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Towards probing Planck scale physics with quantum optics: A feasibility study

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"Towards probing Planck scale physics with quantum optics: A feasibility study"

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Abstract

In this thesis I discuss and analyze a scheme that allows one to measure Planck scale deformations of the commutator with the help of quantum optomechanics, which was introduced by [19]. I will give some motivation for actually performing the experiment and I will give a short introduction to quantum optomechanics and quantum theory. Also I will present the three commutator deformations that can be tested with this scheme. The main target of the thesis is to perform a feasibility study in order to clarify the relevance of parameter regimes of current experiments and to discuss what experimental parameters have to be reached in order to do the experiment.

Zusammenfassung

In dieser Diplomarbeit möchte Ich ein von [19] vorgeschlagenes Experiment diskutieren und analysieren, dass es ermöglichen würde selbst Deformationen des Kommutators auf der Planckskala zu testen. Dazu werde Ich zuerst motivieren warum dieses Experiment wichtig ist und ebenso eine kurze Einführung in die benötigten Techniken aus Quantentheorie und Quantenoptomechanik geben. Ebenso werde ich die drei deformierten Kommutatoren vorstellen die mit diesem Experiment getestet werden können. Das Ziel der Arbeit ist es eine Machbarkeitsstudie durchzuführen um zu ergründen welche experimentellen Anforderungen das Experiment stellt und in wie weit diese schon von Parametern aktueller Experimente erfüllt werden.

Contents

1 Motivation

In this section I will present arguments that motivate this work and outline why there is a need for experimental access to the "Planck scale". I will follow the arguments given in [19] around which this entire work here is based.

Since the beginning of modern physics scientists searched for a consistent set of theories that could explain "everything". So far this search does not seem to be completed. Especially quantum theory and general relativity theory don't seem to be consistent. For this reason there's ongoing research for a unified theory. Many different approaches have been taken over the years in order to merge this two theories into an unified framework of quantum gravity theory, yet no unified framework has been presented. One of the major obstacles is that there is no experimental data on effects that can be ascribed to quantum gravity. Such effects are expected to occur at, or at least near to the Planck-scale. That means at energies as high as the Planck energy $E_p = 1.2 * 10^{19} GeV$ and at length scales as low as the Planck length $L_p = 1.6 * 10^{-33} m$, where space-time itself may be quantized.

But even more challenging: things that are predicted by some new theories, like the "*existence of a minimal length scale*"[19] are thought to be untestable, at least directly. Luckily, the quantization of spacetime can have experimental implications.

A well known result of quantum theory is the Heisenberg uncertainty relation [14] between position x and momentum p of an object: $\Delta\hat{x}\Delta\hat{p} \geq \hbar/2$. In simple words this inequality says that x and p can not simultaneous be known to arbitrary precision. But it is not forbidden to measure for example x alone to an arbitrary precision. On the other hand some proposals for a theory for quantum gravity state that there is a fundamental length scale, for example the Planck length, below which a position can not be defined. In order to take that into account it has been suggested to modify the uncertainty relation in such a way, that it also has a minimum length scale [11]. Other approaches also consider a modified uncertainty principle, for example:

- String theory: For example when *Amati et al.* [7] obtained the explicitly unitary operator for the light (closed) superstring S-matrix above planckian energies, a semi-classical description in an intermediate eikonal region predicted that each string would move in a static Schwarzschild metric.

- The theory of doubly special relativity: In the work of Amelino-Camelia [10] he examined results that involved an observer independent length/momentum scale and also ideas of a minimum wavelength.

- Studies of black holes: Performing a gedanken experiment on micro black holes at the Planck scale of spacetime Scardigli [9] derived a generalized uncertainty principle with a model making use only of Heisenberg's uncertainty principle and the Schwarzschild radius.

This generalized uncertainty relation would also follow from a deformed commutator x and p since $\Delta\hat{x}\Delta\hat{p} \geq \frac{1}{2}|\langle[\hat{x},\hat{p}]\rangle|$. As an example Maggiore [18] considered a general deformed algebra. A deformed algebra is an associative algebra with a commutator that is non-linear in the elements of the algebra, such that for an appropriate limit of the deformation parameter a Lie algebra is recovered. From that he derived a different commutator for x and p and from that a generalized uncertainty principle. The problem in testing these theories is that with current experimental technology it is impossible to prepare and probe quantum states even near the region of the Planck scale. Current suggested experiments or performed experiments to test quantum gravitational effects mainly focus on high energy scattering experiments, but they only reach energies 15 order below the Planck energy; or on the observation of astronomical events like γ -ray bursts to test energy dependent dispersion relations [4], however not yet successful[21]. A direct measurement of the uncertainty relation would be in principle possible, but the best current position measurement is of the order of $\Delta x/L_p \approx 10^{17}$ [13][1] and is therefore insufficient to test quantum gravitational corrections to the uncertainty relation. The following graphic shows a schematic picture of the situation:

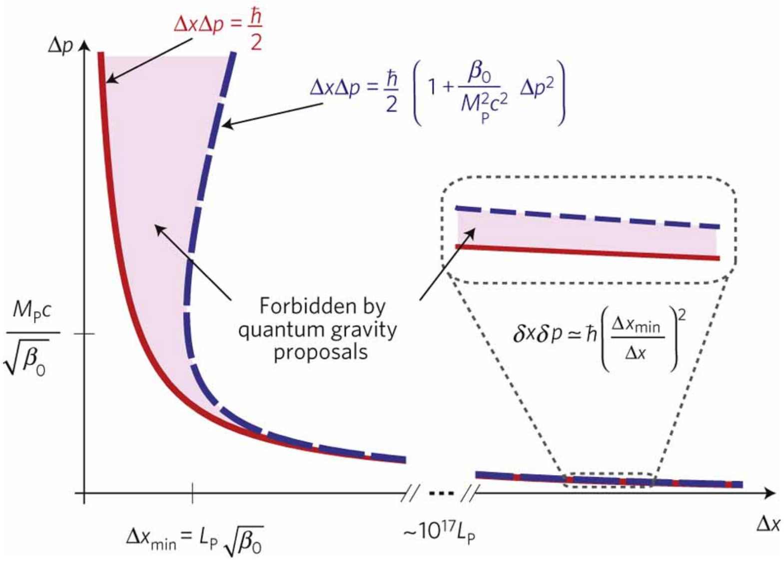

Figure 1: In this picture, the standard uncertainty relation is drawn together with an modified one with a deformation parameter β_0(for more details see following chapters). Due to experimental feasibility it is only possible to measure the commutator in a region where $x \gg \Delta x_{min}$ which would require a measurement precision we don't have.(from [19])

Therefore new methods are needed to test quantum gravitational effects, one of them will be presented in this work and it will be discussed if it is within today's experimental reachability.

2 Introduction

In order to test three different commutation relations given later in this work I will present a scheme proposed by [19]. With this scheme it is within reach, even with current experimental tools, to test if the canonical commutator has to be modified, in principle up to Planck-scale deformations.

The scheme makes use of a quantum optomechanical system, a cavity with one moving massive mirror on a "spring". It relies on a strong optical pulse that interacts four times in a sequence with the center of mass of the mechanical resonator (the mirror) and provides us with the possibility to directly measure the canonical commutator. This sequence maps the commutator of the center of mass mode on to the optical pulse. As we will see, in this case Planck-scale accuracy of position measurement is not necessary to see if there is a deviation from the standard quantum mechanical commutator. To readout the commutator it's only necessary to make a measurement of the mean of the optical field, for example with optical interferometry which supplies high accuracy.

In the end it would even be possible to distinguish between the three types of commutation relations, since they have a different dependence on the systems parameters.

This work presents a feasibility study on the proposed experiment. First I shortly introduce the commutators in question and then provide the theory of the experiment. Then I will discuss the setup of the experiment. Then I put in some realistic numbers for the parameters of the system to show, what results are in reach with already obtained experimental data and compare it to existing bounds on the parameters in the commutation relations.

3 The Planck scale

In this section I will shortly introduce the Planck scale and units, since they seem to play an important role in quantum gravity. I will do this following the arguments of John C. Baez given in the book "Physics Meets Philosophy at the Planck Scale"[6].

To understand the meaning of the Planck scale and units one has to recognize the important roles the constituting units c, G and $\hbar$ of the Planck units play in their original theories.

c, the speed of light:

The important role of c for special relativity comes through its appearance in the Minkowski metric:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{1}$$

The Minkowski metric defines the geometry of spacetime and most importantly the lightcone through each point. Space and time are treated the same, which can be seen as geometrically. All quantities ("local degrees of freedom") we can measure and predict are measured locally in small regions R of spacetime. To predict this quantities we only have to take into account quantities that can be measured in region that are connected to R by a geometrical relation, the causal structure. For example: in order to calculate the value of a field in the region R one only takes into account values in regions that intersect with timelike paths passing through R. The summarizing idea is that nothing travels faster then with the speed of light c. In other words, a theory that contains special relativity should have "*local degrees of freedom propagating causally"*[6].

G, the Newtonian constant of gravity:

G has a very important role in general relativity. General relativity describes gravitational fields as a curvature of spacetime that are unlike special relativity not independent of the other fields. In special relativity the Minkowski metric is only a "back-ground structure". In general relativity the geometry of spacetime itself is a "local degree of freedom". The interaction between the fields and the metric are described by Einstein's equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2}$$

where G is the coupling constant between the Einstein tensor $G_{\mu\nu}$, which is dependent on the metrics curvature and the stress-energy tensor $T_{\mu\nu}$ that describes the flow of energy and momentum caused by the non gravitational fields.

The most important feature of general relativity is that also the causal structure is a "local degree of freedom", the theory is therefore "background-free". Baez say's that "*the great lesson of general relativity is that a good theory of physics should contain no geometrical structures that affect local degrees of freedom while remaining unaffected by them.*"[6]. Therefore it is legitimate to assume that every "good theory" will be "background-free".

$\hbar$, the Planck constant:

Most prominently $\hbar$ is present in the commutation relation between momentum operator $\hat{\mathrm{p}}$ and position operator $\hat{\mathrm{x}}$

$$[\hat{p}, \hat{x}] = \hat{\mathrm{p}}\hat{\mathrm{x}} - \hat{\mathrm{x}}\hat{\mathrm{p}} = -i\hbar \tag{3}$$

and many other commutation relations between other operators. $\hbar$ quantifies our inability to simultaneously measure $\hat{\mathrm{p}}$ and $\hat{\mathrm{x}}$. But $\hbar$ appears in many more occasions in quantum theory. In fact $\hbar$ appears throughout the whole formalism of complex Hilbert spaces and operators. Acknowledging the role of quantum physics and assuming that it is at least valid in a limit one could think that a good theory of the world should be a "quantum theory".

If we now summarize, a good theory should be a theory that is a"*background-free quantum theory with local degrees of freedom propagating causally*"[6]. It would be therefore quite plausible for the constants c, G and $\hbar$ to appear in this theory.

Planck recognized that there is a unique way to build a length unit out of this three constants, the Planck length:

$$L_p = \sqrt{\frac{\hbar G}{c}}. \tag{4}$$

We will now see why the Planck length may could play an important role in a possible theory of quantum gravity.

Quantum field theory knows the so called Compton wavelength L_c:

$$L_c = \frac{\hbar}{mc}. \tag{5}$$

The role of the Compton wavelength is, that if one wants to measure the position of a particle with mass m with precision of L_c this needs so much energy that another particle with the same mass will be created. Since particle creation is a very important feature of quantum field theory one can say, that at length scales of L_c quantum field theory becomes essential if one wants to know the behavior of a particle.

On the other hand in general relativity one knows the Schwarzschild radius L_s:

$$L_s = \frac{Gm}{c^2}. \tag{6}$$

The Schwarzschild radius predicts that if an object of mass m is compressed further down than the Schwarzschild radius this will create a black hole. Therefore the Schwarzshild radius sets the length scale for objects of mass m at which general relativity is essential to understand its behavior.

In quantum gravity it is clear that on one hand general relativity will play a role and on the other quantum theory. It is therefore natural to ask when $L_c = L_s$ is fulfilled and general relativity and quantum field theory are important to understand the behavior of the object. If we set (5)=(6) and solve for m we get

$$M_p = \sqrt{\frac{\hbar c}{G}} \tag{7}$$

the Planck mass, the unique way to combine $c, G\text{and}\hbar$ to get a mass unit and furthermore $L_c = L_s = L_p$. Therefore it is natural to think that one needs general relativity and quantum field theory to understand the physics of an object of the mass $m \approx M_p$ and extension $l \approx L_p$

This not only explains the importance of the Planck scale but also the lack of experimental data:

- The experimental information about general relativity we have mostly comes from the observations of objects with high mass and great extension like stellar objects for which $L_s \gg L_c$.

- Information on quantum field theory mostly comes from measurements on light and small objects, for example electrons or protons for which clearly $L_c \ll L_s$.

- The Planck mass lies somewhere in between these two extremes, comparable to the mass of a flea egg. The Planck length on the other hand is about 10^-20 the radius of a proton, therefore much smaller. There's no principal reason forbidding compressing a flea egg to this size but it is unknown how to accomplish this.

If one wants to probe the Planck length one needs the Planck energy to do so defined by

$$E_p = M_p c^2. \tag{8}$$

but if one confines that energy in a region of the scale L_p one will probably create a black hole.

To get a feeling for the scale I will give here there numbers in conventional units:

$$\begin{array}{l} L_p \approx 1.616 * 10^{-35} m \\ M_p = 2.177 * 10^{-8} kg \\ E_p \approx 1.956 * 10^{9} J \end{array}$$

Table 1: The values of the Planck units in SI-units

4 Introduction to quantum optomechanics

In this section I want to give a short introduction to the optomechanics that are needed in the rest of this work. The introduction follows a lecture of Florian Marquardt given at Les Houches in 2011 [17].

A basic schematic optomechanical setup can be seen in the following figure:

Figure 2: basic optomechanical setup(from [20])

This basic setup consists of a cavity with two mirrors. One of the mirrors is fixed, the other one can move and is coupled harmonically to an other system, here symbolized by a spring.The second mirror is therefore a (massive) mechanical oscillator. The cavity is driven by a laser.

The important system parameters are:

1. "m": the mass of the mechanical oscillator

2. "ω_m": the frequency of the mechanical oscillator

3. "Γ_m": the mechanical damping rate (typically in the range of Hz-MHz)

4. "Q": the quality factor $Q \equiv \omega_m/\Gamma_m$, for good setups values of 10^6 are reachable

5. "L": the mean length of the cavity

6. "λ_l": the wavelength of the light

7. "κ": the cavity decay rate

8. "ω_c": the mean optical resonance frequency of the cavity

9. "$x_0 = \sqrt{\hbar/m\omega_m}$": the width of the ground state of the mechanical oscillator

10. "$g_0 = \omega_c x_0/L$": the optomechanical single photon coupling strength

11. "F": the finesse of the cavity

The system is described by:

1. "$\hat{\mathrm{a}}_l$": the annihilation operator for the light field in the cavity

2. "$\hat{\mathrm{b}}_m$": the annihilation operator for the mechanical oscillator

3. "$\hat{\mathrm{x}}_m = x_0(\hat{\mathrm{b}}_m + \hat{\mathrm{b}}_m^\dagger)$": the operator for the displacement of the mechanical oscillator

4. "$\hat{\mathrm{X}}_m = x_m/x_0$": the dimensionless position operator for the mechanical oscillator.

First we want to consider the basic Hamiltonian for our system:

$$\hat{\mathrm{H}} = \hbar\omega(x_m)\hat{\mathrm{n}}_l + \hbar\omega_m\hat{\mathrm{n}}_m + \dots \tag{9}$$

The first term is the energy of the laser driven cavity. The resonance frequency of the cavity depends on the length of the cavity and therefore on the position of the moveable mirror. The second term gives us the energy of the moveable mirror. The dots represent for example photon and phonon decay and other effects not considered yet.

If we now expand the resonance frequency around $x = 0$

$$\omega(x_m) = \omega_c\left(1 - \frac{x_m}{L} + \dots\right) \tag{10}$$

and insert this into (9) and substitute x_m with the operator x_m then we get

$$\hat{\mathrm{H}} = \hbar\omega_c\hat{\mathrm{n}}_l + \hbar\omega_m\hat{\mathrm{n}}_m - \hbar\omega_c\hat{\mathrm{n}}_l\frac{\hat{\mathrm{x}}_m}{L} + \dots \tag{11}$$

This Hamilton contains now a term with that is force-like $F_f = (\hbar\omega_c/L)\hat{\mathrm{n}}_l$: $-F_f\hat{\mathrm{x}}_m$. We can identify this "force" as the radiation pressure force. Using the definition of g_0 from above the Hamilton has the form

$$\hat{\mathrm{H}} = \hbar\omega_c \hat{\mathrm{n}}_l + \hbar\omega_m \hat{\mathrm{n}}_m - \hbar g_0 \hat{\mathrm{n}}_l (\hat{\mathrm{b}}_m + \hat{\mathrm{b}}_m^\dagger) + \ldots = \hbar\omega_c \hat{\mathrm{n}}_l + \hbar\omega_m \hat{\mathrm{n}}_m - \hbar g_0 \hat{\mathrm{n}}_l \hat{\mathrm{X}}_m + \ldots \quad (12)$$

where we can identify

$$-\hbar\omega_c \hat{\mathrm{n}}_l \frac{\hat{\mathrm{x}}_m}{L} \quad (13)$$

as the optomechanical coupling term.

4.1 Lasercooling

We now go into the regime of "linearized optomechanics". Here we assume a strong laserdrive such that

$$\hat{\mathrm{a}}_l = (\alpha + \delta\hat{\mathrm{a}}_l) e^{-i\omega_l t}. \quad (14)$$

The amplitude of the light field inside the cavity is then given by

$$\alpha = \sqrt{N_p} \quad (15)$$

with N_p being the average photon number. Expanding the photon number operator $\hat{\mathrm{n}}_l$ gives:

$$\hat{\mathrm{n}}_l = \hat{\mathrm{a}}_l^\dagger \hat{\mathrm{a}}_l = \alpha^2 + \alpha(\delta\hat{\mathrm{a}}_l + \delta\hat{\mathrm{a}}_l^\dagger) + \delta\hat{\mathrm{a}}_l^\dagger \delta\hat{\mathrm{a}}_l. \quad (16)$$

We will now ignore the first term because it only gives rise to a classical force and neglect the last one because it is much smaller than the second one and obtain the Hamilton for linearized optomechanics by going to a frame rotating at the laser frequency.

$$\hat{\mathrm{H}} \approx -\hbar g_0 \alpha (\delta\hat{\mathrm{a}}_L + \delta\hat{\mathrm{a}}_L^\dagger)(\hat{\mathrm{b}}_m + \hat{\mathrm{b}}_m^\dagger) + \hbar\omega_m \hat{\mathrm{n}}_m - \hbar\Delta\delta\hat{\mathrm{a}}_l^\dagger \delta\hat{\mathrm{a}}_l + \ldots \quad (17)$$

with

$$\Delta \equiv \omega_l - \omega_c \tag{18}$$

the laser detuning.

The interaction term in (17) is

$$-\hbar g_0 \alpha(\delta\hat{\mathrm{a}}_l + \delta\hat{\mathrm{a}}_l^\dagger)(\hat{\mathrm{b}}_m + \hat{\mathrm{b}}_m^\dagger). \tag{19}$$

This represents two coupled oscillators, the mechanical oscillator $\hat{\mathrm{b}}_m$ at frequency ω_m and the laser driven cavity mode $\delta\hat{\mathrm{a}}_l$ at frequency $-\Delta$. $\delta\hat{\mathrm{a}}_l$ is an oscillator that is in the ground state, because we allready subtracted the coherent state amplitude α, $\delta\hat{\mathrm{a}}_l$ only describes an excitation "*on top of the strongly driven cavity field.*"[17]

Now we go to the red-detuned regime. That is $\Delta = -\omega_m$. If we make the rotating wave approximation (set all off-resonant terms to zero) the coupling is

$$\delta\hat{\mathrm{a}}_l^\dagger\hat{\mathrm{b}}_m + \delta\hat{\mathrm{a}}_l\hat{\mathrm{b}}_m^\dagger. \tag{20}$$

This describes a state transfer between two resonant oscillators and can be used for two different things:

State transfer:

between the photons of the lightfield and the phonons of the mechanical oscillator. For this to happen, $g_0\alpha > \kappa$ has to be fulfilled. In this case a hybridization between $\delta\hat{\mathrm{a}}_l$ and $\hat{\mathrm{b}}_m$ is observed. This could be a peak splitting in the mechanical or the optical response, which is called the "strong-coupling regime". The optical and the mechanical position quadratures are then coupled linear. The coupling is enhanced with α but unfortunatly we loose the nonlinearity of the interaction [12]. In the following plots (picture3) one can see the emission spectra of a driven optomechanical cavity. As one can see, with increasing power of the laser drive, a splitting occurs in the emission spectra of the cavity due to the hybridization:

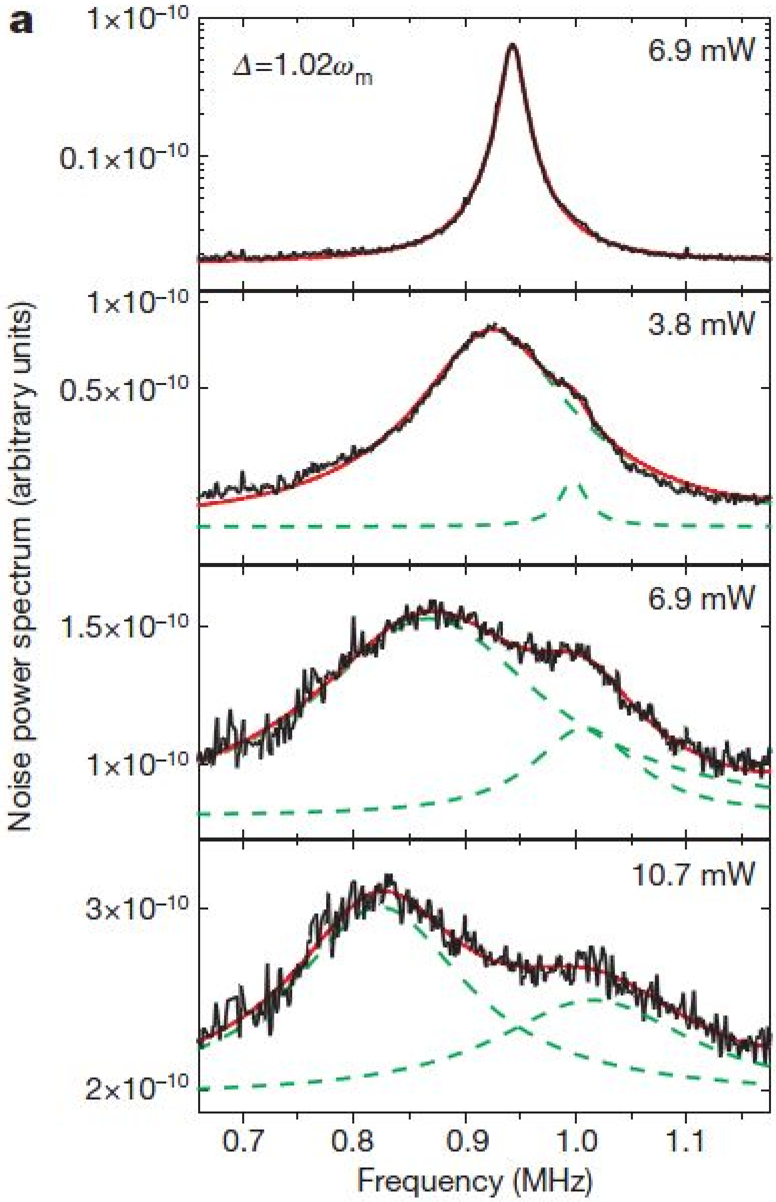

Figure 3: noise power spectrum emission of the cavity plotted against frequency for different laser powers (from [12])

Lasercooling:

Since $\delta\hat{a}_l$ is an oscillator with zero temperature it can take up energy of the mechanical oscillator at according to Fermi's Golden Rule at a cooling rate $\propto (g_0\alpha)^2/\kappa$.[17] In the graphic 4 there's a schematic picture of what happens.

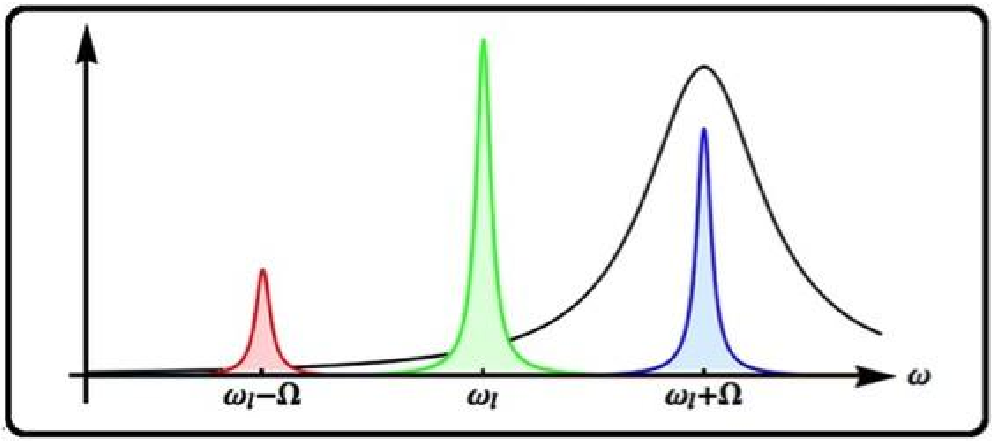

Figure 4: $\Omega \equiv \omega_m$ in the picture. (from [20])

In the picture 4 one can see how cooling works from a sideband point of view. Around the frequency ω_l two sidebands are created due to photons are Doppler shifted by the moving mirror. The black curve is the resonator transmission near it is resonance frequency ω_c. The blue curve is the frequency upshifted anti-Stokes sideband at frequency $\omega_l + \omega_m$ and the red curve is the frequency downshifted Stokes sideband at frequency $\omega_l - \omega_m$. Cooling happens when the anti-Stokes sideband is resonant with the resonator. Therefore the parametric transfer from phonons of the mechanical oscillator to the light field will be dominant over the reverse process(from [20]).

4.2 Quadratures of the light field

Since we will later need the description of the light field in means of so called "quadratures" I want to shortly introduce them. I will do so following an introduction given in [16].

The quantization of the electromagnetic field works because of the perfect formal analogy in how to describe a monochromatic field mode and a harmonic oscillator. Classically the electric field strength of a free single mode field is given by

$$E(r,t) = \epsilon(r,t)A + \epsilon^*(r,t)A^*. \tag{21}$$

Because of the analogy we now just replace the complex amplitude A by the photon annihilation operator â and it is complex conjugated by the photon creation operator $\hat{a}^\dagger$ and get the operator of the electric field strength:

$$\hat{E}(r,t) = C(\epsilon(r,t)\hat{a} + \epsilon^*\hat{a}^\dagger) \tag{22}$$

where C is just a normalization factor and $\epsilon(r,t) = \epsilon(r)e^{-i\omega t}$, which is the same in both descriptions. If we now go to the real representation of the electric field strength instead of the complex one, we can rewrite equation 21 as

$$E(r,t) = X\epsilon_1(r,t) + P\epsilon_2(r,t) \tag{23}$$

with

$$\epsilon_1(r,t) = \sqrt{2}Re[\epsilon^*(r,t)] \ ; \ \epsilon_2(r,t) = \sqrt{2}Im[\epsilon^*(r,t)] \tag{24}$$

and

$$X = \sqrt{2}ReA = \frac{1}{\sqrt{2}}(A^* + A) \ ; \ P = \sqrt{2}ImA = \frac{1}{\sqrt{2}}i(A^* - A). \tag{25}$$

In order to give X and P a bit of physical meaning one may consider a running plain wave

$$E(r,t) = E_0(Xcos(\omega t - kr) + Psin(\omega t - kr)) \tag{26}$$

with angular frequency ω and wavevector k. If we compare this field to a reference field oscillating with $cos(\omega t - kr)$, the X represents the in-phase component and P the out-of-phase component of the field. If we now take again the step to quantum mechanics we get

$$\hat{X} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}) \ ; \hat{P} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}) \tag{27}$$

the so called quadratures.

If we now look at equation 26 again, one can see, that also this would be a legitimate representation of the same physical field:

$$E(r,t) = E_0(X_\phi cos(\omega t - kr - \phi) + P_\phi sin(\omega t - kr - \phi)) \tag{28}$$

with our new X and P quadratures being

$$X_\phi = \frac{1}{\sqrt{2}}(A^*e^{i\phi} + Ae^{-i\phi}) \tag{29}$$

$$P_\phi = \frac{1}{\sqrt{2}}i(A^*e^{i\phi} - Ae^{-i\phi}). \tag{30}$$

We can also write our new so called generalized quadratures in terms of the old ones as

$$X_\phi = cos(\phi)X + sin(\phi)P \tag{31}$$
$$P_\phi = -sin(\phi)X + cos(\phi)P. \tag{32}$$

This can also be done for the quantum mechanical generalize quadratures and we get

$$\hat{\mathrm{X}}_\phi = cos(\phi)\hat{\mathrm{X}} + sin(\phi)\hat{\mathrm{P}} \tag{33}$$
$$\hat{\mathrm{P}}_\phi = -sin(\phi)\hat{\mathrm{X}} + cos(\phi)\hat{\mathrm{P}}. \tag{34}$$

It has to be noted here, that the observable $\hat{\mathrm{X}}_\phi$ corresponds to different physical measurements for different angles ϕ.

4.3 Balanced homodyne detection

In this subsection I want to discuss an important experimental tool in quantum optics, the so called "balanced homodyne detection". I will introduce it following the description given in [16]. In the following picture one can see a basic setup for balanced homodyne detection:

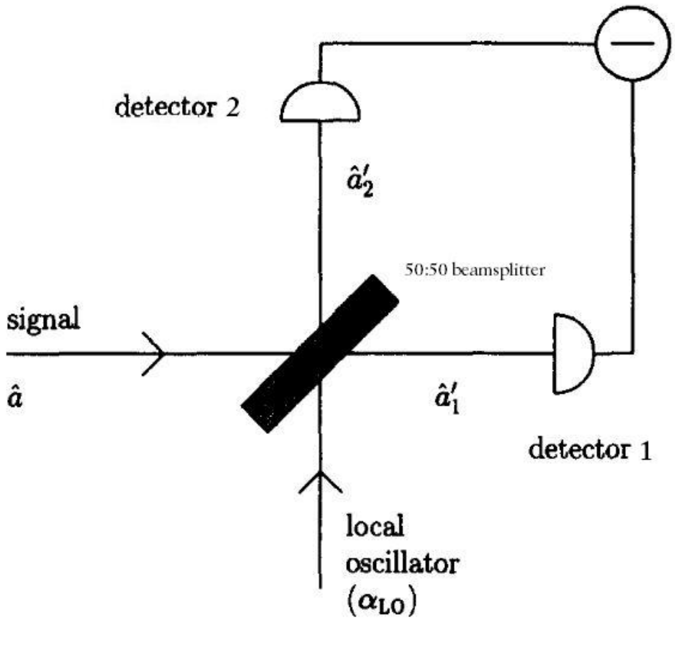

Figure 5: basic setup for balanced homodyne detection (from [16])

As one can see in the picture 5 a signal field is mixed at a 50:50 beamsplitter (from here comes the name "balanced") with the field of a so called "local oscillator". The local oscillator field is used as a reference. The two outgoing fields are then detected at two different detectors(detector1 and detector2) whose outgoing signals are then electronically subtracted. The detectors 1 and 2 measure the intensities

$$\hat{\mathrm{J}}_1 = \hat{\mathrm{a}}_1^{'\dagger}\hat{\mathrm{a}}_1^{'} \tag{35}$$

and

$$\hat{\mathrm{J}}_2 = \hat{\mathrm{a}}_2^{'\dagger}\hat{\mathrm{a}}_2^{'} \tag{36}$$

in means of photonnumbers, where we assume perfect detectors. The operator for the intensity after the subtraction is the given by

$$\Delta\hat{\mathrm{J}} = \hat{\mathrm{J}}_2 - \hat{\mathrm{J}}_1 = \frac{1}{2}(\hat{\mathrm{a}}^\dagger + \hat{\mathrm{a}}_{LO}^\dagger)(\hat{\mathrm{a}} + \hat{\mathrm{a}}_{LO}) - \frac{1}{2}(\hat{\mathrm{a}}^\dagger - \hat{\mathrm{a}}_{LO}^\dagger)(\hat{\mathrm{a}} - \hat{\mathrm{a}}_{LO}) = \hat{\mathrm{a}}^\dagger\hat{\mathrm{a}}_{LO} + \hat{\mathrm{a}}\hat{\mathrm{a}}_{LO}^\dagger. \tag{37}$$

If we now assume that the local oscillator is coherent and strong compared to the signal, then we can make the following substitution

$$\hat{\mathrm{a}}_{LO} \rightarrow \alpha_{LO} = |\alpha_{LO}|e^{i\phi} \tag{38}$$

and can write the photocurrent difference as

$$\Delta\hat{\mathrm{J}} = |\alpha_{LO}|(\hat{\mathrm{a}}e^{-i\phi} + \hat{\mathrm{a}}^\dagger e^{i\phi}) = |\alpha_{LO}|\sqrt{2}\hat{\mathrm{X}}_\phi \tag{39}$$

where $\hat{\mathrm{X}}_\phi$ is the generalized quadrature for the angle ϕ:

$$\hat{\mathrm{X}}_\phi \equiv \hat{\mathrm{X}}cos(\phi) + \hat{\mathrm{P}}sin(\phi). \tag{40}$$

In order to determine the phase of the signal we have to know both $\hat{\mathrm{X}}_\phi$ and $\hat{\mathrm{P}}_\phi$. The scheme that can be seen in the following figure is able to do so:

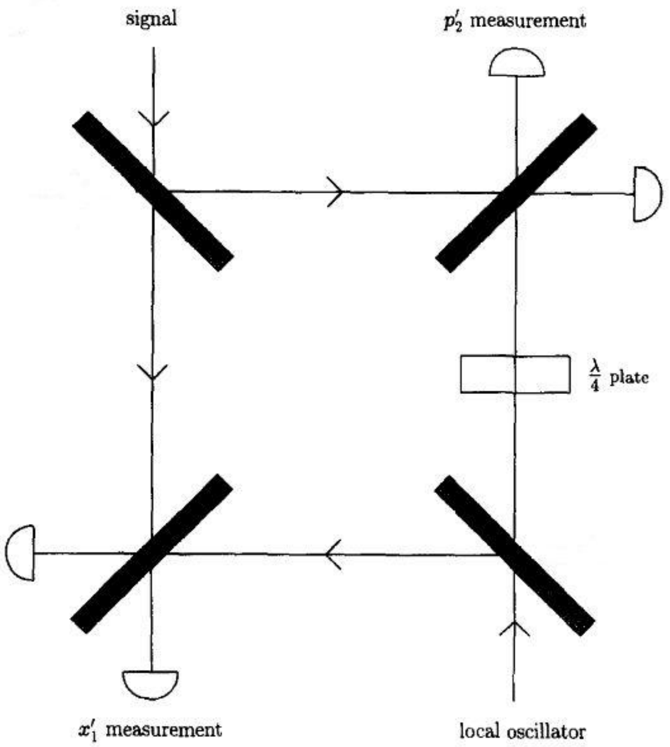

Figure 6: setup to measure both $\hat{\mathrm{X}}_\phi$ and $\hat{\mathrm{P}}_\phi$ (from [16])

As we can see in figure 6 the setup basically consists of two balanced homodyne detections(the needed subtractions at the end of the two times two detectors are not shown). For the upper one we phaseshift the local oscillator by $\pi/2$ with a $\lambda/4$-plate. Therefore the upper balanced homodyne detection allows for a measurement of $\hat{\mathrm{X}}_{\phi+\pi/2} = \hat{\mathrm{P}}_\phi$.

5 The modified commutation relations in consideration

In this section I want to briefly discuss the three commutation relations that we want to test with the proposed experiment.

5.1 β_0-modification of the commutator

The first commutator I want to present is the β_0-modified commutator [19]:

$$[\hat{x}, \hat{p}]_{\beta_0} = i\hbar(1 + \beta_0(\frac{\hat{\mathrm{p}}}{M_p c})^2) \tag{41}$$

This commutator (41) was obtained by Kempf et al. [15]. Their basic idea was to find to find the simplest possible generalized uncertainty relation that leads to a nonzero minimal uncertainty $\Delta\hat{\mathrm{x}}_0$ in position. What they found was

$$\Delta\hat{\mathrm{x}}\Delta\hat{\mathrm{p}} \geq \frac{\hbar}{2}(1 + \beta(\Delta\hat{\mathrm{p}})^2 + \gamma) \tag{42}$$

where β and γ are positive parameters.

This generalized uncertainty relation implies a "minimal length" $\Delta\hat{\mathrm{x}}_0$ as can be seen in the plot (7).

In general an uncertainty relation for two observable is given by

$$\Delta\hat{\mathrm{A}}\Delta\hat{\mathrm{B}} \geq \frac{\hbar}{2}\left|\left\langle[\hat{A}, \hat{B}]\right\rangle\right|. \tag{43}$$

For (42) the commutator looks like

$$[\hat{x}, \hat{p}]_{\beta} = i\hbar(1 + \beta(\hat{\mathrm{p}})^2) \tag{44}$$

with $\gamma = \beta\left\langle\hat{\mathrm{p}}\right\rangle^2$ in (42), using that $(\Delta\hat{\mathrm{p}})^2 = \left\langle(\hat{\mathrm{p}} - \langle\hat{\mathrm{p}}\rangle)^2\right\rangle$.

The uncertainty relation for this commutator is

$$\Delta\hat{\mathrm{x}}\Delta\hat{\mathrm{p}} \geq \frac{\hbar}{2}(1 + \beta(\Delta\hat{\mathrm{p}})^2 + \beta\left\langle\hat{\mathrm{p}}\right\rangle^2) \tag{45}$$

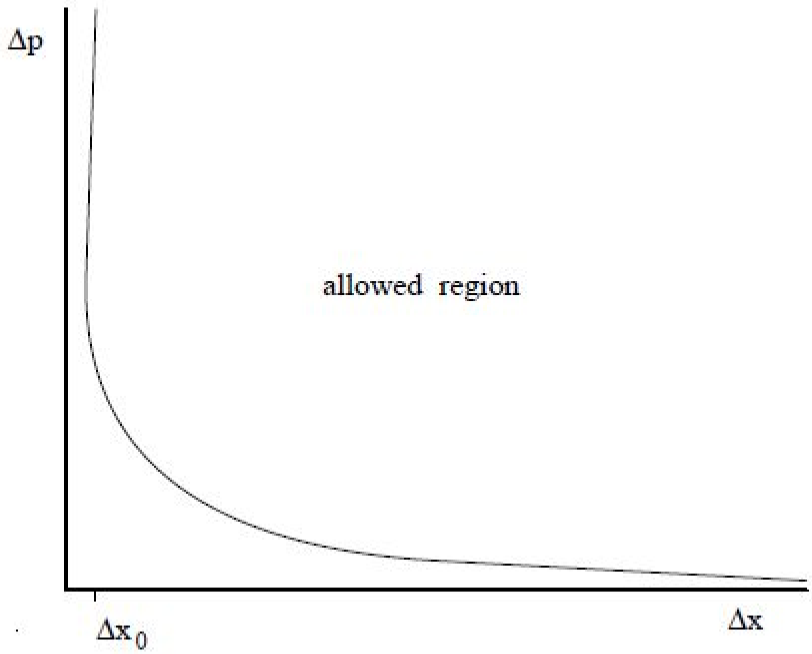

Figure 7: β-modified uncertainty relation (from [15])

Now the curve of the boundary in plot(7) has the form

$$\Delta\hat{p} \equiv \frac{\Delta\hat{x}}{\hbar\beta} \pm \sqrt{\left(\frac{\Delta\hat{x}}{\hbar\beta}\right)^2 - \frac{1}{\beta} - \langle\hat{p}\rangle^2}. \tag{46}$$

By looking at (46) one can see that the minimal uncertainty in position is

$$\Delta\hat{x}_{min}(\langle\hat{p}\rangle) = \hbar\sqrt{\beta}\sqrt{1+\beta\,\langle\hat{p}\rangle^2} \tag{47}$$

.

From this follows, that the absolute smallest value for the position uncertainty is given by

$$\Delta\hat{x}_0 = \hbar\sqrt{\beta}. \tag{48}$$

Since we want to argue, that this value may be the Planck length we set β to

$$\beta \equiv \frac{\beta_0}{M_p^2 c^2} = \frac{\beta_0 L_p^2}{\hbar^2} \tag{49}$$

which can be seen from the definition of Planck mass (7) and Planck length (4).

Now for $\beta_0 = 1$ the minimal position uncertainty becomes the Planck length. The minimal measurable length scale depending on β_0 is now

$$\Delta \hat{\mathrm{x}}_0 = L_p \sqrt{\beta_0} \tag{50}$$

With the definition (49) for β the commutator is the one from the beginning of this section (41) and the modified uncertainty relation we get is

$$\Delta \hat{\mathrm{x}} \Delta \hat{\mathrm{p}} \geq \frac{\hbar}{2} \left(1 + \beta_0 \left(\frac{\Delta \hat{\mathrm{p}}}{M_p c} \right)^2 + \beta_0 \left(\frac{\langle \hat{\mathrm{p}} \rangle}{M_p c} \right)^2 \right). \tag{51}$$

Until now, no effect of a commutator of the form (41) has been seen in an experiment, but at least it was possible to put bounds on the value of β_0. This was done for example by Saurya Das and Elias C. Vagenas [8]. They noted, that if there would be an intermediate length scale $L_{inter} = \sqrt{\beta_0} L_p$ it could not exceed the electroweak length scale $L_{elecrtoweak} \approx 10^{17} L_p$ as otherwise it would have been observed already. This sets an bound on β_0:

$$\beta_0 \leq 10^{34}. \tag{52}$$

Furthermore they calculated the Lamb shift in presence of an β_0-modified commutator and compared it to the standard one. With current measurement precision (no deviation was found) this sets a bound of $\beta_0 \leq 10^{36}$. They also calculated electron tunneling amplitudes under the same conditions. When we assume that we can measure electric currents with a precision of $\delta I \approx 1 fA$ [19] this sets the bound to

$$\beta_0 \leq 10^{33}. \tag{53}$$

5.2 μ_0-modification of the commutator

The next commutator I want to present is the μ_0-modified commutator[19]:

$$[\hat{x},\hat{p}]_{\mu_0} = i\hbar\sqrt{1+2\mu_0(\frac{(\hat{\mathrm{p}}/c)^2+m^2}{M_p^2}} \tag{54}$$

where m is the rest mass and μ_0 a numerical parameter. This result was derived by Michele Maggiore [18]. He wanted to find an algebraic structure that would fit a generalized uncertainty principle of the form

$$\Delta\hat{\mathrm{x}}\Delta\hat{\mathrm{p}} \geq \hbar + const.G\Delta\hat{\mathrm{p}}^2 \tag{55}$$

It has been suggested that measurements in quantum gravity have to obey this generalized uncertainty principle. Since it is clear that (55) can only be produced by a deformed algebra, he looked at the most general that can be constructed from coordinates x_i and p_i with the following restrictions:

1. The 3-d rotation group stays not deformed, therefore the angular momentum $\hat{\mathrm{J}}$ obeys the SU(2) commutation relations and $[\hat{[},\hat{J}]_k, x_j] = \epsilon ijk\hat{\mathrm{x}}_k$ and $[\hat{[},\hat{J}]_k, p_j] = \epsilon ijk\hat{\mathrm{p}}_k$

2. $[\hat{[},\hat{p}]_i, p_j] = 0 \Rightarrow$ the translation group is also not deformed

3. The $[\hat{[},\hat{x}]_i, x_j]$ and $[\hat{[},\hat{x}]_i, p_j]$ are functions of a deformation parameter κ of dimension energy. For $\kappa \to \infty$ the deformation should vanish and the ordinary quantum mechanical commutator should be recovered.

Under these assumptions the most general form of our κ-deformed algebra is:

$$[\hat{x_i},\hat{x_j}] = \frac{c^2\hbar^2 a(E)}{\kappa^2} i\epsilon_{ijk}\hat{\mathrm{J}}_k \tag{56}$$

$$[\hat{x_i},\hat{p_j}] = i\hbar\delta_{ij}f(E) \tag{57}$$

In (56) and (57) $a(E)$ and $f(E)$ are realvalued dimensionless functions of E/κ and $E \equiv p^2c^2 + m^2c^4$ and the angular momentum $\hat{\mathrm{J}}$ is defined dimensionless as well. The i in (56) and (57) has to appear in order to satisfy the hermiticity condition and the powers of c, κ and $\hbar$ are dictated by dimensional analysis. The ϵ_{ijk} tensor in (56) has to appear because of the first restriction we made. The $\hat{\mathrm{J}}_k$ appears because of parity. Because of restriction 3 it is necessary that $f(0) = 1$ and $a(E)$ is less singular than E^{-2} for $E \to 0$. The function $a(E)$ and $f(E)$ are further restricted by

$$[\hat{\mathrm{x}}_i, [\hat{x_j}, \hat{x_k}]] + [\hat{\mathrm{x}}_k, [\hat{x_i}, \hat{x_j}]] + [\hat{\mathrm{x}}_j, [\hat{x_k}, \hat{x_i}]] = 0 \tag{58}$$

a Jacobi Identity. From this we get

$$\frac{da}{dE}\mathbf{p}^*\mathbf{J} = 0 \tag{59}$$

using that $[\hat{\mathrm{x}}_i, E] = i\hbar f(E)\hat{\mathrm{p}}_i/E$ and $[\hat{\mathrm{x}}_i, a(E)] = i\hbar f(E)(\hat{\mathrm{p}}_i/E)da/dE$. From this we are able to conclude that $a(E) = const. \equiv \pm 1$ with a redefinition of κ.

Now we want to satisfy the Jacobi identity

$$[\hat{\mathrm{x}}_i, [\hat{x_j}, \hat{p_k}]] + [\hat{\mathrm{p}}_k, [\hat{x_i}, \hat{x_j}]] + [\hat{\mathrm{x}}_j, [\hat{p_k}, \hat{x_i}]] = 0 \tag{60}$$

which gives us

$$\frac{f(E)}{E}\frac{df}{dE} = \mp\frac{1}{\kappa^2}. \tag{61}$$

The sign on the right hand side corresponds to the choice of the sign of $a(E)$, we will define $a(E) \equiv -$. Using that $f(0) = 1$, the solution for (61) is

$$f(E) = (1 + \frac{E^2}{\kappa^2})^{1/2}. \tag{62}$$

It is quite remarkable that all other Jacobi identities are satisfied automatically.

In the end the algebra Maggiore derived is:

$$[\hat{x_i}, \hat{x_j}] = -\frac{c^2\hbar^2}{\kappa^2} i\epsilon_{ijk}\hat{\mathrm{J}}_k \tag{63}$$

$$[\hat{x_i}, \hat{p_j}] = i\hbar\delta_{ij}\left(1 + \frac{E^2}{\kappa^2}\right)^{1/2} \tag{64}$$

Now we define κ as

$$\kappa \equiv \frac{M_p c^2}{\sqrt{2\mu_0}} \tag{65}$$

to get (64) in the form of (54) as we have it in [19]. This form is practicable for the following reasons[19]:

1. If $m \ll p/c \leq M_p$ and $\mu_0 = \beta_0$, then we recover the β_0-modified commutator (41). (This explains the 2 in (54))

2. For $p/c \ll m \leq M_p$ (54) becomes $[\hat{x}, \hat{p}]_{\mu_0} \approx i\hbar(1 + \mu_0 m^2/M_p^2)$. This can be interpreted as a rescaling $\hbar = \hbar(m)$.

The big difference between (41) and (54) is that (54) has a direct dependence on the mass.

5.3 γ_0-modification of the commutator

The last commutator I want to present is the γ_0-modified commutator [19]:

$$[\hat{x}, \hat{p}]_{\gamma_0} = i\hbar \left(1 - \gamma_0 \frac{\hat{\mathrm{p}}}{M_p c} + \gamma_0^2 \left(\frac{\hat{\mathrm{p}}}{M_p c}\right)^2\right) \tag{66}$$

This commutator was recently proposed by Ahmed Farag Ali , Saurya Das and Elias C. Vagenas in [3] and derived in [2]. The basic idea behind this commutator is that black hole physics and string theory suggest a modification of the commutator that is quadratically dependent on the momentum, whereas doubly special relativity suggests that it is linear dependent on the momentum. The most general deformed algebra that fitted this requirements is

$$[\hat{x_i}, \hat{p_j}] = i\hbar \left(\delta_{ij} + \delta_{ij}\alpha_1\hat{\mathrm{p}} + \alpha_2\frac{\hat{\mathrm{p}}_i\hat{\mathrm{p}}_j}{\hat{\mathrm{p}}} + \beta_1\delta_{ij}\hat{\mathrm{p}}^2 + \beta_2\hat{\mathrm{p}}_i\hat{\mathrm{p}}_j\right). \tag{67}$$

With the assumption that $[\hat{x_i}, \hat{x_j}] = [\hat{p_i}, \hat{p_j}] = 0$ the Jacobi identity gives us

$$-[[x_i, x_j], \hat{\mathrm{p}}_k] = [[x_j, p_k], \hat{\mathrm{x}}_i] + [[p_k, x_i], \hat{\mathrm{x}}_j] = 0. \tag{68}$$

If we do the following:

1. enter (67) in (68)

2. compute $[\hat{x_i}, \hat{p}]$ to order $\mathcal{O}(\hat{\mathrm{p}})$, getting $[\hat{x_i}, \hat{p}] = i\hbar(\hat{\mathrm{p}}_i\hat{\mathrm{p}}^{-1} + (\alpha_1 + \alpha_2)\hat{\mathrm{p}}_i)$

3. compute$[\hat{x_i}, \widehat{p^{-1}}]$ to order $\mathcal{O}(\hat{\mathrm{p}})$, getting $[\hat{x_i}, \widehat{p^{-1}}] = -i\hbar\hat{\mathrm{p}}_i\hat{\mathrm{p}}^{-3}(1 + (\alpha_1 + \alpha_2)\hat{\mathrm{p}})$

4. entering the results of 2. and 3. in the result of 1.

then we get with some simplification

$$0 = [[x_j, p_k], \hat{\mathrm{x}}_i] + [[p_k, x_i], \hat{\mathrm{x}}_j] = \left((\alpha_1 - \alpha_2)\, \hat{\mathrm{p}}^{-1} + \left(\alpha_1^2 + 2\beta_1 - \beta_2\right)\right) \Delta_{ijk} \tag{69}$$

with $\Delta_{ijk} \equiv \hat{\mathrm{p}}_i \delta_{jk} - \hat{\mathrm{p}}_j \delta ik$. Solving (69) gives:

- $\alpha_1 = \alpha_2 \equiv -\alpha$, with $\alpha > 0$
- $\beta_2 = 2\beta_1 + \alpha_1^2$
- for dimensional reasons β has to be proportional to α^2 and we can assume $\beta_1 = \alpha^2 \Longrightarrow \beta_2 = 3\alpha^2$

We now put this in equation (67) and get

$$[\hat{x_i}, \hat{p_j}] = i\hbar \left(\delta_{ij} - \alpha \left(\hat{\mathrm{p}} \delta_{ij} + \frac{\hat{\mathrm{p}}_i \hat{\mathrm{p}}_j}{\hat{\mathrm{p}}} \right) + \alpha^2 \left(\hat{\mathrm{p}}^2 \delta_{ij} + 3 \hat{\mathrm{p}}_i \hat{\mathrm{p}}_j \right) \right). \tag{70}$$

In one dimension this commutator has the form

$$[\hat{x_i}, \hat{p_j}] = i\hbar \left(1 - \alpha \hat{\mathrm{p}} + \alpha^2 \hat{\mathrm{p}}^2\right). \tag{71}$$

Now we set $\alpha \equiv \gamma_0/(M_p c)$ to get (71) in the form of (66).

The special thing about this commutator is that it accounts for a minimal length but also for a maximum measurable momentum $\Delta p_{max} \approx M_p c/\gamma_0$ [2].

Current bounds on γ_0 are $\gamma_0 \leq 10^{10}$ and were obtained through a careful calculation of an effect of the commutator on the Lamb shift (66), which was'nt measured and therefore sets a bound on γ_0 [2].

6 The scheme to measure Planck scale deformations of the commutator

In this section I will present the scheme proposed by Pikovski et al.[19] to measure the the deformations of the commutators presented earlier in this work.

This scheme allows one to map the canonical commutator of a massive object, which in our case will be a mechanical oscillator, with the help of quantum optics on to a light field. To simplify the discussion I will use dimensionless quadrature operators:

- $\hat{\mathrm{X}}_m$ defined through $\hat{\mathrm{x}} = x_0\hat{\mathrm{X}}_m$ and $x_0 = \sqrt{\hbar/(m\omega_m)}$
- $\hat{\mathrm{P}}_m$ defined through $\hat{\mathrm{p}} = p_0\hat{\mathrm{P}}_m$ and $p_0 = \sqrt{\hbar m\omega_m}$

The scheme uses four displacements in the phase space of the mechanical oscillator. A displacement is given by by the displacement operator:

$$D(z/\sqrt{2}) = e^{i(Re[z]\hat{\mathrm{X}}_m - Im[z]\hat{\mathrm{P}}_m)}. \tag{72}$$

This operator displaces the mean of position/momentum of the state it is applied to by Re[z]/Im[z]. Two displacements create in addition to their displacement a phase to the state. For our scheme we utilize four displacements provided by the optical field acting on the mechanical resonator. The sequence of the four displacements displaces the mechanical state around in a loop in phase space. This sequence is described by the following operator:

$$\xi = e^{i\lambda\hat{\mathrm{n}}_l\hat{\mathrm{P}}_m} e^{-i\lambda\hat{\mathrm{n}}_l\hat{\mathrm{X}}_m} e^{-i\lambda\hat{\mathrm{n}}_l\hat{\mathrm{P}}_m} e^{i\lambda\hat{\mathrm{n}}_l\hat{\mathrm{X}}_m} \tag{73}$$

where

- λ is the interaction strength of the optical field with the oscillator
- $\hat{\mathrm{n}}_l$ is the photonnumber operator of the light field
- $\hat{\mathrm{X}}_m$ and $\hat{\mathrm{P}}_m$ are the position and momentum operators of the mechanical oscillator

In classical physics the four displacements would cancel each other since $\hat{\mathrm{X}}_m$ and $\hat{\mathrm{P}}_m$ would "commute" and neither the state of the oscillator nor the state of the light field would be affected. In quantum physics this is not the case. As stated before, two or more displacements do not only displace the state of our system, but also create a phase that will depend on the commutator

$$[X_m, P_m, =]iC_1 \tag{74}$$

Now we rewrite equation (73). The following relation is quite helpful:

$$e^{a\hat{\mathrm{X}}_m}\hat{\mathrm{P}}_m e^{-a\hat{\mathrm{X}}_m} = \sum_{k=0}^{\infty}\left(\frac{i^k a^k}{k!}\right) C_k \tag{75}$$

with $a \equiv -i\lambda\hat{\mathrm{n}}_l$,$iC_k = [\hat{\mathrm{X}}_m, C_{k-1}]$ and $C_0 = \hat{\mathrm{P}}_m$

$$\begin{aligned}
\xi &= e^{-a\hat{\mathrm{P}}_m} e^{a\hat{\mathrm{X}}_m} e^{a\hat{\mathrm{P}}_m} e^{-a\hat{\mathrm{X}}_m} \\
&= e^{-a\hat{\mathrm{P}}_m} e^{a\hat{\mathrm{X}}_m} \left(\sum_{k=0}^{\infty}\left(\frac{i^k a^k}{k!}\right)\hat{\mathrm{P}}_m^k\right) e^{-a\hat{\mathrm{X}}_m} \\
&= e^{-a\hat{\mathrm{P}}_m} e^{a\sum_{k=0}^{\infty}\left(\frac{i^k a^k}{k!}\right)C_k} \\
&= e^{-a\hat{\mathrm{P}}_m} e^{a\sum_{k=0}^{\infty}\left(\frac{(\lambda\hat{\mathrm{n}}_l)^k}{k!}\right)C_k}
\end{aligned}$$

(the following step is only possible when the C_k's are only a function of $\hat{\mathrm{P}}_m$ which in our case is satisfied, even for the deformed commutation relations)

$$\begin{aligned}
&= e^{-a\hat{\mathrm{P}}_m + a\hat{\mathrm{P}}_m + a\sum_{k=1}^{\infty}\left(\frac{(\lambda\hat{\mathrm{n}}_l)^k}{k!}\right)C_k} \\
&= e^{-i\lambda\hat{\mathrm{n}}_l\sum_{k=1}^{\infty}\left(\frac{(\lambda\hat{\mathrm{n}}_l)^k}{k!}\right)C_k} = \xi
\end{aligned}$$

$$\xi = e^{-i\lambda\hat{\mathrm{n}}_l\sum_{k=1}^{\infty}\left(\frac{(\lambda\hat{\mathrm{n}}_l)^k}{k!}\right)C_k} \tag{76}$$

As we see in the end ξ explicitly depends on the commutator of $\hat{\mathrm{X}}_m$ and $\hat{\mathrm{P}}_m$! For the standard quantum mechanical commutator $C_1 = 1$ and $C_k = 0$ for $k > 1$. Therefore ξ becomes

$$\xi = e^{-i\lambda^2 \hat{\mathrm{n}}_l^2} \tag{77}$$

As one can see, the light field in this case experiences an $\hat{\mathrm{n}}_l^2$ operation, a so called self-Kerr nonlinearity. The state of the mechanical oscillator on the contrary stays unaffected. We will now denote the light field by $\hat{\mathrm{a}}_l$. The real and the imaginary part of $\hat{\mathrm{a}}_l$ represent the amplitude and phase quadratures of the optical field. To simplify calculations we assume coherent input state $|\alpha\rangle$ with real amplitude. Furthermore we will neglect a possible deformation of the commutator of the optical field since the effect should be negligible in comparison to the commutator of the massive oscillator.

The mean of the optical field with the displacement for the standard commutator (77) can now be calculated:

$$\langle a_l \rangle = \langle \Psi(t)|\hat{\mathrm{a}}|\Psi(t)\rangle = \langle \alpha|\xi^\dagger \hat{\mathrm{a}} \xi|\alpha\rangle \tag{78}$$

To calculate $\xi^\dagger \hat{\mathrm{a}} \xi$ we use the Baker-Campell-Hausdorff-formula

$$e^{\hat{\mathrm{A}}} \hat{\mathrm{B}} e^{-\hat{\mathrm{A}}} = \hat{\mathrm{B}} + [\hat{A}, \hat{B}] + \frac{1}{2}[\hat{\mathrm{A}}, [\hat{A}, \hat{B}]] + \frac{1}{3!}\dots \tag{79}$$

where in our case $\hat{\mathrm{A}} \equiv i\lambda^2 \hat{\mathrm{n}}_l^2$ and $\hat{\mathrm{B}} \equiv \hat{\mathrm{a}}$. Therefore we have to calculate

$$[\hat{\mathrm{n}}_l^2, \hat{\mathrm{a}}] = \hat{\mathrm{n}}_l[\hat{\mathrm{n}}_l, \hat{\mathrm{a}}] + [\hat{\mathrm{a}}, \hat{\mathrm{n}}_l]\hat{\mathrm{n}}_l = \hat{\mathrm{n}}_l(-\hat{\mathrm{a}}) + (-\hat{\mathrm{a}})\hat{\mathrm{n}}_l = -(2\hat{\mathrm{n}}_l + 1)\hat{\mathrm{a}}. \tag{80}$$

In order to calculate the next term we have to know

$$[\hat{\mathrm{n}}_l^2, [\hat{\mathrm{n}}_l^2, \hat{\mathrm{a}}]] = (2\hat{\mathrm{n}}_l + 1)^2 \hat{\mathrm{a}}. \tag{81}$$

Now we can see that the result of $\xi^\dagger \hat{\mathrm{a}} \xi$ will be

$$\xi^\dagger \hat{\mathrm{a}} \xi = e^{-i\lambda^2(2\hat{\mathrm{n}}_l+1)} \hat{\mathrm{a}}. \tag{82}$$

The only thing left now is to calculate $\left\langle e^{-i\lambda^2(2\hat{n}_l+1)}\right\rangle$. I will make use of $e^{\varphi\hat{n}_l}|\alpha\rangle = |\alpha e^{\varphi}\rangle$ and $\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}+\alpha^*\beta}$:

$$\left\langle e^{-i\lambda^2(2\hat{n}_l+1)}\right\rangle = \langle\alpha|e^{-i\lambda^2(2\hat{n}_l+1)}|\alpha\rangle = \langle\alpha|e^{-i\lambda^2}e^{(-i2\lambda^2\hat{n}_l)}|\alpha\rangle \tag{83}$$

$$= e^{-i\lambda^2}\langle\alpha|e^{(-i2\lambda^2\hat{n}_l)}|\alpha\rangle = e^{-i\lambda^2}\langle\alpha|\alpha e^{-i2\lambda^2}\rangle \tag{84}$$

$$= e^{-i\lambda^2}e^{-|\alpha|^2+\alpha^2}e^{-i2\lambda^2} \tag{85}$$

which finally gives us the mean of the optical field in the case we are dealing only with the standard quantum mechanics commutator:

$$\langle a_l\rangle_{qm} = \alpha e^{-i\lambda^2-N_p(1-e^{-i2\lambda^2})}. \tag{86}$$

We now want to calculate ξ_β. At first we define

$$\beta \equiv \beta_0\hbar\omega_m m/(M_p c)^2 \tag{87}$$

Assumed $\beta \ll 1$ we can calculate the C_k's from (74) to first order in β:

- $C_1 = 1 + \beta\hat{\mathrm{P}}_m^2$
- $C_2 \approx 2\beta\hat{\mathrm{P}}_m$
- $C_3 \approx 2\beta$
- $C_k \approx 0$ for $k > 3$

If we now put this in (76) we get

$$\xi_\beta = e^{-i\lambda^2\hat{n}_l^2}e^{-i\beta(\lambda^2\hat{n}_l^2\hat{\mathrm{P}}_m^2+\lambda^3\hat{n}_l^3\hat{\mathrm{P}}_m+(1/3)\lambda^4\hat{n}_l^4)}. \tag{88}$$

Now we can calculate the mean optical field $\langle a_l\rangle$ and we do that for $N_p \gg 1$ and the state of the mechanical oscillator being thermal and has a mean phonon occupation $\bar{n} \ll \lambda N_p$. In this case the mean optical field becomes

$$\langle a_l \rangle \cong \langle a_l \rangle_{qm} e^{-i\Theta} \tag{89}$$

for $|\Theta| \ll 1$.

The phase Θ caused by the β_0-modification of the commutator is then

$$\Theta(\beta) \cong \frac{4}{3}\beta N_p^3 \lambda^4 e^{-i6\lambda^2}. \tag{90}$$

Next we want to calculate ξ_μ with μ being defined as

$$\mu \equiv \mu_0 \frac{\hbar m^2}{M_p^2}. \tag{91}$$

We will do this in the limit $p/c \ll m \leq M_p$. In this case the only non zero C_k is $C_1 = (1+\mu)$ and if we enter this into (76) we get for ξ_μ

$$\xi_\mu = e^{-i\lambda^2 \hat{n}_l^2 (1+\mu)}. \tag{92}$$

$\langle a_l \rangle$ then has the form

$$\langle a_l \rangle = \alpha e^{-i\lambda^2(1+\mu)} e^{-N_p(1-e^{-i2\lambda^2(1+\mu)})}. \tag{93}$$

For the limit $\mu \leq 1$ and $N_p \gg 1$ we again get (89) with

$$\Theta(\mu) \cong 2\mu N_p \lambda^2 e^{-i2\lambda^2} \tag{94}$$

At last we want to discuss the case of the γ_0-modification of the commutator. With

$$\gamma \equiv \gamma_0 \frac{\sqrt{\hbar m \omega_m}}{M_p c}. \tag{95}$$

the C_k's to first order in $\gamma \ll 1$ are

- $C_1 = (1 - \gamma P_m)$

- $C_2 = \gamma$
- $C_k = 0$ for $k > 2$

The ξ_γ then becomes

$$\xi_\gamma = e^{-i\lambda^2 \hat{\text{n}}_l^2} e^{i\gamma(\lambda^2 \hat{\text{n}}_l^2 \hat{\text{P}}_m + \frac{1}{2}\lambda^3 \hat{\text{n}}_l^3)}. \tag{96}$$

In the limit $N_p \gg 1$ we get

$$\Theta(\gamma) \cong \frac{3}{2}\gamma N_p^2 \lambda^3 e^{-i4\lambda^2}. \tag{97}$$

In the following figure we can see the action of ξ on the optical state:

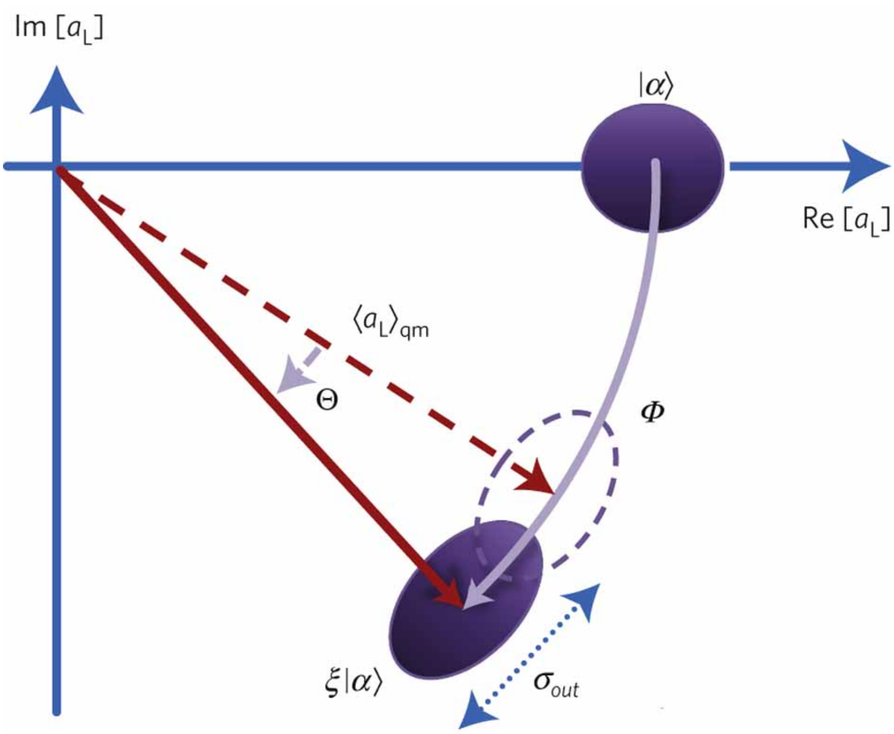

Figure 8: Picture of the rotation of $|\alpha\rangle$ in phase space due to the action of $\xi_{\beta,\gamma,\mu}$. The dotted one is the state in the absence of a deformation. In between these two there's the angle Θ.(from [19])

As we can see ξ rotates the input state $|\alpha\rangle$ by an angle Φ. This angle Φ may possible differ from the angle between the input state and the state we would get through ordinary quantum mechanics by the angle Θ we calculated before for different commutator deformations. It is possible to measure the angle Φ witch interferometric schemes, like homodyne detection, where the fundamental imprecision $\delta(\Phi)$ is given by

$$\delta(\Phi) = \frac{\sigma_{out}}{\sqrt{N_p N_r}} \tag{98}$$

where

- σ_{out} is the quadrature width belonging to the output state. It is assumed that it stay's very near to 1/2 which is the value for coherent states.
- N_p the number of photons in a pulse
- N_r the number of experimental runs

In order to be able to resolve the angle Θ it is clear that

$$\delta(\Phi) < \Theta \tag{99}$$

has to be fulfilled.

7 Experimental implementation of the scheme

In this section I want to discuss the experimental setup developed by [19] that can realize the scheme to measure the modification of the commutator.

It uses a quantum optomechanics setup. This offers a fascinating test bed for quantum gravity for some reason:

- it allows to deal with massive quantum states with a range of masses from picograms up to kilograms
- it allows to laser-cool motional states down to the quantum ground state
- it provides strong coupling and coherent interaction

In our case the main focus is on the pulsed regime of quantum optomechanics.

In the next figure 9 you can see the setup of the experiment:

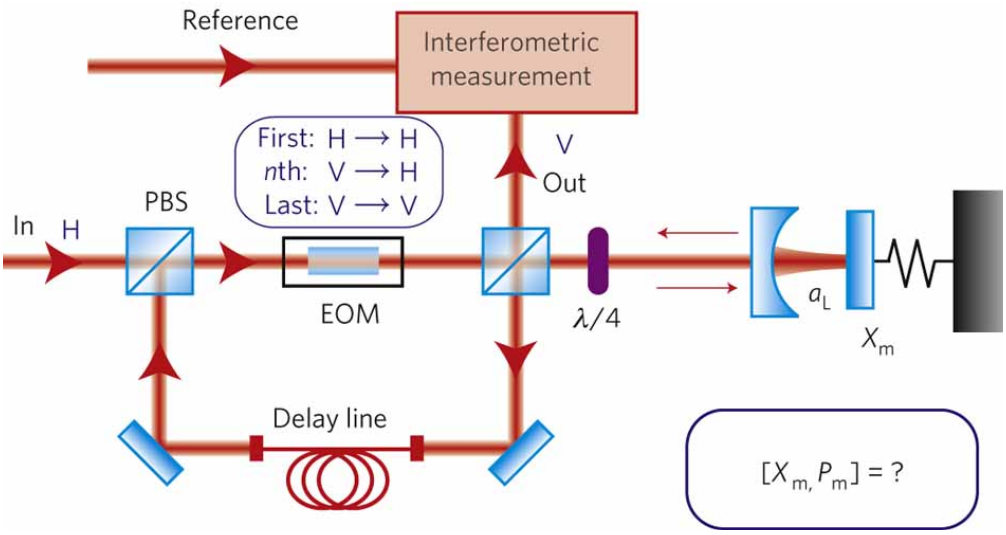

Figure 9: A schematic setup for the experiment. (from [19])

A pulsed beam is splited into a reference beam and an incoming beam with polarization "H". The incoming beam passes through a polarizing beamsplitter(PBS1) and an electro optical modulator(EOM) then another PBS2. After that it passes through a $\lambda/4$-plate ($\lambda/4$P) and then reaches the optomechanical part where it interacts with the massive mechanical oscillator(MMOI). Then the beam passes again the $\lambda/4$P where the polarization of the beam is now rotated to “V” and therefore is reflected down at PBS2 and goes through a delay line(DL) that allows X_m to change role with P_m for the MMOI. The beam is now reflected as PBS1, goes through EOM that changes polarization to “H”. It passes again PBS2, goes through the $\lambda/4$P goes to the MMOI, again through $\lambda/4$P, polarization changed to “V”. The beam again is reflected at PBS2, goes through DL while the roles of P_m and X_m interchange and is reflected at PBS1. Then through EOM where polarization is changed to “H”, passing through PBS2, $\lambda/4$P, get to MMOI, back to the$\lambda/4$P, polarization is changed to “V”, reflected down at PBS2 and goes through the DL while X_m and P_m switch roles. After that it is reflected at PBS1, goes through EOM where the polarization changes to “H”, passes through PBS2, $\lambda/4$P, get to MMOI, back to the$\lambda/4$P, polarization is changed to “V”, reflected down at PBS2 and goes through the DL. Now it is reflected at PBS1, goes through EOM with polarization unchanged and is reflected upwards at PBS2 to the measurement. There the reference beam together with the beam that passed the experiment is measured for their phase difference with a homodyne detection or equivalent interferometric measurement. ted into a reference beam and an incoming beam with polarization “H”. The incoming beam passes through a polarizing beamsplitter(PBS1) and an electro optical modulator(EOM) then another PBS2. After that it passes through a $\lambda/4$-plate ($\lambda/4$P) and then reaches the optomechanical part where it interacts with the massive mechanical oscillator(MMOI). Then the beam passes again the $\lambda/4$P where the polarization of the beam is now rotated to “V” and therefore is reflected down at PBS2 and goes through a delay line(DL) that allows X_m to change role with P_m for the MMOI. The beam is now reflected as PBS1, goes through EOM that changes polarization to “H”. It passes again PBS2, goes through the $\lambda/4$P goes to the MMOI, again through $\lambda/4$P, polarization changed to “V”. The beam again is reflected at PBS2, goes through DL while the roles of P_m and X_m interchange and is reflected at PBS1. Then through EOM where polarization is changed to “H”, passing through PBS2, $\lambda/4$P, get to MMOI, back to the$\lambda/4$P, polarization is changed to “V”, reflected down at PBS2 and goes through the DL while X_m and P_m switch roles. After that it is reflected at PBS1, goes through EOM where the polarization changes to “H”, passes through PBS2, $\lambda/4$P, get to MMOI, back to the$\lambda/4$P, polarization is changed to “V”, reflected down at PBS2 and goes through the DL. Now it is reflected at PBS1, goes through EOM with polarization unchanged and is reflected upwards at PBS2 to the measurement. There the reference beam together with the beam that passed the experiment is measured for their phase difference with a homodyne detection or equivalent interferometric measurement.

In short:

$$|H\rangle_{in} \to PBS1 \to EOM \to PBS2 \to \lambda/4P \to MMOI \to$$
$$\lambda/4P(|H\rangle \to |V\rangle) \to PBS2 \to DL \to$$
$$PBS1 \to EOM(|V\rangle \to |H\rangle) \to PBS2 \to \lambda/4P \to MMOI \to$$
$$\lambda/4P(|H\rangle \to |V\rangle) \to PBS2 \to DL \to$$
$$PBS1 \to EOM(|V\rangle \to |H\rangle) \to PBS2 \to \lambda/4P \to MMOI \to$$
$$\lambda/4P(|H\rangle \to |V\rangle) \to PBS2 \to DL \to$$
$$PBS1 \to EOM(|V\rangle \to |H\rangle) \to PBS2 \to \lambda/4P \to MMOI \to$$
$$\lambda/4P(|H\rangle \to |V\rangle) \to PBS2 \to DL \to PBS1 \to EOM \to$$
$PBS2 \to$ "interferometric measurement together with reference beam"

7.1 Deleterious effects due to the experimental implementation

Now that it is clear how the scheme is experimentally implemented, we have to calculated all the deleterious effects that come with this specific implementation. The derivations are taken from the supplementary information of [19].

7.1.1 The interaction strength λ

The Hamilton operator for the interaction of the light field inside the cavity and the mechanical oscillator is given by

$$\hat{\mathrm{H}} = \hbar\omega_m \hat{\mathrm{n}}_m - \hbar g_0 \hat{\mathrm{n}}_l \hat{\mathrm{X}}_m. \tag{100}$$

In order to also calculate the effects of cavity filling and decay one has to solve the optical Langevin equation:

$$\frac{d\hat{\mathrm{a}}_l}{dt} = (ig_0\hat{\mathrm{X}}_m - \kappa)\hat{\mathrm{a}}_l + \sqrt{2\kappa}(\hat{\mathrm{a}}_l^{(in)} + \sqrt{N_p}\alpha_{in}) \tag{101}$$

with boundary condition

$$\hat{\mathrm{a}}_l^{(in)} + \hat{\mathrm{a}}_l^{(out)} = \sqrt{2\kappa}\hat{\mathrm{a}}_l \tag{102}$$

where κ is the cavity bandwidth and α_{in} is the incident cavity drive which is normalized $\int dt\alpha_{in}^2 = 1$ to the N_p. Because we are in the short pulse regime we neglect the mechanical motion during the pulse interaction and the interaction is given by the effective unitary operator

$$\hat{\mathrm{U}} = e^{i\lambda \hat{\mathrm{n}}_l \hat{\mathrm{X}}_m}. \tag{103}$$

λ can now be determined by the total momentum transfer onto the mechanics by the optical pulse

$$\left\langle \hat{\mathrm{P}}_m \right\rangle = g_0 \int dt \left\langle \hat{\mathrm{n}}_l(t) \right\rangle. \tag{104}$$

$\hat{\mathrm{n}}_l(t)$ is obtained from the solution of (101). In the end one gets

$$\lambda = \zeta \frac{g_0}{\kappa} \tag{105}$$

with ζ for $\hat{\mathrm{U}}$ being

$$\zeta = \int dt e^{-2\kappa t} \kappa^2 \left(\int_{-\infty}^{t} dt' e^{\kappa t'} \alpha_{in}(t') \right)^2. \tag{106}$$

For $\omega_m \ll \tau^{-1} \ll \kappa$, with τ being the pulse duration, $\zeta \approx 1$ and $\lambda \approx g_0/\kappa$.

If the pulse enters the cavity more the one time, the interaction strength will be reduced, but in this regime not to much and we can assume

$$\lambda_i = \eta^{i-1} \lambda \tag{107}$$

for the i-th interaction due to the dominant effect of the loss of light, we assume to be $\eta \approx 0.9$. The four-displacement operator is then given by

$$\xi_\eta = e^{i\lambda_4 \hat{\mathrm{n}}_l \hat{\mathrm{P}}_m} e^{-i\lambda_3 \hat{\mathrm{n}}_l \hat{\mathrm{X}}_m} e^{-i\lambda_2 \hat{\mathrm{n}}_l \hat{\mathrm{P}}_m} e^{i\lambda_1 \hat{\mathrm{n}}_l \hat{\mathrm{X}}_m} \tag{108}$$

which can be written as

$$\xi_\eta = \xi_0^{'} e^{i\lambda(1-\eta^2)\hat{\mathrm{n}}_l \hat{\mathrm{P}}_m} e^{i\lambda(1-\eta^2)\hat{\mathrm{n}}_l \hat{\mathrm{X}}_m}. \tag{109}$$

$\xi_0^{'}$ is the four displacement operator but with different interaction strengths. The interaction for the different commutators will be reduced to:

- β-deformation: $\lambda^4 \rightarrow \eta^7 \lambda^4$
- γ-deformation:$\lambda^3 \rightarrow \eta^5 \lambda^3$
- μ-deformation:$\lambda^2 \rightarrow \eta^3 \lambda^2$

The Θ-contribution to the mean of the outgoing optical state will be therefore reduced by the factor I:

- β-deformation: $I_\beta = 0.5$
- γ-deformation:$I_\gamma = 0.6$
- μ-deformation:$I_\mu = 0.7$

Equation (109) makes the state of the outgoing optical field strongly dependent on the state of the mechanical oscillator. For a "*thermal distribution of the center of mass mode (...)[of the oscillator with according](...) mean phonon occupation $\overline{n}$ the optical mean [will] be reduced by [the factor R]*"[19](Supplemantary information):

$$R = e^{-\overline{n}\lambda^2(1-\eta^2)(1-\eta^4)/2}. \tag{110}$$

Effectively $\sqrt{N_p}$ will be reduced by the factor R.

7.1.2 Mechanical decoherence

An additional effect is mechanical decoherence. The problem is, that in the time between the interactions of the pulse and the center of mass mode of the mechanical oscillator, the center of mass mode couples to the other degrees of freedom within the mechanical oscillator. In order to estimate this effect one considers linear coupling of the mode to an infinite bath of harmonic oscillators. This interaction can be described with the following Hamilton operator:

$$\hat{\mathrm{H}}_{int} = \sum_i \nu_i \left(\hat{\mathrm{b}}_i e^{-i\omega_i t} + \hat{\mathrm{b}}_i^\dagger e^{i\omega_i t} \right) \hat{\mathrm{X}}_m. \tag{111}$$

The $\hat{\mathrm{b}}_i$ are the annihilation operators of the i-th mode of the bath and ω_i is the frequency of this mode. ν_i is the strength which with this mode interacts with the center of mass mode. Introducing the following notation

$$\hat{\mathrm{B}}(t) = \sum_i \nu_i \left(\hat{\mathrm{b}}_i e^{-i\omega_i t} + \hat{\mathrm{b}}_i^\dagger e^{i\omega_i t} \right) \tag{112}$$

one can write the solutions for the position and momentum operator in the following form:

$$\hat{\mathrm{X}}_m(t) = \hat{\mathrm{X}}_m^{(0)}(t, t_0) - \int_{t_0}^{t} dt' \hat{\mathrm{B}}(t') \sin(\omega_m (t - t')) \tag{113}$$

$$\hat{\mathrm{P}}_m(t) = \hat{\mathrm{P}}_m^{(0)}(t, t_0) + \int_{t_0}^{t} dt' \hat{\mathrm{B}}(t') \cos(\omega_m (t - t')). \tag{114}$$

The $\hat{\mathrm{X}}_m^{(0)}(t, t_0)$ and $\hat{\mathrm{P}}_m^{(0)}(t, t_0)$ are given by

$$\hat{\mathrm{X}}_m^{(0)}(t, t_0) = Re \left(\hat{\mathrm{A}}(t_0) e^{i\omega_m (t - t_0)} \right) \tag{115}$$

and

$$\hat{\mathrm{P}}_m^{(0)}(t, t_0) = Im \left(\hat{\mathrm{A}}(t_0) e^{i\omega_m (t - t_0)} \right) \tag{116}$$

with the initial value

$$\hat{\mathrm{A}}(t_0) = \hat{\mathrm{X}}_m(t_0) + i\hat{\mathrm{P}}_m(t_0) \tag{117}$$

and are the position and momentum operator in the absence of decoherence.

Under the assumption that the bath is initially uncorrallated with the center of mass mode our ξ-operator takes the form

$$\xi_B = \xi_0 e^{i\lambda_l \hat{\mathrm{B}}_1} e^{i\lambda_l \hat{\mathrm{B}}_2} e^{i\lambda_l \hat{\mathrm{B}}_3} \tag{118}$$

with the B_i's beeing

$$\hat{\mathrm{B}}_1 = \int_0^{\pi/2\omega_m} dt \hat{\mathrm{B}}(t^{'}) \cos(\omega_m t) \tag{119}$$

$$\hat{\mathrm{B}}_2 = \int_0^{\pi/\omega_m} dt \hat{\mathrm{B}}(t^{'}) \sin(\omega_m t) \tag{120}$$

$$\hat{\mathrm{B}}_3 = -\int_0^{3\pi/2\omega_m} dt \hat{\mathrm{B}}(t^{'}) \cos(\omega_m t) \tag{121}$$

and ξ_0 beeing the one without decoherence.

Under the assumption of a Markovian batch such that

1. $\left\langle \hat{\mathrm{B}}(t) \right\rangle = 0$
2. $\left\langle \hat{\mathrm{B}}(t)\hat{\mathrm{B}}(t^{'}) \right\rangle = \gamma_m \coth(\hbar\omega_m/2k_bT)\delta(t-t^{'})$

that is negligable bath correlation times, we can expand the $e^{\cdots}$ in ξ_B to second order and take the expactation value. If we now write the mechanical dampening rate γ_m as

$$\gamma_m = \frac{\omega_m}{Q} \tag{122}$$

using the mechanical quality factor Q, then we can write the mean of the optical field up to $\mathcal{O}(T/Q)$ as

$$\langle \hat{\mathrm{a}}_l \rangle_B \approx \langle \hat{\mathrm{a}}_l \rangle_0 \left(1 - \lambda^2 \frac{k_b T}{\hbar\omega_m Q}\right). \tag{123}$$

$\langle \hat{\mathrm{a}}_l \rangle_0$ is here the mean of the optical field in the absence of decoherence. Therefore the term

$$D \equiv \left(1 - \lambda^2 \frac{k_b T}{\hbar \omega_m Q}\right). \qquad (124)$$

effectivly reduces $\sqrt{N_p}$ by a factor of D.

8 A feasibility study

In this section I want to perform a feasibility study on the experiment in question. That means putting numbers to the systems parameters that are within current experimental possibilities and compare the results with the parameters needed to test Planck scale deformations of the commutator.

At first we will fit equations (90), (94) and (97) with experimental parameters and insert the definitions for β (87), μ(91) and γ(95). We will use $\lambda = g_0/\kappa = 4Fx_0/\lambda_l$ with F being the finesse of the cavity and λ_l the wavelength of the laser. Then we will take the absolute value.

1. $\|\Theta(\beta)\|$:

$$\|\Theta(\beta)\| = \frac{4}{3}\beta N_p^3\lambda^4 = \beta_0\frac{1024\hbar^3F^4N_p^3}{3M_p^2c^2\lambda_l^4 m\omega_m} \tag{125}$$

2. $\|\Theta(\mu)\|$:

$$\|\Theta(\mu)\| = 2\mu N_p\lambda^2 = \mu_0\frac{32\hbar F^2 m N_p}{M_p^2\lambda_l^2\omega_m} \tag{126}$$

3. $\|\Theta(\gamma)\|$:

$$\|\Theta(\gamma)\| = \frac{3}{2}\gamma N_p^2\lambda^3 = \gamma_0\frac{96\hbar^2F^3N_p^2}{M_p c\lambda_l^3 m\omega_m} \tag{127}$$

Since $\delta(\phi) < \|\Theta(\upsilon)\|$ has to be fulfilled in order to see an effect of an possible deformation it follows for every of our deformations with a deformation parameter υ that

$$\frac{\delta(\phi)}{(\|\Theta(\upsilon)\|/\upsilon)} < \upsilon. \tag{128}$$

On the other hand if we don't see an effect this sets an upper bound on υ:

$$\frac{\delta(\phi)}{(\|\Theta(\upsilon)\|/\upsilon)} \geq \upsilon. \tag{129}$$

Now we also can take into account the deleterious effects considered in the previous section(see definitions(124)(110) and the $I_{\beta,\mu,\gamma}$) and then we get

$$\frac{\delta(\phi)/(R*D)}{(\|\Theta(\upsilon)\|/\upsilon)*I_\upsilon} \geq \upsilon. \tag{130}$$

which describes the bond set on the deformation parameter υ in case of a null result, meaning that one is not able to measure an angle Θ differing from the one that we expect due to ordinary quantum mechanics.

8.1 The experimental parameters in consideration

Here I want to outline two different sets of experimental parameters. The first set only uses system parameters that are already used for an experiment considered in the paper of Vanner et al. [22]. The experiment is currently in progress. The second set is one that should be achievable in reasonable time with reasonable efforts.

The first set:

- the finesse in this case would be $F = 1$, that is using no cavity as a rule of thumb
- the mass of the oscillator would be $m = 300ng = 3*10^{-10}kg$, which is already achieved in various experiments
- the frequency in this case then would be given by $\omega_m = 2\pi kHz$
- the wavelength of the laser will be $\lambda_l = 1064nm$
- the pulse duration will be $1\mu s$ at $1\mu W$ yielding an average photon number of $N_p \approx 5.4*10^6$
- the experiment would take place at approximately room temperature so $T = 300K$

The second set:

- the finesse in this case would be $F = 10^3$
- the mass of the oscillator would be $m = 300ng = 3*10^{-10}kg$
- the frequency in this case then would be given by $\omega_m = 2\pi kHz$

- the wavelength of the laser will be $\lambda_l = 1064nm$
- the pulse duration will be $1\mu s$ at $1\mu W$ yielding an average photon number of $N_p \approx 5.4 * 10^6$
- the experiment would take place at approximately $T = 10K$, which is achievable with standard cooling techniques

8.2 Achievable bounds on β_0

Now we come to the point where we are able to calculate possible bounds on the deformation parameter β_0.

For the first set we can set a bound on β_0

- without deleterious effects of

$$\beta_0 \leq 3.6 * 10^{47} \tag{131}$$

- The deleterious effects for this parameters are:

$$R \approx 1 \tag{132}$$

$$D \approx 1 \tag{133}$$

- also taking into account this deleterious effects and $I_\beta = 0.5$ we get a bound of

$$\beta_0 \leq 7.2 * 10^{47} \tag{134}$$

- therefore we are still 15 orders of magnitude above current bounds for β_0 ($\beta_0 \leq 10^{33}$)

For the second set of parameters we can put a bound on β_0

- without deleterious effects of

$$\beta_0 \leq 3.6 * 10^{35} \tag{135}$$

- The deleterious effects for this parameters are:

$$R = 0.994632 \tag{136}$$

$$D \approx 1 \tag{137}$$

- also taking into account this deleterious effects and $I_\beta = 0.5$ we get a bound of

$$\beta_0 \leq 7.2 * 10^{35} \tag{138}$$

- therefore we are still 3 orders of magnitude above current bounds for β_0 ($\beta_0 \leq 10^{33}$)

As we can see in the following plot 10 the product of $D * R$ decreases rapidly with the increase of the finesse F.

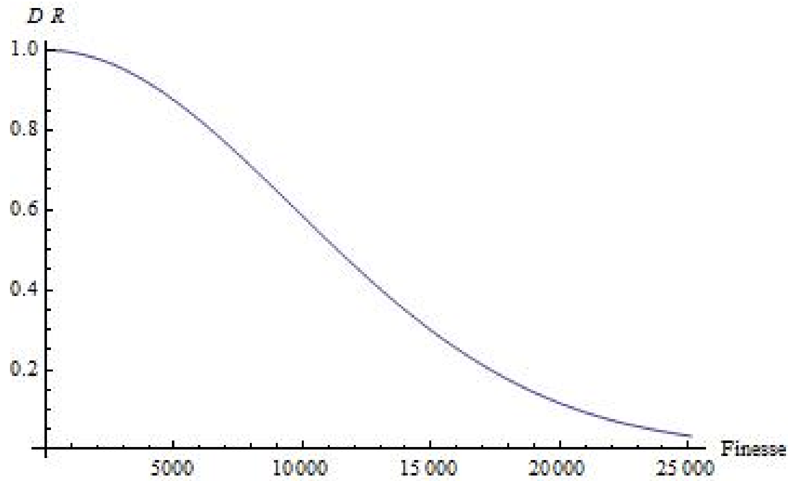

Figure 10: Plot of D*R in dependence of the finesse for temperature $T = 10K$ and according thermal occupation of phonons of the mechanical oscillator

This shows that an increase of of the finesse will not automatically yield an decrease of the bound on β_0. In the following plot11 one can see the bound on β_0 for varying F with fixed temperature $T = 10$ and according phonon occupation:

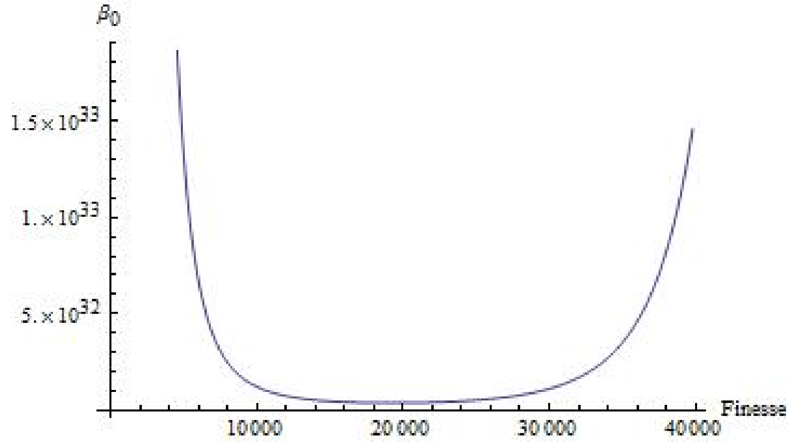

Figure 11: Plot of the bound β_0 in dependence of the finesse for temperature $T = 10K$ and according thermal occupation of phonons of the mechanical oscillator

As one can see in this case it does'nt make sense to push the finesse to more then $F = 2 * 10^4$. For this value of F we could reach a bound for β_0 of

$$\beta_0 \leq 3.9 * 10^{31} \tag{139}$$

This already would improve the bound by nearly 2 orders.

To be able to push the finesse even further one needs to decrease the deleterious effects, for example by cooling the center of mass mode down. Assuming we can cool down the center of mass mode down to a phonon occupation of $\overline{n} = 30$ with the temperature staying at $T = 10K$ we get the following plot 12

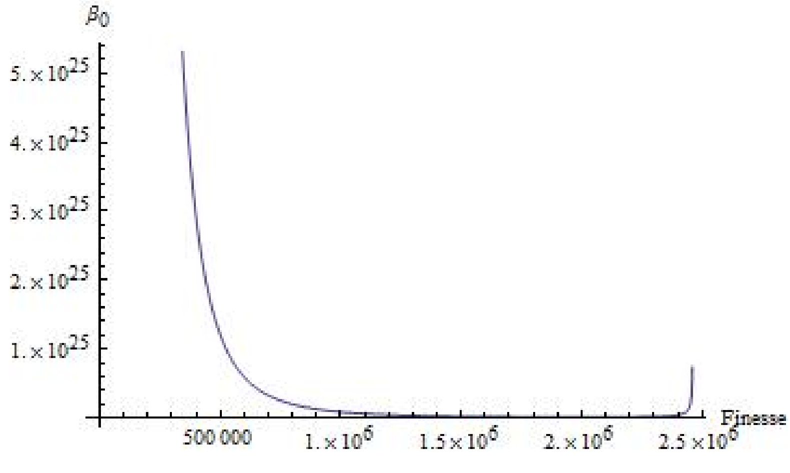

Figure 12: Plot of the bound β_0 in dependence of the finesse for temperature $T = 10K$ and a mean phonon occupation of $\overline{n} = 30$

As one can see in this case we can push F to $F = 2 * 10^6$ getting a bound on β_0 of

$$\beta_0 \leq 1.3 * 10^{23} \tag{140}$$

which would be nearly 10 orders better than the current bounds.

If we now assume we can cool the hole experiment down to $T = 25mK$, which would be achievable via dilution refrigerating, and cool the center of mass mode down to $\overline{n} = 30$ we get the following plot 13

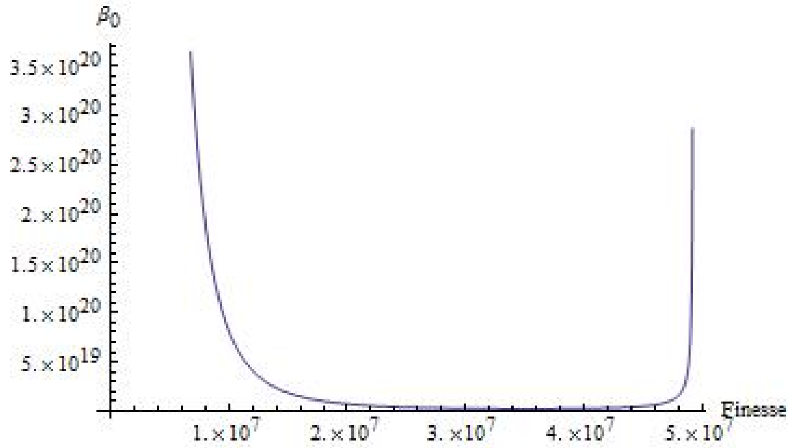

Figure 13: Plot of the bound on β_0 in dependence of the finesse for temperature $T = 25mK$ and a mean phonon occupation of $\overline{n} = 30$

As we can see, even with the best cooling techniques available it is not reasonable to push the finesse higher than $F = 3.5 * 10^7$. The mayor problem is here, that a finesse higher than $F \approx 10^6$ is not within current reach. The reason is, that as a rule of thumb the finesse is given by

$$F \approx \frac{\pi}{\sum L_i} \tag{141}$$

and the mayor loss sources L_i (transmission at the mirrors, scattering off the mirror surface and absorption) can't be brought down to more than 10^{-6}. Therefore the best bound under this conditions($F = 10^6$) we can reach is

$$\beta_0 \leq 7.2 * 10^{23} \tag{142}$$

which is nearly 9 orders better than so far but still insufficient. To reach a bound on β_0 of $\beta_0 \leq 1$ one has to also alter the other system parameter, that increase the number of photons, decrease the laser wavelength, and increase the number of experimental runs, all of which is very challenging.

Following the calculations done in[19] one can find a set of parameters for that even that challenging task is achievable:

- $F = 4.9 * 10^5$
- $m = 10^{-7} kg$
- $\omega_m = 2\pi 10^5 Hz$
- $\lambda_l = 532 nm$
- $N_p = 10^{14}$
- $N_r = 10^6$
- $T = 25 mK$
- $\overline{n} = 30$
- For $T = 10K$ and according phononoccupation one has to increase F to $F = 5.01 * 10^5$.
- if simultaneous using a laser with wavelength $\lambda_l = 1064 nm$ one has to increase F to $F = 1.01 * 10^6$.
- if we now go to $T = 25mK$ and $\overline{n} = 30$ one has to have $F = 9.7 * 10^5$
- if we want to do only one experimental run $N_r = 1$ we would have to increase F to $F = *10^6$ and would only achieve a bound of $\beta_0 \leq 856$,

8.3 Achievable bounds on γ_0

For the first set we can set a bound on γ_0

- without deleterious effects of
$$\gamma_0 \leq 1 * 10^{26} \tag{143}$$
- The deleterious effects for this parameters are:
$$R \approx 1 \tag{144}$$
$$D \approx 1 \tag{145}$$

- also taking into accounts this deleterious effects and the $I_\gamma = 0.6$ we get a bound of
$$\gamma_0 \leq 1.7 * 10^{26} \tag{146}$$

- therefore we are still 17 orders of magnitude above current bounds for γ_0 ($\gamma_0 \leq 10^{10}$)

For the second set of parameters we can put a bound on γ_0

- without deleterious effects of
$$\gamma_0 \leq 1 * 10^{17} \tag{147}$$

- The deleterious effects for this parameters are:
$$R = 0.994632 \tag{148}$$
$$D \approx 1 \tag{149}$$

- also taking into accounts this deleterious effects and the $I_\gamma = 0.6$ we get a bound of
$$\gamma_0 \leq 1.7 * 10^{17} \tag{150}$$

- therefore we are still 8 orders of magnitude above current bounds for γ_0 ($\gamma_0 \leq 10^{10}$)

In the following plot 14 one can see, that pushing the finesse further than $F = 10^4$ doesn't make sense for a temperature $T = 10K$.

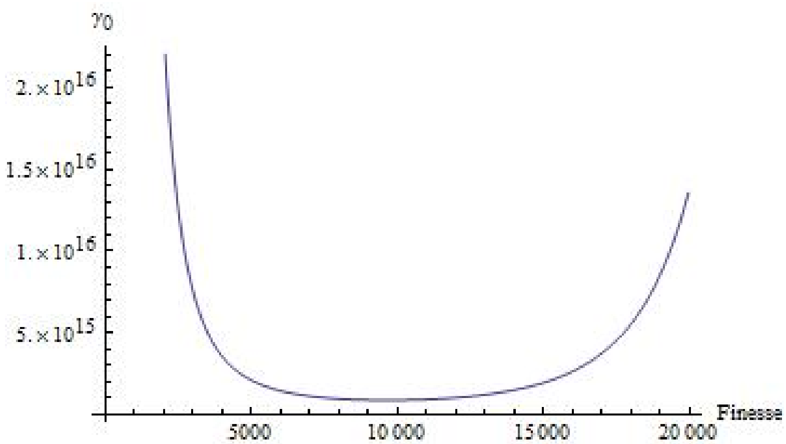

Figure 14: Plot of the bound of γ_0 in dependence of the finesse for temperature $T = 10K$ and according thermal occupation of phonons of the mechanical oscillator

For $F = 10^4$ and $T = 10K$ one gets the following bound on γ_0:

$$\gamma_0 \leq 3 * 10^{14} \tag{151}$$

which is nearly 4 orders away from the current bound.

By going to $T = 0.025K$ and $\overline{n} = 30$ one gets the following plot 15

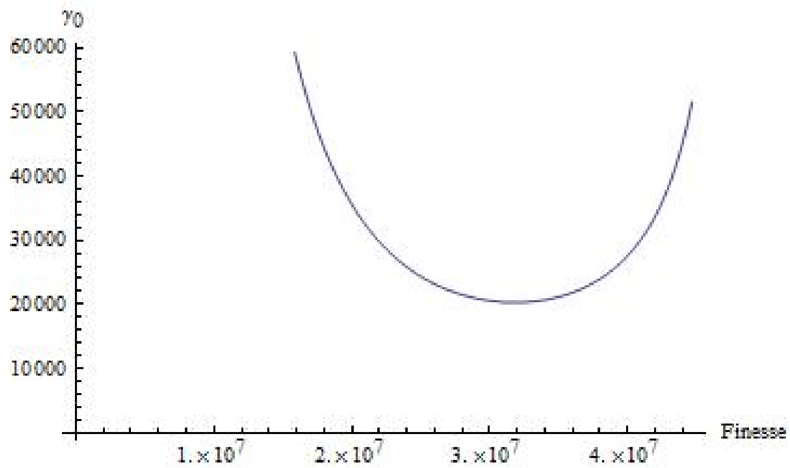

Figure 15: Plot of the bound on γ_0 in dependence of the finesse for temperature $T = 25mK$ and a mean phonon occupation of $\overline{n} = 30$

which tells as, that the optimum for F would be $F = 3.1 * 10^7$ and therefore the best possible finesse would be $F = 10^6$ yielding a bound on γ_0 of

$$\gamma_0 \leq 1.7 * 10^8 \tag{152}$$

which already would be nearly 2 orders better than the current bound. Again, if we want to reach $\gamma_0 \leq 1$ we have to alter the other parameters.

Following the calculations done in[19] one can find a set of parameters for that this task is achievable:

- $F = 2.8 * 10^5$
- $m = 10^{-9}kg$
- $\omega_m = 2\pi 10^5 Hz$
- $\lambda_l = 1064nm$

- $N_p = 5 * 10^{10}$
- $N_r = 10^5$
- $T = 25mK$
- $\overline{n} = 30$
- if we want to do only one experimental run $N_r = 1$ we would have to increase F to $F = 10^6$ and could only reach $\gamma_0 \leq 6.9$, therefor we would have to increase N_p to $N_p = 1.1 * 10^11$ to reach $\gamma_0 \leq 1$

8.4 Achievable bounds on μ_0

For the first set we can set a bound on μ_0

- without deleterious effects of
$$\mu_0 \leq 1.3 * 10^8 \tag{153}$$
- The deleterious effects for this parameters are:
$$R \approx 1 \tag{154}$$
$$D \approx 1 \tag{155}$$
- also taking into accounts this deleterious effects and the $I_\mu = 0.7$ we get a bound of
$$\mu_0 \leq 1.9 * 10^8 \tag{156}$$

For the second set of parameters we can put a bound on μ_0

- without deleterious effects of
$$\mu_0 \leq 135 \tag{157}$$
- The deleterious effects for this parameters are:
$$R = 0.994632 \tag{158}$$
$$D \approx 1 \tag{159}$$

- also taking into accounts this deleterious effects and the $I_\mu = 0.7$ we get a bound of

$$\mu_0 \leq 193 \tag{160}$$

Now we again try to optimize for F as you can see in the next plot 16

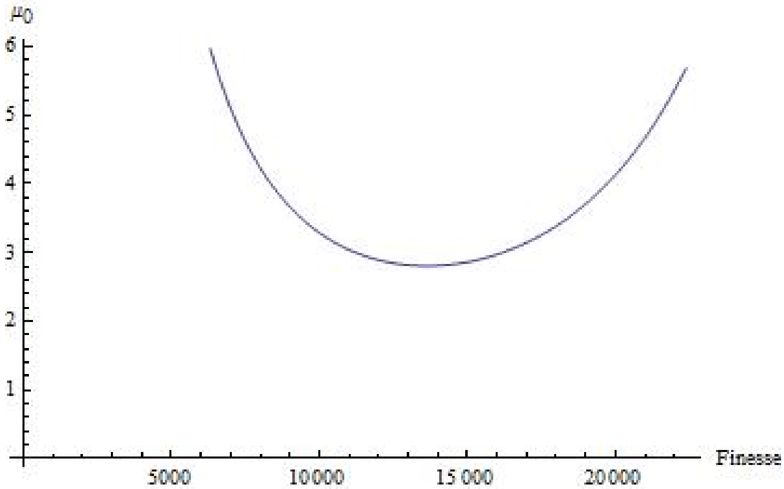

Figure 16: Plot of the bound of μ_0 in dependence of the finesse for temperature $T = 10K$ and according thermal occupation of phonons of the mechanical oscillator

as one can see, this plot has a minimum approximately at $F = 1.36 * 10^4$ and this sets a bound on μ_0 of

$$\mu_0 \leq 2.8 \tag{161}$$

which would be a sensational new bound. By going to the regime of $T = 25mK$ and a phonon occupation of $\overline{n} = 30$ we get the following plot 17

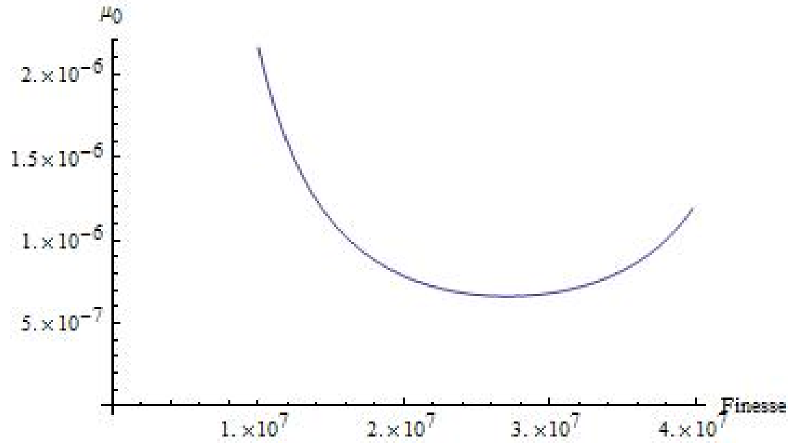

Figure 17: Plot of the bound on μ_0 in dependence of the finesse for temperature $T = 25mK$ and a mean phonon occupation of $\overline{n} = 30$

The best possible F therefore would be $F = 10^6$ yielding

$$\mu_0 \leq 1.9 * 10^{-4} \tag{162}$$

This would be even more then necessary. Even $F = 1.39 * 10^4$ would be sufficient for

$$\mu_0 \leq 0.993 \tag{163}$$

As we can see, putting a bound on μ_0 is a lot less challenging then it is for the other deformation parameters.

For completeness, I will also present here the set of experimental parameters given in [19] to achieve $\mu_0 \leq 1$:

- $F = 8.5 * 10^4$
- $m = 10^{-11} kg$
- $\omega_m = 2\pi 10^5 Hz$
- $\lambda_l = 1064nm$
- $N_p = 5 * 10^8$
- $N_r = 1$
- $T = 25mK$

- $\overline{n} = 30$

8.5 Summary

In this subsection I summarize the results gained in the previous subsections.

At first I will gather the bounds on the deformation parameters gained through the use of the first and second set of parameters given before and compare them to the existing bounds(referred to as ex.bounds):

	set 1	**set 2**	**ex.bounds**	**ooma**
β_0	$7.2 * 10^{47}$	$7.2 * 10^{35}$	10^{33}	3
γ_0	$1.7 * 10^{26}$	$1.7 * 10^{17}$	10^{10}	8
μ_0	$1.9 * 10^{8}$	193	–	–

Table 2: Bounds on the deformation parameters for the different sets of experimental parameters and also the already existing bounds on them. In the last column is how many orders of magnitude away (ooma) the better bound is from the existing bounds

In the next table I gather the bounds on the deformation parameters by using the experimental parameters of set 2 but optimizing the finesse F:

	fob	**ex.bounds**	**ooma**
$\beta_0; F = 2 * 10^4$	$3.9 * 10^{31}$	10^{33}	-2
$\gamma_0; F = 10^4$	$3 * 10^{14}$	10^{10}	4
$\mu_0, F = 1.36 * 10^4$	2.8	–	–

Table 3: Bounds on the deformation parameters with the experimental data set 2 but with a "finesse optimized bound"(fob) and also the already existing bounds on them. In the last column is how many orders of magnitude away (ooma) the bound is from the existing bounds, the minus sign says that it would be an improvement of the bound

In the following table are the bounds we can put on the deformation parameters if we use the experimental data set 2 but decrease the temperature to $T = 25mK$ and cool the center of mass mode of the oscillator for a phonon occupation of $\overline{n} = 30$ and then optimize the finesse:

	fob	**ex.bounds**	**ooma**
$\beta_0; F = 10^6$	$7.2 * 10^{23}$	10^{33}	-9
$\gamma_0; F = 10^6$	$1.7 * 10^8$	10^{10}	-2
$\mu_0, F = 10^6$	$1.9 * 10^{-5}$	–	–

Table 4: Bounds on the deformation parameters with the experimental data set 2 but with $T = 25mK$ and $\overline{n} = 30$. The finesse is optimized again(fob). Also the already existing bounds on them are in the table. In the last column is how many orders of magnitude away (ooma) the bound is from the existing bounds, the minus sign says that it would be an improvement of the bound

In the next tables we see the experimental parameters, that we could use to set a bound of 1 on the deformation parameters

	β_0	γ_0	μ_0
F	$4.9 * 10^5$	$2.8 * 10^5$	$8.5 * 10^4$
m	$10^{-7}kg$	$10^{-9}kg$	$10^{-11}kg$
ω_m	$2\pi 10^5 Hz$	$2\pi 10^5 Hz$	$2\pi 10^5 Hz$
λ_l	$532nm$	$1064nm$	$1064nm$
N_p	10^{14}	$5 * 10^{10}$	$5 * 10^8$
N_r	10^6	10^5	1
T	$25mK$	$25mK$	$25mK$
$\overline{n}$	30	30	30

Table 5: Possible experimental parameters needed to achieve a bound of one on the deformation parameters

	β_0	γ_0
F	10^6	10^6
m	$10^{-7}kg$	$10^{-9}kg$
ω_m	$2\pi 10^5 Hz$	$2\pi 10^5 Hz$
λ_l	$1064nm$	$1064nm$
N_p	$7 * 10^{14}$	$1.1 * 10^{11}$
N_r	1	1
T	$25mK$	$25mK$
$\overline{n}$	30	30

Table 6: Possible experimental parameters needed to achieve a bound of one on the deformation parameters, this time needing only one experimental run. For β_0 the number of photons needed, may be challenging.

The next table shows different sets of parameters, that would allow to set a bound of 1 or less on all three of the deformation parameters:

	first	second
F	10^6	10^5
m	$10^{-10}kg$	$10^{-11}kg$
ω_m	$2\pi 10^5 Hz$	$2\pi 10^5 Hz$
λ_l	$1064nm$	$1064nm$
N_p	10^{14}	10^{14}
N_r	1	10^6
T	$25mK$	$25mK$
$\overline{n}$	30	30

Table 7: 2 different sets for which a bound of at least 1 is reached for all 3 deformation parameters in one single experiment. For the first set the number of photon N_p and the high finesse may be challenging, for the second set also the number of experimental runs N_r but we could ease of for the finesse.

8.6 The case of a non-null result

So far we have only dealed with the case of a null result of the experiment which only sets a bound on the deformation parameters. Now we want to assume that we measure a $\Theta(\nu)$ that differs from the standard quantum mechanical one. In this case we have to distinguish which of the three possible deformed commutators belonging to the deformation parameters it is that causes this deviation. Fortunately $\Theta(\nu)$ scales differently with the experimental parameters for all ν. For now we will assume $\nu = 1$.

We will use the experimental parameters given in table 5 and alter only one of this parameters to see how $\Theta(\nu)$ scales with this parameter for the different ν.

Altering the mass m:

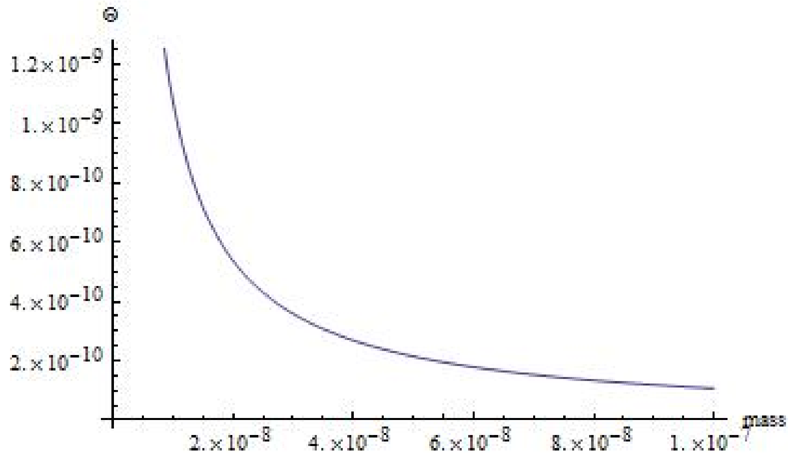

Figure 18: Plot of the dependence of $\Theta(\beta)$ on m

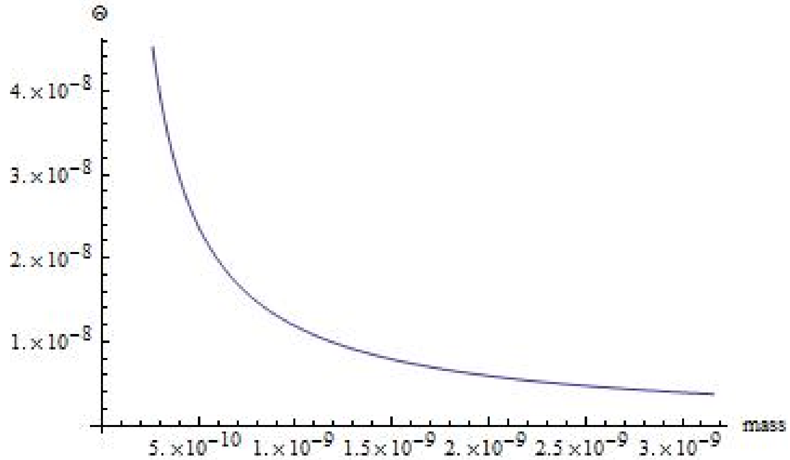

Figure 19: Plot of the dependence of $\Theta(\gamma)$ on m

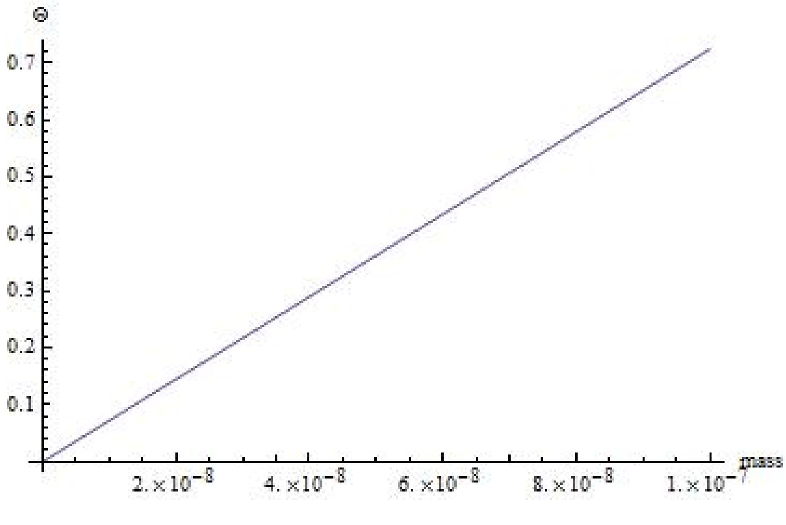

Figure 20: Plot of the dependence of $\Theta(\mu)$ on m

Altering the finesse F:

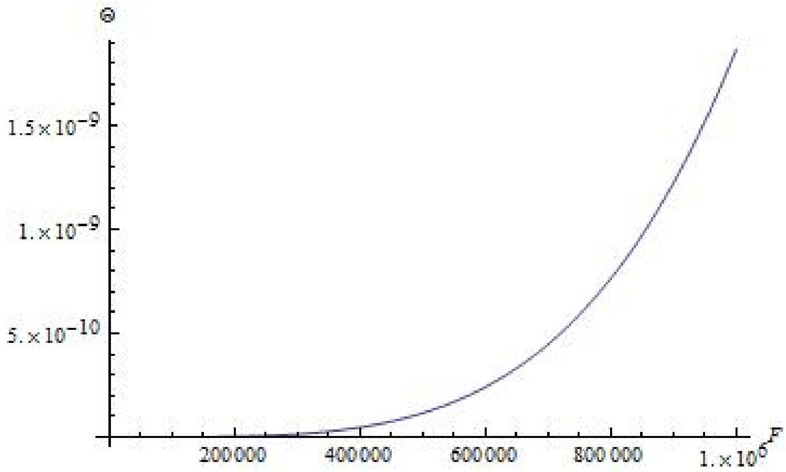

Figure 21: Plot of the dependence of $\Theta(\beta)$ on F

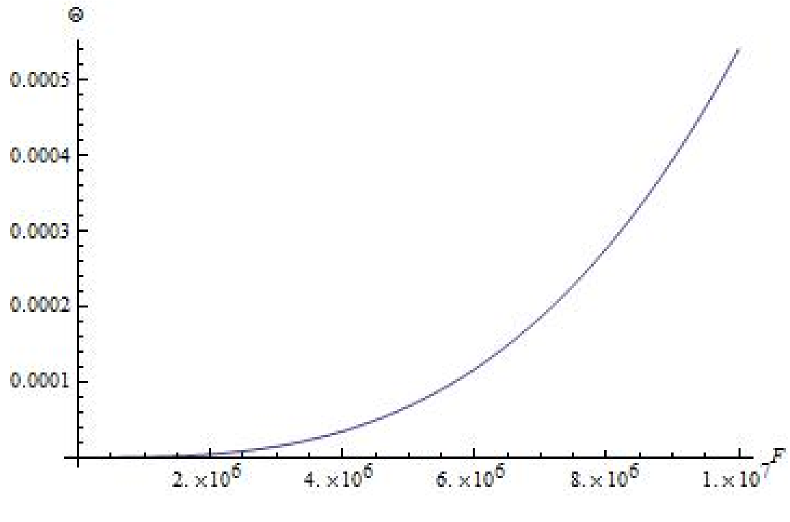

Figure 22: Plot of the dependence of $\Theta(\gamma)$ on F

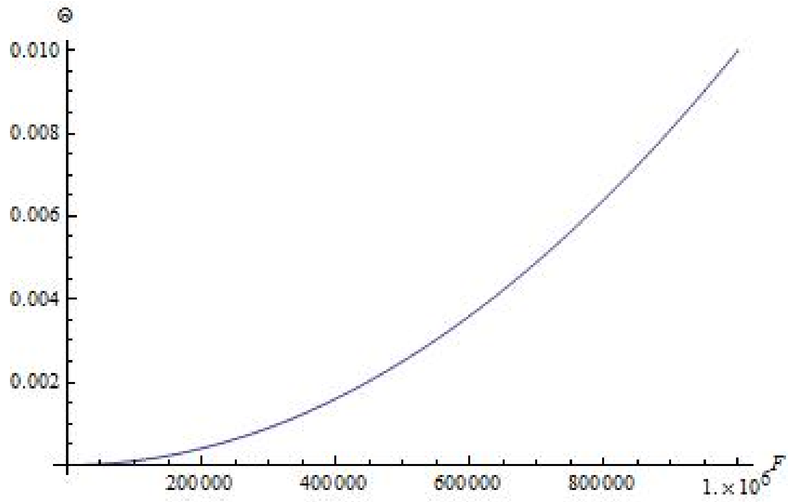

Figure 23: Plot of the dependence of $\Theta(\mu)$ on F

Altering the Photonnumber N_p:

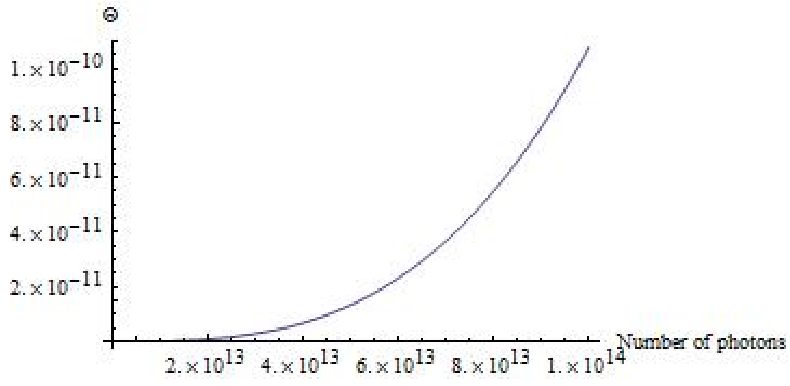

Figure 24: Plot of the dependence of $\Theta(\beta)$ on N_p

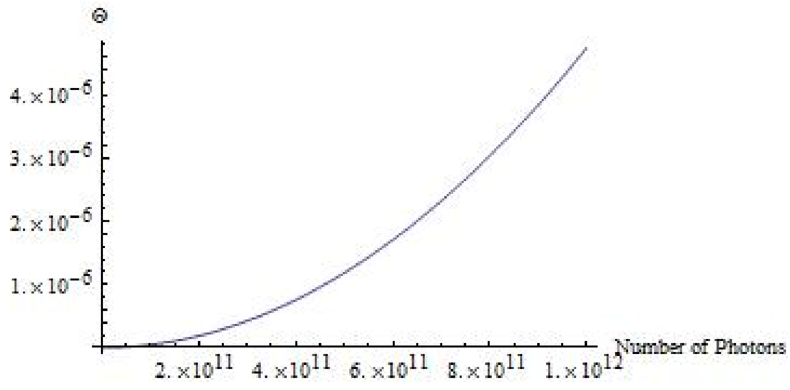

Figure 25: Plot of the dependence of $\Theta(\gamma)$ on N_p

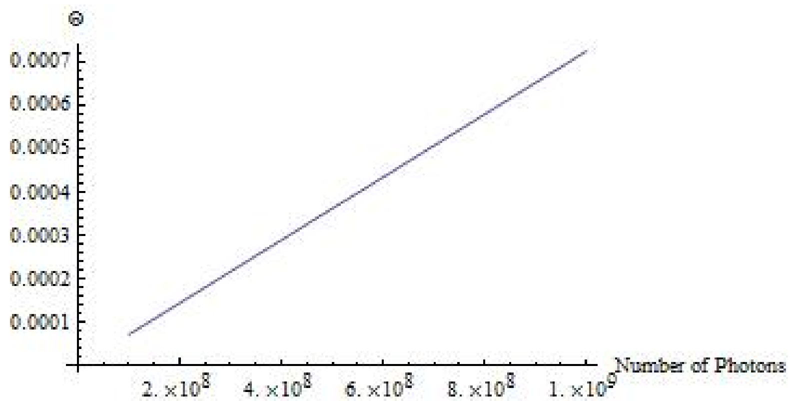

Figure 26: Plot of the dependence of $\Theta(\mu)$ on N_p

As one can see $\Theta(\nu)$ scales quite different with the different parameters. I will now list possible options to distinguish between the different deformations:

- to proof we are dealing with the μ_0-deformation it is sufficient to alter the mass , since it is the only one with a linear dependence on the mass.
- to distinguish between the β_0 and γ_0-deformation, one has to alter the finesse or the photonnumber(which seems easier), since the γ_0-deformation has a F^3 and N_p^2 dependence and the γ_0-deformation has a F^4 and N_p^3 dependence.
- altering the finesse or photonnumber also allows to distinguish between all three deformations since the μ_0-deformation has a F^2 dependence and is linear in N_p.

8.7 Appendix

Based on newest experiments regarding "optomechanical zipper cavities" the following experimental parameters are of interest [5]:

g_0	$2\pi 200kHz$
κ	$2\pi 1Ghz$
Q	200000
N_p	10^{14}
m	$10^{-15}kg$
ω_m	$2\pi 10MHz$
T	$25mK$
$\overline{n}$	30

With this parameters the following bounds on the deformation parameters can be reached:

β_0	$3 * 10^8$
γ_0	$< 1(N_p = 10^8$ would be sufficient)
μ_0	$4 * 10^{34}$

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