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Games between Two Populations with Selfinteraction
under Best Response Dynamics

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Chapter 1

Introduction

We will consider games between two populations, where each player in a population has two pure strategies to choose from and does interact with players from both populations. This was done in [14] under consideration of the replicator dynamics. It was shown, that these dynamics admit diverse behaviour, including the existence of limit cycles.

The question arose, how this class of games would perform under a different kind of dynamics, in particular, the continuous time best response dynamics, which was introduced by Gilboa and Matsui [7] in 1991, studied further by Hofbauer [10], Hofbauer and Sigmund [11] and Cressman [3]. According to Hofbauer [10] the usual interpretation of the continuous time version of the best response dynamics is, that (in an infinite population) in each small time interval a small fraction of boundedly-rational players rethink and change their strategy according to a best-response principle, which produces a set of choices that, from the player's current point of view, maximizes pay-off of the population. Boundedly-rational in this sense means, that players are aware of the strategy distributions in the populations and are able to come up with a best response (in this paper, strategy 0 and/or strategy 1), but do not see a long time ahead. They are myopic in the sense that they cannot anticipate the outcome of their actions. Populations as a whole are at every moment focussing in on a best response, even if, in the long run, the population might be better off doing something else (much like deer staring into the headlights of a car). A best response for a population might be a pure (0 or 1) but might also be a mixed strategy (everything between 0 and 1), which is interpreted as the relative frequencies of the strategies played by the players in the population.

The problem leads to a system of piecewise continuous differential equations of the first order in the plane. Similar equations have been studied by Leon Glass and J.S. Pasternack [8]. $\mathbb{R}^2$ is divided into four disjoint open sets, whose union is dense and in each of which the vector field points towards a different single best response, the focal point of the region. Between the regions the payoff gets maximized by a whole set of different strategies, so the behaviour on the points of discontinuity is naturally captured using the theory of multivalued functions, resulting in a system of differential inclusions

$$\dot{x} \in H(x) - x \tag{1.1}$$

on a domain $D \subseteq \mathbb{R}^2$. The components of the right-hand side are piecewise linear, upper-semicontinuous multivalued functions and $H(x)$ is a compact, convex set for

every $x \in D$, which means that most theory about multivalued functions ([1],[4]), especially existence of solutions, is applicable. Uniqueness of solutions is clearly guaranteed (by Picard-Lindelöf) where each component function of H is a function in the usual sense (not multivalued), but in general that is not the case on the points of discontinuity. Solutions for all positive times can be explicitly given by continuously piecing together solutions of the form

$$x(t) = P + (x_0 - P)e^{-t}$$

where $x_0 \in D$ is the initial condition and $P \in D$ is a focus point. The trajectory of a solution is a continuous combination of points and line segments. Again, a solution with the initial condition x_0 is not uniquely determined in general, so there may be many.

The aim of this paper is to describe most of the possible situations that may occur. There are some restrictions imposed, so that only the general game is studied. Degenerate cases (like for instance abandonment of selfinteraction) will be ignored. We will find that the points of discontinuity form specific sets and that there are three ways to pass these sets. Only one of them, the *sliding motion*, produces non-unique solutions. There is a variety of qualitatively different behaviour, highlights being asymptotically stable Shapley polygons and the formation of a transitive invariant set, a behaviour that has no apparent analogon in the game under the replicator dynamics.

1.1 The Game

Consider two populations X and Y interacting with each other and with themselves. In each population players have two strategies. The state-space is the space of relative frequencies of strategies, the simplex

$\mathbf{S}_2 := \{(x, 1-x) \in \mathbb{R}^2 \mid x \geq 0\}$ for a single population and $\mathbf{S}_2 \times \mathbf{S}_2 \hat{=} \mathbf{Q}_2 := [0, 1] \times [0, 1]$ for the population mix.

Let A, B, C, D be the static payoff-matrices describing interaction between X and X , X and Y , Y and X , Y and Y .

We define the operator $\tau : [0, 1] \rightarrow \mathbf{S}_2$ by $\vec{u} = (u, 1-u)^T$.

1.1.1 Payoff

The payoff for the X -population at state $\vec{s}$ against a population mix at state (x, y) is

$$P_X(s, x, y) = \vec{s} \cdot (A\vec{x} + B\vec{y}) = s(ax + by + e) + r_X(x, y) \quad (1.2)$$

with real numbers a, b, e and a remainder-function r_X , all determined by payoff-matrices A and B , to be precise

$$\begin{aligned} \vec{s} \cdot (A\vec{x} + B\vec{y}) &= a_{22} + b_{22} + (a_{21} - a_{22})x + (b_{21} - b_{22})y + \\ &\quad + s(a_{12} - a_{22} + b_{12} - b_{22} + (a_{11} - a_{12} - a_{21} + a_{22})x + \\ &\quad + (b_{11} - b_{12} - b_{21} + b_{22})y) = \\ &= s(ax + by + e) + r_X(x, y) \end{aligned}$$

The payoff for the Y -population at state $\vec{t}$ against a population mix at state (x, y) is

$$P_Y(t, x, y) = \vec{t} \cdot (C\vec{x} + D\vec{y}) = t(cx + dy + f) + r_Y(x, y) \quad (1.3)$$

with real numbers c, d, f and a remainder-function r_Y , all determined by payoff-matrices C and D .

1.1.2 Best Response Dynamics

We impose on the game the Best Response Dynamics

$$\begin{aligned} \dot{x} &\in \text{BR}_X(x, y) - x \\ \dot{y} &\in \text{BR}_Y(x, y) - y \end{aligned} \quad (1.4)$$

so we are looking for a multivalued function

$$\text{BR}(x, y) = \begin{pmatrix} \text{BR}_X(x, y) \\ \text{BR}_Y(x, y) \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbf{Q}_2 \quad (1.5)$$

which in turn is looking for points on the unit-square $\mathbf{Q}_2$ that are maximizing the payoff at state (x, y) for each population.

So, we have:

$$p \in \text{BR}_X(x, y) \iff P_X(p, x, y) = \max_{s \in [0, 1]} P_X(s, x, y) \quad (1.6)$$

$$q \in \text{BR}_Y(x, y) \iff P_Y(q, x, y) = \max_{t \in [0, 1]} P_Y(t, x, y) \quad (1.7)$$

N is a *Nash equilibrium* of the game, if $N \in \text{BR}(N)$. It is *strict*, if $\text{BR}(N) = \{N\}$.

Clearly, $P_X(s, x, y)$ is maximized by

$$\begin{array}{lll} s = 1 & \text{whenever} & ax + by + e > 0 \\ \text{every } s \in [0, 1] & \text{whenever} & ax + by + e = 0 \\ s = 0 & \text{whenever} & ax + by + e < 0 \end{array}$$

The case of $P_Y(t, x, y)$ is similar, leaving us to consider the differential inclusions

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \in G(x, y) = \begin{pmatrix} H(ax + by + e) - x \\ H(cx + dy + f) - y \end{pmatrix} \quad (1.8)$$

with $(x, y) \in \mathbf{Q}_2$, a multivalued Heaviside-function H

$$H : \mathbb{R} \rightarrow \mathcal{P}([0, 1]), \quad H(u) := \begin{cases} \{0\} & \text{if } u < 0 \\ [0, 1] & \text{if } u = 0 \\ \{1\} & \text{if } u > 0 \end{cases} \quad (1.9)$$

and real numbers a, b, c, d, e, f determined by the payoff-matrices A, B, C, D .

1.1.3 Replicator Dynamics

The replicator dynamics of this game were first analyzed in [14] and further analysis was given in [5]. Under these dynamics, the differential equations have the form

$$\begin{aligned} \dot{x} &= x(1 - x)(ax + by + e) \\ \dot{y} &= y(1 - y)(cx + dy + f) \end{aligned} \quad (1.10)$$

Omitted from the analysis were certain degenerate cases. One of the restrictions imposed, $\Delta := ad - bc \neq 0$, guarantees the existence of an equilibrium $F = (\frac{bf - de}{ad - bc}, \frac{ce - af}{ad - bc})$. One of the first results was, that

$$\Delta < 0 \iff F \text{ is a saddle}$$

$$\Delta > 0 \iff F \text{ is a sink or a source}$$

We will see that the second statement does not hold for the best response dynamics.

Chapter 2

Analysis of the Game

We will restrict our analysis to generic games, by this we mean that the coefficients a through f satisfy the conditions:

$$\begin{array}{llll} a, b, c, d, e, f \neq 0 & a + e \neq 0 & b + e \neq 0 & a + b + e \neq 0 \\ \Delta := ad - bc \neq 0 & c + f \neq 0 & d + f \neq 0 & c + d + f \neq 0 \end{array} \quad (2.1)$$

$$\begin{array}{ll} bf - de \neq 0 & bf - de \neq \Delta \\ ce - af \neq 0 & ce - af \neq \Delta \end{array} \quad (2.2)$$

2.1 Definitions

Definition 2.1.1. Let Φ_1, Φ_2 denote the lines given by

$$\Phi_1 : \phi_1(x, y) := ax + by + e = 0$$

$$\Phi_2 : \phi_2(x, y) := cx + dy + f = 0$$

The restriction to generic games expressed in terms of the lines Φ_1 and Φ_2 means, that none of these lines lie parallel to the axes of the coordinate system ($a, b, c, d = 0$) or parallel to each other ($\Delta = 0$). The other restrictions of (2.1) make it impossible for the lines Φ_1, Φ_2 to contain any of the corners of $\mathbf{Q}_2$. The restrictions (2.2) mean, that the lines Φ_1 and Φ_2 cannot intersect on the boundary of the square.

With the lines Φ_1, Φ_2 in hand, we define

Definition 2.1.2. Regions $\mathbf{A}_i$

$$A_1 := \{(x, y) \in \mathbb{R}^2 \mid \phi_1 > 0, \phi_2 > 0\}$$

$$A_2 := \{(x, y) \in \mathbb{R}^2 \mid \phi_1 < 0, \phi_2 > 0\}$$

$$A_3 := \{(x, y) \in \mathbb{R}^2 \mid \phi_1 < 0, \phi_2 < 0\}$$

$$A_4 := \{(x, y) \in \mathbb{R}^2 \mid \phi_1 > 0, \phi_2 < 0\}$$

Definition 2.1.3. Half-rays $\mathbf{A}_{i,i+1}$ ($=: \mathbf{A}_{i+1,i}$) separating the regions

$$\begin{aligned} A_{1,2} &:= \{(x, y) \in \mathbb{R}^2 \mid \phi_1 = 0, \phi_2 > 0\} \\ A_{2,3} &:= \{(x, y) \in \mathbb{R}^2 \mid \phi_1 < 0, \phi_2 = 0\} \\ A_{3,4} &:= \{(x, y) \in \mathbb{R}^2 \mid \phi_1 = 0, \phi_2 < 0\} \\ A_{1,4} &:= \{(x, y) \in \mathbb{R}^2 \mid \phi_1 > 0, \phi_2 = 0\} \end{aligned}$$

Definition 2.1.4. Corners $\mathbf{P}_i = (p_i^x, p_i^y)$ of the square

$$P_1 := (1, 1), P_2 := (0, 1), P_3 := (0, 0), P_4 := (1, 0)$$

and

Definition 2.1.5. Let

$$\mathbf{B}_{i+1,i} := \mathbf{B}_{i,i+1} := \{sP_i + (1-s)P_{i+1} \mid s \in (0, 1) \subset \mathbb{R}\}, i = 1, 2, 3, 4$$

the open boundary line between P_i and P_{i+1} .

The indices of the above are taken modulo 4, such that $i + 4 = i - 4 = i$.

Since $\Delta \neq 0$, Φ_1 and Φ_2 intersect at

Definition 2.1.6.

$$F = \Phi_1 \cap \Phi_2 = (\hat{x}, \hat{y}) = \left(\frac{bf - de}{ad - bc}, \frac{ce - af}{ad - bc} \right) \quad (2.3)$$

(1.8) can thus be alternatively expressed as

$$\begin{aligned} \dot{x} &\in H(a(x - \hat{x}) + b(y - \hat{y})) - x \\ \dot{y} &\in H(c(x - \hat{x}) + d(y - \hat{y})) - y \end{aligned} \quad (2.4)$$

Obviously, if $F \in \mathbf{Q}_2$ then $\mathbf{0} \in G(F) := \text{BR}(F) - F$ and F is a Nash-equilibrium of the game and a stationary solution of the system of differential inclusions. From 2.2 it follows, that $F \notin \text{bd}(\mathbf{Q}_2)$.

Definition 2.1.7. When F is in the interior of $\mathbf{Q}_2$, let

$$\mathbf{F}_{i,j} := A_{i,j} \cap \text{bd}(\mathbf{Q}_2)$$

denote the intersection of the line separating A_i and A_j with the boundary of the square.

It should be noted, that when F is outside $\mathbf{Q}_2$ it follows that $A_{i,j}$ may intersect $\text{bd}(\mathbf{Q}_2)$ in two points.

2.2 Characterization of Nash Equilibria

Of course, every stationary solution is a Nash equilibrium, as it is a best response to itself. If P_i is a Nash equilibrium, it is also a strict Nash equilibrium, because $\text{BR}(P_i)$ contains only one element in generic games. If $A_{i,j} \cap B_{i,j} \neq \emptyset$, the intersection point S is a Nash equilibrium, because then $S \in \text{BR}(S) = [P_i, P_j] = \overline{B}_{i,j}$. If $F \in \text{int} \mathbf{Q}_2 \Rightarrow S = F_{i,j}$.

Such an equilibrium S does always have an inset on the line segment $A_{i,j}$ and can thus never be a source. The boundary of the square can only contain a source in a non-robust setting, namely if $F \in \text{bd}(\mathbf{Q}_2)$ and F is a source (see figure 2.1). This scenario was ruled out by the restrictions (2.2).

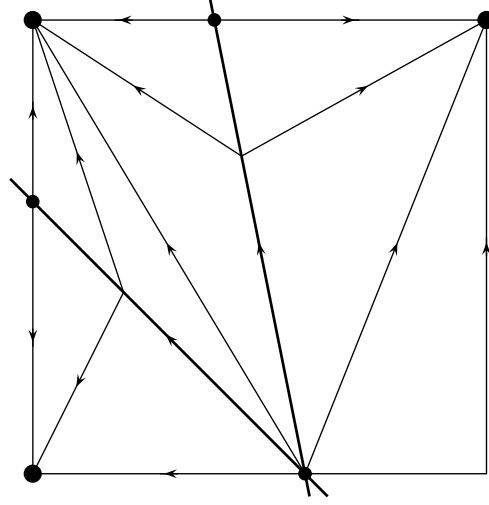


Figure 2.1: Three regions and F on the boundary of $\mathbf{Q}_2$.
 P_1 , P_2 and P_3 are attractors. $F \in B_{34}$ is a source.

2.3 Regional Flow Behaviour

Observe that the focus of the flow in A_i is on P_i . The orbits move on straight lines towards this focal point. When A_i is left, the focus of the orbit changes. The adjoining regions of A_i are A_{i-1} and A_{i+1} . So an orbit starting in A_i can only leave through either F , $A_{i,i-1}$ or $A_{i,i+1}$.

Lemma 2.3.1 (Regional Behaviour of Flow).

- i) If $P_i \in A_i$ then orbits in A_i converge to P_i .
- ii) If $P_i \in A_{i+1}$ (or A_{i-1}) then the entire flow from A_i will move into $A_{i,i+1}$ ($A_{i-1,i}$).
- iii) If $P_i \in A_{i+2}$ then there is an orbit in A_i going straight to F and the rest of the orbits either reach $A_{i-1,i}$ or $A_{i,i+1}$.

Proof. **ad i)** The BR-Path is a straight line towards P_i for every element of A_i . Now, think of a line l starting in $P_i \in A_k$ that is passing through F and consequently entering A_{k+2} .

ad ii) Now if $k = i + 1$ this means that A_i lies entirely on one side of l . Orbits in A_i are straight lines focused on P_i and can thus never touch l inside the square. Since $F \in l$ and $P_i \in A_{i+1}$ this means that A_i in its entirety reaches $A_{i,i+1}$.

ad iii) $k = i + 2$ implies that $l \cap A_i \neq \emptyset$ is an orbit going straight to F . The line l splits A_i in two regions, one closer to A_{i-1} the second adjoining A_{i+1} . The former region flows into $A_{i-1,i}$, the latter into $A_{i,i+1}$ following the same reason as in ii).

□

So the flow in each region A_i either approaches one and leaves the other or leaves both or approaches both its region's "separating lines" $A_{i,i+1}$ and $A_{i,i-1}$. If orbits approach on one side of such a separating line $A_{i,j}$ and leave on the other side, the line just acts as a switch between the vectorfields of A_i and A_j and the orbits are passing through $A_{i,j}$ uniquely.

If the orbits are moving away from $A_{i,j}$, towards P_i on one side, towards P_j on the other, the set of possible focus points on $A_{i,j}$ is $\overline{B}_{i,j}$, which makes it possible to advance on $A_{i,j}$ in the direction of $B_{i,j}$. We will call this type of motion a ***sliding motion on $A_{i,j}$*** . At any point a sliding motion can turn into one of the adjacent regions, so solutions starting here cannot be uniquely determined for positive times.

If, finally, orbits are moving towards $A_{i,j}$ from both A_i and A_j the set of theoretically possible directions is narrowed down to $\lambda \cdot (F - F_{i,j})$ for some $0 \neq \lambda \in \mathbb{R}$ (if F is inside $\mathbf{Q}_2$; if not, the only possible direction is towards $A_{i,j} \cap B_{i,j}$ on the boundary of the square), which means that orbits starting on $A_{i,j}$ will stay there and are moving towards F or $F_{i,j}$. We will call this type of motion a ***coerced motion on $A_{i,j}$*** . A solution starting on or reaching a line with a motion coerced on it is uniquely determined for positive times.

It should be noted, that the common literature on differential inclusions, like [1] and [4], does not differentiate between the above defined terms *coerced* and *sliding motion*. Both these are usually referred to as *sliding motions*.

The four regions may be arranged in two ways around F . Positive, anti-clockwise orientation is equivalent to $\Delta > 0$ (orientation of standard base is preserved), negative or clockwise to $\Delta < 0$ (orientation reversed).

This distinction will be crucial when F is inside the square $\mathbf{Q}_2$ (see sections 4.2, 4.3, 4.4). But first we study the simpler case when F is outside $\mathbf{Q}_2$.

Chapter 3

No Equilibrium inside $\mathbf{Q}_2$

If F lies outside the unit square, Φ_1 and Φ_2 define on $\mathbf{Q}_2$ up to three regions. If $\mathbf{Q}_2$ is covered by just one region A_i , the ω -limit of every orbit is P_i (see figure 3.1).

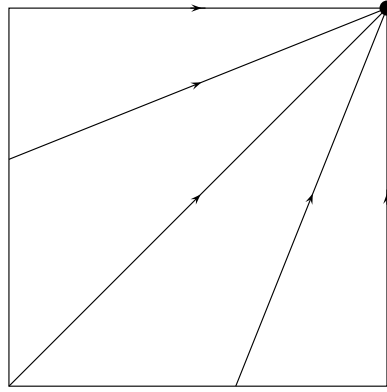


Figure 3.1: F outside $\mathbf{Q}_2$ - Type I

The square is covered by just one region, A_1 . All orbits converge towards P_1 .

3.1 Two Regions

Let the two regions be A_i and A_{i+1} . The corresponding focal points are adjacent/lie next to each other on the square. There are three possibilities:

- IIa $P_i \in A_i$ and $P_{i+1} \in A_{i+1}$: Every orbit outside $A_{i,i+1}$ converges to one of the focal points. There is a sliding motion on $A_{i,i+1}$ towards a saddle on $B_{i,i+1}$. Compare figure 3.2.
- IIb $P_i, P_{i+1} \in A_i$ (or A_{i+1}): The ω -limit of every orbit is P_i (or P_{i+1}). Figure 3.3.
- IIc $P_i \in A_{i+1}$ and $P_{i+1} \in A_i$: Every orbit reaches $A_{i,i+1}$ in finite time on which a coerced motion drives the orbit towards $A_{i,i+1} \cap B_{i,i+1}$, which is a global attractor. Figure 3.4.

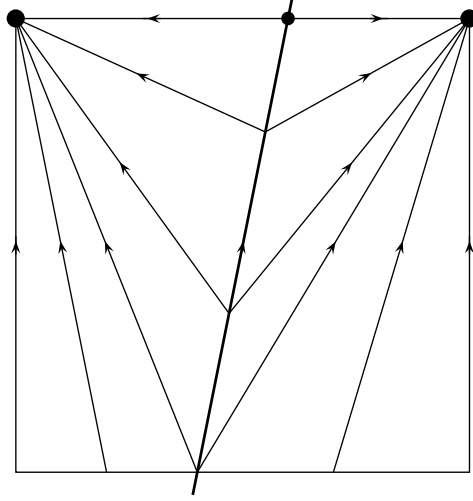


Figure 3.2: F outside $\mathbf{Q}_2$ and the square is covered by two regions.
Type IIa. Each region contains its focus, so there is a sliding motion on the separating ray.

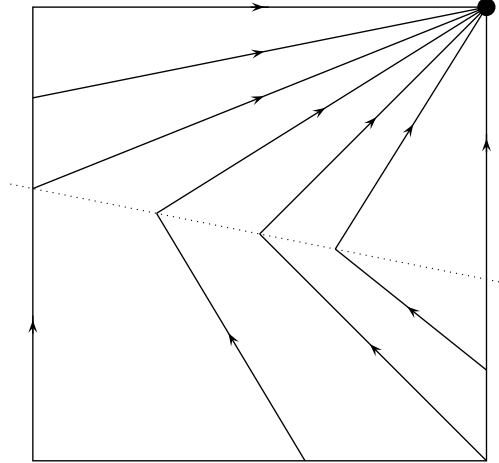


Figure 3.3: Two regions, Type IIb
Region A_1 contains both region's focal points.

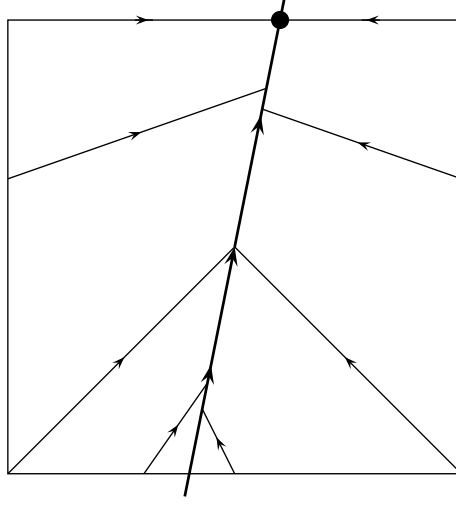


Figure 3.4: Two regions, Type IIc
Neither region contains its focal point. There is a coerced motion on A_{12} towards the Nash equilibrium on the boundary.

3.2 Three Regions

Let the neighbouring regions be A_{i-1} , A_i , A_{i+1} . Then A_{i-1} and A_{i+1} must each contain at least one of the four corners. Here are the possibilities:

- (1) $P_{i-1} \in A_{i-1}$ and $P_{i+1} \in A_{i+1}$. There are two further possibilities:
 - (a) $P_i \in A_i$ Then we get three sinks P_{i-1}, P_i, P_{i+1} , sliding motions on $A_{i-1,i}$ towards $B_{i-1,i}$ and on $A_{i,i+1}$ towards $B_{i,i+1}$ and saddles at the intersection points of those. Compare figure 3.5.
 - (b) $P_i \notin A_i$ Then wlog $P_i \in A_{i-1}$, so P_{i-1} attracts A_{i-1} and A_i . The intersection of $A_{i,i+1}$ with $B_{i,i+1}$ is a saddle and P_{i+1} is the other sink. Figure 3.6.
- (2) $P_{i-1} \in A_{i-1}$ and $P_{i+1} \notin A_{i+1}$, with three further possibilities:
 - (a) $P_i \in A_{i+1}$ There is a coerced motion on $A_{i,i+1}$, $A_{i,i+1} \cap B_{i,i+1}$ is an attractor and attracts everything apart from A_{i-1} and $A_{i-1,i}$. Since $A_{i-1,i} \cap B_{i-1,i} \neq \emptyset$ there is a sliding motion on $A_{i-1,i}$ towards the saddle $A_{i-1,i} \cap B_{i-1,i}$. Figure 3.7.
 - (b) $P_i \in A_{i-1}$ All orbits move into $A_{i-1,i}$ and on to P_{i-1} , the global attractor. Figure 3.8.
 - (c) $P_i \in A_i$ Then there are the two attractors P_{i-1} and P_i , between them the saddle $A_{i-1,i} \cap B_{i-1,i}$, whose inset is $A_{i-1,i}$. Figure 3.9.
- (3) $P_{i-1} \notin A_{i-1}$ It follows that $P_{i-1} \in A_i$ and $P_{i+1} \in A_i$, because the above statement holds with $i-1$ and $i+1$ interchanged. Then the flows from A_{i-1} and A_{i+1} must move into $A_{i-1,i}$ or $A_{i,i+1}$, respectively. There are two possibilities:

-
- The diagram shows a square with vertices marked by black dots. A horizontal line with arrows pointing left and right passes through the top edge. A vertical line with arrows pointing up and down passes through the right edge. A diagonal line with arrows pointing from the top-left towards the bottom-right passes through the square. Several other lines with arrows connect various points on the square's edges and interior, forming a complex network of paths. The diagram is likely a representation of a mathematical proof or a geometric construction, possibly related to the concept of a 'square' in a more abstract sense, such as a square in a vector space or a square in a graph.

We end this section with a classification of the different types of flow by the numbers of sinks, saddles and sources that appear in the phase portraits without an equilibrium in the interior of $\mathbf{Q}_2$.

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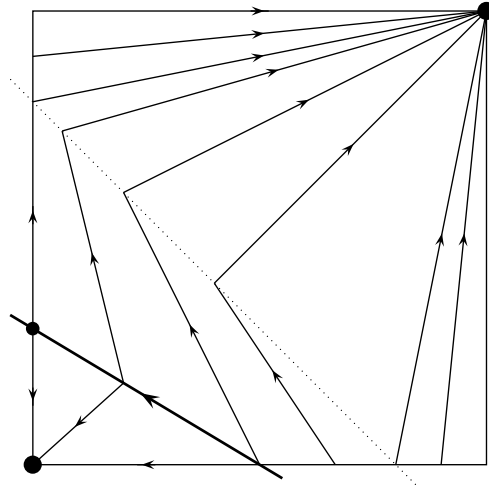


Figure 3.6: Three regions, Type (1b)

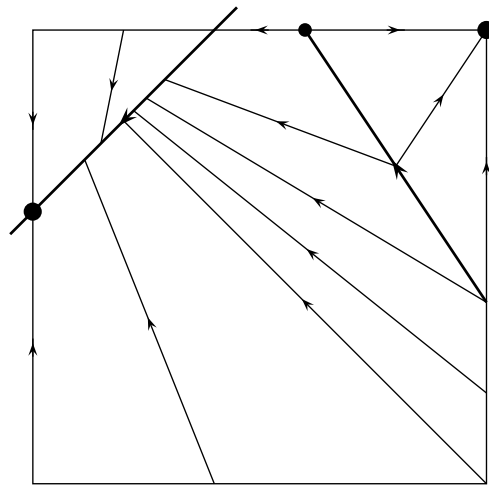


Figure 3.7: Three regions, Type (2a)

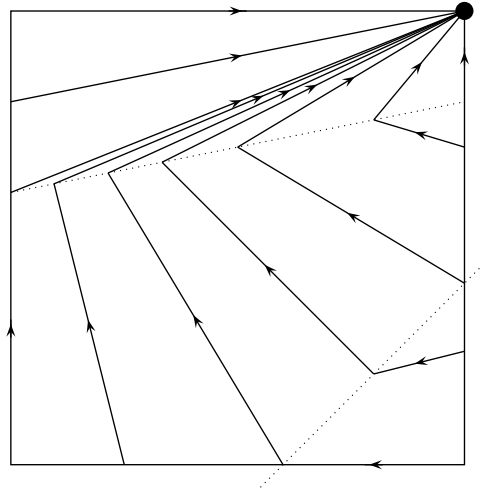


Figure 3.8: Three regions, Type (2b)

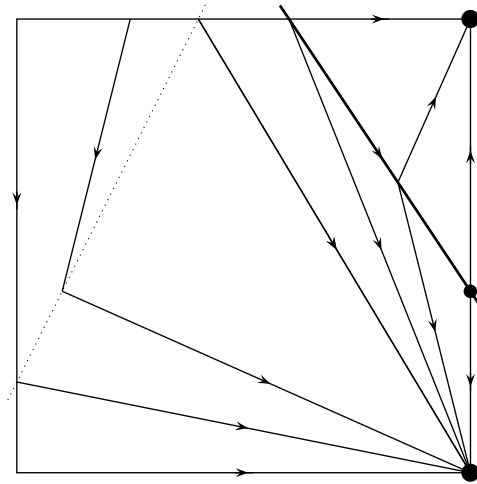


Figure 3.9: Three regions, Type (2c)

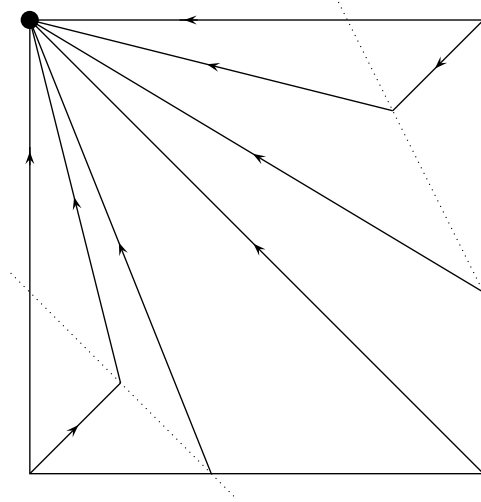


Figure 3.10: Three regions, Type (3a)

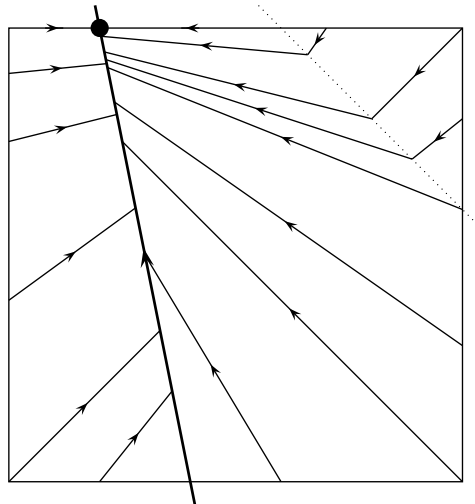


Figure 3.11: Three regions, Type (3b)

Chapter 4

Internal Equilibrium

We will now consider the case, when Φ_1 and Φ_2 intersect in the interior of the square. We begin with some observations about coerced and sliding motions in this setting.

4.1 Coerced and Sliding Motions

Lemma 4.1.1 (Coerced Motions - Necessary Conditions). *For a coerced motion to occur on $A_{i,i+1}$ it is necessary, that*

- i) $P_j \notin A_j$ for $j = i, i + 1$. *The regions must not contain their focal points.*
- ii) $A_{i,i+1} \cap B_{i,i+1} \neq \emptyset$ or $A_{i+2,i+3} \cap B_{i,i+1} \neq \emptyset$. *The line Φ separating the regions must intersect with the line segment between the focal points.*

Proof.

ad i) This is obvious from Lemma 2.3.1. If any of the regions A_i, A_{i+1} contains its focal point it is possible to leave $A_{i,i+1}$.

ad ii) Every motion on $A_{i,i+1}$ moves towards $\overline{B}_{i,i+1}$. That means if $A_{i,i+1}$ and $A_{i+2,i+3}$ both do not intersect with $B_{i,i+1}$, $A_{i,i+1}$ would not point towards $\overline{B}_{i,i+1}$, and thus couldn't have a coerced motion on it. □

Lemma 4.1.2 (Coerced Motions). *Some observations.*

- i) *If $P_i \in A_i$ then there are no coerced motions on $\overline{A}_i$.*
- ii) *Coerced motions move towards F if $\Delta > 0$ and away from F if $\Delta < 0$.*

Proof.

ad i) Starting on any part of the border there is the option of taking a path into A_i . So there is no coerced motion.

ad ii) If there is a coerced motion on $A_{i,i+1}$ it moves towards $\overline{B}_{i,i+1}$. So if $A_{i,i+1} \cap B_{i,i+1} \neq \emptyset$ it would move towards $F_{i,i+1}$ and necessarily $\Delta < 0$. Similarly, if $A_{i+2,i+3} \cap B_{i,i+1} \neq \emptyset$ the coerced motion on $A_{i,i+1}$ moves towards F and Δ must be greater than 0. □

The following theorems show that the necessary conditions for coerced motions are also (almost) sufficient.

Theorem 4.1.1. *Let $\Delta < 0$. Then a coerced motion occurs on $A_{i,i+1}$ if and only if $A_{i,i+1} \cap B_{i,i+1} \neq \emptyset$.*

Proof.

($\implies$) If there is a coerced motion on $A_{i,i+1}$ then the necessary conditions are met. It must hold $A_{i,i+1} \cap B_{i+2,i+3} = \emptyset$ since otherwise this would result in a coerced motion towards F which is only possible when $\Delta > 0$ (see Lemma 4.1.2). So $A_{i,i+1} \cap B_{i,i+1} \neq \emptyset$.
($\impliedby$) Let $A_{i,i+1} \cap B_{i,i+1} \neq \emptyset$. Then since $\Delta < 0 \implies P_i \notin A_i$ and $P_{i+1} \notin A_{i+1}$. So the necessary conditions are met. On the boundary $B_{i,i+1}$ there is movement towards $F_{i,i+1} = A_{i,i+1} \cap B_{i,i+1}$ from both sides. This means there is a coerced motion on $A_{i,i+1}$ and its ω -limit is $F_{i,i+1}$. □

Theorem 4.1.2. *Let $\Delta > 0$. Then a coerced motion occurs on $A_{i,i+1}$ if and only if $A_{i+2,i+3} \cap B_{i,i+1} \neq \emptyset$.*

Proof.

($\implies$) If there is a coerced motion on $A_{i,i+1}$ it follows that $A_{i+2,i+3} \cap B_{i,i+1} \neq \emptyset$.
($\impliedby$) From $\Delta > 0$ and $A_{i+2,i+3} \cap B_{i,i+1} \neq \emptyset$ we deduce that $P_i \notin A_i$ and $P_{i+1} \notin A_{i+1}$, which are the necessary conditions for coerced motions with positive Δ .
Let Φ be the line containing $A_{i,i+1}$ and $A_{i+2,i+3}$. Then P_i is on the other side of Φ for A_i and so is P_{i+1} for A_{i+1} . So orbits from these regions will certainly approach Φ . And since P_i is in either A_{i+1} or A_{i+2} a line from P_i through the equilibrium F does always pass A_i in such a way, that at least a part of A_i moves into $A_{i,i+1}$ (the line can only come out in A_i or A_{i+3} . If it does come out in the latter, then $P_i \in A_{i+1}$ and all of A_i moves into $A_{i,i+1}$). The analogue is true for P_{i+1} and A_{i+1} . □

Slightly less is needed for sliding motions. A region containing its focus does not prevent sliding motions.

Theorem 4.1.3. *For a sliding motion to occur on $A_{i,i+1}$ it is necessary and sufficient, that*

- $A_{i,i+1} \cap B_{i,i+1} \neq \emptyset$ if $\Delta > 0$ or
- $A_{i+2,i+3} \cap B_{i,i+1} \neq \emptyset$ if $\Delta < 0$.

With positive Δ the motion slides towards the saddle $F_{i,i+1} \in B_{i,i+1}$, with negative Δ towards F .

Proof. For the same reason as with coerced motions the line Φ containing $A_{i,i+1}$ and $A_{i+2,i+3}$ must intersect $B_{i,i+1}$.

A sliding motion on $A_{i,i+1}$ moves towards $\overline{B}_{i,i+1}$. If $A_{i,i+1} \cap B_{i,i+1} \neq \emptyset$ then Δ must be greater than zero. If $A_{i+2,i+3} \cap B_{i,i+1} \neq \emptyset$ then Δ must be negative.

Looking at the above proofs about coerced motions, it is obvious that the conditions are also sufficient for a sliding motion to occur. □

4.2 Index in the Context of Multivalued Functions

In $\mathbb{R}^2$ the **index of an isolated singular point** of the continuous vectorfield g is defined as the winding number of a positively oriented circle with sufficiently small radius around the singularity (or alternatively the winding number of a positively oriented closed curve around the singularity). A formula by Bendixson finds the index of a singular point $\hat{p}$ in the plane to be

$$\text{ind}(\hat{p}) = 1 + \frac{e - h}{2}$$

e stands for the number of elliptic sectors, h for the number of hyperbolic sectors around the singularity.

This definition of the index can be extended to differential inclusions, see [4].

The **index of an isolated equilibrium** $\hat{p}$ of the multivalued function G is defined as

$$\text{ind}_G(\hat{p}) := \deg(-G, K_\epsilon(\hat{p}), 0)$$

The **(local) degree of the multivalued function** G on the open ball $K_\epsilon(\hat{p})$ with respect to 0, $\deg(-G, K_\epsilon(\hat{p}), 0)$, is defined through an approximating continuous (C^1 -) function g such that $g(x) \in G_\delta(x)$ and $g(x) \neq 0 \notin G_\delta(x) \forall x \in \text{bd}(K_\epsilon(\hat{p}))$ with $G_\delta(x) := [\text{co}G(x^\delta)]^\delta$ the closed δ -neighbourhood (for suitable δ) of the convex hull of $G(x^\delta)$ and x^δ the closed δ -neighbourhood of the point x .

$$\deg(G, K_\epsilon(\hat{p}), 0) := \deg(g, K_\epsilon(\hat{p}), 0) = \sum_{x \in K_\epsilon(\hat{p}): g(x)=0} \text{sgn}(\det(g'(x)))$$

Such an approximating continuous (even C^∞) function g can always be found, when G is upper-semicontinuous and $G(x)$ is compact and convex for all x in the domain. The degree of G is also independent of the choice of g ([1],[4]).

For the dynamics of this game we get

$$\text{ind}_G(F) = \text{sgn}(\det(-g'(F))) = \text{sgn}(\det(g'(F)))$$

We can approximate the Heaviside-multivalued function H by continuous functions $h_i^\epsilon : \mathbb{R} \rightarrow [0, 1]$, $i = 1, 2$, such that $h_i^\epsilon(-\epsilon) = 0$, $h_i^\epsilon(\epsilon) = 1$ and $h_i^\epsilon(u)$ is C^1 on $[-\epsilon, \epsilon]$ with $h_i^{\epsilon'}(u) > 0 \forall u \in (-\epsilon, \epsilon)$. Lastly, we want $h_1^\epsilon(0) = \hat{x}$ and $h_2^\epsilon(0) = \hat{y}$

We set $f_1(x, y) = a(x - \hat{x}) + b(y - \hat{y})$ and $f_2(x, y) = c(x - \hat{x}) + d(y - \hat{y})$ with $\Delta = ad - bc \neq 0$. The vectorfield $g_\epsilon(x, y)$ approximating $G(x, y)$ in (2.4) then looks like this:

$$g_\epsilon(x, y) = \begin{pmatrix} h_1^\epsilon(f_1(x, y)) - x \\ h_2^\epsilon(f_2(x, y)) - y \end{pmatrix}$$

To find the index of F we first calculate the Jacobi-determinant at F

$$\begin{aligned} \det(g'_\epsilon(F)) &= \det \left(\begin{pmatrix} h_1^{\epsilon'}(0) \cdot a & h_1^{\epsilon'}(0) \cdot b \\ h_2^{\epsilon'}(0) \cdot c & h_2^{\epsilon'}(0) \cdot d \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \\ &= h_1^{\epsilon'}(0) \cdot h_2^{\epsilon'}(0) \cdot \Delta - h_1^{\epsilon'}(0) \cdot a - h_2^{\epsilon'}(0) \cdot d + 1 \end{aligned}$$

and then try to determine the sign

$$\operatorname{sgn}(\det(g'_\epsilon(F))) = \operatorname{sgn}\left(\Delta - \frac{a}{h_2^{\epsilon'}(0)} - \frac{d}{h_1^{\epsilon'}(0)} + \frac{1}{h_1^{\epsilon'}(0) \cdot h_2^{\epsilon'}(0)}\right) \xrightarrow{\epsilon \rightarrow 0} \operatorname{sgn}(\Delta)$$

because $\lim_{\epsilon \rightarrow 0} h_i^{\epsilon'}(0) = +\infty$ for $i = 1, 2$. So the index of the equilibrium in the interior of $\mathbf{Q}_2$ equals the sign of $\Delta \neq 0$.

For the replicator dynamics the Jacobi-determinant at $F = (\hat{x}, \hat{y}) \in \operatorname{int}(\mathbf{Q}_2)$ is

$$\det \left(\begin{pmatrix} a\hat{x}(1-\hat{x}) & b\hat{x}(1-\hat{x}) \\ c\hat{y}(1-\hat{y}) & d\hat{y}(1-\hat{y}) \end{pmatrix} \right) = \Delta \hat{x} \hat{y} (1-\hat{x})(1-\hat{y})$$

So for the best response and also for the replicator dynamics we get

$$\operatorname{ind} F = \operatorname{sgn} \Delta \tag{4.1}$$

4.3 Possible Flows when $\Delta > 0$

When $\Delta > 0$, the regions A_i are oriented positively, in other words they are arranged in the mathematically positive sense around the equilibrium F , like for instance the four quadrants in the plane around the origin.

We will now look at the possible phase-portraits and determine the important points (ie the possible ω -limits), that describe the overall flow. The lines Φ_1, Φ_2 determine the positions of the four regions relative to their focal points. A region can contain its focal point or lie adjacent to the region containing it (it is “near” its focal point) or lie “far away” from its focus (so the adjacent regions do not contain the focus point either).

If every region contains its focal point, that is $P_i \in A_i \forall i$, F is clearly a source. All corners are sinks and there is a saddle on the boundary between every two of them (Figure 4.1).

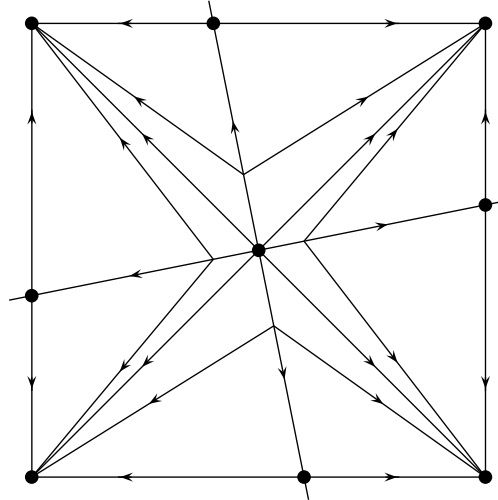


Figure 4.1: $P_i \in A_i \forall i$

If one region contains the maximum of three corners, including its own focus and the corner on the opposite side of the square, the type of the equilibrium inside the square is not so clear.

4.3.1 A Transitive Invariant Region

Theorem 4.3.1. *Let $P_i \in A_i$ and $P_{i+2} \in A_i$ for some i and $\Delta > 0$. Then F is part of a transitive invariant set.*

Proof. From $P_i, P_{i+2} \in A_i$ it follows that either $P_{i+1} \in A_{i+2}$ or $P_{i+3} \in A_{i+2}$. Wlog $P_{i+1} \in A_{i+2}$. Lemma 2.3.1iii) applies to A_{i+2} so there is an orbit going straight to F and all other orbits are leaving A_{i+2} either through $A_{i+1, i+2}$ or $A_{i+2, i+3}$. By Lemma 2.3.1ii) all of A_{i+1} is going to $A_{i+1, i+2}$ and thus there is a coerced motion towards F on $A_{i+1, i+2}$.

Because $F_{i,i+1} \in B_{i,i+1}$ there is a sliding motion towards the border on $A_{i,i+1}$, so orbits can turn into A_i and approach P_i or into A_{i+1} and after finite time merge into the coerced motion on $A_{i+1,i+2}$. Two orbits are leaving $F_{i,i+1}$ on the boundary of the square. One is approaching P_i , the other moves towards $F_{i+1,i+2}$ and reaches it after finite time.

The triangle $FF_{i,i+1}F_{i+1,i+2}$ forms a set in which it is possible to reach every point from any point and, with the exception of the intersection of the triangle with the boundary of the square, it is possible to reach that every point in finite time. It is also possible to spend an infinite amount of time on that triangle. Because of these properties, we call this triangle a **transitive invariant set**.

□

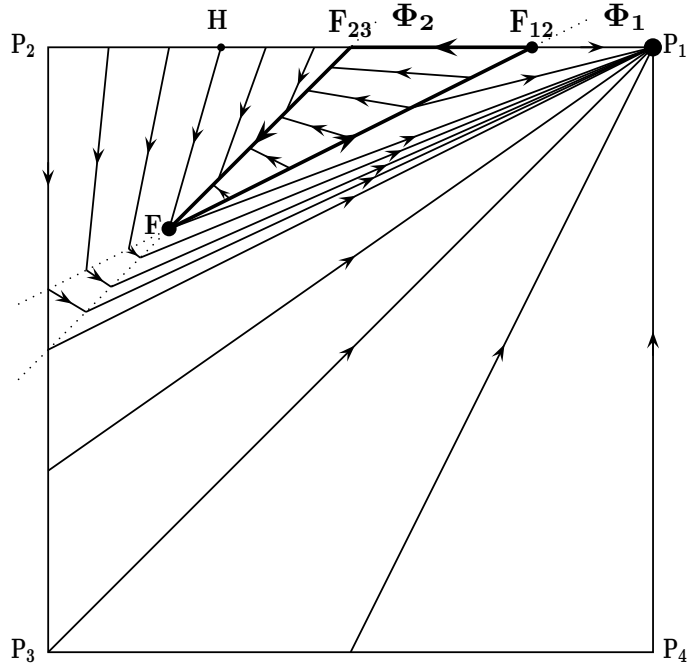


Figure 4.2: Transitive Invariant Set
 P_1 and P_3 lie in region A_1 , where the flow is towards P_1 .

Looking at Figure 4.2 we see that around the equilibrium point F the phase portrait is split into four sectors. The separating orbits are HF , $F_{23}F$, FF_{12} and FP_1 . A_4 and parts of A_1 and A_3 form a hyperbolic sector of orbits going around the equilibrium F towards the sink P_1 on the boundary. The other parts of A_1 and A_3 form two parabolic sectors, one limited by the lines HF and $F_{23}F$. These lines are the two orbits going straight to the internal equilibrium. The other parabolic sector is limited by the two orbits that are leaving the interior equilibrium, FF_{12} and FP_1 . Between the parabolic sectors lies the triangle $FF_{12}F_{23}$, the transitive invariant region, which is an elliptic sector with orbits travelling from F_{12} to F_{23} . The characterization of the different sectors used here, is due to Filippov [4].

The index of F $\text{ind}(F) = 1 = 1 + \frac{e-h}{2}$. F is isolated as an equilibrium, but not as an invariant set. The standard characterization of isolated equilibria in smooth dynamics is not applicable (obviously F is neither sink nor source nor centre and though it does have two stable and two unstable directions, it also doesn't really fit into the idea of a saddle, since the index is not -1).

The corresponding scenario under the replicator dynamics in [14] is boundary flow 2.h with $\Delta > 0$. Not much could be derived for this case. F is either a sink or a source. Limit cycles might occur, but existence could not be proven.

Compare also the example below, which appears in [10]. A similar type of transitive invariant set occurs in a 3×3 symmetric 2-person game with payoff-matrix

$$\begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}$$

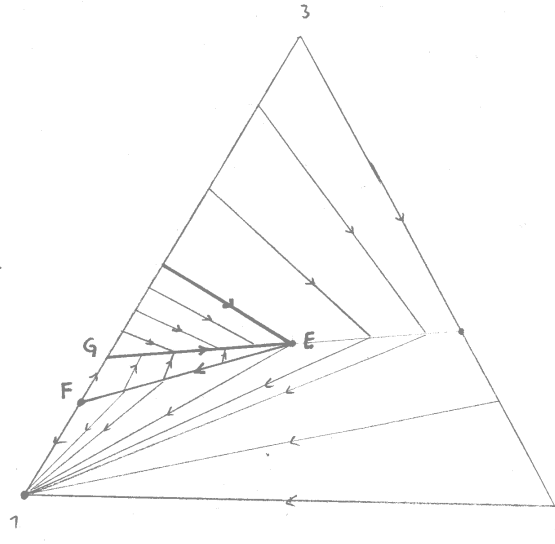


Figure 4.3: Transitive Invariant Set EFG in a 3×3 game from [10]

4.3.2 Sinks and Sources

The following says that when there is a region containing its focal point and the above does not occur, then F must be a source.

Theorem 4.3.2. *If $\forall i : P_i \in A_i \implies P_{i+2} \notin A_i$, $\exists i : P_i \in A_i$ and $\Delta > 0$ then F is a source.*

Proof. Let $P_i \in A_i$ for some i . There are two possibilities:

- $P_{i+2} \in A_{i+2}$ Since $\Delta > 0$ each region is at least near its focus, so the flow in a region does not move towards F , but towards its focal point or the region containing the focal point. The remaining focal points are thus in their own or one of the neighbouring regions. This admits two to four corners to be sinks. The number of saddles on the boundary is the same as the number of sinks. There can be no sinks on the open boundary lines and F is ω -limit for only one solution, $x(t) \equiv F$. See figures 4.4, 4.5 and 4.6.
- $P_{i+2} \notin A_{i+2}$ By assumption $P_{i+2} \notin A_i$. Then let $P_{i+2} \in A_{i+1}$ (the other possibility is $P_{i+2} \in A_{i+3}$, simply replace $i+1$ with $i+3$ in the following). The flow of A_{i+2} moves entirely into A_{i+1} , which itself either contains its focal point or the flow moves on to A_i , which by assumption contains its focus. The behaviour on the other side is similar. A_{i+3} either moves entirely into A_{i+2} or contracts to its focal point if $P_{i+3} \in A_{i+3}$ (notice that $P_{i+3} \in A_i$ is not possible). So anything from one to three sinks in the corners is possible with the same number of saddles on the boundary. See figures 4.7 and 4.8.

So the interior of the square (apart from F itself) converges towards the boundary. It follows that F is a source.

□

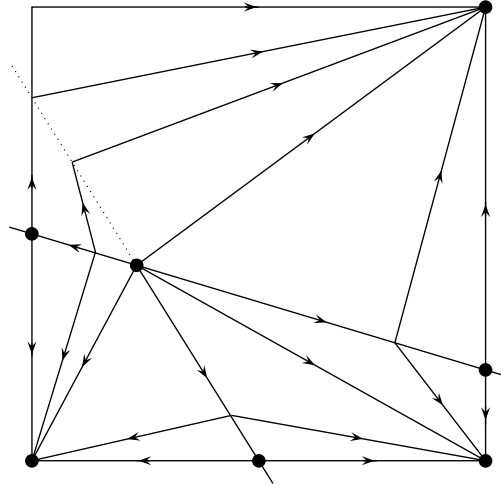


Figure 4.4: $P_i \in A_i$ and $P_{i+2} \in A_{i+2}$ Variant 1

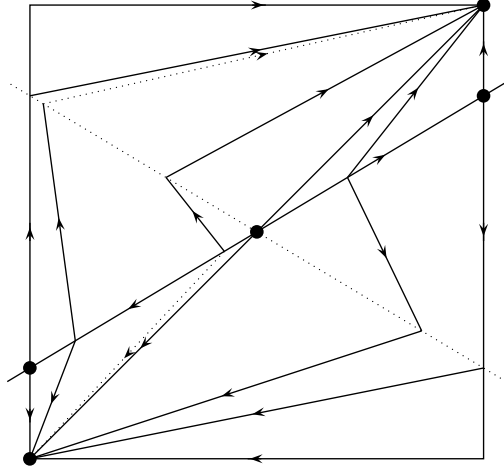


Figure 4.5: $P_i \in A_i$ and $P_{i+2} \in A_{i+2}$ Variant 2

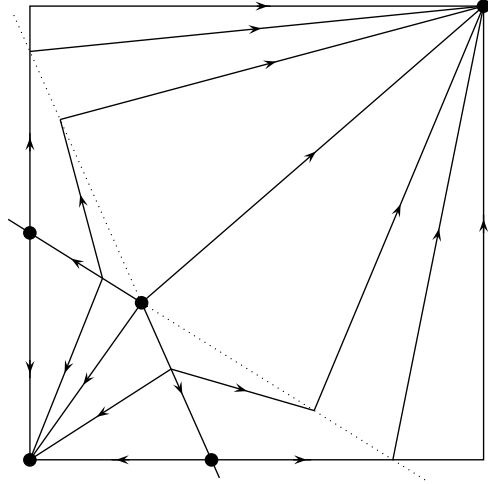


Figure 4.6: $P_i \in A_i$ and $P_{i+2} \in A_{i+2}$ Variant 3

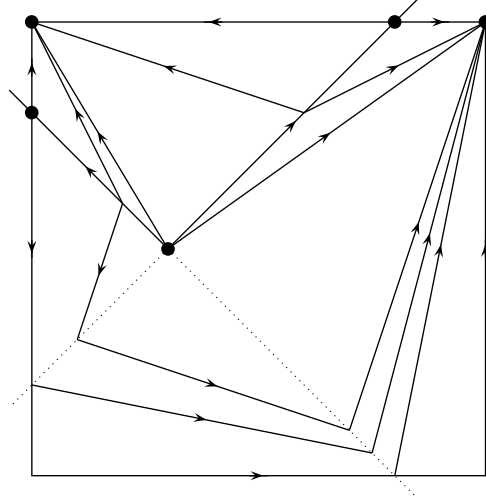


Figure 4.7: $P_i \in A_i$ and $P_{i+2} \notin A_{i+2}$ Variant 1

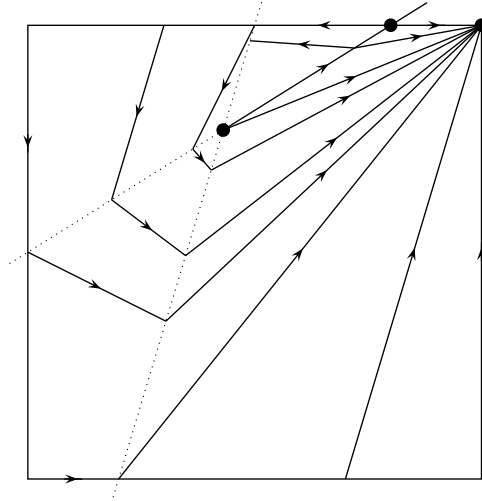


Figure 4.8: $P_i \in A_i$ and $P_{i+2} \notin A_{i+2}$ Variant 2

When none of the regions contains its focus, but every region is still near its focus point and the Φ -lines are placed complaisantly, the flow from the boundary and the orbits coming from the source F asymptotically approach a limit polygon, see 4.3.3. When the Φ -lines are positioned not quite so nicely, the internal equilibrium becomes the global attractor.

Theorem 4.3.3. *Let $P_i \notin A_i \forall i$, $P_i \in A_{i+1}$ for some i and $\Delta > 0$. Then there are two possibilities:*

- a) *Every region contains only one corner: then it depends on a certain constant ρ whether F is a sink or there is an asymptotically stable Shapley polygon and F is a source. This is discussed in section 4.3.3.*
- b) *Some region contains more than one corner: then F is globally asymptotically stable.*

Proof. Part b). If $P_i \in A_{i+1}$ for some i and regions are positively oriented, then the flow from A_i moves into $A_{i,i+1}$ (Lemma 2.3.1). Since $P_{i+1} \notin A_{i+1}$ and $A_{i+1} \cap B_{i,i+1} \neq \emptyset$ there is a motion towards P_{i+1} on the boundary $B_{i,i+1}$, which implies that part of A_{i+1} transitions into $A_{i+1,i+2}$. So we have $A_i \longrightarrow A_{i,i+1}$ and at least a part of $A_{i+1} \longrightarrow A_{i+1,i+2}$. Since none of the regions contain their focal point, there are two possibilities:

- Every region contains only one corner: This is discussed in section 4.3.3.
- Some region contains more than one corner, then either
 - (a) $P_{i+3} \in A_{i+1}$ In this case lemma 2.3.1 says, that A_{i+3} is split in two regions. Orbits on the separating line reach F in finite time. The orbits in these two regions are driven out of A_{i+3} . Also orbits are driven out of A_{i+2} to one or both borderlines. This implies a coerced motion towards F (Lemma 4.1.2) on $A_{i+2,i+3}$ if $P_{i+2} \in A_{i+3}$ or $A_{i+1,i+2}$ if $P_{i+2} \in A_{i+1}$ or both if $P_{i+2} \in A_i$. So we have $\mathbf{A}_{i+3} \rightarrow \mathbf{A}_{i,i+3} \rightarrow \mathbf{A}_i \rightarrow \mathbf{A}_{i,i+1} \leftrightarrow A_{i+1} \rightarrow A_{i+1,i+2} \leftrightarrow A_{i+2} \leftrightarrow \mathbf{A}_{i+2,i+3} \leftarrow \mathbf{A}_{i+3}$. The behaviour in the bold areas is definite, the behaviour in the light areas depends on where the remaining corners lie. But no matter where, there will always be a coerced motion catching the light area's orbits.
 - (b) $P_{i+3} \in A_i$ Then, since some region contains more than one corner, one of these hold
 - * $P_{i+2} \in A_i$, analogous to (a).
 - * $P_{i+2} \in A_{i+3}$ and consequently $P_{i+1} \in A_{i+3}$, so this is also analogous to case (a).

This suffices to show that F is a global sink (there is no outward motion and there are no other static states but F). Compare figures 4.9 through 4.13.

□

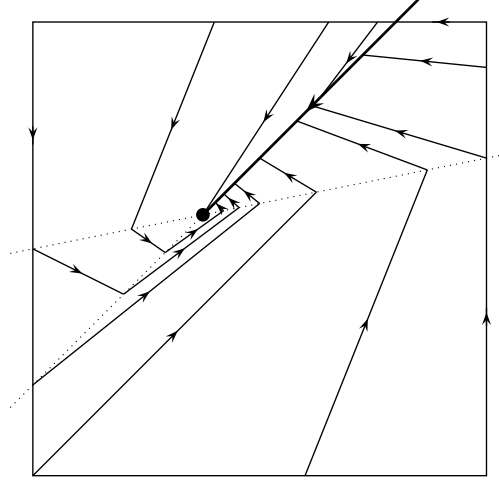


Figure 4.9: $P_i \notin A_i \forall i$ and $P_i \in A_{i+1}$ Type 1

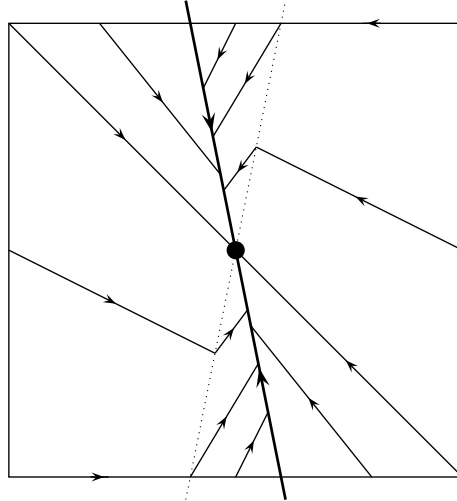


Figure 4.10: $P_i \notin A_i \forall i$ and $P_i \in A_{i+1}$ Type 2

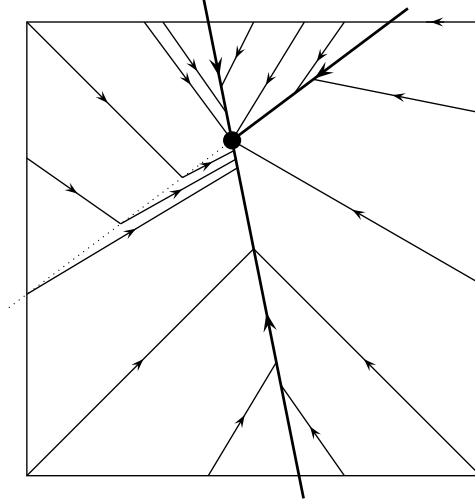


Figure 4.11: $P_i \notin A_i \forall i$ and $P_i \in A_{i+1}$ Type 3

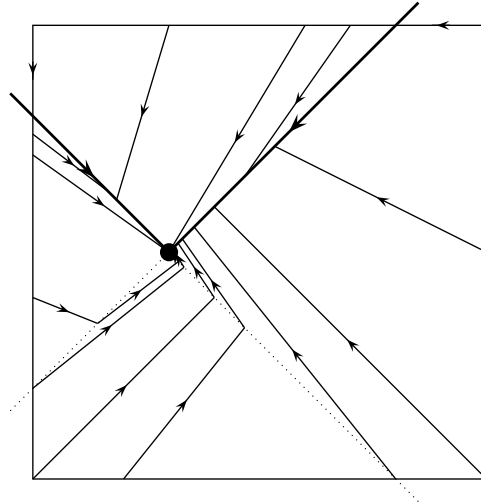


Figure 4.12: $P_i \notin A_i \forall i$ and $P_i \in A_{i+1}$ Type 4

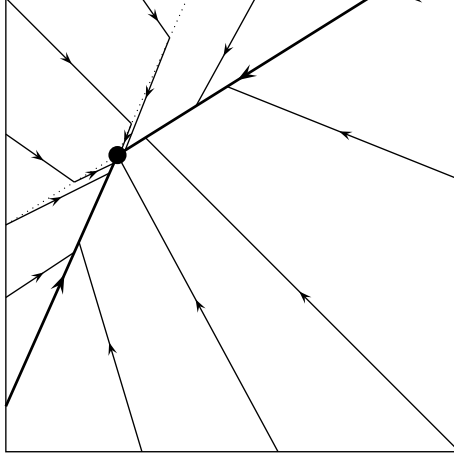


Figure 4.13: $P_i \notin A_i \forall i$ and $P_i \in A_{i+1}$ Type 5

Theorem 4.3.4. *Let $P_i \notin A_i \forall i$, $P_i \notin A_{i+1} \forall i$, $P_i \in A_{i+2}$ for some i and $\Delta > 0$. Then F is globally asymptotically stable.*

Proof. In this scenario either $P_i \in A_{i+2} \forall i$ or $P_i \in A_{i-1}$ for some i . In the second case, the proof reads exactly like the one of theorem 4.3.3b) with $i+1$ and $i-1 = i+3$ interchanged, since this is just the scenario there, but mirrored.

$P_i \in A_{i+2} \forall i$ means that in each region there is an orbit reaching F in finite time while the rest of the orbits are moving out of their respective region. That means we get four coerced motions on the $A_{i,j}$. These are moving towards F since $\Delta > 0$, see figure 4.14.

So F is globally asymptotically stable. □

Theorem 4.3.5. *Let $P_{i+1} \in A_i \forall i$ and $\Delta > 0$. Then it depends on a certain constant ρ whether F is a sink or a source.*

Proof. This is discussed in section 4.3.3. □

So with the exception of the limit-cycles the above observations describe all the possible phase-portraits that occur with positive Δ , since at least one of the above theorems is applicable. The following section takes a look at the scenario of the stable limit cycle.

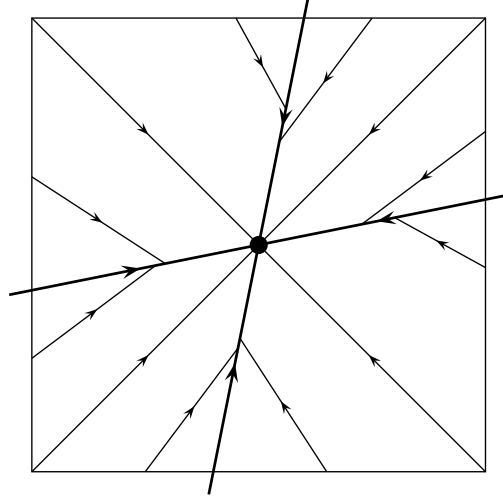


Figure 4.14: $P_i \in A_{i+2} \forall i$

4.3.3 Shapley Polygons

The term Shapley polygon was introduced in [6] and named after American game-theorist Lloyd Shapley, honouring his publication from 1964 [15], where this kind of limit cycle occurred as the limit set of a fictitious play process in a certain 3×3 bimatrix game. The proof of existence using the Poincaré section follows the example of Glass and Pasternack [8]. Shapley polygons also appear in [10] and [9].

Definition 4.3.1.

$$\rho := \frac{f(a+e)(b+e)(c+d+f)}{e(c+f)(d+f)(a+b+e)} = \frac{\phi_2(0,0)\phi_1(1,0)\phi_1(0,1)\phi_2(1,1)}{\phi_1(0,0)\phi_2(1,0)\phi_2(0,1)\phi_1(1,1)} \quad (4.2)$$

Lemma 4.3.1. *If $\Delta > 0$ and $P_i \in A_{i+1} \forall i$, then*

$$\rho > 1 \iff bd(1-\hat{y})\hat{y} < ac(1-\hat{x})\hat{x} \quad (4.3)$$

Proof. This was verified using a computer algebra system. □

The condition on the right hand side is from [14] and indicates, that for the boundary flow 0.c the heteroclinic cycle on the boundary of the square is attractive under the replicator dynamics. The internal equilibrium can be an attractor, too, so the replicator dynamics may admit two attractors for the boundary flow 0.c. Under the best response dynamics the above is the condition for the existence of an (globally) asymptotically stable Shapley polygon, as we will see next. The time averages of the solutions of the replicator dynamics diverging to the boundary converge towards the Shapley polygon, as follows from [12].

The condition of the lemma with “>” instead of “<” means that the boundary must be a repeller under the replicator dynamics [14]. Stable limit cycles can occur. The best response dynamics in this case has the globally asymptotically stable equilibrium F , so does not admit limit cycles.

Theorem 4.3.6. *Let $P_i \notin A_i \forall i$, $P_i \in A_{i+1}$ for some i and $\Delta > 0$. Let every region contain one corner. Then if $\rho > 1$ there is an asymptotically stable Shapley polygon and F is a source. Otherwise, if $\rho \leq 1$, F is globally asymptotically stable.*

or equivalently

Theorem 4.3.7. *Let $P_i \in A_{i+1} \forall i$ and $\Delta > 0$. Then if $\rho > 1$ there is an asymptotically stable Shapley polygon and F is a source. Otherwise, if $\rho \leq 1$, F is globally asymptotically stable.*

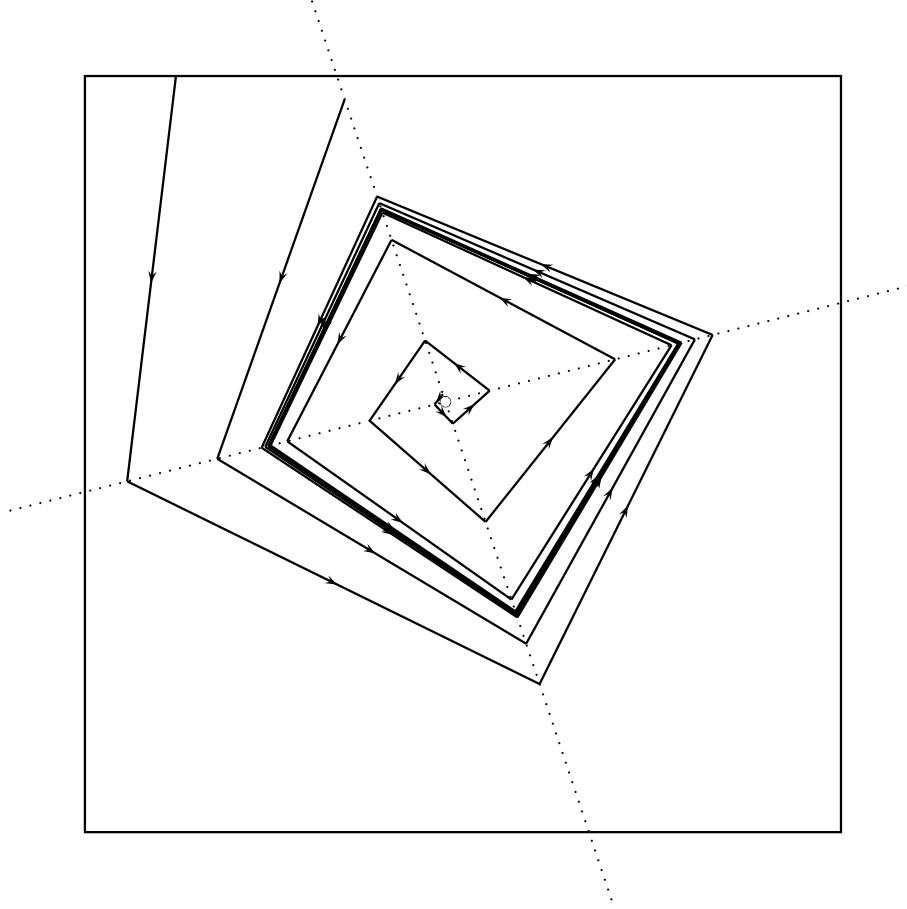


Figure 4.15: Flow converging to a Shapley polygon

Proof. $P_i \in A_{i+1} \forall i$ and $\Delta > 0 \implies P_1 \in A_2, P_2 \in A_3, P_3 \in A_4, P_4 \in A_1 \iff$

$$\begin{array}{lcl} \phi_1(0,0) = & e > 0 & \left| \begin{array}{l} \phi_2(0,0) = f < 0 \\ \phi_2(1,0) = c + f > 0 \\ \phi_2(0,1) = d + f < 0 \\ \phi_2(1,1) = c + d + f > 0 \end{array} \right. \implies \begin{array}{l} b < 0 \\ c > 0 \end{array} \\ \phi_1(1,0) = & a + e > 0 & \\ \phi_1(0,1) = & b + e < 0 & \\ \phi_1(1,1) = & a + b + e < 0 & \end{array}$$

The flow evolves from $A_i \rightarrow A_{i,i+1} \rightarrow A_{i+1} \forall i$, so we get a “circulation” around the equilibrium F . We will now calculate a return map from $A_{i,i+1} \rightarrow A_{i,i+1}$, which follows a solution once around F . The fixed point of this Poincaré map will tell us, where

a solution passing through $A_{i,i+1}$ (which is almost every solution) is going to end up.

To make calculations easier, we will translate the unit square by the vector $-F$ to put the equilibrium in the origin and apply a transformation M so that Φ_1 and Φ_2 become the axes. So we introduce the translation

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad Tv := v - F \quad \text{and its inverse} \quad T^{-1}v = v + F \quad v \in \mathbb{R}^2 \quad (4.4)$$

base transformation matrices

$$M := -\frac{bc}{\Delta} \begin{pmatrix} 1 & \frac{d}{c} \\ \frac{a}{b} & 1 \end{pmatrix} \quad \text{and} \quad M^{-1} = \begin{pmatrix} 1 & -\frac{d}{c} \\ -\frac{a}{b} & 1 \end{pmatrix} \quad (4.5)$$

and operators

$$K := MT \quad \text{and} \quad K^{-1} := T^{-1}M^{-1}. \quad (4.6)$$

The corners or focal points now have the coordinates

$$\begin{aligned} KP_1 &=: (p_1^x, p_1^y) = \left(-\frac{b(c+d+f)}{\Delta}, -\frac{c(a+b+e)}{\Delta} \right) \\ KP_2 &=: (p_2^x, p_2^y) = \left(-\frac{b(d+f)}{\Delta}, -\frac{c(b+e)}{\Delta} \right) \\ KP_3 &=: (p_3^x, p_3^y) = \left(-\frac{bf}{\Delta}, -\frac{ce}{\Delta} \right) \\ KP_4 &=: (p_4^x, p_4^y) = \left(-\frac{b(c+f)}{\Delta}, -\frac{c(a+e)}{\Delta} \right) \end{aligned}$$

In the new coordinates we start the Poincaré map at $R_0 := (0, r_0)$ on the positive y-axis, which reflects a point on Φ_2 that is above F , thus on $A_{2,3}$. From here the flow moves in the direction of KP_3 until it reaches the negative x-axis at $R_1 := (r_1, 0)$. So we have to solve

$$R_1 = R_0 + t(KP_3 - R_0) \quad (4.7)$$

or

$$r_1 = tp_3^x \quad (4.8)$$

$$0 = r_0 + t(p_3^y - r_0) \quad (4.9)$$

which solves to

$$r_1 = -\frac{bfr_0}{ce + (ad - bc)r_0} = -\frac{\frac{bf}{ce}r_0}{1 + \frac{\Delta}{ce}r_0}, \quad t = \dots > 0 \quad (4.10)$$

From R_1 the flow moves to $R_2 := (0, r_2)$ on the negative y-axis. Again

$$R_2 = R_1 + t(KP_4 - R_1) \implies r_2 = \frac{(a+e)fr_0}{ce + ef + (ad - bc)r_0} = \frac{\frac{(a+e)f}{e(c+f)}r_0}{1 + \frac{\Delta}{e(c+f)}r_0} \quad (4.11)$$

From R_2 the flow moves to $R_3 := (r_3, 0)$ on the positive x-axis. We get:

$$R_3 = R_2 + t(KP_1 - R_2) \implies r_3 = -\frac{\frac{b(a+e)f(c+d+f)}{ce(a+b+e)(c+f)}r_0}{1 + \frac{\Delta(c(a+b+e)+(a+e)f)}{ce(a+b+e)(c+f)}r_0} \quad (4.12)$$

And to complete the circulation the flow moves now to $R_4 := (0, r_4)$. We get:

$$R_4 = R_3 + t(KP_2 - R_3) \implies r_4 = \frac{\frac{(a+e)(b+e)f(c+d+f)}{e(a+b+e)(c+f)(d+f)}r_0}{1 + \frac{\Delta(d(a+b+e)+bf)}{e(a+b+e)(c+f)(d+f)}r_0} \quad (4.13)$$

Thus $r_0 \mapsto \frac{\rho r_0}{1+\delta r_0}$ with constants

$$\rho := \frac{(a+e)(b+e)f(c+d+f)}{e(a+b+e)(c+f)(d+f)} \quad \text{and} \quad \delta := \frac{\Delta(d(a+b+e)+bf)}{e(a+b+e)(c+f)(d+f)}. \quad (4.14)$$

We notice $\rho > 0$.

With $s_n := \sum_{j=0}^{n-1} \rho^j$, iterating this map leads to

$$\tau(r) := \frac{\rho r}{1 + \delta r} \quad (4.15)$$

$$\tau \circ \tau(r) = \tau^2(r) = \frac{\rho \frac{\rho r}{1+\delta r}}{1 + \delta \frac{\rho r}{1+\delta r}} = \frac{\rho^2 r}{1 + (1+\rho)\delta r} \quad (4.16)$$

$$\tau^n(r) = \frac{\rho^n r}{1 + (1 + \rho + \rho^2 + \dots + \rho^{n-1})\delta r} \quad (4.17)$$

which we readily verify by taking the induction step

$$\tau \circ \tau^n(r) = \frac{\rho \frac{\rho^n r}{1+s_n \delta r}}{1 + \delta \frac{\rho^n r}{1+s_n \delta r}} = \frac{\frac{\rho^{n+1} r}{1+s_n \delta r}}{\frac{1+\delta r(s_n+\rho^n)}{1+s_n \delta r}} = \quad (4.18)$$

$$= \frac{\rho^{n+1} r}{1 + (1 + \rho + \dots + \rho^{n-1} + \rho^n)\delta r} = \tau^{n+1}(r) \quad (4.19)$$

So

$$\tau^n(r) = \frac{\rho^n r}{1 + \frac{\rho^n - 1}{\rho - 1} \delta r} = \frac{r}{\frac{1}{\rho^n} + \frac{\rho^n - 1}{\rho^n(\rho - 1)} \delta r} = \frac{r}{\frac{1}{\rho^n} + \frac{1 - 1/\rho^n}{\rho - 1} \delta r} \quad \text{if } \rho \neq 1 \quad (4.20)$$

and

$$\lim_{n \rightarrow \infty} \tau^n(r) = \begin{cases} 0 & \text{if } \rho \leq 1 \\ \frac{\rho - 1}{\delta} & \text{if } \rho > 1 \end{cases} \quad (4.21)$$

So if $\rho > 1$ a solution in the transformed coordinates passing through the positive y -axis converges to the Shapley-polygon implied by $(0, \frac{\rho-1}{\delta})$. Every solution $f(t)$ except for $f(t) \equiv \mathbf{0} = KF$ will do so. F is thus a source.

If $\rho \leq 1$ every solution in the transformed coordinates converges to the origin as $t \rightarrow \infty$. F is then globally asymptotically stable.

□

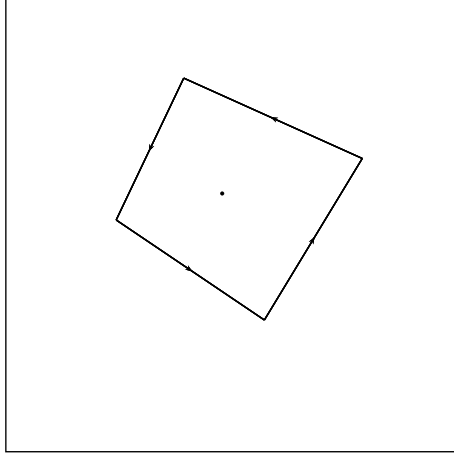


Figure 4.16: An orbit forming a Shapley polygon

Theorem 4.3.8. *Let $P_{i+1} \in A_i \forall i$ and $\Delta > 0$. Then if $\rho > 1$ there is an asymptotically stable Shapley polygon and F is a source. Otherwise, if $\rho \leq 1$, F is globally asymptotically stable.*

This is just the above statement with the orbits now moving clockwise around the equilibrium. This happens when the half-rays of each Φ are interchanged and the regions are thus mirrored, the corner each region contains is now the opposite of the corner in the counter-clockwise situation. The inequalities in the proof are reversed, what was positive is now negative and vice versa. In terms of qualitative behaviour, this does not matter. The return map stays the same, as the four different vector fields are passed just the same, describing where solutions are passing through the positive y-axis in the transformed square. The flow does converge to a Shapley polygon under the same conditions.

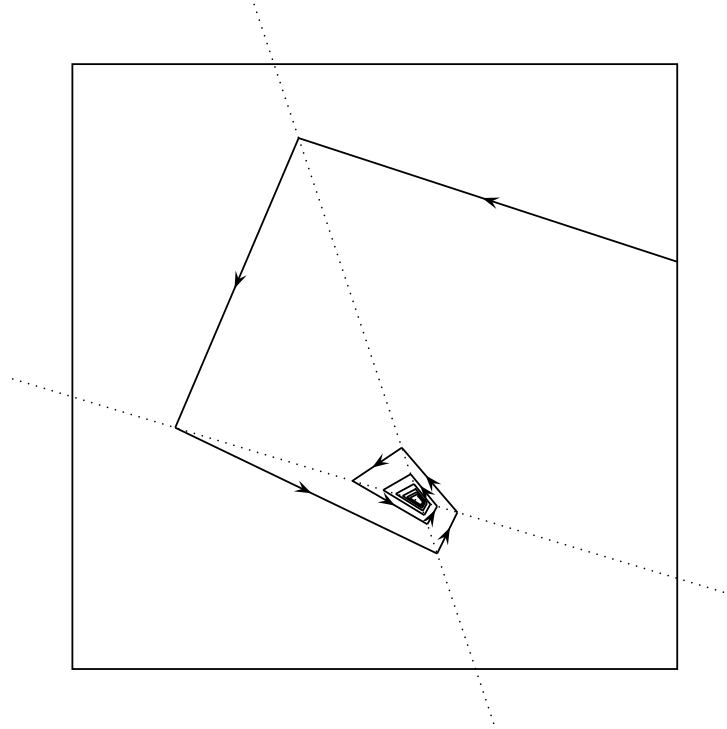


Figure 4.17: Flow converging to the equilibrium in the interior.
Here, $\rho \leq 1$ and orbits are circling into the equilibrium.

4.4 $\Delta < 0$ or F is a Saddle

The behaviour of the flow when $\Delta < 0$ proves much more consistent. We begin by defining when we call the internal equilibrium a ‘saddle’. This seems necessary, considering what we have seen in section 4.3.1.

Definition 4.4.1. When there are two asymptotically stable equilibria on the boundary, whose basins of attraction are separated by the internal equilibrium F and two orbits approaching it and these three are the only orbits not converging to the boundary then we call the equilibrium F a **saddle**.

Theorem 4.4.1. *Let $\Delta < 0$. Then F is a saddle.*

Proof. One of these situations has to occur, when $\Delta < 0$:

- $P_i \in A_i$ for some $i \rightarrow$ Lemma 4.4.1
- $P_i \notin A_i \forall i$ and $P_i \in A_{i+1}$ for some $i \rightarrow$ Lemma 4.4.2
- $P_i \notin A_i \forall i$ and $P_i \in A_{i-1}$ for some $i \rightarrow$ Lemma 4.4.3

The only case not covered by this list is $P_i \in A_{i+2} \forall i$, which is not possible when $\Delta < 0$.

□

Lemma 4.4.1. *Let $\Delta < 0$ and $P_i \in A_i$ for some i . Then F is a saddle.*

Proof. Let $P_i \in A_i$ for some i . P_i is then a sink. Surely $P_{i+1} \notin A_{i+1}$ and $P_{i+3} \notin A_{i+3}$, so the flow is leaving those regions some way or another. Because a borderline can intersect with at most three regions, there is a motion with focus P_{i+2} on $A_{i+2} \cap (B_{i+1,i+2} \cup B_{i+2,i+3}) \neq \emptyset$, which either reaches its goal or one of the rays delimiting A_{i+2} , on the other side of which the flow is also approaching ($\Delta < 0$), creating a coerced motion towards the border on that ray.

F has exactly two stable directions:

If $P_{i+2} \in A_i$ then there is an isolated direct path towards F in A_{i+2} . A_i contains another corner P_j ($j = i+1$ or $i+3$). Then $F_{i+2,j} \in B_{i,j+2}$, so each of the focal points of the regions adjoining $A_{i,j+2}$ lies on a different side of the line Φ containing $A_{i,j+2}$ and $A_{i+2,j}$, allowing an orbit straight to F on $A_{i,j+2}$, which is the other inset of F .

Let now $P_{i+2} \notin A_i$.

If $P_{i+1} \in A_{i+3}$ or $P_{i+3} \in A_{i+1}$ there is an isolated direct path towards F in A_{i+1} or A_{i+3} respectively.

If $P_{i+1} \notin A_{i+3}$ there is a F -convergent orbit on one of the rays $A_{i+1,i+2}$, if $P_{i+1} \in A_i$ or $A_{i,i+1}$, if $P_{i+1} \in A_{i+2}$. This is true because we know where the other half of the line Φ containing the ray in question intersects the border: in-between the focal points of the regions adjoining said ray.

If $P_{i+3} \notin A_{i+1}$ there is a F -convergent orbit on one of the rays $A_{i+2,i+3}$, if $P_{i+3} \in A_i$ or $A_{i,i+3}$, if $P_{i+3} \in A_{i+2}$, following the above reasoning.

Notice that these motions are not coerced, since there is always the possibility of turning into an adjacent region. Every solution not starting on or leaving one of the two orbits going to F will converge to one of the two sinks on the boundary. So F is a saddle. □

Lemma 4.4.2. *Let $\Delta < 0$, $P_i \notin A_i \forall i$ and $P_i \in A_{i+1}$ for some i . Then F is a saddle.*

Proof. $P_{i+1} \notin A_{i+1}$, $P_i \in A_{i+1} \implies A_{i,i+1} \cap B_{i,i+1} \neq \emptyset$, which means there is a coerced motion towards $F_{i,i+1} \in B_{i,i+1}$ on $A_{i,i+1}$. If $A_{i+2,i+3}$ (which is on the same Φ -line as $A_{i,i+1}$) intersects the opposing borderline $B_{i+2,i+3}$, the behaviour of $A_{i,i+1}$ is mirrored and there is a coerced motion towards the border on $A_{i+2,i+3}$.

On the other hand, if $A_{i+2,i+3}$ intersects any of the remaining borderlines, for instance $B_{i,i+3}$, then $A_{i,i+3}$ must also intersect $B_{i,i+3}$ or else $P_{i+3} \in A_{i+3}$, which is ruled out by assumption. So there would be a coerced motion on $A_{i,i+3}$ or $A_{i+1,i+2}$ respectively.

So we find that there are two sinks on the border that split the square into two basins of attraction, since there are no other coerced motions (rays not in place) and $P_i \notin A_i \forall i$, which means the flow doesn't generally stay where it is and only comes to a halt if it is an isolated direct path towards F or if it gets caught in a coerced motion and ends up on the border, see lemma 2.3.1. The orbits separating the two basins are the insets of F . □

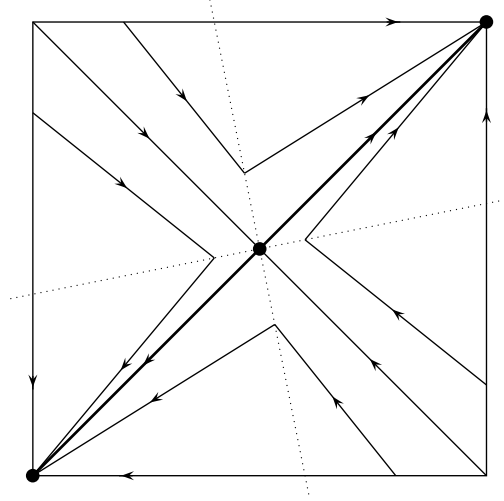


Figure 4.18: $P_i \in A_i$ Type 1 - $P_{i+2} \in A_{i+2}$

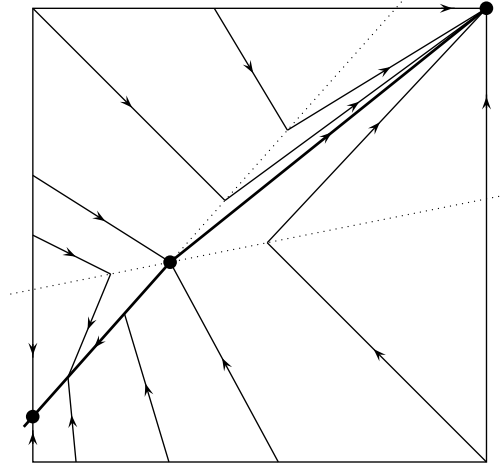


Figure 4.19: $P_i \in A_i$ Type 2 - $P_{i+2} \notin A_{i+2}$

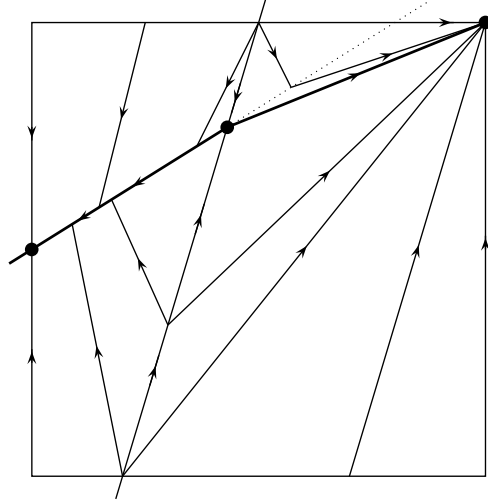


Figure 4.20: $P_i \in A_i$ Type 3

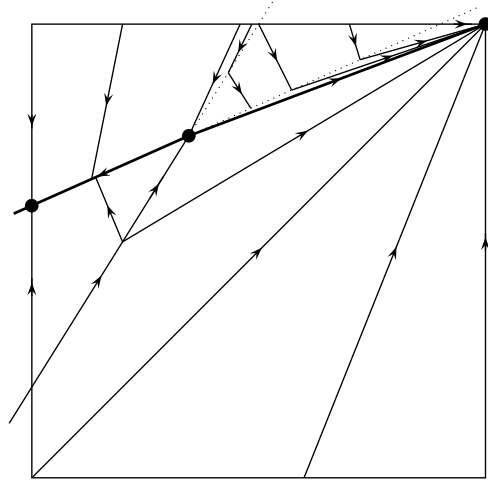


Figure 4.21: $P_i \in A_i$ Type 4

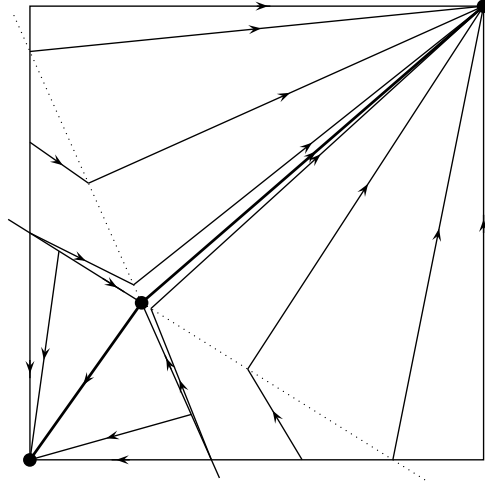


Figure 4.22: $P_i \in A_i$ Type 5

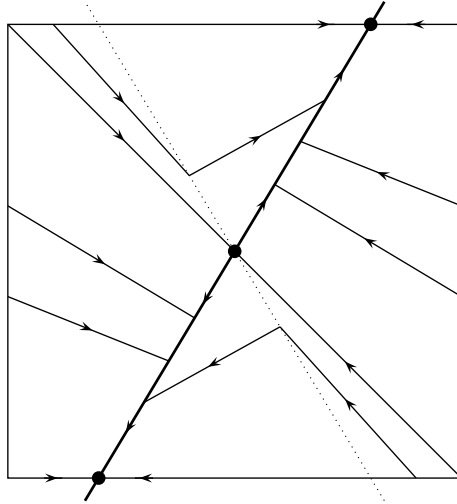


Figure 4.23: $P_i \in A_{i+1}$ Type 1

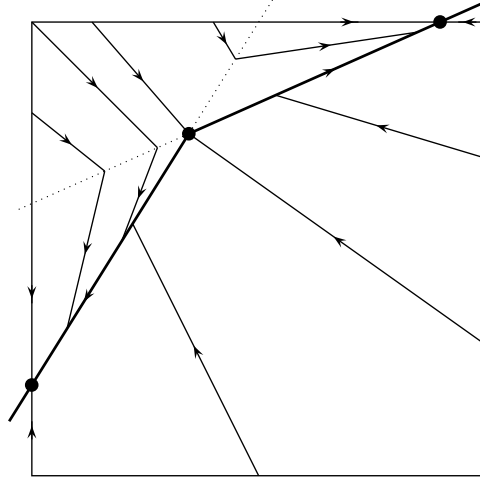


Figure 4.24: $P_i \in A_{i+1}$ Type 2

Lemma 4.4.3. *Let $\Delta < 0$, $P_i \notin A_i \forall i$ and $P_{i+1} \in A_i$ for some i . Then F is a saddle.*

Proof. This is again a mirrored scenario. The proof is analogous to the one above.

□

4.5 Summary

Summarizing the above results, we find that $\Delta > 0$ admits all orbits converging to the boundary as well as to the internal equilibrium. It admits limit-cycles and also something like a limit-region. When $\Delta < 0$ the behaviour is uniformly in that F is always a saddle and there are two attractors on the boundary.

We formulate

Theorem 4.5.1. *Let $\Delta > 0$. Then F has index $+1$.*

F is part of a transitive invariant set, when $P_i \in A_i$ and $P_{i+2} \in A_i$ for some i .

F is globally asymptotically stable, if $P_i \notin A_i \forall i$ and if $\rho \leq 1$ whenever $P_i \in A_{i\pm 1} \forall i$.

F is a source, if $P_i \in A_i$ for some i (and $P_{i+2} \notin A_i \forall i$ with this property) and also if $\rho > 1$ whenever $P_i \in A_{i\pm 1} \forall i$.

and

Theorem 4.5.2. *Let $\Delta < 0$. Then F has index -1 and is a saddle. There are two attracting equilibria on the boundary.*

Proof. These follow from the previous results. □

With the exception of the case $\rho = 1$, the restrictions (2.1) and (2.2) limit the analysis to robust games, meaning the phase portrait does not topologically change under small perturbations of the coefficients of the equations. The restrictions make sure, that there always is interaction within and between the populations. The set of robust games is open and dense.

We end by classifying the different types of flow by the numbers of sinks, saddles and sources that appear in the phase portraits with an equilibrium in the interior of $\mathbf{Q}_2$. We begin with the cases, when F is a source, then the cases when F is a saddle and when F is part of a transitive invariant set. The last line corresponds to the case when F is globally asymptotically stable.

(Sinks, Saddles, Sources)	Figure	
(4, 4, 1)	4.1	$P_i \in A_i \forall i$
(3, 3, 1)	4.4	$P_i \in A_i, P_{i+2} \in A_{i+2}$ Variant 1
(2, 2, 1)	4.5-4.7	
(1, 1, 1)	4.8	$P_i \in A_i, P_{i+2} \notin A_{i+2}$ Variant 2
($\diamond$, 0, 1)	4.15	Shapley polygon
(2, 1, 0)	4.18-4.25	$\Delta < 0$
(1, 1, $\triangle$)	4.2	transitive invariant set
(1, 0, 0)	4.9-4.14, 4.17	

$\diamond \triangleq$ Shapley-polygon

$\triangle \triangleq$ transitive invariant set

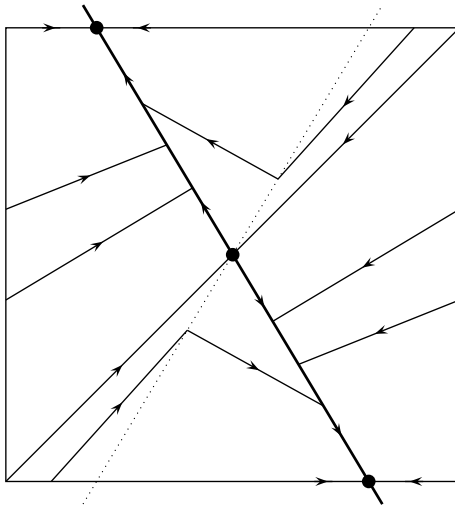


Figure 4.25: $P_i \in A_{i-1}$

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Deutsche Zusammenfassung

Diese Arbeit befasst sich mit Spielen zwischen zwei Populationen, wobei in jeder Population ein Spieler aus zwei reinen Strategien wählen kann. Spieler interagieren mit Spielern aus der anderen, als auch aus der eigenen Population. Diese Klasse von Spielen wurde in [14] bezüglich der Replikatordynamik untersucht. Diese Dynamik liefert eine Vielzahl möglicher Flüsse, insbesondere Grenzzyklen.

Es soll diese Klasse von Spielen für die Best Response Dynamik untersucht werden. Das Problem führt auf ein System von stückweise linearen Differentialgleichungen erster Ordnung in der Ebene. Ähnliche Gleichungen wurden von Leon Glass und J.S. Pasternack [8] untersucht. Die Gleichung bestimmen im $\mathbb{R}^2$ vier disjunkte offene Teilmengen, deren Vereinigung dicht liegt. In jeder dieser Mengen zeigt das Vektorfeld auf eine andere einzelne beste Antwort, den Fokuspunkt der Region. An den Grenzen zwischen den Regionen wird die Auszahlung durch ein ganzes Intervall bester Antworten maximiert. Das Verhalten an diesen Unstetigkeitsstellen wird in natürlicher Weise durch die Theorie der Differentialinklusionen beschrieben, was sodann die Differentialinklusion

$$\dot{x} \in H(x) - x \tag{4.22}$$

auf einem Definitionsbereich $D \subseteq \mathbb{R}^2$ liefert. Die Komponenten der rechten Seite sind stückweise lineare, nach oben halbstetige, mengenwertige Abbildungen und $H(x)$ ist kompakt und konvex für alle $x \in D$. Die Theorie der Differentialinklusionen (etwa [4] oder [1]) liefert damit die Existenz von Lösungen für positive Zeit. Eindeutigkeit der Lösungen ist zumindest dort gegeben, wo jede Komponentenfunktion von H eine Funktion im üblichen Sinne darstellt (nicht mengenwertig), ist aber im Allgemeinen an den Unstetigkeitsstellen nicht gegeben. Lösungen für positive Zeit können explizit angegeben werden und entstehen durch stetige Zusammensetzung von Lösungen der Form

$$x(t) = P + (x_0 - P)e^{-t}$$

wobei $x_0 \in D$ eine Anfangsbedingung und $P \in D$ ein Fokuspunkt sind. Die Trajektorie einer Lösung ist also eine stetige Kombination von Geradenstücken und Punkten. Es sei nochmals darauf hingewiesen, dass Lösungen mit Startwert x_0 im Allgemeinen nicht eindeutig bestimmt sind, weshalb es unter Umständen sehr viele geben kann.

Das Ziel dieser Arbeit ist, die möglichen Szenarien zu beschreiben, allerdings wird die Untersuchung auf generische Fälle beschränkt. Degenerierte Fälle, wie etwa die Abwesenheit von Selbstinteraktion in einer Population, werden ausgelassen. Es ergibt sich eine Anzahl qualitativ unterschiedlicher Phasenportraits. Erwähnenswert scheint die Existenz von asymptotisch stabilen Shapley Polygonen und einer transitiv invarianten Menge, welche unter der Replikatordynamik nicht zu finden war.

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