cf_{,\breve{3}}\right)^2 = 0 \quad (360)$$

$$\Rightarrow E_1 = cf_{,\breve{3}} + \sqrt{m^2 + (p + cf_{,0})^2} \quad (361)$$

$$E_3 = cf_{,\breve{3}} - \sqrt{m^2 + (p + cf_{,0})^2}. \quad (362)$$

The vectors $u_{(1),(3)}$ and $v_{(1),(3)}$ corresponding to the local energies E_1 and E_3 read

$$u_{(1)} = N_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_{(1)} = N_1 \begin{pmatrix} \frac{cf_{,0}+p}{m+\sqrt{m^2+(p+cf_{,0})^2}} \\ 0 \end{pmatrix} \quad \text{and} \quad (363)$$

$$u_{(3)} = N_3 \begin{pmatrix} -\frac{cf_{,0}+p}{m+\sqrt{m^2+(p+cf_{,0})^2}} \\ 0 \end{pmatrix}, \quad v_{(3)} = N_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (364)$$

with

$$N_{1,3} = \frac{1}{\sqrt{2}} \frac{m + \sqrt{m^2 + (p + cf_{,0})^2}}{\sqrt{m^2 + m\sqrt{m^2 + (p + cf_{,0})^2} + (p + cf_{,0})^2}} \quad (365)$$

Analogously we rearrange equation (355) and substitute it into (357) obtaining

$$E_2 = -\,cf_{,\check{3}} + \sqrt{\left(p - cf_{,0}\right)^2 + m^2} \tag{366}$$

$$E_4 = -\,cf_{,\check{3}} - \sqrt{\left(p - cf_{,0}\right)^2 + m^2} \tag{367}$$

with

$$u_{(2)} = N_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_{(2)} = N_2 \begin{pmatrix} 0 \\ \frac{cf_{,0}-p}{m+\sqrt{m^2+(p-cf_{,0})^2}} \end{pmatrix} \quad \text{and} \tag{368}$$

$$u_{(4)} = N_4 \begin{pmatrix} 0 \\ -\frac{cf_{,0}-p}{m+\sqrt{m^2+(p-cf_{,0})^2}} \end{pmatrix}, \quad v_{(4)} = N_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{369}$$

and

$$N_{2,4} = \frac{1}{\sqrt{2}} \frac{m + \sqrt{m^2 + \left(p - cf_{,0}\right)^2}}{\sqrt{m^2 + m\sqrt{m^2 + \left(p - cf_{,0}\right)^2} + \left(p - cf_{,0}\right)^2}}. \tag{370}$$

The solutions ψ_k corresponding to $u_{(k)}$, $v_{(k)}$ via (351) form an orthogonal basis with respect to the inner product $(\psi, \psi') = \int d^3x\, \bar{\psi}\,(t, \vec{x})\, \psi'\,(t, \vec{x})$. The solutions ψ_1 and ψ_3 are polarised in the z-direction whereas the solutions ψ_2 and ψ_4 are polarised in the $-z$-direction.

Thus, the Larmor frequency $\omega_L(t)$ (326) reads

$$\begin{aligned}\omega_L =& \frac{1}{\hbar}\Delta E = \frac{1}{\hbar}\left(E_1 - E_2\right) \\ =& \frac{2}{\hbar}cf_{,\check{3}} + \frac{1}{\hbar}\sqrt{m^2 + \left(p + cf_{,0}\right)^2} - \frac{1}{\hbar}\sqrt{m^2 + \left(p - cf_{,0}\right)^2}.\end{aligned} \tag{371}$$

For $\bar{z} = 4431\,\text{ly}$ giving a maximum of the function $d(\vec{z})$ we obtain $\bar{d} \equiv d(\vec{\bar{z}}) = 3.78 \cdot 10^{-12}$. Recall that $cf_{,0} = \frac{\hbar}{2}\bar{d}\cos(\omega_0 t)\omega_0 \approx 10^{-64}\text{kg}$. In comparison with the rest energy of an electron $E_e = m_e \approx 10^{-30}\text{kg}$ we see that $E_e \gg cf_{,0}$. Thus, we can assume that $m^2 + p^2 \geq |\pm 2pcf_{,0} + \left(cf_{,0}\right)^2|$ is satisfied.

Therefore, we approximate the last two terms on the r.h.s. of equation (371) as follows:

$$\begin{aligned}&\sqrt{m^2+p^2}\sqrt{1 + \frac{\pm 2pcf_{,0} + c^2f_{,0}^2}{m^2+p^2}} = \sqrt{m^2+p^2}\left(1 + \frac{1}{2}\frac{\pm 2pcf_{,0} + c^2f_{,0}^2}{m^2+p^2}\right. \\ &\left. -\frac{1}{8}\left(\frac{\pm 2pcf_{,0} + c^2f_{,0}^2}{m^2+p^2}\right)^2 + \frac{1}{16}\left(\frac{\pm 2pcf_{,0} + c^2f_{,0}^2}{m^2+p^2}\right)^3 - \ldots\right) \\ =& \sqrt{m^2+p^2} + \frac{1}{2}\frac{\pm 2pcf_{,0} + c^2f_{,0}^2}{\sqrt{m^2+p^2}} - \frac{1}{8}\frac{4p^2c^2f_{,0}^2 \pm 4pc^3f_{,0}^3 + f_{,0}^4c^4}{\left(m^2+p^2\right)^{\frac{3}{2}}} + \ldots\end{aligned} \tag{372}$$

using the Taylor series expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \ldots \quad \text{for } |x| \le 1. \tag{373}$$

Subtracting E_2 from E_1 gives

$$\Delta E = 2cf_{,\breve{3}} + \frac{2pcf_{,0}}{(m^2+p^2)^{\frac{1}{2}}} - \frac{pc^3 f_{,0}^3}{(m^2+p^2)^{\frac{3}{2}}} + \frac{3\left(pc^5 f_{,0}^5\right) + 4\left(p^3c^3 f_{,0}^3\right)}{4\,(m^2+p^2)^{\frac{5}{2}}} \ldots \tag{374}$$

We obtain the following Larmor frequency

$$\omega_L = \frac{3}{\Upsilon} f_{,\breve{3}} + \frac{3}{\Upsilon}\frac{pf_{,0}}{(m^2+p^2)^{\frac{1}{2}}} - \frac{27\hbar^2}{8\Upsilon^3}\frac{pf_{,0}^3}{(m^2+p^2)^{\frac{3}{2}}} + \ldots \tag{375}$$

All terms of order $\hbar^2$ or higher are of quantum mechanical origin and drop out in the classical limit $\hbar \to 0$:

$$\lim_{\hbar\to 0} \omega_L = \frac{3}{\Upsilon} f_{,\breve{3}} + \frac{3}{\Upsilon}\frac{pf_{,0}}{(m^2+p^2)^{\frac{1}{2}}} \tag{376}$$

With $p \to \gamma_v m v$, where $\gamma_v = 1/\sqrt{1-v^2}$, the Larmor frequency (375) reads

$$\begin{aligned}\omega_L(t) &= \frac{3}{\Upsilon} f_{,\breve{3}} + \frac{3}{\Upsilon}\frac{\gamma_v m v f_{,0}}{m\sqrt{1+\gamma_v^2 v^2}} = \frac{3}{\Upsilon} f_{,\breve{3}} + \frac{\gamma_v v f_{,0}}{\sqrt{\frac{1-v^2+v^2}{1-v^2}}} \\ &= \frac{3}{\Upsilon} f_{,\breve{3}} + \frac{3}{\Upsilon}\gamma_v^{-1}\gamma_v v f_{,0} = \frac{3}{\Upsilon} f_{,\breve{3}} + \frac{3}{\Upsilon} v f_{,0} \\ &= \frac{3}{\Upsilon} f_{,\breve{3}} + v\frac{3}{\Upsilon} f_{,0} = \bar{h}\cos(\omega_0 t) - v\bar{d}\sin(\omega_0 t)\omega_0 \end{aligned} \tag{377}$$

Integrating the Larmor frequency $\omega_L(t)$ yields

$$\bar{\alpha}_L = \int_0^t \omega_L(t')dt' = \bar{h}\frac{\sin(\omega_0 t)}{\omega_0} + v_{\breve{3}}\bar{d}\,(\cos(\omega_0 t) - 1)\,. \tag{378}$$

In comparison with the pseudo-classical approach we see that the phase $\bar{\alpha}_L(t)$ being the negative difference of the phases of ψ_1 and ψ_2 is equal to the negative rotation angle $\phi(t)$ (241) of the polarization vector. As in the case of a particle at rest the precession frequency displays the opposite sign than in the pseudo-classical approach.

4 Conclusion

The thorough analysis of Hubert Bray's dark matter model [1] reveals that this model is a valid approach to the topic of dark matter and moreover provides an interesting way of deriving an effective gravitational field from the torsion field. However, due to the innovative nature of his approach a number of questions are not dealt with at full depth. For the purpose of this work we chose to focus on the open question of whether and how a torsion field had any measurable effect.

Since torsion couples to the spin of a particle we have computed the effect the torsion field has on a spinning particle using two different approaches. The comparison of the results provided by either approach displays a sign difference which remains an open question. Since the results of the quantum mechanical approach are confirmed by another paper [6] they can be perceived as being valid.

Our approach shows that the torsion field induces the polarization vector to precess about an axis, where the direction of the axis is determined by the torsion field. Since the torsion field is oscillating in time the sense of rotation of the polarization vector is changed every half period. Therefore, the precession results in an oscillatory motion with deflection given by the angle α_L (342) in case of a particle at rest and by the angle $\bar{\alpha}_L$ (378) in case of a moving particle. Thus, the effect is very small and, therefore, hard to measure. Applying a magnetic field on a spinning particle would potentially magnify this effect. Further investigations on this potential magnification have not been conducted by the author.

References

[1] H. L. Bray, "On Dark Matter, Spiral Galaxies, and the Axioms of General Relativity", arxiv:1004.4016v1, 2010, (http://arxiv.org/abs/1004.4016)

[2] A. Liddle, "An Introduction to Modern Cosmology", 2. edition, John Wiley and Sons Ltd., (2003)

[3] F. W. Hehl, "Spin and Torsion in General Relativity: I. Foundations", Gen. Relat. Grav. 4 (1973) pp. 333-349

[4] F. W. Hehl, "Spin and Torsion in General Relativity: II. Geometry and Field Equations", Gen. Relat. Grav. 5 (1974) pp. 491-516

[5] F. W. Hehl, P. v. d. Heyde, D. Kerlick, J. Nester, "General Relativity with Spin and Torsion: Foundations and Prospects", Rev. Mod. Phys. 48 (1976) pp. 393-416

[6] H. Rumpf, "Creation of Dirac Particles in General Relativity with Torsion and Electromagnetism: III. Matter Production in a Model of Torsion", Gen. Relat. Grav. 10 (1979) pp. 647-658

[7] H. Rumpf, "Realtivitätstheorie und Kosmologie II", lecture notes, Vienna, (2008)

[8] M. Roos, "Introduction to Cosmology", 3. edition, John Wiley and Sons Ltd., (2003)

[9] J. A. Peacock, "Cosmological Physics", Cambridge Astrophysics, (1999)

[10] H. Rumpf, "Supersymmetric Dirac Particles in Riemann-Cartan Space-Time", Gen. Relat. Grav. 14 (1982) pp. 773-792

[11] R. Sexl, H. Urbantke "Gravitation und Kosmologie", 5. edition, Spektrum Akademischer Verlag, (2002)

[12] J.-W.Lee, "Is Dark Matter a BEC or Scalar Field" Journal of the Korean Physical Society, Vol. 54, (2009) pp. 2622 (http://arxiv.orf/abs/0801.1442)

[13] R. M. Wald, "General Relativity", The University of Chicago Press, 1984, p.219

[14] L. Ryder, "Introduction to General Relativity", Cambridge University Press, (2009)

[15] J. D. Jackson, "Classical Electrodynamics", 3. edition, John Wiley and Sons Inc., (1998)

[16] L. E. Ballentine, "Quantum Mechanics. A Modern Development.", World Scientific Publishing Co. Pte. Ltd., (1998)

5 Abstract

The current work deals with a possible explanation of the existence of dark matter by introducing a torsion of space-time and the possibility of detecting this torsion by its effects on spinnings particles.

In the first part of this work we analyse the cosmological model [1] proposed by H. Bray, where the author links the phenomenon of dark matter to the existence of a torsion field. The basic idea of [1] is to derive an extension of General Relativity involving a more general connection from particular axioms for the metric and the connection. According to these axioms the gravitational action functional can only take a specific form. However, the Einstein-Hilbert action functional remains a part of the extension of GR. Applying the axioms to the action functional the general connection is reduced to a sum of the Levi-Civita connection and a totally antisymmetric torsion tensor. Since this torsion tensor is totally antisymmetric it is dual to a vector field, in this particular case, to the gradient of a scalar field.

We consider only the vacuum action functional. Its variation leads to Einstein-Klein-Gordon equations. The mass term in the Klein-Gordon equation corresponds to the coupling constant for the torsion. The terms involving the scalar field and its gradient appearing in the Einstein field equations can be interpreted as the effective energy-stress tensor and can in principle be attributed to dark matter.

We solve the Klein-Gordon equation in a spherically symmetric, static space-time in the first post-Newtonian approximation assuming a weak potential as first approximation of a self-consistent solution.

We find that the scalar field satisfying the Klein-Gordon equation is an oscillating function both in time and space. The interference pattern of the spatial oscillations displays a slowly rotating maximum, which resembles a spoke. Inserting the energy density contained in the stress-energy tensor into the Poisson equation yields a Newtonian potential. [1] performs simulations using this Newtonian potential and obtains results resembling a spiral galaxy.

Based on the cosmological model we investigate the measurable effects of the torsion field by analysing the behaviour of a particle with spin-1/2 in the torsion field. The polarization vector of a particle in a torsion field is subject to a torque and hence precesses, as in the case of a spinning particle in a homogeneous magnetic field.

To compute the precession we choose two different approaches: the first one is the supersymmetric approach of [10] that enables us to consistently couple a classical spinning particle to the torsion field.

We compute the precession for both a particle at rest and a moving particle. In the

first case a spinning particle in the torsion field precesses about the axis pointing in the same direction as the spatial gradient of the scalar field. In the second case the particle is moving and is, therefore, subject to an additional torque involving the time component of the gradient of the scalar field leading in addition to a precession about an axis pointing in the direction of motion of the particle.

The second approach is a quantum mechanical one. We solve the Dirac equation minimally coupled to the torsion field. We are able to confirm the results obtained in the pseudo-classical approach with the exception of the sense of rotation of the precession. This discrepancy between both results remains an open question.

In both cases we come to the conclusion that the precession of the polarization vector induced by the torsion field results in an oscillatory motion with the deflection of order of magnitude 10^{-6} rad. The sense of rotation of the precession changes every half period of the time oscillation of the torsion field.

6 Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit dem Versuch die Existenz der dunklen Materie durch eine Raumzeit mit Torsion zu erklären und untersuchen den Effekt, den diese Torsion auf ein Teilchen mit Spin hat.

Im ersten Teil dieser Arbeit befassen wir uns mit dem kosmologischen Modell [1] von Hubert Bray, in der das Phenomän der dunklen Materie mit einem Torsionsfeld in Verbindung gebracht wird. [1] basiert auf der Idee eine Erweiterung der allgemeinen Relativitätstheorie mit einer allgemeineren Konnexion aus Axiomen, die sich auf die Metrik und die Konnexion beziehen, herzuleiten. Diese Axiome fordern eine bestimmte Form des Gravitationswirkungsfunktionals, in welchem die Einstein-Hilbert Wirkung enthalten ist. Die von den Axiomen erlaubte Form des Wirkungsfunktionals reduziert die allgemeine Konnexion zu der Summe aus der Levi-Civita Konnexion und einem total antisymmetrischen Torsionstensors. Letzterer ist dual zu einem Vektorfeld, in unserem Fall zu dem Gradienten des Skalarfeldes.

Zur Einfachheit betrachten wir nur die Vakuumswirkung, deren Variation zu den Einstein-Klein-Gordon Gleichungen führt. Der Massenterm der Klein-Gordon Gleichung entspricht der Kopplungskonstante für die Torsion. Die Feldgleichungen sind um einige Terme, die das Skalarfeld und dessen erste Ableitungen enthalten, erweitert. Diese Terme können als effektiver Energie-Impuls Tensor interpretiert und im Prinzip der dunklen Materie zugeordnet werden.

Wir lösen die Klein-Gordon Gleichung in einer sphärisch symmetrischen, statischen Raumzeit in der ersten post-Newtonschen Näherung, wobei wir ein schwaches Potential als erste Näherung einer selbst-konsistenten Lösung annehmen. Das Skalarfeld, welches die Klein-Gordon Gleichung löst, ist eine räumlich sowie zeitlich oszillierende Funktion, deren räumliches Interferenzmuster ein langsam rotierendes, speichenartiges Maximum zeigt.

Durch Einsetzen der im Energie-Impuls Tensor enthaltene Energiedichte in die Poissongleichgung erhalten wir ein Newtonsches Potential, welches sich, wie Simulationen in [1] zeigen, als Potential für Spiralgalaxien eignet.

Aufbauend auf diesen Ergebnissen widmen wir uns messbaren Effekten des Torsionsfeldes, indem wir dessen Einfluß auf ein Teilchen mit Spin-1/2 untersuchen. Wie im Falle eines Teilchens mit Spin im homogenen Magnetfeld erfährt der Polarisationsvektor des Teilchens im Torsionsfeld ein Drehmoment und präzidiert.

Zur Berechnung der Präzession wählen wir zwei Ansätze: der erste ist ein supersymmetrischer Ansatz von [10], der es ermöglicht ein klassisches Teilchen konsistent an das Torsionsfeld zu koppeln. Wir berechnen die Präzession des Polarisationsvektors für ein ruhendes und bewegtes Teilchen: ein ruhendes Teilchen im Torsionsfeld präzidiert um eine Achse, die in die selbe Richtung zeigt, wie der räumliche Anteil des Gradienten des Skalarfeldes. Im Falle des bewegten Teilchens erfährt der Polarisationsvektor ein weiteres Drehmoment, welches von der zeitlichen Komponente des Gradienten des Skalarfeldes ausgeht und zu einer zusätzlichen Präzession um die Achse führt, die in die Bewegungsrichtung des Teilchens zeigt.

Im zweiten quantenmechanischen Ansatz lösen wir die Diracgleichung minimal gekoppelt an das Torsionsfeld. Wir können die Resultate des pseudo-klassischen Ansatzes bis auf die Präzessionsdrehrichtung bestätigen. Die Diskrepanz zwischen den beiden Resultaten bleibt bis zuletzt eine offene Frage.

In beiden Fällen kommen wir zu dem Ergebnis, dass die Päzession des Polarisationsvektors, die durch das Torsionsfeld verursacht wird, zu einer oszillierenden Bewegung des Polarisationsvektors mit einer Auslenkung von der Größenordnung 10^{-6}rad führt. Dies ist durch die zeitliche Oszillation des Torsionsfeldes zu erklären, da hierdurch die Drehrichtung alle halben Schwingungsdauern umgekehrt wird.

7 Curriculum Vitae

Personal

Name:	Hedda Alice Gressel
Nationality:	Austria

Education

1991 - 1995	Primary School (Volksschule Nonntal, Salzburg)
1995 - 2003	Grammar School (Bundesgymnasium Nonntal Salzburg, Salzburg)
2003	A-Levels (Matura)
2003 - 2004	Bachelor Studies in Mathematics at the University of Salzburg
Since 2004	Diploma Studies in Physics at the University of Vienna

Work

12/2003 - 08/2004	minor employment at Cineplexx Kinobetriebe GmbH, Cineplexx City, Salzburg
10/2005 - 02/2008	minor employment at Cineplexx Kinobetriebe GmbH, Cineplexx Donauplex, Vienna
10/2008 - 01/2012	minor employment at Copyteria, Vienna
03/2010 - 07/2010	minor employment at University of Vienna, Einrichtung für mathematische Physik, Vienna
10/2010 - 02/2011	minor employment at University of Vienna, Einrichtung für mathematische Physik, Vienna
03/2011 - 07/2011	minor employment at University of Vienna, Einrichtung für mathematische Physik, Vienna

8 Acknowledgments

I owe sincere and earnest thankfulness to my advisor Prof. Helmut Rumpf for his support and guidance and for always taking time to answer my questions.

My special thanks go to my family and to all my friends giving me a priceless moral support.

Furthermore, I want to thank the former Copyteria for providing coffee to physics students in need of coffeine and offering a pleasant location for physical and mathematical discussions throughout all semesters.

I'm indebted to my two proofreaders Elisabeth Lucius and Silviu Craciunas.

Besides, I owe my gratitude to Georg Kopsky for the countless hours of physics discussions.

Last I want to thank Michael Nardai for giving me the best moral support I could have hoped for.