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DISSERTATION

Cycles and the cohomology
of arithmetic subgroups of the exceptional group G_2

Verfasser

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Preface

The construction of cycles to prove non-vanishing results for the cohomology of arithmetic subgroups, has a long tradition, traced back three decades to a paper written by J. Millson [Mi] in 1976. In this paper, Millson considered the real hyperbolic n -space H^n together with a torsion-free cocompact arithmetically defined discrete subgroup Γ of the group of isometries of H^n . He constructed a totally geodesic hypersurface of the form $C(\Gamma) := H^{n-1}/\Gamma \cap Iso(H^{n-1})$ in the arithmetic quotient H^n/Γ via the natural map

$$H^{n-1}/\Gamma \cap Iso(H^{n-1}) \rightarrow H^n/\Gamma.$$

In his intricate analysis, Millson proved that one can pass over to a subgroup Γ' of finite index in Γ such that the fundamental class of the geometric cycle $C(\Gamma')$ in the homology $H_{n-1}(H^n/\Gamma'; \mathbb{C})$ is non-trivial.

This procedure can be generalized to algebraic groups, as developed by J. Millson and M.S. Raghunathan in [MiRa], who considered a connected semisimple linear $\mathbb{Q}$ -anisotropic algebraic $\mathbb{Q}$ -group $\mathbf{G}$ (i.e. $\mathbb{Q}$ -rank of $\mathbf{G}$ equals zero) and a connected reductive $\mathbb{Q}$ -subgroup $\mathbf{B}$ of $\mathbf{G}$. Take an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ acting freely and properly discontinuously on the symmetric space X associated with the group of real points of $\mathbf{G}$. This gives rise to a compact complete Riemannian manifold X/Γ . The same holds true for the arithmetic subgroup $\Gamma_{\mathbf{B}} := \Gamma \cap \mathbf{B} \subset \mathbf{B}(\mathbb{Q})$ and the symmetric space $X_{\mathbf{B}}$ associated with the group of real points of $\mathbf{B}$ (i.e. $X_{\mathbf{B}}/\Gamma_{\mathbf{B}}$ is a compact complete Riemannian manifold). According to Raghunathan's results (e.g. [FOR]) (which can be found as an implication in [MiRa, Theorem 3.1]), one can always pass to a subgroup Γ' of finite index in Γ such that the natural inclusion for the (compact) locally symmetric spaces

$$X_{\mathbf{B}}/\Gamma'_{\mathbf{B}} \longrightarrow X/\Gamma'$$

is an imbedding of manifolds. The image $X_{\mathbf{B}}/\Gamma'_{\mathbf{B}}$ in X/Γ' is called a cycle and is denoted by $C_{\mathbf{B}}(\Gamma')$ or $C(\Gamma')$ or just C . In [MiRa], the authors studied the intersection number of a cycle C_1 which intersects a complementary cycle C_2 transverse, to show that the cycles considered in the corresponding arithmetic quotient X/Γ are non-bounding. Moreover, Millson and Raghunathan applied their results in the cases where the group $\mathbf{G}(\mathbb{R})$ is (up to a compact factor) isomorphic to one of the classical real Lie groups $SO(p, q)$, $SU(p, q)$ and $Sp(p, q)$.

Such natural ideas can also be used in a similar way when Γ is a non-cocompact arithmetic lattice in the real Lie group $G = \mathbf{G}(\mathbb{R})$. To analyze intersection numbers, it is sufficient to assume that all connected components of $C_1 \cap C_2$ are compact. This allows one to study modular symbols.

Parts of the findings in [MiRa] were only elaborated for special cycles $C(\sigma)$ and $C(\sigma\theta)$ where σ and θ are rational involutions of $\mathbf{G}$, which commute with each other and θ induce a Cartan involution on $G = \mathbf{G}(\mathbb{R})$. Here, special cycle means that the subgroup $\mathbf{B}$ which induces the cycle is the set of elements in $\mathbf{G}$ which are fixed by an automorphism of finite

order of $\mathbf{G}$.

J. Rohlfs and J. Schwermer [RoSch] considered the intersection number of two special cycles $C(\sigma)$ and $C(\tau)$ coming from two rational automorphisms σ, τ , which commute with each other. They parametrized the connected components of $C(\sigma) \cap C(\tau)$ by a subset of the first non-abelian cohomology set $H^1(\sigma, \tau; \Gamma)$. Rohlfs and Schwermer obtained a computable formula for the intersection number in both transversal and non transversal case. Further, they produced a quite general analysis of intersection numbers of two closed immersed oriented submanifolds. This is the general framework of the geometric construction of non-bounding cycles.

Recall, that one has the following interpretations of cohomology of arithmetic groups. First, the group (or Eilenberg-MacLane) cohomology $H^*(\Gamma; \mathbb{C})$ of Γ with trivial coefficient system $\mathbb{C}$ is isomorphic to the singular cohomology $H^*(X/\Gamma; \mathbb{C})$ of the manifold X/Γ . Second, if $\mathfrak{g}$ is the complexified Lie algebra of G and K is a maximal compact subgroup of G one has the isomorphisms

$$H^*(\Gamma; \mathbb{C}) \cong H^*(X/\Gamma; \mathbb{C}) \cong H^*(\mathfrak{g}, K; C^\infty(G/\Gamma) \otimes \mathbb{C})$$

When X/Γ is compact, so is G/Γ and $C^\infty(G/\Gamma)$ is a subspace of the square integrable functions $L^2(G/\Gamma)$ (in fact $C^\infty(G/\Gamma) = L^2(G/\Gamma)^\infty$ is the space of smooth vectors in $L^2(G/\Gamma)$). By compactness of G/Γ , the unitary G -module $L^2(G/\Gamma)$ (G acts by left translation on functions) decomposes as a (infinite) direct Hilbert sum of irreducible unitary G -modules H_π each occurring with finite multiplicity $m(\pi, \Gamma)$: $L^2(G/\Gamma) = \bigoplus m(\pi, \Gamma) \cdot H_\pi$. Using this, the space $C^\infty(G/\Gamma)$ can be replaced by the space of smooth vectors in $\bigoplus m(\pi, \Gamma) \cdot H_\pi$. From this decomposition, we get Matsushimas formula, i.e. the (finite) direct sum decomposition

$$H^*(X/\Gamma; \mathbb{C}) \cong \bigoplus m(\pi, \Gamma) \cdot H^*(\mathfrak{g}, K; H_\pi^\infty). \quad (*)$$

The representations that could (possibly) contribute to the right hand side of $(*)$ are those π with $H^*(\mathfrak{g}, K; H_\pi^\infty) \neq 0$. These so called *unitary representations with non-zero cohomology* are well understood by the work of many and finally by Vogan and Zuckermann [VZ].

For totally real number fields F of degree $[F : \mathbb{Q}] > 1$, the present work applies this technique to certain F -anisotropic F -forms $\mathbf{G}'$ of the connected split real Lie group of the exceptional type G_2 . This Lie group has real rank 2. Such an algebraic group $\mathbf{G}'$ is realized as the group of automorphisms of a F -octonion algebra $\mathfrak{C}$. Now we obtain a $\mathbb{Q}$ -group $\mathbf{G} := \text{Res}_{F/\mathbb{Q}} \mathbf{G}'$ by restriction of scalars. The octonion algebra $\mathfrak{C}$ is constructed by using a certain basic triple a, b, c , chosen by conditions which guarantee that $\mathbf{G}(\mathbb{R})$ is (up to compact factors) isomorphic to the connected split real Lie group of the exceptional type G_2 . The symmetric space X associated with the group of real points of $\mathbf{G}$ has dimension 8. For an torsion free arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ the locally symmetric space X/Γ is compact. For the compact dual symmetric space X_u of X one has the injective Matsushima map

$$H^*(X_u; \mathbb{C}) \rightarrow H^*(X/\Gamma; \mathbb{C}) \quad (**)$$

in cohomology. Classes which are not in the image of the Matsushima map $(**)$ are called non-invariant classes. The cohomology of the compact dual X_u attached to G_2 does not vanish only in degrees 0, 4 and 8. The following theorem summarizes the results of this thesis; (it will be made more precise in chapter 5 and 8):

Theorem. *For the F -forms $\mathbf{G}'$, there are an arithmetic subgroup Γ and reductive subgroups $\mathbf{B}$, listed below, which are unitary and orthogonal groups of certain hermitian or quadratic spaces (see chapter 4), such that all cycles $C_{\mathbf{B}} = C_{\mathbf{B}}(\Gamma)$, induced by $\mathbf{B}$ are homologically non trivial in $H_*(X/\Gamma; \mathbb{C})$ and their Poincaré dual classes are all non invariant*

classes.

B isomorphic to the group	B ($\mathbb{R}$) isomorphic to	cycle $C_{\mathbf{B}}$	$\dim C_{\mathbf{B}}$
SU (V_a, h_a, τ_a)	$SL(3, \mathbb{R})$	C_a	5
U ($V_a(b), h_a, \tau_a$)	$GL(2, \mathbb{R})$	$C_{a,b}$	3
SU (V_b, h_b, τ_b)	$SL(3, \mathbb{R})$	C_b	5
U ($V_b(a), h_b, \tau_b$)	$GL(2, \mathbb{R})$	$C_{b,a}$	3
SO (V_σ, B_σ)	$SO(2, 2)$	$C(\sigma)$	4
SO ($V_{\sigma\theta}, B_{\sigma\theta}$)	$SO(2, 2)$	$C(\sigma\theta)$	4
SU (V_c, h_c, τ_c)	$SU(2, 1)$	C_c	4
SU ($V_{ab}, h_{ab}, \tau_{ab}$)	$SU(2, 1)$	C_{ab}	4

According to the Vogan-Zuckermann classification of unitary representations with non-zero cohomology [VZ], Γ has only cohomology in degrees 0,3,4,5 and 8. In accordance with Poincaré duality, the homology $H_*(X/\Gamma; \mathbb{C})$ does not vanish exactly in the same degrees 0,3,4,5 and 8. Thus the description of the contribution of cycles to homology is “complete” in the following senses: first, we found in every possible non-vanishing degree 3,4 and 5 a non trivial cycle, and second, we found at least one group of every possible type of subgroup, to get a cycle. We point out that all classes constructed are non-invariant classes.

The present work is divided as follows: In chapters 1 and 2, I present some basic facts regarding locally symmetric spaces and the construction of cycles.

In chapter 3, I review standard facts regarding composition algebras and choose a model for the algebraic group G_2 . Chapter 4 describes some reductive subgroups of this model (at least one of every possible type) and provides a summary of the corresponding cycles.

The main chapter, chapter 5, reviews the most important facts pertaining to intersection numbers of manifolds, in particular, for cycles. In account of the results of [MiRa] on intersection numbers of cycles in the transversal case and the results of [RoSch] on intersection numbers of special cycles. Furthermore, there is a new result on intersection numbers of cycles in a non-transversal case (Theorem 5.21 and the remark after it), obtained by merging the methods of [MiRa] and [RoSch], which, in short, says the following:

Theorem. *Let $\mathbf{G}$ be a connected semisimple algebraic group defined over $\mathbb{Q}$ and $\mathbf{B}_1$ and $\mathbf{B}_2$, two connected reductive $\mathbb{Q}$ -subgroups of $\mathbf{G}$, such that the group $\mathbf{B}_1 \cap \mathbf{B}_2$ is $\mathbb{Q}$ -anisotropic and the natural map on the level of Galois cohomology*

$$H^1(\mathbb{Q}, \mathbf{B}_1 \cap \mathbf{B}_2) \rightarrow H^1(\mathbb{Q}, \mathbf{B}_1) \times H^1(\mathbb{Q}, \mathbf{B}_2)$$

is injective. Furthermore, let Γ be an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Suppose that the group $\mathbf{G}(\mathbb{R})$ resp. $\mathbf{B}_i(\mathbb{R})$ acts orientation preserving on their associated symmetric space X resp. X_i , $i = 1, 2$ and that $\dim X_1 + \dim X_2 = \dim X$ holds. Let $Z = X_1 \cap X_2$, $f = \dim Z$ and Z_u the compact dual to Z . Then there exists a subgroup Γ' of Γ of finite index, such that the intersection number of $[C_1(\Gamma')]$ and $[C_2(\Gamma')]$ is

$$[C_1(\Gamma')][C_2(\Gamma')] = (-1)^{f/2} \langle e(\tilde{\eta}_u), [Z_u] \rangle \cdot \left(\sum c(\gamma) \right),$$

where the $c(\gamma)$'s are positive proportionality factors, corresponding to the connected components of $C_1(\Gamma') \cap C_2(\Gamma')$ and the sum is finite. In particular, the Euler number $\langle e(\tilde{\eta}_u), [Z_u] \rangle$ of a certain “dual” excess bundle $\tilde{\eta}_u$ is non zero, if and only if the intersection number of $C_1(\Gamma')$ and $C_2(\Gamma')$ is non zero.

All of these theories (old and new) are employed to prove the non vanishing of the classes of the cycles listed above in the case G_2 in a case by case manner.

Chapter 6 contains facts on the cohomology ring of the compact dual symmetric space attached to G_2 , the signature of the locally symmetric space to G_2 and a method of showing the non vanishing of cycles in certain degrees, applied to the case G_2 .

Chapter 7 briefly addresses certain facts on irreducible unitary representations with non-zero cohomology and lists these in the case G_2 .

In Chapter 8, I discuss in detail the G -invariance of cycles and recall the result of [MiRa] for the existence of non invariance of special cycles. On the basis of these results, I develop new criteria for non invariance of cycles in the middle degree. One of these criteria stipulates that if the self-intersection number of a cycle in the middle degree is non trivial, then there is a finite covering such that the dual class of the covering cycle is non invariant. Furthermore, I apply these results and criteria to the case G_2 . In particular, this shows the non vanishing property of the multiplicities of automorphic representations in every possible degree.

In appendix A, I have included general facts on characteristic classes and hermitian forms which are essential for the thesis.

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Chapter 1

Preparations

This preparatory chapter discusses basic results from algebraic and arithmetic groups as well as results from locally symmetric spaces, needed in the thesis.

1.1 Algebraic groups, Lie groups and restriction of scalars

Let $\mathbf{G}$ be a semisimple connected (linear) algebraic group over $\mathbb{Q}$. We choose once and for all an imbedding $\mathbf{G} \hookrightarrow \mathbf{GL}_N$ defined over $\mathbb{Q}$. For a commutative $\mathbb{Q}$ -algebra A the group of A -points is denoted by $\mathbf{G}(A)$ and we put $\mathbf{G}(\mathbb{Z}) = \mathbf{G}(\mathbb{Q}) \cap \mathbf{GL}_N(\mathbb{Z})$. The group of real points $G = \mathbf{G}(\mathbb{R})$ is a real semisimple Lie group with finitely many connected components. It is no restriction to assume that $\mathbf{G}$ is defined over $\mathbb{Q}$, because if $\mathbf{G}$ is defined over an algebraic number field F with ring of integers $\mathcal{O}$, restriction of scalars could be used to get an algebraic group $\mathbf{G}' = \text{Res}_{F/\mathbb{Q}} \mathbf{G}$ which is defined over $\mathbb{Q}$. Here $\mathbf{G}$ is equipped with the properties $\mathbf{G}'(\mathbb{Q}) \cong \mathbf{G}(F)$, $\mathbf{G}'(\mathbb{Z}) \cong \mathbf{G}(\mathcal{O})$ and $\mathbf{G}'(\mathbb{R}) \cong \prod_s \mathbf{G}(F^s)$. The product runs over the distinct embeddings $s : F \rightarrow \mathbb{C}$ (i.e. complex embeddings only up to complex conjugation) of F in $\mathbb{C}$ over $\mathbb{Q}$ and $s(\mathbf{G})$ denotes the $s(F)$ -algebraic group obtained by conjugation by s and F^s denotes $\mathbb{R}$ if the embedding s is real and $\mathbb{C}$ otherwise. Moreover, restriction of scalars, is a functor, $\text{Res}_{F/\mathbb{Q}}$, from F -groups and F -morphisms to $\mathbb{Q}$ -groups and $\mathbb{Q}$ -morphisms. The functor, $\text{Res}_{F/\mathbb{Q}}$, defines a bijection of connected F -subgroups of the F -group $\mathbf{G}$ between the connected $\mathbb{Q}$ -subgroups of $\mathbf{G}' = \text{Res}_{F/\mathbb{Q}} \mathbf{G}$ [BoTi, 6.18]. In particular, $F - \text{rank}(\mathbf{G}) = \mathbb{Q} - \text{rank}(\mathbf{G}')$.

For more informations about restriction of scalars see for example [BoTi, 6.17-6.21], [PR, 2.1.2.] or [Sp, 11.4, 12.4, 16.2.6].

1.2 Locally symmetric spaces and arithmetic groups

Again, let $G = \mathbf{G}(\mathbb{R})$ be the group of real points of a semisimple connected algebraic group $\mathbf{G}$ over $\mathbb{Q}$. We assume that G is non compact (i.e. the $\mathbb{R}$ -rank of $\mathbf{G}$ is positive). Let $K \subset G$ be a maximal compact subgroup. Then the homogeneous space $K \backslash G = X$ is a complete Riemannian symmetric space of non compact type, diffeomorphic to an euclidean space. Here the Riemannian metric on X is induced by the Killing form of the Lie algebra $\mathfrak{g}_0 := \text{Lie}(G)$ and G acts isometric on X . More details can be found in [Bo2] and [He]. Let

$\Gamma \subseteq \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup (i.e. Γ is commensurable with $\mathbf{G}(\mathbb{Z})$). Then Γ operates properly on X (i.e. for any given compact subset C of X , the set $\{\gamma \in \Gamma \mid \gamma C \cap C \neq \emptyset\}$ is finite). Every arithmetic subgroup contains a torsion free subgroup of finite index-hence we can assume Γ to be torsion free. Then Γ operates freely and properly on X , and the quotient X/Γ is a smooth connected manifold, a locally symmetric space. Since X is simply connected, it is the universal cover of X/Γ ; so $\pi_1(X/\Gamma) \cong \Gamma$. Moreover, the quotient X/Γ is an Eilenberg-MacLane $K(\Gamma, 1)$ -space. This follows from the fact that K is a deformation retract of G ; hence $X = K \setminus G$ is contractible. The projection $\pi : X \rightarrow X/\Gamma$ is a local isometry, in particular a local diffeomorphism. More facts on locally symmetric spaces can be found in [Ji].

A normal subgroup Γ' of finite index in Γ is also arithmetic and $\text{pr} : X/\Gamma' \rightarrow X/\Gamma$, $x\Gamma' \mapsto x\Gamma$ is a finite regular covering. The group of decktransformations, i.e. those homeomorphisms $\eta : X/\Gamma' \rightarrow X/\Gamma'$ which satisfy $\text{pr} = \text{pr} \circ \eta$ is isomorphic to the finite group Γ/Γ' .

One has the following compactness criterion on the space X/Γ .

Theorem. (Compactness criterion) *Let $\mathbf{G}$ be a semisimple connected linear algebraic group defined over $\mathbb{Q}$, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic subgroup. Then the following conditions are equivalent:*

1. *The quotient space X/Γ is compact.*
2. *$\mathbf{G}(\mathbb{Q})$ does not contain any nontrivial unipotent element.*
3. *The $\mathbb{Q}$ -rank of $\mathbf{G}$ is equal to 0.*

1.3 Vector bundles on (locally) symmetric spaces

Recall some basic facts on homogeneous vector bundles. For more informations see e.g. [MaMu],[Wal],[GHV2],[Tu]. For general facts on vector bundles see also [Milnor Stasheff] and [Hus].

Let G be a Lie group, K be a closed subgroup of G and $M = K \setminus G$ the homogeneous space of right cosets. Let (τ, E) be a finite dimensional representation of K . Then K act from the left on $G \times E$ as follows:

$$k.(g, v) := (kg, \tau(k)v),$$

for $g \in G$, $v \in E$ and $k \in K$. We define the quotient-space $G \times_K E := G \times_\tau E := K \backslash (G \times E)$. This is a vector bundle over M in a natural way (called homogeneous vector bundle), via the projection map

$$p : G \times_K E \rightarrow M, [g, v] \mapsto Kg$$

where $[g, v] = K(g, v)$.

If $M = G/K$ is the homogeneous space of left cosets, then we let K act from the right on $G \times E$ as follows:

$$(g, v).k := (gk, \tau(k)^{-1}v),$$

for $g \in G$, $v \in E$ and $k \in K$. Again, the quotient-space $G \times_K E := G \times_\tau E := (G \times E)/K$ is a vector bundle over M , via

$$p : G \times_K E \rightarrow M, [g, v] \mapsto gK$$

where $[g, v] = (g, v)K$.

Let $\mathfrak{g}$ resp. $\mathfrak{k}$ be the Lie algebra of G resp. K . The adjoint representation Ad_G of G on $\mathfrak{g}$ induces a representation $\text{Ad}_G|_{\mathfrak{k}}$ of K on $\mathfrak{k}$ by restriction to $\mathfrak{k}$. Hence Ad_G induces a representation $(\tau, \mathfrak{g}/\mathfrak{k})$ of K , via $k.(v + \mathfrak{k}) := \text{Ad}_G(k)v + \mathfrak{k}$, where $k \in K$ and $v \in \mathfrak{g}$. Then, it is easy to see, (see e.g. [GHV2, V,3,Prop.III] or [Tu, p.5]), that the tangent bundle of M is isomorphic to $G \times_K \mathfrak{g}/\mathfrak{k}$. More precisely, (in the setup of [Tu, p.5]) let $o = Ke$ and $r_g : M \rightarrow M$, $m \mapsto mg$ the right translation by g . Then r_g is a diffeomorphism and its differential $T_o r_g : T_o M \rightarrow T_{Kg} M$ is an isomorphism. The map

$$G \times_K \mathfrak{g}/\mathfrak{k} \xrightarrow{\sim} TM, [g, v] \mapsto T_o r_g(v)$$

is an isomorphism of vector bundles.

Now we assume, that G is semisimple, K maximal compact in G and $M = X$ the associated symmetric space. Similar as above, the group Γ acts on $X \times E$ from the right via

$$(x, v) \cdot \gamma := (x\gamma, \tau(\gamma)^{-1}v)$$

for $x \in X$, $v \in E$ and $\gamma \in \Gamma$. We define the quotient-space $X \times_{\Gamma} E := (X \times E)/\Gamma$. This is a vector bundle over X in a natural way, via the projection map

$$p : X \times_{\Gamma} E \rightarrow X, [x, v] \mapsto x\Gamma$$

where $[x, v] = (x, v)\Gamma$.

The tangent bundle TX of X is isomorphic to the pullback bundle $\pi^*(T(X/\Gamma))$ of the tangent bundle $T(X/\Gamma)$ of X/Γ relative to the projection $\pi : X \rightarrow X/\Gamma$: Since π is a local diffeomorphism, its differential $T_x \pi : T_x X \xrightarrow{\sim} T_{x\Gamma} X/\Gamma$ is an isomorphism. Hence $\pi^*(T(X/\Gamma)) = \{(x, v) \in X \times T(X/\Gamma) \mid v \in T_{x\Gamma} X/\Gamma \cong T_x X\} \cong TX$.

PREPARATIONS

Chapter 2

Construction of (special) cycles

In the first section of this chapter we describe the construction of cycles in locally symmetric spaces as in [MiRa]. This construction was repeated in [FOR]. In the second section we recall facts on special cycles from [RoSch].

2.1 Construction of cycles

2.1. Let $\mathbf{G}$ be a connected, semisimple linear algebraic group defined over $\mathbb{Q}$. We fix once and for all an imbedding of $\mathbf{G}$ in some $\mathbf{GL}_N$ as a $\mathbb{Q}$ -algebraic subgroup. We set $\mathbf{G}(\mathbb{Z}) = \mathbf{G}(\mathbb{Q}) \cap \mathbf{GL}_N(\mathbb{Z})$ and denote by $\Gamma \subseteq \mathbf{G}(\mathbb{Z})$ a torsionsfree arithmetic subgroup. For an ideal $\underline{a} \subseteq \mathbb{Z}$ we set $\Gamma(\underline{a}) = \{\gamma \in \Gamma \mid \gamma \equiv \mathbb{I}_N \pmod{\underline{a}}\}$. For a $\mathbb{Q}$ -algebraic subgroup $\mathbf{B}$ of $\mathbf{G}$, we set $\Gamma_{\mathbf{B}} = \Gamma \cap \mathbf{B}(\mathbb{Q})$.

Let K be a maximal compact subgroup of $G = \mathbf{G}(\mathbb{R})$ and G^0 the connected component of the identity in G . Suppose that $\mathbf{B}$ is a connected reductive $\mathbb{Q}$ -subgroup of $\mathbf{G}$ such that $K_{\mathbf{B}} = B \cap K$ is a maximal compact subgroup of $B = \mathbf{B}(\mathbb{R})$. Then $X_{\mathbf{B}} = K_{\mathbf{B}} \backslash B$ is in a natural fashion a connected totally geodesic Riemannian submanifold in the Riemannian symmetric space $X = K \backslash G^1$. Since Γ (and therefore also $\Gamma_{\mathbf{B}}$) is torsionsfree, Γ (resp. $\Gamma_{\mathbf{B}}$) acts freely on X (resp. $X_{\mathbf{B}}$). One has evidently a natural map

$$j_{\mathbf{B}} : X_{\mathbf{B}}/\Gamma_{\mathbf{B}} \rightarrow X/\Gamma$$

of manifolds. The real Lie groups G , B have finite many connected components, but in general they are not connected. So they may contain elements which reverse the orientations on X (resp. $X_{\mathbf{B}}$). Consequently X/Γ (resp. $X_{\mathbf{B}}/\Gamma_{\mathbf{B}}$) is a manifold which may not be orientable in general. But for the later study of intersection numbers one has to deal with orientable manifolds. Fortunately, one has the following result due to Rohlfes and Schwermer [RoSch, Prop. 2.2.] and repeated in [FOR, 2.4].

Lemma 2.2. *Let Γ be an arithmetic subgroup of a semisimple algebraic group $\mathbf{G}/\mathbb{Q}$ and $G = \mathbf{G}(\mathbb{R})$. Then there exists a non zero ideal $\underline{a} \subset \mathbb{Z}$ such that $\Gamma(\underline{a}) \subset G^0$. Moreover, if*

¹If $e \in G$ is the neutral element in G , the geodesics through $o := Ke$ have the form $o \cdot \exp tZ \subset X = K \backslash G$, $t \in \mathbb{R}$, $Z \in T_o(X)$. The tangent of this geodesic at o is in $T_o(X_{\mathbf{B}})$ if and only if $Z \in T_o(X_{\mathbf{B}})$; but this means that $X_{\mathbf{B}}$ is geodesic at o . Since B is a group of isometries of X and of $X_{\mathbf{B}}$, and acts transitively on $X_{\mathbf{B}}$, it follows that $X_{\mathbf{B}}$ is geodesic at every arbitrary point $x \in X_{\mathbf{B}}$; hence totally geodesic.

$\mathbf{B}_i, 1 \leq i \leq l$, is any finite collection of reductive subgroups, we can find a non zero ideal $\underline{a}$ such that $\Gamma_{\mathbf{B}_i}(\underline{a}) \subset B_i^0$ for all i .

Suppose now that $\mathbf{B}_1, \mathbf{B}_2$ are two connected reductive $\mathbb{Q}$ -subgroups of $\mathbf{G}$, $X_i := X_{\mathbf{B}_i}$, $i = 1, 2$ and $\Gamma_i = \Gamma_{\mathbf{B}_i}$. Then, the natural maps

$$j_i = j_{\mathbf{B}_i} : X_i/\Gamma_i(\underline{a}) \rightarrow X/\Gamma(\underline{a})$$

are proper² (cf. [Sch3]) immersions of orientable closed manifolds, for $i = 1, 2$ and a suitable non zero ideal $\underline{a}$. In general they are not imbeddings. However, in [FOR], they showed that one can choose an ideal $\underline{a}'$ such that $j_i, i = 1, 2$ are actually imbeddings. This argument relies on the fact that every proper injective immersion is an imbedding. So one has just to prove injectivity. Herefore, we need a lemma.

Lemma 2.3. *Let $\mathbf{B}_1, \mathbf{B}_2$ be two connected reductive $\mathbb{Q}$ -subgroups of $\mathbf{G}$ and $B_i = \mathbf{B}_i(\mathbb{R})$, $i = 1, 2$. Then there is an ideal $\underline{a}' \subset \mathbb{Z}$ such that if $B_1\gamma \cap KB_2 \neq \emptyset$ for $\gamma \in \Gamma(\underline{a}')$, then $\gamma \in \mathbf{B}_1\mathbf{B}_2$.*

Proof. [FOR, Lemma 2.6].

Remark. This lemma is originally proved as theorem 3.1. in [MiRa].

Corollary 2.4. *For the ideal $\underline{a}'$ the maps j_i are imbeddings.*

Proof. It is sufficient to show this for j_1 . Let $\underline{a}'$ as in the lemma. Let $x, y \in B_1$ and $\bar{x}, \bar{y}$ their classes in X_1/Γ_1 . The condition that $j_1(\bar{x}) = j_1(\bar{y})$ is equivalent to the identity of double cosets $Kx\Gamma(\underline{a}') = Ky\Gamma(\underline{a}')$. This implies that there are elements $\gamma \in \Gamma(\underline{a}'), k \in K$ so that $x\gamma = ky \in B_1\gamma \cap KB_1$. By the lemma, this implies $\gamma \in \mathbf{B}_1 \cap \Gamma = \Gamma_1$. Hence $k \in B_1 \cap K$ and $\bar{x} = \bar{y}$. $\square$

If $\mathbf{G}$ is $\mathbb{Q}$ -anisotropic, the reductive subgroups $\mathbf{B}_i, i = 1, 2$ and $\mathbf{H} = \mathbf{G}_1 \cap \mathbf{G}_2$ are $\mathbb{Q}$ -anisotropic too. By the discussion above we can find a non zero ideal $\underline{a}'$ such that the following conditions are satisfied: Let $\Phi \subset \Gamma$ be a subgroup of finite index in $\Gamma(\underline{a}')$ and $\Phi_i = \Phi \cap \Gamma_i(\underline{a}'), i = 1, 2$ and $\Phi_{\mathbf{H}} = \Phi \cap \Gamma_{\mathbf{H}}$; then the maps

$$j_i : X_i/\Phi_i \longrightarrow X/\Phi, \quad i = 1, 2$$

and $j_{\mathbf{H}} : X_{\mathbf{H}}/\Phi_{\mathbf{H}} \longrightarrow X/\Phi$, are smooth imbeddings of connected compact orientable totally geodesic closed manifolds in the compact orientable manifold X/Φ . The image of X_i/Φ_i under j_i will be denoted by C_i resp. $C_i(\Phi)$, $i = 1, 2$ and will be called a *cycle*. If we consider just one single subgroup $\mathbf{B}$ we will denote the cycle $j_{\mathbf{B}}(X_{\mathbf{B}}/\Phi_{\mathbf{B}})$ by $C_{\mathbf{B}}$ or $C_{\mathbf{B}}(\Phi)$. We have induced maps on (singular) homology

$$(j_i)_* : H_*(X_i/\Phi_i; \mathbb{Z}) \longrightarrow H_*(X/\Phi; \mathbb{Z}), \quad i = 1, 2.$$

Let $d_i = \dim X_i/\Phi_i = \dim X_i$, $i = 1, 2$ and $[X_i/\Phi_i]$ a fundamental class of X_i/Φ_i in $H_*(X_i/\Phi_i; \mathbb{Z})$ (i.e. a generator of the group $H_{d_i}(X_i/\Phi_i; \mathbb{Z}) \cong \mathbb{Z}$). We denote by $[C_i]$ the image of $[X_i/\Phi_i]$ under $(j_i)_*$. So C_i represents a homology class $[C_i] \in H_{d_i}(X/\Phi; \mathbb{Z})$ and we call it *the homology class generated by C_i* . Under certain conditions $[C_i]$ is non trivial. Such conditions are discussed in terms of intersection numbers in the chapter 5.

²A map $f : M \rightarrow N$ between topological spaces is called *proper* if $f^{-1}(C)$ is compact for each compact set C of N .

2.2 Construction of special cycles

In this section we recall a decomposition of the fixed points $\text{Fix}(\Theta, X/\Gamma)$ of a finite abelian group Θ acting on X/Γ , parametrized by the first non abelian cohomology set as in [Ro], [RoSch] or [Sch2]. An excellent reference for non abelian cohomology is [Se2].

2.5. Let σ, τ be two $\mathbb{Q}$ -rational automorphisms of a $\mathbb{Q}$ -algebraic semisimple group $\mathbf{G}$ of finite order and assume that σ and τ commute with each other. Then by [He, I 13.5] we can find a maximal compact and $\langle \sigma, \tau \rangle$ -stable subgroup K of $\mathbf{G}(\mathbb{R})$. Then the group $\Theta := \langle \sigma, \tau \rangle$ acts on $K \setminus \mathbf{G}(\mathbb{R})$. Let Γ be a Θ -stable torsion free arithmetic subgroup³ of $\mathbf{G}(\mathbb{Q})$.

If $\gamma = (\gamma_s) \in H^1(\Theta, \Gamma)$, $s \in \Theta$, represents a cocycle, one can define a new γ -twisted Θ -action on $\mathbf{G}$ and Γ by $g \mapsto \gamma_s {}^s g \gamma_s^{-1}$, $g \in \mathbf{G}$, $s \in \Theta$. The operation on X is given by $x \mapsto {}^s x \gamma_s^{-1}$, $x \in X$, $s \in \Theta$. The new operation induced on X/Γ coincides with the previous one. Let $\Gamma(\gamma)$ be the elements fixed by the γ -twisted Θ -action, and let $X(\gamma)$ be the fixed point set of the γ -twisted Θ -action on X . The natural map $\pi_\gamma : X(\gamma)/\Gamma(\gamma) \rightarrow X/\Gamma$ is injective⁴ and its image $F(\gamma) := \text{im} \pi_\gamma \cong X(\gamma)/\Gamma(\gamma)$ lies in the fixed point set $\text{Fix}(\Theta, X/\Gamma)$. Moreover it turns out that the fixed point set $\text{Fix}(\Theta, X/\Gamma)$ is a disjoint union of the connected non empty sets $F(\gamma)$, $\gamma \in H^1(\Theta, \Gamma)$, that is

$$\text{Fix}(\Theta, X/\Gamma) = \bigsqcup_{\gamma \in H^1(\Theta, \Gamma)} F(\gamma).$$

Notice that $F(\gamma)$ depends only on the class of γ in $H^1(\Theta, \Gamma)$. If $\Theta = \langle \mu \rangle$, $\mu = \sigma, \tau$ we have $\text{Fix}(\langle \mu \rangle, X/\Gamma) = \coprod_{\gamma \in H^1(\langle \mu \rangle, \Gamma)} F(\gamma)$ as a disjoint union of connected non empty sets. The connected component corresponding to the base point 1_μ in $H^1(\Theta, \Gamma)$ will be denoted by $C(\mu) = C(\mu, \Gamma)$. It will be called a *special cycle*. Let $\mathbf{G}(\mu)$ resp. $K(\mu)$ be the group of elements in $\mathbf{G}$ resp. K fixed by μ . Clearly $K(\mu)$ is a maximal compact subgroup of $\mathbf{G}(\mu)(\mathbb{R})$. Note that we have an isomorphism of Riemannian symmetric spaces $K(\mu) \setminus \mathbf{G}(\mu)(\mathbb{R}) \cong X(\mu)$; hence a diffeomorphism

$$X(\mu)/\Gamma(\mu) \xrightarrow{\sim} C(\mu)$$

between the locally symmetric spaces $X(\mu)/\Gamma(\mu)$ and the closed submanifold $C(\mu)$.

³Such a group always exist. Obviously if Γ' is arithmetic, $\Gamma := \bigcap_{s \in \Theta} s\Gamma'$ is Θ -stable of finite index in Γ' .

⁴Observe that by corollary 2.4. this in general true, but in this special case it is easier to see. In fact, $x = y\delta$, $x, y \in X(\gamma)$, $\delta \in \Gamma$ implies $x\gamma_s = {}^s x = {}^s y {}^s \delta = y\gamma_s {}^s \delta$ (since $x, y \in X(\gamma)$). Hence $y\delta = x = y(\gamma_s {}^s \delta \gamma_s^{-1})$, but Γ acts freely on X so that $\delta = \gamma_s {}^s \delta \gamma_s^{-1}$ for all $s \in \Theta$. Thus $\delta \in \Gamma(\gamma)$.

Chapter 3

Composition algebras and a model of G_2

In the following we recall some facts concerning composition algebras and use them to describe an algebraic group of type G_2 . For more facts about these considerations see [J2] and [SV].

3.1 Composition algebras

3.1. A *quadratic form* on a vector space V over a field k is a mapping $N : V \rightarrow k$, such that $N(\lambda x) = \lambda^2 N(x)$ for $\lambda \in k, x \in V$ and the mapping $\langle \cdot, \cdot \rangle : V \times V \rightarrow k$, defined by $\langle x, y \rangle = N(x + y) - N(x) - N(y)$ is bilinear. The form N is *nondegenerate*, if $\langle x, y \rangle = 0$ for all $y \in V$, implies $x = 0$.

A *composition algebra* C over a field k , $\text{char}(k) \neq 2$ is a not necessarily associative (finite dimensional) algebra over k with identity element e and with a nondegenerate quadratic form N on C (called *norm*), which *permits composition*, i.e. such that

$$N(xy) = N(x)N(y)$$

for all $x, y \in C^1$. To N we associate a symmetric bilinear form

$$\langle x, y \rangle = N(x + y) - N(x) - N(y),$$

called the *inner product*. A *subalgebra* of C is a linear subspace D , such that $e \in D$, D is closed under multiplication and D is *non singular*, hereby we mean that $\langle \cdot, \cdot \rangle|_{D \times D}$ is non degenerate (i.e. if $x \in D$ and $\langle x, y \rangle = 0$ for all $y \in D$, then $x = 0$). Clearly, if D is a subalgebra of C , then $C = D \oplus D^\perp$. Here $D^\perp$ is the orthogonal complement of D in C with respect to the bilinear form $\langle \cdot, \cdot \rangle$. An *automorphism* of C is a bijective k -linear map $g : C \rightarrow C$ such that $g(xy) = g(x)g(y)$ for all $x, y \in C$. By [SV, 1.2.4] every automorphism g of C is an orthogonal transformation with respect to $\langle \cdot, \cdot \rangle$. The following equations are easily verified

$$N(e) = 1, \tag{3.1}$$

$$\langle x_1 y, x_2 y \rangle = \langle x_1, x_2 \rangle N(y), \tag{3.2}$$

$$\langle xy_1, xy_2 \rangle = N(x) \langle y_1, y_2 \rangle \tag{3.3}$$

¹For $\text{char}(k) = 2$, see [SV].

for all $x, y, x_1, x_2, y_1, y_2 \in C$. The subspace ek is a subalgebra; hence $C = ek \oplus C_0$, where $C_0 := C^\perp$.

For $x = x_0e + x'$, $x_0 \in k, x' \in C_0$, we set $\bar{x} = x_0e - x'$. Then $\bar{\cdot} : C \rightarrow C$ is a k -linear mapping, called *conjugation*. One has

$$\begin{aligned} x\bar{x} &= N(x)e, \\ \overline{xy} &= \bar{y}\bar{x} \\ \bar{\bar{x}} &= x \\ \langle \bar{x}, \bar{y} \rangle &= \langle x, y \rangle \end{aligned}$$

and

$$\langle xy_1, y_2 \rangle = \langle y_1, \bar{x}y_2 \rangle, \quad (3.4)$$

$$\langle x_1y, x_2 \rangle = \langle x_1, x_2\bar{y} \rangle, \quad (3.5)$$

$$\langle x, y \rangle e = x\bar{y} + y\bar{x} \quad (3.6)$$

for all $x, y, x_1, x_2, y_1, y_2 \in C$.

Proposition 3.2. *An element $x \in C$ has an inverse if and only if $N(x) \neq 0$. Then it is uniquely determined and $x^{-1} = N(x)^{-1}\bar{x}$.*

We also have a map $tr : C \rightarrow k$, $x \mapsto x + \bar{x}$, called the *trace*. By (3.6) we have

$$tr(x) = \langle x, e \rangle$$

and

$$tr(x\bar{y}) = \langle x, y \rangle.$$

Proposition 3.3. *Every element x of a composition algebra C satisfies*

$$x^2 - tr(x)x + N(x)e = 0. \quad (3.7)$$

If $x \notin ek$, then $T^2 - tr(x)T + N(x) \in k[T]$ is the minimal polynom of x over k .

Proof. [SV, 1.2.3, 1.2.4]

Let C a composition algebra and $D \neq C$ a subalgebra. Since $\langle \cdot, \cdot \rangle|_{D \times D}$ is non degenerate, we have $C = D \oplus D^\perp$ and $\langle \cdot, \cdot \rangle|_{D^\perp \times D^\perp}$ is also non degenerate; hence there must be a $z \in D^\perp$ with $N(z) \neq 0$. The following proposition is taken from [SV, 1.5.1, 2, 3].

Proposition 3.4.

1) Let C be a composition algebra and $D \neq C$, a composition subalgebra. Then D is associative. If $z \in D^\perp$ with $N(z) \neq 0$, then

$$D(z) := D \oplus Dz$$

is a composition subalgebra of C . Let $z_0 = N(z)$, then product, norm and conjugation on $D(z)$ are given by the formulas

$$(x_1 + y_1z)(x_2 + y_2z) = (x_1x_2 - z_0\bar{y}_2y_1) + (y_2x_1 + y_1\bar{x}_2)z \quad (3.8)$$

$$N(x + yz) = N(x) + z_0N(y) \quad (3.9)$$

$$\overline{x + yz} = \bar{x} - yz \quad (3.10)$$

for $x_1, x_2, y_1, y_2, x, y \in D$, and $\dim D(z) = 2\dim D$.

2) Even the converse is true: Let D be an associative algebra and $z_0 \in k^*$. Define on $D \oplus D$ a product like (3.8) and a norm like (3.9). Then C is a composition algebra.

$D(z)$ (resp. $D \oplus D$) is said to be constructed from D by doubling.

Theorem 3.5. *Every composition algebra (over k) is obtained by repeated doubling, starting from ek . The possible dimensions of a composition algebra are 1, 2, 4 and 8. Composition algebras of dimension 1 or 2 are commutative and associative, those of dimension 4 are associative but not commutative, and those of dimension 8 are neither commutative nor associative.*

Proof. see [SV, 1.6.2]

A composition algebra of dimension 2 over k is either a quadratic field extension of k or isomorphic to $k \oplus k$. A composition algebra of dimension 4 is called a *quaternion algebra* and its elements are called *quaternions*. If the composition algebra C is of dimension 8, it is called a *Cayley algebra* or *octonion algebra*. Most of the time, octonion algebras will be written as $\mathfrak{C}$. Let $\mathfrak{C}$ be an octonion algebra. Its elements are called *octonions*. If $x \in \mathfrak{C}_0 = (ek)^\perp$, it is called a *pure octonion*. By definition these are the elements of trace zero. It is a non associative and non commutative algebra and the center of $\mathfrak{C}$ is ek .

Remark. If C is a composition algebra and D and D' subalgebras of C of the same dimension, then every isomorphism from D to D' can be extended to an automorphism of C (see [SV, 1.7.3]).

Proposition 3.6. *In any octonion algebra $\mathfrak{C}$ over k ($\text{char}(k) \neq 2$) there are elements $a, b, c \in \mathfrak{C}$, such that*

$$N(a), N(b), N(c) \neq 0,$$

and

$$\mathcal{B} = \{e, a, b, ab, c, ac, bc, (ab)c\} \quad (3.11)$$

forms an orthogonal basis of $\mathfrak{C}$ with respect to the associate bilinear form $\langle, \rangle$.

Such a triple a, b, c is called a *basic triple*.

We have the relations

$$\begin{aligned} xy &= -yx, \\ x(yz) &= (yx)z = -(xy)z \end{aligned}$$

for all pairwise different $x, y, z \in \mathcal{B}_0$.

Note that for every triple of numbers a_0, b_0, c_0 in k^* , we obtain a octonion algebra $(\mathfrak{C}, N)$ over k together with an orthogonal basis like (3.11), which satisfies $N(a) = a_0, N(b) = b_0, N(c) = c_0$, by doubling (Proposition 3.4 (2)).

Let $\mathcal{B}_0 = \mathcal{B} - \{e\}$. From a basic triple, one gets immediately quadratic subalgebras L of $\mathfrak{C}$ of the form $F(x), x \in \mathcal{B}_0$. (Since $N(x) \neq 0$, for every $x \in \mathcal{B}_0$, the vectorspace L is non singular.) Let $Q := (x, y)_F$ be the k -linear span of $\{e, x, y, xy\}$ with $x, y \in \mathcal{B}_0$, where $x \neq y$. Since Q is non singular, it is a quaternion subalgebra of $\mathfrak{C}$.

3.7. With respect to the norm N , we can consider two types of octonion algebras: For an octonion algebra $(\mathfrak{C}, N)$ over a field k , we have either N anisotropic (i.e. $N(x) \neq 0$, for every $x \in \mathfrak{C}, x \neq 0$), or N isotropic (i.e. there exist a nonzero $x \in \mathfrak{C}$ with $N(x) = 0$). In the

first case, every element in $\mathfrak{C} - \{0\}$ has an inverse, we call $\mathfrak{C}$ an *octonion division algebra*. In the second case, the Witt index (i.e. the dimension of the maximal totally isotropic subspace with respect to N) must be $\frac{1}{2}\dim\mathfrak{C} = 4$ (cf. [SV, 1.8]). This is shown in [SV, 1.8]. The argument relies on the fact that one can find an element $x \in \mathfrak{C}$ such that $ke \oplus kx$ is an isotropic composition subalgebra of $\mathfrak{C}$; Hence has Witt index 1. By repeatedly applying the doubling procedure, one can see that the Witt index of $\mathfrak{C}$ must be 4. If $(\mathfrak{C}', N')$ is an other isotropic octonion algebra over k , it must be isometric to $(\mathfrak{C}, N)$ as a vector space; hence they are isomorphic [SV, 1.7.1]. Such an octonion algebra is called *split*.

3.8. We wish to give a classification of octonion algebras over certain fields k (see [SV, 1.10]). Let $(\mathfrak{C}, N)$ be an octonion algebra over a field k and $\mathcal{B}$ a orthogonal k -basis as in (3.11). Let $N(a) =: a_0, N(b) =: b_0, N(c) =: c_0 \in k$. Then the norm N has the form

$$N(x) = x_0^2 + x_1^2 a_0 + x_2^2 b_0 + x_3^2 a_0 b_0 + x_4^2 c_0 + x_5^2 a_0 c_0 + x_6^2 b_0 c_0 + x_7^2 a_0 b_0 c_0 \quad (3.12)$$

where $x = x_0 + x_1 a + x_2 b + x_3 ab + x_4 c + x_5 ac + x_6 bc + x_7 (ab)c \in \mathfrak{C}$.

1. If k is algebraically closed, every quadratic form is isotropic; hence there are only split octonion algebras.
2. If $k = \mathbb{R}$, it is easily seen that there are two classes of quadratic forms (3.12). The isotropic ones with signature (4,4) and the positive definite ones. Hence we have both, the split octonion algebra and the octonion division algebra. The latter one is the algebra $\mathbb{O}$ of so called Cayley numbers (real octonion division algebra).
3. If k is a finite field, $\text{char}(k) \neq 2$ every quadratic form in dimension > 2 is isotropic (or equivalent there are no division algebras over k). Hence there are only the split octonion algebras.
4. If k is a complete, discretely valuated field with finite residue class field, then all quadratic forms in dimension > 4 are isotropic, so there is just the split octonion algebra.
5. Let k is an algebraic number field with exactly r real places. Recall Hasse's Theorem, which says that two quadratic forms over k are equivalent if and only if they are so over all local fields k_v , where v runs over all places of k (see also [O'M, §66]). A form like (3.12) is split at all finite places and also at all complex places. So we have to look at the real places. At each of them we have two possibilities. So by Hasse's Theorem there are at most 2^r non-isomorphic octonion algebras over k . These can be realized by a suitable choice of the signs of a_0, b_0, c_0 at the real places. So all of them indeed occur. All of them are division algebras, except the one which is the split at all real places.

3.2 A model for G_2

Here we recall the construction of an algebraic group of type G_2 as the automorphism group of an octonion algebra and use it in a certain case.

3.9. Let $(\mathfrak{C}, N)$ be an octonion algebra over a field k $\text{char}(k) = 0$, $\bar{k}$ an algebraic closure of k and $\mathbf{G}(k) = \text{Aut}(\mathfrak{C})$ the automorphism group of $\mathfrak{C}$ (i.e. the group of bijective k -linear transformations $g : \mathfrak{C} \rightarrow \mathfrak{C}$ which satisfy $g(xy) = g(x)g(y)$, for all $x, y \in \mathfrak{C}$). Moreover we

define for any associative k -algebra A the group of A -points by $\mathbf{G}(A) = \text{Aut}(\mathfrak{C} \otimes_k A)$. (The terminology of tensor products of non-associative algebras is in the usual sense. See [Schafer, p.12].)

In this way we can define the algebraic group $\mathbf{G}$. Denote by $\mathbf{O}(N)$ the algebraic orthogonal group of the quadratic form on $\mathfrak{C}$ defined by N . Then $\mathbf{G}$ is a connected closed subgroup of $\mathbf{O}(N)^2$ and defined over k (cf. [SV, 2.2.3, 2.4.6]). The group $\mathbf{G}$ is a simple algebraic group of type G_2 (cf. [SV, 2.3.5]). Since $\mathbf{G}$ is connected it is contained in $\mathbf{SO}(N) = \mathbf{O}(N)^0$.

For a better understanding of $\mathbf{G}$ we need a detailed description of the automorphisms of $\mathfrak{C}$. Any automorphism of $\mathfrak{C}$ stabilizes e , hence also $ek^\perp = \mathfrak{C}_0$. Let $N_0 = N|_{\mathfrak{C}_0}$ be the restriction of N to $\mathfrak{C}_0$ and $\mathbf{F}$ the stabilizer of e in $\mathbf{O}(N)$. Then $\mathbf{G}$ can be viewed as a subgroup of $\mathbf{F}$; hence $\mathbf{G} \leq \mathbf{SO}(N) \cap \mathbf{F} \cong \mathbf{SO}(N_0)$ (cf. [SV, 2.2.2]).

Example. Let $k = \mathbb{R}$. Then we have two cases:

1. The quadratic form N (3.12) is split, i.e., represents zero non trivially. Then N has signature (4,4); hence $G := \mathbf{G}(\mathbb{R}) \leq SO(3,4)$. In this case G is the connected split real Lie group of type G_2 . The group G is not simply connected and her fundamental group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (cf. [Ti]). Since G is connected, we have $G \leq SO(3,4)^0$.
2. The quadratic form N (3.12) is positive definite; Then N has signature (8,0); hence $G_u := \mathbf{G}(\mathbb{R}) \leq SO(7)$. In this case G_u is the connected compact real Lie group of type G_2 . Further, G_u is simply connected and has trivial center.

A complete description of $\mathbf{G}$ is given as follows (cf. [J2, 4.]). Let $\mathcal{T} : \mathfrak{C}_0 \times \mathfrak{C}_0 \times \mathfrak{C}_0 \rightarrow k$ be the alternating trilinear map, defined by $\mathcal{T}(x, y, z) = \langle x, zy \rangle - \langle x, yz \rangle$.

Proposition 3.10. *An element $g \in \mathbf{SO}(N_0)$ is in $\mathbf{G}$, if and only if g leaves $\mathcal{T}$ invariant, i.e., $\mathcal{T}(gx, gy, gz) = \mathcal{T}(x, y, z)$ for all $x, y, z \in \mathfrak{C}_0$.*

Denote by $\text{Der}_k(\mathfrak{C})$ the derivations of $\mathfrak{C}$, i.e., the k -linear maps d of $\mathfrak{C}$ to $\mathfrak{C}$ such that

$$d(xy) = xd(y) + d(x)y,$$

for all $x, y \in \mathfrak{C}$. The derivations of $\mathfrak{C}$ form a 14-dimensional Lie algebra $\text{Der}_k(\mathfrak{C})$, where the Lie bracket is the usual commutator (i.e. $[d, d'] = d \circ d' - d' \circ d$). It turns out, that the k -Lie algebra $\text{Lie}(\mathbf{G})$ of $\mathbf{G}/k$ is isomorphic to $\text{Der}_k(\mathfrak{C})$. The *associator* of $x, y, z \in \mathfrak{C}$ is defined as

$$\{x, y, z\} = (xy)z - x(yz).$$

It is trilinear and skew symmetric, i.e. for every permutation $\pi \in S_3$: $\{x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}\} = \text{sgn}(\pi)\{x_1, x_2, x_3\}$. By a (not so short) computation, one can show that for any $x, y \in \mathfrak{C}_0$, the map

$$d_{x,y} : \mathfrak{C} \rightarrow \mathfrak{C}, \quad d_{x,y}(v) = [[x, y], v] - 3\{x, y, v\}$$

is a derivation of $\mathfrak{C}$ (cf. [Schafer, (3.70)], see also [Baez, 4.1]) and that every derivation arises in this way. The map

$$d : \mathfrak{C}_0 \times \mathfrak{C}_0 \rightarrow \text{Der}_k(\mathfrak{C}), \quad (x, y) \mapsto d_{x,y}$$

²Since g is k -linear, we have $g(\bar{x}) = x_0e - g(x') = \overline{g(x)}$ and by (3.6) we have $\langle g(x), g(y) \rangle e = g(x\bar{y} + y\bar{x}) = g(\langle x, y \rangle e) = \langle x, y \rangle e$.

is k -bilinear and surjective. If we take the basis $\mathcal{B}_0 = \mathcal{B} - \{e\}$ of $\mathfrak{C}_0$ as in (3.11) we see that $\{d_{x,y} \mid x, y \in \mathcal{B}_0\}$ generates $\text{Der}_k(\mathfrak{C})$ as a k -vector space. Note that this set does not form a basis of $\text{Der}_k(\mathfrak{C})$. In fact, we have $d_{x,y} = -d_{y,x}$ and

$$d_{xy,z} + d_{yz,x} + d_{zx,y} = 0, \quad (3.13)$$

for all $x, y, z \in \mathfrak{C}_0$ -so they can not be linearly independent. The adjoint representation

$$\text{Ad}_{\mathbf{G}} : \mathbf{G} \rightarrow \mathbf{GL}(\text{Der}_k(\mathfrak{C}))$$

is given as usual by $g.d = g \circ d \circ g^{-1}$, $d \in \text{Der}_k(\mathfrak{C})$. Clearly, we have

$$g.d_{x,y} = d_{g(x),g(y)}.$$

By [J1, p.782], the adjoint representation is injective. Moreover, by [J1, p.782], the group of automorphisms $\text{Aut}(\text{Der}_k(\mathfrak{C}))$ of $\text{Der}_k(\mathfrak{C})$ is isomorphic to $\mathbf{G}$, i.e. every automorphism of $\text{Der}_k(\mathfrak{C})$ is inner. Since $\text{char}(k) = 0$, we have $\ker(\text{Ad}_{\mathbf{G}}) = Z(\mathbf{G})$ (see [Milne, 14.8] or [Hum, 13.4 (b)]); thus we get that $Z(\mathbf{G}) = \{e\}$ (see also [SV, 2.3.2]). Hence $\mathbf{G}$ is adjoint. By [PR, p. 63-64] the center of the universal covering of $\mathbf{G}$ is also trivial; thus we conclude that $\mathbf{G}$ is a simply connected, adjoint, connected, simple k -algebraic group.

3.11. We will now specify our discussion to a specific situation. Let F a totally real number field of degree $[F : \mathbb{Q}] = r > 1$. Denote by $V_{\infty} = \{s : F \hookrightarrow \mathbb{R}\}$ the set of real places of F . Instead of $s(F)$ we write F_s . Let $a_0, b_0, c_0 \in F$, $a_0, b_0 < 0, c_0 > 0$ such that

$$s(a_0), s(b_0), s(c_0) > 0, \text{ for all } s \in V_{\infty} - \{\text{id}\}.$$

The existence of such numbers (for every totally real number field F) follows by the weak approximation theorem for number fields. Let $(\mathfrak{C}, N)$ be the octonion algebra over F determined by the F -basic triple defined by $N(a) := a_0, N(b) := b_0, N(c) := c_0$. Now we wish to extend the embeddings $s : F \hookrightarrow \mathbb{R}$ to $\mathfrak{C}$ in a suitable way:

For every $s \in V_{\infty} - \{\text{id}\}$, we define an octonion algebra $(\mathfrak{C}_s, N_s)$ over F_s determined by the F_s -basic triple $\{a_s, b_s, c_s\}$ determined by $N_s(a_s) := s(a_0), N_s(b_s) := s(b_0), N_s(c_s) := s(c_0)$. The map

$$\phi_s : \mathfrak{C} \rightarrow \mathfrak{C}_s, \quad a \mapsto a_s, b \mapsto b_s, c \mapsto c_s$$

induces an s -isomorphism of octonion algebras (i.e. s is an isomorphism of F onto F_s and ϕ_s is a bijective s -linear transformation such that $\phi_s(xy) = \phi_s(x)\phi_s(y)$, $x, y \in \mathfrak{C}$). Hence $\mathfrak{C}_s = \phi_s(\mathfrak{C})$. The norm N_s on $\mathfrak{C}_s$ satisfies $N_s(\phi_s(x)) = s(N(x))$, $x \in \mathfrak{C}$. As expected, the equation (3.12) shows that $\mathfrak{C}_s$ is an F_s -form of the division algebra $\mathbb{O}$ (i.e. $\mathfrak{C}_s \otimes_{F_s} \mathbb{R} \cong \mathbb{O}$), for $s \neq \text{id}$ and an F -form of the split algebra $\mathfrak{C}_{\mathbb{R}}$ (i.e. $\mathfrak{C} \otimes_F \mathbb{R} \cong \mathfrak{C}_{\mathbb{R}}$) otherwise. Let $\mathbf{G}'$ be the F -algebraic group of automorphisms of $\mathfrak{C}$.

If $x \in \mathfrak{C}$ is an isotropic vector, i.e. $N(x) = 0$, then $N_s(\phi_s(x)) = s(N(x)) = 0$, but N_s is an F_s -anisotropic quadratic form, hence $x = 0$. But the F -rank of orthogonal group $\mathbf{SO}(N)$ is positive if and only if N is isotropic; hence the F -rank of $\mathbf{SO}(N)$ must be zero, i.e. the group $\mathbf{SO}(N)$ is anisotropic. Since $F - \text{rank}(\mathbf{G}') \leq F - \text{rank}(\mathbf{SO}(N))$ the group $\mathbf{G}'$ is F -anisotropic. Let $\mathbf{G} := \text{Res}_{F/\mathbb{Q}}(\mathbf{G}')$ the $\mathbb{Q}$ -group obtained from $\mathbf{G}'$ by restriction of scalars. Then, by [Sp, 12.4.7. (3)], $\mathbf{G}$ is connected and, by [Sp, 16.2.7], $\mathbb{Q}$ -anisotropic. The group of real points is

$$G := \mathbf{G}(\mathbb{R}) = \prod_{s \in V_{\infty}} \mathbf{G}'_s(\mathbb{R}) = G' \times G'_u \times \cdots \times G'_u. \quad (3.14)$$

Here the algebraic group $\mathbf{G}'_s$ denotes the automorphism group of $\mathfrak{C}_s$. It is the group $\mathbf{G}'$ conjugated by s . Further $G' = \mathbf{G}'(\mathbb{R}) = \text{Aut}(\mathfrak{C}_{\mathbb{R}})$ is the automorphism group of the split octonion algebra and $G'_u = \mathbf{G}'_s(\mathbb{R}) \cong \text{Aut}(\mathbb{O})$, $s \neq \text{id}$, is the automorphism group of the real division octonion algebra $\mathbb{O}$.

Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an torsion free arithmetic subgroup and X the associated symmetric space to G . By the compactness criterion (see e.g. [Ji]), we see that X/Γ is compact.

3.3 Arithmetic of octonions and arithmetic subgroups of G_2

Here we recall the basic definition of a maximal order in an octonion algebra as in [VS] and some related arithmetic facts.

3.12. Let $\mathfrak{C}$ be an octonion algebra over a number field F and $\mathcal{O} = \mathcal{O}_F$ be the ring of algebraic integers in F . An octonion $x \in \mathfrak{C}$ is called *integral*, if $N(x) \in \mathcal{O}$ and $\text{tr}(x) = \langle x, e \rangle \in \mathcal{O}$. Recall that by (3.7) x satisfies $x^2 - \text{tr}(x)x + N(x)e = 0$. So $x \in \mathfrak{C}$ is integral if it is a root of a monic quadratic polynomial with coefficients in $\mathcal{O}$. A *lattice* in $\mathfrak{C}$ is a finitely generated, free $\mathcal{O}$ -module M in $\mathfrak{C}$ of rank 8. An *order* M of $\mathfrak{C}$ is a lattice in $\mathfrak{C}$ such that M is a ring containing e and consists of integral elements only. A *maximal order* (or *octonion ring*) M of $\mathfrak{C}$ is an order in $\mathfrak{C}$ such that M is a ring containing e , consists of integral elements only and is maximal with respect to these properties. To see that such maximal orders do exist, we have the following

Proposition 3.13. *Every order M in $\mathfrak{C}$ is contained in a maximal order.*

Proof. Assume that M is an order in $\mathfrak{C}$. Take $\{e_i\}_{i=0,\dots,7}$ to be a $\mathcal{O}$ -basis of M and the symmetric matrix $A = (\langle e_i, e_j \rangle)_{i,j=0,\dots,7} \in M_8(\mathcal{O})$. If M' is a bigger order containing $M \subsetneq M'$ and $\{e'_i\}_{i=0,\dots,7}$ is a $\mathcal{O}$ -basis of M' , we get a matrix $T = (t_{ij}) \in M_8(\mathcal{O})$, such that $e_i = \sum_j t_{ij} e'_j$. The matrix $A' = (\langle e'_i, e'_j \rangle)$ has the property that $0 \neq \det(A) = \det(T)^2 \det(A')$. Since M' is bigger than M , $\det(T)$ could not be invertible in $\mathcal{O}$. So the ideal $\det(A)\mathcal{O}$ is properly contained in the ideal $\det(A')\mathcal{O}$. If we repeat this procedure, we get a chain of ideals. But the ring $\mathcal{O}$ is noetherian, hence every such chain is stationary. Thus, M is contained in a maximal order. $\square$

Let M be an order. Note that the conjugation map $x \mapsto \bar{x}$ can be written as $\bar{x} = \langle x, e \rangle e - x$. Hence, since $e \in M$ and $\langle x, e \rangle \in \mathcal{O}$, we have that M is stable under conjugation. By the relation $x^{-1} = N(x)^{-1} \bar{x}$, we get for the invertible elements M^* in M that

$$M^* = \{x \in M \mid N(x) \in \mathcal{O}^*\}.$$

Let $\mathbf{G}'$ be the F -algebraic group of automorphisms of $\mathfrak{C}$ and M be just an arbitrary lattice in $\mathfrak{C}$. Then the group $\Gamma = \{g \in \mathbf{G}' \mid gM = M\}$ is an arithmetic subgroup of $\mathbf{G}'(F)$:

Take an $\mathcal{O}$ -basis $\mathcal{W} = \{w_i\}_{i=0,\dots,7}$ of M . The F -basis $\mathcal{W}$ of $\mathfrak{C}$ induces a F -rational imbedding $\mathbf{G}' \hookrightarrow \mathbf{GL}(\mathfrak{C})$, $g \mapsto [g]_{\mathcal{W}}$ of $\mathbf{G}'$ into the general linear group of the F -vector space $\mathfrak{C}$, with respect to the basis $\mathcal{W}$. Take $\mathcal{E} = \{e_i\}_{i=0,\dots,7}$ to be the canonical F -basis of F^n , then there is an $T \in GL_8(F)$, such that $[g]_{\mathcal{W}} = T[g]_{\mathcal{E}}T^{-1}$ and $\mathcal{W} = T\mathcal{E}$; hence we have an induced F -rational imbedding $\varphi : \mathbf{G}' \hookrightarrow \mathbf{GL}_8$, $g \mapsto [g]_{\mathcal{E}}$ and Γ is precisely the group of $\mathcal{O}$ -points under this imbedding: $\Gamma = \varphi^{-1}(\varphi(\mathbf{G}') \cap GL_8(\mathcal{O})) = \mathbf{G}'(\mathcal{O})$.

3.14 Let $\mathfrak{C}$ and the notation as in 3.11. Let $V_{\infty} = \{\text{id} = s_1, \dots, s_r\}$ and $F_i := F_{s_i}$, $\mathfrak{C}_i := \mathfrak{C}_{s_i}$, $s_i \in V_{\infty}$. Let $\mathcal{B} = \{e, a, b, ab, c, ac, bc, (ab)c\}$ and $\phi_i = \phi_{s_i} : \mathfrak{C} \rightarrow \mathfrak{C}_i$. Assume that

$M \subset \mathfrak{C}$ is an $\mathcal{O}$ -lattice. Define a ring homomorphism

$$R : \mathfrak{C} \rightarrow \prod_{i=1}^r \mathfrak{C}_i, \quad \sum_{m \in \mathcal{B}} x_m m \mapsto \left(\sum_{m \in \mathcal{B}} s_1(x_m) \phi_1(m), \dots, \sum_{m \in \mathcal{B}} s_r(x_m) \phi_r(m) \right),$$

where $x_m \in F$. But $\mathfrak{C}$ -and so $R(\mathfrak{C})$ - is an $8r$ -dimensional $\mathbb{Q}$ -vector space. The linear independence over $\mathbb{R}$ of the mappings $s_1, \dots, s_r$, shows that $R(\mathfrak{C})$ is a $\mathbb{Q}$ -form of the $8r$ -dimensional $\mathbb{R}$ -vectorspace $R(\mathfrak{C}) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathfrak{C} \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{i=1}^r (\mathfrak{C}_i \otimes_{F_i} \mathbb{R})$ and $R(M)$ is a $\mathbb{Z}$ -lattice in $\prod_{i=1}^r (\mathfrak{C}_i \otimes_{F_i} \mathbb{R})$.

Let $\{v_i\}_{i=1, \dots, r}$ be a $\mathbb{Z}$ -basis of $\mathcal{O}$ and $\{w_j\}_{j=0, \dots, 7}$ be a $\mathcal{O}$ -basis of M . Then we view $R(\mathfrak{C})$ resp. $R(M)$ as the $\mathbb{Q}$ -vectorspace resp. $\mathbb{Z}$ -module generated by $R(v_i w_j), i = 1, \dots, r, j = 0, \dots, 7$. If $\mathbf{G}' = \text{Aut}(\mathfrak{C})$, $\mathbf{G} = \text{Res}_{F/\mathbb{Q}}(\mathbf{G}')$, then $\mathbf{G}(\mathbb{Q}) \subset \mathbf{GL}(R(\mathfrak{C}))(\mathbb{Q})$ resp. $\mathbf{G}(\mathbb{Z}) \subset \mathbf{GL}(R(\mathfrak{C}))(\mathbb{Z})$.

Chapter 4

Algebraic subgroups and cycles – the case G_2

In the present chapter we determine reductive subgroups of the group $\mathbf{G} = \text{Res}_{F/\mathbb{Q}} \mathbf{G}'$ found in the last chapter, by looking at an octonion algebra $\mathfrak{C}$. Note that $\text{Res}_{F/\mathbb{Q}}$ defines a bijection from the connected F -subgroups of $\mathbf{G}'$ to the connected $\mathbb{Q}$ -subgroups of $\mathbf{G}$ (see [BoTi, 6.18]). Thus it is sufficient to determine reductive subgroups of the group $\mathbf{G}'$. A closer analysis of the structure of $\mathfrak{C}$ allows us to get a complete overview over the possible types of reductive subgroups of $\mathbf{G}'$. Taking this subgroups we use §2 to obtain the resulting cycles.

4.1 Reductive subgroups of $\mathbf{G}$

We recall our assumptions from §3.2. Let F a totally real number field of degree $r = [F : \mathbb{Q}] > 1$ and V_∞ the set of real places of F . Instead of $s(F)$ we write F_s , $s \in V_\infty - \{\text{id}\}$. Let $a_0, b_0, c_0 \in F$, $a_0, b_0 < 0$, $c_0 > 0$ such that

$$s(a_0), s(b_0), s(c_0) > 0, \text{ for all } s \in V_\infty - \{\text{id}\}.$$

For every $s \in V_\infty - \{\text{id}\}$, we define an octonion algebra $(\mathfrak{C}_s, N_s)$ over F_s determined by the F_s -basic triple $\{a_s, b_s, c_s\}$ determined by $N_s(a_s) := s(a_0)$, $N_s(b_s) := s(b_0)$, $N_s(c_s) := s(c_0)$. The map

$$\phi_s : \mathfrak{C} \rightarrow \mathfrak{C}_s, a \mapsto a_s, b \mapsto b_s, c \mapsto c_s$$

induces an s -isomorphism of octonion algebras. Let $\mathbf{G}'$ be the F -algebraic group of automorphisms of $\mathfrak{C}$. Note, that the group $\mathbf{G}'$ is F -anisotropic. Further, let $\mathbf{G} := \text{Res}_{F/\mathbb{Q}}(\mathbf{G}')$ the $\mathbb{Q}$ -group obtained from $\mathbf{G}'$ by restriction of scalars. For $x, y \in \mathfrak{C}$, we denote by $(x, y)_F$ the subalgebra of $\mathfrak{C}$ generated by x and y .

4.1.1 The subgroups of type $A_1 \times A_1$

4.1. Let $g_\theta \in \mathbf{G}'$ be the (F -linear) automorphism of $\mathfrak{C}$, induced by the assignment

$$a \mapsto -a, b \mapsto -b, c \mapsto c.$$

We obtain a F -rational automorphism θ of $\mathbf{G}'$ of order two by conjugation with g_θ and a $\mathbb{Q}$ -rational automorphism $\text{Res}_{F/\mathbb{Q}}\theta$ of $\mathbf{G}$, for short also denoted by θ . We wish to determine the fixed point set $\mathbf{G}'(\theta) = \{g \in \mathbf{G}' \mid \theta(g) = g\}$.

It is easy to see that $\mathbf{G}'(\theta) = \mathbf{G}'((c, ab)_F) = \{g \in \mathbf{G}' \mid g(c, ab)_F = (c, ab)_F\}$ is the group which leaves the quaternion algebra $(c, ab)_F$ invariant.

For a quaternion subalgebra (i.e. a 4 dimensional composition algebra) $Q < \mathfrak{C}$, we denote the norm 1 elements of Q by $\mathbf{SL}_1(Q)$, the invertible elements by $\mathbf{GL}_1(Q)$ and the automorphism group by $\text{Aut}(Q)$. By doubling (proposition 3.4), we have some $z \in Q^\perp$ such that $\mathfrak{C} = Q \oplus Qz$. Let $g \in \mathbf{G}'(Q, pw)$ in the group of elements in $\mathbf{G}'$ which fix the algebra Q pointwise. Then $g(x_1 + x_2z) = x_1 + (p_g x_2)z$, for all $x_1, x_2 \in Q$ and for one fixed $p_g \in \text{SL}_1(Q)$. The map

$$\mathbf{G}'(Q, pw) \rightarrow \mathbf{SL}_1(Q), \quad g \mapsto p_g \quad (4.1)$$

is an isomorphism (cf. [SV, 2.2.1]). If $g \in \mathbf{G}'(Q)$ we have by Skolem-Noether (cf. [SV, p.26, (2.2)])

$$g(x_1 + x_2z) = c_g x_1 c_g^{-1} + (p_g c_g x_2 c_g^{-1})z \quad (4.2)$$

for all $x_1, x_2 \in Q$, with one fixed $p_g \in \text{SL}_1(Q)$ and one $c_g \in \text{GL}_1(Q)$, unique up to a nonzero element of the center of Q . Thus we have the isomorphism

$$\mathbf{G}'(Q) \cong \mathbf{SL}_1(Q) \rtimes \text{Aut}(Q), \quad g \mapsto (p_g, \text{conj}_{c_g}). \quad (4.3)$$

Since the group of $\mathbb{C}$ -points $\text{GL}_2(\mathbb{C})$ of $\mathbf{GL}_1(Q)$ is connected, the exact sequence

$$1 \rightarrow F^* \rightarrow \mathbf{GL}_1(Q) \rightarrow \text{Aut}(Q) \rightarrow 1$$

shows that $\text{Aut}(Q)$ is connected (see e.g. [Milne, 8.27]). Again [Milne, 8.27] or [Sp, 5.5.9 (1)] and the exact sequence

$$1 \rightarrow \mathbf{SL}_1(Q) \rightarrow \mathbf{G}'(Q) \rightarrow \text{Aut}(Q) \rightarrow 1$$

shows that $\mathbf{G}'(Q)$ is connected (here we use that $\mathbf{SL}_1(Q)$ is connected [SV, p.29]).

Let $V_\theta = \text{span}_F\{a, b, ac, bc\} = \{v \in \mathfrak{C}_0 \mid g_\theta(v) = -v\}$ and $W_\theta = \text{span}_F\{ab, c, (ab)c\} = \{w \in \mathfrak{C}_0 \mid g_\theta(w) = w\}$. Then $\mathfrak{C}_0 \cong V_\theta \oplus W_\theta$ as F -vector spaces. Let $B_\theta := \langle, \rangle|_{V_\theta}$ be the F -bilinear form on V_θ , induced from $\langle, \rangle$ and $\mathbf{SO}(V_\theta, B_\theta)$ the special orthogonal group of the space (V_θ, B_θ) .

Proposition 4.2. *The map*

$$\varrho : \mathbf{G}'(\theta) \rightarrow \mathbf{SO}(V_\theta, B_\theta), \quad g \mapsto g|_{V_\theta}$$

is an F -isomorphism of algebraic groups.

Proof. Let $g \in \mathbf{G}'(\theta)$, then g and g_θ commute; hence g leaves V_θ and W_θ invariant. But g is an orthogonal transformation with respect to $\langle, \rangle$, hence orthogonal with respect to B_θ . So $\varrho(g) \in \mathbf{O}(V_\theta, B_\theta)$. Since $\mathcal{T}$ is an alternating trilinear form and g leaves W_θ invariant, we get $\mathcal{T}(g(ab), g(c), g((ab)c)) = \det(g|_{W_\theta}) \mathcal{T}(ab, c, (ab)c)$. On the other hand $\mathbf{G}'$ leaves $\mathcal{T}$ invariant, hence $\det(g|_{W_\theta}) = 1$. Since $\det g = 1$, it follows that $\det(g|_{V_\theta}) = 1$. So ϱ is well defined. The space V_θ contains a system of generator for the algebra $\mathfrak{C}$, therefore every automorphism of $\mathfrak{C}$ is uniquely determined by its values on V_θ ; hence ϱ is an injective homomorphism.

Counting dimensions in (4.3) we find

$$\dim \mathbf{G}'(Q) = \dim \mathbf{SL}_1(Q) + \dim \text{Aut}(Q) = 3 + 3 = 6 = \dim \mathbf{SO}(V_\theta, B_\theta).$$

Hence ϱ induces a Lie-algebra isomorphism. Since both $\mathbf{G}'(Q)$ and $\mathbf{SO}(V_\theta, B_\theta)$ are connected, this shows by [Sp, 5.3.5.(3)] that ϱ is surjective; hence bijective. $\square$

One sees immediately, that B_θ viewed over $\mathbb{R}$ is negative definite. Thus θ induces a Cartan involution on $G' := \mathbf{G}'(\mathbb{R})$ and that $K' := \mathbf{G}'(\theta)(\mathbb{R}) \cong SO(4)$. Note that θ does **not** induce a Cartan involution on $G := \mathbf{G}(\mathbb{R})$, since the maximal compact subgroup K of $\mathbf{G}(\mathbb{R})$ is $K := SO(4) \times G'_u \times \cdots \times G'_u$ (for the notation of G'_u see (3.14)) and $\mathbf{G}(\theta)(\mathbb{R}) \cong SO(4) \times SO(4) \times \cdots \times SO(4)$. The Cartan involution on $G = G' \times G'_u \times \cdots \times G'_u$ is $(g_1, g_2, \dots, g_r) \mapsto (\theta(g_1), g_2, \dots, g_r)$ ($r = [F : \mathbb{Q}]$).

Let $g_\sigma \in \mathbf{G}'$ be the (F -linear) automorphism of $\mathfrak{C}$, induced by the assignment

$$a \mapsto -a, b \mapsto b, c \mapsto c.$$

Then, as above we obtain an F -rational automorphism σ of $\mathbf{G}'$ of order two by conjugation with g_σ . By the same way (using similar notation with σ instead of θ) we obtain an isomorphism

$$\mathbf{G}'(\sigma) \xrightarrow{\sim} \mathbf{SO}(V_\sigma, B_\sigma), \quad g \mapsto g|_{V_\sigma}.$$

where $V_\sigma = \text{span}_F\{a, ab, ac, (ab)c\}$. Obviously, $\mathbf{G}'(\sigma) \cong \mathbf{G}'((b, c)_F) = \{g \in \mathbf{G}' \mid g(b, c)_F = (b, c)_F\}$ is the group which leaves the quaternion algebra $(b, c)_F$ invariant. The signature of B_σ over $\mathbb{R}$ is $(2, 2)$, hence $\mathbf{G}'(\sigma)(\mathbb{R}) \cong SO(2, 2)$. The automorphisms σ and θ commute. Their composition $\sigma\theta$ is the conjugation by the element $g_{\sigma\theta} := g_\sigma g_\theta$. This element is induced by the assignment

$$a \mapsto a, b \mapsto -b, c \mapsto c.$$

Like above, $\mathbf{G}'(\sigma\theta) \cong \mathbf{G}'((a, c)_F)$ is the group which leaves the quaternion algebra $(a, c)_F$ invariant and $\mathbf{G}'(\sigma\theta) \cong \mathbf{SO}(V_{\sigma\theta}, B_{\sigma\theta})$, where $V_{\sigma\theta} = \text{span}_F\{b, ab, bc, (ab)c\}$ and $\mathbf{G}'(\sigma\theta)(\mathbb{R}) \cong SO(2, 2)$. Note that the group of real points are $\mathbf{G}'(\sigma)(\mathbb{R}) \cong \mathbf{G}'(\sigma\theta)(\mathbb{R}) \cong SO(2, 2)$. The real points of the $\mathbb{Q}$ -groups $\mathbf{G}(\sigma)$ and $\mathbf{G}(\sigma\theta)$ are:

1. $\mathbf{G}(\sigma)(\mathbb{R}) \cong SO(2, 2) \times SO(4) \times \cdots \times SO(4)$
2. $\mathbf{G}(\sigma\theta)(\mathbb{R}) \cong SO(2, 2) \times SO(4) \times \cdots \times SO(4)$

Since θ induces a Cartan involution on $\mathbf{G}'(\mathbb{R})$ we have $K'(\sigma) = K'(\sigma\theta) = (\mathbf{G}'(\sigma) \cap \mathbf{G}'(\sigma\theta))(\mathbb{R}) \cong S(O(2) \times O(2))$. Since θ induces not a Cartan involution on $\mathbf{G}(\mathbb{R})$, we take the maximal compact subgroup $K \subset \mathbf{G}(\mathbb{R})$ and have $K(\sigma) \cong K(\sigma\theta) \cong S(O(2) \times O(2)) \times SO(4) \times \cdots \times SO(4)$, but $K(\sigma) \neq K(\sigma\theta)$. Further, $(\mathbf{G}(\sigma) \cap \mathbf{G}(\sigma\theta))(\mathbb{R}) \subset \mathbf{G}(\theta)(\mathbb{R})$ is compact.

Remark. Of course, there are much more F -rational involutions on $\mathfrak{C}$, than just g_θ and g_σ , which give rise to involutions on $\mathbf{G}$. As example take any quaternion subalgebra Q of $\mathfrak{C}$ and any $z \in Q^\perp$ with $N(z) \neq 0$. Then $\mathfrak{C} = Q \oplus zQ$ and we can define the involution $\varepsilon : \mathfrak{C} \rightarrow \mathfrak{C}$, by $\varepsilon(x) = x$ for $x \in Q$ and $\varepsilon(z) = -z$. However, in the real case we will see in §4.3 that g_θ and g_σ are already all involutions up to conjugacy.

Remark. Since we have found the maximal compact subgroup $K' = SO(4)$ of G' , we are now able to give the locally symmetric space $K' \backslash G' / \Gamma$ a new interpretation as follows. Let $X = K' \backslash G'$ be the symmetric space of the split group of type G_2 and $\mathbb{H}$ be the Hamilton quaternion algebra. Note that $g_\theta \in G'$ and that the algebra A of fixed points by g_θ in $\mathbb{C}_\mathbb{R}$, is isomorphic to $\mathbb{H}$. Then there is a canonical 1 : 1-correspondence

$$\begin{array}{ccc} X & \longleftrightarrow & \left\{ \begin{array}{l} \text{quaternion subalgebras } Q \\ \text{of } \mathbb{C}_\mathbb{R} = \mathfrak{C} \otimes_F \mathbb{R}, \text{ such that } Q \cong \mathbb{H} \end{array} \right\} \\ K'g & \longmapsto & g^{-1}A. \end{array}$$

Proof. If $k \in K' = G'(\theta)$, then $g_\theta k = kg_\theta$ implies that g_θ fixes kA pointwise; hence $kA = A$. So if $g = kg'$, $g, g' \in G'$, $k \in K' = G'(\theta)$, then $g^{-1}A = g'^{-1}k^{-1}A = g'^{-1}A$ and g defines an isomorphism from $g^{-1}A$ to A ; thus $g^{-1}A \cong \mathbb{H}$. So the map is well defined. Obviously, the assignment is injective. Let Q be a subalgebra of $\mathfrak{C}_{\mathbb{R}}$ such that there is an isomorphism $Q \rightarrow \mathbb{H}$. Thereby we get an isomorphism $Q \rightarrow A$. By the remark after theorem 3.5 (resp. [SV, 1.7.3]) we can extend this isomorphism to an automorphism g of $\mathfrak{C}_{\mathbb{R}}$. So $g^{-1}A = Q$, i.e. the map is surjective. $\square$

Now let $\Gamma \subset \mathbf{G}'(F)$ be an arithmetic subgroup. We say that two quaternion subalgebras Q, Q' of $\mathfrak{C}_{\mathbb{R}}$ are Γ -equivalent, if and only if there exists a $\gamma \in \Gamma$ such that $\gamma^{-1}Q = Q'$. We denote the Γ -equivalence class of Q by ΓQ . Then we have a canonical 1 : 1-correspondence

$$\begin{array}{ccc} X/\Gamma & \longleftrightarrow & \left\{ \begin{array}{c} \Gamma\text{-equivalence classes} \\ \text{of quaternion subalgebras } Q \\ \text{of } \mathfrak{C}_{\mathbb{R}} = \mathfrak{C} \otimes_F \mathbb{R}, \text{ such that } Q \cong \mathbb{H} \end{array} \right\} \\ K'g\Gamma & \longmapsto & \Gamma g^{-1}A. \end{array}$$

4.1.2 The subgroups of type 2A_1 and 2A_2

The subgroups of type 2A_2 are determined in [J2, §5] (see also [PR, Proposition 9.7] and [KMRT, p. 507, Ex.6]). They are the stabilizers of vectors in $\mathfrak{C}_0$.

4.3. By our choice of $\mathfrak{C}$, the norm N is F -anisotropic; hence every $z \in \mathfrak{C} - \{0\}$ is anisotropic, so $F(z)$ is either F or a quadratic field extension¹ of F . Assume that $[F(z) : F] = 2$; i.e. $z \in \mathfrak{C}_0 - \{0\}$. Then $F(z)$ is a subalgebra of $\mathfrak{C}$. Denote by τ_z the non trivial Galois automorphism of $F(z)/F$. On the 4-dimensional $F(z)$ -(left) vector space $\mathfrak{C}$ we define a τ_z -hermitian form

$$h_z : \mathfrak{C} \times \mathfrak{C} \rightarrow F(z)$$

by $h_z(x, y) := \langle x, y \rangle + z^{-1}\langle zx, y \rangle$. Clearly, $h_z(x, y) \in F(z)$. Then the calculations

$$\begin{aligned} h_z(zx, y) &= \langle zx, y \rangle + z^{-1}\langle z^2x, y \rangle = zh_z(x, y), \\ h_z(x, zy) &= \langle x, zy \rangle + z^{-1}\langle zx, zy \rangle \\ &\stackrel{(3.3), (3.4)}{=} -\langle zx, y \rangle - z\langle x, y \rangle = -zh_z(x, y) \\ &= h_z(x, y)\tau_z(z) \\ h_z(x, y) &= \langle y, x \rangle + z^{-1}\langle y, zx \rangle \\ &= \langle y, x \rangle - z^{-1}\langle zy, x \rangle = \tau_z(h_z(y, x)) \end{aligned}$$

show that h_z is τ_z -hermitian. Since e, z are F -linear independent, $h_z(x, y) = 0$ if and only if $\langle x, y \rangle = 0$ and $\langle zx, y \rangle = 0$; but this implies that h_z is non-degenerate. If U is a $F(z)$ -vector subspace of $\mathfrak{C}$, then $U^\perp = U^{\perp_{h_z}}$. Let $V_z := (eF(z))^{\perp_{h_z}} = (eF(z))^\perp$. Then V_z is a 3-dimensional $F(z)$ -vector space. Denote by $\mathbf{SU}(V_z, h_z, \tau_z)$ the special unitary group, coming from the hermitian space (V_z, h_z, τ_z) and the stabilizer of z in $\mathbf{G}'$ by $\mathbf{G}'(z) = \{g \in \mathbf{G}' \mid gz = z\}$. We define $\mathbf{G}(z) := \text{Res}_{F/\mathbb{Q}} \mathbf{G}'(z)$

Proposition 4.4. (JACOBSON [J2, Thm 3.]) *We have an isomorphism of algebraic groups*

$$\mathbf{G}'(z) \xrightarrow{\sim} \mathbf{SU}(V_z, h_z, \tau_z), \quad g \mapsto g|_{V_z}.$$

¹Recall that by proposition 3.3, the element z satisfies a quadratic equation.

Two possible cases occur:

1. $F(z) \subset \mathbb{R}$. In this case $\mathbf{G}'(z)(\mathbb{R}) \cong SL(3, \mathbb{R})$ and $\mathbf{G}(z)(\mathbb{R}) \cong SL(3, \mathbb{R}) \times SU(3) \times \cdots \times SU(3)$.
2. $F(z) \not\subset \mathbb{R}$. In this case $\mathbf{G}'(z)(\mathbb{R}) \cong SU(2, 1)$ and $\mathbf{G}(z)(\mathbb{R}) \cong SU(2, 1) \times SU(3) \times \cdots \times SU(3)$.

Now let $z_1, z_2 \in \mathfrak{C}_0$ such that $Q = (z_1, z_2)_F = \text{span}_F\{e, z_1, z_2, z_1 z_2\}$ is a nonsingular subspace with respect to $\langle, \rangle|_Q$. Assume that $\langle z_1, z_2 \rangle = 0$. Then Q is a quaternion algebra and a subalgebra of $\mathfrak{C}$. Further Q is a $F(z_1)$ -vector subspace of $\mathfrak{C}$, hence $V_{z_1}(z_2) := Q^{\perp_{h_{z_1}}} = Q^\perp$ is also a 2-dimensional $F(z_1)$ -vector subspace, moreover it is a $F(z_1)$ -vector subspace of V_{z_1} . Let $\mathbf{G}'(z_1, Q) = \{g \in \mathbf{G}' \mid g z_1 = z_1, gQ = Q\}$ be the stabilizer of z_1 and Q and $\mathbf{G}(z_1, Q) := \text{Res}_{F/\mathbb{Q}} \mathbf{G}'(z_1, Q)$. Let $\mathbf{U}(V_{z_1}(z_2), h_{z_1}|_{V_{z_1}(z_2)}, \tau_{z_1})$ denote the unitary group of the Hermitian space $(V_{z_1}(z_2), h_{z_1}|_{V_{z_1}(z_2)}, \tau_{z_1})$. Since $\langle z_1, z_2 \rangle = 0$, by the assumption we have

$$V_{z_1} = z_2 F(z_1) \oplus V_{z_1}(z_2),$$

as $F(z_1)$ -vectorspace. So the next corollary follows immediately.

Corollary 4.5. *We have isomorphisms of algebraic groups*

$$\mathbf{G}'(z_1, Q) \cong \mathbf{SU}(V_{z_1}, h_{z_1}, \tau_{z_1})(z_2 F(z_1)) \rightarrow \mathbf{U}(V_{z_1}(z_2), h_{z_1}, \tau_{z_1}),$$

where $g \mapsto g|_{V_{z_1}(z_2)}$.

Two possible cases can occur:

1. $F(z_1) \subset \mathbb{R}$. In this case $\mathbf{G}'(z_1, Q)(\mathbb{R}) \cong GL(2, \mathbb{R})$ and $\mathbf{G}(z_2, Q)(\mathbb{R}) \cong GL(2, \mathbb{R}) \times U(2) \times \cdots \times U(2)$.
2. $F(z_1) \not\subset \mathbb{R}$. In this case $\mathbf{G}'(z_1, Q)(\mathbb{R}) \cong U(1, 1)$ or $\mathbf{G}'(z_1, Q)(\mathbb{R}) \cong U(2)$.

Examples. Consider the fields $F(c), F(ab) \not\subset \mathbb{R}$. So we are in case 2. Let $\{a, b, ab\}$ be the ordered $F(c)$ -basis of V_c and $W = \text{span}_{F(c)}\{a, b\}$. We have $\mathbf{G}'(c) \cap \mathbf{G}'(ab) = \mathbf{G}'(c, ab) \cong \mathbf{SU}(W, h_c|_W, \tau_c)$. Since h_c is negative definite on W , $(\mathbf{G}'(c) \cap \mathbf{G}'(ab))(\mathbb{R}) \cong SU(2)$ is compact. Let $K \subset \mathbf{G}(\mathbb{R})$ be the maximal compact subgroup, given as the stabilizer of $(c, ab)_{\mathbb{R}} \times e \times \cdots \times e$ in $\mathfrak{C} \otimes_{\mathbb{Q}} \mathbb{R}$. Then $\mathbf{G}(c)(\mathbb{R}) \cap K \cong \mathbf{G}(ab)(\mathbb{R}) \cap K \cong U(2) \times SU(3) \times \cdots \times SU(3)$ is a maximal compact subgroup in $\mathbf{G}(c)(\mathbb{R})$ resp. $\mathbf{G}(ab)(\mathbb{R})$.

Remark. In some certain situations, it is possible to realize $\mathbf{G}'(c)$ and $\mathbf{G}'(ab)$ as groups, which are the fixed points of rational automorphisms of finite order:

Let F be a totally real number field of degree $r > 1$ and $\mathfrak{C}$ the Cayley algebra with basic triple a, b, c over F as above. Assume further that the field $F(c)$ contains a 3-th root of unity ζ with $N(\zeta) = 1$, e.g. $c^2 = -3$ and $\zeta = -\frac{1}{2} + \frac{c}{2}$ and $N(\zeta) = \zeta \bar{\zeta} = 1$. The map

$$a \mapsto \zeta a, \quad b \mapsto \zeta b, \quad c \mapsto c$$

induces an element $g_\tau \in \mathbf{G}'$ and an automorphism τ of order 3 by conjugation with g_τ . By [Singh, §3.2] we have

$$\mathbf{G}'(\tau) \cong \mathbf{C}_{\mathbf{G}'}(g_\tau) \cong \mathbf{G}'(c).$$

Here $\mathbf{C}_{\mathbf{G}'}(g_\tau)$ denotes the centralizer of g_τ in $\mathbf{G}'$. Moreover we have $a^2, b^2 > 0$, hence $(ab)^2 = -a^2b^2 < 0$. Assume further that $F(ab)$ contains a 3-th root of unity ξ with $N(\xi) = 1$, e.g. $(ab)^2 = -3$ and $\xi = -\frac{1}{2} + \frac{ab}{2}$. Like above, we have a map

$$a \mapsto \xi a, \quad b \mapsto \xi b, \quad c \mapsto \xi c$$

and an induced element $g_\rho \in \mathbf{G}'$. This automorphism maps the element ab to itself. The automorphism ρ obtained by conjugation with g_ρ is of order 3. By [Singh, §3.2] we have

$$\mathbf{G}'(\rho) \cong \mathbf{C}_{\mathbf{G}'}(g_\rho) \cong \mathbf{G}'(ab).$$

Warning. Note that the automorphisms τ and ρ do **not** commute.

4.6 The description of these subgroups is complete in some sense: By Dynkin [D], the complex Lie algebra of exceptional type G_2 has just the semisimple (resp. reductive) subalgebras of type A_1 , $A_1 \times A_1$ and A_2 . So we have found to every possible type of subgroup a representing algebraic subgroup.

4.2 The cycles for G_2 – a summary

4.7. For the reductive subgroups of the group of type G_2 , as constructed in the last section the corresponding cycles look as follows.

algebraic subgroup $\mathbf{B}$	$\mathbf{B}$ isomorphic to the group	$\mathbf{B}(\mathbb{R})$ isomorphic to	cycle $C_{\mathbf{B}}$	dimension of $C_{\mathbf{B}}$	special cycle?
$\mathbf{G}'(a)$	$\mathbf{SU}(V_a, h_a, \tau_a)$	$SL(3, \mathbb{R})$	C_a	5	<i>no</i>
$\mathbf{G}'(a, (a, b)_F)$	$\mathbf{U}(V_a(b), h_a, \tau_a)$	$GL(2, \mathbb{R})$	$C_{a,b}$	3	<i>no</i>
$\mathbf{G}'(b)$	$\mathbf{SU}(V_b, h_b, \tau_b)$	$SL(3, \mathbb{R})$	C_b	5	<i>no</i>
$\mathbf{G}'(b, (b, a)_F)$	$\mathbf{U}(V_b(a), h_b, \tau_b)$	$GL(2, \mathbb{R})$	$C_{b,a}$	3	<i>no</i>
$\mathbf{G}'(\sigma) = \mathbf{G}'((b, c)_F)$	$\mathbf{SO}(V_\sigma, B_\sigma)$	$SO(2, 2)$	$C(\sigma)$	4	<i>yes</i>
$\mathbf{G}'(\sigma\theta) = \mathbf{G}'((a, c)_F)$	$\mathbf{SO}(V_{\sigma\theta}, B_{\sigma\theta})$	$SO(2, 2)$	$C(\sigma\theta)$	4	<i>yes</i>
$\mathbf{G}'(c), \mathbf{G}'(\tau)$	$\mathbf{SU}(V_c, h_c, \tau_c)$	$SU(2, 1)$	C_c	4	<i>yes</i> ¹
$\mathbf{G}'(ab), \mathbf{G}'(\rho)$	$\mathbf{SU}(V_{ab}, h_{ab}, \tau_{ab})$	$SU(2, 1)$	C_{ab}	4	<i>yes</i> ¹

Here C_i is the cycle resulting from $\mathbf{G}(i)$, $i = a, b, c, ab$ and $C_{i,j}$ is the cycle resulting from $\mathbf{G}(i, (i, j)_F)$, $i = a, b$, $j = a, b$.

4.3 A remark on conjugacy classes of certain subgroups

In this subsection we review some results concerning a cohomological description of conjugacy classes of involutions resp. their fixed points. Some of them and also other results on conjugacy classes can be found in [Singh].

Let F be a number field, $\mathfrak{C}$ be an octonion algebra over F and $\mathbf{G}$ be the F -algebraic group of automorphisms of $\mathfrak{C}$. Further, let g_σ be an involution of $\mathfrak{C}$ and $\sigma : \mathbf{G} \rightarrow \mathbf{G}$ be the involution obtained by conjugation with g_σ . Note that $\mathbf{G}(\sigma) = \mathbf{C}_{\mathbf{G}}(g_\sigma)$ is the centralizer of g_σ in $\mathbf{G}$.

¹The occure of an automorphism τ of order 3 of $\mathbf{G}'$, such that $\mathbf{G}'(c)$ resp. $\mathbf{G}'(ab)$ is the set of fixed points $\mathbf{G}'(\tau)$ of τ in $\mathbf{G}'$, depends on $F(c)$ resp. $F(ab)$. So in general their corresponding cycles have not to be special.

Let k be a field containing F , and α a cocycle, representing a class in $H^1(\sigma, \mathbf{G}(k))$. Since σ is an involution, $1 = \alpha_{\sigma^{-1}}(\alpha_{\sigma}) = \alpha_{\sigma} g_{\sigma} \alpha_{\sigma} g_{\sigma}$, hence $\alpha_{\sigma} g_{\sigma}$ is an involution. If α and β are two equivalent cocycles, then there is an $h \in \mathbf{G}(k)$, such that $\alpha_{\sigma} g_{\sigma} = h^{-1} \beta_{\sigma} g_{\sigma} h$, i.e., the involutions $\alpha_{\sigma} g_{\sigma}$ and $\beta_{\sigma} g_{\sigma}$ are conjugate. Conversely, if ε is an involution on $\mathfrak{C} \otimes_F k$, then $\alpha_{\sigma} := g_{\sigma} \varepsilon$ defines a cocycle α and if $\varepsilon' := h \varepsilon h^{-1}$, $h \in \mathbf{G}(k)$, the cocycle $\beta_{\sigma} = g_{\sigma} \varepsilon'$ satisfies $\alpha_{\sigma} = h^{-1} \beta_{\sigma} \sigma(h)$, i.e. they are equivalent.

Thus we have the following 1 : 1-correspondence of sets:

$$\begin{array}{ccc} H^1(\sigma, \mathbf{G}(k)) & \longleftrightarrow & \left\{ \begin{array}{l} \text{Conjugacy classes via } \mathbf{G}(k), \\ \text{of involutions of } \mathfrak{C} \otimes_F k \end{array} \right\} \\ \{\alpha\} & \longrightarrow & \{\alpha_{\sigma} g_{\sigma}\} \\ \{\sigma \mapsto \varepsilon g_{\sigma}\} & \longleftarrow & \{\varepsilon\}. \end{array}$$

If two involutions $\varepsilon, \varepsilon'$ of $\mathfrak{C} \otimes_F k$ are conjugate then the corresponding fixed quaternion subalgebras Q, Q' of $\mathfrak{C} \otimes_F k$ are isomorphic. On the other hand if Q, Q' are isomorphic, then, by the remark after theorem 3.5, we find a $g \in \mathbf{G}(k)$ such that $gQ = Q'$, thus $\varepsilon, \varepsilon'$ are conjugate. The centralizers in $\mathbf{G}(k)$ of these involution are conjugate if and only if the involutions are conjugate. Thus we have the following 1 : 1-correspondence of sets:

$$H^1(\sigma, \mathbf{G}(k)) \longleftrightarrow \left\{ \begin{array}{l} \text{Conjugacy classes via } \mathbf{G}(k), \text{ of centralizers} \\ \mathbf{C}_{\mathbf{G}}(\varepsilon) \text{ of involutions } \varepsilon \text{ of } \mathfrak{C} \otimes_F k \end{array} \right\}.$$

Examples.

1. If $k = \bar{F}$ is an algebraic closure of F , then by [SV, 2.2.4] the group $\mathbf{G}(\bar{F})$ acts transitively on the set of quaternion subalgebras of $\mathfrak{C} \otimes_F \bar{F}$, i.e., up to conjugacy there is just one involution on $\mathfrak{C} \otimes_F \bar{F}$ and the fixed quaternion subalgebra is $M_2(\bar{F})$.
2. If F is the totally real number field as in §4.1 and $k = \mathbb{R}$, then we have up to conjugacy exactly two quaternion subalgebras of $\mathfrak{C}_{\mathbb{R}} = \mathfrak{C} \otimes_F \mathbb{R}$, namely $\mathbb{H}$ and $M_2(\mathbb{R})$. Hence, up to conjugacy over $\mathbf{G}(\mathbb{R})$, there are only the involutions g_{θ} and g_{σ} of §4.1.1, on $\mathfrak{C}_{\mathbb{R}}$.

Chapter 5

Intersection numbers

In §5.1, we recall some general definitions and facts related to intersection numbers for compact manifolds. For more information we refer to, for example, [Bre, VI, §11], [Lü, §8], [RoSch] or [StöZie, §14.6, §15.4]. In §5.2 we analyse the intersection $C_1 \cap C_2$ of cycles C_1 and C_2 as constructed in Chapter 2. This was done by [MiRa] (see also [FOR]) if the cycles intersect transversely. We recall the main facts developed there. In the non transversely case, we use these facts together with the general theory of excess bundles developed in [RoSch], to prove a formula for the intersection number. The transversely case, as well as the non transversely case, occur in our investigation of cycles in arithmetic quotients of the exceptional group of type G_2 . In §5.3 we present the result of Rohlf and Schwermer [RoSch, Thm. 4.11] to compute the intersection numbers of two special cycles and apply their result in the case G_2 .

5.1 General theory

Now and for all the terminus manifold, will be always mean smooth manifold. Let V and M be manifolds. If $i : M \rightarrow V$ an injective immersion, then (M, i) is called an *immersed submanifold* of V and if $i : M \rightarrow i(M)$ is a homeomorphism for the relative topology on $i(M)$, then M is called a *submanifold* of V and i is called an *embedding*.

Now, let V be a v -dimensional connected oriented compact manifold without boundary and $i : M \rightarrow V$ be a m -dimensional closed connected oriented submanifold of V . Then we have an induced map on homology $i_* : H_*(M; \mathbb{Z}) \rightarrow H_*(V; \mathbb{Z})$. We do not differentiate between the fundamental class of M and there image under i_* and write $[M]$ for the homology class *generated by* M ; the fundamental class of V is also denoted by $[V]$. Let $D : H^q(V; \mathbb{Z}) \rightarrow H_{v-q}(V; \mathbb{Z})$ be the Poincaré duality isomorphism (i.e. $D(\alpha) = \alpha \cap [V]$, for $\alpha \in H^q(V; \mathbb{Z})$; see e.g. [StöZie, 14.2.1]) and N be a $v - m$ -dimensional closed connected oriented submanifold of V . Then the *intersection number of M and N* is defined, (see e.g. [StöZie, 15.4.3 (b)]) by

$$[M][N] = \langle D^{-1}([M]) \cup D^{-1}([N]), [V] \rangle.$$

Or more generally, the *intersection pairing*

$$H^q(M; \mathbb{R}) \times H^{n-q}(M; \mathbb{R}) \rightarrow \mathbb{R}, (\alpha, \beta) \mapsto \langle \alpha \cup \beta, [M] \rangle$$

defines a regular¹ bilinear form (see e.g. [StöZie, 14.6.6]).

Note that, if $f : V \rightarrow V$ is a orientation preserving diffeomorphism, the intersection number of $f(M)$ and $f(N)$ is

$$[f(M)][f(N)] = [M][N].$$

This follows immediately from the definition of the intersection number and the properties of the cup-Product.

Two submanifolds M and N intersect *perfectly* if the connected components of $M \cap N$ are submanifolds of V and if for all such components W one has $TW = TM|_W \cap TN|_W$ (i.e. for every $x \in W$ one has $T_xW = T_xM \cap T_xN$).

We say that two submanifolds M, N of V intersect each other *transversely* if for every point $x \in M \cap N$, we have $T_xM + T_xN = T_xV$.

Now assume that the intersection $M \cap N$ consists of one connected manifold W of dimension w , and that M and N intersect perfectly. Let $TM|_W + TN|_W$ be the bundle over W whose fiber over a point $x \in W$ consists of the span of the fibers T_xM and T_xN in the fibre T_xV of the tangent bundle of V . Then there are exact sequences of bundles

$$0 \rightarrow TF \rightarrow TM|_W \oplus TN|_W \rightarrow TM|_W + TN|_W \rightarrow 0$$

and

$$0 \rightarrow TM|_W + TN|_W \rightarrow TV|_W \rightarrow \eta \rightarrow 0.$$

Since M and N intersect perfectly, the last sequence defines a w -dimensional vector bundle η over W given as the quotient bundle of the tangent bundle of V by the sum of the tangent bundles of M and N restricted to W . The bundle η will be called the *excess bundle* of the intersection W (for this notion see, e.g. [Fu]). If the intersection is transversal then the excess bundle is the zero bundle. Suppose that W is an orientable manifold and fix an orientation on W . Then η is an oriented vector bundle in a natural way.

Proposition 5.1. *Let M, N be two closed oriented submanifolds of an oriented manifold V . Suppose that $\dim M + \dim N = \dim V$, that M and N intersect perfectly, and that the intersection $M \cap N$ consists of one connected orientable manifold W of dimension w with excess bundle η . Then one has*

$$[M][N] = \langle e(\eta), [W] \rangle.$$

Here $e(\eta) \in H^w(W; \mathbb{Z})$ is the Euler class² of η , $[W] \in H_w(W; \mathbb{Z})$ the fundamental class of W and the evaluation $\langle e(\eta), [W] \rangle$ is called the Euler number of η .

Proof. [RoSch, Proposition 3.3].

Note that we have fixed an orientation on W . If we choose the opposite orientation on W , the excess bundle clearly changes its orientation. Hence the Euler class $e(\eta)$ changes sign (cf. [Milnor Stasheff, 9.3]). So the Euler number $\langle e(\eta), [W] \rangle$ is independent of the choice of the orientation of W .

¹Let $k = \mathbb{R}$ or $\mathbb{C}$. If M and N are two k -vectorspaces and $s : M \times N \rightarrow k$ is a bilinear form, then s is called *regular*, if the homomorphism $M \rightarrow \text{Hom}(N; k)$, $m \mapsto s(m, -)$ is bijective. Equivalently, there exists a basis $m_1, \dots, m_r$ of M and a basis $n_1, \dots, n_r$ of N , such that $s(m_i, n_j) = \delta_{ij}$, with δ the Kronecker delta.

²The definition of the Euler class and their properties is repeated in Appendix A.1.

Proposition 5.2. *Let M, N be two closed oriented submanifolds of an oriented manifold V . Suppose that $\dim M + \dim N = \dim V$.*

1. If the intersection of the submanifolds M, N of V consists of finitely many points, in which the submanifolds intersect transversely, then

$$[M][N] = \sum_{x \in M \cap N} \varepsilon(x),$$

where $\varepsilon(x) = \begin{cases} +1 & \text{if } T_x(M) \oplus T_x(N) \cong T_x(V) \text{ as oriented vectorspaces,} \\ -1 & \text{otherwise.} \end{cases}$

In such a case $\varepsilon(x)$ is called the local intersection number at x ; symbolically $[M][N]|_x = \varepsilon(x)$.

2. If $M \cap N = \emptyset$, then $[M][N] = 0$.

An idea of the proof of part 1 of proposition 5.2 can be found in [Bre, VI,11.12]. Part 2 is proved in [Bre, VI,11.10]. Both parts will be useful in some cases.

5.2 Intersection numbers of cycles

In the following we analyse the intersection $C_1 \cap C_2$ of cycles C_1 and C_2 . This was done by [MiRa] (see also [FOR]) if the cycles intersect transversely. We recall the main facts developed there. In the non transversely case, we use these facts together with the general theory of excess bundles developed in [RoSch], to prove a formula for the intersection number. The transversely case, as well as the non transversely case, occur in our investigation of cycles in arithmetic quotients of the exceptional group of type G_2 .

5.3. Let $\mathbf{G}$ be a connected, semisimple $\mathbb{Q}$ -anisotropic linear algebraic group defined over $\mathbb{Q}$. Suppose now that $\mathbf{G}_1, \mathbf{G}_2$ are two connected reductive $\mathbb{Q}$ -subgroups of $\mathbf{G}$ and fix a maximal compact subgroup K in $G = \mathbf{G}(\mathbb{R})$ such that $G_i \cap K = K_i$ is a maximal compact subgroup of $G_i = \mathbf{G}_i(\mathbb{R})$, $i = 1, 2$. We set $X_i = K \setminus G_i$ and view X_i as connected totally geodesic submanifold of X . The submanifold $Z = X_1 \cap X_2$ in X is again connected (for two points $x, x' \in X_1 \cap X_2$, take a X_1 -geodesic c_1 in X_1 resp. a X_2 -geodesic c_2 in X_2 joining x to x' , then c_1 and c_2 are X -geodesic; but minimizing geodesics in symmetric spaces of non compact type are unique; hence they are equal) and totally geodesic. Let $\mathbf{H} = \mathbf{G}_1 \cap \mathbf{G}_2$; then $H = \mathbf{H}(\mathbb{R})$ acts transitively on Z ; hence Z is diffeomorphic to $H \cap K \setminus H$. Let $\Gamma \subset \mathbf{G}(\mathbb{Z})$ a torsionsfree arithmetic subgroup, such that Γ (resp. $\Gamma_i = \Gamma_{\mathbf{G}_i}$) acts orientation preserving on X (resp. $X_i = X_{\mathbf{G}_i}$), and $G_1 K G_2 \cap \Gamma \subset \mathbf{G}_1 \mathbf{G}_2$ (such a choice is possible by lemma 2.3) holds for Γ . Furthermore, we assume that $j_i : X_i / \Gamma_i \rightarrow X / \Gamma$ is an imbedding for $i = 1, 2$ and that $\dim X / \Gamma = \dim X_1 / \Gamma_1 + \dim X_2 / \Gamma_2$.

We now wish to examine the intersection of the submanifolds $C_i = j_i(X_i / \Gamma_i)$ in X / Γ , $i = 1, 2$. Let $\pi : X \rightarrow X / \Gamma$ the natural projection. If $x_0 \in C_1 \cap C_2$, we can find $\tilde{x}_i \in X_i$, $i = 1, 2$ and $\gamma \in \Gamma$ such that

$$\tilde{x}_1 \gamma = \tilde{x}_2 \tag{5.1}$$

and $\pi(\tilde{x}_1) = x_0 = \pi(\tilde{x}_2)$; but this shows that we can find $g_i \in G_i$, $i = 1, 2$ and $k \in K$ such that $g_1 k g_2 = \gamma$. Conversely, if $\gamma = g_1 k g_2 \in G_1 K G_2$, it is clear that $\tilde{x}_1 \gamma = \tilde{x}_2$ where $\tilde{x}_1 = K g_1^{-1}$ and $\tilde{x}_2 = K g_2$. So we have found

$$C_1 \cap C_2 = \pi \left(\bigcup_{\gamma \in G_1 K G_2 \cap \Gamma} X_1 \gamma \cap X_2 \right)$$

Proposition 5.4. *There are only finite many connected components of $C_1 \cap C_2$. Each of them is just the image of $X_1\gamma \cap X_2$ under π for a suitable $\gamma \in \Gamma$.*

Proof. By a theorem due to Borel and Harish-Chandra [BoHa], the set of double cosets $\Gamma_1 \backslash \Gamma \cap \mathbf{G}_1 \mathbf{G}_2 / \Gamma_2$ is finite. Since $G_1 K G_2 \cap \Gamma \subset \mathbf{G}_1 \mathbf{G}_2$, it is sufficient to show that there is a one to one correspondence between the connected components of $C_1 \cap C_2$ and the $\Gamma_1 \times \Gamma_2$ -orbits of $G_1 K G_2 \cap \Gamma$. Here Γ_1 (resp. Γ_2) acts obviously from the left (resp. right) on $G_1 K G_2 \cap \Gamma$. But this is proved by [MiRa, Lemma 2.4]. The second statement is [MiRa, Prop.2.3]. $\square$

Let $I(\Gamma) \subset \Gamma$ be a system of representing elements of $\Gamma_1 \backslash G_1 K G_2 \cap \Gamma / \Gamma_2$ and $D(\gamma) := \pi(X_1\gamma \cap X_2)$, $\gamma \in I(\Gamma)$. Then we have

$$C_1 \cap C_2 = \bigsqcup_{\gamma \in I(\Gamma)} D(\gamma). \quad (5.2)$$

5.2.1 Transversal intersections

Lemma 5.5. *The cycles C_1 and C_2 intersect each other transversely at a point x_0 in X/Γ if and only if x_0 is an isolated point in the intersection $C_1 \cap C_2$. In this case, the local intersection number $[C_1][C_2]_{|_{x_0}}$ at x_0 is $+1$ or -1 .*

Proof. Clear, because $C_1 \cap C_2$ is totally geodesic. $\square$

From now on, let us assume that X_1 and X_2 intersect in exactly one point $\tilde{x} \in X_1 \cap X_2 = Z$. Since $Z \cong H \cap K \backslash H$ this is equivalent to say that $H = (\mathbf{G}_1 \cap \mathbf{G}_2)(\mathbb{R}) \subset K$ is compact.

We orient X , X_1 and X_2 such that $[C_1][C_2]_{|\tilde{x}\Gamma} = +1$ (this is always easily achieved by rearranging the orientations on X, X_1 and X_2 (c.f. [RoSch, 4.12])).

Observe that our choice of Γ has been made to ensure that $G_1 K G_2 \cap \Gamma \subset \mathbf{G}_1 \mathbf{G}_2$.

Notation. We denote by G^{or} the elements in G which act orientation preserving on X . This is a subgroup of G containing the identity component G^0 ; hence is of finite index in G . Moreover, clearly G^{or} is a normal subgroup of G .

Proposition 5.6. *Let $x_0 \in C_1 \cap C_2$ and $\gamma \in \Gamma \cap \mathbf{G}_1 \mathbf{G}_2$ such that $x_0 \in \pi(X_1\gamma \cap X_2)$. If $\gamma = g_1 g_2 \in G_1 G_2$, then C_1 and C_2 intersect each other transversely at x_0 . If $\gamma \in G_1^{or} G_2^{or}$, then the local intersection number is $[C_1][C_2]_{|_{x_0}} = +1$.*

Proof. The connected component of $C_1 \cap C_2$, containing x_0 is just the image of $X_1\gamma \cap X_2$ under π (by [MiRa, Prop. 2.3]). Since $\gamma = g_1 g_2$, the manifold $X_1\gamma \cap X_2$ is diffeomorphic to $X_1 \cap X_2 = \{\tilde{x}\}$, via $x \mapsto x g_2^{-1}$; so $X_1\gamma \cap X_2$ is just one point and hence x_0 is isolated in $C_1 \cap C_2$. If $\gamma \in G_1^{or} G_2^{or}$ then, by [MiRa, Lemma 2.5], $[C_1][C_2]_{|_{x_0}} = [C_1][C_2]_{|\tilde{x}\Gamma} = +1$. $\square$

Of course, every connected group acts orientation preserving. But, acting orientation preserving is a weaker condition than to be connected.

Example 1. The group $SO(2, 2)$ has two connected components, but acts orientation preserving on $S(O(2) \times O(2)) \backslash SO(2, 2)$. This can be seen as follows:

Take $T_o(S(O(2) \times O(2)) \backslash SO(2, 2)) = \mathfrak{p}_0 = \{X \in \mathfrak{so}(2, 2) | X - {}^t X = 0\}$ to be the tangent space at $o = S(O(2) \times O(2))\mathbb{I}_4$. Then $S(O(2) \times O(2))$ acts on $\mathfrak{p}_0$ via the adjoint representation Ad. The elements $\mathbb{I}_4, g := \text{diag}(1, -1, 1, -1) \in S(O(2) \times O(2))$ represent the two connected

components of $SO(2, 2)$. So it is sufficient to check that $\text{Ad}g$ acts orientation preserving on $\mathfrak{p}_0$. Take

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & e_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ e_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & e_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Then $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathfrak{p}_0$. Since $ge_i g^{-1} = e_i, i = 1, 4$ and $ge_i g^{-1} = -e_i, i = 2, 3$, we have $\det \text{Ad} \big|_{\mathfrak{p}_0} g = 1$. Thus $SO(2, 2)$ acts orientation preserving.

Example 2. The group $GL(2, \mathbb{R})$ has two connected components and acts orientation preserving on $O(2) \setminus GL(2, \mathbb{R})$. This can be seen as follows:

Take $T_o(O(2) \setminus GL(2, \mathbb{R})) = \mathfrak{p}_0 = \{X \in \mathfrak{gl}(2, \mathbb{R}) | X - {}^t X = 0\}$ to be the tangent space at $o = O(2)\mathbb{I}_2$. Then $O(2)$ acts on $\mathfrak{p}_0$ via the adjoint representation Ad and one sees that $g = \text{diag}(1, -1)$ takes e_1 into e_1 , e_2 into e_2 and e_3 into $-e_3$. Here e_1, e_2, e_3 denotes the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{p}_0$. So $\det \text{Ad} \big|_{\mathfrak{p}_0} g = -1$; hence the component of $GL(2, \mathbb{R})$, which contains g acts orientation reversing.

5.7. The aim of the following discussion is to develop conditions on $\mathbf{G}_1$ and $\mathbf{G}_2$ such that all γ as in (5.1) are in $G_1 G_2$. Then, if $G_1 G_2 = G_1^{or} G_2^{or}$, proposition 5.6 would imply that $[C_1][C_2] > 0$; hence $[C_1]$ and $[C_2]$ would represent non trivial homology classes in $H_*(X/\Gamma, \mathbb{C})$.

Let $\overline{\mathbb{Q}}$ denote an algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and $\mathcal{G}$ the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$. The induced action of $\mathcal{G}$ on $\mathbf{G}(\overline{\mathbb{Q}})$ is used without changing the notation. We write ${}^s g = s(g)$ for $g \in \mathbf{G}(\overline{\mathbb{Q}})$, $s \in \mathcal{G}$. Let $\gamma \in \Gamma \cap \mathbf{G}_1(\mathbb{C})\mathbf{G}_2(\mathbb{C})$; then $\gamma \in \Gamma \cap \mathbf{G}_1(\overline{\mathbb{Q}})\mathbf{G}_2(\overline{\mathbb{Q}})$ (nullstellensatz), i.e. $\gamma = g_1 g_2$, $g_i \in \mathbf{G}_i(\overline{\mathbb{Q}})$, $i = 1, 2$. Since $\gamma \in \mathbf{G}(\mathbb{Q})$, one has ${}^s g_1 {}^s g_2 = {}^s \gamma = \gamma = g_1 g_2$ for all $s \in \mathcal{G}$. Hence $a_s(\gamma) := g_1^{-1} {}^s g_1 = g_2 {}^s g_2^{-1}$ is in $(\mathbf{G}_1 \cap \mathbf{G}_2)(\overline{\mathbb{Q}}) = \mathbf{H}(\overline{\mathbb{Q}})$. The map $a(\gamma) : s \mapsto a_s(\gamma)$ is a 1-cocycle on $\mathcal{G}$ with values in $\mathbf{H}(\overline{\mathbb{Q}})$. For an algebraic $\mathbb{Q}$ -group $\mathbf{B}$, we denote by $H^1(\mathbb{Q}, \mathbf{B})$ the first non abelian Galois cohomology set $H^1(\mathcal{G}, \mathbf{B}(\overline{\mathbb{Q}}))$. The cocycle $a(\gamma)$ is independent of the decomposition of $\gamma = g_1 g_2$: If $\gamma = h_1 h_2$, then $g_1^{-1} {}^s g_1 = g_1^{-1} h_1 h_2 {}^s g_2^{-1} = b^{-1} h_2 {}^s h_2^{-1} b$, where $b = h_1^{-1} g_1 = h_2 g_2^{-1} \in \mathbf{H}(\overline{\mathbb{Q}})$. Moreover, by the definition of $a(\gamma)$, the cocycle $a(\gamma)$ considered as an element in $H^1(\mathbb{Q}, \mathbf{G}_i)$ is trivial for $i = 1, 2$.

Lemma 5.8. Suppose $\mathbf{G}_1, \mathbf{G}_2$ are such that the natural map

$$H^1(\mathbb{Q}, \mathbf{H}) \rightarrow H^1(\mathbb{Q}, \mathbf{G}_1) \times H^1(\mathbb{Q}, \mathbf{G}_2) \quad (5.3)$$

is an injective map of pointed sets. Then $\Gamma \cap \mathbf{G}_1 \mathbf{G}_2 \subset \mathbf{G}_1(\mathbb{Q})\mathbf{G}_2(\mathbb{Q})$.

Proof. The assumptions guarantee that $a(\gamma)$ is trivial in $H^1(\mathbb{Q}, \mathbf{H})$. This means that we can find $h \in \mathbf{H}(\overline{\mathbb{Q}})$ such that $g_1^{-1} {}^s g_1 = a_s(\gamma) = h^{-1} {}^s h$ for all $s \in \mathcal{G}$ leading to $g_1 h^{-1} = s(g_1 h^{-1})$ for all $s \in \mathcal{G}$. Thus $u_1 = g_1 h^{-1} \in \mathbf{G}_1(\mathbb{Q})$. Analogously, $u_2 = h g_2 \in \mathbf{G}_2(\mathbb{Q})$; hence $\gamma = u_1 u_2 \in \mathbf{G}_1(\mathbb{Q})\mathbf{G}_2(\mathbb{Q})$. $\square$

Lemma 5.9. *Suppose $\mathbf{G}_1, \mathbf{G}_2$ are such that the natural map*

$$H^1(\mathbb{R}, \mathbf{H}) \rightarrow H^1(\mathbb{R}, \mathbf{G}_1) \times H^1(\mathbb{R}, \mathbf{G}_2) \quad (5.4)$$

is an injective map of pointed sets. Then $\Gamma \cap \mathbf{G}_1 \mathbf{G}_2 \subset G_1 G_2$. In this case all intersections are transversely.

Proof. Like above or [MiRa, Lemma 3.4 B].

Remark. Obviously, $\gamma \in \mathbf{G}_1(\mathbb{Q})\mathbf{G}_2(\mathbb{Q})$ implies $\gamma \in G_1 G_2$.

Theorem. (MILLSON, RAGHUNATHAN) *Let $\mathbf{G}$ be a connected $\mathbb{Q}$ -anisotropic semisimple algebraic group defined over $\mathbb{Q}$ and $\mathbf{G}_1$ and $\mathbf{G}_2$ two connected reductive $\mathbb{Q}$ -subgroups of $\mathbf{G}$, such that the map*

$$H^1(\mathbb{R}, \mathbf{G}_1 \cap \mathbf{G}_2) \rightarrow H^1(\mathbb{R}, \mathbf{G}_1) \times H^1(\mathbb{R}, \mathbf{G}_2)$$

is injective and the group $\mathbf{G}_1 \cap \mathbf{G}_2$ is $\mathbb{R}$ -anisotropic (i.e. $X_1 \cap X_2$ is a single point). Further, let Γ be an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Suppose that $\dim X_1 + \dim X_2 = \dim X$ holds and that $G = G^{\text{or}}$ and $G_1 G_2 = G_1^{\text{or}} G_2^{\text{or}}$. Then there exists a subgroup Γ' of Γ of finite index, such that the intersection number of $[C_1(\Gamma')]$ and $[C_2(\Gamma')]$ is

$$[C_1(\Gamma')][C_2(\Gamma')] = |\Gamma'_1 \setminus G_1 G_2 \cap \Gamma' / \Gamma'_2|.$$

In particular, the intersection number of $C_1(\Gamma')$ and $C_2(\Gamma')$ is non zero.

Proof. Obviously, the injectivity of (5.4) implies $G_1 K G_2 \cap \Gamma = G_1 G_2 \cap \Gamma$. $\square$

5.11. Application. Let's take a look at the exceptional group of type G_2 : We apply these discussions to some of the reductive subgroups found in chapter 4.

Let $\mathbf{G}'(c), \mathbf{G}'(ab)$ the subgroups of $\mathbf{G}'$ found in chapter 4 and $\mathbf{G}(c) = \text{Res}_{F/\mathbb{Q}} \mathbf{G}'(c)$ and $\mathbf{G}(ab) = \text{Res}_{F/\mathbb{Q}} \mathbf{G}'(ab)$ the subgroups of $\mathbf{G} = \text{Res}_{F/\mathbb{Q}} \mathbf{G}'$. Let $G'_c = \mathbf{G}'(c)(\mathbb{R}), G'_{ab} = \mathbf{G}'(ab)(\mathbb{R})$ and $G' = \mathbf{G}'(\mathbb{R})$, $K' \subset G'$ maximal compact and $G_c = \mathbf{G}(c)(\mathbb{R}), G_{ab} = \mathbf{G}(ab)(\mathbb{R})$ and $G = \mathbf{G}(\mathbb{R})$, $K \subset G$ maximal compact and X_c, X_{ab} the symmetric spaces (resp. C_c, C_{ab} the cycles) resulting from G_c and G_{ab} . Further let $\Gamma_c = \Gamma \cap \mathbf{G}(c), \Gamma_{ab} = \Gamma \cap \mathbf{G}(ab)$. Recall, that $\mathbf{G}'(c) \cap \mathbf{G}'(ab) \cong \mathbf{SU}(W, h_c, \tau_c)$, where $W = \text{span}_{F(c)}\{a, b\}$ and $\mathbf{G}'(c) \cong \mathbf{SU}(V_c, h_c, \tau_c)$ and $\mathbf{G}'(ab) \cong \mathbf{SU}(V_{ab}, h_{ab}, \tau_{ab})$.

Since $(\mathbf{G}'(c) \cap \mathbf{G}'(ab))(\mathbb{R}) \cong \mathbf{SU}(2)$ is compact, X_c and X_{ab} intersects in exactly one point. We have

$$\dim X/\Gamma = 8 = 4 + 4 = \dim C_c + \dim C_{ab}.$$

It is well known (see e.g [KMRT, 29.6] or [Sp, 12.4.7 (1)]) that for an F -algebraic group $\mathbf{B}$ we have

$$H^1(F, \mathbf{B}) \cong H^1(\mathbb{Q}, \text{Res}_{F/\mathbb{Q}} \mathbf{B}).$$

As consequence of Witt's Cancellation law (see Appendix Corollary A.10) the natural map

$$H^1(F, \mathbf{G}'(c) \cap \mathbf{G}'(ab)) \rightarrow H^1(F, \mathbf{G}'(z)), \quad z \in \{c, ab\}$$

is injective. Further G_c resp. G_{ab} is connected, so it acts orientation preserving on X_c resp. X_{ab} . Hence

Corollary 5.12. *There exists an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ such that*

(i) the cycles $C_c(\Gamma)$ and $C_{ab}(\Gamma)$ have a positive intersection number. Hence they represent non trivial homology classes in $H_4(X/\Gamma, \mathbb{C})$.

(ii) The intersection number is $[C_c(\Gamma)][C_{ab}(\Gamma)] = |\Gamma_c \setminus \Gamma \cap G_c G_{ab} / \Gamma_{ab}|$.

Question. Are $[C_c(\Gamma)]$ and $[C_{ab}(\Gamma)]$ linear independent classes in $H_4(X/\Gamma, \mathbb{C})$?

5.2.2 Non transversal intersections

5.13 Let us drop the assumption that the submanifold $Z = X_1 \cap X_2$ of X is just one point. So assume that $H = G_1 \cap G_2$ is not compact. Further, let us assume that the map (5.3) is injective. Let Γ be an arithmetic subgroup of $\mathbf{G}$ like in the discussion 5.3, i.e., suppose w.l.o.g. that $j_i : X_i/\Gamma_i \rightarrow X/\Gamma$ for $i = 1, 2$ are an imbeddings. We denote the cycles by $C_i = j_i(X_i/\Gamma_i)$, $i = 1, 2$ and we choose a representing system $I(\Gamma) \subset \Gamma$ for the connected components of $C_1 \cap C_2$, as in (5.2). Recall that $I(\Gamma) \subset \Gamma \cap \mathbf{G}_1 \mathbf{G}_2$.

Let $\gamma \in \Gamma$. Then $\gamma^{-1} \mathbf{G}_1 \gamma$ is a connected reductive $\mathbb{Q}$ -subgroup of $\mathbf{G}$. Define $\mathbf{H}_\gamma = \gamma^{-1} \mathbf{G}_1 \gamma \cap \mathbf{G}_2$ and $H_\gamma = \mathbf{H}_\gamma(\mathbb{R})$. From now on we assume that $\gamma \in I(\Gamma)$. Then by the injectivity of (5.3), there are $g_i \in \mathbf{G}_i(\mathbb{Q})$, $i = 1, 2$, such that $\gamma = g_1 g_2$. This decomposition is unique up to an element $h \in \mathbf{H}(\mathbb{Q})$: let $g_1 g_2 = \gamma = g'_1 g'_2$ two decompositions; then $g'_1{}^{-1} g_1 = g'_2 g_2{}^{-1} =: h \in \mathbf{H}(\mathbb{Q})$ shows that $g'_1 = g_1 h^{-1}$ and $g'_2 = h g_2$. We have $\mathbf{H}_\gamma = g_2^{-1} \mathbf{H} g_2$ and $H_\gamma = g_2^{-1} H g_2$. Note that $\gamma^{-1} G_1 \gamma$ acts transitively on $X_1 \gamma$. We denote $Z_\gamma := X_1 \gamma \cap X_2$. This is a connected totally geodesic submanifold of X . Let $x, x' \in Z_\gamma$. Since Z_γ is connected we can choose a geodesic $c(t)$, $t \in \mathbb{R}$, in Z_γ , which joins x to x' . Such a geodesic is of the form $c(t) = K \exp(tY)$, $Y \in T_x X$, where $K \exp(0) = x$ and $K \exp(Y) = x'$; but then $h = \exp(Y) \in \gamma^{-1} G_1 \gamma \cap G_2 \subseteq H_\gamma$. So we see that H_γ acts transitively on Z_γ . Hence, if $z \in Z_\gamma$ is a point and K_z the stabilizer of z in G , we have $K_z \cap H_\gamma \setminus H_\gamma = Z_\gamma$.

Recall a group-theoretical lemma.

Lemma 5.14 *Let Q be a subgroup of finite index in a group R and $r_1, \dots, r_t \in R$, such that $R = \bigsqcup_{i=1}^t r_i Q$. Then $Q_0 := \bigcap_{i=1}^t r_i Q r_i^{-1}$ is a normal subgroup of finite index in R , $Q_0 \leq R$, which is contained in Q .*

Note that a submanifold of a compact orientable manifold, need not to be orientable. A simple example are the real projective spaces $\mathbb{R}P^2 \subset \mathbb{R}P^3$. Both of them are compact, but just $\mathbb{R}P^3$ is orientable (see e.g. [Bre, p.218]).

Proposition 5.15 *Let C_1 and C_2 the cycles from 5.13. Then we have:*

(i) *The cycles C_1 and C_2 intersect perfectly.*

(ii) *There exists a normal subgroup of finite index Φ in Γ , such that all connected components $D(\phi) (= \pi(X_1 \phi \cap X_2))$ of $C_1(\Phi) \cap C_2(\Phi)$, $\phi \in I(\Phi)$, are orientable. Moreover, in the notation of 5.13, every $D(\phi)$ is diffeomorphic to $Z_\phi / \Phi_{\mathbf{H}_\phi}$.*

Proof. (i) (the first part is as in [RoSch, Lemma 1.4]) Let the point $x_0 \in C_1 \cap C_2$ be represented by $x \in X$. There is a unique $\gamma \in I(\Gamma)$ such that $x_0 \in D(\gamma)$. Since the assertion as claimed is of local nature, it suffices to prove the corresponding statement for the intersection $X_1 \gamma \cap X_2$ (i.e. we have to show $T_x(X_1 \gamma \cap X_2) = T_x X_1 \gamma \cap T_x X_2$). The injectivity of (5.3) shows that $\gamma = g_1 g_2 \in \mathbf{G}_1(\mathbb{Q}) \mathbf{G}_2(\mathbb{Q})$. Thus $X_1 \gamma \cap X_2$ is diffeomorphic to $X_1 \cap X_2$. Let y be the image of x under this diffeomorphism. From the behavior of the exponential map $\exp : T_y X \rightarrow X$, one sees that $X_1 \cap X_2$ intersect perfectly. So the same holds true for $X_1 \gamma \cap X_2$.

(ii) Now we have to prove the orientability of the connected components by passing to a subgroup of finite index.. First, we prove this for one arbitrary connected component. Let us take the same as above and denote again $\gamma \in I(\Gamma)$, $\gamma = g_1 g_2$ a representing element. We show the existence of a finite covering X/Φ of X/Γ such that every connected component of $C_1(\Phi) \cap C_2(\Phi)$, which covers $D(\gamma)$ is orientable and diffeomorphic to $Z_\phi / \Phi_{\mathbf{H}_\phi}$.

First note, that for a subgroup $B \subset G$ and $g \in G$, obviously $(g^{-1} B g)^0 = g^{-1} B^0 g$ holds.

Define a new arithmetic subgroup $\Gamma' = g_2 \Gamma g_2^{-1} \subset \mathbf{G}(\mathbb{Q})$ and $\Gamma'_{\mathbf{H}} = \Gamma' \cap \mathbf{H}$, $\mathbf{H} = \mathbf{G}_1 \cap \mathbf{G}_2$, $H = \mathbf{H}(\mathbb{R})$. Then there is a normal subgroup $\Delta \trianglelefteq \Gamma' \cap \mathbf{G}(\mathbb{Z})$ of finite index such that for every subgroup $\Phi' \leq \Delta$, we have $\Phi' \cap \mathbf{H} \subset H^0$ (i.e. the manifold $Z/\Phi'_{\mathbf{H}}$ is orientable) and the natural map $j(\Phi')_{\mathbf{H}} : Z/\Phi'_{\mathbf{H}} \rightarrow X/\Phi'$ is an imbedding. Note that Δ needs not to be normal in Γ' . However, by lemma 5.14 we can choose a subgroup $\Phi' \subset \Delta$, which is normal of finite index in Γ' .

The subgroup $\Phi := g_2^{-1} \Phi' g_2 \trianglelefteq \Gamma$ is normal and of finite index in Γ . Let $\phi \in I(\Phi)$ representing a component of $C_1(\Phi) \cap C_2(\Phi)$, which covers $D(\gamma)$. Then there exist elements $\gamma_i \in \Gamma_i$, $i = 1, 2$, such that $\gamma_1 \phi \gamma_2 = \gamma = g_1 g_2$. Since Φ is normal in Γ and, by the definition of $\Phi \subseteq g_2^{-1} \Phi' g_2$ we have that if $h \in \Phi \cap \mathbf{H}_{\phi} = \Phi \cap \gamma_2 g_2^{-1} \mathbf{H} g_2 \gamma_2^{-1}$, it follows that $g_2 \gamma_2^{-1} h \gamma_2 g_2^{-1} \in \Phi' \cap \mathbf{H} \subset H^0$; hence $\Phi_{\mathbf{H}_{\phi}} \subseteq \gamma_2 g_2^{-1} H^0 g_2 \gamma_2^{-1} = H_{\phi}^0$ (i.e. $Z_{\phi}/\Phi_{\mathbf{H}_{\phi}}$ is orientable).

Take K to be a maximal compact subgroup of G such that $K \cap G_i$ is a maximal compact subgroup of G_i , $i = 1, 2$. The element $z := K g_2 \gamma_2^{-1}$ is in Z_{ϕ} . The stabilizer of z in G is $K_z = \gamma_2 g_2^{-1} K g_2 \gamma_2^{-1}$. Recall that Φ' was chosen such that $H \phi' \cap K H \neq \emptyset$, for $\phi' \in \Phi'$, implies that $\phi' \in \mathbf{H}$ (Lemma 2.3). So if $\xi \in \Phi$, such that $H_{\phi} \xi \cap K_z H_{\phi} \neq \emptyset$, then $H \phi' \cap K H \neq \emptyset$, where $\phi' = g_2 \gamma_2^{-1} \xi \gamma_2 g_2^{-1} \in \Phi'$; hence $\xi \in \gamma_2 g_2^{-1} \mathbf{H} g_2 \gamma_2^{-1} = \mathbf{H}_{\phi}$. This shows that $j(\Phi)_{\mathbf{H}_{\phi}} : Z_{\phi}/\Phi_{\mathbf{H}_{\phi}} \rightarrow X/\Phi$ is an imbedding. But the image of this map is $\text{im}(j(\Phi)_{\mathbf{H}_{\phi}}) = D(\phi)$; hence $D(\phi)$ is diffeomorphic to the orientable locally symmetric space $Z_{\phi}/\Phi_{\mathbf{H}_{\phi}}$.

Now let $I(\Gamma) = \{\gamma_1, \dots, \gamma_r\}$ be representing elements for the connected components of $C_1(\Gamma) \cap C_2(\Gamma)$. To every $\gamma_j = g_{1,j} g_{2,j} \in \mathbf{G}_1(\mathbb{Q}) \mathbf{G}_2(\mathbb{Q})$ we can find in the above way a normal subgroup of finite index Φ_j in Γ such that every component of $C_1(\Phi_j) \cap C_2(\Phi_j)$ which covers $D(\gamma_j)$ is diffeomorphic to an orientable manifold $Z_{\phi}/\Phi_{j\mathbf{H}_{\phi}}$, where $\phi \in I(\Phi_j)$.

Further $\Phi_j = g_{2,j}^{-1} \Phi'_j g_{2,j}$, where $\Phi'_j \cap \mathbf{H} \subset H^0$ and $j(\Phi'_j)_{\mathbf{H}} : Z/\Phi'_{j\mathbf{H}} \rightarrow X/\Phi'_j$ is an imbedding.

The subgroup $\Phi := \bigcap_{j=1}^r \Phi_j \trianglelefteq \Gamma$ is normal of finite index. Like above, if $\phi \in I(\Phi)$, we see that $\Phi \cap \mathbf{H}_{\phi} \subset H_{\phi}^0$. Just as above, one proves that $j(\Phi)_{\mathbf{H}_{\phi}} : Z_{\phi}/\Phi_{\mathbf{H}_{\phi}} \rightarrow X/\Phi$ is an imbedding. $\square$

5.16. Now we fix orientations on X/Γ , $C_1(\Gamma)$, and $C_2(\Gamma)$. It follows immediately, by prop.5.1, (5.2) and prop. 5.15, that we can find a subgroup Φ of Γ of finite index in Γ such that the intersection number of $[C_1(\Phi)]$ and $[C_2(\Phi)]$ is

$$[C_1(\Phi)][C_2(\Phi)] = \sum_{\phi \in I(\Phi)} \langle e(\eta_{\phi}), [D(\phi)] \rangle, \quad (5.5)$$

where η_{ϕ} is the excess bundle of $D(\phi)$ in X/Φ .

Corollary 5.17 *In the usual notation, if (5.3) is injective, then the dimensions of the connected components of $C_1 \cap C_2$ are all equal to $\dim(X_1 \cap X_2)$.*

Proof. The assumption that (5.3) is injective implies that $\gamma = g_1 g_2$ for all $\gamma \in I(\Gamma)$; hence the manifold Z_{γ} is diffeomorphic to $X_1 \cap X_2$, via $x \mapsto x g_2^{-1}$ -thus $\dim D(\gamma) = \dim X_1 \cap X_2$. $\square$

5.18 In [RoSch] the authors use non abelian cohomology to describe the objects $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{H}_{\gamma}$, Z_{γ} , etc (see e.g. 2.5). In these set-up they were able to compute the Euler number of the excess bundle by an analysis of the fibers of the excess bundle. These description is only possible in the case of special cycles. However, it turns out that in the case of ordinary

cycles, where the map (5.3) is injective, we can modify of the work of Rohlf and Schwermer [RoSch, 4.1 - 4.6]. This is done as follows.

We choose a torsionfree arithmetic subgroup Γ of $\mathbf{G}(\mathbb{Q})$ such that C_1 and C_2 are orientable submanifolds of the orientable manifold X/Γ , which intersect perfectly and such that all connected components $D(\gamma)$, $\gamma \in I(\Gamma)$, of their intersection are orientable and diffeomorphic to $Z_\gamma/\Gamma_{\mathbf{H}_\gamma}$ (in the notation of Proposition 5.15).

We choose orientations on X/Γ , C_1 and C_2 . Let $\gamma \in I(\Gamma)$ and $x \in Z_\gamma$ and denote by K_x the maximal compact subgroup of $G = \mathbf{G}(\mathbb{R})$ corresponding to x . Let $K_{x,\gamma} := K_x \cap H_\gamma$, then we have $K_{x,\gamma} \backslash H_\gamma \cong Z_\gamma$. The Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G corresponding to the point $x \in Z_\gamma$ is denoted by

$$\mathfrak{g} = \mathfrak{k}_x \oplus \mathfrak{p}_x.$$

Let $\mathfrak{g}_{1,\gamma}$ resp. $\mathfrak{g}_2$ the Lie algebra of $\gamma^{-1}G_1\gamma$ resp. G_2 . The Cartan decompositions corresponding to x are denoted by

$$\mathfrak{g}_{1,\gamma} = \mathfrak{k}_{1,x,\gamma} \oplus \mathfrak{p}_{1,x,\gamma}$$

and

$$\mathfrak{g}_2 = \mathfrak{k}_{2,x} \oplus \mathfrak{p}_{2,x}$$

We obtain an exact sequence endowed with an $K_{x,\gamma}$ -action

$$0 \rightarrow \mathfrak{p}_{1,x,\gamma} + \mathfrak{p}_{2,x} \rightarrow \mathfrak{p}_x \rightarrow \tilde{\eta}_{x,\gamma} \rightarrow 0 \quad (5.6)$$

where, by definition, $\tilde{\eta}_{x,\gamma}$ denotes the cokernel of the inclusion. Since we have $K_{x,\gamma}^0 \backslash H_\gamma^0 \cong K_{x,\gamma} \backslash H_\gamma$, the action of the connected component $K_{x,\gamma}^0$ of $K_{x,\gamma}$ on $\tilde{\eta}_{x,\gamma}$, induced from the adjoint representation Ad_G of K_x on $\mathfrak{p}_x$, determines a bundle

$$\tilde{\eta}_\gamma := H_\gamma^0 \times_{K_{x,\gamma}^0} \tilde{\eta}_{x,\gamma}$$

on Z_γ . Denote $\pi_\gamma : Z_\gamma \rightarrow Z_\gamma/\Gamma_{\mathbf{H}_\gamma}$ the natural projection, then

$$\tilde{\eta}_\gamma = \pi_\gamma^* \eta_\gamma \quad (5.7)$$

is the pullback of the excess bundle η_γ over $Z_\gamma/\Gamma_{\mathbf{H}_\gamma}$ under π_γ .

Let $H_{u,\gamma}$ be the compact real form of H_γ . W.l.o.g. we can assume that $K_{x,\gamma} \subset H_{u,\gamma}$. So we get the compact dual symmetric space $K_{x,\gamma} \backslash H_{u,\gamma} =: Z_{u,\gamma}$.

The $K_{x,\gamma}^0$ -module $\tilde{\eta}_{x,\gamma}$ (via the representation induced by Ad_G) determines a bundle

$$\tilde{\eta}_{u,\gamma} := H_{u,\gamma}^0 \times_{K_{x,\gamma}^0} \tilde{\eta}_{x,\gamma}$$

on $Z_{u,\gamma}$. Orientations of Z_γ and of $\tilde{\eta}_\gamma$ determine orientations on $Z_{u,\gamma}$ and on $\tilde{\eta}_{u,\gamma}$. The dimension of the fibre of $\tilde{\eta}_{u,\gamma}$ is equal to $f := \dim X_1 \cap X_2 = \dim Z_{u,\gamma}$. Therefore, evaluating the Euler class $e(\tilde{\eta}_{u,\gamma})$ of $\tilde{\eta}_{u,\gamma}$ on the fundamental class $[Z_{u,\gamma}]$ of $Z_{u,\gamma}$ gives a well-defined number $\langle e(\tilde{\eta}_{u,\gamma}), [Z_{u,\gamma}] \rangle$.

On the symmetric space $K_{x,\gamma} \backslash H_\gamma = Z_\gamma$, the Killing form of $\text{Lie}(H_\gamma)$, restricted to $\mathfrak{p}_{1,x,\gamma} \cap \mathfrak{p}_{2,x}$ defines a H_γ -invariant metric g_{H_γ} on Z_γ . Similarly on the compact symmetric space $K_{x,\gamma} \backslash H_{u,\gamma} = Z_{u,\gamma}$, the negative of the Killing form of $\text{Lie}(H_{u,\gamma})$, restricted to $i(\mathfrak{p}_{1,x,\gamma} \cap \mathfrak{p}_{2,x})$ defines a $H_{u,\gamma}$ -invariant metric $g_{H_{u,\gamma}}$ on $Z_{u,\gamma}$.

Let μ (resp. μ_u) denote the Riemannian volume element on Z_γ (resp. $Z_{u,\gamma}$) with respect to g_{H_γ} (resp. $g_{H_{u,\gamma}}$). On $K_{x,\gamma}$ we put the normalized Haar measure ν , having mass 1 (i.e.

$\int_{K_{x,\gamma}} d\nu(k) = 1$). On H_γ (resp. $H_{u,\gamma}$) we obtain a Haar measure ω (resp. ω_u) normalized such that if $f \in C_c(G)$ (continuous with compact support) (resp. $f \in C(G_u)$ (continuous)), then

$$\int_{H_\gamma} f(h) d\omega(h) = \int_{Z_\gamma} \left(\int_{K_{x,\gamma}} f(kh) d\nu(k) \right) d\mu(K_{x,\gamma}h)$$

resp.

$$\int_{H_{u,\gamma}} f(h) d\omega_u(h) = \int_{Z_{u,\gamma}} \left(\int_{K_{x,\gamma}} f(kh) d\nu(k) \right) d\mu_u(K_{x,\gamma}h).$$

The measures ω and ω_u are 'corresponding' to another. On $H_\gamma/\Gamma_{\mathbf{H}_\gamma}$ we obtain a measure β from ω . We define $\text{vol}(H_\gamma/\Gamma_{\mathbf{H}_\gamma}) := \int_{H_\gamma/\Gamma_{\mathbf{H}_\gamma}} \beta$ and $\text{vol}(H_{u,\gamma}) := \int_{H_{u,\gamma}} \omega_u$. These are positive finite numbers.

One has the following extension of the Hirzebruch proportionality principle due to Rohlfs and Schwermer [RoSch, Prop. 4.2].

Proposition 5.19. *Under the assumptions of 5.18 and with the notation explained there one has the formula*

$$\langle e(\eta_\gamma), [D(\phi)] \rangle = (-1)^{f/2} \langle e(\tilde{\eta}_{u,\gamma}), [Z_{u,\gamma}] \rangle \cdot c(\gamma),$$

where the proportionality factor is given by

$$c(\gamma) = \frac{\text{vol}(H_\gamma/\Gamma_{\mathbf{H}_\gamma})}{\text{vol}(H_{u,\gamma})} \in \mathbb{Q}$$

If f is odd then $\langle e(\eta_\gamma), [D(\phi)] \rangle = 0$.

Proof. see [RoSch, Prop. 4.2] and replace the notation. $\square$

Recall that Γ has been chosen such that Γ_1, Γ_2 and Γ act orientation preserving on their symmetric spaces X_1, X_2 and X . Let $\gamma, \xi \in I(\Gamma)$, and $\gamma = a_1 a_2$ and $\xi = b_1 b_2$, where $a_1, b_1 \in \mathbf{G}_1(\mathbb{Q})$ and $a_2, b_2 \in \mathbf{G}_2(\mathbb{Q})$. By 5.13 these decompositions are unique up to elements in $\mathbf{H}(\mathbb{Q})$.

Assume that $H \subset G_1^{or} \cap G_2^{or}$ acts orientation preserving on X_1 and X_2 . We get that right translation with a_2 induces an isomorphism

$$X_1 \rightarrow X_1 a_2, \quad x \mapsto x a_2.$$

These map induce an orientation on $X_1 a_2$ such that the map become orientation preserving. Since $H \subset G_1^{or}$ acts orientation preserving on X_1 the induced orientation does not depend on the choice of a_2 . So we get an isomorphism

$$r(a_2^{-1} a_1^{-1} a_2) : X_1 \gamma = X_1 a_1 a_2 \rightarrow X_1 a_2. \quad (5.8)$$

This map is orientation preserving if and only if a_1 acts orientation preserving on X_1 . Using this, we get an isomorphism

$$r(a_2^{-1} a_1^{-1} b_1 b_2) : X_1 \gamma \rightarrow X_1 \xi, \quad x \gamma \mapsto x \xi.$$

This map is orientation preserving if and only if $a_1^{-1} b_1 \in G_1^{or}$, i.e., $a_1 G_1^{or} = b_1 G_1^{or}$. Assume now that $a_1, b_1 \in G_1^{or}$. Then by (5.9) we consider $X_1 \gamma$ resp. $X_1 \xi$ to be $X_1 a_2$ resp. $X_1 b_2$ as oriented manifolds. Hence right translation with $a_2^{-1} b_2$ induces an isomorphism

$$r(a_2^{-1} b_2) : X_1 \gamma \times X_2 \times X \rightarrow X_1 \xi \times X_2 \times X. \quad (5.9)$$

This map acts on the first factor orientation preserving, per definition of the orientation on $X_1\gamma$ resp. $X_2\xi$. On the second factor, $r(a_2^{-1}b_2)$ acts orientation preserving, if and only if $a_2G_2^{or} = b_2G_2^{or}$. So assume now that $a_2, b_2 \in G^{or}$ and that $G = G^{or}$. Then $r(a_2^{-1}b_2)$ acts orientation preserving on all three factors, hence on the whole space.

Proposition 5.20. *Let $\gamma, \xi \in I(\Gamma)$. Assume that $H \subset G_1^{or} \cap G_2^{or}$, $G = G^{or}$ and $I(\Gamma) \subset G_1^{or}G_2^{or}$. Then*

$$\langle e(\tilde{\eta}_{u,\gamma}), [Z_{u,\gamma}] \rangle = \langle e(\tilde{\eta}_{u,\xi}), [Z_{u,\xi}] \rangle.$$

Proof. The proof works as the proof of [RoSch, Prop. 4.4]. For the sake of completeness, we formulate it in our setting: Let $\gamma, \xi \in I(\Gamma)$ and fix an $x \in Z_\gamma$. We have a decompositions $\gamma = a_1a_2$ and $\xi = b_1b_2$ as above. Then $xa_2^{-1}b_2 \in Z_\xi$. We have Cartan decompositions $\mathfrak{g} = \mathfrak{k}_x \oplus \mathfrak{p}_x$ resp. $\mathfrak{g} = \mathfrak{k}_{xa_2^{-1}b_2} \oplus \mathfrak{p}_{xa_2^{-1}b_2}$ and right translation with $a_2^{-1}b_2$ induces an isomorphism $\text{Ad}_G(a_2^{-1}b_2)^{-1} : \mathfrak{p}_x \rightarrow \mathfrak{p}_{xa_2^{-1}b_2}$. Moreover, $\text{Ad}_G(a_2^{-1}b_2)^{-1}$ induces an isomorphism

$$f(a_2^{-1}b_2) : \mathfrak{p}_{1,x,\gamma} \oplus \mathfrak{p}_{2,x} \oplus \mathfrak{p}_x \xrightarrow{\sim} \mathfrak{p}_{1,xa_2^{-1}b_2,\xi} \oplus \mathfrak{p}_{2,xa_2^{-1}b_2} \oplus \mathfrak{p}_{xa_2^{-1}b_2} \quad (5.10)$$

The fiber $F_x(\tilde{\eta}_\gamma)$ of $\tilde{\eta}_\gamma$ over x resp. the fiber $F_{xa_2^{-1}b_2}(\tilde{\eta}_\xi)$ of $\tilde{\eta}_\xi$ over $xa_2^{-1}b_2$, together with the $K_{x,\gamma}^0$ action resp. $K_{xa_2^{-1}b_2,\xi}^0$ action on the fiber, determines the bundle $\tilde{\eta}_\gamma$ resp. $\tilde{\eta}_\xi$. The definition of these fibers in (5.7) shows that $\text{Ad}_G(a_2^{-1}b_2)^{-1}$ induces an isomorphism

$$F(a_2^{-1}b_2) : \tilde{\eta}_\gamma \rightarrow \tilde{\eta}_\xi,$$

where the induced isomorphism on the base spaces $Z_\gamma \rightarrow Z_\xi$, is just right translation by $a_2^{-1}b_2$. Using the exponential map we see that (5.11) induces the map (5.10). Hence $r(a_2^{-1}b_2)$ is orientation preserving if and only if $f(a_2^{-1}b_2)$ is orientation preserving. The tangent space at x to the total space of $\tilde{\eta}_\gamma$ is $T_x(Z_\gamma) \oplus F_x(\tilde{\eta}_\gamma) = (\mathfrak{p}_{1,x,\gamma} \cap \mathfrak{p}_{2,x}) \oplus F_x(\tilde{\eta}_\gamma)$ (see [Lü, Beispiel 11.27]). The orientation of this space is uniquely determined by the orientation of $\mathfrak{p}_{1,x,\gamma} \oplus \mathfrak{p}_{2,x}$ and $\mathfrak{p}_x$. Hence $F(a_2^{-1}b_2)$ is orientation preserving if and only if $f(a_2^{-1}b_2)$ is orientation preserving. By our assumptions and the discussion above, $r(a_2^{-1}b_2)$ acts orientation preserving, so $F(a_2^{-1}b_2)$ is indeed orientation preserving. But $F(a_2^{-1}b_2)$ is orientation preserving if the same holds true for the map induced by $F(a_2^{-1}b_2)$ on the corresponding 'dual'-bundles $\tilde{\eta}_{u,\gamma}$ and $\tilde{\eta}_{u,\xi}$. The rest follows from the fact, that the Euler numbers of two bundles which permit a orientation preserving isomorphism between them, coincide. $\square$

We define the bundle $\tilde{\eta}_u := \tilde{\eta}_{u,e}$ over the manifold $Z_u := Z_{u,e}$.

Main result. Before we formulate our main result, we recall the general setting: Let $\mathbf{G}$ be a connected $\mathbb{Q}$ -anisotropic semisimple algebraic group defined over $\mathbb{Q}$ and $\mathbf{G}_1$ and $\mathbf{G}_2$ two connected reductive $\mathbb{Q}$ -subgroups of $\mathbf{G}$, such that the map

$$H^1(\mathbb{Q}, \mathbf{H}) \rightarrow H^1(\mathbb{Q}, \mathbf{G}_1) \times H^1(\mathbb{Q}, \mathbf{G}_2)$$

is injective (here $\mathbf{H} = \mathbf{G}_1 \cap \mathbf{G}_2$). Let Γ be a torsionfree arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ chosen in such a way that the cycles $C_1(\Gamma)$, $C_2(\Gamma)$ and all connected components of their intersection are orientable (Lemma 2.2, Prop. 5.15). Suppose that $\dim C_1(\Gamma) + \dim C_2(\Gamma) = \dim X/\Gamma$ holds. The subgroup of $\mathbf{G}_i(\mathbb{R})$, which acts orientation preserving on X_i is denoted by G_i^{or} , $i = 1, 2$. In 5.18, a certain bundle $\tilde{\eta}_u := \tilde{\eta}_{u,e}$ over the manifold $Z_u = Z_{u,e}$ was constructed.

We have proved the following theorem:

Theorem 5.21. *Let $\mathbf{G}$, $\mathbf{G}_1$, $\mathbf{G}_2$ and Γ as above. Assume $H \subset G_1^{or} \cap G_2^{or}$ and that G acts orientation preserving on X . Assume further that there exists a subgroup Γ' of Γ , such that $I(\Gamma') \subset G_1^{or} G_2^{or}$, then the intersection number of $[C_1(\Gamma')]$ and $[C_2(\Gamma')]$ is*

$$[C_1(\Gamma')][C_2(\Gamma')] = (-1)^{f/2} \langle e(\tilde{\eta}_u), [Z_u] \rangle \cdot \left(\sum_{\gamma \in I(\Gamma')} c(\gamma) \right),$$

where all $c(\gamma)$, $\gamma \in I(\Gamma')$ are positive³. In particular, the Euler number $\langle e(\tilde{\eta}_u), [Z_u] \rangle$ is non zero, if and only if the intersection number of $C_1(\Gamma')$ and $C_2(\Gamma')$ is non zero.

Remark. In [RoSch], the authors did their computations for special cycles in a more general setting: They only assumed, that the manifold $C_1 \cap C_2$ is compact without any restriction on the $\mathbb{Q}$ -rank of $\mathbf{G}$. The manifold $C_1 \cap C_2$ is compact if and only if all subgroups $\mathbf{H}_\gamma$, $\gamma \in I(\Gamma)$ are $\mathbb{Q}$ -anisotropic; but if (5.3) is injective, all $\mathbf{H}_\gamma$ are conjugate to $\mathbf{H}$; hence $C_1 \cap C_2$ is compact if and only if $\mathbf{H}$ is $\mathbb{Q}$ -anisotropic. So theorem 5.21 holds even, if we replace the assumption $\mathbb{Q}$ -rank $\mathbf{G} = 0$, by the weaker assumption $\mathbb{Q}$ -rank $\mathbf{H} = 0$, by using the more general definition of the intersection number, as in [RoSch, §3] or [Sch2, 3.1].

Let $\mathbf{B} = \mathbf{G}_1$, $X_{\mathbf{B}} = X_1$, $C_{\mathbf{B}} = C_1$.

Corollary. *Let $\dim X = m = 2q$ even and $\dim X_{\mathbf{B}} = q$. If $G = G^{or}$ and $B = B^{or}$ then the self-intersection number of $C_{\mathbf{B}}(\Gamma)$ is*

$$[C_{\mathbf{B}}(\Gamma)][C_{\mathbf{B}}(\Gamma)] = (-1)^{q/2} \langle e(\tilde{\eta}_u), [X_{\mathbf{B},u}] \rangle \cdot \frac{\text{vol}(B/\Gamma_{\mathbf{B}})}{\text{vol}(B_u)}.$$

Further, if $\Gamma' \leq \Gamma$ is a subgroup of finite index, then

$$[C_{\mathbf{B}}(\Gamma)][C_{\mathbf{B}}(\Gamma)] = \frac{1}{[\Gamma_{\mathbf{B}} : \Gamma'_{\mathbf{B}}]} [C_{\mathbf{B}}(\Gamma')][C_{\mathbf{B}}(\Gamma')].$$

Proof. Clear, since $[\Gamma_{\mathbf{B}} : \Gamma'_{\mathbf{B}}] \text{vol}(B/\Gamma_{\mathbf{B}}) = \text{vol}(B/\Gamma'_{\mathbf{B}})$. □

5.22. Application. Now we come back to the case of the exceptional group G_2 (see chapter 3 and 4).

Let $\mathbf{G}'(a) \cong \mathbf{SU}(V_a, h_a, \tau_a)$ and $\mathbf{G}'(b, (b, a)_F) \cong \mathbf{U}(V_b(a), h_b, \tau_b)$ the subgroups of $\mathbf{G}'$ found in chapter 4, $\mathbf{G}(a) = \text{Res}_{F/\mathbb{Q}} \mathbf{G}'(a)$ and $\mathbf{G}(b, (b, a)_F) = \text{Res}_{F/\mathbb{Q}} \mathbf{G}'(b, (b, a)_F)$ the subgroups of $\mathbf{G} = \text{Res}_{F/\mathbb{Q}} \mathbf{G}'$. Further, let $\mathbf{H}' = \mathbf{G}'(a) \cap \mathbf{G}'(b, (b, a)_F) \cong \mathbf{SU}(V_a, h_a, \tau_a)(b) = \mathbf{SU}(W, h_a, \tau_a)$, where $W = \text{span}_{F(a)}\{c, bc\}$. Denote $\mathbf{H} = \text{Res}_{F/\mathbb{Q}} \mathbf{H}'$.

Note that $\text{Res}_{F/\mathbb{Q}}$ defines a bijection from the connected F -subgroups of $\mathbf{G}'$ to the connected $\mathbb{Q}$ -subgroups of $\mathbf{G}$ (see [BoTi, 6.18]).

Let $G = \mathbf{G}(\mathbb{R})$, $G_a = \mathbf{G}(a)(\mathbb{R})$, $G_{b,a} = \mathbf{G}(b, (b, a)_F)(\mathbb{R})$ and $X_{b,a} = K \cap G_{b,a} \setminus G_{b,a}$ the symmetric space (resp. $C_{b,a}$ the cycle) resulting from $\mathbf{G}(b, a)$. Further let $\Gamma_{b,a} = \Gamma \cap \mathbf{G}(b, a)$. We have

$$\dim X/\Gamma = 8 = 5 + 3 = \dim C_a + \dim C_{b,a}.$$

Further $\mathbf{H}'(\mathbb{R}) \cong SL(2, \mathbb{R})$ is not compact. Hence $\dim X_a \cap X_{b,a} = \dim SO(2) \setminus SL(2, \mathbb{R}) = 2$.

³In particular, the conditions that $H \subset G_1^{or} \cap G_2^{or}$ and that there exists a subgroup Γ' of Γ , such that $I(\Gamma') \subset G_1^{or} G_2^{or}$, is always satisfied, if G_1 and G_2 acts orientation preserving on their symmetric spaces X_1 and X_2 .

As a consequence of Witt's Cancellation law (see Appendix, Corollary A.10) the natural map

$$H^1(F, \mathbf{G}'(a) \cap \mathbf{G}'(b, (b, a)_F)) \rightarrow H^1(F, \mathbf{G}'(a))$$

is injective. (Recall (see e.g [KMRT, 29.6] or [Sp, 12.4.7 (1)]) that for an F -algebraic group $\mathbf{B}$ we have $H^1(F, \mathbf{B}) \cong H^1(\mathbb{Q}, \text{Res}_{F/\mathbb{Q}} \mathbf{B})$.)

Let $G' = \mathbf{G}'(\mathbb{R})$ and G'_u be the compact dual of G' . Denote by $\mathfrak{g}' = \text{Lie}(G')$ and $\mathfrak{g}'_u = \text{Lie}(G'_u)$ the Lie algebras of G' and G'_u . Then $G \cong G' \times G'_u \times \dots \times G'_u$ and $\mathfrak{g} := \text{Lie}(G) \cong \mathfrak{g}' \times \mathfrak{g}'_u \times \dots \times \mathfrak{g}'_u$.

Recall from §3.2 (and take the notation used there) that the $\mathbb{R}$ -Lie algebra $\mathfrak{g}'$ of G' is isomorphic to the algebra of derivations $\text{Der}_{\mathbb{R}}(\mathfrak{C}_{\mathbb{R}})$ of $\mathfrak{C}_{\mathbb{R}} := \mathfrak{C} \otimes_F \mathbb{R}$. By [Hum, 13.2], $\mathfrak{g}'(a) := \text{Lie}(\mathbf{G}'(a)(\mathbb{R})) = \{d \in \mathfrak{g}' \mid d(a) = 0\}$, $\mathfrak{g}'(b, (b, a)_{\mathbb{R}}) := \text{Lie}(\mathbf{G}'(b, (b, a)_F)(\mathbb{R})) = \{d \in \mathfrak{g}' \mid d(b) = 0, d((b, a)_{\mathbb{R}}) \subset (b, a)_{\mathbb{R}}\}$ and $\mathfrak{h}' := \text{Lie}(\mathbf{H}'(\mathbb{R})) = \{d \in \mathfrak{g}' \mid d(a) = 0, d(b) = 0\}$. Here $(b, a)_{\mathbb{R}} := (b, a)_F \otimes_F \mathbb{R}$.

The automorphism $g_{\theta} \in G'$ induced by $a \mapsto -a$, $b \mapsto -b$, $c \mapsto c$ defines a Cartan involution on $\mathfrak{g}'$, via $d \mapsto g_{\theta}.d$, with corresponding Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$. Here $g_{\theta}.d = g_{\theta} \circ d \circ g_{\theta}^{-1}$ is the adjoint action of g_{θ} on $d \in \mathfrak{g}'$. We obtain a Cartan involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, $(d_1, d_2, \dots, d_r) \mapsto (g_{\theta}.d_1, g_{\theta}.d_2, \dots, g_{\theta}.d_r)$. The corresponding Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{p}_0,$$

where $\mathfrak{k}_0 = \mathfrak{k}' \times \mathfrak{g}'_u \times \dots \times \mathfrak{g}'_u$ and $\mathfrak{p}_0 = \mathfrak{p}' \times \{0\} \times \dots \times \{0\}$.

In the notation of §3.2, a $\mathbb{R}$ -basis of $\mathfrak{k}'$ is given by

$$\{d_{a,b}, d_{a,ac}, d_{a,bc}, d_{b,ac}, d_{b,bc}, d_{ac,bc}\}$$

and a $\mathbb{R}$ -basis of $\mathfrak{p}'$ is given by

$$\{d_{a,ab}, d_{a,c}, d_{b,ab}, d_{b,c}, d_{ab,ac}, d_{ab,bc}, d_{c,ac}, d_{c,bc}\}.$$

Evaluating of this derivations on a shows that

$$\{d_{b,ab}, d_{c,ac}, c^2 d_{a,ab} - 2a^2 d_{c,bc}, b^2 d_{a,c} - 2d_{ab,bc}, a^2 d_{b,c} - d_{ab,ac}\}$$

is a $\mathbb{R}$ -basis of $\mathfrak{p}'(a) = \{d \in \mathfrak{p}' \mid d(a) = 0\}$. As well as for a , evaluating on b shows that

$$\{d_{a,ab}, d_{c,bc}, c^2 d_{b,ab} + 2b^2 d_{c,ac}, a^2 d_{b,c} + 2d_{ab,ac}, b^2 d_{a,c} + d_{ab,bc}\}$$

is a $\mathbb{R}$ -basis of $\mathfrak{p}'(b) = \{d \in \mathfrak{p}' \mid d(b) = 0\}$. Further, if we apply the condition $d(a) \in \text{span}_{\mathbb{R}}\{a, b, ab\}$, we see that

$$\{d_{a,ab}, d_{c,bc}, c^2 d_{b,ab} + 2b^2 d_{c,ac}\}$$

forms a $\mathbb{R}$ -basis of $\mathfrak{p}'(b, (b, a)_{\mathbb{R}}) = \{d \in \mathfrak{p}' \mid d(b) = 0, d((b, a)_{\mathbb{R}}) \subset (b, a)_{\mathbb{R}}\}$. Therefore

$$\{d_{b,ab}, d_{c,ac}, d_{a,ab}, d_{c,bc}, b^2 d_{a,c} - 2d_{ab,bc}, a^2 d_{b,c} - d_{ab,ac}\}$$

forms a $\mathbb{R}$ -basis of $\mathfrak{p}'(a) + \mathfrak{p}'(b, (b, a)_{\mathbb{R}})$. For a derivation $d \in \mathfrak{p}'$, we denote by $\check{d}$ the coset of d in $\mathfrak{p}'/\mathfrak{p}'(a) + \mathfrak{p}'(b, (b, a)_{\mathbb{R}})$. Hence the cosets

$$S := \{\check{d}_{a,c}, \check{d}_{b,c}\}$$

forms a $\mathbb{R}$ -basis of $\mathfrak{p}'/\mathfrak{p}'(a) + \mathfrak{p}'(b, (b, a)_{\mathbb{R}})$.

Let $K_H = K \cap H$ and $K'_{H'} = K' \cap H'$. Note that $Z_u = K_H \setminus H_u \cong SO(2) \setminus SU(2) \cong U(1) \setminus SU(2) \cong \mathbb{CP}^1$.

5.23. We wish to compute the Euler number $\langle e(\tilde{\eta}_u), [Z_u] \rangle$.

Precisely, $K_H \cong K'_{H'} \times H'_u \times \cdots \times H'_u$. Let $g \in K'_{H'} = \mathbf{G}'(a, b, (c, ab)_{\mathbb{R}})$, $g(ab) = ab$, so $g(c) = zc$, $z = z_1 + z_2 ab \in \mathbb{R}(ab) \cong \mathbb{C}$, $z_1, z_2 \in \mathbb{R}$, $N(z) = |z| = 1$, i.e. $z \in U(1)$. The action of $g \in K'_{H'}$ on $\mathfrak{p}'/\mathfrak{p}'(a) + \mathfrak{p}'(b, (b, a)_{\mathbb{R}})$ is given by the action of g on the basis S :

$$g.\check{d}_{x,c} = \check{d}_{g(x),g(c)} = \check{d}_{x,zc} = z_1 \check{d}_{x,c} + z_2 \check{d}_{x,(ab)c},$$

where $x = a, b$. We compute

$$\begin{aligned} \check{d}_{a,(ab)c} &= \check{d}_{a,b(ac)} = -\check{d}_{b(ac),a} \\ &= \check{d}_{(ac)a,b} + \check{d}_{ab,ac} \text{ (by (3.13))} \\ &= \check{d}_{(ac)a,b} + a^2 \check{d}_{b,c} \text{ (since } \check{d}_{ab,ac} = a^2 \check{d}_{b,c}) \\ &= 2a^2 \check{d}_{b,c}. \end{aligned}$$

Similar we compute

$$\check{d}_{b,(ab)c} = -\frac{1}{2}b^2 \check{d}_{a,c}.$$

Hence the action of g on $\mathfrak{p}'/\mathfrak{p}'(a) + \mathfrak{p}'(b, (b, a)_{\mathbb{R}})$, with respect to the basis S , is given by the matrix

$$[g] = \begin{pmatrix} z_1 & -\frac{1}{2}b^2 z_2 \\ 2a^2 z_2 & z_1 \end{pmatrix}.$$

Recall that $a^2, b^2 > 0$. Let $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$, such that $\alpha^2 = a^2$ and $\beta^2 = b^2$. Further, let $T \in GL_2(\mathbb{R})^0$, be the matrix

$$T = \begin{pmatrix} 1/2\alpha & \beta/2 \\ -1/\beta & \alpha \end{pmatrix}.$$

Then

$$T^{-1}[g]T = \begin{pmatrix} z_1 & -\alpha^2 \beta^2 z_2 \\ z_2 & z_1 \end{pmatrix}.$$

So we can identify the $\mathbb{R}$ -vectorspace $\mathfrak{p}'/\mathfrak{p}'(a) + \mathfrak{p}'(b, (b, a)_{\mathbb{R}})$ with the $\mathbb{R}$ -vectorspace $\mathbb{R}(ab)$, via T , such that the action of $K'_{H'}$ becomes

$$K'_{H'} \times \mathbb{R}(ab) \rightarrow \mathbb{R}(ab), (g, v) \mapsto zv.$$

On the other hand $\mathbb{R}(ab) \xrightarrow{\sim} \mathbb{C}$, $ab \mapsto i\alpha\beta$; so the action of $K'_{H'} \cong U(1)$ becomes

$$U(1) \times \mathbb{C} \rightarrow \mathbb{C}, (x_1 + ix_2, v) \mapsto (x_1 + ix_2)v.$$

Hence

$$\tilde{\eta}_u \cong SU(2) \times_{U(1)} \mathbb{C}.$$

This allows us to compute the Euler number $\langle e(\tilde{\eta}_u), [Z_u] \rangle = \langle e(SU(2) \times_{U(1)} \mathbb{C}), [\mathbb{CP}^1] \rangle$ as the Euler number of $SU(2) \times_{U(1)} \mathbb{C}$ over $\mathbb{CP}^1$.

Recall that the *canonical* or *tautological (complex) line bundle* $\gamma_n^1 = \gamma^1(\mathbb{C}^{n+1})$ over the projective space $\mathbb{CP}^n$, is defined as follows (see [Milnor Stasheff, Problem 13-E and §14]):

Let $l \subset \mathbb{C}^{n+1}$ be a complex line through the origin; hence l is an element in $\mathbb{C}P^n$. The total space $E = E(\gamma_n^1)$ of γ_n^1 is $E = \{(l, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid v \in l\}$ and the projection is

$$p : E \rightarrow \mathbb{C}P^n, (l, v) \mapsto l.$$

The tautological complex line bundle can be viewed as a 2-dimensional real vector bundle over the real $2n$ -dimensional manifold $\mathbb{C}P^n$. The complex projective n -space $\mathbb{C}P^n$ is diffeomorphic to $U(n+1)/(U(1) \times U(n))$ (because $U(n+1)$ acts transitively on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ - hence it acts transitively on the lines through the origin in $\mathbb{C}^{n+1}$ and $U(1) \times U(n)$ is the stabilizer of $\mathbb{C}(1, 0, \dots, 0) \subset \mathbb{C}^{n+1}$).

Lemma 5.24. *The canonical complex line bundle γ_n^1 is isomorphic to*

$$U(n+1) \times_{(U(1) \times U(n))} \mathbb{C} \cong \gamma_n^1,$$

where the action of $U(1) \times U(n)$ on $\mathbb{C}$ is the action of $U(1) \times U(n)$ on $\mathbb{C}(1, 0, \dots, 0)$.

Proof. (like in [Wal, 5.2.3]) For $g \in U(n+1)$ let l_g be the line through the origin and the vector $g(1, 0, \dots, 0)$. Let

$$f : U(n+1) \times_{(U(1) \times U(n))} \mathbb{C} \rightarrow \gamma_n^1, [g, v] \mapsto (l_g, gv).$$

Obviously, f is well-defined. If $(l_g, gv) = (l_h, hw)$, for $g, h \in U(n+1)$, $v, w \in \mathbb{C}$, then $g.v = h.w$; hence $g^{-1}h$ stabilize $\mathbb{C}(1, 0, \dots, 0)$. So $h = gk$, for $k \in U(1) \times U(n)$ and $g.v = h.w = gk.w$. Thus $k^{-1}.v = w$ and f is injective. Because $U(n+1)$ acts transitively on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, it acts transitively on the lines through the origin in $\mathbb{C}^{n+1}$. I.e. if l is a line through the origin in $\mathbb{C}^{n+1}$, there is a $g \in U(n+1)$, such that $l_g = l$ and $l_g = l_{gk}$, for all $k \in U(1) \times U(n)$. If $w \in l$, then $g^{-1}w =: v \in g^{-1}l_g = \mathbb{C}e_1$; hence $f([g, v]) = (l, w)$; i.e. f is surjective. $\square$

In particular, γ_1^1 is isomorphic to $SU(2) \times_{U(1)} \mathbb{C}$.

On the other hand, by [Milnor Stasheff, p.160], the Euler class $e(\gamma_n^1) \in H^2(\mathbb{C}P^n; \mathbb{Z})$, of the real bundle γ_n^1 , generates $H^2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$; hence

$$| \langle e(SU(2) \times_{U(1)} \mathbb{C}), [\mathbb{C}P^1] \rangle | = 1.$$

Now, we are able to prove

Corollary 5.25. *Let $\Gamma \subset \mathbf{G}(\mathbb{Z})$ be an arithmetic subgroup. There exists a subgroup Γ' of finite index in Γ such that the intersection number $[C_a(\Gamma')][C_{b,a}(\Gamma')]$ is non zero. In particular $C_a(\Gamma')$ resp. $C_{b,a}(\Gamma')$ represent non trivial classes in $H_5(X/\Gamma'; \mathbb{C})$ resp. $H_3(X/\Gamma'; \mathbb{C})$.*

Proof. Take $\mathbf{G}_1 = \mathbf{G}(b, (b, a)_F)$ and $\mathbf{G}_2 = \mathbf{G}(a)$. Remark that $f = \dim X_a \cap X_{b,a} = 2$, that $H = \mathbf{H}(\mathbb{R})$ and G are connected. So $H \subset G_1^{or} \cap G_2^{or}$ and $G = G^{or}$. Note that, by §5.2.1 example 2, G_1 does not act orientation preserving on $X_{b,a}$. By theorem 5.21, we have to prove that there exists a subgroup Γ' , such that $I(\Gamma') \subset G_1^{or} G_2^{or}$.

Define the connected reductive F -group $\mathbf{E}' = \mathbf{U}(F(b), \tau_b) = \{x \in F(b) \mid x\tau_b(x) = 1\}$ and consider $\mathbf{E}'$ as subgroup of $\mathbf{G}'$ via the imbedding $\lambda \mapsto g_\lambda$, where g_λ is an automorphism of $\mathcal{C}$ defined by

$$a \mapsto \lambda a, b \mapsto b, ab \mapsto \lambda ab, c \mapsto c, ac \mapsto \tau_b(\lambda)(ac), bc \mapsto bc, (ab)c \mapsto \tau_b(\lambda)((ab)c).$$

Then $\mathbf{E}'$ is a subgroup of $\mathbf{G}'(b, (b, a)_F)$ and $E' := \mathbf{E}'(\mathbb{R}) \cong GL_1(\mathbb{R})$ and $E'^0 \cong \mathbb{R}_{>0}$. Let $\mathbf{E} = \text{Res}_{F/\mathbb{Q}} \mathbf{E}'$ then $E = \mathbf{E}(\mathbb{R}) \cong GL_1(\mathbb{R}) \times U(1) \times \dots \times U(1)$ and $E^0 \cong \mathbb{R}_{>0} \times U(1) \times \dots \times U(1)$.

Let $\mathcal{O} = \mathcal{O}_F$ be the ring of algebraic integers of F . Consider the $\mathcal{O}$ -lattice $M = \bigoplus_{m \in \mathcal{B}} \mathcal{O}m$ of $\mathfrak{C}$, where $\mathcal{B} = \{e, a, b, ab, c, ac, bc, (ab)c\}$. We assume that the embedding of $\mathbf{G}'$ into $\mathbf{GL}(\mathfrak{C})$ is chosen with respect to $\mathcal{B}$; in particular an arithmetic subgroup $\Gamma \subset \mathbf{G}'(\mathcal{O})$, leaves M invariant. By lemma 2.2 we can choose a non zero ideal $\underline{a} \subset \mathbb{Z}$ such that $\mathbf{G}_i(\mathbb{Z})(\underline{a}) \subset G_i^0$, $i = 1, 2$ and $\mathbf{E}(\mathbb{Z})(\underline{a}) \subset E^0$. Now let $\gamma \in I(\Gamma(\underline{a}))$. Then there are $g_i \in \mathbf{G}_i(\mathbb{Q})$, such that $\gamma = g_1 g_2$. Note that $\gamma \in \mathbf{G}(\mathbb{Z}) \cong \mathbf{G}'(\mathcal{O})$ and $g_i \in \mathbf{G}_i(\mathbb{Q}) \cong \mathbf{G}'_i(F)$. Under this point of view, $\gamma a = \lambda a$, where $\lambda \in F(b)$. By $N(\lambda)N(a) = N(\gamma a) = N(a)$ and $\gamma a \in M$, we deduce that $\lambda \in \mathbf{E}'(\mathcal{O}) \cong \mathbf{E}(\mathbb{Z})$.

This shows that we have an assignment

$$I(\Gamma(\underline{a})) \rightarrow \mathbf{E}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{Z})(\underline{a}) = \mathbf{E}(\mathbb{Z})(\underline{a}).$$

Let g'_1 be the image of γ under this assignment. Then $g'_1{}^{-1}\gamma$ is fixing a ; hence $g'_2 := g'_1{}^{-1}\gamma \in G_2 = G_a$. So we get a decomposition $\gamma = g'_1 g'_2$, where $g'_1 \in E^0 \subset G_1^{or}$ and $g'_2 \in G_2 = G_2^{or}$. $\square$

5.26 If we replace the role of a and b , we obtain in the same way, that the cycles C_b and $C_{a,b}$ are non trivial for a suitable arithmetic subgroup.

The cycles C_a and C_b (resp. $C_{b,a}$ and $C_{a,b}$) are linear independent classes in $H_5(X/\Gamma; \mathbb{C})$ (resp. $H_3(X/\Gamma; \mathbb{C})$). This can be easily seen as follows: Since $\mathbf{G}(a, (a, b)_F) \subset \mathbf{G}(a)$, the cycle $C_{a,b}$ is contained in the cycle C_a ; hence the intersection $C_{a,b} \cap C_a$ is just the connected 3-dimensional manifold $C_{a,b}$. But the Euler class of an odd-dimensional vector bundle (in particular, the Euler class of the excess bundle) vanishes [Milnor Stasheff, 9.4]. So $[C_{a,b}][C_a] = 0$. Analogously, we get $[C_{b,a}][C_b] = 0$. But $[C_{b,a}][C_a] \neq 0$ and $[C_{a,b}][C_b] \neq 0$; so $[C_a]$ and $[C_b]$ (resp. $[C_{b,a}]$ and $[C_{a,b}]$) are linearly independent.

5.3 Intersection numbers of special cycles

In this section we present the result of Rohlfs and Schwermer [RoSch, Thm. 4.11] to compute the intersection numbers of two special cycles (in the transversal as in the non transversal case). Then we apply their result in the case G_2 .

5.27. Let $\mathbf{G}$ be a connected semisimple $\mathbb{Q}$ -algebraic group, σ, τ be two $\mathbb{Q}$ -rational automorphisms of $\mathbf{G}$ of finite order which commute with each other. Further, let Γ be a $\langle \sigma, \tau \rangle$ -stable torisonfree arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ chosen such that the special cycles $C(\sigma, \Gamma)$ and $C(\tau, \Gamma)$ and all connected components of their intersection are orientable.⁴ We suppose that $C(\sigma, \Gamma) \cap C(\tau, \Gamma)$ is compact (which is in particular true if $\mathbb{Q}$ -rank($\mathbf{G}$) = 0) and that $\dim C(\sigma, \Gamma) + \dim C(\tau, \Gamma) = \dim X/\Gamma$ holds. If $\mathbf{G}(\mathbb{R})$ resp. $\mathbf{G}(\sigma)(\mathbb{R})$ resp. $\mathbf{G}(\tau)(\mathbb{R})$ act orientation preserving on X resp. $X(\sigma)$ resp. $X(\tau)$, then *condition (Or)* is satisfied.

Theorem 5.28. (ROHLFS, SCHWERMER) *Let $\mathbf{G}$, σ, τ and Γ as above such that (Or) holds. If $X(\sigma)$ and $X(\tau)$ intersect in exactly one point with positive intersection number, then there exists a $\langle \sigma, \tau \rangle$ -stable normal subgroup Γ' of finite index in Γ , such that*

$$[C(\sigma, \Gamma')][C(\tau, \Gamma')] = \sum \chi(F(\gamma))$$

where the sum ranges over the elements γ in the kernel of the map

$$\text{res}_\sigma \times \text{res}_\tau : H^1(\langle \sigma, \tau \rangle, \Gamma') \rightarrow H^1(\langle \sigma \rangle, \Gamma') \times H^1(\langle \tau \rangle, \Gamma')$$

⁴Such a choice is always possible. (see [RoSch, §2] or §2.2 and Prop. 5.15).

and where all Euler characteristics $\chi(F(\gamma))$ are positive.

For the sake of completeness, we briefly give the result of Rohlf and Schwermer in the non transversely case:

Let $\mathcal{H}(\Gamma) := \ker(\text{res}_\sigma \times \text{res}_\tau)$ be the kernel of the map $\text{res}_\sigma \times \text{res}_\tau : H^1(\langle\sigma, \tau\rangle, \Gamma) \rightarrow H^1(\langle\sigma\rangle, \Gamma) \times H^1(\langle\tau\rangle, \Gamma)$. We have the natural map

$$j : H^1(\langle\sigma, \tau\rangle, \Gamma) \rightarrow H^1(\langle\sigma, \tau\rangle, \mathbf{G}(\mathbb{R})).$$

Theorem 5.29. (ROHLFS, SCHWERMER) *Let $\mathbf{G}$, σ, τ and Γ as in 5.27 such that (Or) holds. If the image of $\mathcal{H}(\Gamma)$ under j in $H^1(\langle\sigma, \tau\rangle, \mathbf{G}(\mathbb{R}))$ is trivial then there exists a $\langle\sigma, \tau\rangle$ -stable normal subgroup Γ' of finite index in Γ such that*

$$[C(\sigma, \Gamma')][C(\tau, \Gamma')] = (-1)^{f/2} \langle e(\tilde{\eta}_u), [Z_u] \rangle \cdot \left(\sum_{\gamma \in \mathcal{H}(\Gamma')} c(\gamma) \right)$$

where $f = \dim X(\sigma) \cap X(\tau)$ and $\tilde{\eta}_u$ and Z_u are as in 5.21. The proportionality factors $c(\gamma)$ corresponding to the connected components of $C(\sigma) \cap C(\tau)$ are like 5.19, just parameterized by $\gamma \in \mathcal{H}(\Gamma')$ (see [RoSch, Prop. 1.2 (1)] and [RoSch, Prop. 4.2]). In particular, the Euler number $\langle e(\tilde{\eta}_u), [Z_u] \rangle$ is non zero, if and only if the intersection number of $C(\sigma)$ and $C(\tau)$ is non zero.

Proof. [RoSch, 4.3, 4.4, 4.5], together with the argument in the proof of [RoSch, 4.11], that we can always replace Γ by a $\langle\sigma, \tau\rangle$ -stable subgroup Γ' which is normal in Γ such that all fixed point components with respect to Γ' have the same dimension modulo 4. $\square$

5.30. Application. Now we look at the exceptional group G_2 (see §4):

We use the notation of §4 freely. The group $\mathbf{G}(\sigma)(\mathbb{R})$ resp. $\mathbf{G}(\sigma\theta)(\mathbb{R})$ is not connected but by example 1 (see §5.2.1) it acts orientation preserving on $X(\sigma)$ resp. $X(\sigma\theta)$. Since $\mathbf{G}(\mathbb{R})$ is connected, condition (Or) is satisfied. The manifolds $X(\sigma)$ and $X(\sigma\theta)$ intersect in exactly one point with (possibly after rearranging the orientation on X , $X(\sigma)$ and $X(\sigma\theta)$) positive intersection number and $\dim X(\sigma) + \dim X(\sigma\theta) = \dim X$. Therefore [RoSch, 4.11], implies:

Corollary 5.31. *There exists a $\langle\sigma, \theta\rangle$ -stable torsionfree arithmetic subgroup Γ of $\mathbf{G}(\mathbb{Q})$ such that $[C(\sigma, \Gamma)][C(\sigma\theta, \Gamma)] > 0$; hence $C(\sigma, \Gamma)$ and $C(\sigma\theta, \Gamma)$ represent non trivial homology classes in $H_4(X/\Gamma; \mathbb{C})$.*

Question. Are $[C(\sigma)]$ and $[C(\sigma\theta)]$ resp. $[C(\sigma)]$ and $[C_c]$ linear independent classes in $H_4(X/\Gamma; \mathbb{C})$?

Chapter 6

Compact dual, signature and the method of Lafont and Schmidt

6.1 The compact dual to G_2

The Betti numbers for compact Lie groups of classical type (resp. homogeneous spaces of such one) are well known and published in book form in [GHV3], as Table I-III on p. 492-497. As far as I know, there is no book reference for the Betti numbers of the exceptional groups. The reference which I use, is [BC].

6.1. Let M be a compact manifold and $H^i(M, \mathbb{R})$ the i -th cohomology group with real coefficients. The *Poincaré polynomial* of M is then

$$P(M, t) = \sum_{i \geq 0} \dim_{\mathbb{R}} H^i(M, \mathbb{R}) \cdot t^i$$

Let U be a closed connected subgroup of a compact connected Lie group H , having the same rank r as H . Furthermore, let

$$\begin{aligned} P(H, t) &= (1 + t^{2m_1-1}) \cdot \dots \cdot (1 + t^{2m_r-1}) \\ P(U, t) &= (1 + t^{2n_1-1}) \cdot \dots \cdot (1 + t^{2n_r-1}) \end{aligned}$$

be the Poincaré polynomial¹ of H and U respectively. Then, by Hirsch formula, the Poincaré polynomial of the homogeneous space $U \setminus H$ is

$$P(U \setminus H, t) = (1 - t^{2m_1})(1 - t^{2m_2}) \cdot \dots \cdot (1 - t^{2m_r}) \cdot (1 - t^{2n_1})^{-1} \cdot \dots \cdot (1 - t^{2n_r})^{-1}.$$

Let G_u be the compact real form of G ; G a connected split real Lie group of type G_2 . The maximal compact subgroup K of G is of type $A_1 \times A_1$ (in fact, K is isomorphic to $SO(4)$). Their Poincaré polynomials are (cf. [BC])

$$P(G_u, t) = (1 + t^3)(1 + t^{11}) \tag{6.1}$$

$$P(K, t) = (1 + t^3)(1 + t^3). \tag{6.2}$$

Applying Hirsch formula to $X_u = K \setminus G_u$ together with (6.1) and (6.2), one gets

$$P(X_u, t) = 1 + t^4 + t^8.$$

¹The integers m_i resp. n_i are the so called *primitive invariants*. See [BC].

Now we wish to compute the cohomology ring of $H^*(X_u, \mathbb{R})$.

Remark. Recall that for an orientable compact even dimensional manifold M , of dimension $n = 2q$ the intersection pairing

$$\mathcal{S}_M : H^q(M, \mathbb{R}) \times H^q(M, \mathbb{R}) \rightarrow \mathbb{R}, (\alpha, \beta) \mapsto \langle \alpha \cup \beta, [M] \rangle \quad (6.3)$$

is a regular bilinear form, called the *intersection form*. Hence for every basis $\alpha_1, \dots, \alpha_k$ of $H^q(M, \mathbb{R})$, there is a basis $\dot{\alpha}_1, \dots, \dot{\alpha}_k$ of $H^q(M, \mathbb{R})$ such that $\alpha_i \cup \dot{\alpha}_j = \delta_{ij} [M]^*$. Here δ_{ij} is the usual Kronecker delta and $[M]^* \in H^n(M, \mathbb{R}) \cong \mathbb{R}$ is an generator of $H^n(M, \mathbb{R})$.

In particular, if $H^q(M, \mathbb{R}) \cong \mathbb{R}$, there must be a generator α of $H^q(M, \mathbb{R})$, such that $\alpha \cup \alpha$ generates $H^n(M, \mathbb{R})$.

Proposition 6.2. *The cohomology ring $H^*(X_u, \mathbb{R})$ is isomorphic to the ring $\mathbb{R}[T]/(T^3)$.*

Proof. Let $\alpha \in H^4(X_u, \mathbb{R}) \cong \mathbb{R}$ be a generator of this group. Then $\alpha \cup \alpha$ generates $H^8(X_u, \mathbb{R})$. The assignment $\alpha \mapsto T$ shows the desired. $\square$

6.2 Signature

Let M be a $n = 4k$ -dimensional oriented connected compact manifold. Then the intersection form $\mathcal{S}_M$ is a symmetric regular bilinear form. (If $\alpha \in H^p(M, \mathbb{R})$, $\beta \in H^q(M, \mathbb{R})$, then we have $\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$.)

Let $s : V \times V \rightarrow \mathbb{R}$ be a symmetric regular bilinear form. Choose a basis $\{b_1, \dots, b_m\}$ of V and define the symmetric Matrix $A = (s(b_i, b_j))$. The signature of s is the number of positive eigenvalues of A minus the number of negative eigenvalues of A . The signature depends only on s and not on the chosen basis.

Definition 6.3. The *signature* $\text{sign}(M)$, of a n -dimensional oriented compact manifold M is zero if n is not a multiple of 4, and the signature of $\mathcal{S}_M$, if $n = 4k$, $k \in \mathbb{N}$.

The signature is an important homotopy type invariant of M .

Lemma 6.4. *If M is the boundary ∂V of an $4k + 1$ -dimensional compact manifold V , then $\text{sign}(M) = 0$.*

Proof. [Lü, Lemma 8.43] or [Bre, 10.8].

One can compute the signature of M as a polynomial of the Pontrjagin numbers $p_i(M)$ of M . This was done by Hirzebruch, and is the so called signature theorem. We will formulate the theorem only in the case $n = 8$.

Theorem 6.5. *Let M be an 8-dimensional oriented compact manifold. Then the signature of M , is*

$$\text{sign}(M) = L_2(p_1, p_2) = \frac{1}{45}(7p_2(M) - p_1^2(M)), \quad (6.4)$$

where $p_1^2(M)$ and $p_2(M)$ are the Pontrjagin numbers of M .

Proof. [Milnor Stasheff, 19.4]

Remark. The signature theorem says that the signature of every orientable compact $4k$ -manifold M is the value of an homogeneous polynomial L_k of degree k , in the Pontrjagin classes $p_1, \dots, p_k$ on the fundamental class $[M]$.

Let $\mathbf{G}$ be a semisimple $\mathbb{Q}$ -anisotropic $\mathbb{Q}$ -algebraic group, $G = \mathbf{G}(\mathbb{R})$ and $X = K \backslash G$ be the symmetric space. Further, let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a torsionfree arithmetic group such that X/Γ is orientable and X_u be the compact dual of X . Choose an orientation on X/Γ . If $n = 4k = \dim X$, then, by Hirzebruch proportionality principle, [LaR, theorem A], we have:

Theorem 6.6. *Let $I = i_1, \dots, i_r$ be a partition of k . The I -th Pontrjagin number $p_I(X/\Gamma) = p_{i_1} \dots p_{i_r}(X/\Gamma)$ is zero if and only if $p_I(X_u)$ is zero. Further, there is a rational constant $d(X/\Gamma) = d \neq 0 \in \mathbb{Q}$, depending only on X/Γ such that $dp_I(X_u) = p_I(X/\Gamma)$.*

The previous remark and theorem 6.6 imply that if X/Γ is oriented and compact, then

$$\text{sign}(X/\Gamma) = d(X/\Gamma)\text{sign}(X_u). \quad (6.5)$$

In particular $\text{sign}(X/\Gamma) \neq 0$ if and only if $\text{sign}(X_u) \neq 0$. By theorem 6.5 and 6.6, we can prove the non vanishing of the signature by looking at the Pontrjagin numbers resp. the signature of the compact dual. But these numbers are computed in [BoHi]. Next, we will apply this in the case of G_2 .

6.7. Let $\mathbf{G}$ be (as in §3.2) an $\mathbb{Q}$ -anisotropic $\mathbb{Q}$ -algebraic group of type G_2 , such that $G = \mathbf{G}(\mathbb{R})$ is not compact and $X = K \backslash G \cong SO(4) \backslash G'$, where G' is the connected split simple real Lie group of type G_2 . By [BoHi, §18, p.533] we can orient X_u such that $p_1^2(X_u) = 4$ and $p_2(X_u) = 7$. Let $d = d(X/\Gamma)$ be the constant as in theorem 6.6. Using 6.5 and 6.6 we get

$$\begin{aligned} \text{sign}(X/\Gamma) &= \frac{1}{45}(7p_2(X/\Gamma) - p_1^2(X/\Gamma)) \\ &= \frac{d}{45}(7p_2(X_u) - p_1^2(X_u)) \\ &= \frac{d}{45}(7 \cdot 7 - 4) = d \neq 0. \end{aligned}$$

Corollary 6.8. *In the case G_2 , there is an arithmetic group Γ , such that the signature $\text{sign}(X/\Gamma)$ is equal to the proportionality factor d , in particular non zero. Hence the factor d is always an integer, $d = d(X/\Gamma) \in \mathbb{Z}$. The locally symmetric space X/Γ is not the boundary of a 9-dimensional manifold.*

Remark. Of course, we can compute the signature of X_u , without using [BoHi], by looking at the Poincaré polynomial of X_u . Since $\mathcal{S}_{X_u} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is regular and $\dim H^4(X_u; \mathbb{R}) = 1$, the signature is $+1$ or -1 . Hence after a (possible) change of the orientation the signature is just $+1$.

6.3 The method of Lafont and Schmidt

There is a way (without using intersection numbers) to show that a homology class which is represented by a cycle C (as constructed in §2.1) in a compact locally symmetric space X/Γ , is not homologous to zero. This method was introduced by Lafont and Schmidt [LaS]. However, the method works only in certain cases and not in general. In the case G_2 there is exactly one degree in which such cycles can be discussed by this method.

6.9. Let $\mathbf{B}$ be a reductive $\mathbb{Q}$ -algebraic subgroup of a semisimple $\mathbb{Q}$ -algebraic group $\mathbf{G}$ such that there group of real points B and G are not compact; $K \subseteq G$ a maximal compact subgroup such that $K \cap B$ is maximal compact in B . We choose the notation of §2.1. and the assumptions there; in particular $\mathbb{Q} - \text{rank}(\mathbf{G}) = 0$. We denote by $X = K \setminus G$ resp. $Y = K \cap B \setminus B$ the associated symmetric spaces and by X_u resp. Y_u there compact dual symmetric spaces. As in §2.1. we choose $\Gamma \subset \mathbf{G}(\mathbb{Q})$ such that $i : Y/\Gamma_{\mathbf{B}} \hookrightarrow X/\Gamma$ is an embedding of compact manifolds. We denote the obtained cycle by $i(Y/\Gamma_{\mathbf{B}}) =: C_{\mathbf{B}}$. On the other hand we have the embedding $i_u : Y_u \hookrightarrow X_u$. Let $m = \dim Y/\Gamma_{\mathbf{B}}$. Hence, we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{R} \cong H^m(Y/\Gamma_{\mathbf{B}}, \mathbb{R}) & \xleftarrow{i^*} & H^m(X/\Gamma, \mathbb{R}) \\ j_Y^* \uparrow & & \uparrow j_X^* \\ \mathbb{R} \cong H^m(Y_u, \mathbb{R}) & \xleftarrow{i_u^*} & H^m(X_u, \mathbb{R}) \end{array}$$

Here $j_X^* : H^m(X_u, \mathbb{R}) \rightarrow H^m(X/\Gamma, \mathbb{R})$ (resp. $j_Y^* : H^m(Y_u, \mathbb{R}) \rightarrow H^m(Y/\Gamma_{\mathbf{B}}, \mathbb{R})$) denotes the Matsushima map. Note that, if Γ is cocompact in X (resp. $\Gamma_{\mathbf{B}}$ is cocompact in Y), then the Matsushima map is injective (see e.g. [Bo1, 2.1]); hence $j_Y^* : H^m(Y_u, \mathbb{R}) \rightarrow H^m(Y/\Gamma_{\mathbf{B}}, \mathbb{R})$ is an isomorphism. Obviously, if i_u^* is non zero, i^* couldn't be zero.

Let $[Y/\Gamma_{\mathbf{B}}]$ be the fundamental class of $Y/\Gamma_{\mathbf{B}}$ and ω a generator of $j_X^* H^m(X_u, \mathbb{R})$; then $0 \neq \langle i^* \omega, [Y/\Gamma_{\mathbf{B}}] \rangle = \langle \omega, i_* [Y/\Gamma_{\mathbf{B}}] \rangle$ shows that $i_* [Y/\Gamma_{\mathbf{B}}] =: [C_{\mathbf{B}}]$ is non trivial.

6.10. Let us take a look at G_2 : Take $\mathbf{G}$ to be the group defined in §4.1 and X the associated symmetric space, X_u the compact dual and $\mathbf{B}$ the group $\mathbf{G}(c)$ (resp. $\mathbf{G}(ab)$) as defined in §4.1.2. Then $Y_u = S(U(2) \times U(1)) \setminus SU(3)$. By [T, Thm 8.2], the submanifold $Y_u = S(U(2) \times U(1)) \setminus SU(3) \hookrightarrow X_u$ represents a non trivial homology class in $H_4(X_u, \mathbb{R})$; so i_u^* is non zero. Hence the cycles $C_c \cong C_{ab} \cong S(U(2) \times U(1)) \setminus SU(2, 1)/\Gamma_{\mathbf{G}(c)}$, found in §4.2 represent a non trivial cycle in $H_4(X/\Gamma, \mathbb{R})$.

Question. Is the Poincaré dual class of these cycles invariant under the action of G ? See §7 and §8, for non explained terms and §8.4 for the answer.

Chapter 7

Automorphic representations with non-zero cohomology

7.1 General theory

Here we discuss briefly the contribution of an unitary representation π to the cohomology $H^*(X/\Gamma, \mathbb{C})$ of a torsion free arithmetic group $\Gamma \subseteq \mathbf{G}(\mathbb{Q})$; Γ cocompact in $G = \mathbf{G}(\mathbb{R})$. For more details see as example [Bo1], [BW], [Sch1], [LiSch2] or [V].

Theorem 7.1. (MATSUSHIMA) ([BW, Thm VII.5.2])

$$H^*(X/\Gamma, \mathbb{C}) \cong \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) \cdot H^*(\mathfrak{g}, K; H_{\pi, K}^\infty) \quad (7.1)$$

where $\hat{G}$ is the unitary dual of G , $H_{\pi, K}^\infty$ the Harish-Chandra modul of π and the multiplicity $m(\pi, \Gamma)$ is a non-negative integer for each π . If π_0 is the trivial representation of G , then $m(\pi_0, \Gamma) = 1$.

If $rk_{\mathbb{R}} K = rk_{\mathbb{R}} G$, then (by Harish-Chandra) G has a discrete series representation. Let $\hat{G}_d$ be the set of equivalence classes of irreducible discrete series representations of G . Fix a finite dimensional (complex) rational representation (τ, E) of $\mathbf{G}$. Denote by $\hat{G}_{d, E} = \{[\pi] \in \hat{G}_d \mid \chi_\pi = \chi_{E^*}\}$ the subset of thus elements in $\hat{G}_d$ such that the infinitesimal character of a representative π of a class coincides with the one of the contragredient representation (τ^*, E^*) of E (i.e. $\tau^*(g) := {}^t\tau(g^{-1})$). Then $|\hat{G}_{d, E}| = \frac{|W_G|}{|W_K|}$, where W_G (resp. W_K) denotes the Weyl group of G (resp. K) and for a given π with $[\pi] \in \hat{G}_{d, E}$ one has ([BW, II, 5.3, 5.4], [Sch1, 5.1]),

$$H^q(\mathfrak{g}, K; H_\pi \otimes E) = \begin{cases} \mathbb{C} & q = \frac{1}{2} \dim X \\ 0 & \text{otherwise.} \end{cases}$$

Note that if (τ, E) is the trivial representation $(\pi_0, \mathbb{C})$, then $(\pi_0, \mathbb{C}) = (\pi_0^*, \mathbb{C}^*)$ and $\chi_{\pi_0^*} = \chi_\rho$, (see. [BW, III, 1.5]), where ρ is in the sense of [BW, II, 1.3, (4)] (the half of the sum over all positive roots). By [BW, VII, 6.1], we can restrict the sum of the right hand side of (7.1) to those π with $\chi_\pi = \chi_\rho$.

Definition 7.2. Let M be a smooth manifold acted by a group H from the right via diffeomorphisms. A differential form $\omega \in \Omega(M, \mathbb{C})$ is called *H-right invariant* or simply *invariant* if ω is invariant under the pull back of the action of H ; $h^*\omega = \omega$ for all $h \in H$. The subspace of all H -invariant forms is denoted by $\Omega(M, \mathbb{C})^H$.

If G is a semisimple Lie group and G_u its compact dual, then G resp. G_u acts transitively by right translation r_g , $g \in G$ resp. $g \in G_u$ on $X = K \setminus G$ resp. $X_u = K \setminus G_u$. Let $o = Ke \in X$ resp. $o = Ke \in X_u$. Further let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the Cartan decompositions of $Lie(G)$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ the Lie algebra $Lie(G_u)$. An smooth differential q -form ω on X associates to each $x \in X$ a q -linear map $\omega_x : \bigwedge^q T_x(X) \rightarrow \mathbb{C}$ from the exterior product space $\bigwedge^q T_x(X)$ to $\mathbb{C}$. For a smooth differential q -form ω on X the action of G is given by

$$r_g^* \omega_x(x_1, \dots, x_q) := \omega_{xg}(\text{Ad}_G g^{-1}x_1, \dots, \text{Ad}_G g^{-1}x_q),$$

where $x \in X, x_i \in T_x X$. Since G acts transitively on X , a q -form ω is unique determined by its values on $\bigwedge^q T_o(X) = \bigwedge^q \mathfrak{p}_0$.

Since K is the isotropy group of $o \in X$, the group K stabilizers the vector space $\mathfrak{p}_0$. So we let K operate on $\bigwedge^q \mathfrak{p}_0^*$, sending each q -linear map $\varphi : \bigwedge^q \mathfrak{p}_0 \rightarrow \mathbb{C}$ to $\varphi \circ \text{Ad}_G k : \bigwedge^q \mathfrak{p}_0 \rightarrow \mathbb{C}$, where $\varphi \circ \text{Ad}_G k(x_1, \dots, x_q) = \varphi(\text{Ad}_G kx_1, \dots, \text{Ad}_G kx_q)$, $k \in K$.

Each G -invariant q -form ω on X is in particular K -invariant and the value of $\omega_o \in (\bigwedge^q \mathfrak{p}_0^*)^K$ determines ω uniquely. All that holds true for X_u . Since $\mathfrak{p}_0 \cong i\mathfrak{p}_0$, there are canonical isomorphisms

$$\Omega^q(X, \mathbb{C})^G \cong (\bigwedge^q \mathfrak{p}_0^*)^K \cong \Omega^q(X_u, \mathbb{C})^{G_u},$$

for all $q \geq 0$.

Theorem 7.3. ([Bo1, 1.8], [V, 2.10]) *Let G_u be the compact dual of G and $X_u = K \setminus G_u$. Then the relative Lie algebra cohomology with coefficients in the trivial representation is natural isomorphic to the cohomology of the compact symmetric space X_u ,*

$$H^*(\mathfrak{g}, K; \mathbb{C}) \cong H^*(X_u, \mathbb{C}) \cong H^*(\Omega(X_u, \mathbb{C})^{G_u}).$$

The group Γ was chosen torison free; hence it can be shown that $\Omega(X/\Gamma, \mathbb{C}) \cong \Omega(X, \mathbb{C})^\Gamma$ and that $H^*(X/\Gamma, \mathbb{C}) \cong H^*(\Omega(X, \mathbb{C})^\Gamma)$ [BW, VII, Thm 2.2.]. Obviously, every G -invariant form can be viewed as Γ -invariant form. Therefore the question about the correspondence of the trivial representation to a differential form $\omega \in \Omega(X/\Gamma, \mathbb{C})$ is equivalent to say that this differential form is G -invariant.

7.2 Unitary representations with non zero cohomology-the case of the real Lie group G_2

For the sake of completeness we briefly give the results about the representations with non zero cohomology as treated in [LiSch1, §2] by making explicit the general theory of Vogan-Zuckerman [VZ]. These are the representations that could (possibly) contribute to the right hand side of (7.1)

Let G be the split simple Lie group of type G_2 .

Theorem 7.4.

1. If π_0 is the trivial representation, then $H^i(\mathfrak{g}, K; H_{\pi_0}) = \begin{cases} \mathbb{C} & i = 0, 4, 8 \\ 0 & i \neq 0, 4, 8 \end{cases}$.
2. There are exactly two non-equivalent non-tempered representations π_1, π_2 of G such that $H^i(\mathfrak{g}, K; H_{\pi_j}) = \begin{cases} \mathbb{C} & i = 3, 5 \\ 0 & i \neq 3, 5 \end{cases}$, $j = 1, 2$.

3. *There are exactly three non-equivalent discrete series representations π_3, π_4, π_5 of G such that $H^i(\mathfrak{g}, K; H_{\pi_j}) = \begin{cases} \mathbb{C} & i = 4 \\ 0 & i \neq 4 \end{cases}, j = 3, 4, 5.$*

These exhaust all irreducible unitary representations (π, H_π) of G with non-trivial $(\mathfrak{g}, K)$ -cohomology $H^(\mathfrak{g}, K; H_\pi)$ up to equivalence.*

Remark. Of course, (1) is obvious by the Poincaré polynom of X_u (§6.1) and (3) is clear since $\frac{|W_G|}{|W_K|} = \frac{12}{4} = 3$.

Chapter 8

Non G -invariance of cycles and the growth of Betti numbers

By Matsushimas formula (see §7.1), we see that homological non-trivial cycles can be used to show the non vanishing of multiplicities, i.e. $m(\pi, \Gamma) \neq 0$, of certain irreducible unitary representations π occuring in the spectrum of Γ . We are interested in the multiplicities of representations π which are not the trivial representation. Thus we have to show that the cycles represent non G -invariant classes. This was done by [MiRa, Thm. 2.1] in the case of the special cycles constructed there (i.e. cycles coming from the fixed points of an involution). In §8.1 we review the necessary background in this topic. In §8.2 we will give a detailed account of the proof of [MiRa] and a criterion for the non G -invariance, in the case of special cycles coming from the fixed points of arbitrary automorphisms of finite order. As far as I know, in the most general case of cycles, the non G -invariance is unknown. In §8.3 we give two sufficient criteria for the non G -invariance of cycles which occur in the middle dimension of a $2n$ -dimensional space X/Γ . Thereby we obtain an assessment of the growth of the Betti numbers in the middle degree. In §8.4 we summarize our result regarding the non G -invariance and the growth of Betti numbers in the case G_2 .

8.1 Background

8.1 Let $\mathbf{G}$ be a connected, semisimple linear $\mathbb{Q}$ -anisotropic algebraic group defined over $\mathbb{Q}$, $\mathbf{G}_i$, $i = 1, 2$ reductive connected subgroups of $\mathbf{G}$. We fix once and for all a $\mathbb{Q}$ -rational imbedding of $\mathbf{G}$ in some $\mathbf{GL}_N$ and with respect to this imbedding we define the group of integer points $\mathbf{G}(\mathbb{Z})$. Let $\Gamma \subseteq \mathbf{G}(\mathbb{Z})$ a torsionsfree arithmetic subgroup, such that

$$j_i : X_i/\Gamma_i \longrightarrow X/\Gamma, \quad i = 1, 2$$

are smooth imbeddings of orientable manifolds and $\Gamma \subset \mathbf{G}(\mathbb{R})^0$. The image of X_i/Γ_i under j_i will be denoted by C_i resp. $C_i(\Gamma)$, $i = 1, 2$. For an ideal $\underline{a} \subseteq \mathbb{Z}$ and we denote $\Gamma(\underline{a}) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \pmod{\underline{a}}\}$ and $C_i(\underline{a}) = C_i(\Gamma(\underline{a}))$. If we just have one single connected reductive $\mathbb{Q}$ -subgroup $\mathbf{B}$ of $\mathbf{G}$, we denote $\Gamma \cap \mathbf{B} = \Gamma_{\mathbf{B}}$ and $Y = X_{\mathbf{B}}$ the associated symmetric space to $\mathbf{B}$. Further we denote the cycle obtained, by $C_{\mathbf{B}}$ resp. $C_{\mathbf{B}}(\Gamma)$ and for an ideal $\underline{a} \subseteq \mathbb{Z}$ we denote the cycle $C_{\mathbf{B}}(\Gamma(\underline{a}))$ by $C_{\mathbf{B}}(\underline{a})$.

The locally symmetric space X/Γ , $m = \dim X$ is a closed connected orientable manifold. So we have for all integers q the Poincaré duality isomorphism

$$D_{X/\Gamma} : H^{m-q}(X/\Gamma, \mathbb{C}) \longrightarrow H_q(X/\Gamma, \mathbb{C}), [\omega] \longmapsto [\omega] \cap [X/\Gamma].$$

This isomorphism is natural in the sense, that for a continuous map $\eta : X/\Gamma \rightarrow X/\Gamma$ we have

$$(\eta_* \circ D_{X/\Gamma} \circ \eta^*)([\omega]) = D_{\eta_*(X/\Gamma)}([\omega])$$

for all $[\omega] \in H^{m-q}(X/\Gamma)$. Here η_* (resp. η^*) is the group homomorphism of H_* (resp. H^*), induced by η .

8.2 Let $\underline{a} \subseteq \mathbb{Z}$ be an ideal. We have a $[\Gamma : \Gamma(\underline{a})]$ -fold covering pr: $X/\Gamma(\underline{a}) \rightarrow X/\Gamma$ of locally symmetric spaces. The group of decktransformations $\eta : X/\Gamma(\underline{a}) \rightarrow X/\Gamma(\underline{a})$ with respect to this covering is denoted by $\mathcal{G} := \text{Deck}(X/\Gamma(\underline{a}), \text{pr})$. Since X/Γ (resp. $X/\Gamma(\underline{a})$) is a $K(\Gamma, 1)$ (resp. $K(\Gamma(\underline{a}), 1)$) space and $\Gamma(\underline{a}) \triangleleft \Gamma$ is a normal subgroup, we have $\mathcal{G} = \text{Deck}(X/\Gamma(\underline{a}), \text{pr}) \cong N_\Gamma(\Gamma(\underline{a}))/\Gamma(\underline{a}) = \Gamma/\Gamma(\underline{a})$ and therefore every decktransformation is of the form

$$\eta_\gamma : X/\Gamma(\underline{a}) \rightarrow X/\Gamma(\underline{a}), \quad x\Gamma(\underline{a}) \mapsto x\gamma\Gamma(\underline{a})$$

for a $\gamma \in \Gamma$. By our assumptions above, $\Gamma \subset G^0$ is in the connected component of $G = \mathbf{G}(\mathbb{R})$ and so all decktransformations preserve the orientation. Since η_γ is a continuous map, it induce an homomorphism

$$(\eta_\gamma)_* : H_m(X/\Gamma(\underline{a})) \rightarrow H_m(X/\Gamma(\underline{a})), \quad [X/\Gamma(\underline{a})] \mapsto \deg(\eta_\gamma)[X/\Gamma(\underline{a})]$$

with the degree $\deg(\eta_\gamma) \in \mathbb{Z}$. This is an isomorphism, since η_γ is a homeomorphism, so $\deg(\eta_\gamma) = \pm 1$. But η_γ preserve the orientation, so it must $\deg(\eta_\gamma) = 1$. By this fact we have that the isomorphisms $D_{(\eta_\gamma)_*(X/\Gamma(\underline{a}))}$ and $D_{X/\Gamma(\underline{a})}$ are equal. Thus by 8.1, we obtain

$$\eta_* \circ D_{X/\Gamma(\underline{a})} \circ \eta^* = D_{X/\Gamma(\underline{a})}.$$

We denote by X_u the compact dual of X . Let us recall that if Γ is cocompact in X we have

$$H^*(X_u, \mathbb{C}) \cong H^*(\Omega(X, \mathbb{C})^G) \hookrightarrow H^*(\Omega(X, \mathbb{C})^\Gamma) \cong H^*(\Omega(X/\Gamma, \mathbb{C})) \cong H^*(X/\Gamma, \mathbb{C})$$

(see [Bol] 1.8.(11), 2.1.(2) and [BW] IX 5.6.(i)).

Let $\alpha := D_{X/\Gamma(\underline{a})}^{-1}([C_i(\underline{a})])$ be the Poincaré dual of $[C_i(\underline{a})]$. Assume now that α can be represented by a G -invariant differential form, i.e., $\alpha = [\omega]$ for a G -right invariant differential form $\omega \in \Omega(X, \mathbb{C})^G$. Then $(\eta_\gamma)^*(\alpha) = [(\eta_\gamma)^*\omega] = [(r_\gamma)^*\omega] = [\omega] = \alpha$ for all $\eta_\gamma \in \text{Deck}(X/\Gamma(\underline{a}), \text{pr})$. Here $r_g : G \rightarrow G$ (resp. $r_g : X \rightarrow X$) denotes the right translation by $g \in G$ and $(r_g)^*$ the pullback. By the discussion above we get for all $\eta \in \text{Deck}(X/\Gamma(\underline{a}), \text{pr})$

$$\begin{aligned} \eta_*[C_i(\underline{a})] &= \eta_* D_{X/\Gamma(\underline{a})}(\alpha) \\ &= (\eta_* \circ D_{X/\Gamma(\underline{a})} \circ \eta^*)(\alpha) \\ &= D_{X/\Gamma(\underline{a})}(\alpha) \\ &= [C_i(\underline{a})]. \end{aligned}$$

Summary: We have seen that if $\alpha = D_{X/\Gamma(\underline{a})}^{-1}([C_i(\underline{a})])$ is G -invariant, then the homology class $[C_i(\underline{a})]$ is invariant under all decktransformations.

8.3 Note that for the coverings pr: $X/\Gamma(\underline{a}) \rightarrow X/\Gamma$ and $\text{pr}_i : C_i(\underline{a}) \rightarrow C_i$ (resp. $\text{pr}_\mathbf{B} : C_\mathbf{B}(\underline{a}) \rightarrow C_\mathbf{B}$) we have $\text{pr}|_{C_i(\underline{a})} = \text{pr}_i$, $i = 1, 2$ (resp. $\text{pr}|_{C_\mathbf{B}(\underline{a})} = \text{pr}_\mathbf{B}$). Let $\mathcal{G}_i$ (resp. $\mathcal{B}$) be the group of decktransformations with respect to the covering pr_i (resp. $\text{pr}_\mathbf{B}$). Then $\mathcal{G}_i = \Gamma_i/\Gamma_i(\underline{a}) \cong \Gamma_i/\Gamma(\underline{a}) \leq \mathcal{G}$ (resp. $\mathcal{B} = \Gamma_\mathbf{B}/\Gamma_\mathbf{B}(\underline{a}) \cong \Gamma_\mathbf{B}/\Gamma(\underline{a}) \leq \mathcal{G}$).

The following (well known) lemma 8.4 shows the geometric behaviour of the translates of cycles under decktransformations.

Lemma 8.4. *Let $\delta, \eta \in \mathcal{G}$. Then*

$$\delta C_{\mathbf{B}}(\underline{a}) \cap \eta C_{\mathbf{B}}(\underline{a}) = \emptyset \Leftrightarrow \delta \mathcal{B} \neq \eta \mathcal{B} \text{ in } \mathcal{G}/\mathcal{B}$$

Proof. $\Rightarrow$ Suppose there is a $g \in \mathcal{B}$ such that $\delta = \eta g$. Then $\delta C_{\mathbf{B}}(\underline{a}) = \eta g C_{\mathbf{B}}(\underline{a}) = \eta C_{\mathbf{B}}(\underline{a})$ and $\delta C_{\mathbf{B}}(\underline{a}) \cap \eta C_{\mathbf{B}}(\underline{a}) = \delta C_{\mathbf{B}}(\underline{a}) \neq \emptyset$. This is a contradiction.

$\Leftarrow$ For the converse assume that $\delta C_{\mathbf{B}}(\underline{a}) \cap \eta C_{\mathbf{B}}(\underline{a}) \neq \emptyset$. So there is an element $x \in \nu C_{\mathbf{B}}(\underline{a}) \cap C_{\mathbf{B}}(\underline{a}) \neq \emptyset$, for $\nu := \eta^{-1} \delta \in \mathcal{G}$. Let $y \in C_{\mathbf{B}}(\underline{a})$ such that $\nu(y) = x$. Since $C_{\mathbf{B}}(\underline{a})$ is connected and $\text{pr} : C_{\mathbf{B}}(\underline{a}) \rightarrow C_{\mathbf{B}}$ is a covering, we have a unique lift of the continuous map $\text{pr} : C_{\mathbf{B}}(\underline{a}) \rightarrow C_{\mathbf{B}}$ to a map $\nu' : C_{\mathbf{B}}(\underline{a}) \rightarrow C_{\mathbf{B}}(\underline{a})$ such that $\nu'(y) = x = \nu(y)$.

$$\begin{array}{ccc} & & C_{\mathbf{B}}(\underline{a}) \\ & \nearrow \nu' & \downarrow \text{pr} \\ C_{\mathbf{B}}(\underline{a}) & \xrightarrow{\text{pr}} & C_{\mathbf{B}} \end{array}$$

The lift is uniquely determined by his value in one point; so $\nu = \nu'$; hence ν' is a homeomorphism. But ν' stabilize $C_{\mathbf{B}}(\underline{a})$ and satisfies $\text{pr} \circ \nu' = \text{pr}$; thus $\nu = \nu' \in \mathcal{B}$. This is a contradiction and the lemma is proved. $\square$

Theorem 8.5. (CHEVALLEY) *Every algebraic subgroup of an algebraic group $\mathbf{G}$ arises as the stabilizer of a subspace in some finite-dimensional linear representation of $\mathbf{G}$; the subspace can even be taken to be one-dimensional.*

Proof. [Wat] 16.1.

A lattice Λ in a finite dimensional $\mathbb{Q}$ -vector space is a free $\mathbb{Z}$ -submodule of V of rank $\Lambda = \dim_{\mathbb{Q}} V$ (i.e Λ can be generated as $\mathbb{Z}$ -module by a $\mathbb{Q}$ -basis of V). Let $m \in \mathbb{Z}$ be an integer (resp. $\underline{a}$ an ideal in $\mathbb{Z}$). We denote the submodule of Λ generated by all mv , $v \in \Lambda$ (resp. mv , $m \in \underline{a}$, $v \in \Lambda$) by $m\Lambda$ (resp. $\underline{a}\Lambda$). For two elements $g, h \in GL(V)$, which leave Λ invariant and satisfy $(g - h)\Lambda \subset m\Lambda$ (resp. $(g - h)\Lambda \subset \underline{a}\Lambda$), we write $g \equiv h \pmod{m\Lambda}$ (resp. $\pmod{\underline{a}\Lambda}$). If $\Lambda = \mathbb{Z}^r$, for $r \in \mathbb{N}$, we briefly write $g \equiv h \pmod{m}$ (resp. $\pmod{\underline{a}}$).

Remember the following elementary fact concerning arithmetic groups (see e.g. [Milne, 28.9]).

Proposition 8.6 *Let $\Gamma \subset \mathbf{G}(\mathbb{Z})$ be an arithmetic subgroup and $\rho : \mathbf{G} \rightarrow \mathbf{GL}(V)$ be a rational representation of $\mathbf{G}$. Every lattice Λ' of V is contained in a lattice Λ , which is stable under Γ .*

Lemma 8.7 *Let $\Gamma \subset \mathbf{G}(\mathbb{Z})$ be an arithmetic subgroup and $\rho : \mathbf{G} \rightarrow \mathbf{GL}(V)$ be a rational representation of $\mathbf{G}$. Let Λ be a Γ -stable lattice in V and $m \in \mathbb{Z}$ be an integer. There is an ideal $\underline{a} = \underline{a}(m, \rho)$ in $\mathbb{Z}$, depending only on m and ρ , such that the following holds: If $\alpha = (\alpha_{ij}), \beta = (\beta_{ij}) \in \Gamma$ are two elements, then the property $\alpha_{ij} \equiv \beta_{ij} \pmod{\underline{a}}$, implies that $\rho(\alpha) \equiv \rho(\beta) \pmod{m\Lambda}$.*

Proof. (We follow ideas of [Milne, 28.7]). We choose a basis for Λ , so ρ becomes a homomorphism of algebraic groups $\rho : \mathbf{G} \rightarrow \mathbf{GL}_r$ and $\Lambda \cong \mathbb{Z}^r$. The entries of the matrix $\rho(g)$, $g = (g_{ij})_{i,j=1,\dots,N} \in \mathbf{G}$ are polynomials in the entries of g , i.e., there exists polynomials $P_{kl} \in \mathbb{Q}[X_{11}, \dots, X_{ij}, \dots, X_{NN}, \det(X_{ij})^{-1}]$, such that $\rho(g)_{kl} = P_{kl}(g_{ij})$.

After substitute the variables X_{ij} by $Y_{ij} + \delta_{ij}$, we get polynomials

$$Q_{kl} \in \mathbb{Q}[Y_{11}, \dots, Y_{ij}, \dots, Y_{NN}, \det(Y_{ij})^{-1}],$$

such that

$$\rho(g)_{kl} - \delta_{kl} = Q_{kl}(\dots, g_{ij} - \delta_{ij}, \dots),$$

with δ the Kronecker delta. Since $\rho(\mathbb{I}_N) = \mathbb{I}_r$, we have $0 = Q_{kl}(0)$; so the constant term of Q_{kl} is zero. Let M be the common denominator for the coefficients of the Q_{kl} ; so $MQ_{kl} \in \mathbb{Z}[Y_{11}, \dots, Y_{ij}, \dots, Y_{nn}, \det(Y_{ij})^{-1}]$. Now let $\underline{a} := Mm\mathbb{Z}$ be the ideal generated by Mm . If $\beta^{-1}\alpha \equiv \mathbb{I}_N \pmod{\underline{a}}$, then $Q_{kl}(\dots, (\beta^{-1}\alpha)_{ij} - \delta_{ij}, \dots) \in m\mathbb{Z}$. So $(\rho(\beta^{-1}\alpha) - \mathbb{I}_r)\mathbb{Z}^r \subset m\mathbb{Z}^r$. Since the lattice was Γ -stable, $(\rho(\alpha) - \rho(\beta))\mathbb{Z}^r \subset m\mathbb{Z}^r$. $\square$

Lemma 8.8 *Let Λ be a lattice in V , $\{v_1, \dots, v_r\}$ be a $\mathbb{Z}$ -basis of Λ and $d = \frac{d_1}{d_2} := \det(v_1, \dots, v_r)$, $d_1, d_2 \in \mathbb{Z}$. If $m \in \mathbb{Z}$ is coprime to d_1 and d_2 , then*

$$\dim_{\mathbb{Q}} V = \text{rank}_{\mathbb{Z}/m\mathbb{Z}} \Lambda / m\Lambda.$$

Proof. Let M denote the $r \times r$ -matrix with columns built from the vectors $v_1, \dots, v_r$ and M^* be the adjoint matrix of M , such that $M^*M = \det M \cdot \mathbb{I}_r = d \cdot \mathbb{I}_r$. The matrix M^* induces an isomorphism of $\mathbb{Z}/m\mathbb{Z}$ -modules

$$M^* : \Lambda / m\Lambda \longrightarrow d\mathbb{Z}^r / dm\mathbb{Z}^r, \quad v_i + m\Lambda \longmapsto M^*(v_i + m\Lambda) = de_i + dm\mathbb{Z}^r.$$

Since v_i (resp. e_i) is the i th column in M (resp. $\mathbb{I}_r$). But $m \subseteq \mathbb{Z}$ was chosen coprime to d_1 and d_2 ; so there is a $t \in \mathbb{Z}$ such that $td - 1 \in m\mathbb{Z}$;

$$d\mathbb{Z}^r / dm\mathbb{Z}^r \longrightarrow \mathbb{Z}^r / m\mathbb{Z}^r, \quad dv + dm\mathbb{Z}^r \longmapsto tdv + m\mathbb{Z}^r = v + m\mathbb{Z}^r$$

is an isomorphism of $\mathbb{Z}/m\mathbb{Z}$ -modules. $\square$

The following is actually [MiRa, Lemma 2.6 (first part)].

Lemma 8.9 *Let $\gamma \in \Gamma$ such that $\gamma \notin \Gamma_{\mathbf{B}}$ and $\eta_{\gamma} \in \mathcal{G}$ the corresponding decktransformation. Then there is an ideal $\underline{a} \subseteq \mathbb{Z}$ such that*

$$\eta_{\gamma}\mathcal{B} \neq \mathcal{B},$$

with $\mathcal{B} = \Gamma_{\mathbf{B}} / \Gamma_{\mathbf{B}}(\underline{a})$. Moreover, if $\underline{b} \subset \underline{a}$ is an ideal in $\mathbb{Z}$, the same holds true for $\underline{b}$ replacing $\underline{a}$.

Proof. We show that there is an ideal $\underline{a} \subseteq \mathbb{Z}$, such that $\gamma \not\equiv \beta \pmod{\underline{a}}$ for all $\beta \in \Gamma_{\mathbf{B}}$. This is sufficient, because, if η_{γ} is represented by $\gamma\gamma'$, for a $\gamma' \in \Gamma(\underline{a})$, then $\gamma\gamma' \equiv \gamma \pmod{\underline{a}}$.

By Chevalleys Theorem 8.5, we choose a rational representation $\rho : \mathbf{G} \rightarrow \mathbf{GL}(V)$ such that $\mathbf{B} = \{g \in \mathbf{G} \mid \rho(g)v = \lambda v, \text{ for a } \lambda \in \mathbb{Q}\}$ for a vector $v \in V$. If $\rho(\gamma)v = \lambda v$ for a scalar $\lambda \in \mathbb{Q}$, then by the equation above $\gamma \in \mathbf{B} \cap \Gamma = \Gamma_{\mathbf{B}}$. This is a contradiction to our assumption, so $v, \rho(\gamma)v$ are $\mathbb{Q}$ -linear independent in V . We complete this two vectors to a $\mathbb{Q}$ -basis $\{v, \rho(\gamma)v, w_3, \dots\}$ of V . Let $\Lambda' \subseteq V$ be the $\mathbb{Z}$ -lattice generated by this $\mathbb{Q}$ -basis of V . By 8.6, Λ' is contained in a Γ -stable lattice Λ . Let $\{v_1, \dots, v_r\}$ be a $\mathbb{Z}$ -basis of Λ and $d = \frac{d_1}{d_2} := \det(v_1, \dots, v_r)$, $d_1, d_2 \in \mathbb{Z}$. Let $m \in \mathbb{Z}$, coprime to d_1 and d_2 and choose $\underline{a} = \underline{a}(m, \rho)$ as in lemma 8.7. Now assume that there is a $\beta \in \Gamma_{\mathbf{B}}$, such that $\gamma \equiv \beta \pmod{\underline{a}}$. By 8.7 this implies that $\rho(\gamma) \equiv \rho(\beta) \pmod{m\Lambda}$. Hence $\rho(\gamma)v - \rho(\beta)v = \rho(\gamma)v - \lambda v \in m\Lambda$, for a $\lambda \in \mathbb{Q}$. This is a contradiction to 8.8. So we have proved the lemma. $\square$

8.2 Non G -invariance of special cycles

Assume now, that $\mathbf{G} = \text{Res}_{F/\mathbb{Q}} \mathbf{G}'$ is obtained from a F -algebraic group, via restriction of scalars and assume that $\mathbf{G}'$ contains no connected normal F -subgroups $\mathbf{N}'$ such that

$Res_{F/\mathbb{Q}}\mathbf{N}'(\mathbb{R})$ is compact. In particular this is true if $\mathbf{G}'$ is F -simple. By [Bo3, Thm 3.], this implies that an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q}) \cong \mathbf{G}'(F)$ is Zariski-dense in $\mathbf{G}'$.

Let σ' and τ' be two F -rational involutions of $\mathbf{G}'$ and $\tau' = \sigma' \circ \theta'$, where θ' is an F -rational involution that induce a Cartan involution on $G' = \mathbf{G}'(\mathbb{R})$. We obtain $\mathbb{Q}$ -rational involutions $\sigma = Res_{F/\mathbb{Q}}\sigma'$ and $\tau = Res_{F/\mathbb{Q}}\tau'$. Note that the fixed points satisfy $\mathbf{G}(\sigma) = Res_{F/\mathbb{Q}}(\mathbf{G}'(\sigma'))$ (resp. $\mathbf{G}(\tau) = Res_{F/\mathbb{Q}}(\mathbf{G}'(\tau'))$) and $\Gamma(\mu) = \Gamma_{\mathbf{G}(\mu)} \cong \Gamma \cap \mathbf{G}'(\mu')$, $\mu \in \{\sigma, \tau\}$.

For an ideal $\underline{a} \subset \mathbb{Z}$, let $\mathcal{G} = \Gamma/\Gamma(\underline{a})$, $\mathcal{G}_1 = \Gamma(\sigma)/\Gamma(\sigma)(\underline{a})$ and $\mathcal{G}_2 = \Gamma(\tau)/\Gamma(\tau)(\underline{a})$ be the finite groups of decktransformations of the coverings $X/\Gamma(\underline{a}) \rightarrow X/\Gamma$, $X(\sigma)/\Gamma(\sigma)(\underline{a}) \rightarrow X(\sigma)/\Gamma(\sigma)$ and $X(\tau)/\Gamma(\tau)(\underline{a}) \rightarrow X(\tau)/\Gamma(\tau)$.

Now we give an account of the proof of [MiRa, Thm. 2.1]

Theorem 8.10. (MILLSON, RAGHUNATHAN) *If all intersections of the special cycles $C(\sigma)$ and $C(\tau)$ in X/Γ are of positive multiplicity, then there is an ideal $\underline{a} \subseteq \mathbb{Z}$, such that for all ideals $\underline{b} \subset \underline{a}$, the class $D_{X/\Gamma(\underline{b})}^{-1}([C(\mu)(\underline{b})])$, $\mu \in \{\sigma, \tau\}$ can not be represented by a G -invariant differential form.*

Proof. It is sufficient to show that $\alpha := D_{X/\Gamma(\underline{b})}^{-1}([C(\sigma)(\underline{b})])$ can not be represented by a G -invariant differential form. I.e. we have to show that there is an $\underline{a}$, such that for every $\underline{b} \subset \underline{a}$, the homology class $[C(\sigma)(\underline{b})]$ is not invariant under all decktransformations (see 8.2). Note that it is sufficient to prove this only for one ideal $\underline{a} = \underline{b}$, because if $\underline{b} \subset \underline{a}$, we have a finite covering $X/\Gamma(\underline{b}) \rightarrow X/\Gamma(\underline{a})$, and $C_{\mathbf{B}}(\underline{b})$ projects to $C_{\mathbf{B}}(\underline{a})$. So if there is a $\gamma \in \Gamma$, such that $C_{\mathbf{B}}(\underline{a})$ is not invariant under $\eta_\gamma \in \Gamma/\Gamma(\underline{a})$, then the cycle $C_{\mathbf{B}}(\underline{b})$ is not invariant under $\eta_\gamma \in \Gamma/\Gamma(\underline{b})$.

Lemma 8.11 *There is an ideal $\underline{a} \subseteq \mathbb{Z}$ and a $\gamma \in \Gamma$, such that*

$$\eta_{\tau(\gamma)}\mathcal{G}_1 \neq \eta_\gamma\mathcal{G}_1.$$

Proof. Assume there is a $\gamma \in \Gamma$ such that $\mu := \tau(\gamma)\gamma^{-1} \notin \Gamma(\sigma)$. Then lemma 8.9 implies the desired.

Claim. There exist a $\gamma \in \Gamma$ such that $\tau(\gamma)\gamma^{-1} \notin \Gamma(\sigma)$.

Assume that $\tau(\gamma)\gamma^{-1} \in \Gamma(\sigma)$, for all $\gamma \in \Gamma$. Let $\phi : \mathbf{G}'(\mathbb{C}) \rightarrow \mathbf{G}'(\mathbb{C})$, $g \mapsto \tau'(g)g^{-1}$. By assumption we have $\phi(\Gamma) \subseteq \Gamma(\sigma)$ and since Γ is Zariski-dense in $\mathbf{G}'(\mathbb{C})$, it follows that $\phi(\mathbf{G}'(\mathbb{C})) \subseteq \mathbf{G}'(\sigma')(\mathbb{C})$. Let $\mathfrak{g}' = Lie(\mathbf{G}'(\mathbb{C}))$ and $\mathfrak{g}'_1 = Lie(\mathbf{G}'(\sigma')(\mathbb{C}))$, then $\phi(\mathfrak{g}') \subseteq \mathfrak{g}'_1$ and $\phi(X) = \tau'(X) - X$ for all $X \in \mathfrak{g}'$. Since $\theta'\sigma' = \tau'$ the restricted automorphism $\tau'|_{\mathfrak{g}'_1} = \theta'|_{\mathfrak{g}'_1}$, is the Cartan involution on $\mathfrak{g}'_1$. Let $\mathfrak{g}'_2 = Lie(\mathbf{G}'(\tau')(\mathbb{C}))$ and $\mathfrak{g}'_i = \mathfrak{k}'_i \oplus \mathfrak{p}'_i$, $i = 1, 2$ the Cartan decompositions. Let $x_i \in \mathfrak{p}'_i$, $i = 1, 2$ and $y = [x_1, x_2]$. Then $\phi(y) = [\tau'(x_1), \tau'(x_2)] - [x_1, x_2] = [\theta'(x_1), x_2] - [x_1, x_2] = -2y$, and $\sigma'([x_1, x_2]) = [x_1, \theta'(x_2)] = -y$. By the last equation we see that $y \notin \mathfrak{g}'_1$, so $-2y \notin \mathfrak{g}'_1$, but $y \in \phi(\mathfrak{g}')$ and this is a contradiction. $\square$

The following lemma finishes the proof of 8.10.

Lemma 8.12 *Let γ and $\underline{a}$ be as in lemma 8.11 and let $\eta = \eta_\gamma$ be the decktransformation of the covering $pr: X/\Gamma(\underline{a}) \rightarrow X/\Gamma$ represented by γ . Then $C(\sigma)(\underline{a})$ and $\eta C(\sigma)(\underline{a})$ are not homologous.*

Proof. The intersection number of $[C(\sigma)(\underline{a})]$ and $[C(\tau)(\underline{a})]$ is positive since every intersection of $C(\sigma)(\underline{a})$ and $C(\tau)(\underline{a})$ lies over an intersection of $C(\sigma)$ and $C(\tau)$. A decktransformation $\eta \in \mathcal{G}$ satisfies $pr \circ \eta = pr$, so $\eta C(\sigma)(\underline{a})$ projects to $C(\sigma)$. Since $C(\sigma)$ has no bound, $\eta C(\sigma)(\underline{a})$

has no bound too. Thus it is sufficient to show that the intersection number of $[\eta C(\sigma)(\underline{a})]$ and $[C(\tau)(\underline{a})]$ is zero (then, since $[C(\sigma)(\underline{a})][C(\tau)(\underline{a})] > 0$ it follows $[C(\sigma)(\underline{a})] \neq [\eta C(\sigma)(\underline{a})]$). The required statement follows if we show that $\eta C(\sigma)(\underline{a}) \cap C(\tau)(\underline{a}) = \emptyset$ (then by Proposition 5.2 (2) $[\eta C(\sigma)(\underline{a})][C(\tau)(\underline{a})] = 0$).

Assume that $\eta C(\sigma)(\underline{a}) \cap C(\tau)(\underline{a}) \neq \emptyset$. Then there is a $x \in \eta C(\sigma)(\underline{a})$ such that $\tau(x) = x$ and therefore $x \in \tau(\eta C(\sigma)(\underline{a})) \cap \eta C(\sigma)(\underline{a}) \neq \emptyset$. Thus we have to prove that

$$\tau(\eta C(\sigma)(\underline{a})) \cap \eta C(\sigma)(\underline{a}) = \emptyset.$$

W.l.o.g. $\Gamma(\underline{a})$ is σ and τ stable (if not, we pass to a subgroup of finite index). Let $\varepsilon : \mathcal{G} \rightarrow \mathcal{G}$ be the group automorphism induced by τ .

Let $x\Gamma(\sigma)(\underline{a}) \in C(\sigma)(\underline{a})$. By the short calculation

$$\begin{aligned} \tau(\eta_\gamma(x\Gamma(\sigma)(\underline{a}))) &= \tau(x\gamma\Gamma(\sigma)(\underline{a})) \\ &= \tau(x)\tau(\gamma)\Gamma(\sigma)(\underline{a}) \\ &= \eta_{\tau(\gamma)}(\tau(x)\Gamma(\sigma)(\underline{a})) \\ &= \varepsilon(\eta_\gamma)(\tau(x)\Gamma(\sigma)(\underline{a})), \text{ (per definition of } \varepsilon) \end{aligned}$$

we see that $\tau(\eta_\gamma(C(\sigma)(\underline{a}))) = \varepsilon(\eta_\gamma)(C(\sigma)(\underline{a}))$ (since $X(\sigma)$ is τ stable, $\tau(x)\Gamma(\sigma)(\underline{a}) \in C(\sigma)(\underline{a})$).

The element γ was chosen as in lemma 8.11, so $\eta_\gamma \mathcal{G}_1 \neq \eta_{\tau(\gamma)} \mathcal{G}_1 = \varepsilon(\eta_\gamma) \mathcal{G}_1$. By lemma 8.4, it follows that $\varepsilon(\eta_\gamma)(C(\sigma)(\underline{a})) \cap \eta_\gamma(C(\sigma)(\underline{a})) = \emptyset$. $\square$

Remark. Unfortunately, the proof of [MiRa] works just for $\mathbb{Q}$ -rational involutions σ and τ obtained by F -rational involutions σ' and τ' using restriction of scalars, where $\tau' = \sigma' \circ \theta'$ and θ' is an F -rational involution that induce a Cartan involution on G' . For arbitrary $\mathbb{Q}$ -rational automorphisms of finite order, the theorem holds if we demand a sufficient condition (*) on the automorphisms σ and τ to be discussed in the following.

Let σ (resp. τ) arbitrary $\mathbb{Q}$ -rational automorphisms of $\mathbf{G}$, of finite order s (resp. t) such that σ and τ commute with each other. Further, let θ be a Cartan involution of the Lie algebra $\mathfrak{g}_0$ of G , which commutes with σ and with τ . The Cartan decomposition of $\mathfrak{g}_0$, corresponding to θ is denoted by

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

The complexification of $\mathfrak{k}_0$ (resp. $\mathfrak{p}_0$) is denoted by $\mathfrak{k}$ (resp. $\mathfrak{p}$).

Proposition 8.13 *Let σ (resp. τ) of arbitrary $\mathbb{Q}$ -rational automorphisms of $\mathbf{G}$, of finite order s (resp. t) such that σ and τ commute with each other. If all intersections of the special cycles $C(\sigma)$ and $C(\tau)$ in X/Γ are of positive multiplicity, and the condition*

$$(*) \quad \mathfrak{k}(\sigma) + \mathfrak{k}(\tau) \subsetneq \mathfrak{k}$$

is fulfilled, then there is an ideal $\underline{a} \subseteq \mathbb{Z}$, such that $D_{X/\Gamma(\underline{a})}^{-1}([C(\mu)(\underline{a})])$, $\mu \in \{\sigma, \tau\}$ can not be represented by a G -invariant differential form.

Proof. The proof works exactly as before, except the claim in the proof of lemma 8.8. As replacement we show $\phi(\mathfrak{g}) \subseteq \mathfrak{g}(\sigma) \Rightarrow \mathfrak{k} = \mathfrak{k}(\sigma) + \mathfrak{k}(\tau)$ where $\phi = \tau - \text{id}$. Suppose that $\mathfrak{g}(\sigma) \supseteq \phi(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{g}(\tau)$, where the quotient is considered as a vector space. Therefore

$$\begin{aligned} \phi(\mathfrak{g}) \subseteq \mathfrak{g}(\sigma) &\Leftrightarrow \mathfrak{g} = \mathfrak{g}(\sigma) + \mathfrak{g}(\tau) \\ &\Rightarrow \mathfrak{k} = \mathfrak{k}(\sigma) + \mathfrak{k}(\tau). \end{aligned}$$

$\square$

8.3 Criteria for non G -invariance in the middle degree

Our first sufficient criterion for non G -invariance is the following.

Lemma 8.14 *Let $\dim X = m = 2q$ even and $\dim Y = q$. If the self-intersection number $[C_{\mathbf{B}}][C_{\mathbf{B}}]$ is non zero, then there is an ideal $\underline{a} \subseteq \mathbb{Z}$, such that for all ideals $\underline{b} \subseteq \underline{a}$, the Poincaré dual class*

$$\alpha := D_{X/\Gamma(\underline{b})}^{-1}[C_{\mathbf{B}}(\underline{b})] \in H^{m-q}(X/\Gamma(\underline{b}); \mathbb{R})$$

is a non G -invariant classes. Moreover, we obtain exactly $|\mathcal{G}|/|\mathcal{B}|$ linear independent homology classes in $H_q(X/\Gamma(\underline{b}); \mathbb{R})$ by $[\eta_i(C_{\mathbf{B}}(\underline{b}))]$, $i = 1, \dots, l$. Here $\eta_1, \dots, \eta_l$ is a representing system of the cosets $\mathcal{G}/\mathcal{B}$.

Proof. We prove this just for $\underline{b} = \underline{a}$. If α is a G -invariant class, then $[C_{\mathbf{B}}(\underline{a})]$ is invariant under η , for every $\eta \in \mathcal{G}$ (see 8.2). So we have to show that there is a $\gamma \in \Gamma$ such that $[C_{\mathbf{B}}(\underline{a})]$ is not invariant under $\eta_\gamma := \gamma\Gamma(\underline{a}) \in \mathcal{G}$. Let $\gamma \in \Gamma$ such that $\gamma \notin \Gamma_{\mathbf{B}}$, then by lemma 8.9, we have $\eta_\gamma \mathcal{B} \neq \mathcal{B}$, for an ideals $\underline{a} \subseteq \mathbb{Z}$. By lemma 8.4 this is equivalent to say that $\eta_\gamma(C_{\mathbf{B}}(\underline{a})) \cap C_{\mathbf{B}}(\underline{a}) = \emptyset$. By proposition 5.2 (2), this shows that $[\eta_\gamma(C_{\mathbf{B}}(\underline{a}))][C_{\mathbf{B}}(\underline{a})] = 0$. But $[C_{\mathbf{B}}(\underline{a})][C_{\mathbf{B}}(\underline{a})] \neq 0$, by the corollary to Theorem 5.21; hence $[\eta_\gamma(C_{\mathbf{B}}(\underline{a}))] \neq [C_{\mathbf{B}}(\underline{a})]$. More general, if $\eta_1, \dots, \eta_l$ represents the cosets of $\mathcal{G}/\mathcal{B}$, then

$$[\eta_i C_{\mathbf{B}}(\underline{a})][\eta_j C_{\mathbf{B}}(\underline{a})] = [C_{\mathbf{B}}(\underline{a})][C_{\mathbf{B}}(\underline{a})] \neq 0.$$

On the other hand, $\eta_i \mathcal{B} \neq \eta_j \mathcal{B}$, for $i \neq j$; so, by 8.4 and 5.2 (2), $[\eta_i C_{\mathbf{B}}(\underline{a})][\eta_j C_{\mathbf{B}}(\underline{a})] = 0$, for $i \neq j$; hence

$$[\eta_i C_{\mathbf{B}}(\underline{a})][\eta_j C_{\mathbf{B}}(\underline{a})] = \delta_{ij}[C_{\mathbf{B}}(\underline{a})][C_{\mathbf{B}}(\underline{a})],$$

where δ_{ij} is the usual Kronecker delta; i.e. they form an orthogonal basis of their own linear span. In particular they are linear independent. $\square$

Remark. If q is odd, the self-intersection number is always zero, which can be seen easily, by $\omega \wedge \omega = 0$, for every q -form ω , if q is odd. So the criterion only works if q is even.

Corollary 8.15 *Let $\dim X = m = 2q$ even and $\dim Y = q$. If the self-intersection number $[C_{\mathbf{B}}][C_{\mathbf{B}}]$ is non zero, then there is a constant c and an ideal $\underline{a} \subseteq \mathbb{Z}$, such that for all ideals $\underline{b} \subseteq \underline{a}$, the q -th Betti number $b_q(\Gamma(\underline{b})) := \dim H_q(X/\Gamma(\underline{b}), \mathbb{R})$, satisfies*

$$b_q(\Gamma(\underline{b})) \geq c \frac{\text{vol}(X/\Gamma(\underline{b}))}{\text{vol}(Y/\Gamma_{\mathbf{B}}(\underline{b}))}.$$

Proof. Since $X/\Gamma(\underline{b}) \rightarrow X/\Gamma$ (resp. $Y/\Gamma_{\mathbf{B}}(\underline{b}) \rightarrow Y/\Gamma_{\mathbf{B}}$) is a $|\mathcal{G}|$ -fold covering (resp. $|\mathcal{B}|$ -fold), we have that $\text{vol}(X/\Gamma(\underline{b})) = |\mathcal{G}|\text{vol}(X/\Gamma)$ (resp. $\text{vol}(Y/\Gamma_{\mathbf{B}}(\underline{b})) = |\mathcal{B}|\text{vol}(Y/\Gamma_{\mathbf{B}})$). Define $c := \text{vol}(Y/\Gamma_{\mathbf{B}})/\text{vol}(X/\Gamma)$ and use lemma 8.14. $\square$

Remark. X. Xue in [X] has also considered the growth of Betti numbers in the case of hyperbolic arithmetically defined manifolds by studying intersection numbers of cycles.

Let i_u as in 6.9, the embedding $i_u : Y_u \hookrightarrow X_u$ and i the embedding $i : Y/\Gamma_{\mathbf{B}} \hookrightarrow X/\Gamma$. The second sufficient criterion for non G -invariance is the following.

Lemma 8.16 *Let $\dim X = m = 2q$ even and $\dim Y = q$. Further let $\alpha := D_{X/\Gamma}^{-1}i_*[Y/\Gamma_{\mathbf{B}}] \in H^{m-q}(X/\Gamma; \mathbb{R})$ non trivial. If i_u^* is trivial and $\dim H^q(X_u; \mathbb{C}) = 1$, then α is non G -invariant.*

Proof. Assume that $\alpha \neq 0$ is G -invariant. Since $i^*\alpha \in H^q(Y/\Gamma_{\mathbf{B}})$ and $\dim Y/\Gamma_{\mathbf{B}} = q$, we have a well defined number

$$\begin{aligned} \langle i^*\alpha, [Y/\Gamma_{\mathbf{B}}] \rangle &= \langle \alpha, i_*[Y/\Gamma_{\mathbf{B}}] \rangle \\ &= \langle \alpha, D_{X/\Gamma}\alpha \rangle \\ &= \langle \alpha, \alpha \cap [X/\Gamma] \rangle \\ &= \langle \alpha \cup \alpha, [X/\Gamma] \rangle. \end{aligned}$$

Since $\alpha \neq 0$ is G -invariant and $\dim H^q(X_u; \mathbb{C}) = 1$, the class $\alpha \cup \alpha$ could not be zero and generates $H^{2q}(X/\Gamma; \mathbb{C})$; hence $\langle i^*\alpha, [Y/\Gamma_{\mathbf{B}}] \rangle \neq 0$. So $i^*\alpha \neq 0$. This implies that $i_u^* \neq 0$. $\square$

Remark. This criterion works, like 8.14, just in the case where q is even; because if M is a compact manifold of dimension $2q$ and $\dim H^q(M; \mathbb{C}) = 1$, then a generator β of $H^q(M; \mathbb{C})$ has non trivial self intersection number; i.e. $\langle \beta \cup \beta, [M] \rangle \neq 0$, since the intersection form is regular; but for odd q $\beta \cup \beta = 0$.

Note that the converse of 8.16 is false: As we will see later, the cycle C_c is non G -invariant, but i_u^* is also non trivial in that case, by [T].

8.4 Non G -invariance in the case G_2

Now let $\mathbf{G}$ be the $\mathbb{Q}$ -algebraic group of type G_2 as constructed in §3.2 resp. §4.1. We use the notation of §4 freely.

8.17 In the case of the cycles C_c (resp. C_{ab}), we want to use the criterion 8.14 for non G -invariance in the middle. So we have to compute the self-intersection number. Herefore we recall some facts about a certain bundle.

The *anti-canonical* or *anti-tautological (complex) line bundle* $\gamma_1^n = \gamma^n(\mathbb{C}^{n+1})$ over the projective space $\mathbb{C}P^n$, can be defined as follows:

Let $l \subset \mathbb{C}^{n+1}$ be a complex line through the origin; hence l is an element in $\mathbb{C}P^n$. The total space $E = E(\gamma_1^n)$ of γ_1^n is $E = \{(l, w) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid w \in l^\perp\}$, where $l^\perp \subset \mathbb{C}^{n+1}$ is the subspace of complex dimension n , of all vectors which are orthogonal to l , with respect to the standard Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{n+1}$. The projection is

$$p : E \rightarrow \mathbb{C}P^n, (l, w) \mapsto l.$$

The anti-tautological complex line bundle can be viewed as a $2n$ -dimensional real vector bundle over the real $2n$ -dimensional manifold $\mathbb{C}P^n$. Recall that the complex projective n -space $\mathbb{C}P^n$ is diffeomorphic to $U(n+1)/(U(1) \times U(n))$.

Lemma 8.18. *The anti-canonical complex line bundle γ_1^n is isomorphic to*

$$U(n+1) \times_{(U(1) \times U(n))} \mathbb{C}^n \cong \gamma_1^n,$$

where the action of $U(1) \times U(n)$ on $\mathbb{C}^n$ is the action of $U(1) \times U(n)$ on $\{0\} \times \mathbb{C}^n$.

Proof. For $g \in U(n+1)$, let l_g be the line through the origin and the vector ge_1 , where $e_1 = (1, 0, \dots, 0) \in \mathbb{C}^{n+1}$. Let

$$f : U(n+1) \times_{(U(1) \times U(n))} \mathbb{C}^n \rightarrow \gamma_1^n, [g, v] \mapsto (l_g, gv).$$

Since g is unitary with respect to $\langle, \rangle$, we have $\langle ge_1, gv \rangle = \langle e_1, v \rangle = 0$; so the element (l_g, gv) is in E . Obviously, f is well-defined. If $(l_g, g.v) = (l_h, h.w)$, for $g, h \in U(n+1)$, $v, w \in \{0\} \times \mathbb{C}^n$, then $ge_1 \in l_g = l_h \ni he_1$; hence $g^{-1}h$ stabilize $\mathbb{C}e_1$. So $h = gk$, for $k \in U(1) \times U(n)$ and $g.v = h.w = gk.w$. Thus $k^{-1}.v = w$ and f is injective. Because $U(n+1)$ acts transitive on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, it acts transitive on the lines through the origin in $\mathbb{C}^{n+1}$. I.e. if l is a line through the origin in $\mathbb{C}^{n+1}$, there is a $g \in U(n+1)$, such that $l_g = l$ and $l_g = l_{gk}$, for all $k \in U(1) \times U(n)$. If $w \in l^\perp$, then $g^{-1}w =: v \in g^{-1}l_g^\perp = \mathbb{C}e_1^\perp = \{0\} \times \mathbb{C}^n$; hence $f([g, v]) = (l, w)$; i.e. f is surjective. $\square$

In particular, γ_1^2 is isomorphic to $SU(3) \times_{U(2)} \mathbb{C}^2$.

For short, we denote the cup product $\alpha \cup \beta$ of two cohomology classes α, β , simply by $\alpha\beta$. Let $\varepsilon^n := \mathbb{C}P^n \times \mathbb{C}^{n+1}$ be the trivial bundle.

Lemma 8.19. *The Euler class of the anti-tautological bundle γ_1^n satisfies*

$$e(\gamma_1^n) = (-1)^n e(\gamma_n^1)^n,$$

where γ_n^1 is the tautological bundle (see 5.23).

Proof. First note that the Whitney sum of the tautological bundle γ_n^1 with the anti-tautological bundle is isomorphic to ε^n ; i.e. the map $(l, v) \times (l, w) \mapsto (l, v + w)$ induces an isomorphism $\gamma_n^1 \oplus \gamma_1^n \xrightarrow{\sim} \varepsilon^n$. Then, by the Product Theorem for the Chern classes $c_i(\gamma_n^1)$ and $c_j(\gamma_1^n)$ we have for the k -th Chern class of ε^n sum

$$0 = c_k(\varepsilon^n) = c_k(\gamma_n^1 \oplus \gamma_1^n) = \sum_{i+j=k} c_i(\gamma_n^1) c_j(\gamma_1^n).$$

By definition of the Chern class, $c_0(\gamma_n^1) = 1$. The bundle γ_n^1 has complex dimension 1, so $c_i(\gamma_n^1) = 0$, for $i > 1$. Hereby the above equation reduces to the simple relation $0 = c_k(\gamma_1^n) + c_1(\gamma_n^1) c_{k-1}(\gamma_1^n)$, $0 \leq k \leq n$. If we repeat, this relation, starting by $k = 1$, we get $c_k(\gamma_1^n) = (-1)^k c_1(\gamma_n^1)^k$, $0 \leq k \leq n$. But the top Chern class of a vector bundle is equal to the Euler class; hence $e(\gamma_1^n) = (-1)^n e(\gamma_n^1)^n$. $\square$

Remark. By [Milnor Stasheff, p.160], the Euler class $e(\gamma_n^1) \in H^2(\mathbb{C}P^n; \mathbb{Z})$, of the real bundle γ_n^1 , generates $H^2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$. Moreover, $e(\gamma_n^1)$ generates $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$ as a ring; i.e. the class $e(\gamma_n^1)^i$ generates $H^{2i}(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$ for $i = 1, \dots, n$ and $H^{2i-1}(\mathbb{C}P^n; \mathbb{Z}) = 0$ for $i = 1, \dots, n$. Thus

$$\langle e(\gamma_1^n), [\mathbb{C}P^n] \rangle = (-1)^n \langle e(\gamma_n^1)^n, [\mathbb{C}P^n] \rangle = \pm 1.$$

Now we are able to use 8.14.

Proposition 8.20 *Let C_c (resp. C_{ab}) the cycle resulting from $\mathbf{G}'(c)$ (resp. $\mathbf{G}'(ab)$) (as in §4.2). Then the self-intersection number of C_c (resp. C_{ab}) is non zero. More precisely,*

$$|[C_c][C_c]| = \frac{\text{vol}(G_c/\Gamma_c)}{2^3 3^2 \pi^2}, \text{ resp. } |[C_{ab}][C_{ab}]| = \frac{\text{vol}(G_{ab}/\Gamma_{ab})}{2^3 3^2 \pi^2}.$$

In particular, there is an ideal $\underline{a} \subset \mathbb{Z}$, such that for all ideals $\underline{b} \subset \underline{a}$ the cycles $C_c(\underline{b})$ (resp. $C_{ab}(\underline{b})$) represents non G -invariant classes.

Note that (possible after rearranging the orientation on X/Γ) the self-intersection number of C_c can be assumed to be $[C_c][C_c] = \frac{\text{vol}(G_c/\Gamma_c)}{2^3 3^2 \pi^2}$.

Proof. We prove the statement just for C_c . By theorem 5.21, $[C_c][C_c] = \langle e(\tilde{\eta}_u), [Z_u] \rangle \cdot c(e)$. Let $G_{c,u}$ be the compact dual of G_c . In 5.18, we have fixed a Haar measure ω_u on $H_{u,\gamma} = G_{c,u}$. In [AY], the authors computed the volume of a compact symmetric spaces G_u/K . Their measure on G_u is induced by the negative of the Killing form of $Lie(G_u)$ and their measure on K is induced by the restriction of the negative of the Killing form of $Lie(G_u)$ to K . Since in 5.18, we have chosen a Haar measure ν on $K_{x,\gamma} \subset H_{u,\gamma}$, having mass 1, the volume with respect to our chosen measure is $vol(G_{c,u}) = vol(SU(3)) = vol_{AY}(SU(3)/S(U(2) \times U(1))) = 2^3 3^2 \pi^2$, where vol_{AY} is the volume used and computed in [AY]. So $c(e) = \frac{vol(G_c/\Gamma_c)}{2^3 3^2 \pi^2}$. Like in 5.22 we will now show that $\langle e(\tilde{\eta}_u), [Z_u] \rangle = \pm 1$.

A $\mathbb{R}$ -basis of $\mathfrak{p}'$ is given by

$$\{d_{a,ab}, d_{a,c}, d_{b,ab}, d_{b,c}, d_{ab,ac}, d_{ab,bc}, d_{c,ac}, d_{c,bc}\}.$$

Evaluating of this derivations on c shows that

$$\{b^2 d_{a,c} - d_{ab,bc}, 2c^2 d_{b,ab} + b^2 d_{c,ac}, a^2 d_{b,c} + d_{ab,ac}, 2c^2 d_{a,ab} - a^2 d_{c,bc}\}$$

is a $\mathbb{R}$ -basis of $\mathfrak{p}'(c) = \{d \in \mathfrak{p}' \mid d(c) = 0\}$. For a derivation $d \in \mathfrak{p}'$, we denote by $\check{d}$ the coset of d in $\mathfrak{p}'/\mathfrak{p}'(c)$. Hence the cosets

$$S := \{\check{d}_{a,c}, \check{d}_{c,ac}, \check{d}_{b,c}, \check{d}_{c,bc}\}$$

forms a $\mathbb{R}$ -basis of $\mathfrak{p}'/\mathfrak{p}'(c)$.

Let $K_c = K \cap G_c$ and $K'_c = K' \cap G'_c$. Note that $Z_u = K_c \setminus G_{c,u} \cong S(U(2) \times U(1)) \setminus SU(3) \cong U(2) \setminus SU(3) \cong \mathbb{C}P^2$. Precisely, $K_c \cong K'_c \times G'_{c,u} \times \cdots \times G'_{c,u}$. Let $g \in K'_c = \mathbf{G}'(c, (c, ab)_{\mathbb{R}})(\mathbb{R}) \cong S(U(2) \times U(1)) \cong U(2)$. The action of $g \in K'_c$ on $\mathfrak{G} \otimes_F \mathbb{R}$ is given by the action of g on the elements a, b . We can assume that these action is given as follows

$$\begin{aligned} g(a) &= (z_{11} + z_{12}c)a + (z_{21} + z_{22}c)b, \\ g(b) &= (z_{31} + z_{32}c)a + (z_{41} + z_{42}c)b, \end{aligned}$$

where $z_{ij} \in \mathbb{R}$.

The action of $g \in K'_c$ on $\mathfrak{p}'/\mathfrak{p}'(c)$ with respect to the basis S is

$$[g] = \begin{pmatrix} z_{11} & z_{12}c^2 & z_{31} & z_{32}c^2 \\ z_{12} & z_{11} & z_{32} & z_{31} \\ z_{21} & z_{22}c^2 & z_{41} & z_{42}c^2 \\ z_{22} & z_{12} & z_{42} & z_{41} \end{pmatrix}.$$

So we can identify the $\mathbb{R}$ -vectorspace $\mathfrak{p}'/\mathfrak{p}'(c)$ with the $\mathbb{R}$ -vectorspace $\mathbb{R}(c)^2$, via

$$\check{d}_{a,c} \mapsto (1, 0), \check{d}_{c,ac} \mapsto (c, 0), \check{d}_{b,c} \mapsto (0, 1), \check{d}_{c,bc} \mapsto (0, c),$$

such that the action of K'_c becomes

$$K'_c \times \mathbb{R}(c)^2 \rightarrow \mathbb{R}(c)^2, (g, v) \mapsto \begin{pmatrix} z_{11} + z_{12}c & z_{31} + z_{32}c \\ z_{21} + z_{22}c & z_{41} + z_{42}c \end{pmatrix} v.$$

Let $\delta \in \mathbb{R}$, $\delta > 0$, such that $-\delta^2 = c^2$. The isomorphism $\mathbb{R}(c) \xrightarrow{\sim} \mathbb{C}$, $c \mapsto i\delta$, shows that the action of $K'_c \cong U(2)$ becomes

$$U(2) \times \mathbb{C}^2 \rightarrow \mathbb{C}^2, (g, v) \mapsto gv.$$

Hence we have isomorphisms of bundles

$$\tilde{\eta}_u \cong SU(3) \times_{U(2)} \mathbb{C}^2 \cong \gamma_1^2$$

and the Euler number is $\langle e(\tilde{\eta}_u), [Z_u] \rangle = \langle e(\gamma_1^2), [\mathbb{C}P^2] \rangle = \pm 1$.

□

Corollary 8.21 *There is a constant t and an ideal $\underline{a}$, such that for all ideals $\underline{b} \subset \underline{a}$, the 4-th Betti number $b_4(\Gamma(\underline{b}))$, satisfies*

$$b_4(\Gamma(\underline{b})) \geq t \frac{\text{vol}(X/\Gamma(\underline{b}))}{\text{vol}(X_c/\Gamma_c(\underline{b}))}.$$

8.22 Now the G -invariance and the homological non triviality of the cycles listed in §4.2 is given as follows:

cycle C	dim C	$D_{X/\Gamma}^{-1}[C]$ is G -invariant	C is homological non trivial	type of corresp. representation
C_a	5	no	yes	non-tempered
$C_{a,b}$	3	no	yes	non-tempered
C_b	5	no	yes	non-tempered
$C_{b,a}$	3	no	yes	non-tempered
$C(\sigma)$	4	no	yes	discrete series
$C(\sigma\theta)$	4	no	yes	discrete series
C_c	4	no	yes	discrete series
C_{ab}	4	no	yes	discrete series

The facts of the G -invariance is clear in degree 3 and 5, since the trivial representation does not occur. In degree 4, 8.20 implies that the cycles C_c and C_{ab} give non G -invariant classes. The cycles $C(\sigma)$ and $C(\sigma\theta)$ give non G -invariant classes; this can be seen in three different ways:

1. The group Γ can be chosen such that, the intersection numbers to the connected components of $C(\sigma) \cap C(\sigma\theta)$ are all positive (see the proof of [RoSch, 4.11] and rearranging the orientation on X/Γ if necessary). So [MiRa] (see §8.2) implies that these cycles are not G -invariant.
2. One can show that the self-intersection number of $C(\sigma)$ (resp. $C(\sigma\theta)$) is non zero; so we can use criterion 8.14.
3. The map i_u^* is trivial by [T]; so we can use criterion 8.16.

Appendix A

A.1 Characteristic classes

We repeats a little bit about some certain characteristic classes, which can be found e.g. in [Hus] or [Milnor Stasheff].

A.1 Let ξ be a real n -dimensional vector bundle over the base space B , with total space $E = E(\xi)$ and projection map $p : E \rightarrow B$. For short we write $\xi = (E, p, B)$. For the bundle ξ , let $E_0 = E_0(\xi)$ be the open subset of non zero vectors in E and let $F_{b,0}$ the set of all non-zero vectors in the fiber F_b over $b \in B$. Clearly $F_{b,0} = E_0 \cap F_b$. For $b \in B$, let $j_b : (F_b, F_{b,0}) \hookrightarrow (E, E_0)$, be the inclusion of pairs.

Definition. An *orientation* for ξ is a choice of an orientation of each fiber $F_b = p^{-1}(b)$, $b \in B$, of ξ , such that the following local compatibility condition holds:

For every $b_0 \in B$, there exist a neighborhood $N \subset B$, and a homeomorphism

$$h : N \times \mathbb{R}^n \rightarrow p^{-1}(N),$$

such that for each $b \in N$, the correspondence $x \mapsto h(b, x)$, is an orientation preserving isomorphism between the vector space $\mathbb{R}^n$ and the fiber $F_b = p^{-1}(b)$.

In terms of cohomology, this means that to each point $b \in B$, there is assigned a generator $u_b \in H^n(F_b, F_{b,0}; \mathbb{Z}) \cong \mathbb{Z}$, such that there exist a neighborhood $N \subset B$ of b and a cohomology class $u_N \in H^n(p^{-1}(N), p^{-1}(N) \cap E_0; \mathbb{Z})$, which satisfies

$$j_b^*(u_N) = u_b.$$

An *oriented vector bundle* is a pair consisting of a vector bundle and an orientation on the bundle.

Theorem A.2. *Let ξ be an oriented n -dimensional vector bundle. Then the following holds:*

1. For $i < n$, there is the equation $H^i(E, E_0; \mathbb{Z}) = 0$.
2. The group $H^n(E, E_0; \mathbb{Z})$ contains a unique cohomology class u , such that $j_b^*(u)$ is a fixed generator of $H^n(F_b, F_{b,0}; \mathbb{Z})$.
3. The function $H^i(B; \mathbb{Z}) \rightarrow H^{i+n}(E, E_0; \mathbb{Z})$, $a \mapsto p^*(a) \cup u$, is an isomorphism.

The inclusion $j : (E, \emptyset) \hookrightarrow (E, E_0)$ induces a morphism $j^* : H^i(E, E_0; \mathbb{Z}) \rightarrow H^i(E; \mathbb{Z})$. The projection $p : E \rightarrow B$ induces a morphism $p^* : H^i(E; \mathbb{Z}) \rightarrow H^i(B; \mathbb{Z})$, which is an isomorphism.

Definition. The *Euler class* $e(\xi) \in H^n(B; \mathbb{Z})$ of a real oriented vector bundle ξ over B , is $p^{*-1}j^*(u)$.

A.3 Properties of Euler classes.

- (Naturality) If $f : B \rightarrow B'$ is a homeomorphism, induced by an orientation preserving bundle map $\xi \rightarrow \xi'$, between oriented bundles, then $e(\xi) = f^*e(\xi')$. In particular, the Euler numbers $\langle e(\xi'), [B'] \rangle = \langle e(\xi'), f_*[B] \rangle = \langle f^*e(\xi'), [B] \rangle = \langle e(\xi), [B] \rangle$, agree.
- If we reverse the orientation of ξ , then the Euler class $e(\xi)$ changes sign.
- If the dimension n of ξ , is odd, then $2e(\xi) = 0$ in $H^n(B; \mathbb{Z})$; in particular if the cohomology group of view has no torsion (e.g. if the coefficients are a field of characteristic zero), then $e(\xi) = 0$.
- The Euler class of the Whitney sum is given by $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$.
- If ξ has a everywhere-non-zero cross section, then $e(\xi) = 0$; this holds in particular, if ξ is the trivial bundle of dimension $n \geq 1$.

A.4 Note that a n -dimensional complex vector bundle ξ , viewed as a real $2n$ -dimensional vector bundle $\xi_{\mathbb{R}}$, is orientable.

Let $\xi = (E, p, B)$ be a n -dimensional complex vector bundle and let E_0 be the non-zero vectors in the total space E . Restricting p to E_0 , we obtain a map $p_0 = p|_{E_0} : E_0 \rightarrow B$.

One can construct a new $(n-1)$ -dimensional vector bundle $\xi' = (E', p', E_0)$ with base space E_0 as follows:

A point in E_0 is specified by a fiber F of ξ together with a non-zero vector v in that fiber; denoted by (F, v) . The fiber of ξ' over (F, v) is defined as the quotient vector space $F/\mathbb{C}v$. (Note that we can do this construction inductively; i.e. we can construct a $(n-2)$ -dimensional vector bundle $\xi'' = (E'', p'', E'_0)$ with base space E'_0 as above, a $(n-3)$ -dimensional vector bundle ξ''' and so on.)

It turns out that $p_0^* : H^{2i}(B; \mathbb{Z}) \rightarrow H^{2i}(E_0; \mathbb{Z})$ is an isomorphism for $i < n$.

Definition. The *Chern classes* $c_i(\xi) \in H^{2i}(B; \mathbb{Z})$ are defined, by induction on the complex dimension n of ξ , as follows: The top Chern class $c_n(\xi)$ is the Euler class $e(\xi_{\mathbb{R}})$. For $0 < i < n$, we define

$$c_i(\xi) := p_0^{-1}c_i(\xi').$$

For $i > n$, the Chern classes $c_i(\xi)$ are defined to be zero. Further $c_0(\xi) = 1$.

The formal sum $c(\xi) := 1 + c_1(\xi) + \cdots + c_n(\xi)$ in the ring $H^*(B; \mathbb{Z})$ is called the *total Chern class* of ξ .

A.5 Properties of (total) Chern classes.

- (Naturality) If $f : B \rightarrow B'$ is induced by a bundle map $\xi \rightarrow \xi'$, between complex n -dimensional bundles, then $c_i(\xi) = f^*c_i(\xi')$, for all $i \geq 0$, i.e. $c(\xi) = f^*c(\xi')$.

- (Product Theorem) For two complex vector bundles ξ and η of dimension n and m over B , the relation $c(\xi \oplus \eta) = c(\xi)c(\eta)$ (multiplication in $H^*(B; \mathbb{Z})$) holds; i.e. for $0 \leq k \leq n + m$, the k -th Chern class of the Whitney sum is given by

$$c_k(\xi \oplus \eta) = \sum_{i+j=k} c_i(\xi) \cup c_j(\eta).$$

- If ξ is the trivial complex bundle of dimension $n \geq 1$, then $c_i(\xi) = 0$, for all $i \neq 0$.

A.6 For a real n -dimensional vector bundle ξ , we denote by $\xi \otimes \mathbb{C}$ the complexification of ξ ; i.e. the complex n -dimensional vector bundle, whose fibers are of the form $F \otimes \mathbb{C}$, where F is a fiber of ξ .

Definition. The *Pontrjagin class* of a real vector bundle ξ , denoted by $p_i(\xi)$, is the class $(-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B; \mathbb{Z})$.

Definition. Let M be a compact, oriented manifold of dimension $4m$. For each partition $I = i_1, \dots, i_r$ of m , the I -th *Pontrjagin number* $p_I(M) = p_{i_1} \dots p_{i_r}(M)$, is the value of the cup-product of the Pontrjagin classes $p_{i_1}(TM), \dots, p_{i_r}(TM)$ on the fundamental class of M ; i.e.

$$p_I(M) = \langle p_{i_1}(TM) \dots p_{i_r}(TM), [M] \rangle \in \mathbb{Z}.$$

Here TM is the tangent bundle of M .

Pontrjagin numbers can be used to compute the signature of M (see [Milnor Stasheff, 19.4] and §6.2 for the example of the exceptional group G_2).

Properties of Pontrjagin classes can be found e.g. in [Milnor Stasheff, §15].

A.2 Hermitian forms

Here we have collected some facts on hermitian forms, which are needed in the thesis. These results are essentially taken from [KMRT] and [Scharlau] and there one can find more details.

A.7 Let V be a finite dimensional vector space over an algebraic number field L which is a separable quadratic extension of some subfield F with nontrivial automorphism ι . A ι -hermitian form on V is a map $h : V \times V \rightarrow K$, such that

$$\begin{aligned} h(u + v, w) &= h(u, w) + h(v, w) \\ h(u, v + w) &= h(u, v) + h(u, w) \\ h(av, bw) &= ah(v, w)\iota(b) \\ h(w, v) &= \iota(h(v, w)), \end{aligned}$$

for $u, v, w \in V$ and $a, b \in L$. The form h is called *regular* or *nondegenerated*, if $h(v, w) = 0$, for all $w \in V$, implies that $v = 0$. The pair (V, h) is called a *hermitian space*.

Let $\{e_1, \dots, e_n\}$ be a L -basis of V . Since $h(e_j, e_i) = \iota(h(e_i, e_j))$, we see that the Gram matrix $B := (h(e_i, e_j))_{i,j}$ of h with respect to $\{e_1, \dots, e_n\}$, satisfies $\iota({}^t B) = B$, the determinant $\det B$ is invariant under ι ; hence lies in F . If we choose another L -basis $\{f_1, \dots, f_n\}$ of V , the Gram matrix B' with respect to this basis suffices the equation ${}^t A B' \iota(A) = B$, where $A \in \text{End}_L(V)$ such that $Ae_i = f_i$, for all $i = 1, \dots, n$. Thus $\det B$ is uniquely determined up

to an element in $N(L/F) := N_{L/F}(L^*)$, where $N_{L/F}$, is the norm of the field extension L/F . The *discriminant* of h is the signed determinant:

$$\text{disc } h = (-1)^{n(n-1)/2} \det B \cdot N(L/F) \in F^*/N(L/F).$$

Note that, in general the discriminant is not multiplicative, i.e. for an orthogonal sum $\text{disc } (h_1 \perp h_2) \neq \text{disc } h_1 \text{disc } h_2$.

A hermitian space (V, h) , defines F -algebraic groups; first the unitary group

$$\mathbf{U}(V, h) = \{g \in \mathbf{GL}(V) \mid h(gx, gy) = h(x, y), x, y \in V\}$$

and second the special unitary group

$$\mathbf{SU}(V, h) = \mathbf{U}(V, h) \cap \mathbf{SL}(V)$$

(here $\mathbf{GL}(V)$ and $\mathbf{SL}(V)$ are viewed as F -groups).

A.8 Let $\bar{F}$ be an algebraic closure of F and $\mathcal{G} = \text{Gal}(\bar{F}/F)$ be the absolute Galois group of F . We want to describe the Galois cohomology $H^1(\mathcal{G}, \mathbf{U}(V, h)(\bar{F})) = H^1(F, \mathbf{U}(V, h))$ resp. $H^1(\mathcal{G}, \mathbf{SU}(V, h)(\bar{F})) = H^1(F, \mathbf{SU}(V, h))$, as one can find it in [KMRT, (29.19)].

An *isometry* between two hermitian spaces (V, h) and (V', h') is an injective L -linear map $\varphi : V \rightarrow V'$, such that $h'(\varphi(x), \varphi(y)) = h(x, y)$, for all $x, y \in V$. Two hermitian forms h, h' on the same vectorspace V are isometric, i.e. there is an isometry $(V, h) \rightarrow (V, h')$, if and only if their Gram matrices B, B' with respect to a fixed basis are equivalent, i.e. there is a matrix $A \in GL_n(L)$ such that $AB'\iota({}^tA) = B$. In such a case we write $(V, h) \simeq (V, h')$.

Now and for all, we fix a hermitian space (V, h) and a L -basis $\{e_1, \dots, e_n\}$ of V and identify via this basis, the L -algebra $\text{End}_L(V)$ with $M_n(L)$. Let B be the Gram matrix of h .

We will call a matrix $B' \in M_n(L)$, *hermitian*, if B' satisfies $\iota({}^tB') = B'$. Clearly a hermitian matrix B' , defines a ι -hermitian form h' on V , via $h'(x, y) = {}^txB'\iota(y)$, where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ with respect to $\{e_1, \dots, e_n\}$.

Let $\text{Sym}(V, B, \iota) := \{S \in GL_n(L) \mid B\iota({}^tS)B^{-1} = S\}$ and define an equivalence relation on $\text{Sym}(V, B, \iota)$ via $S \sim S'$ if and only if $S' = ASB\iota({}^tA)B^{-1}$, for an $A \in GL_n(L)$. By [KMRT, (29.19)] we have a canonical bijection of pointed sets

$$\text{Sym}(V, B, \iota) / \sim \longleftrightarrow H^1(F, \mathbf{U}(V, h)).$$

The set $\text{Sym}(V, B, \iota) / \sim$ is also in 1 : 1 correspondence with the set of isometry classes of non degenerated hermitian forms on V , by mapping $S \in \text{Sym}(V, B, \iota)$ to the hermitian form $h_S : V \times V \rightarrow L$ defined by

$$h_S(x, y) = {}^txS^{-1}B\iota(y),$$

for $x, y \in V$, i.e. in every isometry class of non degenerate hermitian forms on V , there is an representing element h' of this class and a matrix $S \in \text{Sym}(V, B, \iota)$ such that the Gram matrix B' of h' satisfies

$$S^{-1}B = B'.$$

This assignment is well-defined and injective, which is obvious. To show surjectivity, take for a hermitian form h' , with Gram matrix B' , the matrix $BB'^{-1} =: S$, which is clearly in $\text{Sym}(V, B, \iota)$.

Therefore, we have a canonical bijection of pointed sets

$$\left\{ \begin{array}{c} \text{isometry classes of non degenerated} \\ \text{hermitian forms on } V \end{array} \right\} \longleftrightarrow H^1(F, \mathbf{U}(V, h)).$$

Further it turns out, that we have a canonical bijection of pointed sets

$$\left\{ \begin{array}{c} \text{isometry classes of non degenerated} \\ \text{hermitian forms } h' \text{ on } V \\ \text{with disc } h' = \text{disc } h \end{array} \right\} \longleftrightarrow H^1(F, \mathbf{SU}(V, h)).$$

In terms of the set $\text{Sym}(V, B, \iota)$, this means the following: Assume we have a hermitian form h_S , with Gram matrix $S^{-1}B$, where $S \in \text{Sym}(V, B, \iota)$, then $\text{disc } h_S = \text{disc } h \cdot \det S^{-1}$ in $F^*/N(L/F)$, hence there exists $z \in L^*$ such that $\det S = N_{L/F}(z)$ if and only if $\text{disc } h_S = \text{disc } h$.

A.9 We repeat the so called *Witt's Cancellation law*, as can be found e.g. in [Scharlau, Chapter 7, §9].

Let F be a field and L be a quadratic field extension of F , with Galois group $\text{Gal}(L/F) = \{1, \iota\}$.

Theorem.(WITT) *Let (V, h) be a nondegenerate ι -hermitian space over L and W a subspace of V . Assume that $\varphi : (W, h_W) \rightarrow (V, h)$ is an isometry. Then there exists an isometry $\psi : V \rightarrow V$ which extends φ , that is $\psi|_W = \varphi$.*

Corollary.(WITT'S CANCELLATION LAW) *Let (V, h) and (V_i, h_i) , $i = 1, 2$, nondegenerate ι -hermitian spaces over L . Then*

$$(V_1, h_1) \perp (V, h) \simeq (V_2, h_2) \perp (V, h) \implies (V_1, h_1) \simeq (V_2, h_2).$$

Proof. Like [Scharlau, 1.5.8]

Corollary A.10. *Let (W, h_W) be a nondegenerated ι -subhermitian space of a nondegenerated ι -hermitian spaces (V, h) over L . Then the natural map*

$$H^1(F, \mathbf{SU}(W, h_W)) \rightarrow H^1(F, \mathbf{SU}(V, h))$$

is injective.

Proof. Since (W, h_W) is nondegenerated, we can decompose the space V as orthogonal sum $V = W \perp W^\perp$ and the hermitian form becomes $h = h_W \oplus u$, where u is the restriction of h to $W^\perp$. To a class $\alpha \in H^1(F, \mathbf{SU}(W, h_W))$ we associate the isometry class of the hermitian form $h_{W,S}$ on W with Gram matrix $S^{-1}B_W$, $S \in \text{Sym}(W, B_W, \iota)$ and $\det S \in N(L/F)$. In these sense, the natural map $H^1(F, \mathbf{SU}(W, h_W)) \rightarrow H^1(F, \mathbf{SU}(V, h))$, maps the isometry class of $h_{W,S}$ to the isometry class of $h_{W,S} \perp u$. Since $\det S \in N(L/F)$, we have $\text{disc } (h_{W,S} \perp u) = \det S^{-1} \cdot \text{disc } (h_W \perp u) = \text{disc } h$ in $F^*/N(L/F)$. If h'_W is a hermitian form on W , such that $\text{disc } h'_W = \text{disc } h_W$ and $h'_W \perp u \simeq h_{W,S} \perp u$, then by Witt's Cancellation law we have $h'_W \simeq h_{W,S}$. This shows the injectivity. $\square$

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Employment:

- Recipient of an DOC-fellowship of the Austrian Academy of Sciences at the Faculty of Mathematics, University of Vienna, January 2007 to December 2008.
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Education:

- Ph.D. in Mathematics, since January 2006. Advisor: Joachim Schwermer.
- Diploma studies at the Faculty of Mathematics, University of Vienna passed with distinction on the 2005-12-15 started on the 2002-03-01.
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Participation

- 5-Day Workshop; Locally Symmetric Spaces; May 18-23 May 23, 2008; Banff International Research Station, Banff, Canada
- Oberwolfach-Seminar; On Arithmetically Defined Hyperbolic Manifolds; November 5-9, 2007; Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach, Germany
- Spectra of arithmetic groups; October 8-11, 2007; Erwin Schrödinger Institute, Vienna
- Summer school and conference on automorphic forms and Shimura varieties; July 9-27, 2007; The Abdus Salam International Center for Theoretical Physics; Trieste, Italy

Talks

- Cycles and the cohomology of arithmetic subgroups of the exceptional group G_2 ; October 10, 2007; ESI Vienna

Zusammenfassung

Das Hauptaugenmerk meiner Dissertation ist die geometrische Konstruktion von (Ko-) Homologieklassen für arithmetische Untergruppen von halbeinfachen, über $\mathbb{Q}$ definierten, $\mathbb{Q}$ -anisotropen algebraischen Gruppen $\mathbf{G}$. Ich untersuche im speziellen eine $\mathbb{Q}$ -Gruppe $\mathbf{G}$, in der der nicht kompakte Faktor der Gruppe der reellen Punkte $\mathbf{G}(\mathbb{R})$, die reelle zerfallende exzeptionelle Lie Gruppe vom Typ G_2 ist. All dies basiert auf dem allgemeinen Ansatz Zykel zu konstruieren (wie von J. Millson und M.S. Raghunathan initiiert und von J. Rohlfs und J. Schwermer weitergeführt). Ich kombiniere diese Ergebnisse und erhalte damit eine neue Formel für die Schnittzahl von zwei Zykeln die einander nicht transversal schneiden. Im Falle der Gruppe G_2 sind Beiträge der Zykeln zur Homologie in folgendem Sinn “Vollständig”: als erstes gibt es in “jedem” kohomologischen Grad einen nicht trivialen Zykel, der einer unitären Darstellungen mit nicht-verschwindender Kohomologie entspricht und zweitens benutzen wir jeden auftretenden Typ von reduktiven Untergruppen von $\mathbf{G}$ um Zykel zu konstruieren. Weiters gebe ich zwei hinreichende Kriterien an Zykel an, die zeigen das die Poincaré duale Klasse zu dem Zykel durch eine Differentialform, die nicht invariant unter $\mathbf{G}(\mathbb{R})$ ist, gegeben ist.

Abstract

The main focus of my doctoral thesis is the geometric construction of (co)homology classes of arithmetic subgroups of semisimple $\mathbb{Q}$ -anisotropic algebraic groups $\mathbf{G}$ defined over $\mathbb{Q}$. In particular, I discuss a $\mathbb{Q}$ -group $\mathbf{G}$ in which the non compact factor of the group of real points $\mathbf{G}(\mathbb{R})$ is the real split exceptional Lie group of type G_2 . All that is based on the general approach of constructing cycles (as initiated by J. Millson and M.S. Raghunathan and pursued by J. Rohlfs and J. Schwermer). Combining their results I get a new formula for the intersection number of two cycles which intersect each other non-transversally. In the case of the group G_2 , the description of the contribution of cycles to the homology is “complete” in the following sense: first, we found in “every” cohomological degree a non trivial cycle which corresponds to a unitary representation of G_2 with non-zero cohomology and second, we found at least one group of every possible type of reductive subgroups of $\mathbf{G}$, to get a cycle. Further, I give two sufficient criteria on cycles, which imply that the Poincaré dual class to the cycle is represented by a differential form which is not invariant under the action of $\mathbf{G}(\mathbb{R})$.