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Detecting and classifying entanglement in multipartite qubit states

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Abstract

This thesis is primarily concerned with the detection and the classification of genuine multipartite entanglement in quantum systems consisting of arbitrary many qubits. Multipartite entanglement is at the heart of many body physics and has potential applications in quantum information processing such as quantum cryptography. In the beginning of this thesis we present a short introduction into finite dimensional quantum systems, describing many particles. We especially emphasize particles with two degrees of freedom, so called qubits, and representations of them. Then we lay our focus on entangled quantum states and how to detect and quantify entanglement. We briefly present the maximally entangled two qubit states and the process of distillation of entanglement. Next we deal with the main topic of this work, multipartite entanglement. We define k -separability and genuine multipartite entanglement, present some genuine multipartite entangled states and concern ourselves with the detection and the classification of genuine multipartite entanglement. In the main part of this thesis we present four different criteria we developed, that were published in two papers in scientific journals. All of these criteria are able to detect genuine multipartite entangled states and three of them are also able to distinguish between different classes of such states. The practicability of the criteria in experiments is shown to be comparatively good, due to their high noise resistance, their low number of required matrix elements, that have to be known of a state to implement a criterion to it and their local realizability, i.e. the possibility to express the criteria in terms of local expectation values of pauli matrices. In the end of this thesis we consider how entanglement can be distributed in an one dimensional, infinite, translational invariant chain of qubits. We present a method to construct states of arbitrary many neighbouring qubits of such a chain. The maximal possible amount of entanglement shared by two neighbouring qubits is the main issue of this topic. When nearest neighbour entanglement is maximized there is no other distribution of entanglement possible in the qubit chain. We show that for not maximized nearest neighbour entanglement, a qubit of the chain can be entangled with its next-next neighbour. By applying the criteria we developed, to a state of four neighbouring qubits, we show that it is even possible to have genuine four particle entanglement in the qubit chain.

Kurzfassung

Diese Diplomarbeit befasst sich hauptsachlich mit dem Nachweis und der Klassifizierung genuiner Vielteilchenverschraenkung in Quantensystemen, die aus beliebig vielen Qubits bestehen. Vielteilchenverschraenkung ist die Grundlage der Vielkoerperphysik und es gibt viele potenzielle Anwendungen fuer sie in der Quanteninformation, etwa Quantenkryptographie. Wir beginnen diese Diplomarbeit mit einer kurzen Einfuehrung in quantenmechanische, endlich dimensionale Vielteilchenzustaende. Wir konzentrieren uns hauptsachlich auf Teilchen mit zwei Freiheitsgraden, sogenannten Qubits, und deren Darstellungen. Anschließend legen wir unseren Fokus auf verschraenkte Zustaende und darauf wie man Verschraenkung detektieren und quantifizieren kann. Wir praesentieren die vier maximal verschraenkten Zustaende, die ein System, bestehend aus zwei Qubits, einnehmen kann und zeigen schematisch den Prozess des Destillierens von Verschraenkung. Dann kommen wir zum Haupthema dieser Arbeit, Vielteilchenverschraenkung. Wir definieren k -separabilitaet und genuine Vielteilchenverschraenkung, stellen einige genuin vielteilchenverschraenkte Zustaende vor und beschaeftigen uns mit der Detektion und der Klassifizierung solcher Zustaende. Im Hauptteil dieser Diplomarbeit praesentieren wir vier, von uns entwickelte Kriterien, die in wissenschaftlichen Journalen veroeffentlicht wurden. Mit all diesen Kriterien kann man genuin vielteilchenverschraenkte Zustaende detektieren und mit drei davon, ist es auch moeglich solche Zustaende zu klassifizieren. Wir zeigen, dass die Zweckmaeßigkeit dieser Kriterien fuer Experimente, wegen ihrer hohen Widerstandsfaehigkeit gegen Stoerungen (noise resistance), der niedrigen Anzahl der Matrixelemente, die man von einem Zustand kennen muss um eines der Kriterien auf ihn anzuwenden und der Moeglichkeit jedes der vier Kriterien in Termen lokaler Erwartungswerte von Paulioperatoren anzuschreiben, vergleichsweise hoch ist. Im letzten Teil dieser Arbeit untersuchen wir, wie Verschraenkung entlang einer eindimensionalen, unendlich langen, translationsinvarianten Kette aus Qubits verteilt sein kann und stellen eine Methode vor um Zustaende beliebig vieler benachbarter Qubits, einer solchen Kette, zu konstruieren. Die maximale Menge an Verschraenkung, die in einem Zustand von zwei benachbarten Qubits stecken kann, ist hier das interessanteste Resultat. Wenn die naechste Nachbar Verschraenkung maximal ist, gibt es keine andere Art der Verschraenkung in dieser Kette. Wir zeigen, dass wenn das nicht der Fall ist, auch Verschraenkung zwischen einem Qubit und seinem ueberuebernaechstem Nachbarn moeglich ist. Außerdem zeigen wir, durch das Anwenden der von uns entwickelten Kriterien auf einen Zustand von vier benachbarten Qubits, dass auch genuine Vierteilchenverschraenkung in der Qubitkette moeglich ist.

List of publications

- P1 Marcus Huber, Hans Schimpf, Andreas Gabriel, Christoph Spengler, Dagmar Bruß,
Beatrix C. Hiesmayr
*Experimentally implementable criteria revealing substructures of genuine multipartite
entanglement*
Phys. Rev. A **83**, 022328 (2011)
- P2 Marcus Huber, Paul Erker, Hans Schimpf, Andreas Gabriel, Beatrix C. Hiesmayr
*Experimentally feasible set of criteria detecting genuine multipartite entanglement in n -qubit Dicke
states and in higher dimensional systems*
Phys. Rev. A **83**, 040301(R) (2011); Phys. Rev. A **84**, 039906(E) (2011)

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Introduction

Entanglement constitutes a fundamental feature of quantum mechanics. Although it is not difficult to define entanglement, one can get a big variety of answers, to the question: “What is entanglement?”. One possible answer is this: If two or more particles are entangled, they cannot be described separately. The composite system contains more information, than the sum of information which is contained in the separated systems.

Until today it is in general still an unsolved problem, to tell if a given quantum state is entangled or not. This problem has been solved for quantum systems consisting of two particles (bipartite systems), where one of the particles has two degrees of freedom (a qubit) and the other particle does not have more than three degrees of freedom (a qubit or a qutrit) (see Ref. [1] and [2]). For quantum systems consisting of more than two particles there have been many attempts to solve the problem of detecting entanglement in such a general multipartite state (for an overview see Ref. [3] and [4]).

Genuine multipartite entanglement is a notion that describes entanglement in quantum systems consisting of more than two particles, where every particle of the system is entangled with all the others. Quantum cryptography is a field where genuine multipartite entanglement already finds an application (see e.g. Ref. [5]). The detection of genuine multipartite entanglement has been subject to many works (see Ref. [6] - [10]). Different genuine multipartite entangled states can behave very differently when one particle is measured, hence there have been attempts to classify them (see Ref. [11] - [17]). The focus of this thesis lies in the detection and the classification of genuine multipartite entangled states.

This work is structured as follows. Section 1 is partitioned in three parts. First we give a brief introduction into quantum states and operations on them. Next we give a mathematical definition of entanglement and show methods to detect and quantify it. In the last part of section 1 we introduce multipartite entanglement, the notions of k -separability and genuine multipartite entanglement and we present ways to detect and classify multipartite entanglement.

Section 2 deals with two papers in which I was able to participate. In “Experimentally implementable criteria revealing substructures of genuine multipartite entanglement” (Ref. [P1]) we introduce different classes of genuine multipartite entanglement and we develop inequalities that are not only able to detect genuine multipartite entanglement but also to distinguish between these classes. In “Experimentally feasible set of criteria detecting genuine multipartite entanglement in n -qubit Dicke states and in higher dimensional systems” (Ref. [P2]) we develop a set of inequalities that are able to detect genuine multipartite entanglement tailored specifically for Dicke states. They work best for general-

ized Dicke states and have an extraordinary high noise resistance.

Section 3 is based on the paper “Maximizing nearest neighbour entanglement in finitely correlated qubit chains” (Ref. [18]). We observe how entanglement can be distributed along the translational invariant spin-chain. First we present the results achieved by the authors of the paper, for the maximal possible amount of entanglement shared by two neighbouring and by two next neighbouring qubits of the spin-chain. Then we present qualitative results we achieved for the entanglement of two next-next neighbouring qubits and for genuine multipartite entanglement in the spin-chain.

1 Foundations

In this section we introduce the mathematical framework for this thesis. The first part is about the Hilbert space, operators, quantum states, operations on quantum states and different basis for them. The second part deals with the topic of entanglement. More precisely with defining, detecting, quantifying and distilling entanglement. The last part is exclusively about multipartite entangled states. We concentrate on states describing more than two particles, we introduce the concepts of k -separability and genuine multipartite entanglement and we give a brief insight to the problem of classifying multipartite entangled states.

1.1 Basics

Here we introduce the Hilbert space, operators and quantum states that live in this Hilbert space. Because of their importance in quantum information theory in general and in this thesis in particular, we then consider quantum states with two degrees of freedom called qubits. Next we lay our attention on states describing more than one particle, the multipartite states. In the middle of this section we give definitions and examples for the operations partial trace and partial transpose acting on quantum states and we show what unitary and LOCC operations are. And finally, in the end of this section we introduce the Pauli matrices as a basis for all qubit states and the generalizations of them, the hermitian Gell-Mann matrices and the unitary Weyl operators, and we give the definition of a matrix norm, the Hilbert Schmidt norm.

1.1.1 The Hilbert space

A Hilbert space is a linear vector space $\mathcal{H}$ over the complex field $\mathbb{C}$, on which an inner product $\langle\psi|\phi\rangle$ is defined. Where $|\psi\rangle$ is a vector in the Hilbert space $\mathcal{H}$, $\langle\phi|$ is a dual vector in the dual space of $\mathcal{H}$ and every inner product $\langle\psi|\phi\rangle$ is a complex number $C \in \mathbb{C}$.

In this thesis we only consider discrete Hilbert spaces $\mathcal{H}^d$ i.e. with finite dimension $d \in \mathbb{N}$. Therefore

$$\mathcal{H}^d = \mathbb{C}^d. \quad (1)$$

1.1.2 Linear operators on the Hilbert space

A linear operator A on the Hilbert space maps all vectors $|\psi\rangle$ of the Hilbert space $\mathcal{H}$ to other vectors $|\phi\rangle$ of the same Hilbert space $\mathcal{H}$, or for the case $A = \mathbb{1}$ to itself.

$$A|\psi\rangle = |\phi\rangle \text{ where } |\psi\rangle, |\phi\rangle \in \mathcal{H} \quad (2)$$

For discrete Hilbert spaces $\mathcal{H} = \mathbb{C}^d$ linear operators can be represented by $d \times d$ -matrices.

Hermitian operators $A = A^\dagger$ on the Hilbert space $\mathcal{H}$ represent quantum mechanical **observables** and their eigenvalues are the possible measurement outcomes.

1.1.3 Pure and mixed quantum states

A state ρ of a quantum system is a positive semidefinite ($\rho \geq 0$), hermitian ($\rho = \rho^\dagger$) operator on the Hilbert space $\mathcal{H}^d$ with $\text{Tr}(\rho) = 1$.

A state is called **pure** iff $\rho^2 = \rho$, otherwise it's called **mixed**.

For pure states ρ_{pure} there always exists a vector $|\psi\rangle$ in the Hilbert space $\mathcal{H}$ such that the outer product of the vector with itself is the state.

$$\rho_{\text{pure}} = |\psi\rangle\langle\psi| \quad (3)$$

There exists no such vector for mixed states ρ_{mixed} , but in general there exist infinitely many decompositions of the form

$$\rho_{\text{mixed}} = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i p'_i |\phi_i\rangle\langle\phi_i| = \dots \quad \text{with } p_i, p'_i > 0 \text{ and } \sum_i p_i = \sum_i p'_i = 1 \quad (4)$$

As a consequence $\text{Tr}(\rho_{\text{mixed}}^2) \leq 1$.

Mixed states correspond to the classical mixture of pure states and $\frac{1}{d}\mathbb{1}_d$ is called the **maximally mixed state** in d dimensions.

Often every vector $|\psi\rangle$ of a Hilbert space is called state and the operator $\rho = |\psi\rangle\langle\psi|$ is called density matrix of the state. When we are considering pure states in this thesis we also refer to vectors $|\psi\rangle$ as states, if $\rho = |\psi\rangle\langle\psi|$.

The **expectation value** $\langle A \rangle$ of an observable A of a quantum system in the state ρ is given by

$$\langle A \rangle = \text{Tr}(\rho A) \quad (5)$$

1.1.4 Qubits

If the dimension of our Hilbert space $\mathcal{H}^d$ is two ($d = 2$) and hence our Hilbert space is a $\mathcal{H}^2 = \mathbb{C}^2$ we have quantum systems with only two degrees of freedom. Such a system is called qubit and its state qubit-state. The pure qubit-states are outer products of normalized superpositions of two orthonormal vectors, which we denote as $|0\rangle$ and $|1\rangle$.

$$|\psi\rangle = a|0\rangle + b|1\rangle \quad \text{with} \quad |a|^2 + |b|^2 = 1 \quad (6)$$

In this thesis we choose $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which is called the computational basis.

From (6) follows the most general pure (one particle) qubit state

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix} \quad (7)$$

In this thesis we mostly concentrate on qubits but in general the states of any discrete Hilbert space $\mathcal{H} = \mathbb{C}^d$ can be described as **Qudits**. Qudit states are outer products of normalized superpositions of d orthonormal vectors.

1.1.5 Multipartite states

When our system consists of more than one particle (or subsystem) we call it multipartite system and its states multipartite states. Its operators and vectors live in a Hilbert space $\mathcal{H}^d$ consisting of the tensor product of its subspaces $\mathcal{H}_A^{d_A}, \mathcal{H}_B^{d_B}, \dots$.

$$\mathcal{H}^d = \mathcal{H}_A^{d_A} \otimes \mathcal{H}_B^{d_B} \otimes \dots \quad (8)$$

where the overall dimension is $d = d_A d_B \dots$.

A **bipartite state** is a state with at least two subsystems, hence every multipartite state is at least bipartite.

If all particles our state describes are qubits ($d_A = d_B = \dots = 2$) and we have a n -partite state, the dimension of the Hilbert space in which our n -partite qubit state lives is $d = 2^n$.

Let's consider a two qubit system. qubit A shall be in the (classical) equally weighted mixture of $|0\rangle$ and $|1\rangle$ (hence ρ_A is the maximally mixed state in two dimensions) and qubit B in an equally weighted superposition of $|0\rangle$ and $|1\rangle$ (hence ρ_B is a pure state). The state describing our two qubit system is the tensor product of the states describing the one qubit systems (ρ_{AB} in (9) is a mixed state).

$$\rho_A = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \mathbb{1}_2, \quad \rho_B = |\psi\rangle\langle\psi| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{with } |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\Rightarrow \rho_{AB} = \rho_A \otimes \rho_B = \frac{1}{4} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad (9)$$

But not all positive semidefinite, hermitian 4×4 -matrices (representing two qubit states) can be achieved by such a tensorproduct of positive semidefinite, hermitian 2×2 -matrices (representing one particle qubit states) or by a superposition of such tensorproducts. Yet their states still describe two qubit systems and they are called **entangled** two qubit states. They can be either pure or mixed and we will have a deeper look at entangled states in section 1.2. Contrary to them ρ_{AB} in (9) is the result of a tensorproduct and therefore is called **separable** two qubit state.

1.1.6 Partial trace

The operation "partial trace" maps operators on a composed Hilbert space $\mathcal{H}^d = \mathcal{H}_A^{d_A} \otimes \mathcal{H}_B^{d_B} \otimes \mathcal{H}_C^{d_C} \otimes \dots \otimes \mathcal{H}_Z^{d_Z}$ to an operator on a subsystem $\mathcal{H}^g = \mathcal{H}_A^{d_A} \otimes \mathcal{H}_C^{d_C} \otimes \dots \otimes \mathcal{H}_Z^{d_Z}$ with $g = d - d_B$.

In general we can expand any state of a multipartite system in a chosen product basis as

$$\rho = \sum_{i,j=0}^{d_A-1} \sum_{k,l=0}^{d_B-1} \rho_{ijkl} |i\rangle_A \langle j|_A \otimes |k\rangle_B \langle l|_B \quad (10)$$

Its **partial trace** is

$$\text{Tr}_A(\rho) = \sum_{i=0}^{d_A-1} \sum_{k,l=0}^{d_B-1} \rho_{iikl} |k\rangle_B \langle l|_B, \quad \text{Tr}_B(\rho) = \sum_{i,j=0}^{d_A-1} \sum_{k=0}^{d_B-1} \rho_{ijkk} |i\rangle_A \langle j|_A \quad (11)$$

Now we take the partial traces of three different two qubit states to see how the partial trace works and what it yields.

- 1) First let us take a state we already know. For ρ_{AB} from eq. (9) the partial trace gives the one particle qubit states.

$$\text{Tr}_A(\rho_{AB}) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \rho_B, \quad \text{Tr}_B(\rho_{AB}) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho_A \quad (12)$$

In this case the original state ρ_{AB} is the tensor product of its partial traces, as we can see in (9).

$$\text{Tr}_A(\rho_{AB}) \otimes \text{Tr}_B(\rho_{AB}) = \rho_{AB} \quad (13)$$

- 2) Next we consider the two qubit state ρ which is an equally weighted mixture of ρ_{AB} from (9) and a state ρ_{01} consisting of a tensor product of two pure states.

$$\begin{aligned} \rho_{01} &= |0\rangle_A \langle 0|_A \otimes |1\rangle_B \langle 1|_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (|0\rangle_A \otimes |1\rangle_B)(\langle 0|_A \otimes \langle 1|_B) := |01\rangle \langle 01| \\ \rho &= \frac{1}{2} \rho_{AB} + \frac{1}{2} \rho_{01} = \frac{1}{8} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{aligned} \quad (14)$$

If we are taking the partial traces of ρ from (14) we end up with two one particle qubit states.

$$\text{Tr}_A(\rho) = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad \text{Tr}_B(\rho) = \frac{1}{4} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \quad (15)$$

The tensor product of these two states does not lead back to the originally state ρ .

$$\mathrm{Tr}_A(\rho) \otimes \mathrm{Tr}_B(\rho) = \frac{1}{16} \begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \neq \rho \quad (16)$$

3) At last we consider the two qubit state ρ_{ϕ^+} , which is the outer product of a vector $|\phi^+\rangle$ of the Hilbert space $\mathbb{C}^4$ with itself (i.e. it is a pure state).

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (17)$$

$$\rho_{\phi^+} = |\phi^+\rangle\langle\phi^+| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

It is not possible to write ρ_{ϕ^+} as a tensor product of states of its subsystems (like in (9)), or as a convex combination of tensor products (like in (14)), hence it is an entangled state. (Actually it is one of the four maximally entangled two qubit states, called the Bell-states.)

Both possible partial traces give the maximally mixed state in two dimensions.

$$\mathrm{Tr}_A(\rho_{\phi^+}) = \mathrm{Tr}_B(\rho_{\phi^+}) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \mathbb{1}_2 \quad (18)$$

The tensor product of the partial traces of ρ_{ϕ^+} gives the maximally mixed state in four dimensions, which is not our original state ρ_{ϕ^+} .

$$\mathrm{Tr}_A(\rho_{\phi^+}) \otimes \mathrm{Tr}_B(\rho_{\phi^+}) = \frac{1}{4} \mathbb{1}_4 \neq \rho_{\phi^+} \quad (19)$$

As we saw above, the tensor product of partial traces only leads back to the original state in special cases. Namely in those, where the multipartite state is a tensor product (and no convex combination of tensor products) of states of its subsystems. We call these states ”**product states**”.

Although all our examples were two qubit states, this also holds for any Qudit state with arbitrary number of subsystems.

1.1.7 Partial transpose

The operation "partial transpose" maps operators from a composed Hilbert space $\mathcal{H}^d = \mathcal{H}_A^{d_A} \otimes \mathcal{H}_B^{d_B} \otimes \dots$ to itself.

Taking the expansion of a multipartite state ρ from eq. (10) the **partial transpose** of ρ is

$$\begin{aligned}\rho^{T^A} &= \sum_{i,j=0}^{d_A-1} \sum_{k,l=0}^{d_B-1} \rho_{\boxed{\text{ji}}_{kl}} |i\rangle_A \langle j|_A \otimes |k\rangle_B \langle l|_B \\ \rho^{T^B} &= \sum_{i,j=0}^{d_A-1} \sum_{k,l=0}^{d_B-1} \rho_{ij\boxed{\text{lk}}} |i\rangle_A \langle j|_A \otimes |k\rangle_B \langle l|_B\end{aligned}\quad (20)$$

Applying the partial transpose operation to the most general two qubit state ρ_{gen} gives

$$\begin{aligned}\rho_{gen} &= \begin{pmatrix} a & b & c & d \\ b^* & e & f & g \\ c^* & f^* & h & i \\ d^* & g^* & i^* & 1 - (a + e + h) \end{pmatrix} \\ \rho_{gen}^{T^A} &= \begin{pmatrix} a & b & c^* & f^* \\ b^* & e & d^* & g^* \\ c & d & h & i \\ f & g & i^* & 1 - (a + e + h) \end{pmatrix}, \quad \rho_{gen}^{T^B} = \begin{pmatrix} a & b^* & c & f \\ b & e & d & g \\ c^* & d^* & h & i^* \\ f^* & g^* & i & 1 - (a + e + h) \end{pmatrix}\end{aligned}\quad (21)$$

Although our example was a two qubit state, this again holds for any Qudit state with arbitrary number of subsystems.

The partial transpose of quantum states plays an important role in detecting whether a state is entangled.

1.1.8 Unitary and LOCC operations

An **unitary operation** acting on a quantum state ρ is achieved by letting an unitary operator U ($U^\dagger = U^{-1}$) act on a quantum state from the left, and its adjoint operator from the right.

$$\rho' = U\rho U^\dagger \quad (22)$$

Such unitary operations describe for example the change of the basis we write our states in, or the evolution of a closed quantum system in time. Every unitary operation is reversible.

A **LOCC operation** on a multipartite quantum state is an operation that can be achieved by local operations (operations just acting on one or a part of the subsystems, in contrast to global operations that act on the whole state) and classical communication (between the subsystems). Therefore LOCC stands for local operations and classical communication and a LOCC operation acting on a n -partite state ρ_n (a state with n subsystems) is described as follows:

$$\begin{aligned} \rho'_n &= \frac{\sum_i (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i) \rho_n (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i)^\dagger}{\text{Tr}(\sum_i (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i) \rho_n (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i)^\dagger)} \\ &\text{with } \sum_i (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i) (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i)^\dagger = \mathbb{1} \\ &\text{and } (J_k^i J_k^{i\dagger}) \geq 0 \end{aligned} \quad (23)$$

Local unitary operations are LOCC operations that have the form

$$\rho'_n = U_{loc} \rho_n U_{loc}^\dagger \quad (24)$$

where $U_{loc} = U_1 \otimes U_2 \otimes \dots \otimes U_n$.

The unitary- and LOCC-operations are completely positive maps (CP-maps, Ref. [2], [4] and [69]), which means that ρ' from (22) and ρ'_n from (23) and (24) are still proper quantum states. The partial trace is also a CP-map, but the partial transpose is not.

1.1.9 Pauli matrices and Bloch sphere

The hermitian and unitary 2×2 matrices with trace zero ($\text{Tr}(\sigma_i) = 0$)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (25)$$

are called **Pauli matrices**. Often $\mathbb{1}_2$ is also called Pauli matrix, denoted by σ_0 .

They fulfill the following algebra

$$\sigma_i \sigma_j = \delta_{ij} \sigma_0 + i \sum_{k=1}^3 \epsilon_{ijk} \sigma_k \quad \text{where } i, j = 1, 2, 3 \quad (26)$$

$$[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = 2i \sum_{k=1}^3 \epsilon_{ijk} \sigma_k, \quad \{\sigma_i, \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \sigma_0 \quad (27)$$

The Pauli matrices form a basis for all 2×2 matrices. Hence any one particle qubit state $\rho_{2 \times 2}$ can be written as a linear combination of Pauli matrices and since all quantum states are hermitian (as are the Pauli matrices) all prefactors are real.

$$\rho_{2 \times 2} = a_0 \sigma_0 + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 \quad \text{with } a_i \in \mathbb{R} \quad (28)$$

Or since we know that $\text{Tr}(\rho_{2 \times 2}) = 1$ we eliminate one parameter and write:

$$\rho_{2 \times 2} = \frac{\sigma_0 + \vec{a} \vec{\sigma}}{2} \quad \text{with } \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and } \vec{a} \vec{\sigma} = a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 \quad (29)$$

$\vec{a}$ is called the **Bloch vector**. All qubit states can be identified with a point in the sphere $|\vec{a}| \leq 1$, the so called **Bloch sphere**, with the pure states on its surface and the maximally mixed state $\frac{1}{2} \mathbb{1}_2$ in the origin ($\vec{a} = \vec{0}$).

1.1.10 Gell-Mann matrices and Weyl operators

There are two generalizations of the Pauli matrices, which together with the unity in d dimensions $\mathbb{1}_d$, form a basis for all $d \times d$ matrices.

The **Gell-Mann matrices** are $(d^2 - 1)$ hermitian $d \times d$ matrices with trace zero. There are symmetric $\lambda_{i,j}^{sym}$, antisymmetric $\lambda_{i,j}^{ant}$ and diagonal λ_k^{dia} Gell-Mann matrices.

$$\lambda_{i,j}^{sym} = |i\rangle\langle j| + |j\rangle\langle i|, \quad \lambda_{i,j}^{ant} = -i|i\rangle\langle j| + i|j\rangle\langle i| \quad \text{with } 0 \leq i < j \leq (d-1)$$

$$\lambda_k^{dia} = \sqrt{\frac{2}{(k+1)(k+2)}} \left(\sum_{l=0}^k |l\rangle\langle l| - (k+1)|k+1\rangle\langle k+1| \right) \quad \text{with } 0 \leq k \leq (d-2) \quad (30)$$

The **Weyl operators** $W_{j,k}$ are d^2 unitary $d \times d$ matrices.

$$W_{j,k} = \sum_{l=0}^{d-1} e^{\frac{2\pi i}{d}lk} |l\rangle\langle (l+j) \bmod d| \quad \text{with } 0 \leq j, k \leq (d-1), \quad W_{0,0} = \mathbb{1}_d \quad (31)$$

For $d = 2$, both the Gell-Mann matrices and the Weyl operators become the Pauli matrices from eq. (25).

1.1.11 Hilbert Schmidt scalar product and norm

The **Hilbert Schmidt scalar product** of two $d \times d$ matrices A and B is the trace of the matrix product of A and $B^\dagger$.

$$\langle A|B \rangle_{HS} = \text{Tr}(AB^\dagger) = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} a_{i,j} b_{i,j} \quad (32)$$

where $a_{i,j}$ is the i -th entry of the j -th column of A .

The **Hilbert Schmidt norm** of a $d \times d$ matrix A is the squareroot the Hilbert Schmidt scalar product of A with itself, i.e. the squareroot of the sum of the square of all matrix elements $a_{i,j}$.

$$\|A\|_{HS} = \sqrt{\langle A|A \rangle_{HS}} = \sqrt{\sum_{i=0}^{d-1} \sum_{j=0}^{d-1} a_{i,j}^2} \quad (33)$$

1.2 Entanglement

Here we first give a definition of entanglement, then we deal with the problem of detecting it. We will have a look at the concept of separability criteria, introduce the Peres Horodecki criterion and show how it works on two examples. Then we explain what entanglement witnesses are and what an optimal entanglement witness is. Next we state what an entanglement measure should fulfill, introduce some measures, give some examples and discuss what they can and cannot carry out. In the second last part of this section we give a very short introduction to the famous Bell states and the last part deals with the concepts of distillation of entanglement and bound entanglement.

1.2.1 Defining entanglement and separability

Formally entanglement can be defined via the exclusion of its opposite: A state ρ_{sep} is separable iff it can be locally produced, which means that it can be written as a tensorproduct or as a convex combination of tensorproducts of states that live in at least two different subsystems.

$$\rho_{sep} = \sum_i p_i \rho_A^i \otimes \rho_B^i \quad (34)$$

$$\text{with } \sum_i p_i = 1, \quad \rho_A^i \in \mathcal{H}_A, \quad \rho_B^i \in \mathcal{H}_B \quad \text{and} \quad \rho_A^i \otimes \rho_B^i \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

If a state can not be written like in (34) it is **entangled**.

1.2.2 Separability criteria

Although the definition of entanglement is not particularly difficult, it is in general, still an unsolved task to say if a given multipartite state is entangled or separable. The purpose of separability criteria is to determine whether a given state is entangled. This means if a state fulfills a separability criteria, we do not know if the state is entangled or separable. But if it violates a separability criteria we know for certain that it is entangled.

The most famous separability criteria (and often used in this thesis) is the Peres Horodecki criterion (Ref. [1] and [2]), also called PPT criterion. **PPT** stands for positive partial transpose. A state ρ is PPT iff its partial transpose has no negative eigenvalues, i.e. it is positive semidefinite.

$$\text{PPT : } \rho^{T^A} \geq 0 \Leftrightarrow \rho^{T^B} \geq 0 \quad (35)$$

If a state is not PPT it is called **NPT**.

Now we state the **Peres Horodecki criterion**:

If ρ is a separable state, then ρ is PPT.

For the two particle systems consisting of two qubits, or of a qubit and a qutrit (a quantum system with three degrees of freedom), the Peres Horodecki criterion is sufficient and necessary, i.e. in such quantum systems we know that if a state is PPT it is separable. But for all higher systems (in this thesis we are especially interested in multipartite qubit states) a state being PPT does not tell us whether it is entangled or separable. When a state is NPT, we know for certain that it is entangled.

If we apply the partial transpose operation on the state ρ from eq. (14) we obtain

$$\rho^{T^A} = \rho^{T^B} = \rho \quad (36)$$

The state ρ from eq. (14) is PPT. The Peres Horodecki criterion tells us that this state is separable. We know this is true because of the way we produced/constructed it.

Applying the partial transpose on $\rho_{\phi+}$ from eq. (17) yields to

$$\rho_{\phi+}^{T^A} = \rho_{\phi+}^{T^B} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (37)$$

The eigenvalues of the matrix from eq. (37) are three times $+\frac{1}{2}$ and one time $-\frac{1}{2}$, so $\rho_{\phi+}$ is NPT. The Peres Horodecki criterion tells us that $\rho_{\phi+}$ is entangled.

The Peres Horodecki criterion is based on the fact that positive, but not complete positive maps (PnCP maps), have the property to preserve the positivity of every separable state ρ_{sep} from eq. (34) but not of every entangled state. Other separability criteria using this property of PnCP maps are the reduction criterion and the extended reduction criterion (Ref. [19]).

Yet other separability criteria are the range criterion (Ref. [20]) or the matrix realignment criterion

(Ref. [21] and [22]), which is based on linear contractions on product states.

1.2.3 Entanglement witnesses

Another way of detecting entanglement is through the concept of entanglement witnesses. All states of a certain Hilbert space build a convex set and the separable states build a closed and convex subset $\mathcal{S}$ of them. Therefore, according to the Hahn-Banach theorem (Ref. [23]), one can define hyperplanes W and V , which separate the set of separable states $\mathcal{S}$ from a part of the entangled states.

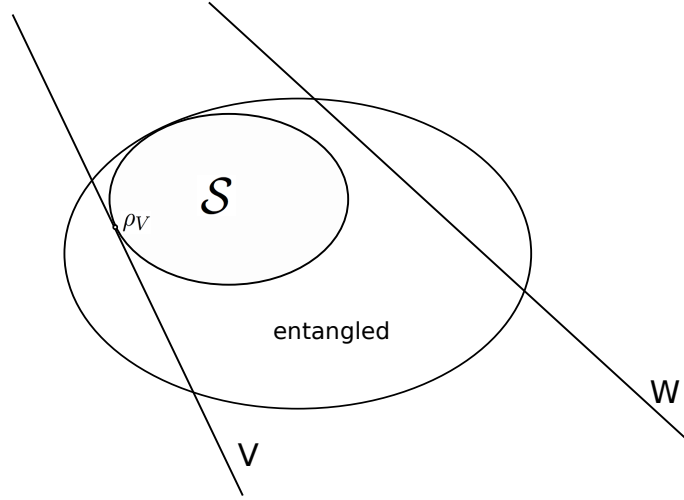


Figure 1: Here we see the (convex) set of all the states of a composed Hilbert space with its (convex) subset of separable states $\mathcal{S}$. The entanglement witnesses defined by the hyperplanes W and V separate a part of the entangled states from the separable ones. V is an optimal witness.

Such hyperplanes are defined by their normal vectors, in discrete Hilbert spaces hermitian operators W and V . The hermitian operator W is called an **entanglement witness** if the scalar product of any separable state ρ_{sep} with W is greater or equal zero for all separable states and if there exist entangled states ρ_{ent} such that the scalar product with W is negative.

$$\langle W, \rho_{sep} \rangle = \text{Tr}(W \rho_{sep}) \geq 0 \quad \forall \rho_{sep}$$

$$\langle W, \rho_{ent} \rangle = \text{Tr}(W \rho_{ent}) < 0 \quad (38)$$

An entanglement witness is an **optimal entanglement witness** (Ref. [2] and [24]) if there exists a separable state $\rho_V \in \mathcal{S}$ for a certain entanglement witness V , such that the scalar product with the

entanglement witness V is zero.

$$\langle V, \rho_V \rangle = \text{Tr}(V \rho_V) = 0 \quad (39)$$

V from figure 1 is such an optimal witness. In figure 1 we can see that V is a tangential hyperplane on $\mathcal{S}$.

The convex structure of the set of all states and its subset of separable states induces that all maximally entangled states ρ_{me} have to be pure, i.e. lie on the boundary of the set of all states.

1.2.4 Entanglement measures

Entanglement measures enable one not only to detect if a state is entangled, but also how much entanglement is in the state. An entanglement measure maps a state $\rho \in \mathcal{H}$ to a positive real number $a \in \mathbb{R}_0^+$. To be justifiably so called, every entanglement measure E has to fulfill some properties (Ref. [25] and [3]).

(I) Normalization

An entanglement measure has to be zero for all separable states ρ_{sep} .

$$E(\rho_{sep}) = 0 \quad \forall \rho_{sep} \in \mathcal{S} \quad (40)$$

An entanglement measure has to be $\log_2(d)$ for every maximally entangled state ρ_{me} .

$$E(\rho_{me}) = \log_2(d) \quad (41)$$

ρ_{ϕ^+} from eq. (17) is the only maximally entangled state we have dealt with so far. The possibility to quantify entanglement, via entanglement measures, makes it clear that there have to be maximally entangled states.

(II) Invariant under local unitary operations

Entanglement does not increase or decrease in a closed system, nor does it increase or decrease under local changes of basis, so an entanglement measure must not either.

$$E(\rho) = E(U_{loc} \rho U_{loc}^\dagger) \quad (42)$$

(III) Non-increasing under LOCC operations

As entanglement in a composed system cannot be created through local operations and classical

communication, an entanglement measure must not increase under LOCC operations .

$$E(\rho) \geq E\left(\frac{\sum_i (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i) \rho (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i)^\dagger}{\text{Tr}(\sum_i (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i) \rho (J_1^i \otimes J_2^i \otimes \dots \otimes J_n^i)^\dagger)}\right) \quad (43)$$

(IV) Convexity

The set of separable states $\mathcal{S}$ is a convex subset of all states with the maximally entangled states on the boundary of the set of all states (see figure 1). One can conclude from this, that a mixture of separable states always gives another separable state and that an entanglement measure of a mixed state ρ_{mixed} can never be higher than the mixture of entanglement measures of those pure states $\rho_{pure,i}$ the mixed state consists of.

$$E(\rho_{mixed}) = E\left(\sum_i p_i \rho_{pure,i}\right) \leq \sum_i p_i E(\rho_{pure,i}) \quad (44)$$

(V) Continuity

If the Hilbert Schmidt norm (eq. (33)) between two states ρ_1 and ρ_2 vanishes, the difference of the entanglement measures of these states has to vanish too.

$$\|\rho_1 - \rho_2\|_{HS} \rightarrow 0 \Rightarrow E(\rho_1) - E(\rho_2) \rightarrow 0 \quad (45)$$

(VI) Additivity

The entanglement measure of a product state (section 1.1.6) has to be equal to the sum of the entanglement measures of the states whose tensor products form the product state, in order to have **strong additivity**.

$$E(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n) = E(\rho_1) + E(\rho_2) + \dots + E(\rho_n) \quad (46)$$

Not all entanglement measures fulfill the above constraint, but some of them do fulfill **subadditivity**, which means that the entanglement measure of a product state is not higher than the sum of the entanglement measures of the states whose tensor product forms the product state.

$$E(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n) \leq E(\rho_1) + E(\rho_2) + \dots + E(\rho_n) \quad (47)$$

If strong additivity of an entanglement measure only holds for product states consisting of the

tensor product of n copies of the same state, we speak of **partial additivity**.

$$E(\rho^{\otimes n}) = nE(\rho) \quad (48)$$

If one has an entanglement measure for pure states ρ_{pure} it is easy to state a formula to expand this measure to mixed states ρ_{mixed} .

$$E(\rho_{mixed}) = \inf_{\rho_{pure,i}, p_i} \sum_i p_i E(\rho_{pure,i}) \quad (49)$$

The infimum is taken over all $\rho_{pure,i}$, p_i with $\rho_{mixed} = \sum_i p_i \rho_{pure,i}$, in other words over all decompositions of ρ_{mixed} .

However in general it is not easy to find the infimum in eq. (49), results are known only for special cases. Constructing entanglement measures in this way is called **Convex Roof Construction**.

The **von Neumann Entropy** (Ref. [26]) of a state ρ is defined as follows:

$$S(\rho) = -\rho \log_2(\rho) = -\sum_i \lambda_i \log_2(\lambda_i), \text{ where } \lambda_i \text{ is the } i\text{-th eigenvalue of } \rho.$$

The sum of the von Neumann Entropy of both partial traces of a bipartite state is an entanglement measure for pure states ρ_{pure} .

$$E_{vN}(\rho_{pure}) = S(\text{Tr}_A(\rho_{pure})) + S(\text{Tr}_B(\rho_{pure})) \quad (50)$$

We test this measure on the pure state $\rho_{\phi+}$ from eq. (17).

First we have to take the partial traces of $\rho_{\phi+}$. In eq. (18) we saw that in both cases it is the maximally mixed state in two dimensions $\frac{1}{2} \mathbb{1}_2$. Now we can write down the von Neumann Entropy of the partial traces of $\rho_{\phi+}$.

$$S(\text{Tr}_A(\rho_{\phi+})) = S(\text{Tr}_B(\rho_{\phi+})) = 2(-\frac{1}{2} \log_2(\frac{1}{2})) = 1 \quad (51)$$

So the measure gives

$$E_{vN}(\rho_{\phi+}) = S(\text{Tr}_A(\rho_{\phi+})) + S(\text{Tr}_B(\rho_{\phi+})) = 2, \quad (52)$$

which is exactly what it should give for a maximally entangled state with $d = 4$.

Except continuity and the fact that its additivity is partial, this entanglement measure fulfills all

the properties listed above for pure states.

One can expand this measure to mixed states ρ_{mixed} by using the convex roof construction. This entanglement measure is called **Entanglement of Formation**, which was introduced in Ref. [27].

$$E_{oF}(\rho_{mixed}) = \inf_{\rho_{pure,i}, p_i} \sum_i p_i E_{vN}(\rho_{pure,i}) \quad (53)$$

This measure fulfills all the properties an entanglement measure should fulfill, though it does not have strong additivity but subadditivity.

An entanglement measure which only holds for bipartite Qudit states is the so called **Concurrence** C . It was first introduced in Ref. [28] and Ref. [29] and for pure bipartite qubit states ρ_{pure} is defined as follows.

$$C(\rho_{pure}) = \frac{d}{d-1} \sqrt{2(1 - \text{Tr}((\text{Tr}_A(\rho_{pure}))^2))} \quad (54)$$

where d is the dimension of one Qudit.

This measure is zero for all separable states like other entanglement measures, but its value cannot become higher than one, which it reaches for maximally entangled states and not $\log_2(d)$.

We can again test the concurrence on the pure state $\rho_{\phi+}$ from eq. (17).

In eq. (18) we took the partial traces and so we can directly compute the Concurrence of $\rho_{\phi+}$.

$$C(\rho_{\phi+}) = \sqrt{2(1 - \text{Tr}((\text{Tr}_A(\rho_{\phi+}))^2))} = \sqrt{2(1 - \text{Tr}(\frac{1}{4}\mathbb{1}_2))} = 1 \quad (55)$$

As the concurrence of $\rho_{\phi+}$ has the value of one, it states that $\rho_{\phi+}$ is a maximally entangled state, exactly the same predication as made by the von Neumann measure.

By the use of the convex roof construction (eq. (49)) the concurrence can be generalized to mixed states ρ_{mixed} .

$$C(\rho_{mixed}) = \inf_{\rho_{pure,i}, p_i} \sum_i p_i C(\rho_{pure,i}) \quad (56)$$

One has again to find the infimum over all decompositions of ρ_{mixed} , which can be very complicated. However, for quantum systems containing only two qubits, Hill and Wootters found an analytic way to compute the concurrence of every two qubit mixed state $\rho_{mixed, 2 \times 2}$.

$$C(\rho_{mixed, 2 \times 2}) = \max\{\sqrt{\mu_1} - \sqrt{\mu_2} - \sqrt{\mu_3} - \sqrt{\mu_4}, 0\} \quad (57)$$

where the μ_i are the eigenvalues of $\rho_{mixed,2 \times 2} \sigma_2 \otimes \sigma_2 \rho_{mixed,2 \times 2}^* \sigma_2 \otimes \sigma_2$ in descending order and $*$ denotes complex conjugation.

Based on these findings, we compute the concurrences of three different two qubit states.

1) First we can again check our result for the concurrence of $\rho_{\phi+}$.

We have to compute $\rho_{\phi+} \sigma_2 \otimes \sigma_2 \rho_{\phi+}^* \sigma_2 \otimes \sigma_2$.

$$\sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{\phi+} = \rho_{\phi+}^* = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \rho_{\phi+} \sigma_2 \otimes \sigma_2 \rho_{\phi+}^* \sigma_2 \otimes \sigma_2 = \rho_{\phi+} \quad (58)$$

As $\rho_{\phi+}$ is a pure state it has only one nonzero eigenvalue which is one. Equation (57) for the concurrence of $\rho_{\phi+}$ then gives

$$C(\rho_{\phi+}) = \max\{1 - 0 - 0 - 0, 0\} = 1, \quad (59)$$

which is the same result that we obtained in eq. (55).

2) Since eq. (57) also works for mixed two qubit states we are able to check if the concurrence of ρ from eq. (14) is zero, as it has to be, since ρ from eq. (14) is the mixture of two separable states and therefore a separable state itself.

$$\rho \sigma_2 \otimes \sigma_2 \rho^* \sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{16} & \frac{1}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{1}{16} & 0 \end{pmatrix} \quad (60)$$

This matrix has two nonzero eigenvalues, which are both $\frac{1}{16}$.

With this result eq. (57) yields

$$C(\rho) = \max\{\frac{1}{4} - \frac{1}{4} - 0 - 0, 0\} = 0, \quad (61)$$

as we expected.

- 3) The third state $\rho_{p.ent}$ of which we compute the concurrence is an equally weighted mixture of the maximally mixed two qubit state $\frac{1}{4}\mathbb{1}_4$ and ρ_{ϕ^+} from eq. (17).

$$\rho_{p.ent} = \frac{1}{2}\left(\frac{1}{4}\mathbb{1}_4 + \rho_{\phi^+}\right) = \frac{1}{8} \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix} \quad (62)$$

Again we have to compute $\rho_{p.ent}\sigma_2 \otimes \sigma_2\rho_{p.ent}^*\sigma_2 \otimes \sigma_2$.

$$\rho_{p.ent}\sigma_2 \otimes \sigma_2\rho_{p.ent}^*\sigma_2 \otimes \sigma_2 = \begin{pmatrix} \frac{13}{64} & 0 & 0 & \frac{3}{16} \\ 0 & \frac{1}{64} & 0 & 0 \\ 0 & 0 & \frac{1}{64} & 0 \\ \frac{3}{16} & 0 & 0 & \frac{13}{64} \end{pmatrix} \quad (63)$$

This matrix has the eigenvalues $\{\frac{25}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}\}$. Now we can compute the concurrence of $\rho_{p.ent}$ by using eq. (57).

$$C(\rho_{p.ent}) = \max\left\{\frac{5}{8} - \frac{1}{8} - \frac{1}{8} - \frac{1}{8}, 0\right\} = \frac{1}{4} \quad (64)$$

So the state $\rho_{p.ent}$ is entangled, but not maximally entangled.

The concurrence fulfills all the properties an entanglement measure should fulfill, although instead of strong additive it is subadditive.

With the analytic result for the concurrence from eq. (57) the Entanglement of Formation of any two qubit state (pure or mixed) becomes

$$E_{oF}(\rho_{mixed,2\times 2}) = H\left[\frac{1}{2}\left(1 + \sqrt{1 - C^2(\rho_{mixed,2\times 2})}\right)\right] \quad (65)$$

$$\text{where } H[f] = -f \log_2(f) - (1-f) \log_2(1-f).$$

Other entanglement measures that are constructed using the convex roof construction, are the "Entanglement Cost" (Ref. [37]) and the "Entanglement of Distillation" (Ref. [36]).

In contrast to the entanglement measures above there are also measures not based on the convex roof construction, but on the distance of a state to the nearest separable state (the distance to $\mathcal{S}$). Such distance measures are for example the "Relative Entropy of Entanglement" (Ref. [32]), the "Hilbert Schmidt measure" (Ref. [33]) or the "Bures Distance measure" (Ref. [34] and [35]).

1.2.5 Maximally entangled two qubit states/Bell states

Originally the Bell states stem from a paper by John Bell published in 1964 (Ref. [31]). Bell tried to test a Gedankenexperiment by Einstein, Podolsky and Rosen published in a paper in 1935 (Ref. [38]), the so called EPR paradoxon. Einstein, Podolsky and Rosen claimed that quantummechanics is not a local realistic theory in their sense and has therefore to be wrong. Bell came up with an idea to test this assumption in an experiment. The outcome of the experiment had to fulfill an inequality, the so called Bell inequality, so that the entire experiment would be describable with local realistic theories. Bell managed to violate this inequality using the four **maximally entangled two qubit states**, the so called **Bell states**.

$$\rho_{\phi^{\pm}} = |\phi^{\pm}\rangle\langle\phi^{\pm}| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_{\psi^{\pm}} = |\psi^{\pm}\rangle\langle\psi^{\pm}| = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (66)$$

$$\begin{aligned} \text{where } |\phi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \\ \text{and } |\psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \end{aligned}$$

1.2.6 Distillation of entanglement and bound entanglement

The aim of distilling entanglement is to produce states containing as much entanglement as possible, in the best case maximally entangled states ρ_{me} . If one has k copies of an entangled, but not maximally entangled, bipartite state and one uses only local operations and classical communication (LOCC operations, see eq. (23)) to increase the entanglement in one of these copies, by so to say shifting entanglement from the other copies into it, we are speaking of **distillation of entanglement**.

$$\rho^{\otimes k} \xrightarrow{\text{LOCC}} \rho_{me} \quad (67)$$

It was first introduced in Ref. [39] and [40]. A lot of different distillation protocols have been developed (see e.g. Ref. [39], [41] and [43]), all of them require a number of copies k of an entangled state ρ and a number of repetitions l of the protocol. In the following the index of ρ denotes the number of iterations the state has passed through.

$$\begin{aligned} & \text{If } \lim_{l \rightarrow \infty} \rho_l = \rho_{me}, \\ & \text{with } \rho_l = \frac{\sum_i (J_1^i \otimes J_2^i) \rho_{l-1}^{\otimes k} (J_1^i \otimes J_2^i)^\dagger}{\text{Tr}(\sum_i (J_1^i \otimes J_2^i) \rho_{l-1}^{\otimes k} (J_1^i \otimes J_2^i)^\dagger)} \quad (68) \\ & \text{and } J_k^i \text{ as in eq. (23),} \end{aligned}$$

then ρ is called distillable.

Every entangled two qubit state is distillable. However in higher dimensional systems, there are entangled states that cannot be distilled to maximally entangled states. In Ref. [42] it was shown that states that are positive under partial transpose (PPT, see section 1.2.2) are not distillable. In systems containing three or more qubits, or in systems being higher dimensional than a system consisting of a qubit and a Qutrit, there exist such states that are PPT but entangled. As they cannot be distilled to maximally entangled states, they are called **bound entangled**. The first example of a bound entangled state can be found in Ref. [44], further examples were given in Ref. [45] - [52]. Until today it is not known if there exist bound entangled states that are NPT, although there have been attempts to prove this (Ref. [53] and [54]).

1.3 Multipartite entanglement

Although most of the things we discussed until now is true for any arbitrary multipartite state, all our examples were quantum states containing only two qubits. In this section our interest lies on multipartite entangled states containing more than two subsystems. In the beginning of this section we define k -separability and genuine multipartite entanglement (GME). Then we introduce two GME states which, as we will make plausible in an example, behave very differently when one subsystem is measured, the W- and the GHZ-state. Next we define two more interesting GME states, namely the Dicke states and the Aharonov states. Section 1.3.4 is about detecting genuine multipartite entanglement, mainly it is about a set of inequalities developed to detect GME states. In the end of this section we will deal with the problem of classifying GME states. We show different ways multipartite states can be classified and give a short insight in our own classification of GME states, which we will introduce in section 2.

1.3.1 k -separability and genuine multipartite entanglement (GME)

If one considers states containing more than two subsystems the concepts of separability and entanglement can be refined. For example a three particle state can describe one particle being entangled with one of the other particles, one particle being entangled with both of the other particles, or all three particles can be separable from one another. This leads to the definition of k -separability.

A pure state ρ_{pure} containing n subsystems is called **k -separable**, iff it can be written in the form

$$\rho_{pure}^k = |\psi^k\rangle\langle\psi^k| \text{ where } |\psi^k\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \dots |\phi_k\rangle \text{ with } k \leq n. \quad (69)$$

If $k = n$ then the state ρ_{pure}^k is fully separable, if $k \neq 2$ then the state is one separable or **genuine multipartite entangled (GME)**.

To make the above definition of genuine multipartite entanglement better comprehensible, we highlight the fact that every k -separable state is also $(k - 1)$ -separable.

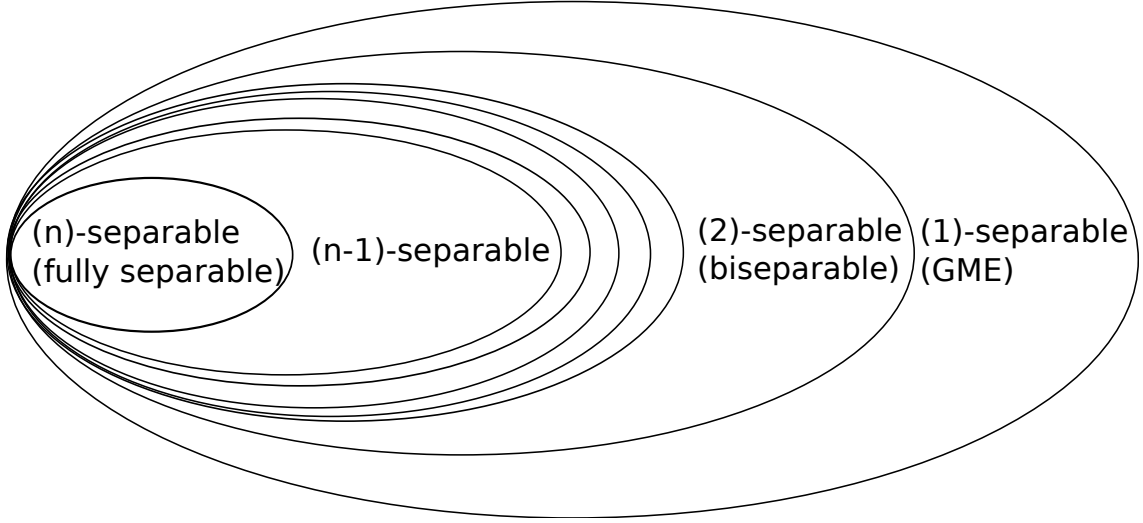


Figure 2: Every k -separable subset of n particle states is convex. With the set of the fully separable states in the innermost and the set of the genuine multipartite entangled states outermost.

It is straightforward to generalize the definition of k -separability for mixed states. A mixed state ρ_{mixed} is k -separable, iff

$$\rho_{mixed}^k = \sum_i p_i \rho_{pure,i}^k \text{ with } p_i \geq 0 \text{ and } \sum_i p_i = 1, \quad (70)$$

where the $\rho_{pure,i}^k$ have to be k -separable, as defined in eq. (69).

One has to be careful with the idea of a mixed state being biseparable ($k = 2$) or higher separable ($k > 2$). Since, unlike the case of a pure state being bi- or higher separable, for mixed states that does not mean necessarily that one can separate the subsystems into two or more parts that are separable from each other.

To give a brief insight into this issue we consider the three particle mixed state ρ_{bisep} which is biseparable.

$$\rho_{bisep} = \sum_i p_i \rho_{12}^i \otimes \rho_3^i + \sum_j q_j \rho_{23}^j \otimes \rho_1^j + \sum_k r_k \rho_{31}^k \otimes \rho_2^k \quad (71)$$

$$\text{with } p_i \geq 0, \quad q_j \geq 0, \quad r_k \geq 0 \text{ and } \sum_i p_i + \sum_j q_j + \sum_k r_k = 1$$

where ρ_{lm} is an entangled state of the subsystems l and m .

Though the state is biseparable ($k = 2$ -separable), there exists no definite partition (e.g. 12|3) to

which it is separable.

1.3.2 1-separable states: The W state and the GHZ state

Though k -separability classifies entanglement properties of multipartite quantum states, it is not fine enough, i.e. two distinct k -separable states can behave very differently. To clarify this we consider two n particle genuine multipartite entangled states (GME states), the Greenberger, Horn, Zeilinger state or GHZ state ρ_{GHZ} (Ref. [55] and [56]) and the W state, supposedly named by Wolfgang Duer, ρ_W (Ref. [57] and [58]). Both states are n particle qubit states. In quantum Information theory these two states have been studied for a long time and are very useful for several tasks.

To make the following definitions of the n particle GHZ and W state better comprehensible we first present the three particle states ρ_{GHZ_3} and ρ_{W_3} .

$$\rho_{GHZ_3} = |GHZ_3\rangle\langle GHZ_3| \text{ with } |GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad (72)$$

$$\rho_{W_3} = |W_3\rangle\langle W_3| = \frac{1}{3} \text{ with } |W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \quad (73)$$

It is rather easy to generalize the above definitions to n particle states.

With the notation

$$|\underbrace{000\dots 0}_n\rangle = \bigotimes_{i=1}^n |0\rangle \text{ and } |\underbrace{111\dots 1}_n\rangle = \bigotimes_{i=1}^n |1\rangle \quad (74)$$

they appear as:

$$\rho_{GHZ} = |GHZ\rangle\langle GHZ| \text{ where } |GHZ\rangle = \frac{1}{\sqrt{2}}\left(\bigotimes_{i=1}^n |0\rangle + \bigotimes_{i=1}^n |1\rangle\right) \quad (75)$$

$$\rho_W = |W\rangle\langle W| \text{ where } |W\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |W_i\rangle \text{ and } |W_i\rangle = \bigotimes_{k \neq i} |0_k\rangle \otimes |1_i\rangle \quad (76)$$

For the case of $n = 2$ they are Bell states (eq. (66)). The GHZ state becomes ρ_{ϕ^+} and the W state becomes ρ_{ψ^+} .

So although both of these states are GME states they differ from each other in important facts:

A single state (no copies, no distillation) that can be converted into a GHZ state via LOCC operations (eq. (23)) cannot be converted into a W state and a single state that can be converted into a W state via LOCC operations cannot be converted into an GHZ state.

Another big difference between the W and the GHZ state is that they behave very different when one of their subsystems is measured. If one measures one particle (in the computational basis) of the n particle GHZ state, the remaining $(n - 1)$ particle state is for certain fully separable. While measuring one particle (in the computational basis) of a n particle W state leaves one with an probability of $\frac{n-1}{n}$ with another $(n - 1)$ particle W state.

Examples: Now we explicitly do the computation of taking partial traces of the three particle W state and of the three particle GHZ state. This corresponds to the case where one particle is "lost" and what is left is the classical mixture of the possible measurement outcomes, of one particle being measured.

$$\rho_{W_3} = |W_3\rangle\langle W_3| = \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (77)$$

$$\text{Tr}_A(\rho_{W_3}) = \text{Tr}_B(\rho_{W_3}) = \text{Tr}_C(\rho_{W_3}) = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (78)$$

To check if this state is entangled we use the Peres Horodecki criterion introduced in section 1.2.2, since this is a two qubit state and for such states the Peres Horodecki criterion is a necessary and sufficient criterion. First we have to take the partial transpose of the state from eq. (78).

$$(\text{Tr}_A(\rho_{W_3}))^{T_A} = (\text{Tr}_A(\rho_{W_3}))^{T_B} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (79)$$

This matrix has the eigenvalues $\{\frac{1}{6}(1 + \sqrt{5}), \frac{1}{3}, \frac{1}{3}, \frac{1}{6}(1 - \sqrt{5})\}$, so $\text{Tr}_A(\rho_{W_3})$ is NPT and we know for certain that it is entangled.

Now we do the same for a three particle GHZ-state.

$$\rho_{GHZ_3} = |GHZ_3\rangle\langle GHZ_3| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (80)$$

$$\text{Tr}_A(\rho_{GHZ_3}) = \text{Tr}_B(\rho_{GHZ_3}) = \text{Tr}_C(\rho_{GHZ_3}) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (81)$$

Again we use the Peres Horodecki criterion to detect if $\text{Tr}_A(\rho_{GHZ_3})$ is entangled. Partial transposition does not change $\text{Tr}_A(\rho_{GHZ_3})$.

$$(\text{Tr}_A(\rho_{GHZ_3}))^{T_A} = (\text{Tr}_A(\rho_{GHZ_3}))^{T_B} = \text{Tr}_A(\rho_{GHZ_3}) \quad (82)$$

As $\text{Tr}_A(\rho_{GHZ_3})$ is a proper quantum state, i.e. it is positive semidefinite, also $(\text{Tr}_A(\rho_{GHZ_3}))^{T_A}$ has no negative eigenvalues. So $\text{Tr}_A(\rho_{GHZ_3})$ is PPT and we know it is for certain separable.

We just saw that tracing out one subsystem of the W-state yields to an entangled state (eq. (79)), while tracing out one subsystem of a GHZ-state yields to a separable state (eq. (81)). So as these partial traces are the classical mixtures of the possible measurement outcomes, we made it plausible that all possible measurement outcomes of a GHZ state are separable and we know for certain that there have to be measurement outcomes of a W state that are entangled.

1.3.3 More types of interesting GME states

Here we present two more types of GME states, which are pure states, like the GHZ and the W state.

The Dicke states were first investigated by Robert Henry Dicke in Ref. [72] with respect to light emission of a cloud of atoms. Nowadays they play an important role in quantum information theory, since they provide a rich resource for quantum information tasks (e.g. open destination teleportation, see Ref. [59]) and they have a remarkable robustness to particle loss. Originally Dicke states were introduced as n particle qubit states with two excited subsystems. Today one often uses a generalization of the original definition to n particle qubit states with m excited subsystems. Such a generalized Dicke state $\rho_{D_m^n}$ is the outer product of the vector $|D_m^n\rangle$, as it is a pure state. It is presented below.

$$\rho_{D_m^n} = |D_m^n\rangle\langle D_m^n|,$$

$$\text{where } |D_m^n\rangle = \frac{1}{N} \sum_{\{\alpha\}} |d_{\{\alpha\}}\rangle, \quad \text{with } N = \binom{n}{m}^{\frac{1}{2}} \quad \text{and} \quad |d_{\{\alpha\}}\rangle = \bigotimes_{i \notin \{\alpha\}} |0\rangle_i \bigotimes_{i \in \{\alpha\}} |1\rangle_i \quad (83)$$

$1 \leq m \leq \frac{n}{2}$ for even n and $1 \leq m \leq \frac{n-1}{2}$ for odd n .

The set $\{\alpha\}$ denotes exactly those indices, whose associated subsystems are excited (i.e. are in the state $|1\rangle$, all others are in the state $|0\rangle$). The sum is taken over all inequivalent sets $\{\alpha\}$, where $|\{\alpha\}| = m$.

With this definition the generalized n particle Dicke state with $m = 1$ is the n particle W state. With respect to local unitary invariance there are $\frac{n}{2}$ different Dicke states $\rho_{D_m^n}$ for a fixed even n and $\frac{n-1}{2}$ for a fixed odd n .

For a better understanding of Dicke states, we explicitly write down the vectors, whose outer product with themselves form the four particle Dicke state with two excitations $\rho_{D_2^4} = |D_2^4\rangle\langle D_2^4|$, the five particle Dicke state with two excitations $\rho_{D_2^5} = |D_2^5\rangle\langle D_2^5|$ and the six particle Dicke state with three excitations $\rho_{D_3^6} = |D_3^6\rangle\langle D_3^6|$.

$$|D_2^4\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle) \quad (84)$$

$$|D_2^5\rangle = \frac{1}{\sqrt{10}}(|00011\rangle + |00101\rangle + |00110\rangle + |01001\rangle + |01010\rangle + \\ + |01100\rangle + |10001\rangle + |10010\rangle + |10100\rangle + |11000\rangle) \quad (85)$$

$$|D_3^6\rangle = \frac{1}{\sqrt{20}}(|000111\rangle + |001011\rangle + |001101\rangle + |001110\rangle + |010011\rangle + \\ + |010101\rangle + |010110\rangle + |011001\rangle + |011010\rangle + |011100\rangle + \\ + |100011\rangle + |100101\rangle + |100110\rangle + |101001\rangle + |101010\rangle + \\ + |101100\rangle + |110001\rangle + |110010\rangle + |110100\rangle + |111000\rangle) \quad (86)$$

The Aharonov states (Ref. [60], [61] and [62]) are the first GME states we consider, which are for more than two particles not realizable with qubits. Actually the number of degrees of freedom is exactly as high as the number of particles ($d = n$), so they are n particle n dimensional states. Aharonov states are also very important in quantum information theory. They provide solutions to the “N-stranger”, “secret sharing”, and “liar detection” problems (Ref. [60]) and they contain a lot of so called nonlocal information (Ref. [61]). The n particle Aharonov state is the totally antisymmetric state and for the case of two qubits ($d = n = 2$) becomes the Bell state ρ_{ψ^-} , see eq. (66).

$$\rho_{\psi_{nA}} = |\psi_{nA}\rangle\langle\psi_{nA}| \quad \text{with} \quad |\psi_{nA}\rangle = \frac{1}{\sqrt{n!}} \sum_{i_1, \dots, i_n}^{n-1} \epsilon_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle \quad (87)$$

where $\epsilon_{i_1, \dots, i_n}$ is the Levi Civita symbol.

As an example we show the vector $|\psi_{3A}\rangle$, whose outer product with itself forms the three particle Aharonov state $\rho_{\psi_{3A}}$.

$$|\psi_{3A}\rangle = \frac{1}{\sqrt{6}}(|012\rangle + |201\rangle + |120\rangle - |021\rangle - |102\rangle - |210\rangle) \quad (88)$$

Another interesting group of states, some of which are GME states, are the graph states (Ref. [63], [64] and [65]). A special graph state is the GHZ state. They are called graph states because every such state corresponds at least to one “graph”, i.e. to a set of vertices and edges. They are mathematical constructed by means of stabilizing operators, one for each vertex, which are tensor products of Pauli matrices (eq. (25)). For several reasons they are important for quantum information theory: they are robust against decoherence (Ref. [63]), occur as codewords of quantum error correcting codes (Ref. [66]) and as a resource for measurement based quantum computation (Ref. [67], [68]). We will not

go deeper into this matter, as graph states (besides the GHZ state) are not connected to the topic of this thesis.

1.3.4 Detecting genuine multipartite entanglement

One of the first who tried successfully to detect some of the GME states were the Horodecki family in 2001 (Ref. [6]). Later others tried more or less successfully (Ref. [7] - [9]), but all these attempts just covered very small areas of the state space. In 2008 Seevinck and Uffink presented inequalities based on nonlinear functions of matrix elements with which it was possible to detect GME states, that could not be detected before (Ref. [10]). Yet they still covered a rather small area of the state space and it became clear that one needs a framework to construct criteria for detecting GME states, in order to bring some light in the structure of the state space.

In 2010 Huber, Mintert, Gabriel and Hiesmayr developed a way of constructing sets of inequalities to detect GME states (Ref. [73]). During the development of these inequalities I could get a first glimpse into scientific research and contribute a little bit to the work.

First they showed that for any separable two particle state $\rho_{sep,2p} \in \mathcal{H}_A \otimes \mathcal{H}_B$ the inequality

$$\sqrt{|\operatorname{Re}(\langle \phi | (\mathbb{1} \otimes \Pi_B) \rho_{sep,2p}^{\otimes m} (\Pi_A \otimes \mathbb{1}) | \phi \rangle)|} \leq \sqrt{\langle \phi | \rho_{sep,2p}^{\otimes m} | \phi \rangle} \quad (89)$$

holds for any positive integer m .

$|\phi\rangle\langle\phi|$ is any fully separable $2m$ -particle state with $|\phi\rangle\langle\phi| \in (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes m}$. The cyclic permutation operator Π_A lives in $\mathcal{H}_A^{\otimes m}$ and acts as follows.

$$\Pi_A |\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_m\rangle = |\psi_2\rangle \otimes |\psi_3\rangle \otimes \dots \otimes |\psi_m\rangle \otimes |\psi_1\rangle \quad (90)$$

Π_B acts on states in $\mathcal{H}_B^{\otimes m}$ the same way, as Π_A acts on states in $\mathcal{H}_A^{\otimes m}$.

For $m = 2$ and $|\phi\rangle = |ijkl\rangle$ eq. (89) becomes

$$|\langle il | \rho_{sep,2p} | kj \rangle| - \sqrt{\langle ij | \rho_{sep,2p} | ij \rangle \langle kl | \rho_{sep,2p} | kl \rangle} \leq 0. \quad (91)$$

To show how this inequality works we give examples below.

We consider the state $\rho_{p.ent}$ from eq. (62) and choose $i = 1$ and $j = k = l = 0$. For that choices the

inequality (91) yields

$$|\langle 10 | \rho_{p.ent} | 00 \rangle| - \sqrt{\langle 10 | \rho_{p.ent} | 10 \rangle \langle 00 | \rho_{p.ent} | 00 \rangle} \leq 0. \quad (92)$$

The first term gives

$$\left| \frac{1}{8} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right| = 0 \quad (93)$$

and the factors under the squareroot give

$$\frac{1}{8} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{8} \quad (94)$$

and

$$\frac{1}{8} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{3}{8}. \quad (95)$$

The inequality becomes

$$0 - \sqrt{\frac{3}{64}} = -\frac{\sqrt{3}}{8} \leq 0, \quad (96)$$

which is true.

So this does not tell us if $\rho_{p.ent}$ is an entangled or a separable state.

We know already from section 1.2.4 that $\rho_{p.ent}$ is an entangled state, so we try again to detect it as an entangled state by using the inequality (91).

Now we choose $i = l = 1$ and $j = k = 0$. For this choice the inequality yields

$$|\langle 11 | \rho_{p.ent} | 00 \rangle| - \sqrt{\langle 10 | \rho_{p.ent} | 10 \rangle \langle 01 | \rho_{p.ent} | 01 \rangle} \leq 0. \quad (97)$$

The first term gives

$$\left| \frac{1}{8} \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right| = \frac{1}{4} \quad (98)$$

and the factors under the squareroot give

$$\frac{1}{8} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{8}. \quad (99)$$

So this time the inequality becomes

$$\frac{1}{4} - \sqrt{\frac{1}{64}} = \frac{1}{8} \leq 0, \quad (100)$$

which is false.

So the inequality (91) is violated and this tells us that $\rho_{p.ent}$ is for certain an entangled state.

Generally, as we just saw, the inequality subtracts products of squareroots of certain diagonal elements (e.g. $\sqrt{\langle 10|\rho|10\rangle\langle 00|\rho|00\rangle}$ for $i = 1$ and $j = k = l = 0$), of the state ρ one wants to investigate, from absolute values of certain off-diagonal elements (e.g. $|\langle 10|\rho|00\rangle|$ for $i = 1$ and $j = k = l = 0$) of the state ρ .

If one wants to check if a given state is entangled by using this inequality one would have to compute all the inequalities for the different values of i, j, k and l . If at least one of them is violated one knows for certain that the state is entangled, if they all are fulfilled then the state can be either entangled or separable.

In Ref. [73] another inequality was derived, based on the inequality (89), which is a generalization to the multipartite case, hence the state one wants to examine with this inequality has in general more than two subsystems.

The inequality

$$\sqrt{\langle \phi | \rho_{bs}^{\otimes 2} \Pi | \phi \rangle} - \sum_i \sqrt{\langle \phi | \mathcal{P}_i^\dagger \rho_{bs}^{\otimes 2} \mathcal{P}_i | \phi \rangle} \leq 0 \quad (101)$$

holds for all (at least) biseparable states ρ_{bs} , if $|\phi\rangle\langle\phi|$ is fully separable.

Where i denotes the i -th bipartition of the $(2^{n-1} - 1)$ bipartitions that are possible for a n particle state. (The order of the bipartitions is arbitrary.) Π is the so called global permutation operator, performing simultaneous permutations on all subsystems. and $\mathcal{P}_i = \Pi_{A_i} \otimes \mathbb{1}_{B_i}$, where A_i and B_i are the subsystems split by the bipartition i . If this inequality is violated we know for certain that the state is genuine multipartite entangled, if it is not we do not know if its entangled or separable. The possibility to detect GME states depends strongly on the choice of $|\phi\rangle\langle\phi|$.

With a certain choice of $|\phi\rangle\langle\phi|$, for the inequality (101), Huber, Mintert, Gabriel and Hiesmayr developed an inequality suited for a certain kind of GME state. They chose $|\phi\rangle$ to be

$$|\phi_{ij}\rangle = |s_i\rangle \otimes |s_j\rangle \text{ with } |s_i\rangle = |xx...xyx...xx\rangle, \quad (102)$$

where $|x\rangle$ and $|y\rangle$ are one particle states and $|y\rangle$ is on the i -th position of $|s_i\rangle$.

Then the inequality

$$\sum_{i \neq j} \sqrt{\langle \phi_{ij} | \rho_{bs}^{\otimes 2} \Pi | \phi_{ij} \rangle} - (n-2) \sqrt{\langle \phi_{ij} | \mathcal{P}_i^\dagger \rho_{bs}^{\otimes 2} \mathcal{P}_i | \phi_{ij} \rangle} \leq 0 \quad (103)$$

holds for all biseparable states ρ_{bs} .

The index i unlike in eq. (101) does not stand for all possible bipartitions, but A_i is the duplicated Hilbert space of the i -th subsystem and B_i is the rest. The permutation operators are defined as above.

The advantages of the criteria from Ref. [73], compared to other criteria, are that the required number of measurements grows polynomially with system size (not exponentially, like a full state tomography does) and their robustness against noise, i.e. robustness against disturbances of the state in the experiment.

Based on these criteria we developed inequalities (Ref. [P1] and [P2]) that can detect genuine multipartite entanglement, suited for special kinds (or classes) of GME states. We will address this matter in section 2.

1.3.5 Different approaches of classifying multipartite qubit states

In the section 1.3.2 we learned that there can be big differences between different GME states. One can conclude that it could be useful to implement different classes for GME states or multipartite states in general. Until today there have been a few attempts of classifying multipartite states, mainly

multipartite qubit states (Ref. [58], [11] -[16]). In these attempts classes were introduced, in such a way that all the states each class contains can be converted into each other with a certain probability by only using local operations and classical communication. Therefore such operations are called stochastic local operations and classical communication or SLOCC operations, see LOCC operations (section 1.1.8).

Below we briefly present two different SLOCC classification methods for qubit states, one for up to four particle states, the other for an arbitrary number of particles.

1) The following classification is presented in Ref. [17].

For the case of bipartite states there are only two SLOCC classes, the class of separable and the class of entangled states. For states of three particles there are four different SLOCC classes, the class of separable states, the class of biseparable states and two classes of GME states, the GHZ class and the W class (Ref. [58]). For four particle states there is an infinite number of different SLOCC classes, but in Ref. [17] Verstraete, Dehaene, De Moor and Verschelde show that they can be gathered into nine continuous families with similar properties of all the states each family contains.

First they point out that

$$SU(2) \otimes SU(2) \simeq SO(4) \text{ and } SL(2, C) \otimes SL(2, C) \simeq SO(4) \quad (104)$$

and that any pure four qubit state can be parametrized by a four index tensor, which can be rewritten as a 4×4 matrix.

Then they prove that every complex $n \times n$ matrix R can be converted into a $n \times n$ matrix $R' = O_1 R O_2$ by two square orthogonal matrices O_1 and O_2 in such a way that R' is a direct sum of so called Jordan blocks and degenerated Jordan blocks. Based on this they show that there are 12 different families of 4×4 matrices consisting of Jordan blocks and degenerated Jordan blocks of certain dimension. Of these 12 families two are equivalent to each other and one can get rid of two more such families due to the fact that the permutation of qubits transforms families into each other, which results in nine different families for pure four qubit states.

(I)

$$G_{abcd} = \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle)$$

(II)

$$L_{abc_2} = \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) + c(|0101\rangle + |0110\rangle + |1010\rangle)$$

(III)

$$L_{a_2b_2} = a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) + |0110\rangle + |0011\rangle$$

(IV)

$$\begin{aligned} L_{ab_3} = & a(|0000\rangle + |1111\rangle) + \frac{a+b}{2}(|0101\rangle + |1010\rangle) + \frac{a-b}{2}(|0110\rangle + |1001\rangle) \\ & + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |1000\rangle + |0100\rangle) \end{aligned} \quad (105)$$

(V)

$$L_{a_4} = a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) + (i|0001\rangle + |0110\rangle - i|1011\rangle)$$

(VI)

$$L_{a_2O_{3\oplus\bar{1}}} = a(|0000\rangle + |1111\rangle) + (|0011\rangle + |0110\rangle) + |0101\rangle$$

(VII)

$$L_{O_{5\oplus\bar{3}}} = |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle$$

(VIII)

$$L_{O_{7\oplus\bar{1}}} = |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle$$

(IX)

$$L_{O_{3\oplus\bar{1}}O_{3\oplus\bar{1}}} = |0000\rangle + |0111\rangle$$

They state that every pure four qubit state up to permutations of the qubits can be transformed into one of these families by SLOCC operations and prove it. The parameters a , b , c and d are complex numbers and the indices of the families (e.g. a_2 , b_2 in $L_{a_2b_2}$) stand for the respective Jordan block structure of the family, i.e. for the Jordan block structure of the transformed 4×4 matrices R' corresponding to the pure four qubit states belonging to the same family.

- 2) In Ref. [11] a classification for pure symmetric n qubit states is presented. These pure symmetric states $|\psi_S\rangle$ are an equally weighted convex combination of tensor products of the vectors which

outer products are qubit states $|\epsilon_i\rangle$ ($|\psi\rangle$ in eq. (6)).

$$|\psi_S\rangle = \mathcal{N} \sum_{1 \leq i_1 \neq \dots \neq i_n \leq n} |\epsilon_{i_1} \dots \epsilon_{i_n}\rangle \quad \text{with} \quad \langle \psi_S | \psi_S \rangle = 1 \quad (106)$$

As shown in Ref. [12] eq. (106) can be obtained by an expansion of the generalized Dicke states (eq. (83)).

They introduce the expressions “degeneracy number”, “degeneracy configuration” and “diversity degree” for symmetric pure states.

The **degeneracy numbers** N_i of a pure symmetric qubit state $|\psi_S\rangle$ are the numbers of $|\epsilon_i\rangle$ in eq. (106) that are identical. For example a symmetric n qubit state with all $|\epsilon_i\rangle$ identical has just one degeneracy number which is $N = n$.

The **degeneracy configuration** $\mathcal{D}_{\{N_i\}}$ of a pure symmetric qubit state $|\psi_S\rangle$ is the list of its degeneracy numbers. A pure symmetric qubit state, with all $|\epsilon_i\rangle$ identical but one, has the degeneracy configuration $\mathcal{D}_{\{n-1,1\}}$.

The **diversity degree** d of a pure symmetric qubit state $|\psi_S\rangle$ is the number of different $|\epsilon_i\rangle$ in other words it's the dimension of the degeneracy configuration, or the number of different degeneracy numbers. For a pure symmetric qubit state, with all $|\epsilon_i\rangle$ identical but two, the possible degeneracy numbers are $d = 2$ for $\mathcal{D}_{\{n-2,2\}}$ and $d = 3$ for $\mathcal{D}_{\{n-2,1,1\}}$.

Then they show that the degeneracy configuration $\mathcal{D}_{\{N_i\}}$ of an arbitrary pure symmetric qubit state is invariant under SLOCC operations and they conclude that pure symmetric qubit states, having a different degeneracy configuration $\mathcal{D}_{\{N_i\}}$, have to belong to different SLOCC classes. Based on this they define families of SLOCC entanglement classes as such, that every family contains all the states with a certain degeneracy configuration. A remarkable result they found is, that all the families with diversity degree three or lower ($d \leq 3$) contain just one SLOCC class, all other families contain infinitely many SLOCC classes.

Next we write down the two different families of SLOCC classes of pure symmetric two particle states ($n = 2$), where each family contains only one SLOCC class, the three families of SLOCC classes of pure symmetric three particle states ($n = 3$), again each containing one SLOCC class, and the five families of SLOCC classes of pure symmetric four particle states ($n = 4$), four of whom contain one SLOCC class, (namely those with diversity degree not higher than three, $d \leq 3$) and one containing infinitely many SLOCC classes (the one family of SLOCC classes of four qubit

states with $d = 4$).

1. $n = 2$

(I) $\mathcal{D}_2$ ($d = 1$)

This family contains the SLOCC class of separable pure symmetric two particle states.

(II) $\mathcal{D}_{1,1}$ ($d = 2$)

This family contains the SLOCC class of entangled pure symmetric two particle states.

2. $n = 3$

(I) $\mathcal{D}_3$ ($d = 1$)

This family contains the SLOCC class of separable pure symmetric three particle states.

(II) $\mathcal{D}_{2,1}$ ($d = 2$)

This family contains exactly one SLOCC class of entangled pure symmetric three particle states, which is called the W class.

(III) $\mathcal{D}_{1,1,1}$ ($d = 3$)

This family also contains one SLOCC class of entangled pure symmetric three particle states, which is called the GHZ class.

3. $n = 4$

(I) $\mathcal{D}_4$ ($d = 1$)

This family contains the SLOCC class of separable pure symmetric four particle states.

(II) $\mathcal{D}_{3,1}$ ($d = 2$)

The one SLOCC class of entangled pure symmetric qubit states contained in this family is the so called W_1 class with the Dicke state $|D_1^4\rangle$ (eq. (83)) as a representative state.

(III) $\mathcal{D}_{2,2}$ ($d = 2$)

The one SLOCC class of entangled pure symmetric qubit states contained in this family is the so called W_2 class with the Dicke state $|D_2^4\rangle$ as a representative state.

(IV) $\mathcal{D}_{2,1,1}$ ($d = 3$)

The one SLOCC class of entangled pure symmetric qubit states contained in this family is the class of states equivalent to the state $(|D_0^4\rangle + |D_2^4\rangle)/\sqrt{2}$.

(V) $\mathcal{D}_{1,1,1,1}$ ($d = 4$)

As this family has a diversity degree d higher than three, it contains a continuous

range of SLOCC classes. The states of this family can be transformed to $(|GHZ_4\rangle + \mu|D_2^4\rangle)/\sqrt{1+|\mu|^2}$, $\mu \in \mathbb{C}$, $\mu \neq \pm\frac{1}{\sqrt{3}}$ by an invertible local operation (ILO, see Ref. [11]). States with different μ belong to different SLOCC classes, hence there is an infinite number of SLOCC classes for pure symmetric four qubit states with a degeneracy configuration of $\mathcal{D}_{1,1,1,1}$.

So the classification in Ref. [11] yields five different families of SLOCC classes, while the classification in Ref. [17] yields nine different families of SLOCC classes of four qubit states. One has to keep in mind, that the classification in Ref. [11] just considers pure symmetric states and therefore the classification in Ref. [17] can be regarded as the more complete classification of four qubit states. The advantage of the classification in Ref. [11] over that one in Ref. [17] is clearly that it holds for an arbitrary number of qubits, while it is not straightforward to generalize the classification in [17] to more than four qubits.

Other than the above classifications of multipartite quantum states with respect to SLOCC invariance, in Ref. [P1] we introduced a classification of GME states with respect to invariance under local unitary operations (eq. (24)) and permutations of the subsystems. In contrast to SLOCC invariant classification, our class definition is Lorentz invariant (Ref. [70]). Furthermore in Ref. [P1] we develop inequalities based on Ref. [73] that are able to distinguish between the different classes in an experimentally feasible way. We will present this classification in section 2.

2 A new method of classifying and detecting genuine multipartite entanglement

This section is based on two papers (Ref. [P1] and [P2]) in which I was able to participate, while I was preparing my diploma thesis. Both are based on the set of inequalities, developed by Marcus Huber, Florian Mintert, Andreas Gabriel and Beatrix Hiesmayr in Ref. [73], which we discussed in section 1.3.4. First we consider Ref. [P1], in which we introduce inequalities capable of classifying and detecting GME states for arbitrary number of particles. Then we consider Ref. [P2] where we introduce a set of inequalities that are able to detect GME states and work at their best for generalized Dicke states. In the last part of this section we show that the criteria from Ref. [P1] and [P2] are experimental feasible for three reasons.

2.1 Revealing substructures of genuine multipartite entanglement

This section is based on Ref. [P1]. While in Ref. [P1] we start out by repeating the definition of k -separability (eq. (69)), in this section we directly start with introducing a definition of equivalence classes of k -separable n particle qubit states with respect to the different decompositions of the state (eq. (4)). Other than in Ref. [P1] we explain how the permutation operators work, that occur in our definition of classes. Next the inequalities, which are able to distinguish between the classes, are introduced. In the end of this section we show in one sketch and two examples how the inequalities work.

2.1.1 Introducing the classes

As stated above in Ref. [P1] we introduce a classification for GME n particle qubit states, which is Lorentz invariant (Ref. [70]). The states of the different classes are not equivalent under SLOCC operations, but they are equivalent under local unitary operations (eq. (24)) and permutations of their subsystems. Here vectors are regarded as states.

A **class** $\mathcal{C}(|\Psi_x^k\rangle)$ we define as all convex combinations of all k -separable states (without all higher separable states, $k' < k$) being local unitary invariant and invariant under the permutation of their subsystems.

$$\mathcal{C}(|\Psi_x^k\rangle) := \left\{ \sum_{i,j} p_i U_{loc}^i \Pi^i |\psi_j\rangle \langle \psi_j| \Pi^{i\dagger} U_{loc}^{i\dagger} \right\} \setminus \bigcup_{i,k > k'} \mathcal{C}(|\Psi_i^{k'}\rangle) \quad (107)$$

Where x denotes the tensor rank of the states in the respective class. The U_{loc}^i are local unitary operators (eq. (24)) and the Π^i are permutation operators, each generating a different permutation of the subsystems in $|\psi_j\rangle \langle \psi_j|$.

The permutation operators Π^i in eq. (107) are not the same permutation operators as those in eq. (89). Here every Π^i generates one of all the possible permutations of the subsystems of a state of a multipartite system $|\phi\rangle = |\phi_1\rangle \otimes \dots \otimes |\phi_n\rangle$. There are $n!$ possibilities to permute n subsystems, therefore, for n subsystems, there are $n!$ permutation operators Π^i as we defined them. So for three particle states there are six different permutation operators. They act as follows

$$\begin{aligned}
 \Pi^A |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle &= |\phi_2\rangle \otimes |\phi_1\rangle \otimes |\phi_3\rangle \\
 \Pi^B |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle &= |\phi_3\rangle \otimes |\phi_2\rangle \otimes |\phi_1\rangle \\
 \Pi^C |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle &= |\phi_1\rangle \otimes |\phi_3\rangle \otimes |\phi_2\rangle \\
 \Pi^D |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle &= |\phi_2\rangle \otimes |\phi_3\rangle \otimes |\phi_1\rangle \\
 \Pi^E |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle &= |\phi_3\rangle \otimes |\phi_1\rangle \otimes |\phi_2\rangle \\
 \Pi^F |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle &= |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle
 \end{aligned} \tag{108}$$

Where Π^F is the identity.

It is not clear how many such local unitary and permutation invariant classes of k -separable states there are. In Ref. [P1] we define three classes $\mathcal{C}(\{|\Psi_x^k\rangle\})$ of one separable or GME n particle qubit states.

- 1) **The class of “double states”** $\mathcal{C}(\{|\Psi_{(2)}^1\rangle\})$
 (a generalization of the GHZ state (eq. (75)))

$$\begin{aligned}
 \{|\Psi_{(2)}^1\rangle\} &:= \{|\psi\rangle \in \mathbb{C}^{2^n} : |\psi\rangle = \lambda_1 \bigotimes_{i=1}^n |0_i\rangle + \lambda_2 \bigotimes_{i=1}^n |1_i\rangle\} \\
 &\text{with } \lambda_1, \lambda_2 \in \mathbb{C}, |\lambda_1|^2 + |\lambda_2|^2 = 1
 \end{aligned} \tag{109}$$

- 2) **The class of “n-tuple states”** $\mathcal{C}(\{|\Psi_{(n)}^1\rangle\})$
 (a generalization of the W state (eq. (76)))

$$\{|\Psi_{(n)}^1\rangle\} := \{|\psi\rangle \in \mathbb{C}^{2^n} : |\psi\rangle = \sum_{i=1}^n \lambda_i |W_i\rangle \} \tag{110}$$

$$\text{with } |W_i\rangle = \bigotimes_{k \neq i} |0_k\rangle \otimes |1_i\rangle \text{ and } \lambda_i \in \mathbb{C}, \sum_i |\lambda_i|^2 = 1$$

3) The class of “(n-1)-tuple states” $\mathcal{C}(\{|\Psi_{(n-1)}^1\rangle\})$

(equivalent to Dicke states (eq. (83)) with two excited subsystems $|D_2^n\rangle$)

$$\{|\Psi_{(n-1)}^1\rangle\} := \{|\psi\rangle \in \mathbb{C}^{2n} : |\psi\rangle = \sqrt{\frac{2}{n(n-1)}} \sum_{i < j} |D_{ij}\rangle \quad (111)$$

$$\text{with } |D_{ij}\rangle = \bigotimes_{k \neq i,j} |0_k\rangle \otimes |1_i\rangle \otimes |1_j\rangle$$

It is not necessary to take two arbitrary orthogonal vectors ($|x\rangle$ und $|\bar{x}\rangle$) to define the different classes as we do in Ref. [P1], since the randomness is given by the local unitary operators U_{loc}^i in eq. (107). So we replace them in the definition of the classes (eq.s (109), (110) and (111)) with the two orthogonal vectors $|0\rangle$ and $|1\rangle$.

It is clear that this classification is not complete, but its advantage, besides its Lorentz invariance, is that one can construct inequalities to distinguish between the classes, which should be possible for any such given class. In the section 2.1.2 we introduce the inequalities we derive in Ref. [P1]. They are able to distinguish between the classes $\mathcal{C}(\{|\Psi_{(2)}^1\rangle\})$, $\mathcal{C}(\{|\Psi_{(n)}^1\rangle\})$ and $\mathcal{C}(\{|\Psi_{(n-1)}^1\rangle\})$, in an experimentally feasible way.

2.1.2 Inequalities to distinguish between the classes

The inequalities we develop in [P1] are based on the work, presented in Ref. [73]. They are not only designed to distinguish between the classes $\mathcal{C}(\{|\Psi_{(2)}^1\rangle\})$ (eq. (109)), $\mathcal{C}(\{|\Psi_{(n)}^1\rangle\})$ (eq. (110)) and $\mathcal{C}(\{|\Psi_{(n-1)}^1\rangle\})$ (eq. (111)), but also to detect if a given multipartite state ρ is genuine multipartite entangled. In each of these inequalities $I(2)$ (eq. (112)), $I(n)$ (eq. (113)) and $I(n-1)$ (eq. (114)) diagonal elements of the matrix of the state ρ are subtracted from real parts of off-diagonal elements of ρ .

1) The “double state” inequality $I(2)$

$$\begin{aligned} & \text{Re}\left\{\sum_{i \neq j} \langle w_i | \rho | w_j \rangle + (-1)^{n+1} \langle \bar{w}_i | \rho | \bar{w}_j \rangle\right\} - (n-2) \sum_i (\langle w_i | \rho | w_i \rangle + \langle \bar{w}_i | \rho | \bar{w}_i \rangle) \\ & - \sum_{i \neq j} (\langle d_{ij} | \rho | d_{ij} \rangle + \langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle) - \frac{n(n-1)}{2} (\langle 0 |^{\otimes n} \rho | 0 \rangle^{\otimes n} + \langle 1 |^{\otimes n} \rho | 1 \rangle^{\otimes n}) \leq 0 \end{aligned} \quad (112)$$

with $|d_{ij}\rangle = |0\rangle^{\otimes(i-1)} \otimes |1\rangle_i \otimes |0\rangle^{\otimes(j-i-1)} \otimes |1\rangle_j \otimes |0\rangle^{\otimes(n-j)}$ and $|w_i\rangle = |0\rangle^{\otimes(i-1)} \otimes |1\rangle_i \otimes |0\rangle^{\otimes(n-i)}$

is true for all (at least) biseparable states and for states of the class $\mathcal{C}(\{|\Psi_{(2)}^1\rangle\})$.

A bar denotes orthonormality in all subsystems, hence all $|0\rangle$ are replaced by $|1\rangle$ and the other way round; e.g. $|\bar{d}_{ij}\rangle = |1\rangle^{\otimes(i-1)} \otimes |0\rangle_i \otimes |1\rangle^{\otimes(j-i-1)} \otimes |0\rangle_j \otimes |1\rangle^{\otimes(n-j)}$.

2) The “n-tuple state” inequality $I(n)$

$$\operatorname{Re}\{\langle 0|^{\otimes n} \rho |1\rangle^{\otimes n}\} - \alpha (1 - \langle 0|^{\otimes n} \rho |0\rangle^{\otimes n} - \langle 1|^{\otimes n} \rho |1\rangle^{\otimes n}) \leq 0 \quad (113)$$

$$\text{with } \alpha = \frac{3}{2} \text{ for } n = 3, \alpha = 1 \text{ for } n = 4 \text{ and } \alpha = \frac{1}{2} \text{ for } n > 4$$

is true for all (at least) biseparable states and for states of the class $\mathcal{C}(\{|\Psi_{(n)}^1\rangle\})$.

3) The “(n-1)-tuple state” inequality $I(n-1)$

$$\begin{aligned} & \operatorname{Re}\left\{\sum_{i \neq j} \langle w_i | \rho | w_j \rangle\right\} - (n-2) \sum_i (\langle w_i | \rho | w_i \rangle) \\ & - (n-2) \sum_{i \neq j} (\langle d_{ij} | \rho | d_{ij} \rangle) - \frac{n(n-1)}{2} (\langle 0|^{\otimes n} \rho | 0\rangle^{\otimes n}) \leq 0 \end{aligned} \quad (114)$$

is true for all (at least) biseparable states and for states of the class $\mathcal{C}(\{|\Psi_{(n-1)}^1\rangle\})$.

If a state ρ violates one of the inequalities $I(2)$ (eq. (112)), $I(n)$ (eq. (113)) and $I(n-1)$ (eq. (114)) one knows for certain that it is a GME state and that it is not in the associated class.

2.1.3 Examples

Here we show by means of a sketch of the set of all five qubit states (figure 3) and of two examples, how the inequalities $I(2)$ (eq. (112)), $I(n)$ (eq. (113)) and $I(n-1)$ (eq. (114)) work.

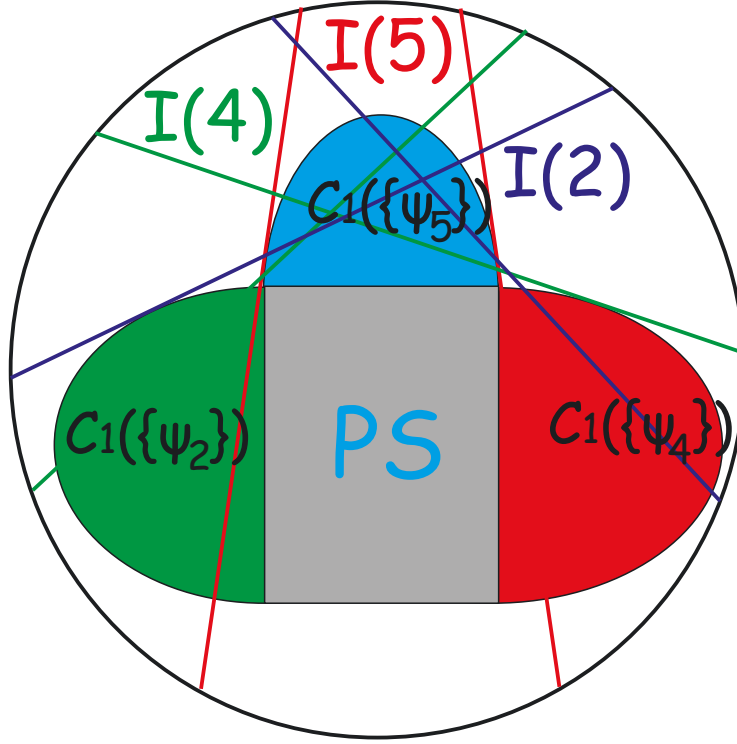


Figure 3: In this sketch of the set of all five qubit states one can see how our criteria work. For example the “5-tuple state” inequality $I(5)$ (orange line, eq. (113)) separates parts of the sets of the “double states” $\mathcal{C}(\{|\Psi_{(2)}^1\rangle\})$ (the green area, eq. (109)) and of the “(4)-tuple states” $\mathcal{C}(\{|\Psi_{(4)}^1\rangle\})$ (the red area, eq. (111)) from the rest of the five qubit states, including the set of the partial separable states (PS, here drawn in grey) and the set of the “5-tuple states” $\mathcal{C}(\{|\Psi_{(5)}^1\rangle\})$ (the blue area, eq. (110)).

In the first example we demonstrate, that a three particle W state $\rho_{W_3} = |W_3\rangle\langle W_3|$ (eq. (77)) violates the “double state” inequality $I(2)$ (eq. (112)). The whole inequality reads

$$\begin{aligned} & \text{Re}\left\{\sum_{i \neq j} \langle w_i | W_3 \rangle \langle W_3 | w_j \rangle + \langle \bar{w}_i | W_3 \rangle \langle W_3 | \bar{w}_j \rangle\right\} - \sum_i (\langle w_i | W_3 \rangle \langle W_3 | w_i \rangle + \langle \bar{w}_i | W_3 \rangle \langle W_3 | \bar{w}_i \rangle) \\ & - \sum_{i \neq j} (\langle d_{ij} | W_3 \rangle \langle W_3 | d_{ij} \rangle + \langle \bar{d}_{ij} | W_3 \rangle \langle W_3 | \bar{d}_{ij} \rangle) - 3(\langle 000 | W_3 \rangle \langle W_3 | 000 \rangle + \langle 111 | W_3 \rangle \langle W_3 | 111 \rangle) \leq 0 \quad (115) \end{aligned}$$

The first term $\langle w_i|W_3\rangle\langle W_3|w_j\rangle$ gives six times $\frac{1}{3}$, e.g.

$$\langle w_1|W_3\rangle\langle W_3|w_2\rangle = \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3}. \quad (116)$$

The second term $\langle \bar{w}_i|W_3\rangle\langle W_3|\bar{w}_j\rangle$ gives zero for all choices of i and j , e.g.

$$\langle \bar{w}_1|W_3\rangle\langle W_3|\bar{w}_2\rangle = \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0. \quad (117)$$

The third term $\langle w_i|W_3\rangle\langle W_3|w_i\rangle$ gives three times $\frac{1}{3}$, e.g.

$$\langle w_1|W_3\rangle\langle W_3|w_1\rangle = \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3}. \quad (118)$$

The fourth term $\langle \bar{w}_i|W_3\rangle\langle W_3|\bar{w}_i\rangle$ as well as all the other remaining ones ($\langle d_{ij}|W_3\rangle\langle W_3|d_{ij}\rangle$, $\langle \bar{d}_{ij}|W_3\rangle\langle W_3|\bar{d}_{ij}\rangle$,

$\langle 000|W_3\rangle\langle W_3|000\rangle$ and $\langle 111|W_3\rangle\langle W_3|111\rangle$ gives zero, e.g.

$$\langle \bar{w}_1|W_3\rangle\langle W_3|\bar{w}_1\rangle = \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0. \quad (119)$$

So the whole “double state” inequality $I(2)$ (eq. (112)), acting on a three particle W state ρ_{W_3} (eq. (77)) becomes

$$6\frac{1}{3} - 3\frac{1}{3} \leq 0, \quad (120)$$

which is false.

The violation of the “double state” inequality $I(2)$ tells us that ρ_{W_3} is definitely a GME state and that ρ_{W_3} is not part of the class of “double states” $\mathcal{C}(\{|\Psi_{(2)}^1\rangle\})$.

Again, like in the inequalities discussed in section 1.3.4, we saw that the inequality works by subtracting certain diagonal elements (e.g. $\langle w_i|W_3\rangle\langle W_3|w_i\rangle$) of the state one wants to investigate from certain offdiagonal elements (e.g. $\langle w_i|W_3\rangle\langle W_3|w_j\rangle$). If the offdiagonal elements of the state are big enough, one can violate the inequality and show that the state is entangled. All the inequalities presented in this section (2) work that way.

In the second example we consider a state ρ_{nr} , consisting of a n particle GHZ state $\rho_{GHZ} = |GHZ\rangle\langle GHZ|$ (eq. (75)), which is to a certain amount mixed with the (fully separable) maximally mixed n qubit state $\frac{1}{2^n}\mathbb{1}_{2^n}$ (also called white noise).

$$\rho_{nr} = (1-p)|GHZ\rangle\langle GHZ| + \frac{p}{2^n}\mathbb{1}_{2^n} \quad \text{with } 0 \leq p \leq 1 \quad (121)$$

$$\text{with } 0 \leq p \leq 1$$

This can be seen as a model of a state that is undergoing disturbances during an experiment. Our goal is to investigate to which amount ρ_{GHZ} can be mixed with white noise, so that it can still be detected

as a GME state by the “n-tuple state” inequality $I(n)$ (eq. (113)). This property of a criterion is called **noise resistance**. To keep the example rather short we restrict ourselves to the case of $n > 4$. With the above choices the whole inequality reads

$$\begin{aligned} & \text{Re}\{\langle 0^{\otimes n} | (1-p) | GHZ \rangle \langle GHZ | 1^{\otimes n} \rangle\} + \text{Re}\{\langle 0 |^{\otimes n} \frac{p}{2^n} \mathbb{1}_{2^n} | 1 \rangle^{\otimes n}\} \\ & - \frac{1}{2} (1 - \langle 0^{\otimes n} | (1-p) | GHZ \rangle \langle GHZ | 0^{\otimes n} \rangle - \langle 1^{\otimes n} | (1-p) | GHZ \rangle \langle GHZ | 1^{\otimes n} \rangle) \\ & - \langle 0 |^{\otimes n} \frac{p}{2^n} \mathbb{1}_{2^n} | 0 \rangle^{\otimes n} - \langle 1 |^{\otimes n} \frac{p}{2^n} \mathbb{1}_{2^n} | 1 \rangle^{\otimes n}) \leq 0 \end{aligned} \quad (122)$$

From the definition of the GHZ state (eq. (75)) we know that $|GHZ\rangle = \frac{1}{\sqrt{2}}(\bigotimes_{i=1}^n |0\rangle + \bigotimes_{i=1}^n |1\rangle)$, hence the first term gives.

$$\text{Re}\{\langle 0^{\otimes n} | (1-p) | GHZ \rangle \langle GHZ | 1^{\otimes n} \rangle\} = \frac{(1-p)}{2}. \quad (123)$$

As the maximally mixed state $\frac{1}{2^n} \mathbb{1}_{2^n}$ does not have any offdiagonal elements the second term is zero. The third and the fourth term both give

$$\langle 0^{\otimes n} | (1-p) | GHZ \rangle \langle GHZ | 0^{\otimes n} \rangle = \langle 1^{\otimes n} | (1-p) | GHZ \rangle \langle GHZ | 1^{\otimes n} \rangle = \frac{(1-p)}{2}. \quad (124)$$

And the fifth and the sixth term both give

$$\langle 0 |^{\otimes n} \frac{p}{2^n} \mathbb{1}_{2^n} | 0 \rangle^{\otimes n} = \langle 1 |^{\otimes n} \frac{p}{2^n} \mathbb{1}_{2^n} | 1 \rangle^{\otimes n} = \frac{p}{2^n}. \quad (125)$$

So the whole inequality yields

$$\frac{(1-p)}{2} - \frac{1}{2} (1 - (1-p) - \frac{p}{2^{n-1}}) \leq 0. \quad (126)$$

From this follows, that the “n-tuple state” inequality $I(n)$ (eq. (113)) is violated by the mixture ρ_{nr} of ρ_{GHZ} with white noise $\frac{1}{2^n} \mathbb{1}_{2^n}$ for

$$p < \frac{1}{2 - \frac{1}{2^{n-1}}}, \quad (127)$$

which for increasing n reaches very fast $\frac{1}{2}$.

In the “n-tuple state” inequality $I(n)$ (eq. (113)) the diagonal elements, that are subtracted from the offdiagonal elements, are $\langle 0 |^{\otimes n} \rho | 0 \rangle^{\otimes n}$ and $\langle 1 |^{\otimes n} \rho | 1 \rangle^{\otimes n}$. There is only one offdiagonal element, which is $\langle 0 |^{\otimes n} \rho | 1 \rangle^{\otimes n}$.

So we just calculated the noise resistance of the “n-tuple state” inequality $I(n)$ and discovered that the boundary for which ρ_{nr} (eq. (121)) still violates the “n-tuple state” inequality $I(n)$ is $p < \frac{1}{2 - \frac{1}{2^{n-1}}}$. As we restricted ourselves to the case of $n > 4$, the maximum value of p not violating $I(n)$ is $p = \frac{16}{31}$ for $n = 5$, for higher n it gets nearer and nearer to the boundary value of $p = \frac{1}{2}$. This means that a GHZ state with arbitrary number of particles can still be detected by the “n-tuple state” inequality $I(n)$ (eq. (113)) as an GME state, belonging to another class than the class of “n-tuple states” $\mathcal{C}(\{|\Psi_{(n)}^1\rangle\})$ (eq. (110)), if it is measured “noisy”, up to an amount of 50% of noise. This is a rather good value for the noise resistance of a separability criterion, when compared to other criteria. We have a deeper look at the noise resistance of our criteria in section 2.3.

2.2 Criteria to detect GME states, working best for Dicke states

This section is based on Ref. [P2]. We present a set of inequalities to detect GME states, developed in Ref. [P2], fitted for generalized Dicke states (eq. (83)). In the first part we define the inequalities and explain how the permutation operators, that occur in it, work. In the second and last part of this section we give simple examples of one of the inequalities acting on two different states.

2.2.1 A set of inequalities to detect GME states, suited for Dicke states

Using the results of Ref. [73], which we discussed in section 1.3.4, in Ref. [P2] we derive inequalities to detect GME n particle states. They work best for generalized Dicke states (eq. (83)) and are experimentally feasible, as we will show in the next section (section 2.3).

The inequalities $I(D)$

$$I_m^n[\rho] = \sum_{\{\gamma\}} \left(|\langle d_{\{\alpha\}} | \rho | d_{\{\beta\}} \rangle| - \sqrt{\langle d_{\{\alpha\}} | \otimes \langle d_{\{\beta\}} | \mathcal{P}_{\{\alpha\}} \rho^{\otimes 2} \mathcal{P}_{\{\alpha\}} | d_{\{\alpha\}} \rangle \otimes | d_{\{\beta\}} \rangle} \right) - N_D \sum_{\{\alpha\}} \langle d_{\{\alpha\}} | \rho | d_{\{\alpha\}} \rangle \leq 0$$

$$\text{where } |d_{\{\alpha\}}\rangle = \bigotimes_{i \notin \{\alpha\}} |0\rangle_i \bigotimes_{i \in \{\alpha\}} |1\rangle_i \quad (128)$$

$$\text{with } \{\gamma\} = \{(\{\alpha\}, \{\beta\}) : |\{\alpha\} \cap \{\beta\}| = m-1\}, \quad 1 \leq m \leq \frac{n}{2} \text{ and } N_D = m(n-m-1)$$

are true for all biseparable states ρ .

The sets of integers $\{\alpha\}$ and $\{\beta\}$ denote exactly those indices, whose associated subsystems are excited (i.e. are in the state $|1\rangle$). They have to have an overlap of $(m-1)$. This means that there have to be $(m-1)$ subsystems being in state $|1\rangle$ in both the vectors $|d_{\{\alpha\}}\rangle$ and $|d_{\{\beta\}}\rangle$ at once, i.e.

$$\{\alpha\} = \{i, j, k\}, \quad \{\beta\} = \{l, j, k\} \text{ with } i \neq j \neq k \neq l \text{ and } 1 \leq i, j, k, l \leq n \quad (129)$$

The permutation operators $\mathcal{P}_{\{\alpha\}}$ are the permutation operators from eq. (101) and (103). They act on the twofold copy of the Hilbert space. For a n particle state there are $(2^{(n-1)} - 1)$ different permutation operators $\mathcal{P}_{\{\alpha\}}$. An example for such a permutation operator is $\mathcal{P}_{\{3,5,6\}}$ occurring in $I_3^7[\rho]$, which acts on the twofold copy of a seven particle state $|\phi\rangle = |\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6\phi_7\rangle \otimes |\phi'_1\phi'_2\phi'_3\phi'_4\phi'_5\phi'_6\phi'_7\rangle$ as follows.

$$\mathcal{P}_{\{3,5,6\}} |\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6\phi_7\rangle \otimes |\phi'_1\phi'_2\phi'_3\phi'_4\phi'_5\phi'_6\phi'_7\rangle = |\phi_1\phi_2\phi'_3\phi_4\phi'_5\phi'_6\phi_7\rangle \otimes |\phi'_1\phi'_2\phi_3\phi'_4\phi_5\phi_6\phi'_7\rangle \quad (130)$$

The permutation operators $\mathcal{P}_{\{\alpha\}}$ always exchange m subsystems of the n particle state $|d_{\{\alpha\}}\rangle$ with m subsystems of the n particle state $|d_{\{\beta\}}\rangle$, respectively with m subsystems of its copy. So there is exactly one subsystems in which a $|0\rangle$ turns into a $|1\rangle$ and exactly one subsystem where it is the other way round, since $\{\gamma\}$ is those set where $\{\alpha\}$ and $\{\beta\}$ overlap in $m - 1$ excited subsystems.

Each of the inequalities $I(D)$ (eq. (128)) is maximally violated by the corresponding Dicke state $\rho_{D_m^n} = |D_m^n\rangle\langle D_m^n|$, with

$$I_m^n[\rho_{D_m^n}] = m. \quad (131)$$

Every state that violates one of the inequalities $I(D)$ (eq. (128)) is with certainty not biseparable and therefore a GME state.

2.2.2 Examples

To make this definition more understandable we show two examples. We choose the most simple of all the inequalities $I(D)$ (eq. (128)), that is suited for Dicke states excluding W states (i.e. $m > 1$), which is $I_2^4[\rho] \leq 0$.

For a general four qubit state ρ , the term $I_2^4[\rho]$ reads

$$I_2^4[\rho] = \quad (132)$$

$$\begin{aligned} & 2(\langle 1100|\rho|1010\rangle - \sqrt{\langle 1000|\rho|1000\rangle\langle 1110|\rho|1110\rangle} + \langle 1100|\rho|1001\rangle - \sqrt{\langle 1000|\rho|1000\rangle\langle 1101|\rho|1101\rangle} \\ & + \langle 1100|\rho|0101\rangle - \sqrt{\langle 1000|\rho|1000\rangle\langle 1101|\rho|1101\rangle} + \langle 1100|\rho|0110\rangle - \sqrt{\langle 0100|\rho|0100\rangle\langle 1110|\rho|1110\rangle} \\ & + \langle 1010|\rho|1001\rangle - \sqrt{\langle 1000|\rho|1000\rangle\langle 1011|\rho|1011\rangle} + \langle 1010|\rho|0110\rangle - \sqrt{\langle 0100|\rho|0100\rangle\langle 1110|\rho|1110\rangle} \\ & + \langle 1010|\rho|0011\rangle - \sqrt{\langle 0010|\rho|0010\rangle\langle 1011|\rho|1011\rangle} + \langle 1001|\rho|0101\rangle - \sqrt{\langle 0001|\rho|0001\rangle\langle 1101|\rho|1101\rangle} \\ & + \langle 1001|\rho|0011\rangle - \sqrt{\langle 0001|\rho|0001\rangle\langle 1011|\rho|1011\rangle} + \langle 0110|\rho|0101\rangle - \sqrt{\langle 0100|\rho|0100\rangle\langle 0111|\rho|0111\rangle} \\ & + \langle 0110|\rho|0011\rangle - \sqrt{\langle 0010|\rho|0010\rangle\langle 0111|\rho|0111\rangle} + \langle 0101|\rho|0011\rangle - \sqrt{\langle 0001|\rho|0001\rangle\langle 0111|\rho|0111\rangle} \\ & - \langle 1100|\rho|1100\rangle - \langle 1010|\rho|1010\rangle - \langle 1001|\rho|1001\rangle - \langle 0110|\rho|0110\rangle - \langle 0101|\rho|0101\rangle - \langle 0011|\rho|0011\rangle) \end{aligned}$$

Here we see that diagonal elements (e.g. $\langle 1001|\rho|1001\rangle$) or products of squareroots of diagonal elements (e.g. $\sqrt{\langle 0010|\rho|0010\rangle\langle 1011|\rho|1011\rangle}$) of the state ρ are subtracted from offdiagonal elements (e.g. $\langle 1100|\rho|0101\rangle$) of ρ , like in the inequalities in the sections 1.3.4 and 2.1.

In the first example, we show that a four particle Dicke state $\rho_{D_2^4} = |D_2^4\rangle\langle D_2^4|$ (eq. (84)) with two excitations violates the inequality $I_2^4 \leq 0$ maximally (see Ref. (131)), i.e. that $I_2^4[\rho_{D_2^4}] = 2$.

In eq. (84) we wrote that $|D_2^4\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle)$, hence each of the offdiagonal elements in $I_2^4[\rho_{D_2^4}]$ gives $\frac{1}{6}$, e.g.

$$\langle 0110 | \rho_{D_2^4} | 0011 \rangle = \frac{1}{6}. \quad (133)$$

Whereas all the products of squareroots of diagonal elements give zero, e.g.

$$\sqrt{\langle 0100 | \rho_{D_2^4} | 0100 \rangle \langle 1110 | \rho_{D_2^4} | 1110 \rangle} = 0. \quad (134)$$

Each of the diagonal elements in $I_2^4[\rho_{D_2^4}]$ (the last line in eq. (132)) gives again $\frac{1}{6}$, e.g.

$$\langle 1001 | \rho_{D_2^4} | 1001 \rangle = \frac{1}{6}. \quad (135)$$

So the whole inequality $I_2^4[\rho_{D_2^4}] \leq 0$ yields (see eq. (132))

$$2\left(\frac{12}{6} - \frac{6}{6}\right) \leq 0, \quad (136)$$

which is false, as we expected.

This tells us that $\rho_{D_2^4}$ is not biseparable and therefore for certain a GME state. We also saw that the corresponding Dicke state, indeed violates $I(D)$ maximally, in our example that $I_2^4[\rho_{D_2^4}] = 2$.

In the second example we show that the (fully separable) maximally mixed 4 qubit state $\frac{1}{16}\mathbb{1}_{16}$ fulfills the inequality $I_2^4[\frac{1}{16}\mathbb{1}_{16}] \leq 0$, i.e. is not detected as an GME state by our inequality.

As $\frac{1}{16}\mathbb{1}_{16}$ does not have any offdiagonal elements it is clear that each of the offdiagonal elements in $I_2^4[\frac{1}{16}\mathbb{1}_{16}]$ gives zero, e.g.

$$\langle 1001 | \frac{1}{16}\mathbb{1}_{16} | 0011 \rangle = 0. \quad (137)$$

Whereas each of the products of squareroots of diagonal elements in $I_2^4[\frac{1}{16}\mathbb{1}_{16}]$, as well as each of the diagonal elements in the last line of eq. (132) gives $\frac{1}{16}$, e.g.

$$\sqrt{\langle 0001 | \frac{1}{16}\mathbb{1}_{16} | 0001 \rangle \langle 0111 | \frac{1}{16}\mathbb{1}_{16} | 0111 \rangle} = \langle 0011 | \frac{1}{16}\mathbb{1}_{16} | 0011 \rangle = \frac{1}{16}. \quad (138)$$

So the inequality becomes

$$2(-\frac{12}{16} - \frac{6}{16}) \leq 0, \quad (139)$$

which is true, as we expected.

This is not very surprising, as all the terms giving diagonal elements in the inequalities $I(D)$ (eq. (128)) have a minus sign in front and just the terms giving offdiagonal elements have plus signs in front of them. Since the maximally mixed state does not have any offdiagonal elements it is obvious that it cannot violate the inequality.

As mentioned above, in the inequalities $I(D)$ (eq. (128)) terms containing diagonal elements are subtracted from offdiagonal elements, as it is done in the “double state” inequality $I(2)$ (eq. (112)), the “n-tuple state” inequality $I(n)$ (eq. (113)) and the “(n-1)-tuple state” inequality $I(n - 1)$ (eq. (114)).

2.3 Experimental feasibility

To be experimentally implementable a criterion should fulfill three requirements. In this section we will show reasons, why the inequalities $I(2)$ (eq. (112)), $I(n)$ (eq. (113)), $I(n-1)$ (eq. (114)) and $I(D)$ (eq. (128)) are comparatively easy experimental feasible. The first requirement is that a criterion should still be working if the state, which is generated in the experiment, is disturbed, i.e. if the generated state is, to a certain amount, mixed with white noise (the maximally mixed state $\frac{1}{d}\mathbb{1}_d$). In other words the noise resistance of the criterion should be reasonable high. We discuss the noise resistance of inequality $I(n)$ and of the inequalities $I(D)$ in the first part of this section. The second requirement for a criterion to be implementable experimentally is, that one can apply it to a state without doing a full state tomography, i.e. that it is useable without measuring all the matrix elements of the state. In the second part of this section we consider the number of measurements required to apply the inequalities $I(2)$, $I(n)$, $I(n-1)$ and $I(D)$. The third requirement for a criterion to be implementable experimentally is, that one is able to apply the criterion to a state by only using results of local measurements. This is especially important for states containing many subsystems, because the higher the number of particles a quantum system contains, the more complex become global measurement operations. In the end of this section we show that all the criteria $I(2)$, $I(n)$, $I(n-1)$ and $I(D)$ are locally implementable.

2.3.1 Noise resistance

An important feature of a criterion is its property to still detect GME states, if they are “disturbed”, i.e. to a certain amount mixed with white noise (the maximally mixed state $\frac{1}{d}\mathbb{1}_d$). This can be seen as a model for the case that a GME state undergoes random disturbances in an experimental setting. The ability to detect GME states, that are mixed with white noise, is called the **noise resistance** of a criterion. Here we first discuss the noise resistance of the “n-tuple state” inequality $I(n)$ (eq. (113)) and of the inequalities $I(D)$ (eq. (128)), then we compare it with the noise resistance of other criteria.

In section 2.1.3 we calculate the noise resistance of the “n-tuple state” inequality $I(n)$ for the state ρ_{nr} , which is consisting of a n particle GHZ state $\rho_{GHZ} = |GHZ\rangle\langle GHZ|$ (eq. (75)), which is to a certain amount mixed with the (fully separable) maximally mixed n qubit state $\frac{1}{2^n}\mathbb{1}_{2^n}$. (see eq. (121))

$$\rho_{nr} = (1-p)|GHZ\rangle\langle GHZ| + \frac{p}{2^n}\mathbb{1}_{2^n} \quad \text{with } 0 \leq p \leq 1 \quad (140)$$

with $0 \leq p \leq 1$

To keep the example rather short we restricted ourselves to the case of $n > 4$. As the result of the example we obtained that the noise resistance parameter p has to be smaller than a certain value to violate the inequality $I(n)$, i.e. so that $I(n)$ still detects ρ_{nr} as a GME state.

$$p < \frac{1}{2 - \frac{1}{2^{n-1}}}, \quad (141)$$

One can easily see that for increasing number of particles the noise resistance goes very fast to the value of $\frac{1}{2}$. The maximum value of the noise resistance parameter, not violating $I(n)$, is $p = \frac{16}{31}$ for $n = 5$, for higher n it decreases and gets nearer and nearer to the boundary value of $p = \frac{1}{2}$.

To complete the calculation of the noise resistance of the “n-tuple state” inequality $I(n)$ (eq. (113)) for the state ρ_{nr} (eq. (121)), we calculate the maximum value of p , still violating the inequality $I(n)$, for the case $n = 3$ and for the case $n = 4$.

For $n = 3$, $I(n)$ of ρ_{nr} yields

$$\frac{(1-p)}{2} - \frac{3}{2}(1 - (1-p) - \frac{p}{2^2}) \leq 0, \quad (142)$$

which leads to

$$p < \frac{4}{13}, \quad (143)$$

so that a three particle ρ_{nr} still violates the inequality $I(n)$ and is therefore detected by the criteria, as a GME state.

For $n = 4$, $I(n)$ of ρ_{nr} yields

$$\frac{(1-p)}{2} - (1 - (1-p) - \frac{p}{2^3}) \leq 0, \quad (144)$$

which leads to

$$p < \frac{4}{11}, \quad (145)$$

so that a four particle ρ_{nr} still violates the inequality $I(n)$.

So the range of the noise resistance parameter p of the “n-tuple state” inequality $I(n)$ is for the three and the four qubit case significantly smaller than for the case $n > 4$. This stems from the fact that in eq. (113) we chose the parameter α with a lot of redundancy. So the “n-tuple state” inequality

$I(n)$ could be fitted much better to the three and four particle case.

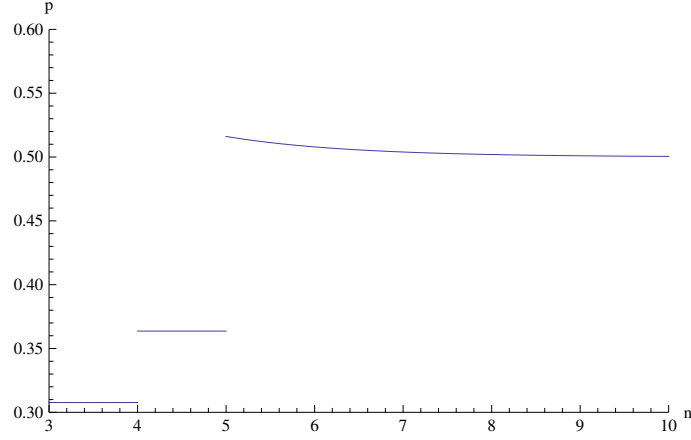


Figure 4: Here we see the noise resistance parameter p of the “n-tuple state” inequality $I(n)$ in terms of the number of qubits n . The leaps in the graph are due to the fact, that the prefactor α in the inequality changes with n , for $n = 3$ it is $\alpha = \frac{3}{2}$, for $n = 4$ it is $\alpha = 1$ and for $n > 4$ it is $\alpha = \frac{1}{2}$. The graph shows how fast p reaches the boundary value of $p = \frac{1}{2}$. This means that a GHZ state with arbitrary number of particles can still be detected by the “n-tuple state” inequality $I(n)$ (eq. (113)) as an GME state, belonging to another class than the class of “n-tuple states” $\mathcal{C}(\{|\Psi_{(n)}^1\rangle\})$ (eq. (110)), if it is measured “noisy”, up to an amount of 50% of noise.

Now we want to discuss the noise resistance of the inequalities $I(D)$ (eq. (128)). In Ref. [P2] we derive the noise resistance of $I(D)$ for the state

$$\rho_{noise} = (1 - p)|D_m^n\rangle\langle D_m^n| + p\frac{1}{2^n}\mathbb{1}. \quad (146)$$

Hence we calculate to which amount a n particle Dicke state $\rho_{D_m^n}$ (eq. (83)) can be mixed with white noise, so that the corresponding inequality $I_m^n[\rho_{noise}] \leq 0$ still detects it as a GME state, i.e. is still violated.

In Ref. [P2] we obtain

$$p < \frac{2^n}{2^n + (-1 - 2m + 2n) \cdot \binom{n}{m}}, \quad (147)$$

so that the corresponding inequality $I_m^n[\rho_{noise}] \leq 0$, is still violated. Hence for $p < \frac{2^n}{2^n + (-1 - 2m + 2n) \cdot \binom{n}{m}}$, the state ρ_{noise} is still detected as a GME state.

Here it is not so easy to see to which value the noise resistance converges, apart from that p is also dependent of the number of excitations m . Nevertheless it turns out that, for large n , the noise

resistance quickly approaches the value of 1 regardless of m . This means that for a high number of particles the Dicke state $\rho_{D_m^n}$ is still detected by the corresponding inequality $I(D)$ (eq. (128)) as a GME state, also if it is nearly totally disturbed, i.e. mixed with white noise, up to an amount of nearly 100%.

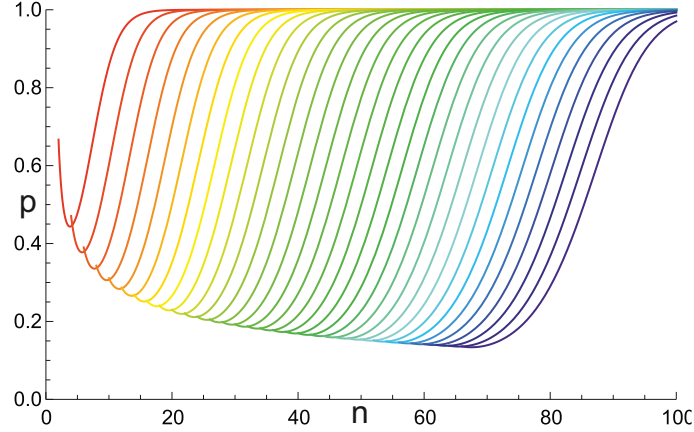


Figure 5: Here we see the noise resistance parameter p of the inequalities $I(D)$ (eq. (128)) in terms of the number of qubits n . The different colors represent different numbers of excited subsystems m , from left to right increasing, from $m = 1$ to $m = 33$. The graphs show, that after a short decrease, the noise resistance parameter very fast reaches the boundary value of $p = 1$. Due to the fact that $m \leq \frac{n}{2}$, respectively, that the number of particles has at least to be double the number of excited subsystems $n \geq 2m$, every graph starts at two times the number of excitations it represents (e.g. the graph on the very right, representing states with $m = 33$, starts at $n = 66$).

Now we compare the noise resistance of the inequalities $I(D)$ (eq. (128)), with the noise resistance of three other criteria.

- 1) In Ref. [30] Otfried Guehne and Michael Seevinck introduce a comparable separability criterion. For three qubits the criterion from Ref. [30] detects the state

$$\rho_{noise3} = (1 - p)|D_1^3\rangle\langle D_1^3| + p\frac{1}{8}\mathbb{1}_8, \quad (148)$$

equivalent to the three particle W state (eq. (76) and (77)) mixed with white noise, for the noise resistance parameter being $p < \frac{8}{17} \approx 0,471$ as a GME state.

The best suited criterion of the inequalities $I(D)$ (eq. (128)), namely $I_1^3[\rho_{noise3}] \leq 0$, detects ρ_{noise3} as a GME state for exactly for the same range of the noise resistance parameter as the criterion from Ref.[30], which is $p < \frac{8}{17} \approx 0,471$.

So for three particles our criteria do not improve the noise resistance compared to the best criteria

so far.

For four qubits the criterion from Ref.[30] detects the state

$$\rho_{noise4} = (1 - p)|D_2^4\rangle\langle D_2^4| + p\frac{1}{16}\mathbb{1}_{16} \quad (149)$$

as a GME state, for the noise resistance parameter being $p < \frac{8}{21} \approx 0,381$.

The best suited criterion of the inequalities $I(D)$ (eq. (128)), namely $I_2^4[\rho_{noise4}] \leq 0$, detects ρ_{noise4} as a GME state for exactly for the same range of the noise resistance parameter, as in the three particle case, which is $p < \frac{8}{17} \approx 0,471$.

So for the four particle case the best suited criterion of the inequalities $I(D)$ (eq. (128)) detects the state ρ_{noise4} as a GME state for a higher noise resistance parameter than the criterion from Ref. [30], i.e. it works for states being exposed to more noise than the criterion from Ref. [30].

In Ref.[74] the authors numerical optimize over a set of observables and thereby achieve to detect the state ρ_{noise4} up to a limit of the noise resistance parameter of $p \approx 0,539$, a higher value than $I(D)$ reaches for four qubits. But one has to keep in mind that a numerical optimization is hardly possible for states describing more than seven qubits.

- 2) In Ref. [77] an experimental method is presented to produce the four qubit phased Dicke state,

$$\rho_{D_6^{ph}} = |D_6^{ph}\rangle\langle D_6^{ph}|, \quad (150)$$

$$\text{with } |D_6^{ph}\rangle = \frac{1}{\sqrt{6}}|0011\rangle + |1100\rangle + |0110\rangle + |1001\rangle - |0101\rangle - |1010\rangle.$$

In Ref. [71] the state

$$\rho_{n.ph} = \rho_{D_6^{ph}} + p\frac{1}{16}\mathbb{1}_{16}, \quad (151)$$

is detected as a GME state by a therein developed criterion, up to a value of the noise resistance parameter of $p < \frac{4}{13} \approx 0,301$.

The inequality $I_2^4[\rho_{n.ph}] \leq 0$ is violated by the four qubit phased Dicke state mixed with white noise up to a value of the noise resistance parameter of $p < \frac{8}{17} \approx 0,471$, which is much better than in the original method.

- 3) With a criterion introduced in Ref.[75] it is possible to detect the state ρ_{Ψ_4} , originally proposed in Ref. [78],

$$\rho_{\Psi_4} = |\Psi_4\rangle\langle\Psi_4| \quad (152)$$

$$\text{with } |\Psi_4\rangle = |0011\rangle + |1100\rangle - \frac{1}{2}(|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle),$$

as a GME state. This criterion detects the state ρ_{Ψ_4} mixed with white noise, up to a noise threshold of $p \approx 0,278$.

The inequality $I_2^4[\rho_{\Psi_4}] \leq 0$ is violated by the four qubit state ρ_{Ψ_4} mixed with white noise up to a value of the noise resistance parameter of $p < \frac{16}{43} \approx 0,372$, which is also much better than in the original method.

So although the inequalities $I(D)$ (eq. (128)) are designed exploiting combinatorial properties of Dicke states, there are a lot of other states that can be detected as GME states, with a high noise resistance.

The noise resistance of the “n-tuple state” inequality $I(n)$ (eq. (113)) and of the inequalities $I(D)$ (eq. (128)) is compared to other criteria rather high, which is an advantage in experiments.

2.3.2 Number of necessary matrix elements

Another important property of a separability criterion to be implementable in an experiment, is the number of necessary matrix elements $\#_{m.e.}(I)$, i.e. the number of matrix entries, that has to be known, so that the criterion can be applied to the state. For systems with a large number of particles this becomes a crucial issue, as the number of local measurements required to do a full state tomography grows exponentially with the number of subsystems. For a n qubit state it is 2^{2n} .

Here we show, that in the inequalities $I(2)$, $I(n)$, $I(n-1)$ and $I(D)$, the number of necessary matrix elements $\#_{m.e.}(I)$ grows polynomially with the number of particles n .

a) The number of matrix elements necessary to apply the “doublestate” inequality $I(2)$ (eq. (112)) to a state ρ is

$$\begin{aligned} \#_{m.e.}(I(2)) &= \#(\langle w_i | \rho | w_j \rangle) + \#(\langle \bar{w}_i | \rho | \bar{w}_j \rangle) + \#(\langle w_i | \rho | w_i \rangle) \\ &\quad + \#(\langle \bar{w}_i | \rho | \bar{w}_i \rangle) + \#(\langle d_{ij} | \rho | d_{ij} \rangle) + \#(\langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle) + 2. \end{aligned} \quad (153)$$

With

$$\#(\langle w_i | \rho | w_j \rangle) = \#(\langle \bar{w}_i | \rho | \bar{w}_j \rangle) = \#(\langle d_{ij} | \rho | d_{ij} \rangle) \#(\langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle) = \frac{n(n-1)}{2} \quad (154)$$

and

$$\#(\langle w_i | \rho | w_i \rangle) = \#(\langle \bar{w}_i | \rho | \bar{w}_i \rangle) = n \quad (155)$$

follows

$$\#_{m.e.}(I(2)) = 2n(n-1) + 2n + 2 = 2(n^2 + 1) \quad (156)$$

- b) The number of matrix elements necessary to apply the “n-tuple state” inequality $I(n)$ (eq. (113)) to a state ρ is

$$\#_{m.e.}(I(n)) = 3. \quad (157)$$

- c) The number of matrix elements necessary to apply one of the inequalities $I(D)$ (eq. (128)) is

$$\begin{aligned} \#_{m.e.}(I(D)) = & \#(|\langle d_{\{\alpha\}}|\rho|d_{\{\beta\}}\rangle|) + \# \left(\sqrt{\langle d_{\{\alpha\}}|\otimes \langle d_{\{\beta\}}|\mathcal{P}_{\{\alpha\}}\rho^{\otimes 2}\mathcal{P}_{\{\alpha\}}|d_{\{\alpha\}}\rangle \otimes |d_{\{\beta\}}\rangle} \right) \\ & + \#(\langle d_{\{\alpha\}}|\rho|d_{\{\alpha\}}\rangle) \end{aligned} \quad (158)$$

With

$$\#(|\langle d_{\{\alpha\}}|\rho|d_{\{\beta\}}\rangle|) = \binom{n}{m} \cdot m \cdot \binom{n-m}{m-1}, \quad (159)$$

$$\# \left(\sqrt{\langle d_{\{\alpha\}}|\otimes \langle d_{\{\beta\}}|\mathcal{P}_{\{\alpha\}}\rho^{\otimes 2}\mathcal{P}_{\{\alpha\}}|d_{\{\alpha\}}\rangle \otimes |d_{\{\beta\}}\rangle} \right) = \binom{n}{m-1} \cdot \binom{n}{m+1} \quad (160)$$

and

$$\#(\langle d_{\{\alpha\}}|\rho|d_{\{\alpha\}}\rangle) = \binom{n}{m} \quad (161)$$

follows

$$\#_{m.e.}(I(D)) = \Gamma[1+n] \left(\frac{1}{\Gamma[1+m]\Gamma[1-m+n]} + \frac{\frac{1}{\Gamma[2-2m+n]} + \frac{\Gamma[m]\Gamma[1+n]}{\Gamma[2+m]\Gamma[-m+n]\Gamma[2-m+n]}}{\Gamma[m]^2} \right). \quad (162)$$

Which for $m = 2$ yields

$$\#_{m.e.}(I_2^n[\rho] \leq 0) = \frac{1}{6}(n^4 + 3n^3 - 13n^2 + 9n) \quad (163)$$

and for $m = 3$

$$\#_{m.e.}(I_3^n[\rho] \leq 0) = \frac{1}{48}(n^6 + 5n^5 - 102n^4 + 420n^3 - 700n^2 + 456n) \quad (164)$$

Above we saw that the number of matrix elements, one has to know of the state ρ , to apply a criterion to it, grows polynomially with the number of subsystems n , in all the inequalities introduced in this chapter. The inequality $I(2)$ and $I(n-1)$ grow with n^2 . For the inequalities $I(D)$ the expression

for the number of necessary matrix elements is rather complicated (see eq. (162)), but they all grow polynomially with system size n . The number of matrix elements one has to know to apply the inequality $I(n)$ is three, independent of n . This really makes these criteria favorable to criteria, where one has to ascertain all the matrix elements of the state. Especially for large systems, since a n qubit state has 2^{2n} matrix elements.

2.3.3 Local realizability

The last property of a separability criterion, that makes it easier to implement in an experiment, which we will deal with, is one's ability to apply the criterion to a state by only using results of local measurements, i.e. the criterion should be locally implementable. All terms of the inequalities $I(2)$ (eq. (112)), $I(n)$ (eq. (113)), $I(n-1)$ (eq. (114)) and $I(D)$ (eq. (128)) can be expressed in terms of local expectation values of Pauli matrices (eq. (25)).

Here we present simple examples of the inequalities and show how they can be rewritten in terms of local expectation values of Pauli matrices. To make the examples more compact, we introduce the notation

$$i_1 i_2 \cdots i_n := \langle \sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n} \rangle, \quad (165)$$

where $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$ and $\sigma_z = \sigma_3$.

a) The three particle “double state” inequality $I(2)$ (eq. (112)) reads

$$\begin{aligned} & \langle 001|\rho|010 \rangle + \langle 001|\rho|100 \rangle + \langle 010|\rho|100 \rangle + \langle 010|\rho|001 \rangle + \langle 100|\rho|001 \rangle + \langle 100|\rho|010 \rangle \\ & + \langle 110|\rho|101 \rangle + \langle 110|\rho|011 \rangle + \langle 101|\rho|011 \rangle + \langle 101|\rho|110 \rangle + \langle 011|\rho|110 \rangle + \langle 011|\rho|101 \rangle \\ & - 2(\langle 001|\rho|001 \rangle + \langle 010|\rho|010 \rangle + \langle 100|\rho|100 \rangle + \langle 110|\rho|110 \rangle + \langle 101|\rho|101 \rangle + \langle 011|\rho|011 \rangle) \\ & - 3(\langle 000|\rho|000 \rangle + \langle 111|\rho|111 \rangle) \leq 0. \end{aligned} \quad (166)$$

Re-written in terms of local expectation values of Pauli matrices, the three particle “double state” inequality $I(2)$ becomes

$$(0xx + xx0 + x0x + 0yy + y0y + yy0) - \frac{9}{32}(3 - zz0 - z0z - 0zz) - \frac{3}{16}(1 - 00z - 0z0 - z00) \leq 0. \quad (167)$$

We can see that we need 12 local measurement settings for the three particle “double state” inequality $I(2)$, which is a lot more feasible than the 63, required for a full state tomography.

b) The three particle “n-tuple state” inequality $I(n)$ (eq. (113)) reads

$$\text{Re}\{\langle 000|\rho|111\rangle\} - \frac{3}{2}(1 - \langle 000|\rho|000\rangle - \langle 111|\rho|111\rangle) \quad (168)$$

Re-expressed in terms of local expectation values of Pauli matrices the three particle “n-tuple state” inequality $I(n)$ yields

$$(xxx - yyx - yxy - xyy) - 3(3 - zz0 - z0z - 0zz) \leq 0, \quad (169)$$

So the three particle “n-tuple state” inequality $I(n)$ requires 7 local measurement settings, also much less than a full state tomography.

c) The four particle “(n-1)-tuple state” inequality $I(n-1)$ (eq. (113)) reads

$$\begin{aligned} & \langle 0001|\rho|0010\rangle + \langle 0001|\rho|0100\rangle + \langle 0001|\rho|1000\rangle + \langle 0010|\rho|0100\rangle + \langle 0010|\rho|1000\rangle + \langle 0100|\rho|1000\rangle \\ & + \langle 0010|\rho|0001\rangle + \langle 0100|\rho|0001\rangle + \langle 1000|\rho|0001\rangle + \langle 0100|\rho|0010\rangle + \langle 1000|\rho|0010\rangle + \langle 1000|\rho|0100\rangle \\ & - 2(\langle 0001|\rho|0001\rangle + \langle 0010|\rho|0010\rangle + \langle 0100|\rho|0100\rangle + \langle 1000|\rho|1000\rangle) \\ & - 2(\langle 1100|\rho|1100\rangle + \langle 1010|\rho|1010\rangle + \langle 1001|\rho|1001\rangle + \langle 0110|\rho|0110\rangle + \langle 0101|\rho|0101\rangle + \langle 0011|\rho|0011\rangle) \\ & - 6(\langle 000|\rho|000\rangle) \leq 0. \end{aligned} \quad (170)$$

Written in terms of local expectation values of Pauli matrices the “(n-1)-tuple state” inequality $I(n-1)$ yields

$$(xxx0 + xx0x + x0xx + 0xxx) - A() - B() - C() \leq 0 \quad (171)$$

The number of local measurement settings required to apply the inequality $I(n-1)$ is 17, which is more than six times less, than one would need for a full state tomography.

d) The four particle inequality $I_2^4[\rho] \leq 0$ of the criteria $I(D)$ (eq. (128)) reads as shown in eq. (132).

Expressed in terms of local expectation values of Pauli matrices it becomes

$$\sum_{\pi} \sum_{i=x,y} \pi(00ii) - \sum_{\pi} \sum_{i=x,y} \pi(iizz) - (3 + 3(zzzz) - \sum_{\pi} \pi(00zz)) - f(z^i z^j z^k z^l) \leq 0, \quad (172)$$

where f is a simple bilinear function and i, j, k and l can be either 0 or 1.

So the number of local measurement settings required to apply the inequality $I_2^4[\rho] \leq 0$ is 25, which is much better than the 255 local measurement settings, one would need for a full state tomography.

We just saw that in all the criteria $I(2)$, $I(n)$, $I(n-1)$ and $I(D)$ can be expressed in terms of local expectation values of Pauli matrices, which is also very important for the experimental implementation, as with increasing number of subsystems global measurement operations become more and more difficult to implement.

3 Entanglement in spin-chains

A field where we can investigate different kinds of multipartite states, are one dimensional arrays of qubits, so called spin-chains. This section is based on the work of Beatrix C. Hiesmayr, Matyas Koniorczyk and Heide Narnhofer “Maximizing nearest neighbour entanglement in finitely correlated qubit chains” (Ref. [18]). They use a method related to the density matrix renormalization group method (DMRG method, see Ref. [92]) to construct an infinite entangled spin-chain, i.e. an infinite spin-chain, where each qubit is at least entangled with its two nearest neighbours. Then they maximize the concurrence of the state describing two neighbouring qubits, meaning they maximize the nearest neighbour entanglement in the qubit chain. Their aim is to find out how high the concurrence of nearest neighbour entanglement can possibly be. In the first part we introduce the spin-chain with its translational invariant states of an arbitrary number of neighbouring qubits. The second part deals with the analytical and numerical results, found by the authors, for the maximal amount of nearest neighbour entanglement and with the comparison of these results to previous results found by William K. Wootters in Ref. [95]. Furthermore, in the second part, it is pointed out that the spin-chain behaves as an ordinary bicycle chain, if nearest neighbour entanglement is maximized. The third part is about some work, that I did, concerning nearest-, next nearest- and next-next nearest neighbour entanglement, as well as multipartite entanglement in the spin-chain.

3.1 Introducing the spin-chain

In Ref. [18] the authors consider an infinite one dimensional qubit chain, where each qubit is at least entangled with its two neighbours. In the first part of this section we show how a general, translational invariant, state of n neighbouring qubits of the spin-chain is constructed in Ref. [18]. The second part of this section is about the explicit choices, made by authors, for matrices occurring in the construction.

3.1.1 Constructing translational invariant states of the spin-chain

As stated above the authors of Ref. [18] use a method, related to the DMRG method (Ref. [92]), to construct the states of the spin-chain. This approach is equivalent to the finitely correlated states (FCS) method (see Ref. [96]). They construct translational invariant states of neighbouring qubits, hence the whole chain is translational invariant. To give a short example, we consider a three qubit translational invariant state $\rho_{[j,j+1,j+2]}$, of three neighbouring qubits of the spin-chain. It has

the property

$$\rho_{[j,j+1,j+2]} = \rho_{[j+k,j+k+1,j+k+2]}, \text{ with } j, k \in \mathbb{Z}. \quad (173)$$

Where the indices in the square brackets denote the positions of the qubits, whose state one considers, in the spin-chain.

To approximate such a translational invariant state of an arbitrary number n of neighbouring qubits $\rho_{[1,\dots,n]}$, the authors consider an auxilliary finite dimensional Hilbert space $\mathcal{H}_B$, with the state ρ_B assigned to it. The state ρ_B represents that part of the spin-chain, which is not described by the state $\rho_{[1,\dots,n]}$. Furthermore they consider a completely positive unital map $\mathbb{E} : \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H}_B$, represented by projectors acting between $\mathcal{H}_B$ and the space in which $\rho_{[1,\dots,n]}$ lives, which we denote by $\mathcal{H}_A$. The completely positive map $\mathbb{E}$ is described by the two $b \times b$ matrices v_1 and v_2 .

The accuracy of the approximation depends on the dimensionality of the auxilliary space $\mathcal{H}_B$, which is b . The three $b \times b$ matrices v_1 , v_2 and ρ_B have to fulfill two conditions.

- 1) The first condition occurs due to the unitality of the CP map $\mathbb{E}(\mathbb{1}_A \otimes \mathbb{1}_B) = \mathbb{1}_B$.

$$v_1 v_1^\dagger + v_2 v_2^\dagger = \mathbb{1} \quad (174)$$

- 2) The second condition exists, because of the required translational invariance of the states $\rho_{[1,\dots,n]}$.

$$\sum_{i=1}^2 v_i^\dagger \rho_B v_i = \rho_B \quad (175)$$

With the above assumptions and conditions a n neighbouring qubits state $\rho_{[1,\dots,n]}$ of the translational invariant spin-chain is constructed as

$$\rho_{[1,\dots,n]} = \sum_{\sigma, \tau} |\sigma\rangle \langle \tau| \text{Tr}(v_\sigma^\dagger \rho_B v_\tau), \quad (176)$$

$$\text{where } |\sigma\rangle = |s_1 \otimes s_2 \otimes \dots \otimes s_n\rangle, \quad |\tau\rangle = |t_1 \otimes t_2 \otimes \dots \otimes t_n\rangle,$$

$$\text{with } v_\sigma = v_{s_1} \cdot v_{s_2} \cdot \dots \cdot v_{s_n}, \quad v_\tau = v_{t_1} \cdot v_{t_2} \cdot \dots \cdot v_{t_n}, \quad s_i = \{1, 2\} \text{ and } t_i = \{1, 2\}.$$

So each different combination of matrix products of n hermitian conjugated $v_1^\dagger$ and $v_2^\dagger$ times ρ_B times a combination of n matrix products of v_1 and v_2 , is a different matrix element of the n qubit state

$\rho_{[1,\dots,n]}$. To make the eq. (176) easier to understand, we give some explicit examples. We write down the general one and two qubit state $\rho_{[1]}$ and $\rho_{[1,2]}$. Remember that we have not chosen v_1 and v_2 and therefore also do not know ρ_B .

$$\rho_{[1]} = \begin{pmatrix} \text{Tr}(v_1^\dagger \rho_B v_1) & \text{Tr}(v_1^\dagger \rho_B v_2) \\ \text{Tr}(v_2^\dagger \rho_B v_1) & \text{Tr}(v_2^\dagger \rho_B v_2) \end{pmatrix} \quad (177)$$

$$\rho_{[1,2]} = \begin{pmatrix} \text{Tr}(v_1^\dagger v_1^\dagger \rho_B v_1 v_1) & \text{Tr}(v_1^\dagger v_1^\dagger \rho_B v_1 v_2) & \text{Tr}(v_1^\dagger v_1^\dagger \rho_B v_2 v_1) & \text{Tr}(v_1^\dagger v_1^\dagger \rho_B v_2 v_2) \\ \text{Tr}(v_2^\dagger v_1^\dagger \rho_B v_1 v_1) & \text{Tr}(v_2^\dagger v_1^\dagger \rho_B v_1 v_2) & \text{Tr}(v_2^\dagger v_1^\dagger \rho_B v_2 v_1) & \text{Tr}(v_2^\dagger v_1^\dagger \rho_B v_2 v_2) \\ \text{Tr}(v_1^\dagger v_2^\dagger \rho_B v_1 v_1) & \text{Tr}(v_1^\dagger v_2^\dagger \rho_B v_1 v_2) & \text{Tr}(v_1^\dagger v_2^\dagger \rho_B v_2 v_1) & \text{Tr}(v_1^\dagger v_2^\dagger \rho_B v_2 v_2) \\ \text{Tr}(v_2^\dagger v_2^\dagger \rho_B v_1 v_1) & \text{Tr}(v_2^\dagger v_2^\dagger \rho_B v_1 v_2) & \text{Tr}(v_2^\dagger v_2^\dagger \rho_B v_2 v_1) & \text{Tr}(v_2^\dagger v_2^\dagger \rho_B v_2 v_2) \end{pmatrix} \quad (178)$$

States being constructed in such a way (eq. (176)), with ρ_B , v_1 and v_2 fulfilling the conditions of eq. (174) and eq. (175), describe n neighbouring qubits of an infinite translational invariant entangled spin-chain. Next we show what the authors of Ref. [18] chose explicitly for the matrices v_1 and v_2 , describing the CP map $\mathbb{E}$, and we derive a state ρ_B of the auxiliary space $\mathcal{H}_B$, describing the rest of the spin-chain.

3.1.2 Explicit choices for v_1 and v_2 and deriving ρ_B

Here we show how the authors of Ref. [18] obtain their choices of v_1 and v_2 , fulfilling the conditions in the eq.s (174). First for the most simplest case of $b = 2$, hence v_1 and v_2 being complex 2×2 matrices, then for the general case. Later we derive the 2×2 real matrix ρ_B , satisfying the condition in eq. (175).

They reason, that the entanglement shared by two neighbouring qubits in a chain cannot be maximal, i.e. the state of two neighbouring qubits of the spin-chain $\rho_{[1,2]}$ (eq. (178)) cannot be one of the four pure Bell states (eq. (66)), because if that was the case, every other qubit of the chain would have to be separable to those two. In multipartite qubit systems there are boundaries to the amount of entanglement shared by two neighbouring qubits, given by the Coffman-Kundu-Wootters inequalities (CKW inequalities, see Ref. [93]), which was proven to be true for more than three particles in Ref. [94]. They also state that it is plausible, that the state $\rho_{[1,2]}$ is orthogonal to the pure state $|00\rangle\langle 00|$, which means that v_1 is nilpotent, hence $v_1 v_1 = 0$.

For the case of $b = 2$, hence v_1 and v_2 are 2×2 matrices, the authors choose without loss of generality

$$v_1 = \begin{pmatrix} 0 & 0 \\ \cos(\alpha) & 0 \end{pmatrix}. \quad (179)$$

This, plugged in eq. (174), yields

$$v_2 = \begin{pmatrix} 1 & 0 \\ 0 & \sin(\alpha) \end{pmatrix} \cdot \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}. \quad (180)$$

Now we present the general case, v_1 and v_2 being $b \times b$ matrices, chosen by Hiesmayr, Koniorczyk and Narnhofer, for arbitrary dimension b of the auxilliary Hilbert space $\mathbb{H}_B$.

Without loss of generality they choose

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \cos(\alpha_1) & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \cos(\alpha_2) & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (181)$$

and again with the condition from eq. (174) follows

$$v_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \sin(\alpha_1) & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \sin(\alpha_2) & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \cdot R. \quad (182)$$

Where R is an arbitrary unitary matrix.

Next we present the matrix ρ_B , which represents the part of the chain, that is not described by the state we want to construct $\rho_{[1,\dots,n]}$, for the simplest case of $b = 2$. We use the condition in eq. (175) to derive a unique ρ_B .

As in Ref. [18] the authors only consider real generators of the interaction, to reduce the set of pa-

rameters, all the matrix elements of ρ_B are real. So, because of its self adjointness ($\rho_B = \rho_B^\dagger$) and because $\text{Tr}(\rho_B) = 1$ we know that ρ_B has just two real parameters, which we denote r_{11} and r_{12} .

$$\rho_B = \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & 1 - r_{11} \end{pmatrix} \quad (183)$$

By plugging this and v_1 from eq. (179) and v_2 from eq. (180) into the translation invariance condition from eq. (175) leads to four conditions for the two real parameters r_{11} and r_{12} . Solving them yields

$$r_{11} = \frac{-10 - 2\text{Cos}[2\phi] + \text{Cos}[2(\phi - \alpha)] + 2\text{Cos}[2\alpha] + \text{Cos}[2(\phi + \alpha)] - 4\text{Sin}[2\phi - \alpha] + 8\text{Sin}[\alpha] + 4\text{Sin}[2\phi + \alpha]}{-14 + 2\text{Cos}[2\phi] + \text{Cos}[2(\phi - \alpha)] + 2\text{Cos}[2\alpha] + \text{Cos}[2(\phi + \alpha)] - 4\text{Sin}[2\phi - \alpha] + 8\text{Sin}[\alpha] + 4\text{Sin}[2\phi + \alpha]} \quad (184)$$

and

$$r_{12} = \frac{4\text{Sin}[2\phi](-1 + \text{Sin}[\alpha])}{-14 + 2\text{Cos}[2\phi] + \text{Cos}[2(\phi - \alpha)] + 2\text{Cos}[2\alpha] + \text{Cos}[2(\phi + \alpha)] - 4\text{Sin}[2\phi - \alpha] + 8\text{Sin}[\alpha] + 4\text{Sin}[2\phi + \alpha]} \quad (185)$$

Now we are able to express a state of n neighbouring qubits of the spin-chain $\rho_{[1, \dots, n]}$, in terms of the two real parameters α and ϕ . However these expressions are very long, even the one qubit state of eq. (177), expressed in terms of α and ϕ , is much too big to be showed here.

Now one is able to calculate the concurrence or other entanglement measures, giving expressions of α and ϕ , and maximize it. Which is what the authors did in Ref. [18] and what we deal with in the section 3.2.

3.2 Results for maximized nearest neighbour entanglement

In Ref. [18] the authors maximize the concurrence of the state $\rho_{[1,2]}$ (eq. (178)), describing two neighbouring qubits of the spin-chain they constructed (see section 3.1). First for the dimension of the auxiliary Hilbert space being $b = 2$ and $b = 3$, giving analytical results. Then, by numerical methods, up to an accuracy of the approximation of the auxiliary Hilbert space being $b = 9$. They nearly reach a value of concurrence of a state of two neighbouring qubits, that was conjectured to be an upper bound to the maximum nearest neighbour concurrence in a spin-chain, by William K. Wootters in Ref. [95]. The first part of this section deals with the analytical results derived in Ref. [18]. The second part of this section shows the numerical results, found by Hiesmayr, Koniorczyk and Narnhofer, for the maximal nearest neighbour entanglement in the spin-chain. In the third and last part of this section we consider some properties of spin-chain, e.g. if it is an ordinary bicycle chain.

3.2.1 Analytical results ($b = 2, b = 3$)

For $b = 2$ we already showed ρ_B , v_1 and v_2 in section 3.1.2. In Ref. [18] the authors do not show any explicit ρ_B , neither for $b = 2$ nor for $b > 2$, but they show the possible solutions for ρ_B in the Bloch sphere (see eq. (29)) for $b = 2$ and $b = 3$, which is what we present here.

1) As for $b = 2$ the state ρ_B is a one particle qubit state it can be decomposed as

$$\rho_B = \frac{\sigma_0 + \vec{a}\vec{\sigma}}{2} \quad \text{with} \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad \vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \quad (186)$$

As solutions for ρ_B Hiesmayr, Koniorczyk and Narnhofer find, with v_1 as in eq. (179) and v_2 as in eq. (180), that the equality

$$\frac{(a_1 - 0)^2}{(\frac{1}{\sqrt{2}})^2} + \frac{(a_3 - \frac{1}{2})^2}{(\frac{1}{\sqrt{2}})^2} = 1 \quad (187)$$

holds for all α and ϕ .

So the possible solutions for ρ_B lie in an ellipse on the x - z -plane of the basis of the Bloch vector $\vec{a}$.

2) For $b = 3$ the 3×3 matrices v_1 and v_2 , describing the CP map $\mathbb{E}$ can be chosen as

$$v_1 = \begin{pmatrix} 0 & 0 & 0 \\ \cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot R \quad (188)$$

(see eq. (181) and (182)), and ρ_B is a Qutrit state. The authors choose R to be an orthogonal matrix, which has three parameters ϕ_1 , ϕ_2 and ϕ_3 .

There is a generalization to the Bloch decomposition for particles having more degrees of freedom than a qubit, using the Gell-Mann matrices λ_i (eq. (30)) instead of the pauli matrices. Using that, the 3×3 matrix ρ_B can be written as

$$\rho_B = \frac{\mathbb{1}_3 + \sqrt{3}\vec{a}\vec{\lambda}}{3} \quad \text{with} \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} \quad \text{and} \quad \vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}, \quad (189)$$

where the λ_i are the 3×3 Gell-Mann matrices and the $a_i \in \mathbb{R}$.

For v_1 and v_2 as in eq. (188) and for of fixed ϕ_1 , ϕ_2 and ϕ_3 the solutions for ρ_B form a five axial ellipsoid. Varying ϕ_1 , ϕ_2 or ϕ_3 gives an ellipsoid with semiaxis and a shifted center.

Now we write down the concurrence (eq. (54) and (57)) of a state of two neighbouring qubits of the chain, for dimension of the auxiliary Hilbert space $\mathcal{H}_B$ being $b = 2$. This is an expression in terms of the parameters α and ϕ , which we maximize over all α and ϕ .

For v_1 from eq. (179), v_2 from eq. (180) and ρ_B from eq. (183), the concurrence of a state of two neighbouring qubits $\rho_{[1,2]}$ (eq. (178)) of the spin-chain is

$$C(\rho_{[1,2]}) = \frac{\cos^2(\alpha)(1 + \sin(\alpha))\sin^2(\phi)\cos^2(\phi)}{\cos^2(\alpha)\cos^2(\phi)(-1 + \sin(\alpha)) - 2(1 + \sin(\alpha))\sin^2(\phi)}. \quad (190)$$

By maximizing this expression over all α and ϕ , one obtains a value of $C_{max}(\rho_{[1,2]}) = \sqrt{2} - 1 = 0.41421$, which is reached for $\alpha = 0.427079$ and $\phi = 0.571859$.

With these choices for α and ϕ , the state of two neighbouring qubits of the spin-chain $\rho_{[1,2]}$ (eq.

(178)) reads

$$\rho_{[1,2]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.292893 & 0.207107 & 0.174155 \\ 0 & 0.207107 & 0.292893 & 0.174155 \\ 0 & 0.174155 & 0.174155 & 0.414213 \end{pmatrix} \quad (191)$$

One can easily see that this state is orthogonal to the pure state $|00\rangle\langle 00|$ as stated in section 3.1.2.

In Ref. [95] William K. Wootters conjectures $C_W(\rho_{[1,2]}) = 0.434467$ to be an upper bound to the concurrence of two neighbouring qubits of a translational invariant entangled spin-chain. So by maximizing eq. (190) one nearly reaches this suggested limit. By increasing the dimension b and thereby the accuracy of the approximation the authors of Ref. [18] are trying to get closer to that limit. We present this numerical approach in the next section (section 3.2.2).

3.2.2 Numerical results ($b > 3$)

Here we briefly present the numerical results obtained by Hiesmayr, Koniorczyk and Narnhofer, for maximum nearest neighbour entanglement and compare them to results obtained by Wootters.

By plugging v_1 from eq. (181) and v_2 from eq. (182) into the condition from eq. (175), for each dimension of the auxiliary space, up to $b = 9$, the authors of Ref. [18] obtain a function for the concurrence of the state of two neighbouring qubits of the spin-chain $\rho_{[1,2]}$ (eq. (178)), depending on the parameters $(\alpha_1, \alpha_2, \dots)$ and $(\phi_1, \phi_2, \dots)$. They evaluate the maximum of $C(\alpha_1, \alpha_2, \dots, \phi_1, \phi_2, \dots)$ numerically, by the simulated annealing method (Ref. [97]). They use this method as $C(\alpha_1, \alpha_2, \dots, \phi_1, \phi_2, \dots)$ is not a convex function, and it might not be differentiable at every point. The simulated annealing method is known to be effective for such functions. Nevertheless, they confess that it is not totally certain, though very likely, that the so obtained maxima are global.

Note that Hiesmayr, Koniorczyk and Narnhofer restricted themselves to general orthogonal matrices R , they use in eq. (182) to construct v_2 for $b > 6$. However, for $b \leq 6$ the use of a general unitary matrix R instead of an orthogonal matrix R , in eq. (182), does not improve the approximation.

As stated above, in Ref. [95] William K. Wootters has found the maximum value of the concurrence of two neighbouring qubits in a translational invariant entangled spin-chain to be $C_W(\rho_{[1,2]}) = 0.434467$, which is conjectured to be an upper bound to the maximum nearest neighbour entanglement possible, in an entangled spin-chain. Beatrix C. Hiesmayr, Matyas Koniorczyk and Heide Narnhofer nearly

reach that value and get closer and closer to it for increasing b .

In table 1 we show the different maxima of $C(\rho_{[1,2]})$ and their relative difference to $C_W(\rho_{[1,2]})$, for the different accuracy of approximation, i.e. for different dimensions of the auxiliary Hilbert space $\mathcal{H}_B$, starting at $b = 2$ and going up to $b = 9$.

b	$C_{max}(\rho_{[1,2]})$	$(1 - \frac{C_{max}(\rho_{[1,2]})}{C_W(\rho_{[1,2]})})$
2	0.41421	4.66%
3	0.41825	3.73%
4	0.43200	0.57%
5	0.43247	0.46%
6	0.43336	0.25%
7	0.43381	0.15%
8	0.43406	0.09%
9	0.43447	0.08%

Table 1: Here we see the maximal concurrences of the state of two neighbouring qubits for the different dimensions b and the relative difference to the conjectured upper bound of Wootters.

The authors remark, that due to the facts that $C_W(\rho_{[1,2]})$ is a conjecture, they do not know for certain that the obtained maxima are global, and that they cannot check the case of $b \rightarrow \infty$, it could also be possible to reach values of maximal concurrence of a state of two neighbouring qubits, of a translational invariant entangled spin-chain, that are higher than $C_W(\rho_{[1,2]})$. At least they state, that they can prove, that $C_W(\rho_{[1,2]})$ can be reached by FCS methods.

3.2.3 Properties of the spin-chain

Now we consider the properties of such an infinitely translational invariant entangled qubit chain with maximized nearest neighbour entanglement.

In Ref. [18] the authors ask the question, if this chain is an ordinary bicycle chain, where each link is only connected to its nearest neighbours or if the connection goes further, to the next and next-next nearest neighbour. In other words they are interested, if one qubit of the chain is entangled to more than its two nearest neighbours.

They consider the state of three neighbouring qubits in the spin-chain

$$\rho_{[1,2,3]} = \sum_{\sigma, \tau} |s_1 s_2 s_3\rangle \langle t_1 t_2 t_3| \text{Tr}(v_{s_1}^\dagger v_{s_2}^\dagger v_{s_3}^\dagger \rho_B v_{t_1} v_{t_2} v_{t_3}), \quad (192)$$

with σ , τ , s_i and t_i as in eq. (176).

By applying the partial trace (see section 1.1.6) operation to the second subsystem, i.e. tracing out the qubit in the middle, one is left with the state of two next nearest neighbours of the spin-chain, which we will denote $\rho_{[1,,3]}$.

$$\rho_{[1,,3]} = \text{Tr}_2(\rho_{[1,2,3]}) = \sum |s_1 s_3\rangle \langle t_1 t_3| \left[\text{Tr}(v_{s_1}^\dagger v_1^\dagger v_{s_3}^\dagger \rho_B v_{t_1} v_1 v_{t_3}) + \text{Tr}(v_{s_1}^\dagger v_2^\dagger v_{s_3}^\dagger \rho_B v_{t_1} v_2 v_{t_3}) \right] \quad (193)$$

The sum runs over all different combinations of s_1 , s_3 , t_1 and t_3 .

As v_1 is nilpotent (see section 3.1.2) the term $\text{Tr}(v_{s_1}^\dagger v_1^\dagger v_{s_3}^\dagger \rho_B v_{t_1} v_1 v_{t_3})$ vanishes for all choices of s_1 , s_3 , t_1 and t_3 , except for $s_1 = s_3 = t_1 = t_3 = 2$. This makes it much easier to calculate the next nearest neighbour entanglement, which is what the authors did. They found, that for the parameters maximizing nearest neighbour entanglement it is zero for all dimensions b of the auxiliary space.

So one can conclude that the spin-chain, with maximized nearest neighbour entanglement, is indeed an “ordinary bicycle chain”, just neighbouring qubits are linked via entanglement.

However, in Ref. [18] it is shown that for $b=2$ it is possible to find parameters, for which next nearest neighbour entanglement is nonzero. The maximum of possible concurrence of the state of two next nearest neighbours of the spin-chain (eq. (193)) is $C_{max}(\rho_{[1,,3]}) = 0.169470$ for the parameters being $\alpha = 0.88563$ and $\phi = 0.25066$.

In the next section we deal with entanglement of next and next-next nearest neighbours in the qubit chain.

3.3 Next nearest neighbour-, next-next nearest neighbour- and multipartite entanglement in a spin-chain

In this section we deal with states of the spin-chain, constructed as shown in section 3.1 and 3.2, and their entanglement. We plug v_1 from eq. (179), v_2 from eq. (180) and ρ_B from eq. (183) into eq. (176) to construct such a state, i.e. the dimension of the auxiliary space is $b = 2$, in this whole section. In general we do not maximize over the parameters α and ϕ . The first part of this section is about the concurrence of the state of two neighbouring, two next neighbouring and two next-next neighbouring qubits. The second part of this section deals with qualitative results of the entanglement in spin-chain states, achieved by applying the PPT criterion (see section 1.2.2). We present three dimensional plots of the PPT criterion applied to different states of the spin-chain. The third and last part of this section is about detecting multipartite entanglement in the spin-chain, via the criteria presented in section 2. Again three dimensional plots of this are presented.

3.3.1 Concurrence of nearest-, next nearest- and next-next nearest neighbour states

In section 3.2 we obtained the maximal concurrence of nearest neighbour states of the spin-chain and of next nearest neighbour states of the spin-chain, and the values of the parameters α and ϕ , maximizing them. Here we present the concurrences of nearest, next nearest, and next-next nearest neighbour states for the values of the parameters α and ϕ maximizing the concurrences of the nearest and next nearest neighbour entanglement.

To obtain such nearest-, next nearest-, and next-next nearest neighbour states of the infinitely, entangled spin-chain, with the parameters α and ϕ we used the program “Wolfram Mathematica 8”. First we constructed the states of neighbouring qubits $\rho_{[1,2]}$, $\rho_{[1,2,3]}$ and $\rho_{[1,2,3,4]}$ by plugging v_1 from eq. (179), v_2 from eq. (180) and ρ_B from eq. (183) into eq. (176). Then we took the necessary partial traces (see section 1.1.6) to obtain the next nearest neighbour state $\rho_{[1,,3]}$ and the next-next nearest neighbour state $\rho_{[1,,,4]}$.

$$\rho_{[1,,3]} = \text{Tr}_2(\rho_{[1,2,3]}) \text{ and } \rho_{[1,,,4]} = \text{Tr}_2(\text{Tr}_2(\rho_{[1,2,3,4]})) = \text{Tr}_2(\rho_{[1,,3,4]}) \quad (194)$$

As $\rho_{[1,2]}$, also $\rho_{[1,,3]}$ and $\rho_{[1,,,4]}$, with open parameters α and ϕ , contain much to long expressions to be shown here.

Now we show the different values the concurrence reaches for $C(\rho_{[1,2]})$, $C(\rho_{[1,,3]})$ and $C(\rho_{[1,,,4]})$, first

for the values of α and ϕ maximizing $C(\rho_{[1,2]})$, then for the values of α and ϕ maximizing $C(\rho_{[1,,3]})$.

- 1.) The parameters maximizing the concurrence of the nearest neighbour state $C(\rho_{[1,2]})$ of the translational invariant, infinite, entangled spin-chain are $\alpha = 0.427079$ and $\phi = 0.571859$, giving a maximum value of the concurrence of $C_{max}(\rho_{[1,2]}) = 0.4121$.

The nearest neighbour state $\rho_{[1,2]}$ (eq. (178)) with these parameters plugged in is shown in eq. (191).

The concurrence of the nearest neighbour state, with α and ϕ maximized for nearest neighbour entanglement, is given above.

The next nearest neighbour state $\rho_{[1,,3]}$ with the parameters being $\alpha = 0.427079$ and $\phi = 0.571859$ is

$$\rho_{[1,,3]} = \begin{pmatrix} 0.0710678 & 0.0845143 & 0.0845143 & 0.0147185 \\ 0.0845143 & 0.221825 & 0.100505 & 0.0896408 \\ 0.0845143 & 0.100505 & 0.221825 & 0.0896408 \\ 0.0147185 & 0.0896408 & 0.0896408 & 0.485281 \end{pmatrix}, \quad (195)$$

and the concurrence of the next nearest neighbour state, with α and ϕ maximized for nearest neighbour entanglement, is $C(\rho_{[1,,3]}) = 0$.

The next-next nearest neighbour state $\rho_{[1,,,4]}$ with these parameters plugged in is

$$\rho_{[1,,,4]} = \begin{pmatrix} 0.100505 & 0.0597606 & 0.0597606 & 0 \\ 0.0597606 & 0.192388 & 0.0355338 & 0.114394 \\ 0.0597606 & 0.0355338 & 0.192388 & 0.114394 \\ 0 & 0.114394 & 0.114394 & 0.514719 \end{pmatrix}, \quad (196)$$

and the concurrence of the next-next nearest neighbour state, with α and ϕ maximized for nearest neighbour entanglement, is $C(\rho_{[1,,4]}) = 0$.

So we saw that for maximized nearest neighbour entanglement the spin-chain is indeed an “ordinary bicycle chain”, i.e. each qubit is just entangled to its nearest neighbours, as stated in section 3.2.3.

- 2.) The parameters maximizing the concurrence of the next nearest neighbour state $C(\rho_{[1,,3]})$ of the

translational invariant, infinite, entangled spin-chain are $\alpha = 0.88563$ and $\phi = 0.25066$, giving a maximum value of the concurrence of $C_{max}(\rho_{[1,,3]}) = 0.169470$.

The nearest neighbour state $\rho_{[1,2]}$ (eq. (178)) with the parameters being $\alpha = 0.88563$ and $\phi = 0.25066$ is

$$\rho_{[1,2]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.144202 & 0.13533 & 0.150815 \\ 0 & 0.13533 & 0.144202 & 0.150815 \\ 0 & 0.150815 & 0.150815 & 0.711596 \end{pmatrix}, \quad (197)$$

and the concurrence of the nearest neighbour state, with α and ϕ maximized for next nearest neighbour entanglement, is $C(\rho_{[1,2]}) = 0.27066$.

The next nearest neighbour state $\rho_{[1,,3]}$ with these parameters plugged in is

$$\rho_{[1,,3]} = \begin{pmatrix} 0.00355273 & 0.0201634 & 0.0201634 & 0.00964889 \\ 0.0201634 & 0.140649 & 0.114437 & 0.130652 \\ 0.0201634 & 0.114437 & 0.140649 & 0.130652 \\ 0.00964889 & 0.130652 & 0.130652 & 0.715148 \end{pmatrix}, \quad (198)$$

and the concurrence of the next nearest neighbour state, with α and ϕ maximized for next nearest neighbour entanglement, is given above.

The next-next nearest neighbour state $\rho_{[1,,,4]}$ with the parameters being $\alpha = 0.88563$ and $\phi = 0.25066$ is

$$\rho_{[1,,,4]} = \begin{pmatrix} 0.0104965 & 0.0303892 & 0.0303892 & 0.0068433 \\ 0.0303892 & 0.133706 & 0.087982 & 0.120426 \\ 0.0303892 & 0.087982 & 0.133706 & 0.120426 \\ 0.0068433 & 0.120426 & 0.120426 & 0.722092 \end{pmatrix}, \quad (199)$$

and the concurrence of the next-next nearest neighbour state, with α and ϕ maximized for next nearest neighbour entanglement, is $C(\rho_{[1,,,4]}) = 0.0758173$.

So for the concurrence of the next nearest neighbour state maximized, the spin-chain does not behave like an ordinary bicycle chain. Each qubit is at least entangled with its six nearest neigh-

bours.

In this section we learned that the entanglement in the spin-chain can reach from one qubit at least to its two next-next nearest neighbours and presumably much further. We deal with multipartite entanglement in the spin-chain in section 3.3.3.

3.3.2 Qualitative results for entanglement in the spin-chain

Here we present some qualitative results concerning entanglement in the spin-chain, achieved by applying the PPT criterion (see section 1.2.2) to certain states of the spin-chain. First we show three dimensional plots of the PPT criterion applied to the (multipartite) states $\rho_{[1,2,3]}$ and $\rho_{[1,2,3,4]}$, where it is not necessary and sufficient. Then we apply it to the two qubit states $\rho_{[1,2]}$, $\rho_{[1,,3]}$ and $\rho_{[1,,,4]}$, where the PPT criterion is necessary and sufficient, and show three dimensional plots thereof.

We apply the partial transpose operation (section 1.1.7) to the state $\rho_{[1,2,3]}$, for all α and ϕ . Then we plot the lowest eigenvalue of the so obtained matrix with the program “Wolfram Mathematica 8”.

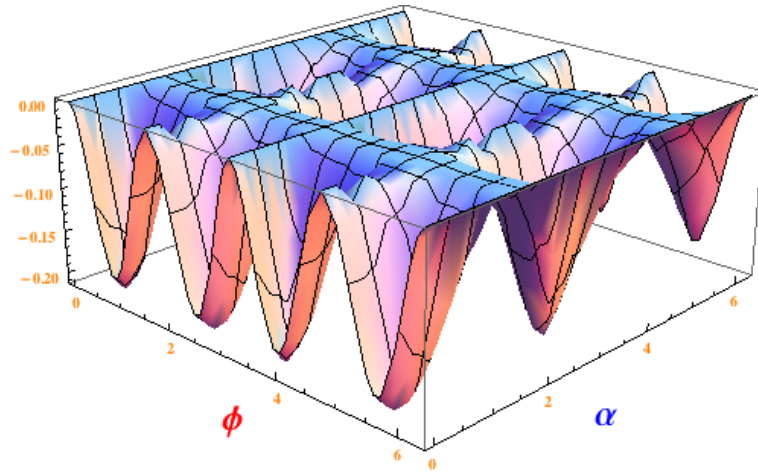


Figure 6: Here we can observe the lowest eigenvalue of a matrix obtained by applying the partial transpose operation to the state $\rho_{[1,2,3]}$, plotted for all α and ϕ .

In figure 6 one can observe that $\rho_{[1,2,3]}$ is NPT for all choices of α and ϕ , except for $\phi = k \cdot \frac{\pi}{2}$, with $k \in \mathbb{N}$. Therefore we know for certain that it is entanglement in the state $\rho_{[1,2,3]}$ for nearly all choices of α and ϕ .

Next we apply the partial transpose operation (section 1.1.7) to the state $\rho_{[1,2,3,4]}$, for all α and ϕ . Again we plot the lowest eigenvalue of the so obtained matrix.

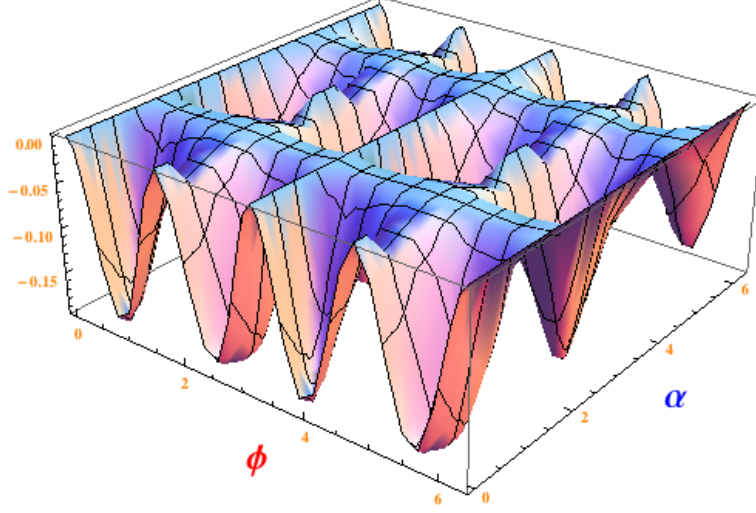


Figure 7: Here we can observe the lowest eigenvalue of a matrix obtained by applying the partial transpose operation to the state $\rho_{[1,2,3,4]}$, plotted for all α and ϕ .

Again in figure 7 we can observe that $\rho_{[1,2,3,4]}$ is NPT for all choices of α and ϕ , except for $\phi = k \cdot \frac{\pi}{2}$, with $k \in \mathbb{N}$. So again we know for certain that there exists entanglement in the state $\rho_{[1,2,3,4]}$ for all choices of α and ϕ , except for $\phi = k \cdot \frac{\pi}{2}$.

So from the above results one can conclude that there has to be some entanglement in the spin-chain, either between nearest neighbours or distributed in another way.

Now we apply the PPT criterion to the two qubit nearest-, next nearest- and next-next nearest neighbour states of the spin-chain $\rho_{[1,2]}$, $\rho_{[1,,3]}$ and $\rho_{[1,,,4]}$, where it is a necessary and sufficient criterion. We present three dimensional plots, in which one can observe that there are areas of α and ϕ for which $\rho_{[1,,3]}$ and $\rho_{[1,,,4]}$ are with certainty separable and other areas where they are with certainty entangled. $\rho_{[1,2]}$ is NPT for nearly all choices of α and ϕ .

First we apply the partial transpose operation (section 1.1.7) to the state $\rho_{[1,2]}$, for all α and ϕ . Then we plot the lowest eigenvalue of the so obtained matrix with the program “Wolfram Mathematica 8”.

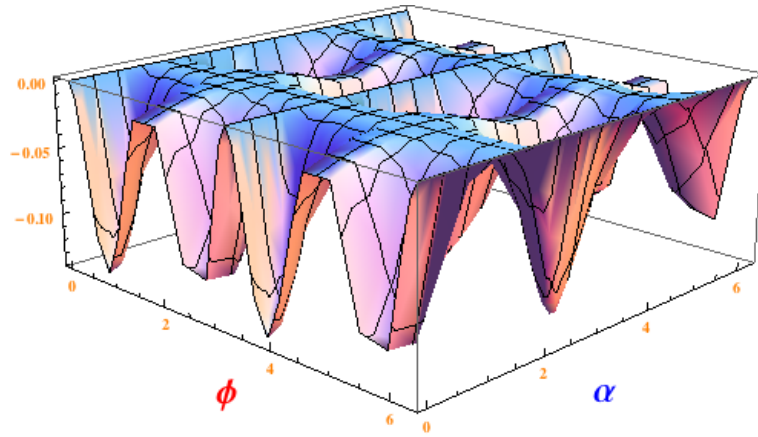


Figure 8: Here we can observe the lowest eigenvalue of a matrix obtained by applying the partial transpose operation to the state $\rho_{[1,2]}$, plotted for all α and ϕ .

Like in the figures 6 and 7, in figure 8 one can observe that $\rho_{[1,2]}$ is NPT for all choices of α and ϕ , except for $\phi = k \cdot \frac{\pi}{2}$, with $k \in \mathbb{N}$. This tells us again that in $\rho_{[1,2]}$, there exists nearest neighbour entanglement for nearly all choices of α and ϕ . This is not surprising, as it was a condition in the construction, that every qubit has to be entangled, at least, to its two nearest neighbours.

Next we apply the partial transpose operation (section 1.1.7) to the next nearest neighbour state $\rho_{[1,,3]}$, for all α and ϕ . Then we plot the lowest eigenvalue of the so obtained matrix with the program “Wolfram Mathematica 8”.

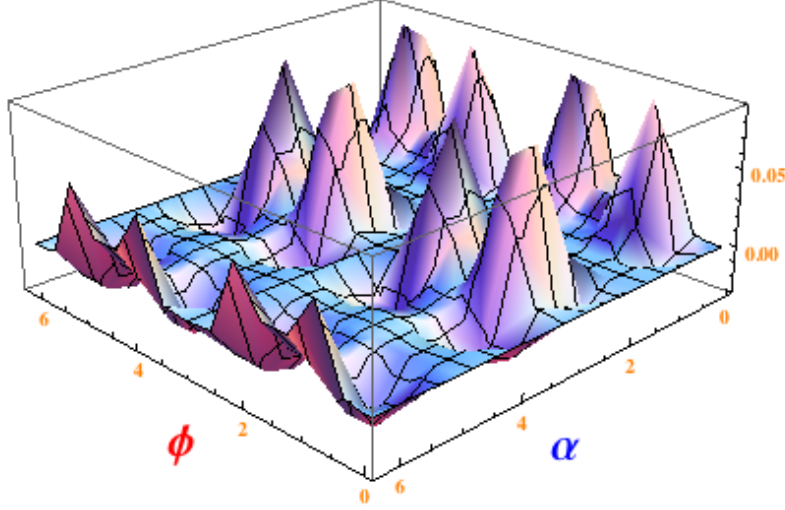


Figure 9: Here we can observe the lowest eigenvalue of a matrix obtained by applying the partial transpose operation to the state $\rho_{[1,,3]}$, plotted for all α and ϕ .

In figure 9 we can see that the next nearest neighbour state $\rho_{[1,,3]}$ is PPT for most choices of α and ϕ , but there are areas, e.g. around $\alpha = 5.6318$ and $\phi = 5.5022$ (a local maxima of the concurrence of $\rho_{[1,,3]}$), for which $\rho_{[1,,3]}$ is for certain entangled.

Finally we apply the partial transpose operation (section 1.1.7) to the next-next nearest neighbour state $\rho_{[1,,,4]}$, for all α and ϕ . And again we plot the lowest eigenvalue of the so obtained matrix with the program “Wolfram Mathematica 8”.

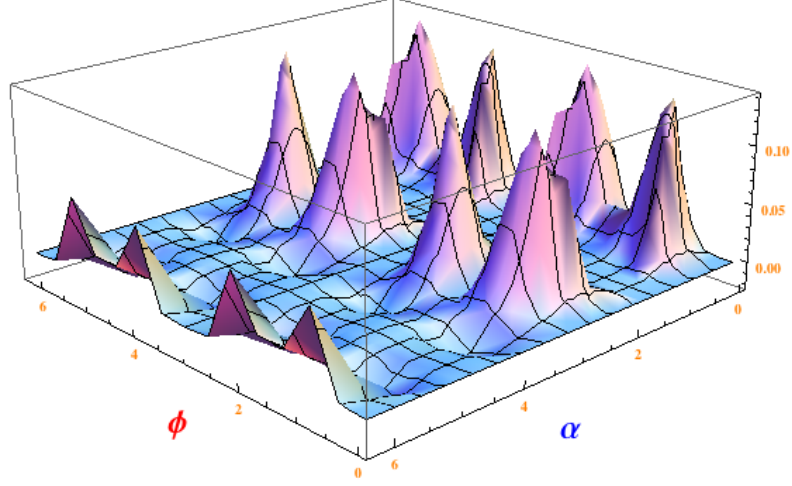


Figure 10: Here we can observe the lowest eigenvalue of a matrix obtained by applying the partial transpose operation to the state $\rho_{[1,,,4]}$, plotted for all α and ϕ .

In figure 10 we can see that $\rho_{[1,,,4]}$ is PPT for most choices of α and ϕ , but again there are areas for which $\rho_{[1,,,4]}$ is NPT. These regions of parameters seem to be around the same points as in figure 10, so one could suggest that there are certain values of α and ϕ , for which the entanglement reaches very far.

However we still do not know if the entanglement in the spin-chain exists just in pairs of qubits or if there exists multipartite entanglement. In the next section we apply criteria capable to detect (genuine) multipartite entanglement in states of four neighbouring qubits of the spin-chain.

3.3.3 Detecting multipartite entanglement in the spin-chain

In this section we apply the criteria, presented in section 2 to the state of four neighbouring qubits of the spin-chain $\rho_{[1,2,3,4]}$, to detect if it can be a GME state (see section 1.3.1) for certain choices of α and ϕ .

We indeed reach a violation of one of the criteria of section 2 for a certain unitary transformed state of four neighbouring qubits. We present a three dimensional plot of this criteria, namely the “(n-1)-tuple state” inequality $I(n-1)$ (eq. (114)), applied to a unitary transformed state of four neighbouring qubits. Furthermore we present a plot of those regions of α and ϕ for which the “(n-1)-tuple state” inequality is violated by this unitary transformed $\rho_{[1,2,3,4]}$.

First we try to find a violation of one of the criteria presented in section 2 by applying them to the state of four neighbouring qubits of the spin-chain $\rho_{[1,2,3,4]}$. We apply the “double state” inequality $I(2)$ (eq. (112)), the “n-tuple state” inequality $I(n)$ (eq. (113)), the “(n-1)-tuple state” inequality $I(n-1)$ (eq. (114)) and the inequality $I_2^4 \leq 0$ of the inequalities $I(D)$ (eq. (128)) to the state $\rho_{[1,2,3,4]}$. Unfortunately we observe that none of the criteria detects $\rho_{[1,2,3,4]}$ as a GME state, i.e. we find no violation of the inequalities listed above by applying them to $\rho_{[1,2,3,4]}$. This is somehow unexpected, as we found some hints in this chapter that there exists multipartite entanglement in the translational invariant, infinitely, entangled spin-chain.

So the last attempt to detect $\rho_{[1,2,3,4]}$ as a GME state is to apply a general unitary transformation (eq. (22)) to $\rho_{[1,2,3,4]}$ (this corresponds to the case of arbitrary changes of bases in the subsystems of $\rho_{[1,2,3,4]}$) and maximize the different inequalities $I(2)$, $I(n)$, $I(n-1)$, and $I(D)$ over all α and ϕ and over all parameters occurring in the general unitary matrix.

We did this by using the parametrization of general unitary matrices presented in Ref. [98] and [99]. We did not reach a violation of the inequalities $I(2)$, $I(n)$ and $I(D)$, but for inequality $I(n-1)$ we obtained a value of 0.21788 for the parameters being $\alpha = 0.395782$ and $\phi = 0.177926$ and for certain parameters that occur in the unitary parametrization of the general unitary matrix. So by applying the “(n-1)-tuple state” inequality $I(n-1)$ (eq. (114)) we could indeed detect that $\rho_{[1,2,3,4]}$ is a GME state for some α and ϕ .

We plot the “(n-1)-tuple state” inequality applied to $\rho_{[1,2,3,4]}$, generally unitary transformed by a unitary matrix with the parameters we obtained via the maximization.

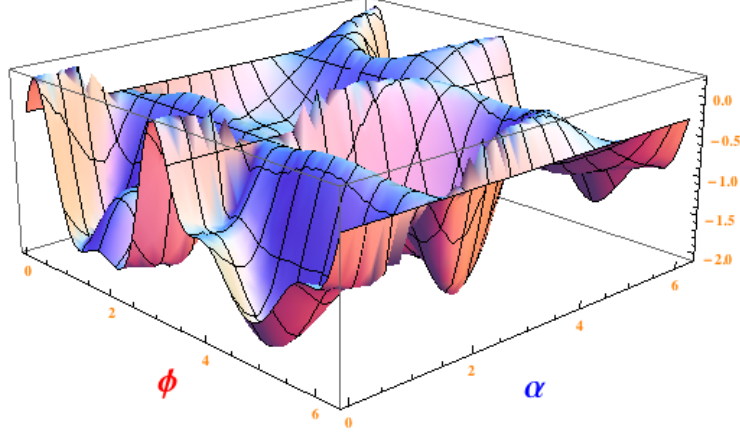


Figure 11: Here we can observe values obtained by applying the “(n-1)-tuple state” inequality to a unitary transformed $\rho_{[1,2,3,4]}$, plotted for all α and ϕ .

In figure 11 we can observe that there are certain regions of α and ϕ for which a specific unitary transformed $\rho_{[1,2,3,4]}$ violates the “(n-1)-tuple state” inequality $I(n-1)$ (eq. (114)). Therefore we know that, for certain choices of α and ϕ , there indeed exists genuine four qubit entanglement, in the states of four neighbouring qubits $\rho_{[1,2,3,4]}$ in the translational invariant, infinite, entangled spin-chain.

To make the regions of α and ϕ , for which the unitary transformed $\rho_{[1,2,3,4]}$ violates the “(n-1)-tuple state” inequality better observable we present a two dimensional plot of those areas.

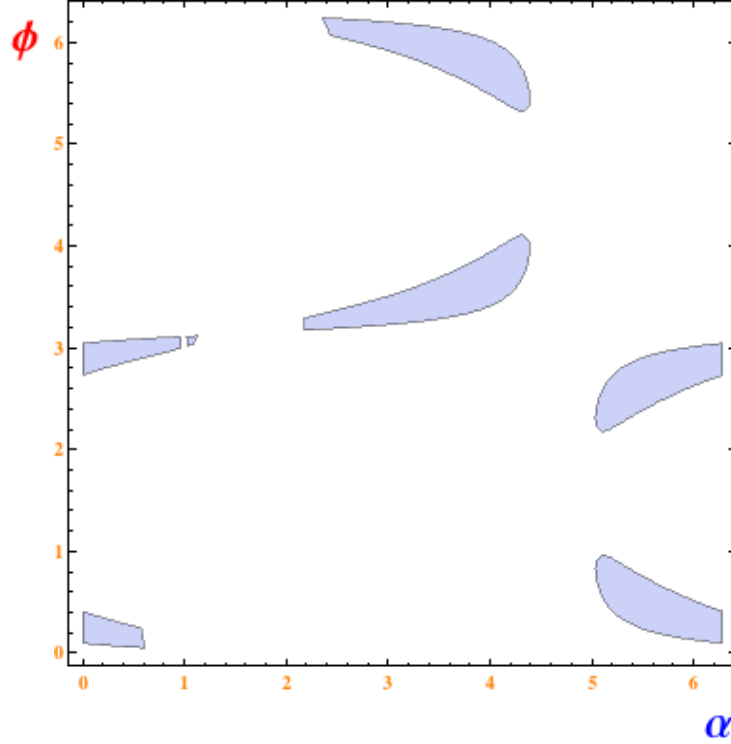


Figure 12: Here the regions of α and ϕ for which the unitary transformed $\rho_{[1,2,3,4]}$ is detected as genuine four particle entangled, by the “(n-1)-tuple state” inequality $I(n-1)$ (eq. (114)) are presented in blue.

In figure 12 we can observe that, for the values $\alpha = 0,42709$ and $\phi = 0,571859$ which maximize the nearest neighbour entanglement, the unitary transformed $\rho_{[1,2,3,4]}$ does not violate the “(n-1)-tuple state” inequality. For the unitary transformed $\rho_{[1,2,3,4]}$ and the values of α and ϕ maximizing nearest neighbour entanglement, the left hand side of the “(n-1)-tuple state” inequality gives $-1,35164$.

We also can observe that, for the values $\alpha = 0,88563$ and $\phi = 0,25066$ which maximize the next nearest neighbour entanglement the unitary transformed $\rho_{[1,2,3,4]}$ does not violate the “(n-1)-tuple state” inequality either. For the unitary transformed $\rho_{[1,2,3,4]}$ and the values of α and ϕ maximizing next nearest neighbour entanglement, the left hand side of the “(n-1)-tuple state” inequality yields $-0,393773$.

So the values of the left hand side of the “(n-1)-tuple state” inequality, achieved by plugging α and ϕ for maximized nearest and maximized next nearest neighbour entanglement into the the unitary transformed $\rho_{[1,2,3,4]}$ are between the minimal value of $-2,04923$ (for $\alpha = 0,334722$ and $\phi = 1,5708$)

and the maximal value of 0,21788 (for $\alpha = 0.395782$ and $\phi = 0.177926$).

We did not eliminate the possibility that there is genuine four particle entanglement in the spin-chain for the parameters that maximize nearest- and next nearest neighbour entanglement, we just showed that for other choices of α and ϕ there exists genuine four particle entanglement in the spin-chain with certainty.

Summary and Outlook

In the last years multipartite entanglement has been studied more and more intensively and has already found applications in quantum cryptography (see e.g. Ref. [5]). Doubtlessly there is a potential for further applications of multipartite entangled states in quantum information processing. Therefore it is of great importance to develop tools with which it is possible to detect if a given multipartite state is entangled. An exceptional form of multipartite entanglement is the so called genuine multipartite entanglement. In the course of the study of entanglement it became apparent, that there are different classes of multipartite entangled states in general and in specific of genuine multipartite entangled states. Until today it is not known how many classes there are for a quantum state consisting of arbitrary many qudits. It is a task at hand to identify these classes and to develop tools to differentiate between these classes.

Two papers (Ref. [P1] and [P2]) emerged during the work on this thesis and were published in scientific journals. In Ref. [P1] we presented a possible classification scheme for genuine multipartite entangled qubit states and introduced three different classes of such states. The advantage of this classification, compared to others (see Ref. [11] - [17]), is that we also developed criteria that are not only able to detect genuine multipartite entanglement but also to distinguish between these three classes, both in an experimental feasible way.

In Ref. [P2] we presented a set of criteria to detect genuine multipartite entanglement, tailored for generalized Dicke states (see Ref. [72]). These criteria have the advantages to previous criteria (see Ref. [6] - [10]) that their noise resistance is extraordinarily high and is growing with system size. Furthermore in all the criteria from Ref. [P1] and [P2], the number of matrix elements that have to be known of a state, to implement a criterion to it, grows polynomially (and not exponentially) with system size and it is possible to express the criteria in terms of local expectation values of pauli matrices, which makes them local realizable. These two properties are very convenient in experiments.

In the end of this thesis (section 3) we deal with the distribution of entanglement along an one dimensional, translational invariant, infinite chain of qubits, that was presented in Ref. [18]. We discover the possibility of next-next nearest neighbour entanglement in the chain, by applying the Peres Horodecki criterion (see section 1.2.2 or Ref. [1] and [2]) to a state of two next-next neighbouring qubits of the chain. By applying the criteria from Ref. [P1] and [P2] to a state of four neighbouring qubits in the chain, we show that genuine four particle entanglement is possible in this translational invariant infinite chain.

Based on the criteria from Ref. [P1] and [P2] and also on the criteria from Ref. [73] four works resulted, that are either already published in scientific journals or have been submitted recently. In Ref. [76] a criteria to detect k -separability (hence also 1-separability or genuine multipartite entanglement) in multipartite states with continuous variables, was developed. A measure for genuine multipartite entanglement was introduced in Ref. [79]. The number of degrees of freedom that takes part at the genuine multipartite entanglement in multipartite qubit states is subject to Ref. [80]. In Ref. [81] it was shown that it is possible to detect genuine multipartite entanglement in many body systems, via the measurement of macroscopic observables.

The works Ref. [P1] and [P2] have been cited e.g. in Ref. [82] - [91].

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Experimentally implementable criteria revealing substructures of genuine multipartite entanglement

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We present a general framework that reveals substructures of genuine multipartite entanglement. Via simple inequalities it is possible to discriminate different sets of multipartite qubit states. These inequalities are beneficial regarding experimental examinations as only local measurements are required. Furthermore, the number of observables scales favorably with system size. In exemplary cases we demonstrate the noise resistance and discuss implementations.

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I. INTRODUCTION

Quantum entanglement has been subject to intense studies in various directions for several decades and has been found in many different physical systems, including systems not consisting of ordinary matter and light (see, e.g., Refs. [1,2]). It gave rise to several concepts for new technologies such as quantum cryptography (see, e.g., Ref. [3]), quantum computation (e.g., Ref. [4]), or quantum teleportation (e.g., Ref. [5]). Despite these efforts, entanglement is still far from being completely understood (for an overview, see, e.g., Ref. [6]) and particularly in multipartite entanglement (see, e.g., Refs. [7,8]) there are many unanswered questions remaining. One of these open questions concerns the problem of classification of multipartite entanglement. This is usually addressed in terms of separability properties (which has been studied, e.g., in Refs. [9–11]) or, in more detail, in terms of density matrix decomposition equivalence classes. The focus of this work is the latter.

While for bipartite entanglement, states can be characterized comparatively easily by means of Schmidt numbers [12], multipartite entangled states have a much more complicated structure, which is only known explicitly for very special cases. For example, for three qubits, there are four distinct classes of states [13,14], namely separable ones, biseparable ones (i.e., states with two entangled particles which are separable from the third), and the two well-known classes of genuinely multipartite entangled states, the Greenberger-Horne-Zeilinger (GHZ) state and the W state (see Ref. [13]). For four qubits there are nine different classes (see Ref. [15]) and beyond that there is no known classification scheme.

Employing and classifying the substructure of (genuine) entanglement is of interest as current experiments are continuously improving in controlling larger and larger multipartite entangled systems, e.g., in quantum optics with photons (see, e.g., Refs. [16–18]) or with ions (see, e.g., Refs. [19,20]) or

in circuit QED of superfluid systems (see, e.g., Refs. [21,22]). However, it is also important to be able to determinate the entanglement class as, e.g., the security of secret sharing protocols rely on them (see Ref. [23]).

In this manuscript, we use the framework introduced in Ref. [24] in order to derive general inequalities, capable of distinguishing between different classes for multipartite qubit systems. So far, classes were usually defined with respect to stochastic local operations and classical communication (SLOCC) equivalence (for details see Refs. [25–29]), which captures whether given states can be converted into each other via stochastic local operations and classical communication. Here we pursue a different approach, defining certain equivalence classes that arise via local unitaries and permutations. This has the advantage that this class definition is Lorentz invariant (see Ref. [30]) and we are able to develop inequalities to distinguish between these classes in an experimentally feasible way. This manuscript is organized as follows: In Sec. II the basic terminology and definitions are introduced, so in Sec. III criteria for discriminating different classes of states can be presented, which is the main result of this work. These criteria are then tested and illustrated for a particular class of states in Sec. IV and thoroughly discussed in Sec. V.

II. CLASSIFYING MULTIPARTITE ENTANGLED STATES

Let us first repeat the definition for a multipartite state to be k -separable [6]. A pure n -partite state $|\Psi\rangle$ is called k -separable if it can be written as a product

$$|\Psi^k\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_k\rangle, \quad k \leq n, \quad (1)$$

of k states $|\phi_i\rangle$, each of which corresponds to a single subsystem or a group of subsystems. If $k = n$, then the state is fully separable. If $k = 1$, i.e., there is no such form with at least two factors, then $|\Psi\rangle$ is 1-separable or genuinely n -partite entangled. For example, the well-known GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and the W state $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ are genuinely multipartite entangled while, e.g., the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|0\rangle$ is 2-separable or biseparable.

For mixed states the definition of k separability is straightforward: A state is k separable if and only if the state can be

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decomposed into pure states

$$\varrho = \sum_i p_i |\Psi_i^k\rangle\langle\Psi_i^k|, \quad p_i > 0 \quad \text{and} \quad \sum_i p_i = 1, \quad (2)$$

wherein all contained pure states are at least k separable and at least one pure state is k separable (i.e., there exists no decomposition with $(k+1)$ -separable states and no $(k-1)$ -separable state is needed in the decomposition). Note that a mixed state can still be partially separable, even if the k subsystems cannot be split into two groups that are not entangled with each other. For example, the following tripartite state is biseparable (2-separable)

$$\varrho = \sum_i p_i \rho_{12}^i \otimes \rho_3^i + \sum_i q_i \rho_{13}^i \otimes \rho_2^i + \sum_i r_i \rho_{23}^i \otimes \rho_1^i$$

with $p_i, q_i, r_i > 0$ and $\sum_i p_i + \sum_j q_j + \sum_k r_k = 1$, (3)

where ρ_{xy}^i are pure entangled bipartite states. Even though there is no bipartite splitting with respect to which the state is separable, it is considered biseparable since it can be prepared through a statistical mixture of bipartite entangled states. The nested structure of k separability is shown in Fig. 1.

For pure states the question of k separability can be answered by common separability criteria for bipartite systems, simply by considering all segmentations of the k -partite system into two parts. However, the same question becomes significantly more difficult to answer for mixed states ϱ . In Ref. [31] separability criteria for arbitrary k in form of simple inequalities have been introduced.

An equivalence class (with respect to density matrix decompositions) is defined as follows:

Definition 1. A class $\mathcal{C}(|\Psi_x^k\rangle)$ is defined as the set of convex mixtures of local unitary equivalents and permutational equivalents of k -separable states $|\psi_j\rangle \in \{|\Psi_x^k\rangle\}$, excluding all higher separable states ($k' > k$) within that set

$$\mathcal{C}(|\Psi_x^k\rangle) := \left\{ \sum_{i,j} p_i U_{\text{loc}}^i \Pi^i |\psi_j\rangle\langle\psi_j| \Pi^{i\dagger} U_{\text{loc}}^{i\dagger} \right\} \setminus \bigcup_{i,k'>k} \mathcal{C}(|\Psi_{x'}^{k'}\rangle). \quad (4)$$

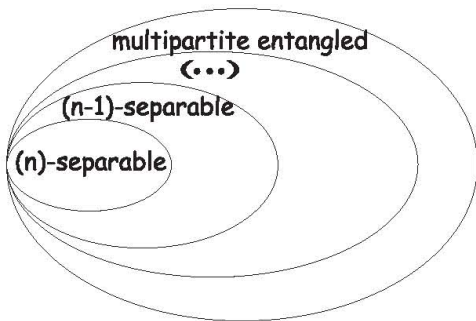


FIG. 1. Here the nested convex structure of k separability in multipartite systems is illustrated.

Here, the index x labels a defining property for the set $\{|\Psi_x^k\rangle\}$, e.g., the tensor rank, as in the following. The $U_{\text{loc}}^i = U_1^i \otimes U_2^i \otimes (\dots) \otimes U_n^i$ are local unitary operators and Π^i are permutation operators exchanging arbitrary subspaces in $|\psi_j\rangle\langle\psi_j|$.

In this article we are interested in all classes of genuinely multipartite entangled states, i.e., $\mathcal{C}(|\Psi_x^1\rangle)$. It is an open question how many of such equivalence classes are needed in general for a decomposition of a given ρ , as in Eq. (2). The task at hand is to identify all possible families of states $\{|\Psi_x^k\rangle\}$, which in a convex sum can build up a given density matrix ρ . For n qubits we can introduce at least three such classes for arbitrary n , which can be distinguished with the mathematical framework we introduce. These three classes are defined via generalizations of the famous GHZ state (see Ref. [32]), the W state (see Ref. [13]), and the Dicke state with two excitations (see Ref. [33]). In order to avoid confusion with the previous definition and to put them into a more general framework we label them according to their respective tensor rank (for more details on the tensor rank of multipartite states see, e.g., Ref. [34]). The above definition also includes all previous classification schemes, depending on the choice of $\{|\Psi_x^k\rangle\}$ (e.g., the classification from Ref. [14] with the two appropriate choices).

Definition 2. We define the class $\mathcal{C}(|\Psi_{(2)}^1\rangle)$ of “double states” for n qubits (a generalization of the GHZ state [32]) via the set

$$\{|\Psi_{(2)}^1\rangle\} := \left\{ |\psi\rangle \in \mathbb{C}^{2^n} : |\psi\rangle = \lambda_1 \bigotimes_{i=1}^n |x_i\rangle + \lambda_2 \bigotimes_{i=1}^n |\bar{x}_i\rangle \right\}, \quad (5)$$

where $|x_i\rangle$ are arbitrary normalized one-qubit states and $\langle\bar{x}_i|x_i\rangle = 0$ such that $|x_i\rangle$ and $|\bar{x}_i\rangle$ form orthonormal bases of each single-qubit system, $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ and $|\lambda_1|^2 + |\lambda_2|^2 = 1$.

Definition 3. We define the class $\mathcal{C}(|\Psi_{(n)}^1\rangle)$ of “ n -tuple states” for n qubits (a generalization of the W state [13]) via the set

$$\{|\Psi_{(n)}^1\rangle\} := \left\{ |\psi\rangle \in \mathbb{C}^{2^n} : |\psi\rangle = \sum_{i=1}^n \lambda_i |W_i\rangle \right\}, \quad (6)$$

where $|W_i\rangle = \bigotimes_{k \neq i} |x_k\rangle \otimes |\bar{x}_i\rangle$ and $\lambda_i \in \mathbb{C} \setminus \{0\}$, satisfying $\sum_i |\lambda_i|^2 = 1$.

Definition 4. We define the class $\mathcal{C}(|\Psi_{(n-1)}^1\rangle)$ of “ $(n-1)$ -tuple states” for n qubits (equivalent to the Dicke state for n qubits [33] with two excitations) via the set

$$\{|\Psi_{(n-1)}^1\rangle\} := \left\{ |\psi\rangle \in \mathbb{C}^{2^n} : |\psi\rangle = \sqrt{\frac{2}{n(n-1)}} \sum_{i < j} |D_{ij}\rangle \right\}, \quad (7)$$

where $|D_{ij}\rangle = \bigotimes_{k \neq i,j} |x_k\rangle \otimes |\bar{x}_i\rangle \otimes |\bar{x}_j\rangle$. In Ref. [34] it was shown that this state has tensor rank $(n-1)$.

Note that our definition of these classes differs from the one introduced in the context of SLOCC equivalence in Ref. [14], where the classes are defined to be nested convex such that $W \subset \text{GHZ}$, whereas $\mathcal{C}(|\Psi_{(n)}^1\rangle) \subset \mathcal{C}(|\Psi_{(2)}^1\rangle)$ is not true. Also, in our definition there are more classes than in Ref. [14],

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since, for example, the most general pure three-qubit state

$$|\Psi\rangle = \lambda_1|000\rangle + \lambda_2|111\rangle + \lambda_3|001\rangle + \lambda_4|010\rangle + \lambda_5|100\rangle \quad (8)$$

 (see also Ref. [35]) is contained in neither of the classes defined above when $|\lambda_i| > 0 \forall i$. This is discussed in more detail in Sec. V.

In the following we will show how we can distinguish our introduced classes of genuine multipartite entanglement by certain inequalities, followed by a section showing how these inequalities are experimentally implementable.

III. DISTINGUISHING CLASSES OF STATES

In the following we present our main results. All inequalities are proven in detail in the Appendix (Sec. VI).

 The “(n)-tuple state” inequality $I(n)$. The inequality

$$\text{Re}\{\langle 0|^{\otimes n} \rho | 1 \rangle^{\otimes n}\} - \alpha(1 - \langle 0|^{\otimes n} \rho | 0 \rangle^{\otimes n} - \langle 1|^{\otimes n} \rho | 1 \rangle^{\otimes n}) \leq 0 \quad (I(n))$$

 is satisfied for all biseparable states and by all states of the class $\mathcal{C}(|\Psi_{(n)}^1\rangle)$, where $\alpha = \frac{3}{2}$ for $n = 3$, $\alpha = 1$ for $n = 4$, and $\alpha = \frac{1}{2}$ for $n > 4$.

 The “double state” inequality $I(2)$. The inequality

$$\begin{aligned} & \text{Re} \left\{ \sum_{i \neq j} \langle w_i | \rho | w_j \rangle + (-1)^{n+1} \langle \bar{w}_i | \rho | \bar{w}_j \rangle \right\} \\ & - (n-2) \sum_i (\langle w_i | \rho | w_i \rangle + \langle \bar{w}_i | \rho | \bar{w}_i \rangle) \\ & - \sum_{i \neq j} (\langle d_{ij} | \rho | d_{ij} \rangle + \langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle) \\ & - \frac{n(n-1)}{2} (\langle 0|^{\otimes n} \rho | 0 \rangle^{\otimes n} + \langle 1|^{\otimes n} \rho | 1 \rangle^{\otimes n}) \leq 0 \quad (I(2)) \end{aligned}$$

 is satisfied for all biseparable states and by all states of the class $\mathcal{C}(|\Psi_{(2)}^1\rangle)$, where $|d_{ij}\rangle = |0\rangle^{\otimes(i-1)} \otimes |1\rangle_i \otimes |0\rangle^{\otimes(j-i-1)} \otimes |1\rangle_j \otimes |0\rangle^{\otimes(n-j)}$, $|w_i\rangle = |0\rangle^{\otimes(i-1)} \otimes |1\rangle_i \otimes |0\rangle^{\otimes(n-i)}$ and an overline denotes orthonormality in all subsystems, e.g., $|\bar{d}_{ij}\rangle = |1\rangle^{\otimes(i-1)} \otimes |0\rangle_i \otimes |1\rangle^{\otimes(j-i-1)} \otimes |0\rangle_j \otimes |1\rangle^{\otimes(n-j)}$.

 The “(n-1)-tuple state” inequality $I(n-1)$. The inequality

$$\begin{aligned} & \text{Re} \sum_{i \neq j} \{\langle w_i | \rho | w_j \rangle\} - (n-2) \sum_i \langle w_i | \rho | w_i \rangle \\ & - (n-2) \sum_{i \neq j} (\langle d_{ij} | \rho | d_{ij} \rangle) - \frac{n(n-1)}{2} (\langle 0|^{\otimes n} \rho | 0 \rangle^{\otimes n}) \leq 0 \quad (I(n-1)) \end{aligned}$$

 is satisfied for all biseparable states and by all states of the class $\mathcal{C}(|\Psi_{(n-1)}^1\rangle)$.

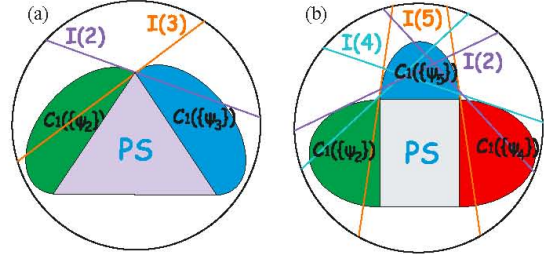
 As these inequalities constitute only necessary conditions, their yield depends strongly on the chosen basis. So in case of unknown input states it is necessary to optimize over all local unitary operators $U = U_1 \otimes U_2 \otimes \dots \otimes U_n$ to obtain optimal results. A parametrization of unitary operators which is suited very well for this task can be found in Ref. [36]. In Fig. 2 we illustrate the substructure of multipartite entangled states, together with the corresponding inequalities, for three- and five-qubit states.


FIG. 2. (Color online) Here the structure of (a) three- and (b) five-qubit mixed states is illustrated. The three entanglement classes $\mathcal{C}(|\Psi_{(2)}^1\rangle)$, $\mathcal{C}(|\Psi_{(3)}^1\rangle)$, and $\mathcal{C}(|\Psi_{(n-1)}^1\rangle)$ of course form convex sets together with the partially separable states, which can be excluded by our inequalities $I(2)$, $I(n)$, and $I(n-1)$ and inequality II from Ref. [24]. The partially separable states are subsumed as PS. The different sets are drawn completely disjoint, although it might still be possible that there is a nonvanishing overlap. Numerical analysis of tripartite systems suggests otherwise, but even if there was, it would not change any of the presented results. Also note that a two-dimensional picture will always fail to incorporate all essential properties of multipartite entanglement. This illustration should facilitate the understanding of how the inequalities work.

IV. EXPERIMENTAL IMPLEMENTATION

 It is crucial for any criteria for large systems to be implementable experimentally without having to resort to quantum state tomography, i.e., in large systems it is beneficial if criteria are examinable without knowing all entries of a density matrix. This stems from the fact that a full state tomography for n qubits requires 2^{2n} measurement settings which for large n is unfeasible. It is also very important that the criteria are locally implementable, as in large systems global measurement operations become more complex and all particles may not be available for global manipulation.

All three criteria to distinguish the three defined subclasses of genuine multipartite entanglement are locally implementable in experiments. This can be shown as our inequalities can be rewritten in terms of local expectation values of Pauli operators. This directly follows from the fact that our inequalities consist of a linear combination of density matrix elements, where each of which can of course be expressed in terms of local expectation values. To that end let us first introduce a compact notation in order to provide the inequalities in an elegant way:

$$i_1 i_2 \dots i_n := (\sigma_{i_1} \otimes \sigma_{i_2} \otimes (\dots) \otimes \sigma_{i_n}) \quad (9)$$

 where $\sigma_1 := \mathbb{1}$.

In this notation the elements of density matrices can be written in a compact way, e.g., for three qubits

$$\text{Re}\{\langle 000 | \rho | 111 \rangle\} = xxx - yyx - yxy - xyy. \quad (10)$$

So for three qubits the “double state” inequality reads

$$(xxx - yyx - yxy - xyy) - 3(3 - zz1 - z1z - 1zz) \leq 0, \quad (11)$$

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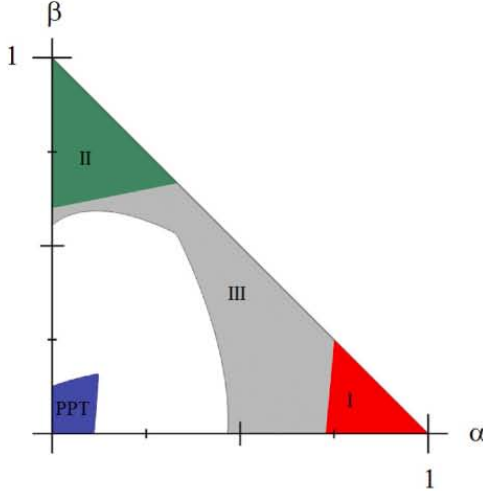
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FIG. 3. (Color online) Here the entanglement classes for the four-qubit state $\rho = \alpha|\phi_1\rangle\langle\phi_1| + \beta|\phi_2\rangle\langle\phi_2| + \frac{1-\alpha-\beta}{16}\mathbb{1}$ are illustrated, with $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \in \mathcal{C}(|\Psi_{(2)}^1\rangle)$ and $|\phi_2\rangle = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle) \in \mathcal{C}(|\Psi_{(4)}^1\rangle)$. For the parameter region I (red) the state is not in $\mathcal{C}(|\Psi_{(2)}^1\rangle)$, for region II (green) it is not in $\mathcal{C}(|\Psi_{(4)}^1\rangle)$, and for region III (gray) it is genuinely multipartite entangled, as detected by the criteria introduced in Ref. [24]. The region labeled PPT (blue) contains all states which are positive under partial transposition for the two distinct bipartitions.

and the “three-tuple-state” inequality yields

$$\begin{aligned} & (1xx + xx1 + x1x + 1yy + y1y + yy1) \\ & - \frac{9}{32}(3 - zz1 - z1z - 1zz) \\ & - \frac{3}{16}(1 - 11z - 1z1 - z11) \leq 0, \end{aligned} \quad (12)$$

so we see that seven local measurement settings are required for the “double state” inequality and 12 for the “triple state” inequality as opposed to the 63 measurement settings required for a full state tomography. In this place we notice some correspondence to the class definition via SLOCC. The witness in Ref. [37] uses the same observables to distinguish between the W and GHZ class (however, in a different combination). The advantage of our criteria is that they generalize for systems with more parties in a straightforward way. In Fig. 3 we illustrate the membership to the various entanglement classes for an example of a family of four-qubit states.

V. DISCUSSION AND CONCLUSION

We have identified three classes of genuinely multipartite entangled states of n qubits and introduced criteria which enable a simple computational and an experimental discrimination between those classes. However, as shown in Refs. [14,15], these classes do not sufficiently characterize the set of all genuinely multipartite entangled states. In order to have a complete characterization one can use criteria to detect genuine multipartite entanglement (such as, e.g., Refs. [9,10,14,24]) to also include the complementing classes.

One possibility to define the complementing class is to write down the most general form of an n -qubit state:

$$|\psi_t\rangle := \sum_{i_1, i_2, \dots, i_n} c_{i_1, i_2, \dots, i_n} |i_1 i_2 \dots i_n\rangle. \quad (13)$$

All n -qubit states are local unitarily equivalent to this state for some choice of $c_{i_1, i_2, \dots, i_n}$. The equivalence class $\mathcal{C}(|\psi_t\rangle)$ contains all $\mathcal{C}(|\psi^1\rangle)$ for some choice of $c_{i_1, i_2, \dots, i_n}$. So if we exclude the parameter choices for the three known classes, i.e., $\mathcal{C}(|\psi_t^1\rangle) \setminus \{\mathcal{C}(|\psi_{(2)}^1\rangle), \mathcal{C}(|\psi_{(n)}^1\rangle), \mathcal{C}(|\psi_{(n-1)}^1\rangle)\}$, we have the complementing set containing all other genuinely multipartite entangled states. It is completely unknown in general how many inequivalent choices for $c_{i_1, i_2, \dots, i_n}$ are possible for n -partite systems, but with our proposed framework it should be possible to design an inequality for any given class.

In summary, we have defined three different classes of genuine multipartite entanglement for n qubits, the double states (a generalization of GHZ states), the (n) -tuple states (a generalization of W states), and then $(n-1)$ -tuple states (a generalization of certain Dicke states) which possess different physical properties and therefore provide different applications. These substructures of genuine entangled states are equivalence classes that arise via local unitaries and permutations. We have presented three simple computable inequalities, each of which is satisfied for all biseparable states and for the class of genuine multipartite entanglement that corresponds to the inequality. Therefore, any violation of the inequality detects that the given state is not of a certain class. This certainly can be used for different applications as, e.g., the success of secure sharing protocols or quantum algorithms rely on certain class of entangled states. We have further shown that all presented criteria can be rewritten by local expectation values, thus are experimentally implementable and require far less measurement settings than full state tomography, i.e., scale favorably with the system size. Last but not least, we want to stress that we presented a framework which may be generalized to any n -partite qudit system, where until now there were no results concerning classification.

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APPENDIX

The general concept behind all upcoming proofs is a calculation of the inequalities for the most general pure state. As all inequalities are convex the validity for mixed states is automatically given. All inequalities consist of the real parts of a certain off-diagonal density matrix elements from which diagonal elements are subtracted. In order to prove the inequalities we will first show that certain real parts of the diagonal elements cancel each other, so we can then use them freely to complete negative squares with the real parts from the off-diagonal elements and thus prove the inequality.

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1. Proof of the “double state” inequality

The “double state” inequality, (I(2)),

$$\begin{aligned} & \sum_{i \neq j} \operatorname{Re}[(\langle w_i | \rho | w_j \rangle + (-1)^{n+1} \langle \bar{w}_i | \rho | \bar{w}_j \rangle)] \\ & - \alpha \sum_i (\langle w_i | \rho | w_i \rangle + \langle \bar{w}_i | \rho | \bar{w}_i \rangle) \\ & - \beta (\langle 00 \cdots 0 | \rho | 00 \cdots 0 \rangle + \langle 11 \cdots 1 | \rho | 11 \cdots 1 \rangle) \\ & - \gamma \sum_{i \neq j} (\langle d_{ij} | \rho | d_{ij} \rangle + \langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle) \leq 0 \end{aligned} \quad (\text{A1})$$

is satisfied for all states of the form

$$|\psi_{(2)}\rangle = \lambda_1 |x_1 x_2 \cdots x_n\rangle + \lambda_2 |y_1 y_2 \cdots y_n\rangle, \quad (\text{A2})$$

where

$$|x_i\rangle = a_i |0\rangle + \bar{a}_i |1\rangle \quad (\text{A3})$$

$$|y_i\rangle = \bar{a}_i^* |0\rangle - a_i^* |1\rangle \quad (\text{A4})$$

 for some $\alpha, \beta, \gamma > 0$.

Proof 1. The individual scalar products yield

$$\langle w_i | \psi_{(2)} \rangle = \lambda_1 \prod_{n \neq i} (a_n) \bar{a}_i - \lambda_2 \prod_{n \neq i} (\bar{a}_n^*) a_i^*, \quad (\text{A5})$$

$$\langle \bar{w}_i | \psi_{(2)} \rangle = \lambda_1 \prod_{n \neq i} (\bar{a}_n) a_i + (-1)^{n+1} \lambda_2 \prod_{n \neq i} (a_n^*) \bar{a}_i^*, \quad (\text{A6})$$

$$\langle d_{ij} | \psi_{(2)} \rangle = \lambda_1 \prod_{n \neq i, j} (a_n) \bar{a}_i \bar{a}_j + \lambda_2 \prod_{n \neq i, j} (\bar{a}_n^*) a_i^* a_j^*, \quad (\text{A7})$$

$$\langle \bar{d}_{ij} | \psi_{(2)} \rangle = \lambda_1 \prod_{n \neq i, j} (a_n) \bar{a}_i \bar{a}_j + (-1)^n \lambda_2 \prod_{n \neq i, j} (\bar{a}_n^*) a_i^* a_j^*, \quad (\text{A8})$$

$$\langle 00 \cdots 0 | \psi_{(2)} \rangle = \lambda_1 \prod_n a_n + \lambda_2 \prod_n \bar{a}_n^*, \quad (\text{A9})$$

$$\langle 11 \cdots 1 | \psi_{(2)} \rangle = \lambda_1 \prod_n \bar{a}_n + (-1)^n \lambda_2 \prod_n a_n^*, \quad (\text{A10})$$

which results in

$$\begin{aligned} \langle w_i | \psi_{(2)} \rangle \langle \psi_{(2)} | w_j \rangle &= \underbrace{|\lambda_1|^2 \prod_{n \neq i, j} (|a_n|^2) \bar{a}_i a_i^* \bar{a}_j a_j^*}_{\mathcal{W}_{ij}^0} + \underbrace{|\lambda_2|^2 \prod_{n \neq i, j} (|\bar{a}_n|^2) a_i^* \bar{a}_i \bar{a}_j^* a_j}_{\mathcal{W}_{ij}^1} \\ &- \left(\lambda_1 \lambda_2^* \prod_{n \neq i, j} (\bar{a}_n a_n) \bar{a}_i^2 a_j^2 + \lambda_1^* \lambda_2 \prod_{n \neq i, j} (\bar{a}_n^* a_n^*) a_i^{*2} \bar{a}_j^{*2} \right) \end{aligned} \quad (\text{A11})$$

and

$$\begin{aligned} \langle \bar{w}_i | \psi_{(2)} \rangle \langle \psi_{(2)} | \bar{w}_j \rangle &= \underbrace{|\lambda_1|^2 \prod_{n \neq i, j} (|\bar{a}_n|^2) \bar{a}_i^* a_i \bar{a}_j^* a_j}_{\bar{\mathcal{W}}_{ij}^0} + \underbrace{|\lambda_2|^2 \prod_{n \neq i, j} (|a_n|^2) \bar{a}_i a_i^* \bar{a}_j^* a_j}_{\bar{\mathcal{W}}_{ij}^1} \\ &+ (-1)^{n+1} \left(\lambda_1 \lambda_2^* \prod_{n \neq i, j} (\bar{a}_n a_n) a_i^2 \bar{a}_j^2 + \lambda_1^* \lambda_2 \prod_{n \neq i, j} (\bar{a}_n^* a_n^*) \bar{a}_i^{*2} a_j^{*2} \right) \end{aligned} \quad (\text{A12})$$

and

$$|\langle w_i | \psi_{(2)} \rangle|^2 = |\lambda_1|^2 \prod_{n \neq i} (|a_n|^2) |\bar{a}_i|^2 + |\lambda_2|^2 \prod_{n \neq i} (|\bar{a}_n|^2) |a_i|^2 - 2 \operatorname{Re} \left[\lambda_1 \lambda_2^* \prod_n (\bar{a}_n a_n) \right] \quad (\text{A13})$$

$$|\langle \bar{w}_i | \psi_{(2)} \rangle|^2 = |\lambda_1|^2 \prod_{n \neq i} (|\bar{a}_n|^2) |a_i|^2 + |\lambda_2|^2 \prod_{n \neq i} (|a_n|^2) |\bar{a}_i|^2 + (-1)^{n+1} 2 \operatorname{Re} \left[\lambda_1 \lambda_2^* \prod_n (\bar{a}_n a_n) \right] \quad (\text{A14})$$

and

$$|\langle 00 \cdots 0 | \psi_{(2)} \rangle|^2 = \underbrace{|\lambda_1|^2 \prod_n (|a_n|^2)}_{|M_0|^2} + \underbrace{|\lambda_2|^2 \prod_n (|\bar{a}_n|^2)}_{|M_1|^2} + 2 \operatorname{Re} \left[\lambda_1 \lambda_2^* \prod_n (a_n \bar{a}_n) \right] \quad (\text{A15})$$

$$|\langle 11 \cdots 1 | \psi_{(2)} \rangle|^2 = \underbrace{|\lambda_1|^2 \prod_n (|\bar{a}_n|^2)}_{|\bar{M}_0|^2} + \underbrace{|\lambda_2|^2 \prod_n (|a_n|^2)}_{|\bar{M}_1|^2} + (-1)^n 2 \operatorname{Re} \left[\lambda_1 \lambda_2^* \prod_n (a_n \bar{a}_n) \right] \quad (\text{A16})$$

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and, finally,

$$|\langle d_{ij} | \psi_{(2)} \rangle|^2 = |\lambda_1|^2 \underbrace{\prod_{n \neq i,j} (|a_n|^2) |\bar{a}_i|^2 |\bar{a}_j|^2}_{|D_{ij}^0|^2} + |\lambda_2|^2 \underbrace{\prod_{n \neq i,j} (|\bar{a}_n|^2) |a_i|^2 |a_j|^2}_{|D_{ij}^1|^2} + 2\text{Re} \left[\lambda_1 \lambda_2^* \prod_n (\bar{a}_n a_n) \right] \quad (\text{A17})$$

$$|\langle \bar{d}_{ij} | \psi_{(2)} \rangle|^2 = |\lambda_1|^2 \underbrace{\prod_{n \neq i,j} (|\bar{a}_n|^2) |a_i|^2 |a_j|^2}_{|\bar{D}_{ij}^0|^2} + |\lambda_2|^2 \underbrace{\prod_{n \neq i,j} (|a_n|^2) |\bar{a}_i|^2 |\bar{a}_j|^2}_{|\bar{D}_{ij}^1|^2} + (-1)^n 2\text{Re} \left[\lambda_1 \lambda_2^* \prod_n (\bar{a}_n a_n) \right]. \quad (\text{A18})$$

 This results for the real parts $R = 2\text{Re}[\lambda_1 \lambda_2^* \prod_n (\bar{a}_n a_n)]$ in

$$-n\alpha R + n\alpha(-1)^{n+1}R + \beta R + (-1)^n \beta R + n(n-1)\gamma R + (-1)^n n(n-1)\gamma R \quad (\text{A19})$$

$$= R[n\alpha[(-1)^{n+1} - 1] + \beta[1 + (-1)^n] + n(n-1)\gamma[1 + (-1)^n]], \quad (\text{A20})$$

 which is zero for odd n . For even n and the choice $\alpha = \frac{n-2}{2}$, $\beta = \frac{n(n-2)}{4}$, and $\gamma = \frac{n-2}{4(n-1)}$ it yields

$$= R \left[n \frac{n-2}{2} (-2) + \frac{n(n-2)}{4} (2) + n(n-1) \frac{n-2}{4(n-1)} (2) \right] = 0. \quad (\text{A21})$$

Using

$$\begin{aligned} & \text{Re}[\langle w_i | \psi_{(2)} \rangle \langle \psi_{(2)} | w_j \rangle] - \frac{n-2}{2(n-1)} (\langle w_i | \psi_{(2)} \rangle \langle \psi_{(2)} | w_i \rangle + \langle w_j | \psi_{(2)} \rangle \langle \psi_{(2)} | w_j \rangle) \\ &= \frac{1}{n-1} \text{Re}[\langle w_i | \psi_{(2)} \rangle \langle \psi_{(2)} | w_j \rangle] - \underbrace{\frac{n-2}{2(n-1)} |\langle w_i | \psi_{(2)} \rangle - \langle w_j | \psi_{(2)} \rangle|^2}_{\bar{X}_{ij}} \end{aligned} \quad (\text{A22})$$

and

$$\begin{aligned} & \text{Re}[\langle \bar{w}_i | \psi_{(2)} \rangle \langle \psi_{(2)} | \bar{w}_j \rangle] - \frac{n-2}{2(n-1)} (\langle \bar{w}_i | \psi_{(2)} \rangle \langle \psi_{(2)} | \bar{w}_i \rangle + \langle \bar{w}_j | \psi_{(2)} \rangle \langle \psi_{(2)} | \bar{w}_j \rangle) \\ &= \frac{1}{n-1} \text{Re}[\langle \bar{w}_i | \psi_{(2)} \rangle \langle \psi_{(2)} | \bar{w}_j \rangle] - \underbrace{\frac{n-2}{2(n-1)} |\langle \bar{w}_i | \psi_{(2)} \rangle - \langle \bar{w}_j | \psi_{(2)} \rangle|^2}_{\bar{X}_{ij}} \end{aligned} \quad (\text{A23})$$

the inequality now reads

$$\begin{aligned} & \sum_{i \neq j} \left\{ \frac{1}{n-1} \text{Re}[W_{ij}^0 + W_{ij}^1 + (-1)^{n+1}(\bar{W}_{ij}^0 + \bar{W}_{ij}^1)] - (X_{ij} + \bar{X}_{ij}) - \frac{n-2}{4(n-1)} (|D_{ij}^0|^2 + |D_{ij}^1|^2 + |\bar{D}_{ij}^0|^2 + |\bar{D}_{ij}^1|^2) \right\} \\ & - \frac{n(n-2)}{4} (|M^0|^2 + |M^1|^2 + |\bar{M}^0|^2 + |\bar{M}^1|^2) \leq 0 \end{aligned} \quad (\text{A24})$$

or, equivalently,

$$\begin{aligned} & \sum_{i \neq j} \frac{1}{n-1} \left(\text{Re}[W_{ij}^0] - \frac{n-2}{4} |D_{ij}^0|^2 - \frac{n-2}{4} |M^0|^2 \right) \\ & \sum_{i \neq j} \frac{1}{n-1} \left(\text{Re}[W_{ij}^1] - \frac{n-2}{4} |D_{ij}^1|^2 - \frac{n-2}{4} |M^1|^2 \right) \\ & \sum_{i \neq j} \frac{1}{n-1} \left(\text{Re}[\bar{W}_{ij}^0] - \frac{n-2}{4} |\bar{D}_{ij}^0|^2 - \frac{n-2}{4} |\bar{M}^0|^2 \right) \\ & \sum_{i \neq j} \frac{1}{n-1} \left(\text{Re}[\bar{W}_{ij}^1] - \frac{n-2}{4} |\bar{D}_{ij}^1|^2 - \frac{n-2}{4} |\bar{M}^1|^2 \right) \leq 0. \end{aligned} \quad (\text{A25})$$

Now we can use

$$|M^0 - D_{ij}^0|^2 = |M^0|^2 + |D_{ij}^0|^2 - 2\text{Re}[W_{ij}^0], \quad (\text{A26})$$

$$|M^1 - D_{ij}^1|^2 = |M^1|^2 + |D_{ij}^1|^2 - 2\text{Re}[W_{ij}^1], \quad (\text{A27})$$

$$|\bar{M}^0 + (-1)^{n+1} \bar{D}_{ij}^0|^2 = |\bar{M}^0|^2 + |\bar{D}_{ij}^0|^2 + (-1)^{n+1} 2\text{Re}[\bar{W}_{ij}^0], \quad (\text{A28})$$

$$|\bar{M}^1 + (-1)^{n+1} \bar{D}_{ij}^1|^2 = |\bar{M}^1|^2 + |\bar{D}_{ij}^1|^2 + (-1)^{n+1} 2\text{Re}[\bar{W}_{ij}^1], \quad (\text{A29})$$

 (A25) which proves the inequality for $n \geq 4$. ■

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Corollary 2.

$$\begin{aligned} & \sum_{i \neq j} \text{Re}[(\langle w_i | \rho | w_j \rangle + (-1)^{n+1} \langle \bar{w}_i | \rho | \bar{w}_j \rangle)] \\ & - \alpha \sum_i (\langle w_i | \rho | w_i \rangle + \langle \bar{w}_i | \rho | \bar{w}_i \rangle) \\ & - \beta (\langle 00 \dots 0 | \rho | 00 \dots 0 \rangle + \langle 11 \dots 1 | \rho | 11 \dots 1 \rangle) \\ & - \gamma \sum_{i \neq j} (\langle d_{ij} | \rho | d_{ij} \rangle + \langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle) \leq 0 \end{aligned} \quad (\text{A30})$$

 is satisfied for all biseparable states for some $\alpha, \beta, \gamma > 0$.

Proof. The following inequality is satisfied by all biseparable states (as proven in Ref. [24]):

$$\begin{aligned} & \sum_{i \neq j} \text{Re}[\langle w_i | \rho | w_j \rangle] - \sum_{i \neq j} \sqrt{\langle 00 \dots 0 | \rho | 00 \dots 0 \rangle \langle d_{ij} | \rho | d_{ij} \rangle} \\ & - (n-2) \sum_i \langle w_i | \rho | w_i \rangle \leq 0 \end{aligned} \quad (\text{A31})$$

or, equivalently,

$$\begin{aligned} & \sum_{i \neq j} \text{Re}[\langle \bar{w}_i | \rho | \bar{w}_j \rangle] - \sum_{i \neq j} \sqrt{\langle 11 \dots 1 | \rho | 11 \dots 1 \rangle \langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle} \\ & - (n-2) \sum_i \langle \bar{w}_i | \rho | \bar{w}_i \rangle \leq 0 \end{aligned} \quad (\text{A32})$$

and

$$\begin{aligned} & - \sum_{i \neq j} \text{Re}[\langle \bar{w}_i | \rho | \bar{w}_j \rangle] - \sum_{i \neq j} \sqrt{\langle 11 \dots 1 | \rho | 11 \dots 1 \rangle \langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle} \\ & - (n-2) \sum_i \langle \bar{w}_i | \rho | \bar{w}_i \rangle \leq 0. \end{aligned} \quad (\text{A33})$$

Using

$$\begin{aligned} & \sqrt{\langle 00 \dots 0 | \rho | 00 \dots 0 \rangle \langle d_{ij} | \rho | d_{ij} \rangle} \\ & \leq \frac{1}{2} (\langle 00 \dots 0 | \rho | 00 \dots 0 \rangle + \langle d_{ij} | \rho | d_{ij} \rangle) \end{aligned} \quad (\text{A34})$$

we arrive at

$$\begin{aligned} & \sum_{i \neq j} \text{Re}[\langle w_i | \rho | w_j \rangle + (-1)^{n+1} \langle \bar{w}_i | \rho | \bar{w}_j \rangle] - \sum_{i \neq j} (\langle d_{ij} | \rho | d_{ij} \rangle + \langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle) - (n-2) \sum_i (\langle w_i | \rho | w_i \rangle + \langle \bar{w}_i | \rho | \bar{w}_i \rangle) \\ & - \frac{n(n-1)}{2} (\langle 00 \dots 0 | \rho | 00 \dots 0 \rangle + \langle 11 \dots 1 | \rho | 11 \dots 1 \rangle) \leq 0, \end{aligned} \quad (\text{A35})$$

 where we have to choose $\alpha = (n-2)$, $\beta = \frac{n(n-1)}{2}$, and $\gamma = 1$.

Corollary 3.

$$\begin{aligned} & \sum_{i \neq j} \text{Re}[(\langle w_i | \rho | w_j \rangle + (-1)^{n+1} \langle \bar{w}_i | \rho | \bar{w}_j \rangle)] - \alpha \sum_i (\langle w_i | \rho | w_i \rangle + \langle \bar{w}_i | \rho | \bar{w}_i \rangle) \\ & - \beta (\langle 00 \dots 0 | \rho | 00 \dots 0 \rangle + \langle 11 \dots 1 | \rho | 11 \dots 1 \rangle) - \gamma \sum_{i \neq j} (\langle d_{ij} | \rho | d_{ij} \rangle + \langle \bar{d}_{ij} | \rho | \bar{d}_{ij} \rangle) \leq 0 \end{aligned} \quad (\text{A36})$$

 is satisfied by both biseparable states and $\mathcal{C}(|\Psi_{\mathcal{D}}^1\rangle)$ for $\alpha = (n-2)$, $\beta = \frac{n(n-1)}{2}$ and $\gamma = 1$.

 Proof. It is evident that inequality (A36) is satisfied for all biseparable states as proven in Proof 2. As α , β , and γ are larger than what is needed for all “double states,” we can conclude that subtracting even more positive terms does not change the validity of inequality (A36) for these states.

2. Proof of the “n-tuple state” inequality

The “n-tuple state” inequality (I(n)) is satisfied by all states of the form

$$|\psi_{(n)}\rangle := \sum_i \lambda_i |W_i\rangle, \quad (\text{A37})$$

where

$$|W_i\rangle := |x_1 x_2 \dots x_{i-1} y_i x_{i+1} \dots x_n\rangle \quad (\text{A38})$$

and

$$\begin{aligned} |x_i\rangle &= a_i |0\rangle + \bar{a}_i |1\rangle \\ |y_i\rangle &= \bar{a}_i^* |0\rangle - a_i^* |1\rangle. \end{aligned} \quad (\text{A39})$$

Proof 2. First observe that

$$\begin{aligned} \langle 0 |^{\otimes n} |\psi_{(n)}\rangle &= \sum_i \lambda_i \bar{a}_i^* \left(\prod_{n \neq i} a_n \right), \\ \langle 1 |^{\otimes n} |\psi_{(n)}\rangle &= - \sum_i \lambda_i a_i^* \left(\prod_{n \neq i} \bar{a}_n \right), \end{aligned} \quad (\text{A40})$$

 such that we obtain for the first term of the “n-tuple state” inequality with $\rho = \sum_{i,j} \lambda_i \lambda_j^* |W_i\rangle \langle W_j|$

$$\begin{aligned} & \text{Re}[\langle 0 |^{\otimes n} \rho | 1 \rangle^{\otimes n}] \\ &= -\text{Re} \left[\prod_n a_n \bar{a}_n^* + \sum_{i \neq j} \lambda_i \lambda_j^* \bar{a}_i^* a_j^2 \left(\prod_{n \neq i,j} a_n \bar{a}_n^* \right) \right]. \end{aligned} \quad (\text{A41})$$

 Let us introduce an index set $m_k = \{i_1 i_2 \dots i_k\}$ and denote the complement by m_k^C . We can then with

$$|d_{m_k}\rangle := |0\rangle_{m_k^C}^{\otimes n-k} |1\rangle_{m_k}^{\otimes k} \quad (\text{A42})$$

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 rewrite the second term of the “ n -tuple state” inequality by

$$\begin{aligned}
 & 1 - \langle 0|^{\otimes n} \rho |0\rangle^{\otimes n} - \langle 1|^{\otimes n} \rho |1\rangle^{\otimes n} \\
 &= \sum_{k=1}^{n-1} \sum_{m_k} \langle d_{m_k} | \rho | d_{m_k} \rangle \\
 &= \sum_{k=1}^{n-1} \sum_{m_k} \sum_{i,j} \lambda_i \lambda_j^* \langle d_{m_k} | W_i \rangle \langle W_j | d_{m_k} \rangle, \quad (\text{A43})
 \end{aligned}$$

 i.e., we have to calculate all diagonal terms minus the first and last one. We have two different cases ($m_k/i \dots$ denotes the index set m_k without the index i)

$$\langle d_{m_k} | W_i \rangle = \begin{cases} -a_i^* \prod_{n \in m_k^c} (a_n) \prod_{l \in m_k/i} (\bar{a}_l), & \text{if } i \in m_k \\ \bar{a}_i^* \prod_{n \in m_k^c/i} (a_n) \prod_{l \in m_k} (\bar{a}_l), & \text{if } i \notin m_k \end{cases} \quad (\text{A44})$$

and therefore four possibilities for the products

$$\langle W_j | d_{m_k} \rangle \langle d_{m_k} | W_i \rangle = \begin{cases} \prod_{n \in m_k^c} (|a_n|^2) \prod_{l \in m_k/i,j} (|\bar{a}_l|^2) a_i^* a_j \bar{a}_i^* \bar{a}_j, & \text{if } i \in m_k, j \in m_k \\ -\prod_{n \in m_k^c/j} (|a_n|^2) \prod_{l \in m_k/i} (|\bar{a}_l|^2) a_i^* a_j \bar{a}_i^* \bar{a}_j, & \text{if } i \in m_k, j \notin m_k \\ -\prod_{n \in m_k^c/i} (|a_n|^2) \prod_{l \in m_k/j} (|\bar{a}_l|^2) a_i^* a_j \bar{a}_i^* \bar{a}_j, & \text{if } i \notin m_k, j \in m_k \\ \prod_{n \in m_k^c/i,j} (|a_n|^2) \prod_{l \in m_k} (|\bar{a}_l|^2) a_i^* a_j \bar{a}_i^* \bar{a}_j, & \text{if } i \notin m_k, j \notin m_k \end{cases} \quad (\text{A45})$$

We expect that certain real parts cancel each other; in detail, we find that the following relation holds

$$X_{m_k}^{ij} := \langle W_j | d_{m_k} \rangle \langle d_{m_k} | W_i \rangle = -\langle W_j | d_{m'_k} \rangle \langle d_{m'_k} | W_i \rangle \quad (\text{A46})$$

if

$$k' = k + 1 \quad (\text{A47})$$

and

$$m_k \cup m'_{k+1} \setminus m_k \cap m'_{k+1} = \{i\}. \quad (\text{A48})$$

 This can be proven explicitly by computing the expressions. Now we can complete all corresponding $|\langle W_j | d_{m_k} \rangle|^2 + |\langle W_i | d_{m_k} \rangle|^2$ to complete negative squares.

Explicitly we observe

$$\sum_i |\lambda_i|^2 |\langle d_{[i]} | W_i \rangle|^2 = \prod_n |a_n|^2 \quad (\text{A49})$$

$$\sum_i |\lambda_i|^2 |\langle d_{[i]^c} | W_i \rangle|^2 = \prod_n |\bar{a}_n|^2 \quad (\text{A50})$$

 such that for the case $i = j$ we obtain the following negative square for the inequality under investigation

$$\begin{aligned}
 & -\sum_i |\lambda_i|^2 \text{Re}[\langle 0|^{\otimes n} W_i \rangle \langle W_i | 1 \rangle^{\otimes n}] - \alpha \left(\sum_i |\lambda_i|^2 |\langle d_{[i]} | W_i \rangle|^2 + \sum_i |\lambda_i|^2 |\langle d_{[i]^c} | W_i \rangle|^2 \right) \\
 &= -\text{Re} \left[\prod_n a_n \bar{a}_n^* \right] - \alpha \left(\prod_n |a_n|^2 + \prod_n |\bar{a}_n|^2 \right) = -\frac{1}{2} \left| \prod_n a_n + \prod_n \bar{a}_n \right|^2 \leq 0, \quad (\text{A51})
 \end{aligned}$$

 where the last equation holds for $\alpha = \frac{1}{2}$. Thus we have proven that all terms with $i = j$ are negative.

 For the terms $i \neq j$ we can proceed in a similar way

$$\begin{aligned}
 & \sum_{i,j} (-\text{Re}[\lambda_i \lambda_j^* \langle 0|^{\otimes n} W_i \rangle \langle W_j | 1 \rangle^{\otimes n}] - \alpha (|\lambda_i|^2 |\langle d_{[ij]} | W_i \rangle|^2 + |\lambda_j|^2 |\langle d_{[ij]^c} | W_j \rangle|^2)) \\
 &= \sum_{i,j} \left\{ -\text{Re} \left[\lambda_i \lambda_j^* \bar{a}_i^2 a_j^2 \left(\prod_{n \neq i,j} a_n \bar{a}_n^* \right) \right] - \alpha \left(|\lambda_i|^2 \left| (-) a_i^* \bar{a}_j \prod_{n \neq i,j} a_n \right|^2 + |\lambda_j|^2 \left| a_i \bar{a}_j^* \prod_{n \neq i,j} \bar{a}_n \right|^2 \right) \right\} \\
 &= \sum_{i,j} \left[-\frac{1}{2} \left| \lambda_i \bar{a}_i^* a_j \left(\prod_{n \neq i,j} a_n \right) + \lambda_j \bar{a}_j^* a_i \left(\prod_{n \neq i,j} \bar{a}_n \right) \right|^2 \right] \leq 0, \quad (\text{A52})
 \end{aligned}$$

 where the last equation holds for $\alpha = \frac{1}{2}$ (note that as the sum goes over all i and j in any sum the role of i, j can be interchanged). Now we can combine both cases and the proof is complete. However, note that in the case $n = 4$ the set $\{d_{[ij]}\}$ and its complement are identical such that terms used to complete the negative squares for the case $i = j$ are

 not available for the case $i \neq j$, this we can compensate by choosing $\alpha = 1$. Thus we have proven that the “ n -tuple state” inequality for all n with $\alpha = \frac{1}{2}$ except for $n = 4$, where one has to choose $\alpha = 1$ holds for any state of the set $\mathcal{C}(\{|\Psi_n^1\rangle\})$.

That the inequality holds for biseparable states follows from inequality (II) presented in Ref. [24]. ■

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3. Proof of the “(n-1)-tuple state” inequality

The inequality

$$\begin{aligned} & \sum_{i \neq j} \operatorname{Re}[(\langle w_i | \rho | w_j \rangle) - (n-2) \sum_i (\langle w_i | \rho | w_i \rangle)] \\ & - \frac{n(n-1)}{2} (\langle 00 \dots 0 | \rho | 00 \dots 0 \rangle) \\ & - (n-2) \sum_{i \neq j} (\langle d_{ij} | \rho | d_{ij} \rangle) \leq 0 \end{aligned} \quad (\text{A53})$$

is satisfied for all states of the form

$$|\psi_{(n-1)}\rangle = \sqrt{\frac{2}{n(n-1)}} \sum_{i < j} |D_{ij}\rangle, \quad (\text{A54})$$

where

$$|D_{ij}\rangle = \bigotimes_{k \neq i,j} |x_k\rangle \otimes |y_i\rangle \otimes |y_j\rangle \quad (\text{A55})$$

$$|x_i\rangle = a_i|0\rangle + \bar{a}_i|1\rangle \quad (\text{A56})$$

$$|y_i\rangle = \bar{a}_i^*|0\rangle - a_i^*|1\rangle. \quad (\text{A57})$$

Proof 3. To prove the inequality we first take a look at the individual products

$$\langle w_i | D_{xy} \rangle = \begin{cases} \bar{a}_i \bar{a}_x \bar{a}_y \prod_{k \neq i,x,y} a_k, & \text{if } i \neq x, y \\ -a_x^* \bar{a}_y \prod_{k \neq x,y} a_k, & \text{if } i = x \\ -\bar{a}_x^* a_y \prod_{k \neq x,y} a_k, & \text{if } i = y \end{cases} \quad (\text{A58})$$

and

$$\langle d_{ij} | D_{xy} \rangle = \begin{cases} \bar{a}_i \bar{a}_j \bar{a}_x \bar{a}_y \prod_{k \neq i,j,x,y} a_k, & \text{if } i, j \neq x, y \\ -\bar{a}_j a_x \bar{a}_y \prod_{k \neq j,x,y} a_k, & \text{if } i = x, j \neq y \\ -\bar{a}_i a_x \bar{a}_y \prod_{k \neq i,x,y} a_k, & \text{if } i = y, j \neq x \\ -\bar{a}_i \bar{a}_x a_y \prod_{k \neq i,x,y} a_k, & \text{if } i \neq x, j = x \\ -\bar{a}_i \bar{a}_x a_y \prod_{k \neq i,x,y} a_k, & \text{if } i \neq x, j = y \\ a_x^* a_y \prod_{k \neq x,y} a_k, & \text{if } i = x, j = y \end{cases} \quad (\text{A59})$$

Now the two products

$$|\langle \psi_{(n-1)} | w_i \rangle|^2 = \left| \sum_{x < y} \langle w_i | D_{xy} \rangle \right|^2 \quad (\text{A60})$$

and

$$|\langle \psi_{(n-1)} | d_{ij} \rangle|^2 = \left| \sum_{x < y} \langle d_{ij} | D_{xy} \rangle \right|^2 \quad (\text{A61})$$

again have real parts which cancel each other out:

$$\langle d_{ij} | D_{xy} \rangle \langle D_{x'y'} | d_{ij} \rangle = -\langle w_k | d_{mn} \rangle \langle d_{m'n'} | w_k \rangle \quad (\text{A62})$$

if

$$\{ijjxyx'y'\} \setminus \{ \{ \{ij\} \cap \{xy\} \} \cup \{ \{ij\} \cap \{x'y'\} \} \} \quad (\text{A63})$$

$$= \{kkmm'm'n'\} \setminus \{ \{ \{k\} \cap \{mn\} \} \cup \{ \{k\} \cap \{m'n'\} \} \}, \quad (\text{A64})$$

from which follows that the according squared terms may be used to complete negative squares with the first part of the inequality. The condition reads

$$\operatorname{Re}[\langle w_i | d_{xy} \rangle \langle d_{x'y'} | w_j \rangle] - \frac{1}{2} (|\langle d_{ab} | D_{mn} \rangle|^2 + |\langle d_{ab} | D_{m'n'} \rangle|^2) \quad (\text{A65})$$

$$= -|\langle d_{ab} | D_{mn} \rangle + \langle d_{ab} | D_{m'n'} \rangle|^2 \quad (\text{A66})$$

when

$$\{ijjxyx'y'\} \setminus \{ \{ \{i\} \cap \{xy\} \} \cup \{ \{j\} \cap \{x'y'\} \} \} \quad (\text{A67})$$

$$= \{abcdmnm'n'\} \setminus \{ \{ \{ab\} \cap \{mn\} \} \cup \{ \{cd\} \cap \{m'n'\} \} \} \quad (\text{A68})$$

also, where the appropriate terms are not available one can use the corresponding $|\langle w_i | D_{xy} \rangle|^2$ from the inequality. By explicit, yet cumbersome, calculation one can in a straightforward way show that the inequality is indeed satisfied for all states of the class $\mathcal{C}(|\Psi_{(n-1)}^1\rangle)$, analogously to the previous proofs. As this inequality is just the linearized version of inequality (III) from Ref. [24] it also holds for all biseparable states. ■

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Experimentally feasible set of criteria detecting genuine multipartite entanglement in n -qubit Dicke states and in higher-dimensional systems

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We construct a set of criteria detecting genuine multipartite entanglement in arbitrary dimensional multipartite systems. These criteria are optimally suited for detecting multipartite entanglement in n -qubit Dicke states with m excitations, as shown in exemplary cases. Furthermore, they can be employed to detect multipartite entanglement in different states related to quantum cloning, decoherence-free communication, and quantum secret sharing. In a detailed analysis, we show that the criteria are also more robust to noise than any other criterion known so far, especially with increasing system size. Furthermore, it is shown that the number of required local observables scales only polynomially with size, thus making the criteria experimentally feasible.

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The feature of many-body entanglement has recently been recognized as a fundamental property of a broad variety of systems. It appears in quantum phase transitions (e.g., [1]) and ionization procedures (e.g., [2]). Even biological systems have raised questions as to whether multipartite entanglement might be responsible for their astonishing transport efficiency (e.g., [3,4]). There are already numerous examples of possible applications of this quantum feature. In quantum information processing, it facilitates quantum computation (e.g., [5]), enables multiparty cryptography (e.g., [6,7]), and also plays a fundamental role in many popular quantum algorithms (e.g., [8]).

In contrast to the well-studied bipartite case (for this, see, e.g., [9]), the structure of multipartite entanglement is far more intricate. Recently, tools have been developed to find out whether a given quantum state is multipartite entangled (see, e.g., [10–14]). In multipartite systems, it is of high importance that criteria detecting multipartite entanglement are experimentally accessible without having to resort to a full state tomography since that would require an unfeasible number of measurements. For multipartite qubit systems, there was some recent progress in locally measurable criteria in Refs. [15,16]. These were further developed for two types of multipartite qubit states in Ref. [17]. In multipartite systems, many different classes of genuinely multipartite entangled states exist (see, e.g., Refs. [18–24]). The previously introduced criteria only detect a very limited number of such classes to be genuinely multipartite entangled. In Refs. [25,26], we have provided a general framework for constructing experimentally feasible criteria in higher-dimensional systems. In this Rapid Communication, we introduce a complete set of criteria optimally suited for all n -qubit Dicke states (originally introduced in Ref. [27]) with an arbitrary number of excitations. In addition, we show that these criteria can also be employed to detect different multipartite entangled states related to quantum secret sharing [28], decoherence-free quantum communication [29], and quantum cloning [30].

For this purpose, let us first briefly review the definition of genuine multipartite entanglement in discrete Hilbert spaces. State vectors $|\psi\rangle$, which can be written as a tensor product

$$|\psi_{A|B}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \quad (1)$$

with respect to some bipartition $A|B$, are called biseparable. State vectors that are not biseparable with respect to *any* bipartition are genuinely multipartite entangled. This generalizes for mixed states in a straightforward way. Any state ρ which can be decomposed into a convex sum of biseparable pure states

$$\rho = \sum_i p_i |\psi_{A_i|B_i}\rangle \langle \psi_{A_i|B_i}| \quad (2)$$

is biseparable. Again, any state that is not biseparable is called genuinely multipartite entangled. Note, however, that the elements of the pure state decomposition need not be biseparable with respect to the same bipartition (i.e., $A_i|B_i \neq A_j|B_j$ is permitted in such decompositions). Therefore, a multipartite state can be biseparable without there existing a bipartition with respect to which it is separable. This gives rise to the difficulty of detecting genuine multipartite entanglement in mixed states. Whereas in the pure state case it is sufficient to check for biseparability with respect to certain bipartitions (e.g., with the Peres-Horodecki criterion [9]), this is no longer possible and new tools for detecting entanglement need to be derived.

Famous examples of multipartite entangled states are the GHZ (Greenberger, Horne, and Zeilinger) states and the W state, for which the first developed criteria were optimally suited (see Refs. [17,25] for details). However, in multibody quantum physics, there exist a vast and still unknown number of types of multipartite entangled states. One prominent family of multipartite entangled states are the so-called Dicke states. They were first studied in the context of coherent light emission (see Ref. [27]). In recent experiments, various different Dicke states were created (see, e.g., Refs. [31–34]), and different tools were used to show that indeed a genuinely multipartite state has been created (see, e.g., Refs. [35–38]). Furthermore, Dicke states provide a rich resource for quantum communication tasks. They can be converted into different kinds of resource states (such as GHZ and W) via local operations and classical

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communication (see Refs. [31,33]) and can be used for various tasks such as, e.g., open destination teleportation (see Ref. [32]). Also, they exhibit a remarkable robustness of entanglement to particle loss (see, e.g., Ref. [39]).

In this Rapid Communication, we provide a simple set of inequalities, which work for all kinds of Dicke states and improve experimental detection, both in terms of noise robustness and feasibility. To this end, we first introduce a generalized notion of the originally proposed states for m excitations in n -qubit systems, which also incorporates the W state (for the case $m = 1$)

$$|D_m^n\rangle = \frac{1}{N} \sum_{\{\alpha\}} |d_{\{\alpha\}}\rangle, \quad (3)$$

$$I_m^n[\rho] = \sum_{\{\gamma\}} \left(\underbrace{|\langle d_{\{\alpha\}} | \rho | d_{\{\beta\}} \rangle|}_{O_{\{\alpha\},\{\beta\}}} - \underbrace{\sqrt{\langle d_{\{\alpha\}} | \otimes \langle d_{\{\beta\}} | \Pi_{\{\alpha\}} \rho^{\otimes 2} \Pi_{\{\alpha\}} | d_{\{\alpha\}} \rangle \otimes | d_{\{\beta\}} \rangle}}_{F_{\{\alpha\},\{\beta\}}} \right) - N_D \sum_{\{\alpha\}} \underbrace{\langle d_{\{\alpha\}} | \rho | d_{\{\alpha\}} \rangle}_{D_{\{\alpha\}}} \leq 0 \quad (5)$$

holds for all biseparable states, where $\Pi_{\{\alpha\}}$ is the cyclic permutation operator acting on the twofold copy Hilbert space (as introduced in Ref. [25]) $\{\gamma\} = \{(\{\alpha\}, \{\beta\}) : |\{\alpha\} \cap \{\beta\}| = 1, 1 < m < \frac{n}{2}, \text{ and } N_D \in \mathbb{N}\}$. For every m , the inequality is always maximally violated by the corresponding Dicke state $|D_m^n\rangle$ (with the number of excitations m equal to $|\{\alpha\}|$), with a value of

$$I_m^n[|D_m^n\rangle\langle D_m^n|] = m[(n-m) - N_D], \quad (6)$$

which yields, e.g., $I_2^n[|D_2^n\rangle\langle D_2^n|] = 2$ independent of n .

For a possible experimental implementation, there are two essential issues. One is the resistance to noise and the other is the feasibility in terms of required local measurements. Before we continue to discuss these issues in detail, let us quickly prove that Eq. (5) indeed holds for all biseparable states. To this end, let us examine each off-diagonal element $O_{\{\alpha\},\{\beta\}}$ separately for a given bipartition $A|B$:

$$O_{\{\alpha\},\{\beta\}} \leq \begin{cases} P_{\{\alpha\},\{\beta\}} & \text{if } x \in A, y \in B, \\ \frac{1}{2}(D_{\{\alpha\}} + D_{\{\beta\}}) & \text{if } x, y \in A \vee x, y \in B, \end{cases} \quad (7)$$

where $x = \{\alpha\} \setminus \{\{\alpha\} \cap \{\beta\}\}$ and $y = \{\beta\} \setminus \{\{\alpha\} \cap \{\beta\}\}$. The first statement can be directly proven with equation (I) from Ref. [25] (as $\Pi_{\{\alpha\}} = \Pi_A$ in this case) and the second statement is always true, as it follows from the positivity of the density matrix. Now, one just needs to count the number N_D of required $D_{\{\alpha\}}$ elements for the worst case scenario of all possible bipartitions, and the inequality holds. This can be analytically determined in a straightforward way. For n qubits and m excitations,

$$N_D = m \binom{n-m-1}{m-1}. \quad (8)$$

Now let us continue to discuss the noise resistance of Eq. (5), first in a few exemplary cases where comparable criteria exist, and then in the general case.

with

$$N = \binom{n}{m}^{-\frac{1}{2}}, \quad (4)$$

$$|d_{\{\alpha\}}\rangle = \bigotimes_{i \notin \{\alpha\}} |0\rangle_i \bigotimes_{i \in \{\alpha\}} |1\rangle_i$$

where $\{\alpha\}$ denotes a set of indices corresponding to the respective subsystems of excitation, and the sum is taken over all inequivalent sets $\{\alpha\}$ satisfying $|\{\alpha\}| = m$.

Let us now proceed to construct a set of inequalities that are optimally suited for detecting genuine multipartite entanglement in such states. The set of inequalities

Four qubits: For four qubits, there exists a comparable criterion, introduced in Ref. [17]. To compare, let us consider a four-qubit Dicke state with two excitations mixed with white noise

$$\rho_{ex1} = (1-p)|D_2^4\rangle\langle D_2^4| + p\frac{1}{16}\mathbb{1}. \quad (9)$$

In this case, the criterion from Ref. [17] detects the state as genuinely multipartite entangled up to a noise threshold of $p < \frac{8}{21} \approx 0.381$, whereas our criterion works up to a noise threshold of $p < \frac{8}{17} \approx 0.471$ and, thus, has a significantly larger detection range. Recently, an approach based upon semidefinite programming has been proposed in Ref. [40]. By numerically optimizing over a set of observables, they achieve a noise threshold of $p \approx 0.539$, which, for four qubits, is even better than our proposed criterion. However, such a numerical optimization is hardly possible beyond seven qubits. This shows that our criterion is already close to the border of biseparable states, as we achieve comparable thresholds without any numerical optimization involved.

For the phased Dicke state introduced in Ref. [34], the criterion developed in Ref. [38] detects the state mixed with white noise to be genuinely multipartite entangled for $p < 0.4$, whereas Eq. (5) also detects the state up to a noise threshold of $p < \frac{8}{17} \approx 0.471$, which also provides a great improvement.

For the state originally proposed for quantum telecloning in Ref. [30], which can be employed also for decoherence-free quantum communication [29] and quantum secret sharing [28] mixed with white noise, we can detect genuine multipartite entanglement up to a noise threshold of $p < \frac{16}{43} \approx 0.372$. The original methods used in Ref. [24] can only detect this state up to a noise threshold of $p \approx 0.278$. This is remarkable as it shows that, even though the inequalities were designed exploiting combinatorial facts about Dicke states, many other states can also be detected in a very noise-robust fashion. Furthermore, for this state, there exists a feasible experimental proposal in Ref. [41], which has since been realized in three different experiments (see Refs. [42–44]).

EXPERIMENTALLY FEASIBLE SET OF CRITERIA ...

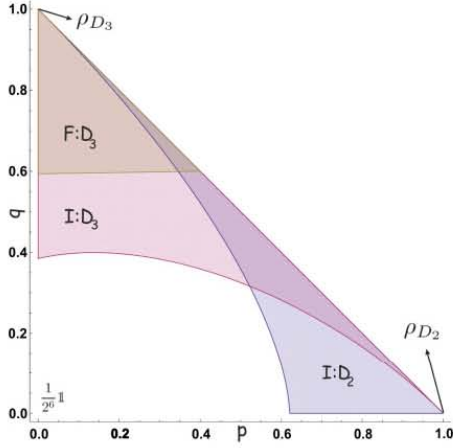
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FIG. 1. (Color online) This plot shows the detection quality of Eq. (5) for the six-qubit state $\rho = p\rho_{D_2} + q\rho_{D_3} + \frac{1-p-q}{2^6}\mathbb{1}$, where $\rho_{D_2} = |D_2^6\rangle\langle D_2^6|$ and $\rho_{D_3} = |D_3^6\rangle\langle D_3^6|$. Region $I : D_2$ represents the parameter region for which the state is detected to be genuinely multipartite entangled by Eq. (5) corresponding to $|D_2^6\rangle$ and, analogously, $I : D_3$ corresponds in the same way to Eq. (5) using $|D_3^6\rangle$. The region $F : D_3$ depicts the parameter region detected by the fidelity witness used in Ref. [33], which demonstrates the improvement provided by Eq. (5).

Six qubits: If we proceed to the six-qubit case, Ref. [17] provides no comparable criterion, but we can compare Eq. (5) to fidelity-based criteria from Ref. [35], as done in Fig. 1.

n qubits: Let us continue by deriving a general expression for white noise resistance. For n -qubit Dicke states with m excitations mixed with white noise, the noise threshold works out as

$$p < \frac{1-m}{1-m-\frac{1}{2^n}\binom{n}{m}(2n-3m+1)}, \quad (10)$$

which for the large n quickly approaches 1 regardless of the chosen m . So, the white noise tolerance increases rapidly in large systems (e.g., for 20 qubits and $m = 2$ it is already over 99%), which is illustrated in Fig. 2.

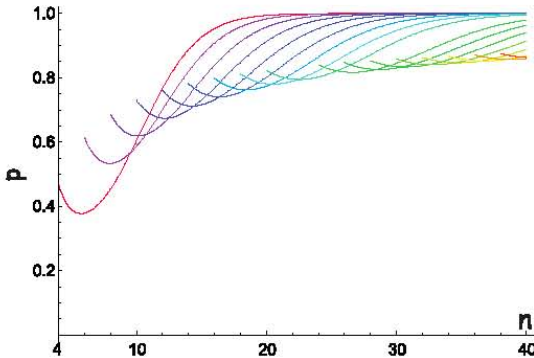


FIG. 2. (Color online) The plot shows the white noise resistance parameter p in terms of the number of qubits n , where, from left to right, the resistance is plotted for $m = 2$ to 20 in ascending order.

Another crucial issue for experimental implementation comes in terms of feasibility of measurements. With growing system size, a full state tomography becomes unachievable since the number of local measurements required grows exponentially (with 2^{2n} for n qubits). The number of density matrix elements that need to be ascertained for Eq. (5) grows only polynomially with system size (e.g., for $m = 2$ with n^3). Also, it is clear that any density matrix element can be re-expressed in terms of local expectation values of Pauli operators (as explicitly done in Refs. [7,23] for similar inequalities). Expressed in terms of local expectation values of Pauli operators, e.g., the inequality for four qubits and $m = 2$ reads as

$$I_2^4[\rho] = \sum_{\pi} \sum_{i=x,y} \langle \pi(1_2 1_2 \sigma_i \sigma_i) \rangle - \sum_{\pi} \sum_{i=x,y} \langle \pi(\sigma_i \sigma_i \sigma_z \sigma_z) \rangle - \left[3 + 3 \langle \sigma_z \sigma_z \sigma_z \sigma_z \rangle - \sum_{\pi} \langle \pi(1_2 1_2 \sigma_z \sigma_z) \rangle - f(\langle \sigma_z^i \sigma_z^j \sigma_z^k \sigma_z^l \rangle) \right] \quad (11)$$

where $i, j, k, l \in \{0, 1\}$, π denotes a permutation, and f is a simple bilinear function of its arguments. For this example, we have replaced $|O_{\{\alpha\}, \{\beta\}}\rangle$ by $\text{Re}[O_{\{\alpha\}, \{\beta\}}]$, which does not affect the detection quality for standard Dicke states. So, one can explicitly see here that, for four qubits, there are 39 local measurement settings required, which is a lot more feasible than the 255 required for a full state tomography. The bilinear function f stems from the $P_{\{\alpha\}, \{\beta\}}$ term in the inequality and can straightforwardly be calculated solving the linear equation associating density matrix elements with expectation values. For example, for $P_{|0101\rangle, |0011\rangle} = \sqrt{\langle 0111 | \rho | 0111 \rangle \langle 0001 | \rho | 0001 \rangle}$, we can compute

$$\langle 0111 | \rho | 0111 \rangle = \frac{1}{8} \left[1 - \langle \sigma_z \sigma_z \sigma_z \sigma_z \rangle - \sum_{\pi} \langle 1\pi(11\sigma_z) \rangle + \langle \sigma_z \pi(1\sigma_z \sigma_z) \rangle - \langle 1\pi(11\sigma_z) \rangle + \langle \sigma_z \pi(1\sigma_z \sigma_z) \rangle + \langle \sigma_z 111 \rangle - \langle 1\sigma_z \sigma_z \sigma_z \rangle \right], \quad (12)$$

and, of course, also all other elements. They all use the same expectation values, but yield a rather cumbersome expression, which is why we chose to abbreviate using f . In conclusion, the introduced set of criteria are the most general in this field so far. They detect every possible n -qubit Dicke state with m excitations to be genuinely multipartite entangled in a very noise-robust and experimentally feasible way. In addition, they also detect genuine multipartite entanglement in states used in many practical quantum information processing applications. This provides the opportunity to re-examine already performed experiments as well as designing novel ones. They possibly provide a foundation for experimental classification of multipartite entanglement, as done with similar inequalities in Ref. [23].

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