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DISSERTATION

Non-standard Operators in Almost Grassmannian Geometry

Verfasser

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Contents

Abstract

An almost Grassmannian geometry of type (p, q) is a parabolic geometry modelled on the Grassmann manifold $Gr_p(\mathbb{R}^{p+q})$ of p-dimensional subspaces in $\mathbb{R}^{p+q}$. It is well-known that for $p = q = 2$ the structure is equivalent to the four-dimensional conformal geometry. In view of the equivalence, there exists an analogue of the conformally invariant Paneitz operator, transforming functions to densities. This invariant operator is known as the non-standard operator for the almost Grassmannian geometry of type $(2, 2)$. In this thesis we deal with almost Grassmannian geometries of type $(2, q)$ with $q > 2$. It follows from the general theory of parabolic geometries that the complete obstruction against its local flatness (i.e. isomorphism to the homogeneous model $Gr_2(\mathbb{R}^{2+q})$) are two invariants - a torsion and a curvature. Vanishing of the first one ensures existence of a torsion-free connection. In such a case there exists a family of non-standard invariant operators of order four between exterior forms. We use curved Casimir operators to construct the first of these operators, transforming functions to sections of an irreducible subbundle of four-forms. Next we find a formula for this operator, which is analogous to the formula for Paneitz operator in the sense that it has the same form but contractions with metric are replaced by projections to subbundles of exterior forms. In particular, it shows that the operator factorizes through one-forms and three-forms. The main result of the thesis is theorem 3.15, where we prove that this operator can be extended to an invariant operator on arbitrary almost Grassmannian geometries, including torsion. We also give an explicit formula for this corrected operator and we prove that in the presence of torsion this operator does not factorize as in the torsion-free case.

Introduction

An almost Grassmannain structure on a smooth manifold M (briefly an AG-structure) is given by a fixed identification of the tangent bundle TM with the tensor product of two auxiliary vector bundles, together with the identification of their top degree exterior powers. It turns out that this is a specific example of a so called parabolic geometry, i.e. a Cartan geometry of type (G, P), where $P \subset G$ is a parabolic subgroup of a semi-simple Lie group G, and so the rich set of tools of parabolic geometries summarized in **[4]** applies. The AG-structure is one of the simplest examples of parabolic geometries since the Lie algebra corresponding to the group G comes endowed with a $|1|$-grading. Such structures are known under the name almost Hermitian symmetric structures (or briefly AHS-structures). They are also known under the name generalized conformal structures since the prototypical structure from this class is the conformal structure - probably at most studied structure since it is the natural setting for the physics of massless particles and other theories in Physics.

In this text, we will deal with AG-structures for which one of the defining vector bundles has rank two and the other has rank $q \geq 2$. Such a structure is called the AG-structure of type $(2, q)$. It is a generalization of conformal structures in the above sense and moreover, it is well-known that the AG-structure on a four-dimensional manifold M (the case $q = 2$) is even equivalent to the conformal structure on M. This equivalence can be seen from the description of the two structures as classical first-order G-structures. Namely, the identification of TM with the tensor product of two bundles of rank two together with the identification of the top-degree forms yields the reduction of the frame bundle to the structure group $G_0 = S(GL(2, \mathbb{R}) \times GL(2, \mathbb{R})) \subset GL(4, \mathbb{R})$, and this group is known to be isomorphic to the conformal spin group $CSpin(2, 2)$. Indeed, the adjoint action of $(A, B) \in G_0$ on $T_xM \cong M_2(\mathbb{R})$ is given by $X \mapsto BXA^{-1}$, and it is an easy observation that the determinant defines a quadratic form of signature $(2, 2)$ on $M_2(\mathbb{R})$ and satisfies $\det(BXA^{-1}) = \det(B)^2 \det(X)$ since $\det(A)\det(B) = 1$. Hence the adjoint action defines a homomorphism $G_0 \to CSO(2, 2)$, which is evidently a two-fold covering of the connected component of the identity of the conformal group, i.e. $G_0 \cong CSpin(2, 2)$. Of course, AG-structures of type $(2, q)$ for $q > 2$ are not equivalent to conformal structures. One of the main differences is that this structure generally does not allow the existence of a torsion-free connection. Nevertheless, we may still try to generalize concepts from conformal geometry. One of the important concepts is that of a conformally invariant differential operator, like the conformal Laplacian (sometimes called the Yamabe operator), the Maxwell operator, or the Paneitz operator. Such operators act on sections of natural vector bundles, and may be defined by universal natural formulae which depend only on the conformal structure and not on any choice of metric tensor from the conformal class. Similarly, invariant differential operators for AG-structures are defined by universal natural formulae which depend only on the structure and not on any particular choice. It turns out that most of conformally invariant operators have their invariant analogues for AG-structures. The aim of this text is to construct an analogue of the Paneitz operator.

The construction of an invariant operator is a difficult task in general. An easier situation is in the case of a manifold which is locally isomorphic to a homogeneous space G/P, so called locally flat manifold. There the problem of the existence and the construction of invariant operators can be reduced to a purely algebraical problem. Namely, invariant operators are constructed in an unique way from homomorphisms of so called generalized Verma modules. The structure of such homomorphisms is well understood in the literature, and so the full classification of invariant operators on locally flat manifolds is available. In particular, in the case of a locally flat manifold M equipped with the AG-structure there is exactly one non-trivial operator (up to scalar multiples) of order four acting on the bundle of functions $\mathcal{E}$, see e.g. **[18]**. Precisely, given an identification $TM = E^* \otimes F$ where E is a rank two vector bundle and F is a rank $q \geq 2$ vector bundle, the there is an unique invariant operator

$$\mathcal{E} \to \boxplus F^*[-2].$$

The bundle $\boxplus F^*[-2] := \boxplus F^* \otimes \mathcal{E}[-2]$ is an irreducible subbundle of the bundle of differential four-forms which is induced by the representation $\boxplus \mathbb{R}^{q*}$ of $SL(q, \mathbb{R})$, and the one-dimensional representation $\mathbb{R}[-2] := (\Lambda^2\mathbb{R}^2)^2$. It follows from the theory of Verma modules that there is actually a whole family of invariant operators, namely for each $0 \leq k \leq q-2$ we have an operator

$$k\updownarrow\begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline\end{array} F^*[-k] \to k+2\updownarrow\begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline\end{array} F^*[-k-2].$$

These operators are known as non-standard invariant operators for (flat) AG-structures.

In the thesis, we will mainly deal with the non-standard operator on functions. It is easy to see that in the dimension four, where the AG-structure is equivalent to a conformal structure, the line bundle $\mathcal{E}[-2]$ corresponds exactly to the bundle of functions with conformal weight -2, and that $\boxplus F^*$ is the line bundle $(\Lambda^2 F^*)^2$ which is by the second defining property of the AG-structure isomorphic to $(\Lambda^2 F^*)^2 \cong (\Lambda^2 E)^2 = \mathcal{E}[-2]$. Hence the non-standard operator on functions in dimension four is an invariant operator $\mathcal{E} \to \mathcal{E}[-4]$. It is well known that there is a unique invariant operator between the respective bundles over flat structures, namely the second power of the Laplacian. Moreover, it is also well known that it admits a unique curved analogue - the Paneitz operator. Thus we see that the non-standard operator on functions in dimension four exists also on curved manifolds, and coincides with the Paneitz operator. In this sense, we may view the non-standard operator on functions in higher dimensions as an analogue of the Paneitz operator.

There is a large class of invariant operators on curved manifolds which have a similar behavior as the invariant operators on locally flat structures, so called strongly invariant operators. These operators can be also constructed from homomorphisms of induced modules, called semi-holonomic Verma modules. In particular, this is the case of all invariant operators of order at most two. However there exist higher-order invariant operators on curved manifolds which do not have this property, and are of much more subtle nature. One of them is by **[10]** the conformally invariant critical power of the Laplacian, i.e. conformally invariant operator on manifold of even dimension n with principal part $\Delta^{\frac{n}{2}}$. This power is called critical since the all lower powers are strongly invariant, and moreover there are no conformally invariant operators with principal part Δ^k for $k > \frac{n}{2}$. This was proved by Graham for $n = 4$ and $k = 3$ in **[14]**, and generalized by Gover and Hirachi in **[17]**. We observe that the Paneitz operator in dimension four is exactly the critical power and thus is not strongly invariant. This can be also shown for the non-standard operator for AG-structures in higher dimensions, see **[19]**, which indicates that the construction of a curved analogue will be difficult.

Nevertheless, the curved analogue is known to exist on a class of curved manifolds. Precisely, the existence of all the non-standard invariant operators on curved manifolds which admit a torsion-free connection was proved by Gover and Slovák **[11]**. The proof is via an invariant local tractor calculus developed therein. We give an alternative construction of the non-standard operator on functions via curved Casimir operators, originally introduced in **[5]** in the setting of general parabolic geometries. The curved Casimir operator is a basic invariant differential operator acting between smooth sections of associated natural vector bundles, which on locally flat manifolds reduces to the action of the quadratic Casimir element well-known from the representation theory. It may be expressed by a Laplacian like formula in terms of the fundamental derivative and so it inherits its strong naturality properties. In contrast to the Laplace operator, the order of Casimir operator is at most one. In particular, it acts by a scalar on any irreducible bundle, and this scalar (in sequel called the Casimir eigenvalue) can be computed from representation theory data. These properties allow to use the curved Casimir operators to construct in a conceptual way higher order invariant differential operators. In [**6**] it was demonstrated on the construction of several conformally invariant operators, among them the Paneitz operator. We will proceed along the lines of the construction, and we will construct a curved analogue of the non-standard operator on functions. The construction works in the torsion-free case only. However, we can use explicit formulae obtained from the construction to prove the existence of an invariant operator also in the case of non-vanishing torsion.

Structure of the text. In the first chapter, we give a short introduction to the theory of parabolic geometries and we summarize basic facts, which we will need in the course of the thesis. We also introduce the curved Casimir operators. The second chapter is devoted to the description of the almost Grassmannian geometry and related structures. We make explicit some of the general results from the first chapter. The crucial part of the thesis is the third chapter, which has two parts. In the first one we use the curved Casimirs to construct the non-standard operator on functions for AG-structures admitting a torsion-free connection. In the second one we prove the main result that there exist correction terms involving torsion which make the non-standard operator into an invariant operator for AG-structures with arbitrary torsion. Some of the technical computations concerning Bianchi identity is placed into an appendix. The second part of the appendix contains a description of an alternative construction of the non-standard operator in torsion-free case, which then leads to an analogue of the conformal Q-curvature.

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CHAPTER 1

Background On Parabolic Geometries

1.1. Cartan geometries

The concept of Cartan geometries was introduced by E. Cartan under the name "generalized spaces" in order to make a connection between differential geometry and geometry in the sense of F. Klein's Erlangen program. This concept associates to an arbitrary homogeneous space G/H a differential geometric structure on smooth manifolds whose dimension equals the dimension of G/H. A manifold endowed with such a geometry is called Cartan geometry of type (G, H) and it can be considered as a curved analog of the homogeneous space G/H.

1.1.1. Basic concepts. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $H \subset G$ be a closed subgroup. The basic idea behind Cartan geometries is to endow the homogeneous space G/H with a geometric structure, whose automorphisms are exactly the left actions of the elements of G. It turns out that the right ingredient to recognize these automorphisms is the *Maurer Cartan form* $\omega^{MC} \in \Omega^1(G, \mathfrak{g})$ which gives a trivialization of the tangent bundle TG by left translations. For $\xi \in T_gG$, it is defined by

$$\omega^{MC}(g)(\xi) = T\lambda_{g^{-1}} \cdot \xi \in T_eG = \mathfrak{g},$$

where $\lambda_{g^{-1}}$ denotes left translation by $g^{-1} \in G$. One can then prove for connected homogeneous spaces G/H that the left translations λ_g are exactly the principal bundle automorphisms of $G \to G/H$ which pull back ω^{MC} to itself, c.f. proposition 1.5.2. in [**4**].

Before generalizing this picture to the case of a general manifold, let us recall some basic properties of the Maurer-Cartan form. Namely, it follows from its definition that $(\rho^g)^*\omega^{MC} = \mathrm{Ad}(g^{-1}) \circ \omega^{MC}$ where ρ^g is the principal right action on the bundle $G \to G/H$ given by the right translation by g. The next obvious property is that $\omega^{MC}(L_X) = X$ for all $X \in \mathfrak{g}$ where L_X is left invariant vector field generated by left translations of X. The third property says that for all vector fields ξ and η on G, the following holds

$$d\omega^{MC}(\xi, \eta) + [\omega^{MC}(\xi), \omega^{MC}(\eta)] = 0,$$

This equation is known under the name *Maurer-Cartan equation* and it follows from the definition of the exterior derivative and the definition of the Lie bracket on $\mathfrak{g}$.

The definition of a Cartan geometry is now obtained by replacing the cannonical principal bundle $G \to G/H$ by an arbitrary principal H–bundle and ω^{MC} by a form which has all the properties of ω^{MC} that make sense in the more general setting. Let ζ_X denote a fundamental vector field defined by the formula $\zeta_X(x) = \frac{d}{dt}|_0 \rho_{\exp tX}(x)$, for all $x \in M$, $X \in \mathfrak{h}$. The fundamental vector field thus provides an infinitesimal version of the principal right action and it coincide with the left-invariant vector field L_X on the cannonical bundle $G \to G/H$. Then the Cartan geometry is defined as follows.

Definition 1.1. (1) A *Cartan geometry* of type (G, H) on a smooth manifold M is a principal H–bundle $p : \mathcal{G} \to M$ together with a one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, called the *Cartan connection*, such that

- $(r^h)^*\omega = \mathrm{Ad}(h)^{-1} \circ \omega$ for all $h \in H$.
- $\omega(\zeta_X) = X$ for all $X \in \mathfrak{h}$.
- $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

(2) A *morphism* between two Cartan geometries $(\mathcal{G} \to M, \omega)$ and $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ is a principal bundle homomorphism $\Phi : \mathcal{G} \to \tilde{\mathcal{G}}$ such that $\Phi^*\tilde{\omega} = \omega$.
(3) The *curvature form* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, H) is defined by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$. Since the Cartan connection ω trivializes the tangent space $T\mathcal{G}$, the curvature form K is determined by its values on the constant vector fields $\omega^{-1}(X)$. Therefore, the complete information about K is given by the *curvature function* $\kappa : \mathcal{G} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ defined by $\kappa(u)(X, Y) = K(\omega^{-1}(X), \omega^{-1}(Y))$. And since K is horizontal, the curvature function may be viewed as a map $\mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$. The formula for the exterior derivative d then yields

$$\kappa(u)(X, Y) = [X, Y] - \omega(u)([\omega^{-1}(X), \omega^{-1}(Y)])$$

By definition $(G \to G/H, \omega^{MC})$ is a Cartan geometry of type (G, H), and by (2) in the proposition 1.5.2 in **[4]** the automorphisms of this geometry are exactly the left translations by elements of G. This geometry is called the *homogeneous model* of Cartan geometries of type (G, H). Since the Cartan connection coincides with the Maurer-Cartan form in such case, the curvature vanishes and so the homogenous case is also reffered to as the flat case. Moreover, the following proposition shows that the curvature measures exactly the local difference to the flat model of the given geometry. For more details, see **[4]**.

Proposition 1.1. *The curvature of a Cartan geometry* $(\mathcal{G} \to M, \omega)$ *vanishes identically if and only if* $(\mathcal{G} \to M, \omega)$ *is locally isomorphic to the homogeneous model.*

1.1.2. Natural vector bundles. The name "natural" comes from the general definition in [**15**] which defines a natural bundle F as a regular functor on the cathegory of n-dimensional smooth manifolds and local diffeomorphisms assigning to any manifold a fibre bundle and to any local diffeomorphism a bundle map. In the case of the Cartan geometries, a simpler concept of natural bundles is sufficient. Namely, it can be easily shown that the left multiplication on G/H by elements in G lifts to $F(G/H)$ which means that the structure of the whole bundle $F(G/H)$ is determined by the H-action on its standard fibre $S := F_0(G/H)$.

Therefore, given a H-action on a manifold S and a Cartan geometry $(\mathcal{G} \to M, \omega)$, we define the *natural bundle* as the associated bundle $\mathcal{G} \times_H S$. In sequel, we consider exclusively the case when the standard fibre is a vector space V. In such a case, the H-module V gives rise to *natural vector bundle* $VM = \mathcal{G} \times_H V$. Recall that

$$\mathcal{G} \times_H V := \mathcal{G} \times V/\sim,$$

where $\sim$ denotes the equivalence $(u, v) \sim (u \cdot h, h^{-1} \cdot v)$ for all $h \in H$. The functorial properties of associated vector bundles guarantee that it is also the natural bundle in the sense described above. The space $\Gamma(VM)$ of smooth sections of a natural vector bundle is identified with the space $C^\infty(\mathcal{G}, V)^H$ of smooth H-equivariant functions. The isomorphism is given by the map which assigns to an equivariant function f the equivalence class of $(u, f(u))$ in $\mathcal{G} \times_H V$.

We also observe that natural vector bundles depends only on the principal Cartan bundle $\mathcal{G}$ but not on the Cartan connection ω. Nevertheless, ω is necessary to identify the natural bundles with traditional objects like tensor bundles. We clarify this on the example of the tangent bundle.

Let $(p : \mathcal{G} \to M, \omega)$ be a general Cartan geometry of type (G, H). We show that the tangent bundle TM may be identified via ω with the natural vector bundle $\mathcal{G} \times_H \mathfrak{g}/\mathfrak{h}$. Consider the mapping $\mathcal{G} \times \mathfrak{g} \to TM$ defined by $(u, X) \mapsto T_u p \cdot \omega_u^{-1}(X)$. This map factors to $\mathcal{G} \times (\mathfrak{g}/\mathfrak{h})$ since for $X \in \mathfrak{h}$, the field $\omega^{-1}(X)$ is the fundamental field ζ_X and thus vertical. Fixing u, one gets a linear isomorphism $\mathfrak{g}/\mathfrak{h} \to T_{p(u)}M$ and the equivariancy of ω immediately implies that this factors to a bundle map $\mathcal{G} \times_H \mathfrak{g}/\mathfrak{h} \to TM$. This map induces a linear isomorphism in each fiber and covers the identity on M and so it is an isomorphism of vector bundles. Likewise, the cotangent bundle T^*M may be identified with the natural bundle corresponding to $(\mathfrak{g}/\mathfrak{h})^*$ and hence all tensor bundles may be viewed as natural bundles induced by an appropriate tensor product of copies of $(\mathfrak{g}/\mathfrak{h})$ and $(\mathfrak{g}/\mathfrak{h})^*$.

1.1.3. Tractor bundles. *Tractor bundles* form a special class of natural vector bundles for which the inducing H-representation V is a restriction of a representation of the whole group G. The main reason for the importance of such bundles is that the Cartan connection induces a linear connection on them. So the tractor bundles come equipped with cannonical linear connections, called *tractor connections.*

The fundamental example of a tractor bundle is the *adjoint tractor bundle.* This is a tractor bundle denoted by $\mathcal{A}M$ which is induced by a restriction of the adjoint action $\mathrm{Ad} : G \to GL(\mathfrak{g})$, i.e. $\mathcal{A}M = \mathcal{G} \times_H \mathfrak{g}$. According to the previous identification of the tangent bundle, the short exact sequence $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ of H-modules induces a short exact sequence of natural vector bundles

$$0 \to \mathcal{G} \times_H \mathfrak{h} \to \mathcal{A}M \to TM \to 0.$$

Therefore, there is a natural surjective bundle map $\Pi : \mathcal{A}M \to TM$ and so the adjoint tractor bundle may be viewed as an extension of the tangent bundle. Now let us summarize some important properties which we will need later. For a proof, see the section 1.5.7. in [**4**].

Proposition 1.2. *Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, H), $\mathcal{A}M \to M$ its adjoint tractor bundle and $\Pi : \mathcal{A}M \to TM$ the natural projection. Let VM be the tractor bundle corresponding to a representation of G on V.*

(1) The curvature κ of the Cartan connection ω can be naturally interpreted as a two-form κ on M with values in $\mathcal{A}M$.

(2) There is a natural bundle map $\{\ ,\ \} : \mathcal{A}M \times \mathcal{A}M \to \mathcal{A}M$, which makes each fibre $\mathcal{A}_xM$ into a Lie algebra isomorphic to $\mathfrak{g}$.

(3) There is an isomorphism between the space $\Gamma(\mathcal{A}M)$ of smooth sections of $\mathcal{A}M$ and the space $\mathfrak{X}(\mathcal{G})^H$ of vector field on $\mathcal{G}$ which are invariant under the principal right action of H. This induces a Lie bracket $[\ ,\ ]$ on $\Gamma(\mathcal{A}M)$. For $s_1, s_2 \in \Gamma(\mathcal{A}M)$, one has $\Pi([s_1, s_2]) = [\Pi(s_1), \Pi(s_2)]$, where one has the Lie bracket of vector fields on the right-hand side.

(4) There is a natural bundle map $\bullet : \mathcal{A}M \times \mathcal{V}M \to VM$. For each point $x \in M$, this makes the fibre $\mathcal{V}_xM$ into a module over the Lie algebra $\mathcal{A}_xM$. In particular, for sections $s_1, s_2 \in \Gamma(\mathcal{A}M)$ and $t \in \Gamma(\mathcal{V}M)$, we get

$$\{s_1, s_2\} \bullet t = s_1 \bullet (s_2 \bullet t) - s_2 \bullet (s_1 \bullet t)$$

(5) The operations introduced in (2) and (4) are parallel for the cannonical tractor connections. Denoting them by $\nabla^{\mathcal{A}}$ and $\nabla^{\mathcal{V}}$ we get

$$\nabla^{\mathcal{A}}_{\xi}\{s_1, s_2\} = \{\nabla^{\mathcal{A}}_{\xi}s_1, s_2\} + \{s_1, \nabla^{\mathcal{A}}_{\xi}s_2\}$$
$$\nabla^{\mathcal{V}}_{\xi}(s \bullet t) = (\nabla^{\mathcal{A}}_{\xi}s) \bullet t + s \bullet (\nabla^{\mathcal{V}}_{\xi}t)$$

1.1.4. Fundamental derivative. The property (3) from the previous proposition which says that smooth sections $\Gamma(\mathcal{A}M)$ of the adjoint bundle are identified with H-invariant vector fields $\mathfrak{X}(\mathcal{G})^H$ has an important consequence. Namely, it gives rise to a family of natural differential operators on arbitrary natural bundles. Indeed, any section σ of natural vector bundle VM (which is not ncessarily tractor bundle) corresponds to an equivariant function $f : \mathcal{G} \to V$, which we we can differentiate along the invariant vector field ξ corresponding to section s of $\mathcal{A}M$, and we get a function $\xi \cdot f : \mathcal{G} \to V$. Since

$$\xi(u \cdot h) \cdot f = (Tr^h \cdot \xi(u)) \cdot f = \xi(u) \cdot (f \circ r^h) = h^{-1} \cdot (\xi(u) \cdot f),$$

this function is equivariant and thus corresponds to a section of VM which we dentote by $D_s\sigma$. Hence we get an differential operator $D : \Gamma(\mathcal{A}M) \times \Gamma(VM) \to \Gamma(VM)$, called *fundamental derivative* which is by construction linear over $C^\infty(M, \mathbb{R})$ in s.

It follows straight from the definition of the fundamental derivative that it has strong naturality properties which we describe below. We also show that although its action is defined by a differntiation, it is tensorial on a subbundle of $\mathcal{A}M$. Indeed, let $\bullet : (\mathcal{G} \times_H \mathfrak{h}) \times VM \to VM$ denotes the bundle map arising from the derivative of the representation V inducing VM (which is a H-equivariant map $\mathfrak{h} \times V \to V$), i.e. it coincides with the map from the part (4) in the proposition 1.2 in the case that VM is a tractor bundle. Then the following proposition summarizes the basic prpoperties of the fundamental derivative. For a proof, see 1.5.8. in [**4**].

Proposition. *(1) For a smooth function $f : M \to \mathbb{R}$ and $s \in \Gamma(\mathcal{A}M)$ we get $D_s f = \Pi(s) \cdot f$.*

(2) If s is a section of the subbundle $\mathcal{G} \times_H \mathfrak{h} \subset \mathcal{A}M$, then $D_s\sigma = -s \bullet \sigma$ for any $\sigma \in \Gamma(VM)$.

(3) The fundamental derivative is compatible with all natural bundle maps coming from H-equivariant maps between the inducing representations. In particular, for natural vector bundles VM and $\mathcal{W}M$, the dual $\mathcal{V}^*M$ of VM, sections $\sigma \in \Gamma(VM)$, $\tau \in \Gamma(\mathcal{W}M)$ and $\beta \in \Gamma(\mathcal{V}^*M)$ and a function $f \in$

$C^\infty(M,\mathbb{R})$ *we get*

$$D_s(f\sigma) = (\Pi(s)\cdot f)\sigma + fD_s\sigma$$

$$D_s(\sigma\otimes\tau) = D_s\sigma\otimes\tau + \sigma\otimes D_s\tau$$

$$\Pi(s)\cdot(\beta(\sigma)) = (D_s\beta)(\sigma) + \beta(D_s\sigma)$$

Of course, the fundamental derivative may be also viewed as a differential operator $\Gamma(VM)\to\Gamma(\mathcal{A}^*M\otimes VM)$. Hence it alows iterating, i.e. for $k\in\mathbb{N}$ we have $D^k:\Gamma(VM)\to\Gamma(\otimes^k\mathcal{A}^*M\otimes VM)$. In a sense, the fundamental derivative is an analog of the covariant derivative by the Levi-Civita connection in Riemannian geometry. Its differential part also satisfies (analog of) Bianchi and Ricci identities which are obtained by differentiating the basic identities for the curvature of the Cartan connection.

Observe that the dual of the natural projection Π gives an inclusion $T^*M\to\mathcal{A}^*M$ which means that differential forms on M cannonically extend to the adjoint tractor bundle. In sequel, we omitt writting this inclusion and we simply allow appearence of sections of $\mathcal{A}M$ in arguments of differential forms. Then the Bianchi and Ricci identities for the fundamental derivative have the following form. For a proof, see 1.5.9. in [**4**].

Proposition 1.3. *Let $(p:\mathcal{G}\to M,\omega)$ be a Cartan geometry of type (G,H) with the curvature $\kappa\in\Omega^2(M,\mathcal{A}M)$ and $\{\ ,\ \}$ the algebraic bracket on $\mathcal{A}M$.*

(1) (Bianchi-identity) The curvature κ satisfies

$$\sum_{cycl}\Big(\{s_1,\kappa(s_2,s_3)\} - \kappa(\{s_1,s_2\},s_3) + \kappa(\kappa(s_1,s_2),s_3) + (D_{s_1}\kappa)(s_2,s_3)\Big) = 0$$

for all $s_i\in\Gamma(\mathcal{A}M)$, where the sum is over all cyclic permutations of the arguments.

(2) (Ricci-identity) For any natural vector bundle E and any section $\sigma\in\Gamma(EM)$, the alternation of the second order fundamental derivative is given by

$$(D^2\sigma)(s_1,s_2) - (D^2\sigma)(s_2,s_1) = -D_{\kappa(s_1,s_2)}\sigma + D_{\{s_1,s_2\}}\sigma$$

Let us mention that in the case that VM is a tractor bundle, we have also another natural operation available, namely the covariant derivative by the tractor connection. Its relation to the fundamental derivative is given by

$$\nabla^{\mathcal{V}}_{\Pi(s)}t = D_st + s\bullet t$$

for $s\in\Gamma(\mathcal{A}M)$ and $t\in\Gamma(VM)$. For more details, see theorem 1.5.8. in [**4**].

1.2. Parabolic geometries

Parabolic geometries are Cartan geometries which are of type (G, P), where the group G is a semisimple Lie group and P is a parabolic subgroup in G. A parabolic subgroup may be defined in several ways. For our purpose, the most convenient way is the definition below via $|k|$-gradings. The important feature of parabolic geometries is that we can use a rich set of algebraic tools from the theory of semisimple Lie algebras.

1.2.1. $|k|$-gradings. In order to define parabolic geometries, we need first to define $|k|$-gradings on semisimple Lie algebras.

Definition 1.2. Let $\mathfrak{g}$ be a complex or real semisimple Lie algebra. A $|k|$–*grading* on $\mathfrak{g}$ is a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and such that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$.

One of the basic properties of $|k|$-graded Lie algebras is the existence of a unique element E in the center of the Lie subalgebra $\mathfrak{g}_0$, called the grading element, which satisfies $[E, X] = iX$ for $X \in \mathfrak{g}_i$ for $i = -k, \dots, k$. The next important property is that the Killing form of $\mathfrak{g}$ induces an isomorphism $\mathfrak{g}_i \cong \mathfrak{g}^*_{-i}$ of $\mathfrak{g}_0$-modules for each $i = 1, \dots, k$. It also induces an isomorphism of $\mathfrak{p}$-modules $(\mathfrak{g}/\mathfrak{p})^*$ and $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. The Lie algebras $\mathfrak{g}_0$ and $\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ can be characterized by

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} : ad(X)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for } i = -k, \dots, k\}$$

$$\mathfrak{p} = \{X \in \mathfrak{g} : ad(X)(\mathfrak{g}^i) \subset \mathfrak{g}^i \text{ for } i = -k, \dots, k\},$$

where $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. A proof of these basic facts about $|k|$-graded Lie algebras can be found in section 3.1.2. of [**4**]. It can be also shown that the Lie subalgebra $\mathfrak{p}$ contains the Borel subalgebra $\mathfrak{b}$, i.e. the maximal solvable subalgebra in $\mathfrak{g}$. Such subalgebras $\mathfrak{p} \leq \mathfrak{g}$ are called *parabolic subalgebras.* For details, see e.g. section 3.2.1. in [**4**].

Closed subgroup $P \subset G$ with Lie algebra $\mathfrak{p}$ is called a *parabolic subgroup* (corresponding to the given $|k|$-grading). This is satisfied for the group

$$P := \{g \in G : Ad(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \text{ for } i = -k, \dots, k\}$$

since it has Lie algebra $\mathfrak{p}$ and it is the intersection of the normalizers of $\mathfrak{g}^i$ in G and thus closed. Any other parabolic subgroup with Lie algebra $\mathfrak{p}$ then lies between this group and its connected component of identity. The closed subgroup defined by

$$G_0 := \{g \in P : Ad(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for } i = -k, \dots, k\}$$

has Lie algebra $\mathfrak{g}_0$ and it is called *Levi subgroup* of P. The relation of the groups G_0 and P is described by the theorem 3.1.3. in **[4]** which basically says that the map $(g_0, Z) \mapsto g_0 \exp(Z)$ defines a diffeomorphism $G_0 \times \mathfrak{p}_+ \to P$. Moreover, $P_+ := \exp(\mathfrak{p}_+)$ is a closed nilpotent subgroup of G and $P/P_+ \cong G_0$.

After introducing parabolic subgroups, we are now able to formulate the definition of a parabolic geometry.

Definition 1.3. A *parabolic geometry* is a Cartan geometry of type (G, P), where G is a semisimple Lie group with Lie algebra $\mathfrak{g}$ equipped with a $|k|$-grading and P is a parabolic subgroup corresponding to this grading.

For our purposes, we switch now to the case of $|1|$-gradings, although most of the concepts we are going to introduce apply to a general parabolic geometry corresponding to a $|k|$-grading. Hence we assume that the Lie algebra $\mathfrak{g}$ is endowed with a grading of the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

We refer to a parabolic geometry with such a Lie algebra as $|1|$-graded parabolic geometry. As we shall see, $|1|$-graded parabolic geometries are equivalent to so called the almost Hermitian symmetric structures, or briefly the AHS structures. These structures were intensively studied, see e.g. **[7]**, **[8]**, **[9]**.

1.2.2. Complex $|1|$-gradings. In the case of a simple complex Lie algebra $\mathfrak{g}$, the meaning of a $|1|$-grading is easy to describe. We know from above about the existence of the grading element E which acts by i on $\mathfrak{g}_i$ for $i = -1, 0, 1$. Since the action is diagonizable, we can extend $\mathbb{C}E$ by semisimple elements of $\mathfrak{g}_0$ to a Cartan subalgebra. Conjugating by an appropriate inner automorphism, we may therefore assume that $E \in \mathfrak{h}$. Moreover, the grading element E lies in a real form of $\mathfrak{h}$ on which all roots are real and thus it defines a set of positive roots Δ^+ by requiring $\alpha(E) \geqq 0$ for all $\alpha \in \Delta^+$. The corresponding root spaces then lie in $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which means that $\mathfrak{p}$ contains the standard Borel subalgebra

$$\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

and thus is a parabolic subalgebra. Next to the Borel subalgebra, $\mathfrak{p}$ also contains a direct sum of root spaces corresponding to a subset of negative roots.

In order to describe possibilities for such subsets and hence the structure of $|1|$-gradings, let us consider the action of the grading element on the highest root of $\mathfrak{g}$. If we choose a set of simple roots Δ^0, the highest root is equal to $\alpha_0 = \sum_i a_i\alpha_i$ where $\alpha_i \in \Delta^0$ and the coefficients a_i are strictly positive integers. The highest root space obviously lies in the top component of the gradation, i.e. $\mathfrak{g}_1$, and thus the evaluation of the highest root on the grading element then yields the equation $\sum_i a_i\alpha_i(E) = \alpha_0(E) = 1$. Since $a_i \geqq 1$ and $\alpha_i(E)$ may only have values 0 or 1, the equation is satisfied if and only if $a_k = 1$ and $\alpha_k(E) = 1$ for an index k and $\alpha_i(E) = 0$ for all $i \neq k$. Hence the all root spaces corresponding to simple roots must lie in $\mathfrak{g}_0$ exept the root space $\mathfrak{g}_{\alpha_k}$ which is in $\mathfrak{g}_1$.

In analogy with a general case of $|k|$-grading, we define a map $\mathrm{ht} : \Delta \to \{-1, 0, 1\}$ by associating to a root $\alpha = \sum_i a_i\alpha_i$ the coefficient a_k corresponding the root α_k from above. Then the set $\{\alpha : \mathrm{ht}(\alpha) = 0\}$ is spanned by simple roots $\alpha \in \Delta^0 \setminus \{\alpha_k\}$ and the $\mathfrak{g}_0$-component of the $|1|$-grading has the form

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\mathrm{ht}(\alpha)=0} \mathfrak{g}_\alpha$$

while the $\mathfrak{g}_1$-part of the grading is given by a direct sum of root spaces $\mathfrak{g}_\alpha$ for which $\mathrm{ht}(\alpha) = 1$. Obviously, the $\mathfrak{g}_{-1}$-part is a direct sum of root spaces with $\mathrm{ht}(\alpha) = -1$.

The subalgebras $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are irreducible $\mathfrak{g}_0$-modules and the reductive subalgebra $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$ has one-dimensional center $\mathfrak{z}(\mathfrak{g}_0) = \mathbb{C}E$. It can also be easily shown that the grading element E is a multiple of the coroot $H_{\alpha_k} \in \mathfrak{h}$ and thus $\mathfrak{g}_{-\alpha_k}, \mathfrak{z}(\mathfrak{g}_0), \mathfrak{g}_{\alpha_k}$ form a $\mathfrak{sl}(2, \mathbb{C})$ tripple in $\mathfrak{g}$. The Cartan subalgebra is equal to $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}''$ where $\mathfrak{h}''$ is spanned by H_α with $\alpha \in \Delta^0 \setminus \{\alpha_k\}$ and $\mathfrak{g}_0^{ss} = \mathfrak{h}'' \oplus \bigoplus_{\mathrm{ht}(\alpha)=0} \mathfrak{g}_\alpha$ is the root decomposition of the semisimple part of $\mathfrak{g}_0$. For details, see section 3.2.1. in [**4**], where the general case of a $|k|$-grading is treated.

So we obtained that any $|1|$-grading, or equivalently any parabolic subalgebra, is given by a choice of a simple root α_k. Thus the $|1|$-gradings are classified by Dynkin diagrams, where the simple root α_k in $\mathfrak{g}_1$ is emphasized. We will use the usual notation with a cross instead of the corresponding node. Due to the condition $a_k = 1$, not every simple root can be crossed. Going through the list of Dynkin diagrams and looking at the corresponding highest roots, one immediately sees which simple roots satisfy the condition on the corresponding coefficient and so one obtains the full classification of $|1|$-gradings. Namely, for A_l series, an arbitrary simple root can be crossed since the highest root is the sum of all simple roots and thus all the coefficient are equal to one. For D_l series, there are two different options while for B_l, C_l, E_6, E_7 there is an unique choice of a simple root which may appear in $\mathfrak{g}_1$. The table with highest roots corresponding to simple complex Lie algebras can be found in appendix B of [**4**]. It follows straight from the root decomposition of the semisimple part $\mathfrak{g}_0^{ss}$ that the corresponding Dynkin diagram is obtained from the diagram for $\mathfrak{g}$ by removing the crossed node and all edges connected to this node. The detailed description of all $|1|$-gradings is given in the section 3.2.2. of [**4**].

1.2.3. Real $|1|$-gradings. In the case of a real Lie algebra $\mathfrak{g}$, one can proceed similarly. A subalgebra $\mathfrak{p}$ of a real (semi)simple $\mathfrak{g}$ is called *parabolic*, if its complexification $\mathfrak{p}^{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Now we recall the form of the root decomposition of $\mathfrak{g}$ in the real case and then we give a description of real $|1|$-gradings.

Choosing a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$, the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ into eigenspaces to the eigenvalues 1 and -1 respectively. Recall that then $\mathfrak{k}$ is a subalgebra, $[\mathfrak{k}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k}$. Now let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a} := \mathfrak{h} \cap \mathfrak{q}$ has maximal dimension from all θ-stable Cartan subalgebras. One can easily show that $ad(A)$ for $A \in \mathfrak{a}$ is simultaneously diagonisable. The eigenvalues are functionals $\lambda : \mathfrak{a} \to \mathbb{R}$, called *restricted roots* and denoted by Δ_r and $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta_r} \mathfrak{g}_\lambda$$

By construction, the elements of Δ_r are restrictions to $\mathfrak{a}$ of the root system of $\mathfrak{g}^{\mathbb{C}}$. If the positive roots Δ^+ form an admissible subsystem in Δ, i.e. for a $\alpha \in \Delta^+$ we have either $\sigma^*\alpha = -\alpha$ or $\sigma^*\alpha \in \Delta^+$, where $\sigma^* : \Delta \to \Delta$ is the involutive automorphism defined by the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$, then the restriction gives a subsystem Δ_r^+ of positive restricted roots and also a subset of simple restricted root Δ_r^0.

Since we described $|1|$-gradings in the complex case, it is now easy to see that $\mathfrak{g}_0$-component of any $|1|$-grading on a real $\mathfrak{g}$ is given by

$$\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \bigoplus_{\mathrm{ht}(\lambda)=0} \mathfrak{g}_\lambda$$

where $\lambda \in \Delta_r$ is a restricted root and the map $\mathrm{ht} : \Delta_r \to \{-1, 0, 1\}$ is the restriction of the map from the complex case. The $\mathfrak{g}_{\pm 1}$-component is a direct sum of root spaces $\mathfrak{g}_{\pm\lambda}$ for restricted roots satisfying $\mathrm{ht}(\lambda) = \pm 1$. There is one more restriction on the root α_k in the real case. Namely, the simple restricted root given by $\sigma^*\alpha_k$ must lie in the same grading component. Hence if $\sigma^*\alpha_k \neq \alpha_k$ on $\mathfrak{a}$ then we cannot obtain $|1|$-grading on $\mathfrak{g}$.

The real simple Lie algebras are classified by *Satake diagrams* which one obtains from Dynkin diagrams of the complexifications by denoting the simple roots in Δ_r^0 by a white dot $\circ$ and the other, which vanish under the restriction to $\mathfrak{a}$, called compact roots, by a black dot $\bullet$. Moreover, the simple restricted roots α and $\sigma^*\alpha$ are connected with an arrow if they do not coincide. A table with Satake diagrams for real simple Lie algebras can be found in table B4 in [**4**]. Similarly to the complex case, the real $|1|$-gradings are denoted by crossing an admissible restricted root which then defines $\mathfrak{g}_1$.

1.2.4. Underlying structures. Let us fix a $|1|$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, a Lie group G with Lie algebra $\mathfrak{g}$, parabolic subgroup $P \subset G$ corresponding to the grading and consider a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P). From above we know that we have the reductive subgroup $G_0 \subset P$ and the nilpotent normal subgroup $P_+ \subset P$. Since P acts freely on the Cartan bundle $\mathcal{G}$, the same is true for P_+ and so we can form the orbit space $\mathcal{G}_0 := \mathcal{G}/P_+$. One concludes from the construction of $\mathcal{G}_0$ that the projection p factors to a smooth map $p_0 : \mathcal{G}_0 \to M$ which defines a smooth principal bundle with structure group $G_0 = P/P_+$, called the underlying bundle.

Now observe that $T^i\mathcal{G} := \omega^{-1}(\mathfrak{g}^i)$ for $i = -1, 0, 1$ defines a filtration on $T\mathcal{G}$ which is stable under the principal right action. Since $T^1\mathcal{G}$ is vertical subbundle of $\mathcal{G} \to \mathcal{G}_0$, we have $T^1\mathcal{G}_0 = \{0\}$ and so the filtration descends to a filtration $T\mathcal{G}_0 = T^{-1}\mathcal{G}_0 \supset T^0\mathcal{G}_0 \supset T^1\mathcal{G}_0 = \{0\}$. Since ω reproduces the generators of fundamental vector fields, the bundle $T^0\mathcal{G}_0$ is spanned by fundamental vector fields with generators in $\mathfrak{g}^0 = \mathfrak{p}$ and therefore it is exactly the vertical bundle of the underlying bundle $p_0 : \mathcal{G}_0 \to M$. One can also show that the definition $\omega^0_{-1}(u)(\xi) := \omega_{-1}(\tilde{u})(\tilde{\xi})$ for $u = \pi(\tilde{u})$ and $\xi = T\pi(\tilde{\xi})$, where π is the natural projection $\mathcal{G} \to \mathcal{G}_0$ and ω_{-1} is the $\mathfrak{g}_{-1}$-component of ω, does not depend on the choice of $\tilde{u}$ and $\tilde{\xi}$ and thus gives a map $\omega^0_{-1} : T\mathcal{G}_0 \to \mathfrak{g}_{-1}$. Moreover, it turns out that the kernel of this map is $T^0\mathcal{G}_0$ and the map is G_0-equivariant. Now fixing a point $u \in \mathcal{G}_0$, one obtains a linear isomorphism $T_u\mathcal{G}_0/T^0_u\mathcal{G}_0 \to \mathfrak{g}_{-1}$. On the other hand, the tangent map T_up induces a linear isomorphism $T_u\mathcal{G}_0/T^0_u\mathcal{G}_0 \to T_xM$, where $x = p_0(u)$. Hence we may interpret $\omega^0_{-1}(u)$ as a linear isomorphism $T_xM \to \mathfrak{g}_{-1}$ and so it defines an element in the fibre over x of the linear frame bundle $\mathcal{P}^1M$. And since ω^0_{-1} is G_0-equivariant, it defines a bundle map $\mathcal{G}_0 \to \mathcal{P}^1M$ over the inclusion $G_0 \to GL(\mathfrak{g}_{-1})$ defined by the adjoint action which covers the identity on M. But this is exactly the definition of a reduction of the frame bundle of M to the structure group G_0, i.e. a G-sructure with structure group G_0.

Thus we get that any $|1|$-graded parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) gives rise to a classical first-order G-structure on M with structure group the Levi subgroup $G_0 \subset P$. A similar result holds for all $|k|$-graded parabolic geometries. The induced underlying structures then are so called infinitesimal flag structures. For details, see sections 3.1.5. and 3.1.6. in [**4**].

In the rest of this section, we discuss the opposite direction, i.e. construction of a |1|-graded parabolic geometry from a G-structure. It turns out that such construction is possible and, under certain conditions it is even unique and thus defines an one to one correspondence between a class of |1|-graded parabolic geometries and G-structures.

1.2.5. Kostant codifferential and Hodge decomposition. In order to formulate the condition on |1|-graded parabolic geomeries which then leads to the equivalence with G-structures, we need to introduce the Kostant codifferential.

First recall that for an abelian Lie algebra $\mathfrak{m}$ and its representation V, $\Lambda^k\mathfrak{m}^* \otimes V$ is the space $C^k(\mathfrak{m}, V)$ of k-cochains on $\mathfrak{m}$ with values in V, and the coboundary operator $C^k(\mathfrak{m}, V) \to C^{k+1}(\mathfrak{m}, V)$ is defined by

$$\partial\phi(X_0, \dots, X_k) := \sum_{i=0}^{k} X_i \cdot \phi(X_0, \dots \widehat{X_i} \dots, X_k) \tag{1.1}$$

Since it is a differential, it defines the cohomology $H^*(\mathfrak{m}, V)$ of $\mathfrak{m}$ with coefficients in V. Now let consider the complex $(C^*(\mathfrak{g}_{-1}, V), \partial)$ which computes the Lie algebra cohomology $H^*(\mathfrak{g}_{-1}, V)$ of the abelian Lie algebra $\mathfrak{g}_{-1}$. It is easy to see that the cochaines $C^k(\mathfrak{g}_{-1}, V)$ are $\mathfrak{g}_0$-modules and that the codifferentials are $\mathfrak{g}_0$-homomorphisms and hence $H^k(\mathfrak{g}_{-1}, V)$ are naturally $\mathfrak{g}_0$-modules. Although the identification of $\mathfrak{g}_{-1}$ and $\mathfrak{g}/\mathfrak{p}$ makes the cochains even into $\mathfrak{p}$-modules, the codifferential ∂ is not $\mathfrak{p}$-homomorphism. On the other hand, one can prove using the formula above that it is a $\mathfrak{p}$-homomorphism if we consider ∂ as the coboundary operator in the complex $(C^*(\mathfrak{g}_1, V^*), \partial)$. The space $C^k(\mathfrak{g}_1, V^*)$ is dual to $C^k(\mathfrak{g}/\mathfrak{p}, V)$ since the Killing form induces an isomorphism $\mathfrak{g}_1 \cong (\mathfrak{g}/\mathfrak{p})^*$ of $\mathfrak{p}$-modules and therefore, dualizing the $\mathfrak{p}$-homomorphism ∂ gives a $\mathfrak{p}$-homomorphism $\partial^* : C^{k+1}(\mathfrak{g}/\mathfrak{p}, V) \to C^k(\mathfrak{g}/\mathfrak{p}, V)$, called the *Kostant codifferential.* For $V = \mathfrak{g}$, we get a map $\partial^* : \Lambda^{k+1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ and using the identification $\mathfrak{g}_1 = (\mathfrak{g}/\mathfrak{p})^*$, we easily obtain the following formula on decomposable elements.

$$\partial^*(Z_0 \wedge \dots \wedge Z_l \otimes A) = \sum_{i=0}^{l} (-1)^{i+1} Z_0 \wedge \dots \wedge \widehat{Z_i} \wedge \dots \wedge Z_l \otimes [Z_i, A]$$

Next, one defines the associated *Kostant Laplacian* by $\Box = \partial \circ \partial^* + \partial^* \circ \partial$. By construction, it is a $\mathfrak{g}_0$-automorphism of $C^k(\mathfrak{g}_{-1}, \mathfrak{g})$. One can construct an inner product with respect to which the operators ∂ and ∂^* ar adjoint and use this to prove the following version of Hodge decomposition. For a proof and results for $|k|$-grading, see sections 3.1.11. and 3.3.1. in [**4**].

Proposition 1.4. *For each $n \geq 0$, the chain space $C^k(\mathfrak{g}_{-1}, \mathfrak{g})$ naturally splits into a direct sum of G_0-submodules as*

$$C^k(\mathfrak{g}_{-1}, \mathfrak{g}) = \operatorname{im}(\partial^*) \oplus \ker(\Box) \oplus \operatorname{im}(\partial),$$

with the first two summands adding up to $\ker(\partial^*)$ *and the last two summands adding up to* $\ker(\partial)$.

It is evident from the proposition that we may naturally identify the G_0-module $H^k(\mathfrak{g}_{-1}, \mathfrak{g})$ with $\ker(\Box) \subset C^k(\mathfrak{g}_{-1}, \mathfrak{g})$. We may also view $H^k(\mathfrak{g}_{-1}, \mathfrak{g})$ as $\ker(\partial^*)/\operatorname{im}(\partial^*)$, which naturally makes the cohomology groups into P-modules since ∂^* is a $\mathfrak{p}$-homomorphism and thus P-equivariant. Moreover, one can compute using the formula for ∂^* that the action of $\mathfrak{g}_1$ maps $\ker(\partial^*)$ to $\operatorname{im}(\partial^*)$ and thus acts trivially on the quotient. This shows that that the action of $\mathfrak{g}_1$ and hence the action of $P_+ = \exp(\mathfrak{g}_1)$ on the cohomology groups $H^k(\mathfrak{g}_{-1}, \mathfrak{g})$ is trivial.

The cohomology groups also possess a $\mathfrak{g}_0$-invariant grading induced by the grading of $\mathfrak{g}$. Namely, the spaces $H^k(\mathfrak{g}_{-1}, \mathfrak{g})$ decompose into direct sums $\oplus_\ell H^k(\mathfrak{g}_{-1}, \mathfrak{g})_\ell$ according to homogeneous degrees of representative cocycles. Of course, we also have the associated $\mathfrak{p}$-invariant filtration given by $H^k(\mathfrak{g}_{-1}, \mathfrak{g})^\ell = \oplus_{i \geqq \ell} H^k(\mathfrak{g}_{-1}, \mathfrak{g})_i$.

A complete description of the $\mathfrak{g}_0$-module structure of the cohomology spaces $H^*(\mathfrak{g}_{-1}, \mathfrak{g})$ in the complex case is given by Kostant's version of the Bott-Borel-Weil theorem. This description is in terms of the Hasse diagram which provides an algorithm for computing the cohomologies. For details, see section 3.3. in [**4**].

1.2.6. Normal parabolic geometries. The constructions from the previous subsection can be directly carried over to a manifold endowed with a $|1|$-graded parabolic geometry. Suppose that $(p : \mathcal{G} \to M, \omega)$ is such a geometry of type (G, P) and consider the bundle $\Lambda^k T^*M \otimes \mathcal{A}M = \Omega^k(M, \mathcal{A}M)$, whose sections are k-forms with values in the adjoint tractor bundle. By definition, the adjoint tractor bundle is induced by $\mathfrak{g}$ and the cotangent bundle is induced by $(\mathfrak{g}/\mathfrak{p})^*$. Consequently, the bundle $\Omega^k(M, \mathcal{A}M)$ is induced by a P-module $L(\Lambda^k(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$ of linear maps $\Lambda^k(\mathfrak{g}/\mathfrak{p}) \to \mathfrak{g}$ and since the Kostant codifferential ∂^* is P-equivariant, it gives rise to bundle maps $\partial^* : \Lambda^{k+1} T^*M \otimes \mathcal{A}M \to \Lambda^k T^*M \otimes \mathcal{A}M$ for any k. A formula for this map is obtained directly from the explicit formula for the P-equivariant map ∂^* by replacing the elements of $\mathfrak{g}_1$ by one forms ϕ_i and the Lie bracket by the algebraic bracket $\{\ ,\ \}$ on $\mathcal{A}M$ introduced in the proposition 1.2.

$$\partial^*(\phi_0 \wedge \cdots \wedge \phi_l \otimes s) = \sum_{i=0}^{l} (-1)^{i+1} \phi_0 \wedge \cdots \wedge \widehat{\phi_i} \wedge \cdots \wedge \phi_l \otimes \{\phi_i, s\} \quad (1.2)$$

Now consider the curvature function of the Cartan connection ω. By definition, it is a P-equivariant map $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ and so it defines a section in the bundle $\mathcal{G} \times_P \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \cong \Lambda^2 T^*M \otimes \mathcal{A}M$. Hence we may view the curvature κ as a two-form with values in the adjoint tractor bundle and we may act on it by the Kostant codifferential ∂^*.

Definition 1.4. Let $(p : \mathcal{G} \to M, \omega)$ be a (real or complex) parabolic geometry on a manifold M, and let $\kappa \in \Omega^2(M, \mathcal{A}M)$ be the curvature of ω. Then the parabolic geometry is called *normal* if $\partial^*\kappa = 0$.

It turns out that the normality is the right condition to uniquely associate a $|1|$-graded parabolic geometry to an underlying structure. Namely, one can show that starting with a G-structure, one can construct a Cartan bundle and a normal Cartan connection. Moreover, this connection is under certain cohomological condition unique. Consequently, one obtains the following theorem which relates $|1|$-graded parabolic geometries and G-structures. It is a special case of the theorem 3.1.14. in **[4]**, where also a proof can be found.

Theorem 1.5. *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $|1|$-graded semisimple Lie algebra such that none of the simple ideals of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$, and such that $H^1(\mathfrak{g}_{-1}, \mathfrak{g})$ is concentrated in homogeneous degrees $\leqq 0$. Suppose that G is a Lie group with a Lie algebra $\mathfrak{g}$, and $P \subset G$ is a parabolic subgroup corresponding to the grading with Levi subgroup $G_0 \subset P$.*

Then associating to parabolic geometry of type (G, P) its underlying G_0-structure and to any morphism of parabolic geometries the induced morphism of the underlying G_0-structures defines an equivalence between the category of normal parabolic geometries of type (G, P) and the category of G-structures with structure group G_0.

Using the Kostant's version of the Bott-Borel-Weil theorem, one can compute the first cohomology in individual examples. Then it turns out that $H^1(\mathfrak{g}_{-1}, \mathfrak{g})^2 \neq 0$ only if $\mathfrak{g}$ contains a simple summand isomorphic to $\mathfrak{sl}(2)$. Moreover, $H^1(\mathfrak{g}_{-1}, \mathfrak{g})^1 \neq 0$ only if a simple factor of $\mathfrak{g}$ is a simple $|1|$-graded Lie algebra from A_ℓ series which corresponds to classical projective structures. In this case, one needs to choose additional data on the level of the G-structure to specify a normal parabolic geometry. Namely, one obtains an equivalence of torsion-free projective structures with corresponding normal geometries.

In all other case, the cohomological condition from the theorem is satisfied and the normal geometries are equivalent to their underlying structures. Therefore, we will use freely both the names "structure" and "geometry" in sequel. A panorama of structures corresponding to $|1|$-gradings is given in section 4.1. in [**4**].

1.2.7. Harmonic curvature. The simplest parabolic geometries are those which are locally isomorphic to their homogenous model. These geometries are called *locally flat*, and as mentioned above, they are characterized exactly by $\kappa = 0$. As we shell see, it is not necessary to look at the whole curvature since a simpler object, called the harmonic curvature, is already a complete obstruction to a local flatness.

The bundle maps $\partial^* : \Lambda^{k+1}T^*M \otimes \mathcal{A}M \to \Lambda^k T^*M \otimes \mathcal{A}M$ induced by the Kostant codifferential give rise to smooth subbundles $\operatorname{im}(\partial^*) \subset \ker(\partial^*) \subset \Lambda^k T^*M \otimes \mathcal{A}M$. Therefore, we may form the quotient bundles $\ker(\partial^*)/\operatorname{im}(\partial^*)$. We know from above that these bundles are induced by the P-modules $H^k(\mathfrak{g}_{-1}, \mathfrak{g})$, and since the action of P_+ on the cohomologies is trivial, we get the isomorphism

$$\ker(\partial^*)/\operatorname{im}(\partial^*) \cong \mathcal{G}_0 \times_{G_0} H^k(\mathfrak{g}_{-1}, \mathfrak{g}).$$

In particular, the bundles depend on the underlying G-structure only and any smooth form $\varphi \in \Omega^k(M, \mathcal{A}M)$ such that $\partial^*\varphi = 0$ may be projected to a section φ_H of the bundle $\mathcal{G}_0 \times_{G_0} H^k(\mathfrak{g}_{-1}, \mathfrak{g})$. Now we apply this to the curvature of a normal parabolic geometry.

Definition 1.5. Let $(p : \mathcal{G} \to M, \omega)$ be a normal parabolic geometry with curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$. Then the *harmonic curvature* κ_H is defined to be the image of κ in the space of sections of the bundle $\mathcal{G}_0 \times_{G_0} H^2(\mathfrak{g}_{-1}, \mathfrak{g})$.

The harmonic curvature is a much simpler object then the curvature since it is a section of a bundle depending only on the underlying G-structure. It turns out however that it contains the complete information about the curvature. The explicit relation between the curvature and its harmonic part is given in the following theorem which is a special case of theorem 3.1.12. in [**4**].

Before stating the theorem, recall that $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ is graded according to degrees of the representative cocycles. Since this grading is $\mathfrak{g}_0$-invariant, it translates to a grading on $\mathcal{G}_0 \times_{G_0} H^2(\mathfrak{g}_{-1}, \mathfrak{g})$. Hence the harmonic curvature has three components $(\kappa_H)_\ell$ of homogeneity $\ell = 1, 2, 3$. By definition, the same grading is on the bundle $\mathcal{G}_0 \times_{G_0} L(\Lambda^2\mathfrak{g}_{-1}, \mathfrak{g})$. Since the isomorphism $\mathfrak{g}_{-1} \cong \mathfrak{g}/\mathfrak{p}$ of $\mathfrak{g}_0$-modules gives the identification $TM \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$, this bundle may be identified with $\Lambda^2 T^*M \otimes \operatorname{gr}(\mathcal{A}M)$, where $\operatorname{gr}(\mathcal{A}M) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}$ is the associated graded bundle to

$$\mathcal{A}M = \mathcal{A}^{-1}M \supset \mathcal{A}^0 M \supset \mathcal{A}^1 M \cong T^*M$$

defined by $\mathrm{gr}_i(\mathcal{A}M) = \mathcal{A}^i M/\mathcal{A}^{i-1}M$. Thus a section $\mathrm{gr}(\kappa)$ of the bundle $\Lambda^2 T^*M \otimes \mathrm{gr}(\mathcal{A}M)$ has also three components $\mathrm{gr}_\ell(\kappa)$ according to homogeneities $\ell = 1, 2, 3$. Now we are ready to formulate the following.

Theorem 1.6. *Let $(p : \mathcal{G} \to M, \omega)$ be a normal parabolic geometry of type (G, P) and κ its curvature. Then the component $\mathrm{gr}_\ell(\kappa)$ of lowest homogeneity ℓ is a section of the subbundle $\ker(\Box) \subset \Lambda^2 TM^* \otimes \mathcal{A}M$, and under the natural identification of this bundle with $\ker(\partial^*)/\operatorname{im}(\partial^*)$, the section $\mathrm{gr}_\ell(\kappa)$ coincides with the homogenous component of degree ℓ of κ_H. In particular, for normal parabolic geometries vanishing of the harmonic curvature κ_H implies vanishing of the curvature κ.*

1.3. Weyl structures

As we saw in the previous section, most of the normal $|1|$-graded parabolic geometries are equivalent to their underlying G-structures. Now the natural task is to describe the Cartan connection and its curvature in terms of data on the underlying G-structure $\mathcal{G}_0 \to M$. This is the theme of this section. At the end, we also obtain the Bianchi and Ricci identities from proposition 1.3 in terms of geometric objects associated to the G-structure.

1.3.1. Weyl structures. Let us consider the same setting as in the previous section. That is, let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $|1|$-graded semisimple Lie algebra, G a Lie group with Lie algebra $\mathfrak{g}$, let $P \subset G$ be a parabolic subgroup for the given grading and $G_0 \subset P$ the Levi subgroup. Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P), and consider its underlying principal G_0-bundle $p_0 : \mathcal{G}_0 \to M$. By definition, $\mathcal{G}_0 = \mathcal{G}/P_+$ and thus there is a natural projection $\pi : \mathcal{G} \to \mathcal{G}_0$, which is a principal bundle with structure group $P_+ = \exp(\mathfrak{g}_1)$ (which is always trivial by topological reasons).

Definition 1.6. A (local) *Weyl structure* for the parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is a (local) smooth G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the projection $\pi : \mathcal{G} \to \mathcal{G}_0$.

To formulate a proposition on the existence of global Weyl structures, let us recall that the tangent bundle TM is the associated bundle $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$ and the cotangent bundle is isomorphic to $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_1$. In particular, smooth sections of T^*M can be identified with smooth G_0-equivariant functions $\mathcal{G}_0 \to \mathfrak{g}_1$. Now the following proposition shows that global Weyl structures always exist and that they form an affine space. For a proof, see 5.1.1. in [**4**].

Proposition 1.7. *For any $|1|$-graded parabolic geometry $(p : \mathcal{G} \to M, \omega)$, there exist a global Weyl structure $\sigma : \mathfrak{g}_0 \to \mathcal{G}$.*

Fixing one Weyl structure σ, there is a bijective correspondence between the set of all Weyl structures and the space $\Gamma(T^*M)$ of smooth sections of the cotangent bundle. Explicitly, this correspondence is given by mapping $\Upsilon \in \Gamma(T^*M)$ viewed as a function $\Upsilon : \mathcal{G}_0 \to \mathfrak{g}_1$ to the Weyl structure $\hat{\sigma}(u) := \sigma(u)\exp(\Upsilon(u))$.

1.3.2. Weyl connections and Rho-tensor. Given a Weyl structure $\sigma : \mathfrak{g}_0 \to \mathcal{G}$ on a $|1|$-graded parabolic geometry $(p : \mathcal{G} \to M, \omega)$, we can consider the pullback $\sigma^*\omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$ of the Cartan connection. Equivariancy of σ then implies G_0-equivariancy of $\sigma^*\omega$ and since the G_0-module $\mathfrak{g}$ decomposes as $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, we get a G_0-invariant decomposition $\sigma^*\omega = \sigma^*\omega_{-1} + \sigma^*\omega_0 + \sigma^*\omega_1$. Then using the properties of ω, we easily prove that the component $\sigma^*\omega_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ defines a principal connection on the bundle $p_0 : \mathcal{G}_0 \to M$ and the component $\sigma^*\omega_1$ determine a T^*M-valued one-form $\mathsf{P} \in \Omega^1(M, T^*M)$. For details, see 5.1.2. in [**4**]. These facts lead to the following definitions.

Definition 1.7. Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be a Weyl structure for a $|1|$-graded parabolic geometry $(p : \mathcal{G} \to M, \omega)$.
(1) The principal connection $\sigma^*\omega_0$ on the bundle $\mathcal{G}_0 \to M$ is called the *Weyl connection* associated to the Weyl structure σ.
(2) The one-form $\mathsf{P} \in \Omega^1(M, T^*M)$ induced by the positive component of $\sigma^*\omega$ is called the *Rho-tensor* associated to the Weyl structure σ.

Since $\sigma^*\omega_0$ is a principal connection on the bundle $\mathcal{G}_0$, it gives rise to an induced linear connection on any vector bundle associated to $\mathcal{G}_0$. All these induced connections will be refered to as the Weyl connections corresponding to the Weyl structure σ. Moreover, it is easy to see that the map $\sigma \times \mathrm{id}_V : \mathcal{G}_0 \times V \to \mathcal{G} \times V$ composed with the projection $\mathcal{G} \times V \to \mathcal{G} \times_P V$ factors to an isomorphism of vector bundles

$$\mathcal{G}_0 \times_{G_0} V \xrightarrow{\cong} \mathcal{G} \times_P V.$$

Hence each natural vector bundle is via σ associated to the principal bundle $\mathcal{G}_0$ and the Weyl connection defines a linear connection on it.

For most natural vector bundles, we can formulate this result in a more coceptual way. Namely, consider that V is a finite dimensional representation of P which is completely reducible as a representation of G_0. Now let us put $V^0 := V$ and $V^i := \mathfrak{g}_1 \cdot V^{i-1}$ for $i > 0$. Then it is easy to prove that each of the subspaces V^i is P-invariant. Essentialy, this follows from the fact that P is diffeomorphic to $G_0 \times \mathfrak{g}_1$. Considering the action of the grading element, we also find that for each i the subspace V^{i+1} is strictly smaller than V^i. Hence we get a P-invariant filtration $V = V^0 \supset V^1 \supset \cdots \supset V^N \supset \{0\}$ such that for each i, the action of $\mathfrak{g}_1$ maps V^i to V^{i+1}. As a G_0-module, V splitts into a direct sum of quotient spaces V^i/V^{i+1}, which is known as associated graded module $\mathrm{gr}(V)$. Such a filtration is called the *composition series* of the representation V and we write it as

$$V = V^0/V^1 \oplus\!\!\!\!+ V^1/V^2 \oplus\!\!\!\!+ \cdots \oplus\!\!\!\!+ V^{N-1}/V^N \oplus\!\!\!\!+ V^N.$$

In this notation, $V^N \subset V$ is a P-submodule, V^{N-1}/V^N is a submodule in V/V^N, etc. Now since the filtration is P-invariant, it induces a filtration V^iM on any natural bundle $VM = \mathcal{G} \times_P V$. Moreover, the infinitesimal action of $\mathfrak{g}_1$ defines a P-equivariant map $\mathfrak{g}_1 \times V^i \to V^{i+1}$ inducing a bundle map $\bullet : T^*M \times V^iM \to V^{i+1}M$ since $T^*M = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_1$ and thus the composition series for V induces an analogous composition series for the bundle VM.

The associated graded bundle $\mathrm{gr}(VM) = \oplus_i(V^iM/V^{i+1}M)$ is equal to $\mathcal{G} \times_P \mathrm{gr}(V)$ and it may be identified with $\mathcal{G}_0 \times_{G_0} \mathrm{gr}(V) = \mathcal{G}_0 \times_{G_0} V$ since the action of the nilpotent part of P on $\mathrm{gr}(V)$ is trivial and $V \cong \mathrm{gr}(V)$ as G_0-modules. So the previous result which says that any Weyl structure induces an isomorphism $VM \cong \mathcal{G}_0 \times_{G_0} V$ may be reformulated in the way that any Weyl structure induces an isomorphism

$$VM \cong \mathrm{gr}(VM),$$

which defines a splitting of the filtration. Let us note that a complete reducibility of V is rather a weak condition. In particular, it is satisfied for all representations which are restrictions of a representation of G.

1.3.3. The effect of a change of Weyl structures. Analyzing the effect of a change of Weyl structure to the isomorphism $VM \cong \mathrm{gr}(VM)$, to the associated Weyl connection and Rho-tensor, we easily obtain the following proposition which is a special case of the propositions 5.1.5, 5.1.6 and 5.1.8 in [**4**]. The bracket $\{\ ,\ \}$ appearing in the proposition is the algebraic bracket from 1.2. It is induced by the Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0$ and $\mathfrak{g}_1 \times \mathfrak{g}_0 \to \mathfrak{g}_1$ respectively. The symbol $\bullet$ is used for both the bundle maps $\mathrm{gr}_0(\mathcal{A}M) \times VM \to VM$ and $T^*M \times VM \to VM$ since they are induced by the same infinitesimal action of $\mathfrak{p}$ on V (restricted to $\mathfrak{g}_0$ and $\mathfrak{g}_1$ respectively).

Proposition 1.8. *Let $\hat\sigma$ and σ be two Weyl structures related by $\hat\sigma(u) = \sigma(u)\exp(\Upsilon(u))$ for an one-form Υ. For a representation V of P which is completely reducible as a representation of G_0, we have:*
(1) The isomorphisms $VM \ni s \mapsto (s_0, \dots, s_N) \in \mathrm{gr}(VM)$ corresponding to $\hat\sigma$ and σ are related by

$$\hat{s}_k = \sum_{i=0}^{k} \frac{(-1)^i}{i!}(\Upsilon\bullet)^i s_{k-i} \tag{1.3}$$

(2) For a smooth section s of a bundle VM and $\xi \in TM$, the Weyl connections $\hat{\nabla}$ and ∇ are related by

$$\hat{\nabla}_\xi s = \nabla_\xi s - \{\Upsilon, \xi\} \bullet s \tag{1.4}$$

(3) The Rho-tensors $\hat{\mathsf{P}}$ and P associated to $\hat{\sigma}$ and σ are related by

$$\hat{\mathsf{P}}(\xi) = \mathsf{P}(\xi) + \nabla_\xi \Upsilon + \frac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\} \tag{1.5}$$

Remark 1.1. For any $|1|$-graded parabolic geometry $(\mathcal{G} \to M, \omega)$ one defines *bundles of scales* L^λ as the natural line bundle associated to an one-dimensional representation $\lambda : G_0 \to \mathbb{R}$. Its derivative $\lambda' : \mathfrak{g}_0 \to \mathbb{R}$ is a Lie algebra homomorphism vanishing on the semisimple part of $\mathfrak{g}_0$, so it is just a linear functional on the (one-dimensional) center $\mathfrak{z}(\mathfrak{g}_0)$. Hence the bundle of scales corresponds to multiples of the grading element E. From the property (2) in the previous theorem we have $\hat{\nabla} = \nabla$ implies $\Upsilon = 0$ which by proposition 1.7 implies $\hat{\sigma} = \sigma$. Thus the induced linear connections on bundles of scales are in bijective correspondence with Weyl structures. This correspondence leads to definitions of closed and an exact Weyl structures. Namely, the *closed Weyl structure* is a Weyl structure inducing a flat connection on L^λ and the *exact Weyl structure* corresponds to a connection induced by a global trivialization of L^λ. The corresponding connections on the tangent space form an affine space modelled on closed respectively exact one-forms. For details, see section 5.1.4 and corollary 5.1.7 in [**4**].

1.3.4. Description of $|1|$-graded geometries via Weyl structures. The goal of this section is to give a description of $|1|$-graded normal parabolic geometries in terms of data associated to a Weyl structure. Such a description is provided by theorem 5.2.3 in [**4**] which we are ready to formulate.

Theorem 1.9. *Let $p_0 : \mathcal{G}_0 \to M$ be a G-structure with a structure group $G_0 \subset P \subset G$ such that $H^1(\mathfrak{g}_{-1}, \mathfrak{g})^1 = 0$, and let $(p : \mathcal{G} \to M, \omega)$ be the unique normal parabolic geometry of type (G, P) extending this G-structure, κ its curvature and κ_H its harmonic curvature.*
(1) The Weyl connections associated to Weyl structures $\sigma : \mathcal{G}_0 \to G$ are exactly the principal connections on $\mathcal{G}_0$ with ∂^*-closed torsion. All these connections have the same torsion T, which coincides with the homogenous component of degree one of κ_H.
(2) For such a Weyl connection let R be the curvature. Then the Rho-tensor satisfies $\mathrm{gr}_0(\partial^*)(R + \partial\mathsf{P}) = 0$ and is uniquely determined by $\Box\mathsf{P} = -\partial^* R$. The remaining components of the harmonic curvature are given by the components in $\ker(\Box)$ of R respectively the covariant exterior derivative Y of P.

(3) The Cartan curvature κ of ω is represented by

$$(T, U, Y) \in \Omega^2(M, TM \oplus \mathrm{End}_0(TM) \oplus T^*M),$$

where T and Y are defined in (1) and (2) and $U := R + \partial \mathsf{P}$.

According to (3), the curvature $U = R + \partial\mathsf{P}$ is the $\mathfrak{g}_0$-component of the Cartan curvature. This curvature is called the *Weyl curvature* due to the analogy with the conformal case, where this quantity coincides exactly with the (invariant) conformal Weyl curvature. In contrast to the conformal case, the Weyl curvature is not invariant in general. The invariant piece is the irreducible component corresponding to the $\mathfrak{g}_0$-component of the harmonic curvature.

Using the data associated to a Weyl structure, we can also express the two cannonical objects we defined, namely the fundamental derivative and the tractor conection. Proposition 5.1.10 in [**4**] specialized to the $|1|$-graded case yields

Proposition 1.10. *Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be a Weyl structure, ∇ and P the corresponding Weyl connection and Rho-tensor, $VM = \mathcal{G} \times_P V$ a natural vector bundle and $\mathcal{A}M$ the adjoint tractor bundle. For $s \in \Gamma(\mathcal{A}M)$ and $t \in \Gamma(VM)$ let $s \mapsto (s_{-1}, s_0, s_1)$ and $t \mapsto (t_0, \dots, t_N)$ be the isomorphisms with associated graded bundles given by σ.*
(1) The fundamental derivative is given by

$$(D_s t)_i = \nabla_{s_{-1}} t_i + \mathsf{P}(s_{-1}) \bullet t_{i-1} - s_0 \bullet t_i - s_1 \bullet t_{i-1}, \qquad (1.6)$$

(2) If t is a section of a tractor bundle and $\xi \in TM$, then the tractor connection $\nabla^{\mathcal{T}}$ is given by

$$(\nabla^{\mathcal{T}}_\xi t)_i = \nabla_\xi t_i + \mathsf{P}(\xi) \bullet t_{i-1} + \xi \bullet t_{i+1} \qquad (1.7)$$

1.3.5. Bianchi and Ricci identities. Given a full description of the Cartan curvature by the previous theorem and the expression for the fundamental derivative by the previous proposition, we can reformulate the Bianchi identities and the Ricci identity from proposition 1.3 and write these identities in terms of geometric objects associated to the Weyl structure.

Proposition 1.11. *Let ∇ be a Weyl connection given by a choice of Weyl structure, T and R its torsion and curvature, let P be the corresponding Rho-tensor and U the Weyl curvature. Then for $X, Y, Z \in TM$ we have:*
(1) Algebraic Bianchi identity

$$\sum_{cycl.} -U(Y,Z)(X) + T(T(X,Y),Z) + (\nabla_X T)(Y,Z) = 0$$

(2) Differential Bianchi identity

$$\sum_{cycl.}(\nabla_X R)(Y,Z) + U(T(X,Y),Z) - \{\mathsf{P}(T(X,Y)),Z\} = 0$$

(3) Ricci identity

$$(\nabla^2 Z)(X,Y) - (\nabla^2 Z)(Y,X) = R(X,Y)(Z) - \nabla_{T(X,Y)}Z$$

Proof. (1) The algebraic Bianchi identity follows by taking the part in $\mathrm{gr}_{-1}(\mathcal{A}M) \cong TM$ of the Bianchi identity (1) in proposition 1.3. Then for $s_1, s_2, s_3 \in TM$ the expression appearing in the sum in this equation has the form

$$\{s_1, \kappa_0(s_2,s_3)\} - \kappa_{-1}(\{s_1,s_2\},s_3) + \kappa_{-1}(\kappa_{-1}(s_1,s_2),s_3) + (\nabla_{s_1}\kappa_{-1})(s_2,s_3)$$

One can easily deduce that the algebraic bracket of $\mathrm{gr}_0(\mathcal{A}M) \cong \mathrm{End}(TM)$ and $\mathrm{gr}_{-1}(\mathcal{A}M)$ is given simply by applying the endomorphism. Hence the first summand is equal to $-\kappa_0(s_2,s_3)(s_1)$ which is equal to $-U(s_2,s_3)(s_1)$ according to the previous theorem. The theorem also implies that the second term vanishes and that $\kappa_{-1}(\kappa_{-1}(s_1,s_2),s_3) = T(T(s_1,s_2),s_3)$ and $(\nabla_{s_1}\kappa_{-1})(s_2,s_3) = (\nabla_{s_1}T)(s_2,s_3)$. The result then follows by renaming the sections.

(2) The differential Bianchi identity follows by taking the part of the Bianchi identity (1) in proposition 1.3 corresponding to $\mathrm{gr}_0(\mathcal{A}M) \cong \mathrm{End}_0(TM)$. Then the interior of the sum appearing in the equation has the form

$$\{s_1, \kappa_1(s_2,s_3)\} - \kappa_0(\{s_1,s_2\},s_3) + \kappa_0(\kappa_{-1}(s_1,s_2),s_3) + (\nabla_{s_1}\kappa_0)(s_2,s_3)$$

since the part in $\mathrm{gr}_0(\mathcal{A}M)$ of the fundamental derivative of κ_0 with respect to $s_1 \in TM$ coincides with the Weyl connection according to (1.6). Using the theorem and that $s_1, s_2, s_3 \in TM$, this is equal to

$$\{s_1, Y(s_2,s_3)\} + U(T(s_1,s_2),s_3) + (\nabla_{s_1}U)(s_2,s_3),$$

where $Y = d^\nabla \mathsf{P}$. From the definition of the exterior derivative d^∇, the definition of the torsion T and the differential ∂, we conclude

$$\sum_{\mathrm{cycl.}}\{s_1, Y(s_2,s_3)\} = -\sum_{\mathrm{cycl.}}((\nabla_{s_1}\partial\mathsf{P})(s_2,s_3) + \{\mathsf{P}(T(s_1,s_2)),s_3\}).$$

Now inserting this into the equation (1) and using that $U = R + \partial\mathsf{P}$, the result (2) follows.

(3) A formula for the Ricci identity in terms of Weyl connection is obtained by taking the part of Ricci identity from 1.3 corresponding to $\mathrm{gr}_{-1}(\mathcal{A}M) \cong TM$. By definition, we have $D^2\sigma(s_1,s_2) = D_{s_1}D_{s_2}\sigma - D_{D_{s_1}s_2}\sigma$ and using (1.6) we find that its part in TM is equal to

$$\nabla_{s_1}\nabla_{s_2}\sigma - \nabla_{\nabla_{s_1}s_2}\sigma + \{\mathsf{P}(s_1),s_2\} \bullet \sigma.$$

Alternating in s_1 and s_2 and the use of definition of the differential ∂ then leads to the equation

$$D^2\sigma(s_1,s_2) - D^2\sigma(s_2,s_1) = \nabla^2\sigma(s_1,s_2) - \nabla^2\sigma(s_2,s_1) + \partial\mathsf{P}(s_1,s_2)\bullet\sigma.$$

The term $D_{\{s_1,s_2\}}\sigma$ on the right-hand side of (2) in 1.3 evidently vanishes and the part in TM of the first term $-D_{\kappa(s_1,s_2)}\sigma$ is equal to

$$-\nabla_{\kappa_{-1}(s_1,s_2)}\sigma + \kappa_0(s_1,s_2)\bullet\sigma = -\nabla_{T(s_1,s_2)}\sigma + U(s_1,s_2)\bullet\sigma,$$

due to (1.6) and the theorem. And since $U = R + \partial\mathsf{P}$, the result of (3) follows. □

Remark 1.2. The curvature term $U(Y,Z)(X)$ in the algebraic Bianchi identity can be replaced by $R(Y,Z)(X)$ without any change since

$$\sum_{\text{cycl.}} \partial\mathsf{P}(Y,Z)(X) = (\partial^2\mathsf{P})(Y,Z,X) = 0.$$

This follows from the definition of ∂ and the fact that the algebraic bracket $\{\ ,\ \}$ is induced by the Lie bracket and thus satisfies Jacobi identity.

1.4. Construction of invariant operators via curved Casimirs

For any natural vector bundle associated to a parabolic geometry, there is a curved Casimir operator which acts on the space of smooth sections of the bundle and which reduces to the canonical action of the quadratic Casimir element on the homogeneous model of the geometry. The essential properties of the curved Casimir operator are that it is natural in a strong sense and that it acts by a scalar on a bundle associated to an irreducible representation. One concludes from these properties that one can use the curved Casimir operator systematically to construct invariant differential operators. This is a difficult task in general, since since the classification via homomorphisms of induced modules of the invariant operators on locally flat parabolic geometries does not apply to the case of a general ($|1|$-graded) parabolic geometry.

1.4.1. Invariant differential operators on homogeneous spaces. Linear invariant differential operators are linear differential operators intrinsic to the given structure. In the case of the homogeneous model $M = G/P$, it is evident what it means. The invariant operators are those which intertwine the G-action on sections of homogenous vector bundles induced by the left action of G on G/P. Formally, given two homogeneous vector bundles $VM = G\times_P V$ and $\mathcal{W}M = G\times_P W$ on $M = G/P$, *invariant differential operator* is a differential operator $D : \Gamma(VM) \to \Gamma(\mathcal{W}M)$ such that $D(g\cdot s) = g\cdot D(s)$ for all $s\in\Gamma(VM)$ and $g\in G$.

It is well-known that an action on sections of a homogeneous vector bundle VM extends to an action on the jet prolongations J^kVM which makes J^kVM into a homogeneous bundle. Then it is easy to see that any invariant differential opeartor D of order $\leqq k$ between homogeneous bundles VM and $\mathcal{W}M$ corresponds to a morphism $\tilde{D} : J^kVM \to \mathcal{W}M$ of homogeneous bundles. Next, one defines the k-th order *symbol* of D as the vector bundle map $S^kT^*M \otimes VM \to \mathcal{W}M$ given by the restriction of $\tilde{D}$ to the kernel of the projection $\pi^k_{k-1} : J^kVM \to J^{k-1}VM$.

Since the k-th jet prolongation J^kVM is a homogeneous vector bundle, it is induced by a representation which we denote by J^kV, i.e. $J^kVM = G \times_P J^kV$, and any invariant operator $\tilde{D} : J^kVM \to \mathcal{W}M$ corresponds to a P-module homomorphism $J^kV \to W$. Hence the problem of classification of linear invariant differential operators on $M = G/P$ boils down to an algebraic problem of classification of homomorphisms of P-modules. Although the representations J^kV inducing the jet bundles are quite complicated, the structure of such homomorphisms is known completely.

Let us remark that he proof of this classification is via a duality relating these homomorphism to homomorphisms of induced modules. Namely, considering the infinite jet prolongation $J^\infty VM = G \times_P J^\infty V$, which is defined as the direct limit of the system $\cdots \to J^{k+1}VM \to J^kVM \to \cdots$ and hence allows to let the order k free, one finds that

$$J^\infty V = (\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V^*)^*,$$

where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. In the case that V is an irreducible representation with highest weight λ, the induced module $M_\mathfrak{p}(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V^*$ is called *generalized Verma module.* The duality is obtained essentialy by identifying $J^\infty V$ with $J^\infty_e(G, V)$ and $\mathfrak{g}$ with left-invariant vector fields on G which gives an identification of $\mathcal{U}(\mathfrak{g})$ with the space of left-invariant differential operators on $C^\infty(G, \mathbb{R})$. Thus in the dual picture, the invariant operators between irreducible bundles $\Gamma(VM) \to \Gamma(\mathcal{W}M)$ correspond to P-module homomorphisms $W^* \to M_\mathfrak{p}(\lambda)$. And since the left action of $\mathfrak{g}$ on generalized Verma modules makes them into $(\mathfrak{g}, P)$-modules, the algebraic version of Frobenius reciprocity implies that P-homomorphisms $W^* \to M_\mathfrak{p}(\lambda)$ correspond to $(\mathfrak{g}, P)$-module homomorphisms $M_\mathfrak{p}(\mu) \to M_\mathfrak{p}(\lambda)$, where μ is the highest weight of W.

The question of existence of such homomorphisms is a purely representation theoretical task and there is a complete answer in terms of the highest weights of representations. The homomorphisms between (ordinary) Verma modules (where $\mathfrak{p}$ is the Borel algebra), have been completely characterized by Verma and Bernstein-Gelfand-Gelfand, cf. [**1**]. The generalization to an arbitrary parabolic $\mathfrak{p} \subset \mathfrak{g}$ have been given by Lepowsky and Boe, cf. [**2**] and

[**3**]. Let us mention that the existence of a non-zero homomorphism between Verma modules induces a homomorphism between generalized Verma modules, called the standard homomorphism. It may be zero, and even when this happens, there might be some other homomorphism, which is then called a non-standard homomorphism.

1.4.2. Invariant operators on curved manifolds. In the case of a parabolic geometry on a manifold M which is not locally flat and thus is not isomorphic to a homogeneous space, there are several definitions of operators intrinsic to the given structure. The definition from the flat case does not apply since there is no induced action of G on sections of natural vector bundles and hence the G-equivariancy has no sense. Nevertheless, we may use the fact that left multiplications of G are exactly the automorphisms of the flat geometry $G \to G/P$, and hence replace the G-equivariancy property by the property of commuting with automorphisms of the given geometry. This leads to the notion of natural operators defined in [**15**]. But, this naturality requirement is too weak in our case since there are nearly no morphisms of geometries on general manifolds. Therefore, the following stronger restriction of the class of operators is mostly specified.

By an *invariant operator* for geometry $(\mathcal{G} \to M, \omega)$, we mean a linear differential operator acting between natural bundles on M which one can write out by a universal formula in terms of Weyl connection and its curvature such that the formula is independent of the choice of Weyl structure. Obviously, such invariant operators are natural in the sense as above and thus, restricted to the subcathegory of locally flat geometries, they coincide with the invariant differential operators on homogeneous spaces defined in the previous subsection. The definition of invariant operators may be viewed as a generalization of the ususal notion of invariant operators from the conformal geometry. These are defined as formal expressions in Levi-Civita connection which remain invariant under rescalings of the metric.

In contrast to the case of a locally flat manifold, higher order invariant operators $J^kVM \to WM$ on a curved manifold M cannot be constructed in the same algebraic way as in the flat case since the jet prolongations J^kVM are not associated to the Cartan bundle $\mathcal{G}$ for $k > 1$. Nevertheless, there is a class of invariant differential operators which correspond to homomorphisms of P-modules just like in the flat case, with the exception that the P-modules are different. Namely, one can consider so called semi-holonomic jet prolongation $\bar{J}^kVM$ of the bundle VM, see [**10**]. This vector bundle comes with a natural inclusion $J^kVM \hookrightarrow \bar{J}^kVM$ and the important property is that, in contrast to normal (holonomic) jets, this bundle is associated to the Cartan bundle and thus is induced by a P-module $\bar{J}^kV$. Hence any homomorphism

$\bar{J}^k V \to W$ gives rise to an operator

$$J^k VM \hookrightarrow \bar{J}^k VM \to WM.$$

The operators arising in such a way are called *strongly invariant operators*. For instance, this is the case of all first-order operators since $\bar{J}^1 VM = J^1 VM$. Strongly invariant operators behave similarly to invariant operators on homogeneous spaces. In particular, they can be translated to other bundles by means of the curved translation principle, see [**16**].

In the dual picture, strongly invariant operators arise from homomorphisms of induced modules, called semi-holonomic Verma modules. In contrast to the holonomic case, they do not provide the classification of invariant operators since it may happen that for a non-zero homomorphism, the composition above defining the operator vanishes. On the other hand, it may also happen that despite the fact that there is no non-zero homomorphism of semi-holonomic Verma modules, there still exists an invariant operator between the corresponding bundles, as we shall see.

1.4.3. Curved Casimir operator. The curved Casimir operator is an invariant differential operator acting between natural vector bundles, which is a generalization of the well-known Casimir element from the representation theory. First let us recall its definition. The Casimir element for a semisimple Lie algebra $\mathfrak{g}$ is an element from the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ defined by $\mathcal{C} := \sum_\ell \xi^\ell \cdot \xi_\ell$, where $\{\xi^\ell\}$ is a basis of $\mathfrak{g}$ and $\{\xi_\ell\}$ is the dual basis with respect to the Killing form. This is well-defined since the (non-degenerate) Killing form induces an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ of G-modules and $\mathcal{C}$ is independent of the choice of the basis $\{\xi^\ell\}$. It is also easy to see that $\mathcal{C}$ acts on any representation of G by a G-equivariant map. Now the idea is to let the Casimir operator act on the space of smooth sections of a homogeneous bundle $G \times_P V \to G/P$. This is possible since there is an induced action of G on the sections and thus also the infinitesimal action of $\mathfrak{g}$. One can easily compute the form of this action. Namely, for $X \in \mathfrak{g}$ and a section f viewed as a P-equivariant function $G \to V$, we get $X \cdot f = -R_X \cdot f$, where $R_X \in \mathfrak{X}(G)$ is the right invariant vector field generated by X. Hence the action of $\mathcal{C}$ on a section f is given by $\mathcal{C}(f) = \sum_\ell R_{\xi^\ell} \cdot R_{\xi_\ell} \cdot f$.

The generalization to the case of a general parabolic geometry is obtained by rewritting this formula in terms of the fundamental derivative. Concretely, since the right invariant vector field R_X is in particular invariant under P, it defines a section s_X in the adjoint tractor bundle $\mathcal{A}(G/P)$. And since the differentiation of an equivariant f with respect to such a vector field is exactly the definition of the fundamental derivative, we can write

$\mathcal{C}(f) = \sum_\ell D_{s_{\xi^\ell}} \cdot D_{s_{\xi_\ell}} \cdot f$. Now this expression defines the curved Casimir operator on general parabolic geometries. Using the definition of the iteration D^2 and naturality of D, one finds that $\mathcal{C}$ can be expressed as the following composition. For details, see proposition 1 in [**5**].

$$\Gamma(VM) \xrightarrow{D^2} \Gamma(\otimes^2 \mathcal{A}^*M \otimes VM) \xrightarrow{B\otimes \mathrm{id}_V} \Gamma(VM)$$

Definition 1.8. On general parabolic geometries, the *curved Casimir operator* is the operator $\mathcal{C} := (B \otimes \mathrm{id}) \circ D^2$.

An immediate consequence of the defining formula is that the curved Casimir operator inherits strong naturality properties of the fundamental derivative. Namely, it commutes with with all natural vector bundle morphisms. In particular, it preserves sections of natural subbundles and restricts to the curved Casimir of the subbundle. Similarly, the induced operator on sections of a natural quotient bundle coincides with the curved Casimir on that bundle.

Further properties of the curved Casimir follows from a fromula in terms of suitably chosen local dual frames for the adjoint tractor bundle

$$\mathcal{A}M = TM \oplus \mathrm{End}_0(TM) \oplus T^*M.$$

Namely, let us choose a local frame φ^ℓ for the subbundle T^*M. This frame is automatically orthogonal with respect to the bilinear form on $T^*M \times T^*M$ induced by the Killing form B since T^*M is induced by $\mathfrak{g}_1$ and B vanishes on $\mathfrak{g}_1 \times \mathfrak{g}_1$. On the other hand, B gives a duality $\mathfrak{g}_1 \cong \mathfrak{g}/\mathfrak{p}$ of P-modules and thus induces a non-degenerate bilinear form on $T^*M \times TM$. Therefore, we can choose a basis ψ_ℓ for TM which is dual to φ^ℓ. Moreover, one can easily prove that ψ_ℓ may be chosen such that it is orthogonal, see lemma 1 in [**5**].

Now let D denote the fundamental derivative and let $\bullet$ denote the action of T^*M on VM induced by the representation of $\mathfrak{g}_1$ on V. Further, let $\langle\ ,\ \rangle$ be the inner product on weights of $\mathfrak{g}_0$, resp. $\mathfrak{g}_0^{\mathbb{C}}$, induced by the Killing form. Then the curved Casimir can be expressed using the local dual frames ψ_ℓ and φ^ℓ as follows.

Theorem 1.12. *For any ($|1|$-graded) parabolic geometry $(\mathcal{G} \to M, \omega)$ and any natural vector bundle VM, the action of the curved Casimir on a $s \in \Gamma(VM)$ has the form*

$$\mathcal{C}(s) = c(s) - 2\sum_\ell \varphi^\ell \bullet D_{\psi_\ell} s$$

where $c : VM \to VM$ is a tensorial bundle map which acts on each irreducible component $WM \subset VM$ by multiplication by c_W. If $-\lambda$ is the lowest *weight of the inducing irreducible representation W, then this scalar is given by*

$$c_W = \langle \lambda, \lambda + 2\rho \rangle$$

Proof. The theorem is a combination of proposition 3 and theorem 1 in [**5**]. □

In particular, the theorem shows that the curved Casimir is a differential operator of order at most one and that it acts by multiplication by a scalar on irreducible bundles, which can be easily computed from representation theory data. This scalar is refered to as Casimir eigenvalue.

Fixing a Weyl structure σ, we can express the formula from the theorem for the curved Casimir in terms of data on the underlying G-structure associated to σ. Namely, inserting $(s)_\sigma = (\psi_\ell, 0, 0)$ into equation (1.6) for the fundamental derivative, we get

$$D_{\psi_\ell} s = \nabla_{\psi_\ell} s + \mathsf{P}(\psi_\ell) \bullet s.$$

Therefore, the formula for the curved Casimir operator in terms of the Weyl connection and the Rho-tensor reads as

$$\mathcal{C}(s) = c(s) - 2 \sum_\ell \varphi^\ell \bullet \nabla_{\psi_\ell} s - 2 \sum_\ell \varphi^\ell \bullet \mathsf{P}(\psi_\ell) \bullet s \tag{1.8}$$

1.4.4. Construction of invariant operators. Finally, we show how to apply the curved Casimir operator to construct invariant operators. Let us explain the idea in the simplest case first. Therefore, assume a tractor bundle $VM = \mathcal{G} \times_P V$ induced by a G-representation V with a simple composition series

$$V = V^0/V^1 \oplus V^1.$$

and assume that the representations V^0/V^1 and V^1 are irreducible. Then the curved Casimir acts by scalars c_0 and c_1 on the corresponding bundles VM/V^1M and V^1M respectively. Now consider the operator $\mathcal{C} - c_1$. By definition, it is an invariant operator $\Gamma(VM) \to \Gamma(VM)$ which vanishes on V^1M. Hence if $c_0 \neq c_1$, then it defines a splitting $\Gamma(VM/V^1M) \to \Gamma(VM)$ of the tensorial projection $\Gamma(VM) \to \Gamma(VM/V^1M)$. In the case that the eigenvalues c_0 and c_1 coincide, we get an invariant operator $\Gamma(VM/V^1M) \to \Gamma(V^1M)$.

Now this idea can be implemented to the case of a longer filtration of the representation V. Consider a tractor bundle $VM = \mathcal{G} \times_P V$ induced by a G-representation V with a composition series

$$V = V^0/V^1 \oplus V^1/V^2 \oplus \cdots \oplus V^{N-1}/V^N \oplus V^N.$$

For such a V, each of the subquotients V^i/V^{i+1} splits into a direct sum of irreducible representations of P. Let $c^i_1, \dots, c^i_{n_i}$ denote the different eigenvalues of the curved Casimir operator on the bundles induced by these irreducible representations. Next, let us form an operator $L_i := \Pi^{n_i}_{j=1}(\mathcal{C} - c^i_j)$ which then maps $\Gamma(V^iM) \to \Gamma(V^{i+1}M)$ and hence $L := L_{i+1} \circ \dots \circ L_N$ maps $\Gamma(V^{i+1}M) \to 0$. This means that if the curved Casimir eigenvalue c_W on an irreducible bundle $WM \subset V^iM/V^{i+1}M$ is different from all of the numbers $c^i_1, \dots, c^i_{n_i}$, then L, restricted to WM, defines a splitting $\Gamma(WM) \to \Gamma(V^iM)$.

Now let us assume that c_W coincides with an eigenvalue corresponding to an irreducible bundle $W'M \subset V^kM/V^{k+1}M$ for a $k > i$. Such a case is of the main interest since it is easy to see that the operator $L := L_{i+1} \circ \dots \circ L_k$ restricted to WM then defines an invariant operator $\Gamma(WM) \to \Gamma(W'M)$. Indeed, the naturality of the curved Casimir implies that the factors in L can be commuted freely and so we may act first with the factor $\mathcal{C} - c_W$ from L_k to get a map $\Gamma(WM) \to \Gamma(V^{i+1}M)$. Next, the composition with $L_{i+1} \circ \dots \circ L_{k-1}$ yields an operator $\Gamma(WM) \to \Gamma(V^kM)$ by definition. Acting further with the rest of L_k, everything from $\Gamma(V^kM)$ is mapped to $\Gamma(V^{k+1}M)$ up to the bundle $W'M$. Hence projecting to $V^kM/V^{k+1}M$, we obtain an induced operator $\Gamma(WM) \to \Gamma(W'M)$.

The conclusion is that whenever there is a coincidence of the Casimir eigenvalues occuring somewhere in the composition series, there is an invariant operator between the respective bundles and we even know a formula for this operator. Fixing a Weyl structure, the formula can be expressed in terms of Weyl connection by use of the explicit form of $\mathcal{C}$ given by equation (1.8) in the previous subsection.

Let us mention that it can also happen that the induced invariant operator vanishes. This may be detected by computing the principal symbol of the operator which is easier than the computation of the whole formula for the operator. An algorithm for the symbol computation can be deduced from the construction of the operators. For general $|1|$-graded semisimple Lie algebras it was developed by Čap and Gover in a forthcoming article. A detailed description of the construction of invariant operators vie curved Casimirs in the case of a conformal geometry can be found in [**6**].

CHAPTER 2

Almost Grassmannian geometry

2.1. Introduction

In this introductory part, we give a definition of an almost Grassmannian geometry (in sequel AG-geometry), we introduce the corresponding structure and basic natural bundles. We also collect some basic properties which we will need later on.

2.1.1. Almost Grassmannian geometry. Let us consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. It can be viewed as the so called split real form of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$, characterized by the property that there exists a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$ such that all roots of $\mathfrak{g}^{\mathbb{C}}$ restricted to $\mathfrak{h}$ are real-valued. Of course, this $\mathfrak{h}$ is the subspace of tracefree real diagonal $n \times n$-matrices. Taking the usual choice of basis of $\mathfrak{h}$ and denoting by $e_i : \mathfrak{h} \to \mathbb{C}$ the linear functional which extracts the ith entry on the diagonal, the restricted roots of $\mathfrak{sl}(n, \mathbb{R})$ are $\Delta_r = \{e_i - e_j : 1 \leq i, j \leq n, i \neq j\}$ and its subsystem of positive roots is $\Delta_r^+ = \{e_i - e_j : i < j\}$. The set of simple roots is then $\{\alpha_1, \dots, \alpha_{n-1}\}$, where $\alpha_i = e_i - e_{i+1}$. In terms of the simple roots, the positive roots are exactly the combinations of the form $\alpha_i + \cdots + \alpha_j$ for $1 \leq i < j \leq n-1$. Obviously, the Satake diagram for $\mathfrak{sl}(n, \mathbb{R})$ has white nodes only and thus coincides with the Dynkin diagram A_{n-1}. For more details on the structure theory of $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{R})$, see sections 2.2. and 2.3. in [**4**].

Now consider a parabolic geometry of type (G, P), where the Lie algebra of G is equal to $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ and the parabolic subgroup $P \subset G$ is determined by a $|1|$-grading of $\mathfrak{g}$. As we described in the previous chapter, any $|1|$-grading of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ is given by a choice of one simple root, say α_p, and has the form

$$\mathfrak{g} = \bigoplus_{\mathrm{ht}(\alpha)=1} \mathfrak{g}_{-\alpha} \oplus (\mathfrak{h} \oplus \bigoplus_{\mathrm{ht}(\alpha)=0} \mathfrak{g}_{\alpha}) \oplus \bigoplus_{\mathrm{ht}(\alpha)=1} \mathfrak{g}_{\alpha},$$

where $\mathrm{ht}(\alpha) = 1$ if and only if α contains the simple root α_p as a summand, i.e. $\alpha = \alpha_i + \cdots + \alpha_p + \cdots + \alpha_j$. The corresponding diagram then has the pth node crossed. The grading is easily visible in block form with blocks of sizes $p, n-p$:

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \quad \mathfrak{g}_0 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad \mathfrak{g}_1 = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

Definition 2.1. A parabolic geometry of type (G, P) on a manifold M with $\dim(M) = pq$, where $p, q > 1$ and the parabolic subgroup P is determined by the grading of $\mathfrak{sl}(p+q, \mathbb{R})$ as above is called the *almost Grassmanian geometry of type* (p, q).

In sequel, we exclusively consider the case of almost Grassmanian geometries of type $(2, q)$, i.e. the $|1|$-graded parabolic geometry characterized by the diagram

$$\circ \text{———} \times \text{———} \circ \text{———} \dots \text{———} \circ \,.$$

The simple root which corresponds to the crossed node is $\alpha_2 = e_2 - e_3$ and so it is easy to see that the $\mathfrak{g}_1$-part of the grading is formed by root spaces corresponding to roots $\alpha_1 + \alpha_2 + \cdots + \alpha_i$ and $\alpha_2 + \cdots + \alpha_i$ for $3 \leq i \leq q+1$. In the form of block matrices, the elements of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2+q, \mathbb{R})$ may be viewed as $\left(\begin{smallmatrix} U & Z \\ Y & V \end{smallmatrix}\right)$ with block sizes 2 and q respectively. Then $\mathfrak{g}_0$ consists of block diagonal matrices $\left(\begin{smallmatrix} U & 0 \\ 0 & V \end{smallmatrix}\right)$, the subalgebra $\mathfrak{g}_1$ consists of those matrices for which only the Z-block is non-zero and similarly, $\mathfrak{g}_{-1}$ consists of those for which only the Y-block is non-zero. Hence we see that

$$\mathfrak{g}_{-1} = \mathbb{R}^{2*} \boxtimes \mathbb{R}^q, \quad \mathfrak{g}_0 = \mathfrak{s}(\mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{gl}(q, \mathbb{R})), \quad \mathfrak{g}_1 = \mathbb{R}^2 \boxtimes \mathbb{R}^{q*},$$

where $\boxtimes$ denotes the outer tensor product. Obviously, the component $\mathfrak{g}_{-1}$ may be also viewed as the space of linear maps $\mathbb{R}^2 \to \mathbb{R}^q$ while $\mathfrak{g}_1$ may be identified with linear maps $\mathbb{R}^q \to \mathbb{R}^2$.

As a Lie group G with Lie algebra $\mathfrak{g}$ we take $SL(2+q, \mathbb{R})$ and as the parabolic subgroup P, we take the stabilizer of $\mathbb{R}^2$ in $\mathbb{R}^{2+q}$. Since $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is exactly the stabilizer of this subspace in $\mathfrak{g}$, it is easy to see that this is the maximal parabolic subgroup for this grading, In terms of matrices, P is the subgroup of block upper triangular matrices with block sizes 2 and q. The resulting Levi subgroup $G_0 \subset P$ is then the group of block diagonal matrices with these block sizes, i.e. $G_0 = S(GL(2, \mathbb{R}) \times GL(q, \mathbb{R})) \subset SL(2+q, \mathbb{R})$. From the definition of P we conclude that the homogeneous model G/P of almost Grassmannian geometry is the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^{2+q})$ of 2-dimensional subspaces of $\mathbb{R}^{2+q}$.

2.1.2. Underlying structure. From the description above we see that the underlying G_0-structure $p_0 : \mathcal{G}_0 \to M$ of an almost Grassmannian geometry of type $(2, q)$ is given by the reduction of $GL(2q, \mathbb{R})$ to the structure group

G_0 formed by block diagonal matrices with blocks $(C_1, C_2) \in GL(2, \mathbb{R}) \times GL(q, \mathbb{R})$ such that $\det(C_1)\det(C_2) = 1$. Viewing elements $Y \in \mathfrak{g}_{-1}$ as linear maps $\mathbb{R}^2 \to \mathbb{R}^q$, the adjoint action $\mathrm{Ad} : G_0 \to GL(\mathfrak{g}_{-1})$ defining the G_0-structure is immediately seen to be given by $\mathrm{Ad}(C_1, C_2)(Y) = C_2 Y C_1^{-1}$.

The standard representations of $GL(2, \mathbb{R})$ and $GL(q, \mathbb{R})$ define the basic representations $\mathbb{R}^2$ and $\mathbb{R}^q$ of G_0, which then induce the basic natural vector bundles associated to $\mathcal{G}_0$: a rank 2 vector bundle $E \to M$ and a rank q vector bundle $F \to M$. Since the tangent bundle TM can be identified with $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$ and $\mathfrak{g}_{-1} = L(\mathbb{R}^2, \mathbb{R}^q)$, we get an isomorphism

$$TM \cong E^* \otimes F \tag{2.1}$$

Moreover, since $(C_1, C_2) \in G_0$ acts by $\det(C_1)^{-1}$ on the one-dimensional representation $\Lambda^2\mathbb{R}^{2*}$ and by the same scalar $\det(C_2) = \det(C_1)^{-1}$ on the one-dimensional representation $\Lambda^q\mathbb{R}^q$, we get an isomorphism $\Lambda^2\mathbb{R}^{2*} \cong \Lambda^q\mathbb{R}^q$ of G_0-modules, which then results into an isomorphism

$$\Lambda^2 E^* \cong \Lambda^q F. \tag{2.2}$$

Conversely, let us assume that on a manifold M of dimension $2q$ we have given vector bundles E and F of rank 2 respectively q and isomorphisms as in (2.1) and (2.2). Then it is easy to show that these data define a first order G_0-structure $p_0 : \mathcal{G}_0 \to M$ with structure group G_0. Indeed, we consider the fibered product $GL(\mathbb{R}^2, E) \times_M GL(\mathbb{R}^q, F)$ of the linear frame bundles of E and F. The fiber of this bundle over $x \in M$ is formed by pairs (ψ_1, ψ_2) of isomorphisms $\psi_1 : \mathbb{R}^2 \to E_x$ and $\psi_2 : \mathbb{R}^q \to F_x$. Then we define $\mathcal{G}_0$ to be a subspace in this bundle consisting of those pairs which respect the isomorphism (2.2) in the sense that the following diagram commutes

$$\begin{array}{ccc} \Lambda^2 E^* & \xrightarrow{\cong} & \Lambda^q F \\ {\scriptstyle \Lambda^2\psi_1}\uparrow & & \uparrow{\scriptstyle \Lambda^q\psi_2} \\ \Lambda^2\mathbb{R}^{2*} & \xrightarrow{\cong} & \Lambda^q\mathbb{R}^q \end{array}$$

The group G_0 acts on $\mathcal{G}_0$ by composition from the right. Visibly, the action is free and transitive on the fibers of the natural projection $p_0 : \mathcal{G}_0 \to M$ and hence $\mathcal{G}_0$ becomes a smooth principal G_0-bundle. The reduction $i : \mathcal{G}_0 \to \mathcal{P}^1 M$ is then given by assigning to a frame $(\psi_1, \psi_2) \in \mathcal{G}_0$ the isomorphism $T_x M \to \mathfrak{g}_{-1}$ defined by $\xi \to \psi_2^{-1} \circ \xi \circ \psi_1 \in \mathbb{R}^{2*} \otimes \mathbb{R}^q = \mathfrak{g}_{-1}$, where the vector $\xi \in T_x M$ is viewed as a linear map $E \to F$ according to isomorphism (2.1).

Hence we conclude that a choice of E, F, and isomorphisms (2.1) and (2.2) is equivalent to a first order G_0-structure $p_0 : \mathcal{G}_0 \to M$. Such a structure is called an *almost Grassmanninan structure of type* $(2, q)$. And since it is easy to check that the cohomological condition in theorem 1.5 is satisfied, we know that such a structure is equivalent to a normal almost Grassmannian geometry on M.

It is obvious that for the homogeneous model, the auxilliary bundles E and F are the two tautological bundles over the Grassmannian. The bundle E is the subbundle in $\mathrm{Gr}(2, \mathbb{R}^{2+q}) \times \mathbb{R}^{2+q}$ whose fiber over a 2-dimensional subspace is given by that subspace, while F is the quotient of the trivial $\mathbb{R}^{2+q}$-bundle by the subbundle E.

2.1.3. Abstract index notation. In sequel, we use an abstract index notation, which is similar to the spinorial abstract index notation used in 4-dimensional conformal geometry. For the basic vector bundles, we put $E := \mathcal{E}^{A'}$ and $F := \mathcal{E}^{A}$ and follow the usual convention that dualizing makes upper indices into lower indices and vice versa, and concatenation of indices corresponds to tensor products of bundles. Then the isomorphisms (2.1) and (2.2) have the form

$$TM = \mathcal{E}_{A'} \otimes \mathcal{E}^{A} = \mathcal{E}^{A}_{A'} \quad \text{respectively} \quad \Lambda^2 \mathcal{E}_{A'} \cong \Lambda^q \mathcal{E}^A.$$

For the tangent and cotangent bundle we also use the usual tensor inices, i.e. $TM = \mathcal{E}^a$ and $T^*M = \mathcal{E}_a$. The tensorial and spinorial notation will be used alternately in sequel.

We demonstrate the efficiency of the abstract index formalism on computing the explicit form of the algebraic bracket from 1.2, which we will need later on. Let us recall that the algebraic bracket acts on the adjoint tractor bundle $\mathcal{A}M$, which under a choice of Weyl structure is a graded bundle with grading components induced by representations $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ of G_0. These are isomorphic to the bundles TM, $\mathrm{End}_0(TM)$ and T^*M respectively.

By definition, the algebraic bracket is induced by the Lie bracket on the grading components of the matrix algebra $\mathfrak{g}$. If we adopt the same abstract index notation for the representations $\mathbb{R}^2$ and $\mathbb{R}^q$ inducing E and F, then we can write typical elements $Y \in \mathfrak{g}_{-1} = \mathbb{R}^{2*} \boxtimes \mathbb{R}^q$, $(U, V) \in \mathfrak{g}_0 = \mathfrak{s}(\mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{gl}(q, \mathbb{R}))$ and $Z \in \mathfrak{g}_1 = \mathbb{R}^{q*} \boxtimes \mathbb{R}^2$ as

$$Y = y^{A}_{A'}, \quad (U, V) = (u^{A'}_{B'}, v^{A}_{B}), \quad Z = z^{A'}_{A}.$$

Now the form of the Lie bracket in these abstract indices can be read off its form in block matrices. Namely, the Lie bracket $\mathfrak{g}_0 \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ has the form

$$\left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ VY - YU & 0 \end{pmatrix},$$

which leads to the following formula in abstract indices

$$[(U, V), Y]^{A}_{A'} = v^{A}_{B} y^{B}_{A'} - u^{B'}_{A'} y^{A}_{B'}.$$

Hence viewing $\mathfrak{g}_0$ as a submodule of endomorphisms of $\mathfrak{g}_{-1}$, an element $(u_{B'}^{A'}, v_B^A) \in \mathfrak{g}_0$ is identified with the map $(u,v)_{A\,B'}^{A'B} = v_A^B\delta_{B'}^{A'} - u_{B'}^{A'}\delta_A^B$. Passing to bundles, we see that a section of $\mathcal{A}_0M \cong \mathrm{End}_0(TM)$ may be written as

$$(\alpha,\beta)_{A\,B'}^{A'B} = \beta_A^B\delta_{B'}^{A'} - \alpha_{B'}^{A'}\delta_A^B$$

for some $\alpha \in \mathrm{End}(\mathcal{E}^{A'})$ and $\beta \in \mathrm{End}(\mathcal{E}^A)$. Using such an identification, we have

Lemma 2.1. *For $\xi_{A'}^A \in TM$, $\varphi_A^{A'} \in T^*M$ and $(\alpha,\beta)_{A\,B'}^{A'B} = \beta_A^B\delta_{B'}^{A'} - \alpha_{B'}^{A'}\delta_A^B \in \mathrm{End}_0(TM)$ the algebraic bracket has the following form*

$$\begin{aligned}
(1)\quad \{(\alpha,\beta),\xi\}_{A'}^A &= -\alpha_{A'}^{C'}\xi_{C'}^A + \beta_C^A\xi_{A'}^C\\
(2)\quad \{(\alpha,\beta),\varphi\}_A^{A'} &= \alpha_{C'}^{A'}\varphi_A^{C'} - \beta_A^C\varphi_C^{A'}\\
(3)\quad \{\varphi,\xi\}_{A\,B'}^{A'B} &= -\varphi_A^{C'}\xi_{C'}^B\delta_{B'}^{A'} - \varphi_C^{A'}\xi_{B'}^C\delta_A^B
\end{aligned}$$

Proof. By construction, the bracket $\mathrm{End}_0(TM)\times TM \to TM$ is simply given by the application of endomorphisms, while the bracket $\mathrm{End}_0(TM) \times T^*M \to T^*M$ is given by applying the negative of the dual of an endomorphism. The equations (1) and (2) then follow from the identification $(\alpha,\beta)_{A\,B'}^{A'B} = \beta_B^A\delta_{A'}^{B'} - \alpha_{A'}^{B'}\delta_B^A$. The most interesting point is the bracket $TM \times T^*M \to \mathrm{End}_0(TM)$, which can be deduced from the form of Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0$. In block matrices, we have

$$\left[\begin{pmatrix}0 & Z\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ Y & 0\end{pmatrix}\right] = \begin{pmatrix}ZY & 0\\ 0 & -YZ\end{pmatrix},$$

which yields $[Z,Y] = (z_C^{A'}y_{B'}^C, -z_A^{C'}y_{C'}^B)$ in abstract indices. Viewing $\mathfrak{g}_0$ as $\mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$, this element corresponds to $-z_A^{C'}y_{C'}^B\delta_{B'}^{A'} - z_C^{A'}y_{B'}^C\delta_A^B$. □

2.1.4. Irreducible bundles. By definition, each irreducible natural vector bundle on an almost Grassmannian geometry $(\mathcal{G} \to M, \omega)$ is an associated bundle to $\mathcal{G}$, induced by an irreducible representation of the parabolic subgroup $P \subset G = SL(2+q,\mathbb{R})$. We will deal here with real representations only. The description of the complex irreducible representations coincides with the real ones since any complex irreducible representations of $SL(2+q,\mathbb{R})$ is the complexification of a real irreducible representation, and the same is true for P.

In sequel, we consider representations of P, which are completely reducible as representations of the corresponding Lie algebra $\mathfrak{p}$, since this is the case of our interest. In such a case, there is a complete description available. Namely, any finite dimensional completely reducible representation V of $\mathfrak{p}$ is obtained by trivially extending a completely reducible representation of $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(q,\mathbb{R})$ to $\mathfrak{p}$, see e.g. section 3.2.10 in [**4**] for details. The irreducible representation of $\mathfrak{g}_0$ is given by the irreducible representation of its semisimple part $\mathfrak{g}_0^{\mathrm{ss}} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(q,\mathbb{R})$ and a linear functional on its center $\mathfrak{z}(\mathfrak{g}_0) = \mathbb{R}$.

It follows from from the theory of semisimple Lie algebras that irreducible representations of $\mathfrak{g}_0^{\mathrm{ss}}$ are classified by highest weights of its simple factors, see e.g. [**13**] or chapter 2 in [**4**]. In our case, the simple factors are $\mathfrak{sl}(2,\mathbb{R})$ respectively $\mathfrak{sl}(q,\mathbb{R})$, and so the highest weights are functionals on the corresponding Cartan subalgebras $\mathfrak{h}_{\mathfrak{sl}(2)}$ and $\mathfrak{h}_{\mathfrak{sl}(q)}$. Since evidently the Cartan subalgebra of $\mathfrak{g}$ is equal to $\mathfrak{h} = \mathfrak{h}_{\mathfrak{sl}(2)} \oplus \mathfrak{h}_{\mathfrak{sl}(q)} \oplus \mathfrak{z}(\mathfrak{g}_0)$, specifying the highest weights of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(q,\mathbb{R})$ and the functional on $\mathfrak{z}(\mathfrak{g}_0)$ (ie. specifying the irreducible representation of $\mathfrak{p}$) defines a weight of $\mathfrak{g}$. We call this weight the highest weight of the irreducible representation of $\mathfrak{p}$. It does not need to be $\mathfrak{g}$-dominant since the functional on $\mathfrak{z}(\mathfrak{g}_0)$ may have negative values. We will denote highest weights in the usual way by labeled Dynkin diagrams. That is, if $\lambda = a_1\lambda_1 + \cdots + a_{q+1}\lambda_{q+1}$ is the expression of the highest weight λ in terms of the fundamental weights $\lambda_1, \ldots \lambda_{q+1}$ of $\mathfrak{g}$, then for each $1 \le i \le q+1$ we write the number a_i over a node of the diagram corresponding to the simple root α_i.

$$\lambda = \overset{a_1}{\circ}\!-\!\!\!-\!\overset{a_2}{\times}\!-\!\!\!-\!\overset{a_3}{\circ}\!-\!\!\!-\!\overset{a_4}{\circ}\!-\!\!\!-\, \cdots \,-\!\!\!-\!\overset{a_{q+1}}{\circ}$$

This weight is a highest weight of a representation of $\mathfrak{p}$ iff it is $\mathfrak{p}$-dominant, i.e. the coefficients a_i are non-negative for all $i \neq 2$. We also assume that the all coefficients are intergral since exactly these representations integrate to representations of P.

Now let us describe the highest weights of the representations $\mathbb{R}^2$ and $\mathbb{R}^q$ inducing the basic bundles $\mathcal{E}^{A'}$ respectively $\mathcal{E}^{A}$. We will use the well-known fact that any irreducible representation of $\mathfrak{p}$ with highest weight λ for a $\mathfrak{g}$-dominant weight λ is given as the subspace in the $\mathfrak{g}$-irreducible representation with highest weight λ on which $\mathfrak{g}_1$ acts trivially, see e.g. 3.2.11. in [**4**]. So to get the highest weight of $\mathbb{R}^2$, let us consider the standard representation $\mathbb{R}^{2+q}$ of $\mathfrak{g}$. Of course, it is an irreducible $\mathfrak{g}$-representation with highest weight λ_1 and it is an easy observation that $\mathbb{R}^2$ is the subspace in $\mathbb{R}^{2+q}$ on which $\mathfrak{g}_1$ acts trivially. So we conclude that the highest weight of $\mathfrak{p}$-representation $\mathbb{R}^2$ is λ_1, i.e.

$$HGW(\mathbb{R}^2) = \overset{1}{\circ}\!-\!\!\!-\!\overset{0}{\times}\!-\!\!\!-\!\overset{0}{\circ}\!-\!\!\!-\!\overset{0}{\circ}\!-\!\!\!-\, \cdots \,-\!\!\!-\!\overset{0}{\circ}$$

Next, consider irreducible $\mathfrak{g}$-representations $\Lambda^k\mathbb{R}^{2+q}$ corresponding to fundamental weights λ_k for $k > 1$. Since the standard representation decomposes as $\mathbb{R}^{2+q} = \mathbb{R}^2 \oplus \mathbb{R}^q$ as a $\mathfrak{g}_0$-module, the k-th exterior power of the standard representation decomposes as $\Lambda^k\mathbb{R}^{2+q} = \oplus_{i+j=k}\Lambda^i\mathbb{R}^2 \otimes \Lambda^j\mathbb{R}^q$. It is an easy observation that the subspace on which $\mathfrak{g}_1$ acts trivially is given by the part in which i is as large as possible. Hence for $k = 2$, we obtain that weight λ_2 corresponding to the crossed node is the highest weight of one-dimensional representation $\Lambda^2\mathbb{R}^2 \cong \Lambda^q\mathbb{R}^{q*}$, where only the center of $\mathfrak{g}_0$ acts non-trivially. For $k = 3$, we obtain $\Lambda^2\mathbb{R}^2 \otimes \mathbb{R}^q$ and so the highest weight of $\mathbb{R}^q$ is $\lambda_3 - \lambda_2$. Similarly, we deduce that the highest weight of $\Lambda^k\mathbb{R}^q$ is $\lambda_{k+2} - \lambda_2$. Diagrammaticly:

$$HGW(\mathbb{R}^q) = \overset{0}{\circ}-\overset{-1}{\times}-\overset{1}{\circ}-\overset{0}{\circ}-\cdots-\overset{0}{\circ}$$

$$HGW(\Lambda^2\mathbb{R}^q) = \overset{0}{\circ}-\overset{-1}{\times}-\overset{0}{\circ}-\overset{1}{\circ}-\cdots-\overset{0}{\circ}$$

$$HGW(\Lambda^{q-1}\mathbb{R}^q) = \overset{0}{\circ}-\overset{-1}{\times}-\overset{0}{\circ}-\overset{0}{\circ}-\cdots-\overset{1}{\circ}$$

The description of highest weights of the duals of these basic representations and their exterior powers follows from the isomorphism $\mathbb{R}^{2*} \cong \mathbb{R}^2 \otimes (\Lambda^2\mathbb{R}^2)^*$ induced by the wedge product. From here we conclude that the highest weight of $\mathbb{R}^{2*}$ is $\lambda_1 - \lambda_2$. Similarly, for $1 \leq k < q$, we get that $\Lambda^k\mathbb{R}^{q*}$ is irreducible with highest weight λ_{q+2-k}. Thus we have

$$HGW(\mathbb{R}^{2*}) = \overset{1}{\circ}-\overset{-1}{\times}-\overset{0}{\circ}-\overset{0}{\circ}-\cdots-\overset{0}{\circ}$$

$$HGW(\mathbb{R}^{q*}) = \overset{0}{\circ}-\overset{0}{\times}-\overset{0}{\circ}-\overset{0}{\circ}-\cdots-\overset{1}{\circ}$$

Let us next look at representations inducing the tangent bundle $TM = \mathcal{E}^A_{A'}$ and the cotangent bundle $T^*M = \mathcal{E}^{A'}_A$, i.e. the adjoint representations $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ respectively. If we use the isomorphisms $\mathfrak{g}_{-1} \cong \mathbb{R}^{2*}\boxtimes\mathbb{R}^q$ $\mathfrak{g}_1 \cong \mathbb{R}^{q*}\boxtimes\mathbb{R}^2$, we conclude from above that these are the irreducible representations of highest weight $\lambda_1 - 2\lambda_2 + \lambda_3$ respectively $\lambda_1 + \lambda_{q+1}$, i.e.

$$HGW(\mathfrak{g}_{-1}) = \overset{1}{\circ}-\overset{-2}{\times}-\overset{1}{\circ}-\overset{0}{\circ}-\cdots-\overset{0}{\circ}$$

$$HGW(\mathfrak{g}_1) = \overset{1}{\circ}-\overset{0}{\times}-\overset{0}{\circ}-\overset{0}{\circ}-\cdots-\overset{1}{\circ}$$

2.1.5. Weights. The one-dimensional representations of center $\mathfrak{z}(\mathfrak{g}_0)$ can be obviously labelled by scalars which we will call weights. These representations give rise to line bundles over M, which we call bundles of *densities*. We fix the labelling such that representation $\Lambda^2\mathbb{R}^2$ corresponding to the fundamental weight λ_2 has weight -1. The induced line bundle $\Lambda^2\mathcal{E}^{A'} \cong \Lambda^q\mathcal{E}_A$ of weighted functions of weight -1 will be denoted $\mathcal{E}[-1]$. The dual bundle $\Lambda^2\mathcal{E}_{A'} \cong \Lambda^q\mathcal{E}^A$ is identified with $\mathcal{E}[1]$ and via tensor product, we get bundle

$\mathcal{E}[w]$ of weighted functions of weight w for any $w \in \mathbb{Z}$. Since we know from the previous section that $\Lambda^2\mathbb{R}^2$ has highest weight λ_2, the highest weight of the representation inducing $\mathcal{E}[w]$ in Dynkin diagram notation is

$$\overset{0}{\circ}\!-\!\overset{-w}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}\ .$$

This labelling fits together with the classical notion of conformal weight in the sense that both notions coincide in the case of a four-dimensional manifold, where our structure is eqiuvalent to the conformal structure.

Of course, we can tensorize any bundle $\mathcal{F}$ with the bundle $\mathcal{E}[w]$ of weighted functions. We obtain so called weighted bundle, which we denote simply by $\mathcal{F}[w]$. In the notation of Dynkin diagrams, $\mathcal{F}[w]$ is evidently obtained from $\mathcal{F}$ by adding $-w$ to the label over the crossed node.

We write $\epsilon_{A'B'}$ for the section of $\Lambda^2\mathcal{E}_{A'}[-1]$ which gives the identification $\mathcal{E}[1] \cong \Lambda^2\mathcal{E}_{A'}$ via $f \mapsto f\epsilon_{A'B'}$ for $f \in \mathcal{E}[1]$. The inverse mapping is given by $\Lambda^2\mathcal{E}_{A'} \ni v_{A'B'} \mapsto -\frac{1}{2}v_{A'B'}\epsilon^{A'B'}$, where $\epsilon^{A'B'}$ is inverse to $\epsilon_{A'B'}$. These objects can be used to raise and lower primed indices similarly as a metric is used in conformal geometry. One must be careful though since $\epsilon_{A'B'}$ is antisymmetric. We use the convention for lowering and raising of primed indices such that $v_{B'} = v^{A'}\epsilon_{A'B'}$ and $v^{B'} = v_{A'}\epsilon^{A'B'}$. We have also a similar object $\epsilon^{A_1A_2\dots A_q}$ (with inverse $\epsilon_{A_1A_2\dots A_q}$) which gives the identification $\Lambda^q\mathcal{E}_A \cong \mathcal{E}[1]$. But the raising of unprimed indices given by this object is quite different from the previous case. Only in the case that $q = 2$, we have ϵ_{AB} and ϵ^{AB} analogueical to $\epsilon^{A'B'}$ and $\epsilon_{A'B'}$. Then the object $\epsilon_{AB}\epsilon^{A'B'}$ is a section of $S^2T^*M[2]$ and thus it defines a conformal metric.

2.1.6. Harmonic curvature. Recall that the harmonic curvature κ_H is represented by a P-equivariant function $\mathcal{G} \to H^2(\mathfrak{g}_{-1}, \mathfrak{g})$. The cohomology $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ can be computed (as a P-module) by Kostant's version of the Bott-Borel-Weil theorem, c.f. theorem 3.3.5 in [**4**]. This theorem gives a bijective correspondence between the irreducible components of $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ and certain elements in so called Hasse diagram of the parabolic $\mathfrak{p}$. For details, see sections 3.2 and 3.3 in [**4**]. It turns out that $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ consits of two irreducible components with highest weights

$$\overset{3}{\circ}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}\!-\cdots-\!\overset{1}{\circ}\ ,\qquad \overset{0}{\circ}\!-\!\overset{0}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{3}{\circ}\ .$$

The corresponding highest weight modules can be described explicitely as follows. The first one is the intersection of the kernels of the two possible contractions of

$$(S^2\mathbb{R}^2 \otimes \mathbb{R}^{2*}) \boxtimes (\Lambda^2\mathbb{R}^{q*} \otimes \mathbb{R}^q) \subset \Lambda^2\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1},$$

while the second is the kernel of a unique contraction on the space

$$\Lambda^2\mathbb{R}^2 \boxtimes S^3\mathbb{R}^{q*} \otimes \mathbb{R}^q \subset \Lambda^2\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0.$$

This information on $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ directly translates into a description of the harmonic curvature components. Namely, we obtain two components of two different homogeneities. The first one is of homogeneous degree one and lies in $\Gamma((S^2E \otimes E^*) \boxtimes (\Lambda^2F^* \otimes F)) \subset \Omega^2(M, TM)$. According to theorem 1.9, it coincides with the torsion of any Weyl connection. The second component is of homogeneous degree two and is a section of $\Lambda^2E \boxtimes (S^2F^* \otimes \mathfrak{sl}(F))$. Since it is a section of a subbundle of

$$\mathcal{G}_0 \times_{G_0} (\Lambda^2\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0) \cong \Lambda^2T^*M \otimes \mathrm{End}(TM),$$

it may be interpretated as a part of the curvature of a linear connection on the tangent bundle. The explicit description of the harmonic curvature in terms of data associated to the Weyl structure is given in the next section.

2.1.7. Basic tractor bundles. The description of the basic tractor bundles is simple. Let $\mathcal{E}^\alpha$ denote the standard tractor bundle, i.e. the bundle induced by the standard representation of $G = SL(2+q, \mathbb{R})$ on $\mathbb{R}^{2+q}$. By definition, the parabolic subgroup $P \subset G$ is the stabilizer of $\mathbb{R}^2$ in $\mathbb{R}^{2+q}$, and the associated bundle to the P-representation $\mathbb{R}^2$ is $\mathcal{E}^{A'}$. This means that $\mathcal{E}^{A'}$ is a smooth subbundle of $\mathcal{E}^\alpha$. On the other hand, the quotient P-representation $\mathbb{R}^{2+q}/\mathbb{R}^2$ is the trivial extension of the representation $\mathbb{R}^q$ of the subgroup $G_0 \subset P$. This corresponds to the bundle $\mathcal{E}^A$, so we get a short exact sequence

$$0 \to \mathcal{E}^{A'} \to \mathcal{E}^\alpha \to \mathcal{E}^A \to 0 \tag{2.3}$$

of natural vector bundles. Of course, this sequence does not admit a natural splitting since the P-representation $\mathbb{R}^{2+q}$ is indecomposable. On the other hand, we can view $\mathcal{E}^{A'} \subset \mathcal{E}^\alpha$ as a filtration of the vector bundle $\mathcal{E}^\alpha$, and so we have $\mathcal{E}^\alpha = \mathcal{E}^A \oplus \mathcal{E}^{A'}$ in the notation introduced in the previous chapter. Since any irreducible representation of $\mathfrak{sl}(2+q, \mathbb{R})$ is isomorphic to a subrepresentation of some tensor power of the standard representation, any tractor bundle corresponding to an irreducible representation can be found in some tensor power of the standard tractor bundle. Thus the filtration $\mathcal{E}^A \oplus \mathcal{E}^{A'}$ gives rise to a filtration of any tractor bundle. As an example, consider the tractor bundles $\Lambda^k\mathcal{E}^\alpha$ for $k = 2, \ldots, q+1$ which correspond to the other fundamental representations. For these bundles we get

$$\Lambda^k\mathcal{E}^\alpha = \Lambda^k\mathcal{E}^A \oplus \mathcal{E}^{A'} \boxtimes \Lambda^{k-1}\mathcal{E}^A \oplus \Lambda^2\mathcal{E}^{A'} \boxtimes \Lambda^{k-2}\mathcal{E}^A.$$

2.1.8. Almost quaternionic structures. At the end of this section, we briefly introduce almost quaternionic geometries and the related structures. These geometries are similar to Grassmannian geometries in many aspects due to the fact that both are |1|-graded parabolic geometries with the corresponding Lia algebra $\mathfrak{g}$ being a real form of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$. In particular, analogues of our results will apply also to these structures, although we will mention the almost Grassmannian case exclusively.

The real form $\mathfrak{g}$ which corresponds to the almost quaternionic geometry is the Lie algebra $\mathfrak{sl}(n, \mathbb{H})$ of quaternionic $n \times n$ matrices with vanishing real trace. The Satake diagram of $\mathfrak{sl}(n, \mathbb{H})$ has $2n-1$ dots, alternatingly black and white starting and ending with a black dot. Similarly to the Grassmannian case, the almost quaternionic geometry is defined by a parabolic subgroup given by crossing of the second root, i.e. it corresponds to the diagram

$$\bullet - \times - \bullet - \circ - \bullet - \circ - \cdots - \bullet \; .$$

In quaternion block matrices, the parabolic has the form $\left(\begin{smallmatrix} a & Z \\ X & A \end{smallmatrix}\right)$ with blocks of size 1 and $q := n-1$, i.e. $X \in \mathbb{H}^q$, $Z \in \mathbb{H}^{q*}$, $a \in \mathbb{H}$ and $A \in M_q(\mathbb{H})$, where $\mathrm{re}(a) + \mathrm{re}(\mathrm{tr}(A)) = 0$. The entries a and A span $\mathfrak{g}_0$, X spans $\mathfrak{g}_{-1} \cong \mathbb{H}^q$ and Z spans $\mathfrak{g}_1 \cong \mathbb{H}^{q*}$. As the group G we choose the group $PGL(1+q, \mathbb{H})$, the quotient of all invertible quaternionic linear endomorphisms of $\mathbb{H}^{1+q}$ by the closed normal subgroup of all real multiples of the identity. Then we define the parabolic $P \subset G$ to be the stabilizer of the quaternionic line spanned by the first basis vector. Thus the homogeneous space G/P can be identified with the quaternionic projective space $\mathbb{H}P^q$ of quaternionic lines in $\mathbb{H}^{1+q}$. The Levi subgroup $G_0 \subset P$ is the quotient of block diagonal matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & A \end{smallmatrix}\right)$ with $a \neq 0$ and $A \in GL(q, \mathbb{H})$ by real multiples of the identity and the adjoint action of G_0 on $\mathfrak{g}_{-1} \cong \mathbb{H}^n$ is given by $X \mapsto A(Xa^{-1})$. From this description, we can deduce the geometric interpretation of the quaternionic structure. Namely, having a manifold M of dimension $4n$, it is equivalent to a rank three subbundle $\mathcal{Q} \subset L(TM, TM)$ such that locally around each point of M we can find with a local smooth frame $\{I, J, K\}$, satisfying the usual relations of the standard basis $\{i, j, k\}$ of $\mathrm{im}(\mathbb{H})$. For details, see 4.1.7 in [**4**].

By the above description of the Levi subgroup we have $G_0 \cong GL(1, \mathbb{H}) \times_{\mathbb{R}^*} GL(q, \mathbb{H})$. We can multiply on the right by any real multiple of the identity, so without loss of generality we can reduce the first factor to $Sp(1)$. Thus we see that the almost quaternionic geometry is a first-order structure with a reduction to $G_0 = Sp(1) \times_{\mathbb{Z}_2} GL(q, \mathbb{H})$. The group $\tilde{G}_0 = S(GL(1, \mathbb{H}) \times GL(q, \mathbb{H}))$ is the universal cover of G_0 and the choice of the structure group $\tilde{G}_0$ makes no difference locally. Hence the complexified tangent bundle $T^{\mathbb{C}}M = TM \times_{\mathbb{R}} \mathbb{C}$ is equipped by the reduction of its structure group to $\tilde{G}_0^{\mathbb{C}} = S(GL(2,\mathbb{C}) \times GL(2q,\mathbb{C}))$. In other words, $T^{\mathbb{C}}M$ satisfies the fundamental identifications (2.1) and (2.2) for complex bundles $E = \mathcal{E}^{A'}$ $F = \mathcal{E}^{A}$. So the almost quatrnionic structures correspond to (complexified) almost Grassmannian structures of type $(2, 2q)$. We may therefore include them into our framework if we deal with the complex P-modules and the complexified tangent bundle.

2.2. Weyl structures for AG geometries

Let $(p : \mathcal{G} \to M, \omega)$ be a normal almost Grassmannian geometry and let us choose a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$. By definition, the Weyl connection ∇ on the tangent bundle TM is a linear connection induced by the principal connection $\sigma^*\omega$ on $\mathcal{G}_0$. In this section, we give a detailed description of these connections on AG-structures, we also give a geometric description of the harmonic curvature of Cartan connection and we show how various geometric objects transform under a change of Weyl structure.

2.2.1. Weyl connections for AG geometries. According to theorem 1.9, Weyl connections are exactly those linear connections which have ∂^*-closed torsion. And moreover, all Weyl connections have the same torsion T which coincides with the homogeneous component of degree one of the harmonic curvature. We know from the above description of the harmonic curvature that this component is a section of $(S^2E \otimes E^*) \boxtimes (\Lambda^2 F^* \otimes F)$. Thus we conclude that the torsion of Weyl connection satisfies

$$T_{A\ B\ C'}^{A'B'C} \in \mathcal{E}_{[A\ B]\ C'}^{(A'B')C} \tag{2.4}$$

Now let us describe the ∂^*-closedness condition in more detail. By (1.2), a general formula for the bundle map induced by the Kostant codifferential $\partial^* : \Lambda^2\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$ on decomposable elements has the form

$$\partial^*(\phi_0 \wedge \phi_1 \otimes \xi) = \phi_0 \otimes \{\phi_1, \xi\} - \phi_1 \otimes \{\phi_0, \xi\}.$$

The formula for the algebraic bracket in lemma 2.1 yields

$$\begin{aligned}\partial^*(\phi_0 \wedge \phi_1 \otimes \xi)_{A\ B\ C'}^{A'B'C} = -(\phi_0)_A^{A'}(\phi_1)_B^{I'}\xi_{I'}^{C}\delta_{C'}^{B'} - (\phi_0)_A^{A'}(\phi_1)_I^{B'}\xi_{C'}^{I}\delta_B^{C} \\ + (\phi_1)_A^{A'}(\phi_0)_B^{I'}\xi_{I'}^{C}\delta_{C'}^{B'} + (\phi_1)_A^{A'}(\phi_0)_I^{B'}\xi_{C'}^{I}\delta_B^{C}.\end{aligned}$$

Hence we see that in the case of the AG geometry, the condition $\partial^*T = 0$ is equivalent to

$$T_{A\ B\ I'}^{A'I'C}\delta_{C'}^{B'} + T_{A\ I\ C'}^{A'B'I}\delta_B^{C} = 0,$$

and this is visibly equivalent to vanishing of both the traces appearing in this formula. Thus the full characterization of the Weyl connections is that its torsion is totally trace-free and enjoys the symmetry (2.4). Equivalently, in the Young diagram notation the torsion is a section of

$$T \in \Gamma(\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}\,\mathcal{E}^{A'} \otimes \; q-1 \updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array}\,\mathcal{E}_A[2]). \tag{2.5}$$

In sequel, we will deal with Weyl connections which correspond to exact Weyl structures as introduced briefly in remark 1.1. The bundle of scales for an almost Grassmannian geometry is the bundle $\mathcal{E}[1] = \Lambda^2\mathcal{E}^{A'} \cong \Lambda^q\mathcal{E}_A$. Thus any section of $\mathcal{E}[1]$ defines an exact Weyl structure which in turn induces exact Weyl connections on all natural bundles. The advantage of using exact Weyl connections is that they preserve $\epsilon^{A'B'} \cong \epsilon_{A\ldots B}$ and so we obtain simplier formulae. Moreover, the difference of two such connections is an exact one-form. Observe that the class of exact Weyl connections is an analogueue of Levi-Civita connections from the conformal geometry. Indeed, recall that the Levi-Civita connection of a metric g is a torsion-free connection which preserves g. In almost Grassmannian geometry, the exact Weyl connection of a scale ϵ is a connection with a totally trace-free torsion with the above symmetry which preserves ϵ. Of course, in dimension four these two notions coincide.

2.2.2. Curvature of Weyl connections. Our conventions for the torsion $T_{ab}{}^c$ and curvature $R_{ab}{}^c{}_d$ of a connection ∇_a on the tangent bundle TM are determined by Ricci identity from proposition 1.11 which in abstract indices has the form

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)u^c = R_{ab}{}^c{}_e u^e - T_{ab}{}^e\nabla_e u^c.$$

In almost Grassmaniann geometry, we have $TM = \mathcal{E}^a = \mathcal{E}^A_{A'}$ and thus $\nabla_a = \nabla^{A'}_A$, $T_{ab}{}^c = T^{A'B'C}_{A\ B\ C'}$ and for the curvature $R_{ab}{}^c{}_d = R^{A'B'C\ D'}_{A\ B\ C'D}$, we have

$$R^{A'B'C\ D'}_{A\ B\ C'D} = R^{A'B'C}_{A\ B\ \ \ D}\delta^{C'}_{D'} - R^{A'B'D'}_{A\ B\ \ \ C'}\delta^C_D,$$

where the components $R^{A'B'C}_{A\ B\ \ \ D}$ and $R^{A'B'D'}_{A\ B\ \ \ C'}$ are the curvatures of the connection $\nabla^{A'}_A$ on the bundles $\mathcal{E}^A$ and $\mathcal{E}^{A'}$ respectively. Since the Weyl connection preserves the isomorphism $\epsilon^{A'B'} \cong \epsilon_{A\ldots B}$, we conclude that the curvatures satisfy $R^{A'B'I}_{A\ B\ \ \ I} = -R^{A'B'I'}_{A\ B\ \ \ I'}$. Moreover, for an exact Weyl connection the volume forms are parallel and thus these traces vanish.

By theorem 1.9, the curvature of a Weyl connection decomposes as $R = U - \partial\mathsf{P}$, where the Weyl part U satisfies $\mathrm{gr}_0(\partial^*)(U) = 0$ and $\partial\mathsf{P}$ is the part containing only the Rho-tensor. Let us first look in more detail at the latter part of the curvature. From the formula (1.1) for the Lie algebra differential $\partial : \mathfrak{g}^*_{-1}\otimes\mathfrak{g}_1 \to \Lambda^2\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_0$, we conclude that the action of the induced bundle map ∂ on P has the form $\partial\mathsf{P}(\xi,\zeta) = \{\mathsf{P}(\xi),\zeta\} - \{\mathsf{P}(\zeta),\xi\}$. According to the explicit form of the algebraic bracket in AG geometry from lemma 2.1, this equation reads as

$$\{\mathsf{P}(\xi),\zeta\}^{C\ D'}_{C'D} = -\mathsf{P}(\xi)^{I'}_{D}\zeta^{C}_{I'}\delta^{D'}_{C'} - \mathsf{P}(\xi)^{D'}_{I}\zeta^{I}_{C'}\delta^{C}_{D}.$$

If we fix the abstract index notation which we use for Rho tensor P in such a way that $\mathsf{P}(\xi)^{A'}_{A} = \mathsf{P}^{I'A'}_{I\ A}\xi^{I'}_{I}$, then the previous equation may be written as

$$\{\mathsf{P}(\xi),\zeta\}^{C\ D'}_{C'D} = -(\mathsf{P}^{A'B'}_{A\ D}\delta^{C}_{B}\delta^{D'}_{C'} + \mathsf{P}^{A'D'}_{A\ B}\delta^{B'}_{C'}\delta^{C}_{D})\xi^{A}_{A'}\zeta^{B}_{B'}.$$

Therefore, for the part $\partial\mathsf{P}$ of the curvature we get

$$(\partial\mathsf{P})^{A'B'C\ D'}_{A\ B\ C'D} = -\mathsf{P}^{A'D'}_{A\ B}\delta^{B'}_{C'}\delta^{C}_{D} + \mathsf{P}^{B'D'}_{B\ A}\delta^{A'}_{C'}\delta^{C}_{D} - \mathsf{P}^{A'B'}_{A\ D}\delta^{C}_{B}\delta^{D'}_{C'} + \mathsf{P}^{B'A'}_{B\ D}\delta^{C}_{A}\delta^{D'}_{C'} \quad (2.6)$$

Now let us describe the Weyl curvature U. By definition, it is a two-form with values in the bundle $\mathrm{End}_0(\mathcal{E}^{A}_{A'})$ of trace-free endomorphisms of the tangent bundle. This bundle is induced by $\mathfrak{g}_0$ and thus decomposes with respect to the action of G_0 as $\mathrm{End}_0(\mathcal{E}^{A}_{A'}) = \mathcal{E} \oplus \mathrm{End}_0(\mathcal{E}^{A}) \oplus \mathrm{End}_0(\mathcal{E}_{A'})$. Since $(\mathcal{E}^{B'}_{A'})_0 \cong \mathcal{E}^{(A'B')}[1]$ and since two-forms obviously decompose as

$$\mathcal{E}^{A'}_{A} \wedge \mathcal{E}^{B'}_{B} = \mathcal{E}^{(A'B')}_{[AB]} \oplus \mathcal{E}_{(AB)}[-1],$$

the domain of U viewed as a subbundle of $\mathcal{E}^{A'B'C'D}_{ABCD'}$ decomposes into irreducibles as follows:

$$\begin{aligned}\Lambda^2\mathcal{E}^{A'}_{A} \otimes \mathrm{End}_0(\mathcal{E}^{C}_{C'}) = 2\cdot\mathcal{E}^{(A'B')}_{(AB)} &\oplus 2\cdot\mathcal{E}_{[AB]}[-1] \oplus 2\cdot\mathcal{E}_{(AB)}[-1] \oplus 3\cdot\mathcal{E}^{(A'B')}_{[AB]}\\ &\oplus \mathcal{E}^{D}_{(ABC)_0}[-1] \oplus \mathcal{E}^{(A'B'C'D')}_{[AB]}[1] \oplus \mathcal{E}^{(A'B')D}_{\boxplus(ABC)_0}\\ &\oplus \mathcal{E}^{D}_{\boxplus(ABC)_0}[-1] \oplus \mathcal{E}^{(A'B')D}_{[ABC]_0}.\end{aligned} \quad (2.7)$$

Moreover, the Weyl curvature U lies in the kernel of $\mathrm{gr}_0(\partial^*)$, and so we need to make this condition explicit in order to describe U. The bundle map $\mathrm{gr}_0(\partial^*)$ is induced by the Kostant codifferential $\Lambda^2\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_0 \to \mathfrak{g}^*_{-1}\otimes\mathfrak{g}_1$, and thus on decomposable elements by (1.2) has the form

$$\mathrm{gr}_0(\partial^*)(\phi_0\wedge\phi_1\otimes A) = \phi_0\otimes\{\phi_1,A\} - \phi_1\otimes\{\phi_0,A\}.$$

According to (2.1), this yields

$$\begin{aligned}\mathrm{gr}_0(\partial^*)(\phi_0\wedge\phi_1\otimes A)^{A'B'}_{A\ B} &= (\phi_0)^{A'}_{A}((\phi_1)^{B'}_{I}A^{I}_{B} - (\phi_1)^{I'}_{B}A^{B'}_{I'})\\ &- (\phi_1)^{A'}_{A}((\phi_0)^{B'}_{I}A^{I}_{B} - (\phi_0)^{I'}_{B}A^{B'}_{I'}),\end{aligned}$$

and so the equation $\mathrm{gr}_0(\partial^*)(U) = 0$ reads

$$U^{A'B'I}_{A\ I\ \ B} - U^{A'I'B'}_{A\ B\ \ I'} = 0, \quad (2.8)$$

where these components of U are the Weyl curvatures of ∇ viewed as a connection on $\mathcal{E}^{A}$ and $\mathcal{E}^{A'}$ respectively, i.e.

$$U^{A'B'C\ D'}_{A\ B\ C'D} = U^{A'B'C}_{A\ B\ \ D}\delta^{D'}_{C'} - U^{A'B'D'}_{A\ B\ \ C'}\delta^{C}_{D}. \quad (2.9)$$

Since one can easily conclude from proposition 1.4 that the map $\mathrm{gr}_0(\partial^*) : \Lambda^2\mathcal{E}_A^{A'} \otimes \mathrm{End}_0(\mathcal{E}_{B'}^B) \to \mathcal{E}_{A\ B}^{A'B'}$ is surjective and we have

$$\mathcal{E}_{A\ B}^{A'B'} = \mathcal{E}_{(AB)}^{(A'B')} \oplus \mathcal{E}_{[AB]}^{(A'B')} \oplus \mathcal{E}_{[AB]}[-1] \oplus \mathcal{E}_{(AB)}[-1],$$

we see that the ∂^*-closedness condition (2.8) on U translates into four equations for irreducible components corresponding to these bundles. Thus the Weyl curvature splits according to (2.7), but the multiplicity of the "trace" bundles is less. Namely, each irreducible bundle has multiplicity one with one exeption - the bundle $\mathcal{E}_{[AB]}^{(A'B')}$ has multiplicity two. The explicit form of equation $\mathrm{gr}_0(\partial^*)(U) = 0$ in terms of irreducible components of U is given in lemma A.1 in appendix.

The next ingredients which reveal the structure of the curvature, are the Bianchi identities. Indeed, we know from 1.6 that all components of the curvature κ of the Cartan connection can be expressed in terms of components of the harmonic curvature. By 1.9, this means that we can express in terms of torsion all components of U, up to its harmonic part. According to the description of the harmonic curvature given in the previous section, it is the part of U in $\mathcal{E}^D_{(ABC)_0}[-1]$. The list of formulae for the other components, obtained from the Bianchi identity, is given in lemma A.2 in appendix.

2.2.3. Linearized transformations. Assume we are given two Weyl structures σ and $\hat{\sigma}$, related by $\hat{\sigma}(u) = \sigma(u)\exp(\Upsilon(u))$ for an one-form Υ. The full transformation of the associated objects for arbitrary one-graded geometry is described in proposition 1.8. In the sequel, we will deal with the linearized transformation only. It is sufficient for our purposes since if the linearized transformation vanishes then it follows by integration that the full transformation vanishes. Hence the transformation formula (1.3) giving the isomorphism $\mathcal{V}M \ni s \mapsto (s_0, \dots, s_N) \in \mathrm{gr}(\mathcal{V}M)$ simplifies to $\hat{s}_k = s_k - \Upsilon \bullet s_{k-1}$. The linearized transformation of the covariant derivative of a section $s \in \mathcal{V}M$ obviously coincides with the full formula (1.4), i.e. $\hat{\nabla}_\xi s = \nabla_\xi s - \{\Upsilon, \xi\} \bullet s$, where $\xi \in TM$ and $\bullet$ is the bundle map induced by the action of $\mathfrak{g}_0$ on $\mathcal{V}M$. By (1.5), the linearized transformation of Rho-tensor is evidently equal to $\hat{\mathsf{P}}(\xi) = \mathsf{P}(\xi) + \nabla_\xi \Upsilon$.

Now let us work out in detail the transformations which we will need in sequel. Let us start with the induced connection on the tangent bundle. In this case, the action $\bullet$ is induced by the bracket of $\mathfrak{g}_0$ and $\mathfrak{g}_{-1}$ and thus is given by the algebraic bracket, i.e. $\hat{\nabla}_\xi \zeta = \nabla_\xi \zeta - \{\{\Upsilon, \xi\}, \zeta\}$ for $\xi, \zeta \in TM$. In abstract index notation for the AG geometry, by lemma 2.1 we get

$$\begin{aligned}\{\{\Upsilon, \xi\}, \zeta\}^A_{A'} &= \{\Upsilon, \xi\}^A_I \zeta^I_{A'} - \{\Upsilon, \xi\}^{I'}_{A'} \zeta^A_{I'} \\ &= -\Upsilon^{I'}_I \xi^A_{I'} \zeta^I_{A'} - \Upsilon^{I'}_I \xi^I_{A'} \zeta^A_{I'},\end{aligned}$$

and from here, we conclude

$$\hat{\nabla}^{A'}_{A}\zeta^{B}_{B'} = \nabla^{A'}_{A}\zeta^{B}_{B'} + \Upsilon^{I'}_{A}\zeta^{B}_{I'}\delta^{A'}_{B'} + \Upsilon^{A'}_{I}\zeta^{I}_{B'}\delta^{B}_{A}$$

Similarly, the linearized transformation of the covariant derivative on $\varphi \in T^*M$ has the form $\hat{\nabla}_\xi\zeta = \nabla_\xi\zeta - \{\{\Upsilon,\xi\},\varphi\}$ and its explicit form can be deduced by applying lemma 2.1. Namely, we have

$$\{\{\Upsilon,\xi\},\varphi\}^{A'}_{A} = \Upsilon^{A'}_{I}\xi^{I}_{I'}\varphi^{I'}_{A} + \Upsilon^{I'}_{A}\xi^{I}_{I'}\varphi^{A'}_{I},$$

and thus we obtain

$$\hat{\nabla}^{A'}_{A}\varphi^{B'}_{B} = \nabla^{A'}_{A}\varphi^{B'}_{B} - \Upsilon^{B'}_{A}\varphi^{A'}_{B} - \Upsilon^{A'}_{B}\varphi^{B'}_{A}$$

These basic formulae for linearized transformations for covariant derivatives on the tangent and cotangent bundle can be easily extended to arbitrary tensor bundles. For instance, for the covariant derivative of a torsion $T \in \Lambda^2T^*M \otimes TM$, we get

$$\begin{aligned}\hat{\nabla}^{A'}_{A}T^{B'C'D}_{B\,C\,D'} &= \nabla^{A'}_{A}T^{B'C'D}_{B\,C\,D'} - \Upsilon^{B'}_{A}T^{A'C'D}_{B\,C\,D'} - \Upsilon^{A'}_{B}T^{B'C'D}_{A\,C\,D'}\\ &- \Upsilon^{C'}_{A}T^{B'A'D}_{B\,C\,D'} - \Upsilon^{A'}_{C}T^{B'C'D}_{B\,A\,D'} + \Upsilon^{I'}_{A}T^{B'C'D}_{B\,C\,I'}\delta^{A'}_{D'} + \Upsilon^{A'}_{I}T^{B'C'I}_{B\,C\,D'}\delta^{D}_{A}.\end{aligned} \tag{2.10}$$

The formula for linearized transformation of the covariant derivative of a weighted function $f \in \mathcal{E}[w]$ can be deduced from the transformation of the connection induced on $\mathcal{E}^{[B'C']} \cong \mathcal{E}[-1]$. We compute

$$\hat{\nabla}^{A'}_{A}\epsilon^{B'C'} = \nabla^{A'}_{A}\epsilon^{B'C'} - \Upsilon^{B'}_{A}\epsilon^{A'C'} - \Upsilon^{C'}_{A}\epsilon^{B'A'} = \nabla^{A'}_{A}\epsilon^{B'C'} - \Upsilon^{A'}_{A}\epsilon^{B'C'},$$

since $\Upsilon^{[A'}_{A}\epsilon^{B'C']} = 0$, and hence for a $f \in \mathcal{E}[w]$ we conclude

$$\hat{\nabla}^{A'}_{A}f = \nabla^{A'}_{A}f + w\Upsilon^{A'}_{A}f.$$

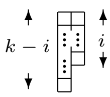
Likewise, for a section s of a weighted bundle $\mathcal{F}[w] = \mathcal{F}\otimes\mathcal{E}[w]$, the linearized transformation of ∇s is $\hat{\nabla}s + w\Upsilon \cdot s$, where $\hat{\nabla}s$ is the transformation for $\mathcal{F}$. The linearized transformation of the Rho-tensor in abstract indices evidently has the following simple form

$$\hat{\mathsf{P}}^{A'B'}_{A\,B} = \mathsf{P}^{A'B'}_{A\,B} + \nabla^{A'}_{A}\Upsilon^{B'}_{B}. \tag{2.11}$$

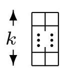
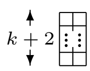
2.3. Invariant operators for AG geometry

As we mentioned in section 1.4.1, invariant linear operators between natural bundles over locally flat geometries are in bijective correspondence with homomorphisms of generalized Verma modules and thus they are well-known from representation theory. In this section, we give their description in the case of a (locally flat) almost Grassmannian geometry. We introduce the non-standard invariant operators and known results about their curved analogues.

2.3.1. Invariant operators on flat AG geometires. It is well-known from the literature that all the standard operators (corresponding to standard homomorphisms between Verma modules) for a locally flat almost Grassmannian structure are obtained from the de Rham resolution of the sheaf of constant functions. The de Rham resolution splits according to the decomposition of bundles of differential forms into irreducible G_0-modules and the standard invariant operators are exactly components of the exterior derivative and their non-zero compositions. Let us look in a detail at the de Rham sequence for almost Grassmannian structure of type $(2, q)$. Obviously, for any k the bundle of k-forms is irreducible as a $SL(2q, \mathbb{R})$-module but it decomposes into irreducibles with respect to the action of $SL(2, \mathbb{R}) \times SL(q, \mathbb{R})$. This decomposition is easy to deduce if we look at how the primed indices inherit the symmetry of unprimed indices and vice versa. The fact that $\mathcal{E}^{A'}$ is a rank two bundle and thus each alternation over more than two primed indices vanishes yields that any k-form which is symmetric in more then two unprimed indices vanishes and thus the Young diagram of $SL(q, \mathbb{R})$-factor must have two columns at most. Moreover, the number of rows with two boxes is equal to the number of contracted primed indices. Hence we conclude that the components of $\Lambda^k \mathcal{E}_A^{A'}$ are

$$S^{k-2i}\mathcal{E}^{A'}[-i] \otimes$$  $$\mathcal{E}_A,$$

where $0 \leq i \leq \frac{k}{2}$. Of course, if $k - i > q$ then the component vanishes, and if $k - i = q$ then we get a one more copy of $\mathcal{E}[-1] \cong \Lambda^q \mathcal{E}_A$ and the Young diagram contains q boxes less. In particular, we see that for $k \leq q$, the bundle of k-forms decomposes into $[\frac{k}{2}] + 1$ components, while for $k > q$ we have $q - [\frac{k}{2}] + 1$ components. Thus the split de Rham sequence has a triangular pattern. The case of a locally flat eight dimensional manifold with AG structure is shown in figure 1. All standard invariant operators are components of the exterior derivative (short arrows) and their non-zero compositions. There are two possible compositions in each square. These compositions coincide up to the sign and are denoted by the longer arrows. Hence the standard operators are of order one and two rspectively.

In addition to the standard operators, there is by the theory of Verma modules a family of non-standard operators, which are of our main interest. In figure 1, they are the long arrows on the left. They are of order four and transform sections of  $\mathcal{E}_A[-k]$ to sections of  $\mathcal{E}_A[-k-2]$. In terms of the flat connection, they are given by

$$s \mapsto \epsilon_{A'B'}\epsilon_{C'D'}\nabla_A^{A'}\nabla_B^{B'}\nabla_C^{C'}\nabla_D^{D'}s,$$

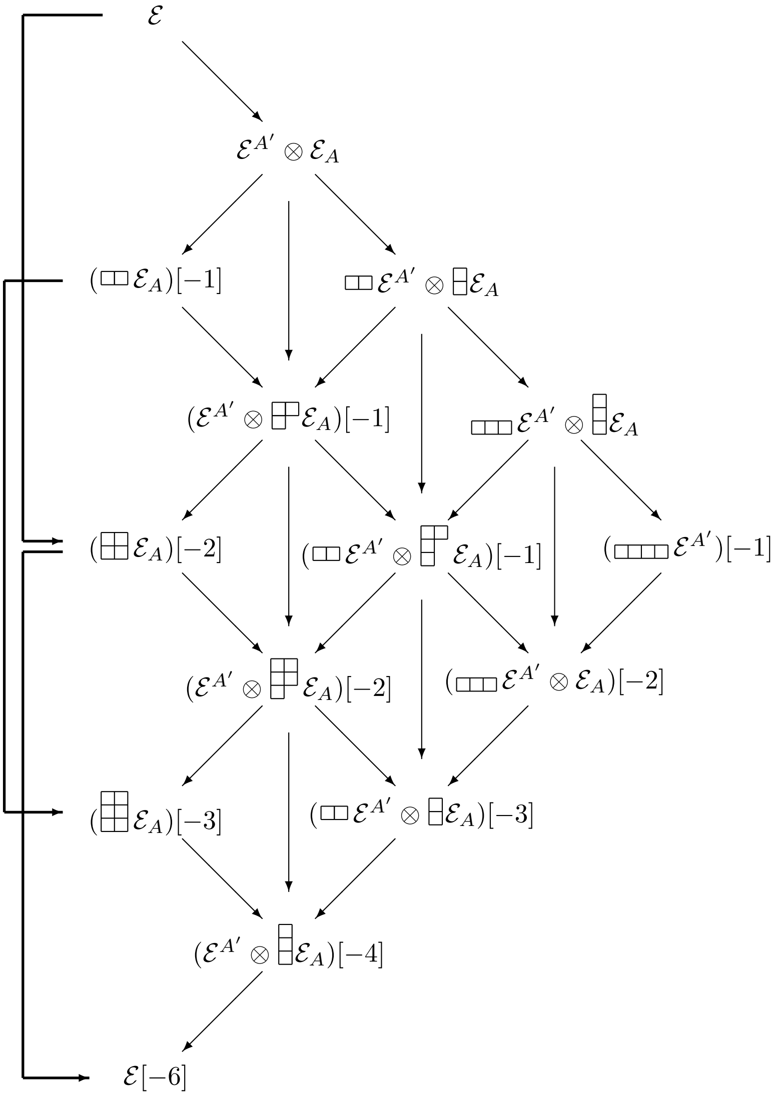


FIGURE 1. Non-standard operators and the de Rham resolution on an eight-dimensional manifold equipped with the AG-structure.

followed by the unique projection to the target bundle. Indeed, one can easily check that for flat connection this formula remains invariant under the change of Weyl structure.

Definition 2.2. For each q and $0 \leq k \leq q-2$ the fourth-order invariant operators

$$\updownarrow k \,[\text{Young diagram: two columns, } k \text{ rows}]\,\mathcal{E}_A[-k] \to \updownarrow k+2 \,[\text{Young diagram: two columns, } k+2 \text{ rows}]\,\mathcal{E}_A[-k-2]$$

are called the *non-standard operators* on locally flat AG geometries.

2.3.2. Invariant operators on curved AG geometries. As mentioned in 1.4.2, first-order operators are strongly invariant and thus the whole triangular pattern of the De Rham sequence lifts to the case of a general manifold with the AG structure. In contrast to components of the exterior derivative, the non-standard operators are not strongly invariant, c.f. [**19**]. Therefore, their existence on a curved manifold is not obvious. A partial answer to the question of their existence is given in theorem 5.1 in [**11**]:

Theorem 2.2. *Let M be a torsion free AG-structure of type $(2, q)$. For each integer k such that $0 \leq k \leq q-2$ there is a fourth order invariant operator,*

$$\Box_{ABCD} : {\scriptstyle k}\updownarrow \boxplus \, \mathcal{E}_E[-k] \to {\scriptstyle k+2}\updownarrow \boxplus \, \mathcal{E}_E[-k-2],$$

which coincides with the corresponding non-standard operator on flat structures.

We give an alternative proof of a part of this theorem in the next chapter. Namely, we construct the first operator from this family (the one acting on functions) via Curved Casimir operators. We also get an explicit formula for this operator, and we prove that the operator can be extended to an invariant operator on AG-structures with non-vanishing torsion.

Before doing this, we compute explicit formulae for some of the components of the exterior derivative. Namely, we will need in sequel the formulae for those components appearing in the beginning of the sequence, i.e. the upper corner in firgure 1. It is more usual to draw the diagram rotated by 90 degrees, such that the non-standard operators appear in the bottom row. Then the beginning of the sequence corresponds to the left corner. The notation for components of the exterior derivative appearing in this part of De Rham resolution is shown in figure 2. The components are basically indexed by Young diagrams of the $SL(q, \mathbb{R})$-factor of the target bundles.

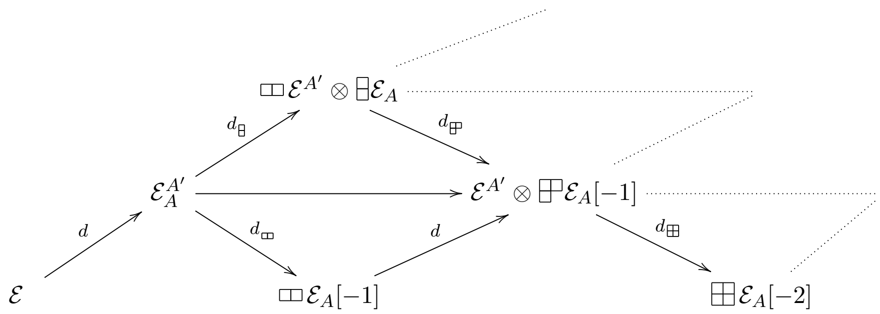


FIGURE 2. Notation for components of the exterior derivatives.

Realizing bundle $\yng(2,1)\mathcal{E}_A$ as the kernel of alternation map $\mathcal{E}_{[AB]C} \to \mathcal{E}_{[ABC]}$ (which we denote also by $\mathcal{E}_{\yng(2,1)(ABC)}$) and bundle $\yng(2,2)\mathcal{E}_A$ as the kernel of $\mathcal{E}_{[AB]} \vee \mathcal{E}_{[CD]} \to \mathcal{E}_{[ABCD]}$, (denoted also by $\mathcal{E}_{\yng(2,2)(ABCD)}$ in sequel), we get the following.

Lemma 2.3. *For $f \in \mathcal{E}$, $\mu_A^{A'} \in \mathcal{E}_A^{A'}$, $\alpha_{AB} \in \mathcal{E}_{(AB)}[-1]$, $A_{A\,B}^{A'B'} \in \mathcal{E}_{[AB]}^{(A'B')}$ and $\nu_{ABC}^{A'} \in \mathcal{E}_{\yng(2,1)(ABC)}^{A'}$ we have*

$$
\begin{aligned}
(df)_A^{A'} &= \nabla_A^{A'} f \\
(d_{\yng(2)}\mu)_{AB} &= 2\nabla_{(A}^{A'}\mu_{B)}^{B'}\epsilon_{A'B'} \\
(d_{\yng(1,1)}\mu)_{A\,B}^{A'B'} &= 2\nabla_{[A}^{(A'}\mu_{B]}^{B')} + T_{A\,B\,I'}^{A'B'I}\mu_I^{I'} \\
(d\alpha)_{ABC}^{A'} &= \tfrac{3}{2}\nabla_{[A}^{A'}\alpha_{B]C} \\
(d_{\yng(2,1)}A)_{ABC}^{A'} &= \tfrac{3}{2}\nabla_C^{C'}A_{A\,B}^{A'B'}\epsilon_{B'C'} + \tfrac{3}{2}T_{A\,B\,I'}^{A'B'I}A_{I\,C}^{I'C'}\epsilon_{B'C'} \\
&\quad - \tfrac{3}{2}\nabla_{[C}^{C'}A_{A\,B]}^{A'B'}\epsilon_{B'C'} - \tfrac{3}{2}T_{[A\,B|I'}^{A'B'I}A_{I|C]}^{I'C'}\epsilon_{B'C'} \\
(d_{\yng(2,2)}\nu)_{ABCD} &= -2\nabla_{[A}^{A'}\nu_{|CD|B]}^{B'}\epsilon_{A'B'} - 2\nabla_{[C}^{A'}\nu_{|AB|D]}^{B'}\epsilon_{A'B'}
\end{aligned}
$$

Proof. The formulae for components of the exterior derivative are obtained essentially as follows. First we embed the given component into the bundle of exterior forms, then we apply the exterior derivative and then we project to the target bundle. Thus we need to make explicit the embeddings and projections. The formula for the exterior derivative is well-known from the literature. On a $(p-1)$-form ω it is defined by

$$(d\omega)_{a_1\dots a_p} = p\nabla_{[a_1}\omega_{a_2\dots a_p]} + \frac{p(p-1)}{2}T_{[a_1a_2}{}^{e}\omega_{|e|a_3\dots a_p]}.$$

Since the first equation is obvious, let us start with the second and third. The embedding is trivial in this case and so the components are given by the two projections of

$$(d\mu)_{A\,B}^{A'B'} = \nabla_A^{A'}\mu_B^{B'} - \nabla_B^{B'}\mu_A^{A'} + T_{A\,B\,I'}^{A'B'I}\mu_I^{I'}.$$

The result then follows from the symmetry of the torsion, (2.4). The fourth and fifth equation follow from the formula for the exterior derivative on two-forms:

$$
\begin{aligned}
(d\omega)_{A\,B\,C}^{A'B'C'} &= \nabla_A^{A'}\omega_{B\,C}^{B'C'} + \nabla_B^{B'}\omega_{C\,A}^{C'A'} + \nabla_C^{C'}\omega_{A\,B}^{A'B'} \\
&\quad + T_{A\,B\,I'}^{A'B'I}\omega_{I\,C}^{I'C'} + T_{B\,C\,I'}^{B'C'I}\omega_{I\,A}^{I'A'} + T_{C\,A\,I'}^{C'A'I}\omega_{I\,B}^{I'B'}.
\end{aligned}
\tag{2.12}
$$

The form of the isomorphism $\mathcal{E}[-1] \cong \Lambda^2\mathcal{E}^{A'}$ implies that the embedding of $\mathcal{E}_{(AB)}[-1]$ into $\Lambda^2\mathcal{E}_A^{A'}$ is given by $\omega_{A\,B}^{A'B'} = -\frac{1}{2}\alpha_{AB}\epsilon^{A'B'}$. Inserting this into (2.12) and projecting to $\mathcal{E}_{[AB]C}^{A'}[-1]$, the second torsion-term vanishes due to the symmetry of the torsion (2.4) and the other two vanish due to its trace-freeness, and so we immediately obtain the equation

$$(d\alpha)^{A'}_{[AB]C} = \frac{3}{2}\nabla^{A'}_{[A}\alpha_{B]C}.$$

This formula coincides with the projection to $\mathcal{E}^{A'}_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}(ABC)}[-1]$ since the total alternation of unprimed indices vanishes due to the symmetry of α. The fifth equation is obtained by setting $\omega^{A'B'}_{A\ B} = A^{A'B'}_{A\ B}$ in (2.12) and projecting to $\mathcal{E}^{A'}_{[AB]C}[-1]$. We get

$$(dA)^{A'B'C'}_{[A\ B]C}\,\epsilon_{B'C'} = (\nabla^{\ B'}_{[B}A^{C'A'}_{|C|A]} - \nabla^{B'}_{C}A^{C'A'}_{A\ B} + T^{A'B'I}_{A\ B\ I'}A^{I'C'}_{I\ C} - T^{A'B'I}_{C[A|I'}A^{I'C'}_{I|B]})\epsilon_{B'C'}$$

and since the first term on the right may be written as

$$\nabla^{\ B'}_{[B}A^{C'A'}_{|C|A]} = -\frac{1}{2}\nabla^{B'}_{C}A^{C'A'}_{A\ B} + \frac{3}{2}\nabla^{\ B'}_{[C}A^{C'A'}_{A\ B]},$$

and we may also rewrite in the same way the last term on the right, we obtain

$$(dA)^{A'B'C'}_{[A\ B]C}\,\epsilon_{B'C'} \equiv \frac{3}{2}\nabla^{C'}_{C}A^{A'B'}_{A\ B}\,\epsilon_{B'C'} + \frac{3}{2}T^{A'B'I}_{A\ B\ I'}A^{I'C'}_{I\ C}\,\epsilon_{B'C'}$$

modulo terms in $\mathcal{E}^{A'}_{[ABC]}[-1]$. Hence the projection of dA to $\mathcal{E}^{A'}_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}(ABC)}[-1]$ yields the resulting equation. The last equation is obtained by applying the formula for the exterior derivetive on

$$\omega^{A'B'C'}_{A\ B\ C} = -\frac{4}{9}(\nu^{A'}_{A(BC)}\epsilon^{B'C'} + \nu^{B'}_{B(CA)}\epsilon^{C'A'} + \nu^{C'}_{C(AB)}\epsilon^{A'B'}),$$

(since this is the embedding inverse to the projection $\Lambda^3\mathcal{E}^{A'}_{A} \to \mathcal{E}^{A'}_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}(ABC)}[-1]$), and then projecting to $\mathcal{E}_{\boxplus(ABCD)}[-2]$. Again, there is no contribution of torsion. This can be proved either by a direct computation or by the following represenatation-theoretical argumentation. By (2.5), the torsion may be viewed as totaly symmetric in primed indices. Thus its action on $\nu^{A'}_{ABC}$ must be symmetric in two or four indices. In particular, there is no non-zero complete contraction and so there is no contribution to $\mathcal{E}_{\boxplus(ABCD)}[-2]$. $\square$

Remark 2.1. The argument from the end of the proof can be generalized to the whole bottom row of the figure 2. We conclude that formulae for all components of the exterior derivative appearing there do not depend on the torsion. Namely, for $s \in \mathcal{E}^{A'} \boxtimes {\scriptstyle k\updownarrow}\begin{smallmatrix}\square&\square\\\vdots&\vdots\\\square&\end{smallmatrix}\mathcal{E}_A[-k+1] \subset \mathcal{E}^{A'}_{[A_1\ldots A_k][B_1\ldots B_{k-1}]}[-k+1]$ we deduce

$$(d_{\,k\updownarrow\boxplus}s)_{A_1\ldots A_k B_1\ldots B_k} = \nabla^{A'}_{[A_1}s^{B'}_{|B_1\ldots B_k|A_2\ldots A_k]}\epsilon_{A'B'} + \nabla^{A'}_{[B_1}s^{B'}_{|A_1\ldots A_k|B_2\ldots B_k]}\epsilon_{A'B'}$$

up to a scalar multiple, and for $s \in {\scriptstyle k\updownarrow}\begin{smallmatrix}\square&\square\\\vdots&\vdots\\\square&\square\end{smallmatrix}\mathcal{E}_A[-k] \subset \mathcal{E}_{[A_1\ldots A_k][B_1\ldots B_k]}[-k]$ we get

$$(ds)^{A'}_{A_1\ldots A_{k+1}B_1\ldots B_k} = \nabla^{A'}_{[A_1}s_{A_2\ldots A_{k+1}]B_1\ldots B_k}.$$

CHAPTER 3

Construction of the non-standard operator

3.1. Construction in the torion-free case

We use the curved Casimir operator to construct a curved anologue of the non-standard operator $\mathcal{E} \to \boxplus\mathcal{E}[-2]$ for the Grassmannian geometry, i.e. for an almost Grassmannian geometry which allows existence of a torsion-free connection. We show that although the direct application of the procedure of the construction via curved Casimirs described in the first chapter yields a vanishing operator, the operator can be obtained by a modification of this procedure, similarly as the square of the conformal Laplacian is constructed in [**6**].

3.1.1. Suitable tractor bundle. In order to construct the nonstandard operator $\mathcal{E} \to \boxplus\mathcal{E}_A[-2]$, we need a suitable tractor bundle first. In particular, the trivial representation and the representation $\boxplus\mathcal{E}_A$ must occur in the composition series of such tractor bundle. It turns out that the right tractor bundle is a weighted version of $\boxplus\mathcal{E}_\alpha$, i.e. the tractor bundle induced by the irreducible representation $\boxplus\mathbb{C}^{(2+q)*}$ of $\mathfrak{sl}(2+q)$.

The composition series of this bundle is obtained in the following way. The standard tractor bundle $\mathcal{E}^\alpha$ by (2.3) has a natural subbundle $\mathcal{E}^{A'}$ and the quotient $\mathcal{E}^\alpha/\mathcal{E}^{A'}$ is isomorphic to $\mathcal{E}^A$. Therefore, the dual bundle $\mathcal{E}_A$ is a subbundle of the cotractor bundle $\mathcal{E}_\alpha := (\mathcal{E}^\alpha)^*$ and $\mathcal{E}_\alpha/\mathcal{E}_A$ is isomorphic to $\mathcal{E}_{A'}$. Hence we have a filtration of $\mathcal{E}_\alpha$ which we describe by the composition series $\mathcal{E}_\alpha = \mathcal{E}_{A'} \oplus \mathcal{E}_A$. Let us write Y_α^A for the cannonical section of $\mathcal{E}_\alpha^A$ which gives the injecting morphism $\mathcal{E}_A \to \mathcal{E}_\alpha$. We have the following exact sequence

$$0 \longrightarrow \mathcal{E}_A \xrightarrow{Y} \mathcal{E}_\alpha \underset{\xi}{\overset{}{\rightleftarrows}} \mathcal{E}_{A'} \longrightarrow 0,$$

which splitts when we choose a Weyl structure. We denote the splitting $\mathcal{E}_{A'} \to \mathcal{E}_\alpha$ by $\xi_\alpha^{A'}$. Every section of the standard cotractor bundle can be then written either as an expression in the "injectors" Y, ξ or simply as a "row vector" as follows

$$v_\alpha = v_{A'}\xi_\alpha^{A'} + v_A Y_\alpha^A = \begin{pmatrix} v_{A'} & v_A \end{pmatrix}.$$

From the composition series of the standard cotractor bundle, we can easily deduce the form of the composition series of $\mathcal{E}_{[\alpha\beta]}$:

$$\mathcal{E}_{[\alpha\beta]} = \mathcal{E}[1] \oplus \mathcal{E}_{AB'} \oplus \mathcal{E}_{[AB]}.$$

Here we used the isomorphism $\mathcal{E}[1] \cong \mathcal{E}_{[A'B']}$ given by $\epsilon_{A'B'}$. A section $v_{\alpha\beta} \in \mathcal{E}_{[\alpha\beta]}$ can be expressed as

$$v_{\alpha\beta} = \sigma\epsilon_{A'B'}\xi^{A'}_{[\alpha}\xi^{B'}_{\beta]} + \mu_{AB'}Y^A_{[\alpha}\xi^{B'}_{\beta]} + \rho_{AB}Y^A_{[\alpha}Y^B_{\beta]}$$

or shortly displayed as a row vector $v_{\alpha\beta} = \begin{pmatrix}\sigma & \mu_{AB'} & \rho_{AB}\end{pmatrix}$ with $\rho_{AB} = \rho_{[AB]}$. From the composition series of $\mathcal{E}_{[\alpha\beta]}$, we conclude the composition series for its second symmetric power:

$$\mathcal{E}_{[\alpha\beta]} \vee \mathcal{E}_{[\gamma\delta]} = \mathcal{E}[2] \oplus \mathcal{E}_{AB'}[1] \oplus \begin{matrix}\mathcal{E}_{[AB]}[1] \\ \oplus \\ \mathcal{E}_{AB'} \vee \mathcal{E}_{CD'}\end{matrix} \oplus \mathcal{E}_{[AB]CD'} \oplus \mathcal{E}_{[AB]} \vee \mathcal{E}_{[CD]}.$$

Now we use the isomorphism $\mathcal{E}_{A'} \cong \mathcal{E}^{A'}[1]$ to raise the all primed indices. Then we get

$$\mathcal{E}_{[\alpha\beta]} \vee \mathcal{E}_{[\gamma\delta]} = \mathcal{E}[2] \oplus \mathcal{E}_A^{A'}[2] \oplus \begin{matrix}\mathcal{E}_{[AB]}[1] \\ \oplus \\ \mathcal{E}_A^{A'} \vee \mathcal{E}_B^{B'}[2]\end{matrix} \oplus \mathcal{E}^{A'}_{[AB]C}[1] \oplus \mathcal{E}_{[AB]} \vee \mathcal{E}_{[CD]}.$$

If we write its section $v_{\alpha\beta\gamma\delta}$ as a "transposed matrix" for better readibility, we have

$$v_{\alpha\beta\gamma\delta} = \begin{pmatrix} & \sigma & \\ & \mu_B^{A'} & \\ B_{AB}^{A'B'} & & \alpha_{AB} \\ & \nu^{A'}_{ABC} & \\ & \rho_{ABCD} & \end{pmatrix},$$

where $B_{AB}^{A'B'} = B_{BA}^{B'A'}$, $\alpha_{AB} = \alpha_{[AB]}$, $\nu^{A'}_{ABC} = \nu^{A'}_{[AB]C}$, $\rho_{ABCD} = \rho_{[AB][CD]} = \rho_{[CD][AB]}$. The representatives in the interior of this matrix are chosen in such way that we have

$$\begin{aligned}
v_{\alpha\beta\gamma\delta} &= \sigma\epsilon_{A'B'}\epsilon_{C'D'}\xi^{A'}_{[\alpha}\xi^{B'}_{\beta]}\vee\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]} + \mu_A^{A'}\epsilon_{A'B'}\epsilon_{C'D'}Y^A_{[\alpha}\xi^{B'}_{\beta]}\vee\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]} \\
&+ B_{AB}^{A'B'}\epsilon_{A'C'}\epsilon_{B'D'}Y^A_{[\alpha}\xi^{C'}_{\beta]}\vee Y^B_{[\gamma}\xi^{D'}_{\delta]} + \alpha_{AB}\epsilon_{C'D'}Y^A_{[\alpha}Y^B_{\beta]}\vee\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]} \\
&+ \nu^{C'}_{ABC}\epsilon_{C'D'}Y^A_{[\alpha}Y^B_{\beta]}\vee Y^C_{[\gamma}\xi^{D'}_{\delta]} + \rho_{ABCD}Y^A_{[\alpha}Y^B_{\beta]}\vee Y^C_{[\gamma}Y^D_{\delta]}.
\end{aligned}$$

We are using here a shortened notation

$$Y^A_{[\alpha}\xi^{B'}_{\beta]}\vee\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]} = \tfrac{1}{2}(Y^A_{[\alpha}\xi^{B'}_{\beta]}\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]} + Y^A_{[\gamma}\xi^{B'}_{\delta]}\xi^{C'}_{[\alpha}\xi^{D'}_{\beta]})$$
$$= \tfrac{1}{4}(Y^A_{\alpha}\xi^{B'}_{\beta}\xi^{C'}_{\gamma}\xi^{D'}_{\delta} - Y^A_{\beta}\xi^{B'}_{\alpha}\xi^{C'}_{\gamma}\xi^{D'}_{\delta} + Y^A_{\gamma}\xi^{B'}_{\delta}\xi^{C'}_{\alpha}\xi^{D'}_{\beta} - Y^A_{\delta}\xi^{B'}_{\gamma}\xi^{C'}_{\alpha}\xi^{D'}_{\beta}).$$

So this is a map $\mathcal{E}_{AB'C'D'} \to \mathcal{E}_{[\alpha\beta]}\vee\mathcal{E}_{[\gamma\delta]}$ which factorizes through $\mathcal{E}_{AB'[C'D']}$. Similarly, the other terms give other injections into $\mathcal{E}_{[\alpha\beta]}\vee\mathcal{E}_{[\gamma\delta]}$ and the symmetries of tractor indices translates to the indices of the same kind (primed or unprimed), i.e. $Y^A_{[\alpha}Y^B_{\beta]} = Y^{[A}_{[\alpha}Y^{B]}_{\beta]}$, $\xi^{A'}_{[\alpha}\xi^{B'}_{\beta]} = \xi^{[A'}_{[\alpha}\xi^{B']}_{\beta]}$ etc.

We are finally approaching the composition series of the tractor bundle $\boxplus\mathcal{E}_\alpha$. We realize this bundle as the kernel of the alternation map $\mathcal{E}_{[\alpha\beta]}\vee\mathcal{E}_{[\gamma\delta]} \to \mathcal{E}_{[\alpha\beta\gamma\delta]}$ and denote by $\mathcal{E}_{\boxplus(\alpha\beta\gamma\delta)}$. Since the fourth skewsymmetric power of the standard tractor bundle has the simple composition series

$$\mathcal{E}_{[\alpha\beta\gamma\delta]} = \mathcal{E}_{[AB]}[1] \oplus\!\!\!\!- \ \mathcal{E}^{A'}_{[ABC]}[1] \oplus\!\!\!\!- \ \mathcal{E}_{[ABCD]},$$

from the composition series of $\mathcal{E}_{[\alpha\beta]}\vee\mathcal{E}_{[\gamma\delta]}$ we read off

$$\mathcal{E}_{\boxplus(\alpha\beta\gamma\delta)} = \mathcal{E}[2] \oplus\!\!\!\!- \ \mathcal{E}^{A'}_{A}[2] \oplus\!\!\!\!- \ \begin{matrix}\mathcal{E}_{[AB]}[1]\\ \oplus \\ \mathcal{E}^{(A'B')}_{(AB)}[2]\end{matrix} \oplus\!\!\!\!- \ \mathcal{E}^{A'}_{\boxminus(ABC)}[1] \oplus\!\!\!\!- \ \mathcal{E}_{\boxplus(ABCD)}, \qquad (3.1)$$

where $\mathcal{E}_{\boxminus(ABC)}$ is the kernel of $\mathcal{E}_{[AB]C} \to \mathcal{E}_{[ABC]}$, and $\mathcal{E}_{\boxplus(ABCD)}$ is the kernel of $\mathcal{E}_{[AB]}\vee\mathcal{E}_{[CD]} \to \mathcal{E}_{[ABCD]}$. By definition of $\mathcal{E}_{\boxplus(\alpha\beta\gamma\delta)}$, its section is exactly a section $v_{\alpha\beta\gamma\delta}$ of $\mathcal{E}_{[\alpha\beta]}\vee\mathcal{E}_{[\gamma\delta]}$ for which $v_{[\alpha\beta\gamma\delta]} = 0$. Looking at its expression via the injectors given above, we find that this condition is equivalent to the equation

$$B^{A'B'}_{AB}\epsilon_{A'C'}\epsilon_{B'D'}Y^A_{[\alpha}\xi^{C'}_{\beta}Y^B_{\gamma}\xi^{D'}_{\delta]} + \alpha_{AB}\epsilon_{C'D'}Y^A_{[\alpha}Y^B_{\beta}\xi^{C'}_{\gamma}\xi^{D'}_{\delta]}$$
$$+\nu^{C'}_{ABC}\epsilon_{C'D'}Y^A_{[\alpha}Y^B_{\beta}Y^C_{\gamma}\xi^{D'}_{\delta]} + \rho_{ABCD}Y^A_{[\alpha}Y^B_{\beta}Y^C_{\gamma}Y^D_{\delta]} = 0.$$

The other terms do not appear because the alternation over three primed indices vanishes. This equation is obviously equivalent to the following system of three equations

$$\begin{aligned} -B^{[A'B']}_{[A\ B]}\epsilon_{A'C'}\epsilon_{B'D'} + \alpha_{AB}\epsilon_{C'D'} &= 0,\\ \nu^{C'}_{[ABC]} &= 0,\\ \rho_{[ABCD]} &= 0. \end{aligned}$$

Tensor $B^{[A'B']}_{[A\ B]}$, which is by the first equation determined by α_{AB}, is one of the two irreducible components of (in pairs) symmetric tensor $B^{A'B'}_{A\ B} =$

$B^{B'A'}_{B\ A}$. The second irreducible component will be denoted $A^{A'B'}_{A\ B} := B^{(A'B')}_{(A\ B)}$ in sequel. A section $v_{\alpha\beta\gamma\delta} \in \boxplus\mathcal{E}_\alpha$ is then given by the expression

$$\begin{aligned} v_{\alpha\beta\gamma\delta} &= \sigma\epsilon_{A'B'}\epsilon_{C'D'}\xi^{A'}_{[\alpha}\xi^{B'}_{\beta]}\vee\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]} + \mu^{A'}_{A}\epsilon_{A'B'}\epsilon_{C'D'}Y^{A}_{[\alpha}\xi^{B'}_{\beta]}\vee\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]} \\ &+A^{A'B'}_{A\ B}\epsilon_{A'C'}\epsilon_{B'D'}Y^{A}_{[\alpha}\xi^{C'}_{\beta]}\vee Y^{B}_{[\gamma}\xi^{D'}_{\delta]} + \alpha_{AB}\epsilon_{C'D'}(Y^{A}_{[\alpha}Y^{B}_{\beta]}\vee\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]} + Y^{A}_{[\alpha}\xi^{C'}_{\beta]}\vee Y^{B}_{[\gamma}\xi^{D'}_{\delta]}) \\ &+\nu^{C'}_{ABC}\epsilon_{C'D'}Y^{A}_{[\alpha}Y^{B}_{\beta]}\vee Y^{C}_{[\gamma}\xi^{D'}_{\delta]} + \rho_{ABCD}Y^{A}_{[\alpha}Y^{B}_{\beta]}\vee Y^{C}_{[\gamma}Y^{D}_{\delta]}, \end{aligned}$$

where $A^{A'B'}_{A\ B} = A^{(A'B')}_{(A\ B)}$, $\alpha_{AB} = \alpha_{[AB]}$, $\nu^{C'}_{ABC} = \nu^{C'}_{[AB]C}$, $\nu^{C'}_{[ABC]} = 0$, $\rho_{ABCD} = \rho_{[AB][CD]} = \rho_{[CD][AB]}$ and $\rho_{[ABCD]} = 0$. We will display such section as a matrix

$$\begin{pmatrix} \sigma \\ \mu^{A'}_{A} \\ A^{A'B'}_{A\ B} \mid \alpha_{AB} \\ \nu^{C'}_{ABC} \\ \rho_{ABCD} \end{pmatrix}$$

The action of the nilpotent part $\mathfrak{g}_1$ of the algebra $\mathfrak{g}$ gives rise to the action of one-form on the standard tractor bundle which maps the component in $\mathcal{E}_{A'}$ to $\mathcal{E}_A$ and the component in $\mathcal{E}_A$ to zero. This action can be immediately computed from the matrix representation of $\mathfrak{g}$. For $\varphi^{A'}_A \in T^*M$ and a standard cotractor $v_\alpha = \begin{pmatrix} v_{A'} & v_A \end{pmatrix}$, we have

$$(\varphi \bullet v)_\alpha = -\begin{pmatrix} v_{A'} & v_A \end{pmatrix} \begin{pmatrix} 0 & \varphi^{A'}_{A} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\varphi^{A'}_{A} v_{A'}, \end{pmatrix}$$

which we can write using the second notation as $(\varphi \bullet v)_\alpha = -\varphi^{A'}_A v_{A'} Y^A_\alpha$. This shows the form of the action of a one-form φ on the injectors Y, ξ:

$$(\varphi \bullet \xi)^{A'}_{\alpha} = -\varphi^{A'}_{A} Y^{A}_{\alpha}, \quad \varphi \bullet Y = 0.$$

Now we use these basic relations to compute the action of a one-form on a tractor $v_{\alpha\beta\gamma\delta} \in \mathcal{E}_{\boxplus(\alpha\beta\gamma\delta)}$ to obtain the following.

Lemma 3.1. *Displaying sections as matrices, the action of T^*M on a section of $\mathcal{E}_{\boxplus(\alpha\beta\gamma\delta)}$ has the form*

$$\varphi \bullet \begin{pmatrix} \sigma \\ \mu_A^{A'} \\ A_{A\ B}^{A'B'} \mid \alpha_{AB} \\ \nu_{ABC}^{C'} \\ \rho_{ABCD} \end{pmatrix} = \begin{pmatrix} 0 \\ -4\varphi_A^{A'}\sigma \\ -2\varphi_{(A}^{(A'}\mu_{B)}^{B')} \mid -\varphi_{[A}^{A'}\mu_{B]}^{B'}\epsilon_{A'B'} \\ -2\varphi_{[B}^{B'}A_{A]C}^{A'C'}\epsilon_{A'B'} - 2\varphi_C^{C'}\alpha_{AB} - 2\varphi_{[B}^{C'}\alpha_{A]C} \\ -\frac{1}{2}(\nu_{AB[C}^{C'}\varphi_{D]}^{D'} + \nu_{CD[A}^{C'}\varphi_{B]}^{D'})\epsilon_{C'D'} \end{pmatrix} \tag{3.2}$$

Proof. The action on the first term in the compositon serie of the tractor $v_{\alpha\beta\gamma\delta}$ equals

$$\begin{aligned} &\sigma\epsilon_{A'B'}\epsilon_{C'D'}((\varphi\bullet\xi)_{[\alpha}^{A'}\xi_{\beta]}^{B'}\vee\xi_{[\gamma}^{C'}\xi_{\delta]}^{D'} + \xi_{[\alpha}^{A'}(\varphi\bullet\xi)_{\beta]}^{B'}\vee\xi_{[\gamma}^{C'}\xi_{\delta]}^{D'} \\ &+\xi_{[\alpha}^{A'}\xi_{\beta]}^{B'}\vee(\varphi\bullet\xi)_{[\gamma}^{C'}\xi_{\delta]}^{D'} + \xi_{[\alpha}^{A'}\xi_{\beta]}^{B'}\vee\xi_{[\gamma}^{C'}(\varphi\bullet\xi)_{\delta]}^{D'}) \\ &= \sigma\epsilon_{A'B'}\epsilon_{C'D'}(-\varphi_A^{A'}Y_{[\alpha}^A\xi_{\beta]}^{B'}\vee\xi_{[\gamma}^{C'}\xi_{\delta]}^{D'} - \xi_{[\alpha}^{A'}\varphi_{|A|}^{B'}Y_{\beta]}^A\vee\xi_{[\gamma}^{C'}\xi_{\delta]}^{D'} \\ &-\xi_{[\alpha}^{A'}\xi_{\beta]}^{B'}\vee\varphi_A^{C'}Y_{[\gamma}^A\xi_{\delta]}^{D'} - \xi_{[\alpha}^{A'}\xi_{\beta]}^{B'}\vee\xi_{[\gamma}^{C'}\varphi_{|A|}^{D'}Y_{\delta]}^A) \end{aligned}$$

When we use the displayed symmetries of tractor indices and we rename some of the summing indices, we see that the all four summands are equal and therefore the action on the first term results in $-4\varphi_A^{A'}\sigma\epsilon_{A'B'}\epsilon_{C'D'}Y_{[\alpha}^A\xi_{\beta]}^{B'}\vee\xi_{[\gamma}^{C'}\xi_{\delta]}^{D'}$. Proceeding in the same way term by term we find that

$$\begin{aligned} &(\varphi\bullet v)_{\alpha\beta\gamma\delta} = \\ &-4\varphi_A^{A'}\sigma\epsilon_{A'B'}\epsilon_{C'D'}Y_{[\alpha}^A\xi_{\beta]}^{B'}\vee\xi_{[\gamma}^{C'}\xi_{\delta]}^{D'} - 2\varphi_B^{B'}\mu_A^{A'}\epsilon_{A'C'}\epsilon_{B'D'}Y_{[\alpha}^A\xi_{\beta]}^{C'}\vee Y_{[\gamma}^B\xi_{\delta]}^{D'} \\ &-\varphi_B^{B'}\mu_A^{A'}\epsilon_{A'B'}\epsilon_{C'D'}Y_{[\alpha}^AY_{\beta]}^B\vee\xi_{[\gamma}^{C'}\xi_{\delta]}^{D'} - 2\varphi_B^{B'}A_{A\ C}^{A'C'}\epsilon_{A'B'}\epsilon_{C'D'}Y_{[\alpha}^AY_{\beta]}^B\vee Y_{[\gamma}^C\xi_{\delta]}^{D'} \\ &-2\varphi_C^{C'}\alpha_{AB}\epsilon_{C'D'}Y_{[\alpha}^AY_{\beta]}^B\vee Y_{[\gamma}^C\xi_{\delta]}^{D'} - 2\varphi_B^{C'}\alpha_{AC}\epsilon_{C'D'}Y_{[\alpha}^AY_{\beta]}^B\vee Y_{[\gamma}^C\xi_{\delta]}^{D'} \\ &-\varphi_D^{D'}\nu_{ABC}^{C'}\epsilon_{C'D'}Y_{[\alpha}^AY_{\beta]}^B\vee Y_{[\gamma}^CY_{\delta]}^D. \end{aligned}$$

The symmetry of the tractor indices in the second term translate into the symmetry of the pairs A, C' and B, D'. Therefore,it decomposes into two components: one symmetric in A, B and symmetric in C', D' and the second skewsymmetric in A, B and skewsymmetric in C', D'. The symmetry in C', D' translates to A', B' and thus we get

$$\begin{aligned} &-2\varphi_B^{B'}\mu_A^{A'}\epsilon_{A'C'}\epsilon_{B'D'}Y_{[\alpha}^A\xi_{\beta]}^{C'}\vee Y_{[\gamma}^B\xi_{\delta]}^{D'} \\ &= -2\varphi_{(B}^{(B'}\mu_{A)}^{A')}\epsilon_{A'C'}\epsilon_{B'D'}Y_{[\alpha}^A\xi_{\beta]}^{C'}\vee Y_{[\gamma}^B\xi_{\delta]}^{D'} - 2\varphi_{[B}^{[B'}\mu_{A]}^{A']}\epsilon_{A'C'}\epsilon_{B'D'}Y_{[\alpha}^A\xi_{\beta]}^{C'}\vee Y_{[\gamma}^B\xi_{\delta]}^{D'}. \end{aligned}$$

The last term can be replaced by $-\varphi^{B'}_{[B}\mu^{A'}_{A]}\epsilon_{A'B'}\epsilon_{C'D'}Y^A_{[\alpha}\xi^{C'}_{\beta]}\vee Y^B_{[\gamma}\xi^{D'}_{\delta]}$ because our convention for raising and lowering of primed indices yields equations $\epsilon^{B'A'}\epsilon_{A'C'}\epsilon_{B'D'}=\epsilon_{C'D'}$ and $\epsilon^{B'A'}\epsilon_{A'B'}=2$. Inserting this into the formula for the action of φ on $v_{\alpha\beta\gamma\delta}$ gives

$$
\begin{aligned}
&(\varphi\bullet v)_{\alpha\beta\gamma\delta}\\
&=-4\varphi^{A'}_{A}\sigma\epsilon_{A'B'}\epsilon_{C'D'}Y^A_{[\alpha}\xi^{B'}_{\beta]}\vee\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]}-2\varphi^{(B'}_{(B}\mu^{A')}_{A)}\epsilon_{A'C'}\epsilon_{B'D'}Y^A_{[\alpha}\xi^{C'}_{\beta]}\vee Y^B_{[\gamma}\xi^{D'}_{\delta]}\\
&-\varphi^{B'}_{[B}\mu^{A'}_{A]}\epsilon_{A'B'}\epsilon_{C'D'}(Y^A_{[\alpha}Y^B_{\beta]}\vee\xi^{C'}_{[\gamma}\xi^{D'}_{\delta]}+Y^A_{[\alpha}\xi^{C'}_{\beta]}\vee Y^B_{[\gamma}\xi^{D'}_{\delta]})\\
&-2(\varphi^{B'}_{[B}A^{A'C'}_{A]C}\epsilon_{A'B'}+\varphi^{C'}_{C}\alpha_{AB}+\varphi^{C'}_{[B}\alpha_{A]C})\epsilon_{C'D'}Y^A_{[\alpha}Y^B_{\beta]}\vee Y^C_{[\gamma}\xi^{D'}_{\delta]}\\
&-\tfrac{1}{2}(\nu^{C'}_{AB[C}\varphi^{D'}_{D]}+\nu^{C'}_{CD[A}\varphi^{D'}_{B]})\epsilon_{C'D'}Y^A_{[\alpha}Y^B_{\beta]}\vee Y^C_{[\gamma}Y^D_{\delta]},
\end{aligned}
$$

and the result follows by rewritting in the matrix notation. ☐

3.1.2. Casimir eigenvalues. To construct the non-standard operator we need to determine the eigenvalues of the curved Casimir operator on tractor bundle $\mathcal{E}_{\boxplus(\alpha\beta\gamma\delta)}[w]$. Its composition series is obtained from the series (3.1) for $\mathcal{E}_{\boxplus(\alpha\beta\gamma\delta)}$ by twisting each component of the series by the weight w:

$$
\mathcal{E}_{\boxplus(\alpha\beta\gamma\delta)}[w]=
$$

$$
\mathcal{E}[w+2]\;\subset\!\!\!\!+\;\mathcal{E}^{A'}_{A}[w+2]\;\subset\!\!\!\!+\;\begin{matrix}\mathcal{E}_{[AB]}[w+1]\\ \oplus\\ \mathcal{E}^{(A'B')}_{(AB)}[w+2]\end{matrix}\;\subset\!\!\!\!+\;\mathcal{E}^{A'}_{\boxplus(ABC)}[w+1]\;\subset\!\!\!\!+\;\mathcal{E}_{\boxplus(ABCD)}[w].
\tag{3.3}
$$

By 1.12 the formula for Casimir eigenvalues is $c_i=\langle\lambda_i,\lambda_i+2\rho\rangle$ where ρ is the lowest form which is from definition given by the sum of all fundamental weights (or equivalently the half of the sum of positive roots), i.e. in Dynkin diagram notation

$$
\rho=\overset{1}{\circ}\!-\!\overset{1}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{1}{\circ}\!-\cdots-\!\overset{1}{\circ}\;.
$$

The weights $-\lambda_i$ are the lowest weights of the irreducible $\mathfrak{p}$-representations inducing the bundles appearing in the composition series. Since we will work with tensor representations only, the lowest weight $-\lambda_i$ of a $\mathfrak{g}$-representation coincides with the highest weight of the dual $\mathfrak{g}$-representation. Hence the lowest weights of representations inducing the basic bundles $\mathcal{E}^{A'}$ and $\mathcal{E}^{A}$ can be easily deduced from the description of highest weights in section 2.1.4.

Namely, in Dynkin diagram notation they are

$$-LOW(\mathcal{E}^{A'}) = \overset{1}{\circ}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}$$

$$-LOW(\mathcal{E}^{A}) = \overset{0}{\circ}\!-\!\overset{0}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{1}{\circ}\ .$$

For the duals we conclude

$$-LOW(\mathcal{E}_{A'}) = \overset{1}{\circ}\!-\!\overset{0}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}$$

$$-LOW(\mathcal{E}_{A}) = \overset{0}{\circ}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}\ ,$$

and thus for the tangent and cotangent bundle we get

$$-LOW(\mathcal{E}^a = \mathcal{E}^A_{A'}) = \overset{1}{\circ}\!-\!\overset{0}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{1}{\circ}$$

$$-LOW(\mathcal{E}_a = \mathcal{E}^{A'}_A) = \overset{1}{\circ}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}\ .$$

From the lowest weights of the basic bundles we deduce the lowes weights of all representations appearing in composition series (3.1) of $\boxplus\mathcal{E}_a$. And since the convention for the bundle of densities fixed in section 2.1.5 in Dynkin diagram notation reads

$$-LOW(\mathcal{E}[w]) = \overset{0}{\circ}\!-\!\overset{w}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}\ ,$$

we also easily get the expressions for lowest weights appearing in the composition series (3.3) of $\boxplus\mathcal{E}_a[w]$. Namely, in terms of minus lowest weights $-\lambda_i$, the composition series has the form

$$\overset{0}{\circ}\!-\!\overset{w+2}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ}$$

$$= \left(\begin{array}{ccc} & \overset{0}{\circ}\!-\!\overset{w+2}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ} & \\ & \overset{1}{\circ}\!-\!\overset{w}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ} & \\ \overset{2}{\circ}\!-\!\overset{w-2}{\times}\!-\!\overset{2}{\circ}\!-\!\overset{0}{\circ}\!-\cdots-\!\overset{0}{\circ} & | & \overset{0}{\circ}\!-\!\overset{w}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}\!-\cdots-\!\overset{0}{\circ} \\ & \overset{1}{\circ}\!-\!\overset{w-2}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{1}{\circ}\!-\cdots-\!\overset{0}{\circ} & \\ & \overset{0}{\circ}\!-\!\overset{w-2}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{2}{\circ}\!-\cdots-\!\overset{0}{\circ} & \end{array} \right)$$

Now we choose the usual basis $e^1, e^2, e^3, \dots, e^{q+2}$ of $\mathfrak{h}^*$ with the property that $e^1 + e^2 + e^3 + \cdots + e^{q+2} = 0$ and we express the highest weights in this basis. The advantage of this expression is that the inner product of weights then coincides with the standard inner product in $\mathbb{R}^{q+2}$.

With this choice of a basis, the fundamental weights equal

$$e^1 + e^2 + e^3 + \cdots + e^i \quad \text{for} \quad i = 1, 2, \dots, q+1$$

which implies that the lowest form equals

$$\rho = (q+1)e^1 + qe^2 + (q-1)e^3 + \cdots + e^{q+1}$$

This can be written as a vector in $\mathbb{R}^{q+2}$

$$\rho = (q+1, q, q-1, \dots, 1, -\frac{(q+1)(q+2)}{2})$$

For the weights λ_i, we get

$$\lambda_0 = \overset{0}{\circ}\!-\!\overset{w+2}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{0}{\circ}\!- \cdots -\!\overset{0}{\circ} = (w+2)(e^1+e^2)$$
$$= (w+2, w+2, 0, \dots, 0, -2w-4)$$

$$\lambda_1 = \overset{1}{\circ}\!-\!\overset{w}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}\!- \cdots -\!\overset{0}{\circ} = e^1 + w(e^1+e^2) + e^1 + e^2 + e^3$$
$$= (w+2, w+1, 1, 0, \dots, 0, -2w-4)$$

$$\lambda_2^1 = \overset{2}{\circ}\!-\!\overset{w-2}{\times}\!-\!\overset{2}{\circ}\!-\!\overset{0}{\circ}\!- \cdots -\!\overset{0}{\circ} = 2e^1 + (w-2)(e^1+e^2) + 2(e^1+e^2+e^3)$$
$$= (w+2, w, 2, 0, \dots, 0, -2w-4)$$

$$\lambda_2^2 = \overset{0}{\circ}\!-\!\overset{w}{\times}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}\!- \cdots -\!\overset{0}{\circ} = w(e^1+e^2) + e^1 + e^2 + e^3 + e^4$$
$$= (w+1, w+1, 1, 1, 1, 0, \dots, 0, -2w-4)$$

$$\lambda_3 = \overset{1}{\circ}\!-\!\overset{w-2}{\times}\!-\!\overset{1}{\circ}\!-\!\overset{1}{\circ}\!- \cdots -\!\overset{0}{\circ} = e^1 + (w-2)(e^1+e^2) + 2e^1 + 2e^2 + 2e^3 + e^4$$
$$= (w+1, w, 2, 1, 0, \dots, 0, -2w-4)$$

$$\lambda_4 = - = (w, w, 2, 2, 0, \dots, 0, -2w-4)$$

Now it is easy to calculate the Casimir eigenvalues. We have

$$c_0 = (w+2)(w+2+2(q+1)) + (w+2)(w+2+2q) + (2w+4)(2w+4+\frac{(q+1)(q+2)}{2})$$

If we put

$$p(w,q) = (2w+4)(2w+4+\frac{(q+1)(q+2)}{2})$$

then we get

$$c_0 = 2w^2 + 10w + 4qw + 8q + 12 + p(w,q)$$

Similarly

$$
\begin{aligned}
c_1 &= (w+2)(w+2+2(q+1)) + (w+1)(w+1+2q) \\
&\quad +(1+2(q-1)) + p(w,q) \\
&= 2w^2 + 4qw + 8w + 8q + 8 + p(w,q) \\
c_2^1 &= (w+2)(w+2+2(q+1)) + w(w+2q) \\
&\quad +2(2+2(q-1)) + p(w,q) \\
&= 2w^2 + 4qw + 6w + 8q + 8 + p(w,q) \\
c_2^2 &= (w+1)(w+1+2(q+1)) + (w+1)(w+1+2q) + 1 + 2(q-1) \\
&\quad +1 + 2(q-2) + p(w,q) \\
&= 2w^2 + 4qw + 6w + 8q + p(w,q) \\
c_3 &= (w+1)(w+1+2(q+1)) + w(w+2q) + 2(2+2(q-1)) \\
&\quad +1 + 2(q-2) + p(w,q) \\
&= 2w^2 + 4qw + 4w + 8q + p(w,q) \\
c_4 &= w(w+2(q+1)) + w(w+2q) + 2(2+2(q-1)) \\
&\quad +2(2+2(q-2)) + p(w,q) \\
&= 2w^2 + 4qw + 2w + 8q - 4 + p(w,q)
\end{aligned}
$$

Hence the differences of Casimir eigenvalues $c_0 - c_i$ are

$$
\begin{pmatrix}
 & 0 & \\
 & -2w-4 & \\
-4w-4 & | & -4w-12 \\
 & -6w-12 & \\
 & -8w-16 &
\end{pmatrix}. \tag{3.4}
$$

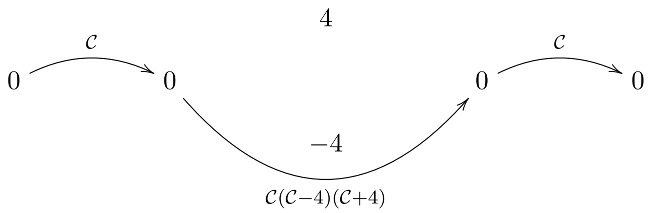
3.1.3. Operators obtained from the curved Casimir construction. As we can observe from (3.4), the coincidence of the Casimir eigenvalues in the top-slot and the bottom-slot happens for the weight $w = -2$. So this is the case when the construction yields an invariant operator between the respective bundles, as explained in section 1.4. Hence the right tractor bundle to consider is $\boxplus\mathcal{E}_\alpha[-2]$. It follows from (3.3) that its composition series has the form

$$\boxplus\mathcal{E}_\alpha[-2] =$$

$$\mathcal{E} \;\text{⊕}\; \mathcal{E}_A^{A'} \;\text{⊕}\; \begin{matrix} \boxminus\mathcal{E}_A[-1] \\ \oplus \\ \square\!\square\,\mathcal{E}^{A'} \otimes \square\!\square\,\mathcal{E}_A \end{matrix} \;\text{⊕}\; \mathcal{E}^{A'} \otimes \boxplus\mathcal{E}_A[-1] \;\text{⊕}\; \boxplus\mathcal{E}_A[-2] \tag{3.5}$$

Obviously, the curved Casimir operator acts by zero on the bundle of functions $\mathcal{E}$ which appears in the injecting slot and so we have $c_0 = 0$. Then it follows from (3.4) that the next Casimir eigenvalues are $c_1 = c_3 = c_4 = 0$ and $c_2^1 = 4$, $c_2^2 = -4$. Hence we observe that this case is highly degenerate in the sense that four out of six eigenvalues coincide. This indicates that the curved Casimir construction yields more invariant operators. In view of the composition series (3.5), the situaton can be schematically displayed as

(3.6)



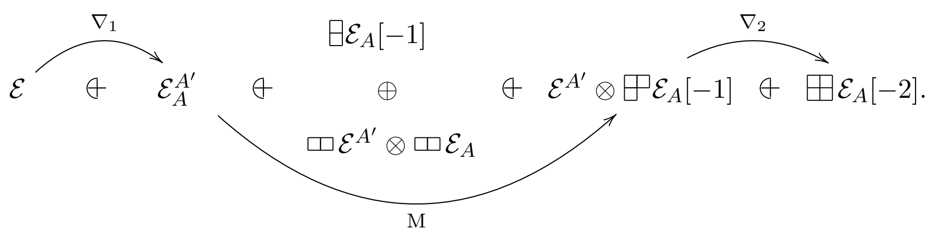
This shows that the curved Casimir operator $\mathcal{C}$ itself gives rise to invariant operators

$$\nabla_1 : \mathcal{E} \to \mathcal{E}_A^{A'} \quad \text{and} \quad \nabla_2 : \mathcal{E}^{A'} \otimes \boxplus\mathcal{E}_A[-1] \to \boxplus\mathcal{E}_A[-2],$$

and the composition $\mathcal{C}(\mathcal{C}-4)(\mathcal{C}+4)$ gives rise to a "middle" operator

$$\mathrm{M} : \mathcal{E}_A^{A'} \to \mathcal{E}^{A'} \otimes \boxplus\mathcal{E}_A[-1].$$

These induced invariant operators can be drawn as



By construction, the composition $\mathcal{C}^2(\mathcal{C}-4)(\mathcal{C}+4)$ induces invariant operators $\mathcal{E} \to \mathcal{E}^{A'} \otimes \boxplus\mathcal{E}_A[-1]$ and $\mathcal{E}_A^{A'} \to \boxplus\mathcal{E}_A[-2]$. It is easy to see that these operators are given by compositions $\mathrm{M}\nabla_1$ and $\nabla_2\mathrm{M}$ respectively. And finally, we also get an invariant operator $\mathcal{E} \to \boxplus\mathcal{E}_A[-2]$. It is induced by $\mathcal{C}^3(\mathcal{C}-4)(\mathcal{C}+4)$ and it is obviously given by the composition $\nabla_2\mathrm{M}\nabla_1$.

Although the construction guarantees that this operator is invariant, we do not know its order (or even whether it is non-zero) and so we do not know whether it is an analog of the non-standard operator. Before computing a full formula, it is helpful to look at the homogenous model. If we assume that the underlying manifold M is locally flat, then there is an easy general algorithm for computing formulae for the induced operators from the knowledge of the Casimir eigenvalues and the action of T^*M on the given tractor bundle. Applying this algorithm, we can easily find that both the compositions $\mathrm{M}\nabla_1$ and $\nabla_2\mathrm{M}$ vanish in the locally flat case and hence so does $\nabla_2\mathrm{M}\nabla_1$. We do not describe this algorithm here because we are going to compute the full formulae for ∇_1, ∇_2, M on a general (curved) manifold M in the course of this section and then the vanishing of $\mathrm{M}\nabla_1$ and $\nabla_2\mathrm{M}$ on flat manifolds will become obvious.

Hence we conclude that $\nabla_2\mathrm{M}\nabla_1$ is not the curved analog that we wanted to construct. Nevertheless, vanishing of $\mathrm{M}\nabla_1$ and $\nabla_2\mathrm{M}$ implies the existence of another invariant operator in the following sense. Let us consider once more the composition $\mathcal{C}^2(\mathcal{C}-4)(\mathcal{C}+4)$ of the curved Casimir operators acting on the tractor bundle $\ydiagram{2,2}\mathcal{E}_\alpha[-2]$. In view of the Casimir eigenvalues displayed in (3.6), it is easy to see that it acts trivially on the all exept the first two irreducible pieces of the composition series (3.5), and so the operator $\mathcal{C}^2(\mathcal{C}-4)(\mathcal{C}+4)$ descends to an invariant operator

$$\mathcal{E} \;\mathbin{\subset\!\!\!\!+}\; \mathcal{E}_A^{A'} \;\to\; \mathcal{E}^{A'} \otimes \ydiagram{2,1}\mathcal{E}_A[-1] \;\mathbin{\subset\!\!\!\!+}\; \ydiagram{2,2}\mathcal{E}_A[-2]\ .$$

Now the main observation is that the image in $\mathcal{E}^{A'}\otimes\ydiagram{2,1}\mathcal{E}_A[-1]$ by construction depends only on $\mathcal{E}$, and not on $\mathcal{E}_A^{A'}$. Concretely, choosing a Weyl structure σ, this operator is by definitions of ∇_1, ∇_2, M given by

$$\begin{pmatrix} f \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{M}\nabla_1(f) \\ \nabla_2\mathrm{M}(\mu) + \mathsf{D}_\sigma(f) \end{pmatrix}, \tag{3.7}$$

where D_σ is an operator $\mathcal{E} \to \ydiagram{2,2}\mathcal{E}_A[-2]$, which is not invariant in general. The subscript stresses that it depends on the choice of the Weyl structure σ. But in the same time, it is obvious that D_σ becomes invariant when the operators $\mathrm{M}\nabla_1$ and $\nabla_2\mathrm{M}$ happen to vanish. Indeed, the exact dependence of D_σ on a choice of σ is easy to compute. Let us consider a change of Weyl structure $\sigma(u) \mapsto \hat{\sigma}(u) = \sigma(u)\exp(\Upsilon(u))$. Then the invariance of (3.7) yields

$$\begin{pmatrix} \mathrm{M}\nabla_1(f) \\ \nabla_2\mathrm{M}(\mu - \Upsilon \bullet f) + \mathsf{D}_{\hat{\sigma}}(f) \end{pmatrix} = \begin{pmatrix} \mathrm{M}\nabla_1(f) \\ \nabla_2\mathrm{M}(\mu) + \mathsf{D}_\sigma(f) - \Upsilon \bullet \mathrm{M}\nabla_1(f) \end{pmatrix},$$

and from here we read off the transformation formula for D_σ:

$$\mathsf{D}_{\hat\sigma}(f) = \mathsf{D}_\sigma(f) - \Upsilon \bullet \mathrm{M}\nabla_1(f) + \nabla_2\mathrm{M}(\Upsilon \bullet f) \tag{3.8}$$

In particular, D_σ becomes an invariant operator D whenever the compositions $\mathrm{M}\nabla_1$ and $\nabla_2\mathrm{M}$ vanish. This is the case for a locally flat manifold and it turns out that then D coincides with the non-standard operator. In the rest of this section, we show that $\mathrm{M}\nabla_1$ and $\nabla_2\mathrm{M}$ vanish identically also in the torsion-free case and so we will prove the following

Theorem 3.2. *In the case of a manifold endowed with Grassmannian (i.e. torsion-free almost Grassmannian) structure, the action of $\mathcal{C}^2(\mathcal{C}-4)(\mathcal{C}+4)$ on the tractor bundle $\boxplus\mathcal{E}_\alpha[-2]$ gives rise to an invariant operator $\mathsf{D}:\mathcal{E}\to\boxplus\mathcal{E}_A[-2]$ which is a curved analog of the non-standard operator on functions.*

In the rest of this section, we compute formulae for the operators ∇_1, ∇_2, M and D_σ and we use them to prove this theorem.

3.1.4. Abstract formulae for ∇_1, ∇_2, M. We compute formulae for ∇_1, ∇_2, M in an abstract way first. This means that we do not use the explicit form of the action of T^*M. Let us display again the formula (1.8) for the curved Casimir operator

$$\mathcal{C}(s) = c(s) - 2\sum_\ell \varphi^\ell \bullet \nabla_{\psi_\ell} s - 2\sum_\ell \varphi^\ell \bullet \mathsf{P}(\psi_\ell)\bullet s.$$

For the sake of simplicity, we replace $\mathcal{C}$ by $-\frac{1}{2}\mathcal{C}$. Then we can write the formula for this modified Casimir operator in a simple shortened way as

$$\mathcal{C} = -\frac{1}{2}c. + \nabla\bullet + \mathsf{P}\bullet\bullet.$$

Now we consider its action on our tractor bundle $\boxplus\mathcal{E}_\alpha[-2]$. In the previous subsections, we have computed how this tractor bundle decomposes (3.5) and we computed the eigenvalues on the individual irreducible pieces, cf. 3.6. Although we have also computed the explicit form of the action $\bullet$, we do not use it now and we write the action of $\mathcal{C}$ on $\boxplus\mathcal{E}_\alpha[-2]$ analogous to the previous equation as

$$\mathcal{C}\begin{pmatrix} f \\ \mu \\ A \mid \alpha \\ \nu \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ \nabla\bullet f \\ -2A + (\nabla\bullet\mu)_1 + (\mathsf{P}\bullet\bullet f)_1 \;\mid\; 2\alpha + (\nabla\bullet\mu)_2 + (\mathsf{P}\bullet\bullet f)_2 \\ \nabla\bullet A + \nabla\bullet\alpha + \mathsf{P}\bullet\bullet\mu \\ \nabla\bullet\nu + \mathsf{P}\bullet\bullet A + \mathsf{P}\bullet\bullet\alpha \end{pmatrix} \tag{3.9}$$

where the brackets $()_1$ and $()_2$ denote the projections to the corresponding irreducible representations in the middle slot.

Now the operators ∇_1, ∇_2 are defined just by this action of $\mathcal{C}$ and so we get

$$\nabla_1(f) = \nabla \bullet f \quad \text{and} \quad \nabla_2(\nu) = \nabla \bullet \nu \tag{3.10}$$

The operator M is by definition induced by $\mathcal{C}(\mathcal{C}+2)(\mathcal{C}-2)$ (the eigenvalues are divided by -2 now) and since we can permute the factors in this composition freely, its action on μ equals

$$(\mathcal{C}-2)(\mathcal{C}+2)\mathcal{C}\begin{pmatrix} 0 \\ \mu \\ 0 \mid 0 \\ 0 \\ 0 \end{pmatrix} = (\mathcal{C}-2)(\mathcal{C}+2)\begin{pmatrix} 0 \\ 0 \\ (\nabla \bullet \mu)_1 \mid (\nabla \bullet \mu)_2 \\ \mathsf{P} \bullet \bullet \mu \\ 0 \end{pmatrix}$$

according to (3.9). Acting with the next factor, we get

$$(\mathcal{C}-2)\begin{pmatrix} 0 \\ 0 \\ 0 \mid 4(\nabla \bullet \mu)_2 \\ 2\mathsf{P} \bullet \bullet \mu + \nabla \bullet (\nabla \bullet \mu)_1 + \nabla \bullet (\nabla \bullet \mu)_2 \\ \nabla \bullet (\mathsf{P} \bullet \bullet \mu) + \mathsf{P} \bullet \bullet (\nabla \bullet \mu)_1 + \mathsf{P} \bullet \bullet (\nabla \bullet \mu)_2 \end{pmatrix}$$

and this equals

$$\begin{pmatrix} 0 \\ 0 \\ 0 \mid 0 \\ -4\mathsf{P} \bullet \bullet \mu - 2\nabla \bullet (\nabla \bullet \mu)_1 + 2\nabla \bullet (\nabla \bullet \mu)_2 \\ -2\mathsf{P} \bullet \bullet (\nabla \bullet \mu)_1 + 2\mathsf{P} \bullet \bullet (\nabla \bullet \mu)_2 + \nabla \bullet \nabla \bullet (\nabla \bullet \mu)_1 + \nabla \bullet \nabla \bullet (\nabla \bullet \mu)_2 \end{pmatrix}$$

where we used (3.9) again. Now the expression in the upper slot gives a formula for M. We divide it by the factor -2 for the sake of simplicity and so we get

$$\mathrm{M}(\mu) = \nabla \bullet (\nabla \bullet \mu)_1 - \nabla \bullet (\nabla \bullet \mu)_2 + 2\mathsf{P} \bullet \bullet \mu \tag{3.11}$$

3.1.5. Explicit formulae for ∇_1, ∇_2, M. Now we use the explicit form of the action $\bullet$ of T^*M on the tractor bundle $\mathcal{E}_{\boxplus(\alpha\beta\gamma\delta)}[-2]$ computed in (3.2) to make the formulae (3.10) and (3.11) for the operators ∇_1, ∇_2 and M explicit. From now on, we omit the brackets while expressing the action of these operators on sections. We write simply $\nabla_1 f$ instead of $\nabla_1(f)$, $\mathrm{M}\mu$ instead of $\mathrm{M}(\mu)$ etc.

The case of the operators ∇_1 and ∇_2 is very easy. We obtain the corresponding formulae directly from (3.2). Namely, we have

$$(\nabla_1 f)_A^{A'} = (\nabla \bullet f)_A^{A'} = -4\nabla_A^{A'}\sigma$$

for a function $f \in \mathcal{E}$ and

$$(\nabla_2\nu)_{ABCD} = (\nabla \bullet \nu)_{ABCD} = -\frac{1}{2}(\nabla^{A'}_{[A}\nu^{B'}_{|CD|B]} + \nabla^{A'}_{[C}\nu^{B'}_{|AB|D]})\epsilon_{A'B'}$$

for a section $\nu^{A'}_{ABC} = \nu^{A'}_{[AB]C} \in \mathcal{E}^{A'}_{\boxplus(ABC)}[-1]$. These expressions may look familiar. Indeed, they differ from the expressions for the exterior derivative from lemma 2.3 just by a scalar multiple. Hence we have proved the following

Proposition 3.3. *Up to a non-zero scalar multiple, the operators ∇_1, ∇_2 coincide with components of the exterior derivative.*

In order to make the formula (3.11) for the operator M explicit, we first compute what is the result of going from the second slot to the fourth one (counting from the top) along the two possible paths. Concretely, we compute $\varphi \bullet (\psi \bullet \mu)_1$ and $\varphi \bullet (\psi \bullet \mu)_2$.

Lemma 3.4. *Let $\varphi, \psi, \mu \in \mathcal{E}^{A'}_A$. Then the following holds*

$$\begin{aligned}
\varphi \bullet (\psi \bullet \mu)_1{}^{C'}_{ABC} &= (2\varphi^{A'}_{[A}\psi^{B'}_{B]}\mu^{C'}_C + 2\varphi^{A'}_{[A}\psi^{C'}_{|C|}\mu^{B'}_{B]} - \varphi^{C'}_C\psi^{A'}_{[A}\mu^{B'}_{B]} \\
&\quad +3\varphi^{C'}_{[C}\psi^{A'}_A\mu^{B'}_{B]})\epsilon_{A'B'} \\
\varphi \bullet (\psi \bullet \mu)_2{}^{C'}_{ABC} &= (3\varphi^{C'}_C\psi^{A'}_{[A}\mu^{B'}_{B]} - 3\varphi^{C'}_{[C}\psi^{A'}_A\mu^{B'}_{B]})\epsilon_{A'B'}
\end{aligned}$$

Proof. According to (3.2), the left-hand path gives

$$\begin{aligned}
\varphi \bullet (\psi \bullet \mu)_1{}^{C'}_{ABC} &= -2\varphi^{A'}_{[A}(\psi \bullet \mu)_1{}^{B'C'}_{B]C}\epsilon_{A'B'} \\
&= 2\varphi^{A'}_A\psi^{(B'}_{(B}\mu^{C')}_{C)}\epsilon_{A'B'} - 2\varphi^{A'}_B\psi^{(B'}_{(A}\mu^{C')}_{C)}\epsilon_{A'B'}
\end{aligned}$$

When we expand the symmetrizations and we keep the alternation over A and B, we obtain

$$\varphi \bullet (\psi \bullet \mu)_1{}^{C'}_{ABC} = (\varphi^{A'}_{[A}\psi^{B'}_{B]}\mu^{C'}_C + \varphi^{A'}_{[A}\psi^{C'}_{B]}\mu^{B'}_C + \varphi^{A'}_{[A}\psi^{B'}_{|C|}\mu^{C'}_{B]} + \varphi^{A'}_{[A}\psi^{C'}_{|C|}\mu^{B'}_{B]})\epsilon_{A'B'}$$

This can be written also in the following way

$$\begin{aligned}\varphi\bullet(\psi\bullet\mu)_{1}{}^{C'}_{ABC} &= (2\varphi^{A'}_{[A}\psi^{B'}_{B]}\mu^{C'}_{C}+2\varphi^{A'}_{[A}\psi^{[C'}_{B]}\mu^{B']}_{C}+2\varphi^{A'}_{[A}\psi^{[B'}_{|C|}\mu^{C']}_{B]}\\ &\quad +2\varphi^{A'}_{[A}\psi^{C'}_{|C|}\mu^{B'}_{B]})\epsilon_{A'B'}\end{aligned}$$

The two terms in the middle can be replaced by $2\varphi^{A'}_{C}\psi^{[B'}_{[A}\mu^{C']}_{B]}-6\varphi^{A'}_{[C}\psi^{[B'}_{A}\mu^{C']}_{B]}$ and contracting with $\epsilon_{A'B'}$ gives $(-\varphi^{C'}_{C}\psi^{[A'}_{[A}\mu^{B']}_{B]}+3\varphi^{C'}_{[C}\psi^{[A'}_{A}\mu^{B']}_{B]})\epsilon_{A'B'}$. Therefore, we get

$$\begin{aligned}\varphi\bullet(\psi\bullet\mu)_{1}{}^{C'}_{ABC} &= (2\varphi^{A'}_{[A}\psi^{B'}_{B]}\mu^{C'}_{C}+2\varphi^{A'}_{[A}\psi^{C'}_{|C|}\mu^{B'}_{B]}-\varphi^{C'}_{C}\psi^{A'}_{[A}\mu^{B'}_{B]}\\ &\quad +3\varphi^{C'}_{[C}\psi^{A'}_{A}\mu^{B'}_{B]})\epsilon_{A'B'}\end{aligned}$$

as we claimed. The right-hand path gives

$$\begin{aligned}&\varphi\bullet(\psi\bullet\mu)_{2}{}^{C'}_{ABC} = -2\varphi^{C'}_{C}(\psi\bullet\mu)_{2AB}-2\varphi^{C'}_{[B}(\psi\bullet\mu)_{2A]C}\\ &=2\varphi^{C'}_{C}\psi^{A'}_{[A}\mu^{B'}_{B]}\epsilon_{A'B'}+\varphi^{C'}_{B}\psi^{A'}_{[A}\mu^{B'}_{C]}\epsilon_{A'B'}-\varphi^{C'}_{A}\psi^{A'}_{[B}\mu^{B'}_{C]}\epsilon_{A'B'}\end{aligned}$$

The expansion of the alternations in the last two terms (and keeping the alternation over A and B) yields the following expression for the right-hand path

$$\varphi\bullet(\psi\bullet\mu)_{2}{}^{C'}_{ABC}=(2\varphi^{C'}_{C}\psi^{A'}_{[A}\mu^{B'}_{B]}+\varphi^{C'}_{[B}\psi^{A'}_{A]}\mu^{B'}_{C}-\varphi^{C'}_{[B}\psi^{A'}_{|C|}\mu^{B'}_{A]})\epsilon_{A'B'}$$

The last two terms can be replaced by $\varphi^{C'}_{C}\psi^{A'}_{[A}\mu^{B'}_{B]}-3\varphi^{C'}_{[C}\psi^{A'}_{A}\mu^{B'}_{B]}$ and thus we obtain

$$\varphi\bullet(\psi\bullet\mu)_{2}{}^{C'}_{ABC}=(3\varphi^{C'}_{C}\psi^{A'}_{[A}\mu^{B'}_{B]}-3\varphi^{C'}_{[C}\psi^{A'}_{A}\mu^{B'}_{B]})\epsilon_{A'B'}$$

□

From the second equation in the previous lemma we conclude

Lemma 3.5. *For any one-form μ the Rho-tensor satisfies*

$$(\mathsf{P}\bullet\bullet\mu)^{C'}_{ABC}=-2\partial\mathsf{P}(\mu)^{C'A'B'}_{C[AB]}\epsilon_{A'B'}+2\partial\mathsf{P}(\mu)^{C'A'B'}_{[CAB]}\epsilon_{A'B'}.$$

Proof. The action of P on μ is defined by $\mathsf{P}\bullet\bullet\mu=\sum_\ell\varphi^\ell\bullet\mathsf{P}(\psi_\ell)\bullet\mu=\sum_\ell(\varphi^\ell\bullet(\mathsf{P}(\psi_\ell)\bullet\mu)_1+\varphi^\ell\bullet(\mathsf{P}(\psi_\ell)\bullet\mu)_2)$. So this term is given by the sum of the two possible paths and thus by lemma 3.4 we get

$$(\mathsf{P}\bullet\bullet\mu)^{C'}_{ABC}=2(\mathsf{P}^{A'B'}_{[AB]}\mu^{C'}_{C}+\mathsf{P}^{A'C'}_{[A|C|}\mu^{B'}_{B]}+\mathsf{P}^{C'A'}_{C[A}\mu^{B'}_{B]})\epsilon_{A'B'}. \tag{3.12}$$

On the other hand, $\partial\mathsf{P}$ viewed as a two-form with values in endomorphisms of the tangent bundle has been computed in (2.6). From this equation, we deduce that the value of $\partial\mathsf{P}$ on the one-form μ is equal to

$$\partial\mathsf{P}(\mu)^{C'A'B'}_{C\,A\,B}=-\mathsf{P}^{A'B'}_{A\,C}\mu^{C'}_{B}+\mathsf{P}^{C'B'}_{C\,A}\mu^{A'}_{B}-\mathsf{P}^{A'C'}_{A\,B}\mu^{B'}_{C}+\mathsf{P}^{C'A'}_{C\,B}\mu^{B'}_{A}. \tag{3.13}$$

Alternating indices A and B and contracting A' and B' immediately yiekds

$$\partial \mathsf{P}(\mu)_{C[AB]}^{C'A'B'}\epsilon_{A'B'} = (-\mathsf{P}_{[A|C|}^{A'B'}\mu_{B]}^{C'} + \mathsf{P}_{C[A}^{C'B'}\mu_{B]}^{A'} - \mathsf{P}_{[AB]}^{A'C'}\mu_{C}^{B'} + \mathsf{P}_{C[B}^{C'A'}\mu_{A]}^{B'})\epsilon_{A'B'}.$$

Since the alternation over three primed indices vanishes, we observe that the second term on the right-hand side can be written as

$$\mathsf{P}_{C[A}^{C'B'}\mu_{B]}^{A'}\epsilon_{A'B'} = \mathsf{P}_{C[A}^{B'C'}\mu_{B]}^{A'}\epsilon_{A'B'} + \mathsf{P}_{C[A}^{A'B'}\mu_{B]}^{C'}\epsilon_{A'B'}$$

Replacing the term by this expression we get

$$\partial \mathsf{P}(\mu)_{C[AB]}^{C'A'B'}\epsilon_{A'B'} = (-\mathsf{P}_{[A|C|}^{A'B'}\mu_{B]}^{C'} + \mathsf{P}_{C[A}^{B'C'}\mu_{B]}^{A'} + \mathsf{P}_{C[A}^{A'B'}\mu_{B]}^{C'} - \mathsf{P}_{[AB]}^{A'C'}\mu_{C}^{B'} + \mathsf{P}_{C[B}^{C'A'}\mu_{A]}^{B'})\epsilon_{A'B'}.$$

Now the sum of the first and the third term on the right can be written as

$$-\mathsf{P}_{[A|C|}^{A'B'}\mu_{B]}^{C'}\epsilon_{A'B'} + \mathsf{P}_{C[A}^{A'B'}\mu_{B]}^{C'}\epsilon_{A'B'} = -\mathsf{P}_{[AB]}^{A'B'}\mu_{C}^{C'}\epsilon_{A'B'} + 3\mathsf{P}_{[AB}^{A'B'}\mu_{C]}^{C'}\epsilon_{A'B'},$$

and similarly

$$\mathsf{P}_{C[A}^{B'C'}\mu_{B]}^{A'}\epsilon_{A'B'} - \mathsf{P}_{[AB]}^{A'C'}\mu_{C}^{B'}\epsilon_{A'B'} = -\mathsf{P}_{[A|C|}^{A'C'}\mu_{B]}^{B'}\epsilon_{A'B'} + 3\mathsf{P}_{[AC}^{A'C'}\mu_{B]}^{B'}\epsilon_{A'B'}.$$

The two terms skew-symmetric in all unprimend indices in the two previous equations sum up to $-3\mathsf{P}_{[CA}^{C'A'}\mu_{B]}^{B'}\epsilon_{A'B'}$, and thus we obtain

$$\partial \mathsf{P}(\mu)_{C[AB]}^{C'A'B'}\epsilon_{A'B'} = (-\mathsf{P}_{[AB]}^{A'B'}\mu_{C}^{C'} - \mathsf{P}_{[A|C|}^{A'C'}\mu_{B]}^{B'} - \mathsf{P}_{C[A}^{C'A'}\mu_{B]}^{B'} - 3\mathsf{P}_{[CA}^{C'A'}\mu_{B]}^{B'}\epsilon_{A'B'})\epsilon_{A'B'}.$$

Taking next the alternation over all unprimed indices in (3.13) yields

$$\partial \mathsf{P}(\mu)_{[CAB]}^{C'A'B'}\epsilon_{A'B'} = (-\mathsf{P}_{[A\ C}^{A'B'}\mu_{B]}^{C'} + \mathsf{P}_{[CA}^{C'B'}\mu_{B]}^{A'} - \mathsf{P}_{[AB}^{A'C'}\mu_{C]}^{B'} + \mathsf{P}_{[CB}^{C'A'}\mu_{A]}^{B'})\epsilon_{A'B'}.$$

Since the alternation over three primed indices vanishes, the first and the third term on the right-hand side sum up to $-\mathsf{P}_{[CA}^{C'A'}\mu_{B]}^{B'}\epsilon_{A'B'}$, so we conclude

$$\partial \mathsf{P}(\mu)_{[CAB]}^{C'A'B'}\epsilon_{A'B'} = -3\mathsf{P}_{[CA}^{C'A'}\mu_{B]}^{B'}\epsilon_{A'B'}.$$

Hence the right-hand side of the equation in the lemma is

$$-2\partial \mathsf{P}(\mu)_{C[AB]}^{C'A'B'}\epsilon_{A'B'} + 2\partial \mathsf{P}(\mu)_{[CAB]}^{C'A'B'}\epsilon_{A'B'} = (2\mathsf{P}_{[AB]}^{A'B'}\mu_{C}^{C'} + 2\mathsf{P}_{[A|C|}^{A'C'}\mu_{B]}^{B'} + 2\mathsf{P}_{C[A}^{C'A'}\mu_{B]}^{B'})\epsilon_{A'B'},$$

and this is equal to the formula (3.12) for $\mathsf{P} \bullet \bullet \mu$. □

The latter two lemmas motivate a simplier notation which makes the formulae similar to formulae known from conformal geometry. Namely, we may view $\mathcal{E}^{A'} \otimes \boxplus\mathcal{E}_A[-1]$ as the image of the bundle map $\Pi_{\boxplus} : \mathcal{E}_{abc} \to \mathcal{E}_{\boxplus(ABC)}^{A'}[-1]$ defined by

$$\Pi_{\boxplus}(s)_{ABC}^{C'} := (s_{[AB]C}^{A'B'C'} - s_{[ABC]}^{A'B'C'})\epsilon_{A'B'}, \tag{3.14}$$

and represent each section $s_{ABC}^{A'} \in \mathcal{E}_{\boxplus(ABC)}^{A'}[-1]$ by its preimage $s_{abc} \in \otimes^3\mathcal{E}_a$, which we denote by the same symbol but with tensor indices. Of course, this correspondence is not bijective. For two sections s^1_{abc} and s^2_{abc} of $\mathcal{E}_{abc}$ which have the same image under the map $\Pi_{\boxplus}$ and thus represent the same section of $\mathcal{E}^{A'}_{\boxplus(ABC)}[-1]$, we use the notation $s^1_{abc} \equiv s^2_{abc}$ modulo $\mathrm{Ker}(\Pi_{\boxplus})$. The map $\Pi_{\boxplus}$ can be thought of as a projection to a subbundle. Indeed, it is a composition of the projection from $\otimes^3\mathcal{E}_A$ to our realization of $\boxplus\mathcal{E}_A$ as the kernel of Alt : $\mathcal{E}_{[AB]C} \to \mathcal{E}_{[ABC]}$ with a contraction of two primed indices such that it factorizes through $\mathcal{E}_{(ab)c}$. It is obvious that it coincides with the contraction with the conformal metric $g_{ab} := \epsilon_{A'B'}\epsilon^{AB}$ in the case of a four-dimensional manifold ($q = 2$). Hence the projection $\Pi_{\boxplus}$ can be viewed as a replacement for that. Using this projection we may reformulate the previous two lemmas as follows.

Lemma 3.6. *The following equations hold modulo* $\mathrm{Ker}(\Pi_{\boxplus})$*:*

$$(\varphi \bullet (\psi \bullet \mu)_1)_{abc} \equiv 2\varphi_a\psi_b\mu_c + 2\varphi_a\psi_c\mu_b - \varphi_c\psi_a\mu_b,$$
$$(\varphi \bullet (\psi \bullet \mu)_2)_{abc} \equiv 3\varphi_c\psi_a\mu_b,$$
$$(\mathsf{P} \bullet \bullet \mu)_{abc} \equiv -2\partial\mathsf{P}(\mu)_{cab} \equiv 2\mathsf{P}_{ab}\mu_c + 2\mathsf{P}_{ac}\mu_b + 2\mathsf{P}_{ca}\mu_b. \tag{3.15}$$

PROOF. By definition (3.14) of projection $\Pi_{\boxplus}$, the image of the riht-hand side of the first equation under $\Pi_{\boxplus}$ is equal to

$$(2\varphi^{A'}_{[A}\psi^{B'}_{B]}\mu^{C'}_{C} + 2\varphi^{A'}_{[A}\psi^{C'}_{|C|}\mu^{B'}_{B]} - \varphi^{C'}_{C}\psi^{A'}_{[A}\mu^{B'}_{B]})\epsilon_{A'B'}$$
$$- (2\varphi^{A'}_{[A}\psi^{B'}_{B}\mu^{C'}_{C]} + 2\varphi^{A'}_{[A}\psi^{C'}_{C}\mu^{B'}_{B]} - \varphi^{C'}_{[C}\psi^{A'}_{A}\mu^{B'}_{B]})\epsilon_{A'B'}.$$

By lemma 3.4, this is evidently equal to $\varphi \bullet (\psi \bullet \mu)_1{}^{C'}_{ABC}$ modulo terms in $\mathcal{E}^{C'}_{[ABC]}$. And since both terms lie by construction in the kernel of the complete alterantion of unprimed indices, they are equal. One can also check directly that the second line in the previous expression sum up to $3\varphi^{C'}_{[C}\psi^{A'}_{A}\mu^{B'}_{B]}\epsilon_{A'B'}$. Analogously, the second and the third equation can be derived directly from lemma 3.4 respectively lemma 3.5 (and equation (3.12) in the proof) by use of the definition of the map $\Pi_{\boxplus}$. □

In sequel, we will need the expression for the difference and the sum of the two paths from the previous lemma since exactly these terms appear in the formula (3.11) for the operator M. One verifies directly from the lemma that modulo $\mathrm{Ker}(\Pi_{\boxplus})$ we have

$$(\varphi \bullet (\psi \bullet \mu)_1 - \varphi \bullet (\psi \bullet \mu)_2)_{abc} \equiv 8\varphi_{[a}\psi_{c]}\mu_b + 4\varphi_a\psi_{[b}\mu_{c]} \tag{3.16}$$
$$(\varphi \bullet (\psi \bullet \mu)_1 + \varphi \bullet (\psi \bullet \mu)_2)_{abc} \equiv 2\varphi_a\psi_c\mu_b + 2\varphi_a\psi_b\mu_c + 2\varphi_c\psi_a\mu_b. \tag{3.17}$$

.

Using the latter lemma we immediately obtain an explicit form of the equation for the invariant operator M. Recall that U denotes the Weyl curvature defined in section 1.3.4.

Proposition 3.7. *Up to a non-zero scalar multiple, the operator* M *is given by the composition* $\mathrm{M} = \Pi_{\boxplus} \circ \tilde{\mathrm{M}}$*, where*

$$(\tilde{\mathrm{M}}\mu)_{abc} = \nabla_a \nabla_{[b}\mu_{c]} + U_{acb}{}^e \mu_e - T_{ac}{}^e \nabla_e \mu_b. \tag{3.18}$$

Proof. It is a direct consequence of equations (3.15) and (3.16) that the formula (3.11) for the operator M can be written as $\Pi_{\boxplus}$ applied to

$$(\tilde{\mathrm{M}}\mu)_{abc} = 8\nabla_{[a}\nabla_{c]}\mu_b + 4\nabla_a\nabla_{[b}\mu_{c]} + 4\partial \mathsf{P}(\mu)_{acb}.$$

Now we use Ricci identity to the first term and we get

$$(\tilde{\mathrm{M}}\mu)_{abc} = 4(R_{acb}{}^e\mu_e - T_{ac}{}^e\nabla_e\mu_b + \nabla_a\nabla_{[b}\mu_{c]} + \partial\mathsf{P}(\mu)_{acb}).$$

But $R + \partial\mathsf{P}$ is the Weyl curvature U and the result follows. □

Formula (3.18) for the operator M simplifies in the torsion-free case. Namely, we show in this case that M is a composition of components of the exterior derivative, and hence coincides with the standard operator between corresponding bundles in the split de Rham resolution.

Corollary 3.8. *In the case that the torsion of* ∇ *vanishes, the operator* M *coincides with the composition* $dd_{\square\square}$ *of components of the exterior derivative up to a non-zero scalar multiple.*

Proof. Let us look first at the lower order terms in the formula (3.18) for the operator $\tilde{\mathrm{M}}$. The term involving torsion vanishes by assumption. The term involving Weyl curvature is non-zero but it vanishes when projected to $\mathcal{E}^{A'} \otimes \boxplus \mathcal{E}_A[-1]$. This can be explained as follows. According to corollary A.3 in appendix, vanishing of the torsion of ∇ implies that the Weyl curvature $U_{acb}{}^e$ is irreducible, and equal to

$$U_{acb}{}^e = \varphi_{A\ C\ B}{}^E \epsilon^{A'C'}\delta^{B'}_{E'},$$

where $\varphi_{A\ C\ B}{}^E \in \mathcal{E}^E_{(ACB)_0}[-1]$. Hence we have $U_{acb}{}^e\mu_e = \varphi_{A\ C\ B}{}^E \mu^{B'}_E \epsilon^{A'C'}$, and this is mapped to zero when projected under $\Pi_{\boxplus}$ to $\mathcal{E}^{A'}_{\boxplus(ABC)}[-1]$. Thus the operator M is given simply by the projection $\Pi_{\boxplus}$ of $\nabla_a\nabla_{[b}\mu_{c]} = \nabla_a(d\mu)_{bc}$. On the other hand, we have from lemma 2.3 the following formula for the exterior derivative

$$(dd_{\square\square}\mu)^{A'}_{ABC} = \frac{3}{2}\nabla^{A'}_{[A}(d_{\square\square}\mu)_{B]C}.$$

In the torsion-free case, by the same lemma we also have

$$(d_{\boxplus}d_{\boxminus}\mu)^{A'}_{ABC} = \frac{3}{2}(\nabla^{C'}_{C}(d_{\boxminus}\mu)^{A'B'}_{A\ B} - \nabla^{C'}_{[C}(d_{\boxminus}\mu)^{A'B'}_{A\ B]})\epsilon_{B'C'}.$$

These formulae can be rewritten as

$$
\begin{aligned}
(dd_{\square\square}\mu)^{C'}_{ABC} &= \tfrac{3}{2}\nabla^{C'}_{[A}(d_{\square\square}\mu)^{A'B'}_{B]C}\,\epsilon_{A'B'} = -3\nabla^{A'}_{[A}(d_{\square\square}\mu)^{B'C'}_{B]C}\,\epsilon_{A'B'} \\
(d_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mu)^{C'}_{ABC} &= \tfrac{3}{2}(\nabla^{B'}_{C}(d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mu)^{C'A'}_{A\ B} - \nabla^{B'}_{[C}(d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mu)^{C'A'}_{A\ B]})\epsilon_{A'B'} \\
&= 3(\nabla^{A'}_{A}(d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mu)^{B'C'}_{B\ C} - \nabla^{A'}_{[A}(d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mu)^{B'C'}_{B\ C]})\epsilon_{A'B'}
\end{aligned}
$$

and so we see that the equations for exterior derivatives can be written with the help of the projection $\Pi_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}$ simply as

$$dd_{\square\square}\mu = -3\Pi_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}(\nabla d_{\square\square}\mu)$$

$$d_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mu = 3\Pi_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}(\nabla d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\mu).$$

And since $d = d_{\square\square} + d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$, we get $dd_{\square\square} - d_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = -3\Pi_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}} \circ \nabla d = -3\mathrm{M}$, which shows that M is given by the difference of the two possible paths in the De-Rham sequence from one-forms to three-forms. Since $d^2 = 0$, these two paths give the same operator up to the sign and hence we end up with

$$\mathrm{M} = -\frac{2}{3}dd_{\square\square} = \frac{2}{3}d_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

$\square$

Now it is a straightforward consequence of this corollary and the proposition 3.3 that the compositions $\mathrm{M}\nabla_1$ and $\nabla_2\mathrm{M}$ vanish. Indeed, $\mathrm{M}\nabla_1 = dd_{\square\square}d = 0$ since $d_{\square\square}d$ is the projection of d^2 to $\square\square\mathcal{E}_A[-1]$ and $\nabla_2\mathrm{M} = d_{\boxplus}dd_{\square\square} = 0$ since $d_{\boxplus}d$ is the projection of d^2 to $\boxplus\mathcal{E}_A[-2]$. So the existence of the invariant operator D from theorem 3.2 is proved.

3.1.6. A formula for the operator D. In order to complete the proof of theorem 3.2, we need to show that, in the case of a locally flat manifold M, the operator D coincides with the non-standard operator on functions. This becomes obvious after we find an explicit formula for D. By definition, D is induced by an action of $\mathcal{C}^2(\mathcal{C}-4)(\mathcal{C}+4)$ on the tractor bundle $\boxplus\mathcal{E}_\alpha[-2]$ in the torsion-free case, see 3.1.3. Hence we need to express this operator formed by curved Casimirs in terms of data associated to a Weyl structure. We first use formula (3.9) for the curved Casimir operator to obtain an abstract formula for D in terms of the action $\bullet$ of T^*M on the tractor bundle $\boxplus\mathcal{E}_\alpha$, whose explicit form is given by equation (3.2) and then we make this formula explicit.

Lemma 3.9. *In terms of the action $\bullet$, we have*

$$
\begin{aligned}
\mathsf{D}(f) = {}& \nabla\bullet\nabla\bullet(\nabla\bullet\nabla\bullet f)_1 + \nabla\bullet\nabla\bullet(\nabla\bullet\nabla\bullet f)_2 \\
& - 2\mathsf{P}\bullet\bullet(\nabla\bullet\nabla\bullet f)_1 + 2\mathsf{P}\bullet\bullet(\nabla\bullet\nabla\bullet f)_2 \\
& - 2\nabla\bullet\nabla\bullet(\mathsf{P}\bullet\bullet f)_1 + 2\nabla\bullet\nabla\bullet(\mathsf{P}\bullet\bullet f)_2
\end{aligned}
\tag{3.19}
$$

Proof. The operator D is defined by equation (3.7) for the action of $\mathcal{C}^2(\mathcal{C}-4)(\mathcal{C}+4)$ on a section of the tractor bundle $\boxplus\mathcal{E}_\alpha[-2]$. By this equation, a formula for D is obtained by acting on a section that has a non-zero injecting slot and the all other slots zero. Using formula (3.9) for the curved Casimir operator (divided by -2) and the fact that individual factors of the composition of curved Casimirs commute, we get

$$\mathcal{C}^2(\mathcal{C}+2)(\mathcal{C}-2)\begin{pmatrix} f \\ 0 \\ 0 \mid 0 \\ 0 \\ 0 \end{pmatrix} = \mathcal{C}(\mathcal{C}-2)(\mathcal{C}+2)\begin{pmatrix} 0 \\ \nabla \bullet f \\ (\mathsf{P}\bullet\bullet f)_1 \mid (\mathsf{P}\bullet\bullet f)_2 \\ 0 \\ 0 \end{pmatrix}.$$

We have already computed the action of $(\mathcal{C}-2)(\mathcal{C}+2)\mathcal{C}$ on a section with a one-form μ in the second slot from the top and zero in the other slots while computing the abstract formula for the operator M in the section 3.1.4. There we obtained $\mathrm{M}(\mu)$ in the slot above the bottom, and in the bottom-slot we obtained the following formula

$$-2\mathsf{P}\bullet\bullet(\nabla\bullet\mu)_1 + 2\mathsf{P}\bullet\bullet(\nabla\bullet\mu)_2 + \nabla\bullet\nabla\bullet(\nabla\bullet\mu)_1 + \nabla\bullet\nabla\bullet(\nabla\bullet\mu)_2.$$

Hence inserting $\mu = \nabla\bullet f$ into this formula, we get the formula appearing in the bottom slot of $\mathcal{C}(\mathcal{C}-2)(\mathcal{C}+2)(\nabla\bullet f)$. Explicitly, this formula reads

$$-2\mathsf{P}\bullet\bullet(\nabla\bullet\nabla\bullet f)_1 + 2\mathsf{P}\bullet\bullet(\nabla\bullet\nabla\bullet f)_2 + \nabla\bullet\nabla\bullet(\nabla\bullet\nabla\bullet f)_1 + \nabla\bullet\nabla\bullet(\nabla\bullet\nabla\bullet f)_2.$$

These terms are already four terms from the formula (3.19) and thus we only need to show that the action of $\mathcal{C}(\mathcal{C}-2)(\mathcal{C}+2)$ on the middle slot in the equation above yields the remaining two terms. Applying the factor $\mathcal{C}+2$ first, by (3.9) we get

$$\mathcal{C}(\mathcal{C}-2)(\mathcal{C}+2)\begin{pmatrix} 0 \\ 0 \\ (\mathsf{P}\bullet\bullet f)_1 \ \mid \ (\mathsf{P}\bullet\bullet f)_2 \\ 0 \\ 0 \end{pmatrix}$$

$$=\mathcal{C}(\mathcal{C}-2)\begin{pmatrix} 0 \\ 0 \\ 0 \ \mid \ 4(\mathsf{P}\bullet\bullet f)_2 \\ \nabla\bullet(\mathsf{P}\bullet\bullet f)_1+\nabla\bullet(\mathsf{P}\bullet\bullet f)_2 \\ \mathsf{P}\bullet\bullet(\mathsf{P}\bullet\bullet f)_1+\mathsf{P}\bullet\bullet(\mathsf{P}\bullet\bullet f)_2 \end{pmatrix}.$$

Now we apply $\mathcal{C}-2$ and we get zeros everywhere up to the two slots in the bottom. We do not need to care about the bottom slot since this will be killed by the remaining factor $\mathcal{C}$. The slot above the bottom equals

$$-2\nabla\bullet(\mathsf{P}\bullet\bullet f)_1+2\nabla\bullet(\mathsf{P}\bullet\bullet f)_2$$

and after an application of $\mathcal{C}$, we get in the bottom slot

$$-2\nabla\bullet\nabla\bullet(\mathsf{P}\bullet\bullet f)_1+2\nabla\bullet\nabla\bullet(\mathsf{P}\bullet\bullet f)_2.$$

□

Having abstract formula (3.19) for D, it suffices to use equation (3.2) for the action $\bullet$ to make formula (3.19) explicit. But a straightforward use of this equation leads to complicated formulae with many indices. In order to reduce their number, we are going to find an explicit formula in a compact form, which is similar to the formula (3.18) for M. Namely, we will write $\mathsf{D}_\sigma=\Pi_\boxplus\circ\tilde{\mathsf{D}}_\sigma$ where $\tilde{\mathsf{D}}_\sigma$ is a (non-invariant) operator $\mathcal{E}\to\otimes^4\mathcal{E}_a$ and $\Pi_\boxplus$ is a projection $\otimes^4\mathcal{E}_a\to\boxplus\mathcal{E}_A[-2]$. It turns out that a suitable projection is the one defined for any $\rho\in\otimes^4\mathcal{E}_a$ by

$$\Pi_\boxplus(\rho)_{ABCD}=(\frac{1}{2}\rho^{A'B'C'D'}_{[AB][CD]}+\frac{1}{2}\rho^{A'B'C'D'}_{[CD][AB]}-\rho^{A'B'C'D'}_{[A\ B\ C\ D]})\epsilon_{A'B'}\epsilon_{C'D'}. \quad (3.20)$$

It is easy to see that the map $\Pi_\boxplus$ is a combination of the projection of $\mathcal{E}_{ABCD}$ to $\mathcal{E}_{\boxplus(ABCD)}$, realized as the the kernel of Alt : $\mathcal{E}_{[AB]}\vee\mathcal{E}_{[CD]}\to\mathcal{E}_{[ABCD]}$, with a contraction of primed indices such that the resulting map factors through $\mathcal{E}_{(ab)}\vee\mathcal{E}_{(cd)}$. In the case of a four-dimensional manifold M, this map obviously coincides with the contraction of two pairs of indices with conformal metric. Hence $\Pi_\boxplus$ may be viewed as a replacement of that.

Similarly to the case of the map $\Pi_{\yng(2,1)}$, for two sections $\rho_1, \rho_2 \in \otimes^4 \mathcal{E}_a$ that have the same image under $\Pi_{\boxplus}$ we write $\rho_1 \equiv \rho_2 \mod \operatorname{Ker}(\Pi_{\boxplus})$. The following lemma summarizes basic properties of the projection $\Pi_{\boxplus}$, which will be used systematically throughout the text.

Lemma 3.10. *Let $\varphi \in \mathcal{E}_a$, $\nu \in \otimes^3 \mathcal{E}_a$ and $\rho \in \otimes^4 \mathcal{E}_a$. Then the following identities hold modulo* $\operatorname{Ker}(\Pi_{\boxplus})$.

(1) $\varphi \bullet \Pi_{\yng(2,1)}(\nu) = -\Pi_{\boxplus}(\nu \otimes \varphi)$, *in particular* $\varphi_d \nu_{abc} \equiv 0$ *if* $\Pi_{\yng(2,1)}(\nu) = 0$,

(2) $\rho_{abcd} \equiv \rho_{bacd} \equiv \rho_{abdc} \equiv \rho_{cdab}$,

(3) $\rho_{a(bc)d} \equiv \rho_{(a|bc|d)}$, $\rho_{a[bc]d} \equiv \rho_{[a|bc|d]}$,

(4) $\rho_{[A\ B\ C]D}{}^{A'B'C'D'} \equiv \rho_{[A\ B\ C\ D]}{}^{A'B'C'D'} \equiv 0$.

(5) $2\rho^{[A'C']B'D'}{}_{A\ B\ C\ D} \equiv \rho_{A\ B\ C\ D}^{A'B'C'D'} \equiv 2\rho_{[A\ C]\ B\ D}^{A'B'C'D'}$

Proof. (1): By equation (3.2) for the action $\bullet$, we have

$$\varphi \bullet \Pi_{\yng(2,1)}(\nu)_{ABCD} = -\frac{1}{2}(\Pi_{\yng(2,1)}(\nu)_{AB[C}^{A'}\varphi_{D]}^{B'} + \Pi_{\yng(2,1)}(\nu)_{CD[A}^{A'}\varphi_{B]}^{B'})\epsilon_{A'B'},$$

and this is by definition (3.14) of the projection $\Pi_{\yng(2,1)}$ equal to

$$\begin{aligned}\varphi \bullet \Pi_{\yng(2,1)}(\nu)_{ABCD} = &-\tfrac{1}{2}\nu_{[AB][C}^{E'F'A'}\varphi_{D]}^{B'}\epsilon_{E'F'}\epsilon_{A'B'} - \tfrac{1}{2}\nu_{[CD][A}^{E'F'A'}\varphi_{B]}^{B'}\epsilon_{E'F'}\epsilon_{A'B'} \\ &+ \tfrac{1}{4}(\nu_{[ABC]}^{E'F'A'}\varphi_D^{B'} - \nu_{[ABD]}^{E'F'A'}\varphi_C^{B'} + \nu_{[CDA]}^{E'F'A'}\varphi_B^{B'} - \nu_{[CDB]}^{E'F'A'}\varphi_A^{B'})\epsilon_{E'F'}\epsilon_{A'B'}.\end{aligned}$$

The formula in the second line is evidently skew-symmetric in all unprimed indices. It can be written as $\nu_{[ABC}^{E'F'A'}\varphi_{D]}^{B'}\epsilon_{E'F'}\epsilon_{A'B'}$, and thus the formula for $\varphi \bullet \Pi_{\yng(2,1)}(\nu)$ up to the sign coincides with the defining formula (3.20) for $\Pi_{\boxplus}$ applied on $\nu \otimes \varphi$. The consequence of the first equation says in other words that the projection $\Pi_{\boxplus}$ factorizes through $\Pi_{\yng(2,1)} \otimes \mathrm{id}$.

(2): The statement says that $\Pi_{\boxplus}$ factorizes also through $\mathcal{E}_{(ab)} \vee \mathcal{E}_{(cd)}$ which follows directly from the definition (3.20) of the map $\Pi_{\boxplus}$.

(3): The claim can be easily shown from (2). Namely, we compute

$$\rho_{a(bc)d} \equiv \frac{1}{2}\rho_{abcd} + \frac{1}{2}\rho_{acbd} \equiv \frac{1}{2}\rho_{abcd} + \frac{1}{2}\rho_{cadb} \equiv \frac{1}{2}\rho_{abcd} + \frac{1}{2}\rho_{dbca} \equiv \rho_{(a|bc|d)},$$

and one can analogously verify the second equation.

(4): Follows directly from definition (3.20) of $\Pi_{\boxplus}$. It can be also deduced from the fact that $\Pi_{\boxplus}$ is the projection on irreducible bundle $\mathcal{E}_{\boxplus(ABCD)}$ and this bundle does not appear in the decomposition of the tensor product $\mathcal{E}_{[ABC]} \otimes \mathcal{E}_D$.

(5): By definition, $\Pi_{\boxplus}$ acts on ρ_{abcd} by contracting A' and B' and alternating over A and B. On the other hand, since the alternation over A', B' and C' vanishes, we have

$$2\rho^{[A'C']B'D'}{}_{A\ B\ C\ D}\,\epsilon_{A'B'} = \rho_{A\ B\ C\ D}^{A'B'C'D'}.$$

And since the alternation over indices A and B of $2\rho^{A'B'C'D'}_{[A\ C]\ B\ D}$ can be written as

$$\rho^{A'B'C'D'}_{[A|C|B]\ D} - \rho^{A'B'C'D'}_{C[A\ B]D} = \rho^{A'B'C'D'}_{[A\ B]C\ D} - 3\rho^{A'B'C'D'}_{[A\ B\ C]D},$$

the result follows by applying (4). $\square$

Now we use projections $\Pi_{\boxplus}$ and $\Pi_{\boxplus}$ and their basic properties summarized in the previous lemmas to make the abstract formula (3.19) for D explicit. Denoting by $\mathsf{S}_{ab} := \mathsf{P}_{(ab)}$ the symmetrization of Rho-tensor, we get

Proposition 3.11. *The invariant operator* D *induced by curved Casimirs in the sense of equation* (3.7) *is given by projection* $\Pi_{\boxplus}$ *of*

$$(\tilde{\mathsf{D}}f)_{abcd} \equiv 3\nabla_{(abcd)}f - 8\mathsf{S}_{ad}\nabla_{bc}f + 8\mathsf{S}_{cd}\nabla_{ab}f \qquad (3.21)$$

Proof. Let us start with the leading part of equation (3.19) defining operator D. We see that it is given by a sum of the two possible paths from the top slot to the bottom slot. Namely, it is given by an action of $\nabla\bullet$ on $\nu := \nabla\bullet(\nabla\bullet\nabla\bullet f)_1 + \nabla\bullet(\nabla\bullet\nabla\bullet f)_2$. By equation (3.17), this sum satisfies

$$\nu_{abc} \equiv 2\nabla_a\nabla_b(\nabla\bullet f)_c + 2\nabla_a\nabla_c(\nabla\bullet f)_b + 2\nabla_c\nabla_a(\nabla\bullet f)_b$$

modulo $\mathrm{Ker}(\Pi_{\boxplus})$. Since $(\nabla\bullet f)_a = -4\nabla_a f$ according to (3.2), this is equal to

$$\nu_{abc} \equiv -8(\nabla_a\nabla_b\nabla_c f + \nabla_a\nabla_c\nabla_b f + \nabla_c\nabla_a\nabla_b f).$$

Now the leading part is given by an action of $\nabla\bullet$ on this tensor. According to lemma 3.10, we have $(\nabla\bullet\nu)_{abcd} \equiv -\nabla_d\nu_{abc}$ modulo $\mathrm{Ker}(\Pi_{\boxplus})$ and so the leading part of D is given by projection $\Pi_{\boxplus}$ of

$$(\nabla\bullet\nu)_{abcd} \equiv 8(\nabla_d\nabla_a\nabla_b\nabla_c f + \nabla_d\nabla_a\nabla_c\nabla_b f + \nabla_d\nabla_c\nabla_a\nabla_b f).$$

It is easy to see that the total symmetrization of an element of $\mathcal{E}_{(ab)}\vee\mathcal{E}_{(cd)}$ is given exactly by the sum of permutations appearing on the right-hand side of the previous equation, and hence we conclude $(\nabla\bullet\nu)_{abcd} \equiv 24\nabla_{(a}\nabla_b\nabla_c\nabla_{d)}f$ modulo $\mathrm{Ker}(\Pi_{\boxplus})$.

The lower order terms in abstarct formula (3.19) are of two kinds. In both cases, there occurs a difference of the two possible paths from up to down. Let us strat with the first pair of terms. By definition we have

$$\begin{aligned} &-2\mathsf{P}\bullet\bullet(\nabla\bullet\nabla\bullet f)_1 + 2\mathsf{P}\bullet\bullet(\nabla\bullet\nabla\bullet f)_2 \\ &= -2\textstyle\sum_\ell \varphi^\ell\bullet(\mathsf{P}(\psi_\ell)\bullet(\nabla\bullet\nabla\bullet f)_1 - \mathsf{P}(\psi_\ell)\bullet(\nabla\bullet\nabla\bullet f)_2) \end{aligned}$$

and according to (3.2) and (3.16), we get

$$
\begin{aligned}
&(\mathsf{P}(\psi_\ell)\bullet(\nabla\bullet\nabla\bullet f)_1-\mathsf{P}(\psi_\ell)\bullet(\nabla\bullet\nabla\bullet f)_2)_{abc}\\
&\equiv 2\mathsf{P}(\psi_\ell)_a\nabla_b(\nabla\bullet f)_c+2\mathsf{P}(\psi_\ell)_a\nabla_c(\nabla\bullet f)_b-4\mathsf{P}(\psi_\ell)_c\nabla_a(\nabla\bullet f)_b\\
&=-8\mathsf{P}(\psi_\ell)_a\nabla_b\nabla_c f-8\mathsf{P}(\psi_\ell)_a\nabla_c\nabla_b f+16\mathsf{P}(\psi_\ell)_c\nabla_a\nabla_b f
\end{aligned}
$$

modulo $\mathrm{Ker}(\Pi_{\boxplus})$. Substituting corresponding terms in the equation above and using lemma 3.10 leads to

$$
\begin{aligned}
&(-2\mathsf{P}\bullet\bullet(\nabla\bullet\nabla\bullet f)_1+2\mathsf{P}\bullet\bullet(\nabla\bullet\nabla\bullet f)_2)_{abcd}\\
&\equiv -16(\mathsf{P}_{da}\nabla_b\nabla_c f+\mathsf{P}_{da}\nabla_c\nabla_b f-2\mathsf{P}_{dc}\nabla_a\nabla_b f)\quad \mathrm{mod}\,\mathrm{Ker}(\Pi_{\boxplus}).
\end{aligned}
$$

Now consider the other pair of lower order terms in (3.19). By definition of $\mathsf{P}\bullet\bullet$ we have

$$
\begin{aligned}
&-2\nabla\bullet\nabla\bullet(\mathsf{P}\bullet\bullet f)_1+2\nabla\bullet\nabla\bullet(\mathsf{P}\bullet\bullet f)_2\\
&=-2\nabla\bullet\textstyle\sum_\ell(\nabla\bullet(\varphi^\ell\bullet\mathsf{P}(\psi_\ell)\bullet f)_1-\nabla\bullet(\varphi^\ell\bullet\mathsf{P}(\psi_\ell)\bullet f)_2).
\end{aligned}
$$

Similarly to the previous case, we apply (3.16) to the difference occuring in the sum. Then by lemma 3.10 we obtain

$$
\begin{aligned}
&(-2\nabla\bullet\nabla\bullet(\mathsf{P}\bullet\bullet f)_1+2\nabla\bullet\nabla\bullet(\mathsf{P}\bullet\bullet f)_2)_{abcd}\\
&\equiv -16\nabla_d(\nabla_a(\mathsf{P}_{bc}f)+\nabla_a(\mathsf{P}_{cb}f)-2\nabla_c(\mathsf{P}_{ab}f))\quad \mathrm{mod}\,\mathrm{Ker}(\Pi_{\boxplus}).
\end{aligned}
$$

Putting the partial results together (and removing the overall factor 8) yields the following formula for the operator $\tilde{\mathsf{D}}$ which is a preimage of D under $\Pi_{\boxplus}$.

$$
\begin{aligned}
(\tilde{\mathsf{D}}f)_{abcd}&\equiv 3\nabla_{(a}\nabla_b\nabla_c\nabla_{d)}f-2\mathsf{P}_{da}\nabla_b\nabla_c f-2\mathsf{P}_{da}\nabla_c\nabla_b f+4\mathsf{P}_{dc}\nabla_a\nabla_b f\\
&\quad-2\nabla_d\nabla_a(\mathsf{P}_{bc}f)-2\nabla_d\nabla_a(\mathsf{P}_{cb}f)+4\nabla_d\nabla_c(\mathsf{P}_{ab}f)
\end{aligned}
$$

By lemma 3.10, this can be rewritten as

$$
\begin{aligned}
&(\tilde{\mathsf{D}}f)_{abcd}\equiv 3\nabla_{(a}\nabla_b\nabla_c\nabla_{d)}f-8\mathsf{P}_{(bc)}\nabla_a\nabla_d f+8\mathsf{P}_{ab}\nabla_c\nabla_d f\\
&-8(\nabla_a\mathsf{P}_{(bc)})\nabla_d f+8(\nabla_c\mathsf{P}_{ab})\nabla_d f-4(\nabla_a\nabla_d\mathsf{P}_{(bc)})f+4(\nabla_c\nabla_d\mathsf{P}_{ab})f
\end{aligned}
$$

Hence we see that there appears automatically only the symmetrization S of Rho-tensor in the formula for D obtained from the curved Casimir construction. Namely, we can equivalently write

$$
\begin{aligned}
&(\tilde{\mathsf{D}}f)_{abcd}\equiv 3\nabla_{(a}\nabla_b\nabla_c\nabla_{d)}f-8\mathsf{S}_{bc}\nabla_a\nabla_d f+8\mathsf{S}_{ab}\nabla_c\nabla_d f\\
&-8(\nabla_a\mathsf{S}_{bc}-\nabla_c\mathsf{S}_{ab})\nabla_d f-4(\nabla_d\nabla_a\mathsf{S}_{bc}-\nabla_d\nabla_c\mathsf{S}_{ab})f\quad \mathrm{mod}\,\mathrm{Ker}(\Pi_{\boxplus}).
\end{aligned}
\tag{3.22}
$$

The result then follows by observing that the whole second line in this formula vanishes in the torsion-free case. Indeed, lemma A.5 in the appendix shows that the differential Bianchi identity implies

$$\nabla_A^{A'} \mathsf{S}_{C\ B}^{C'B'} - \nabla_C^{C'} \mathsf{S}_{A\ B}^{A'B'} = Q_{A\ C\ B}^{A'C'B'} = \frac{1}{1-q} \epsilon^{A'C'} \nabla_E^{B'} W_{A\ C}{}^{E}{}_{B}$$

But since the right-hand side lies in $\mathcal{E}_{(ABC)}^{A'}[-1]$, it vanishes when projected to $\mathcal{E}^{A'} \otimes \boxplus \mathcal{E}_A[-1]$ by $\Pi_{\boxplus}$ and so we have

$$\nabla_a \mathsf{S}_{bc} \equiv \nabla_c \mathsf{S}_{ab} \quad \text{mod Ker}(\Pi_{\boxplus}). \tag{3.23}$$

Then the second line in (3.22) vanishes since by lemma (3.10) $\Pi_{\boxplus}$ factorizes through $\Pi_{\boxplus} \otimes \text{id}$. □

We can read off the formula (3.21) for D that it is a true fourth order operator with the principal part equal to $\Pi_{\boxplus} \circ \text{Symm}(\nabla^4)$. In the case of a locally flat manifold, the operator can be written in terms of a flat connection for which the covariant derivatives may be commuted freely and thus by definition of $\Pi_{\boxplus}$, D descends to

$$\mathsf{D}^{\text{flat}}_{ABCD} = (\nabla_{[A}^{A'} \nabla_{B]}^{B'} \nabla_{[C}^{C'} \nabla_{D]}^{D'} - \nabla_{[A}^{A'} \nabla_{B}^{B'} \nabla_{C}^{C'} \nabla_{D]}^{D'}) \epsilon_{A'B'} \epsilon_{C'D'},$$

which is the non-standard operator on functions. This completes the proof of theorem 3.2.

3.1.7. Factorization of D. At the end of this section, we find a simple formula for the invariant operator D. Although (3.21) is already quite simple formula, we can find even a better one in the sense that it is analogous to the formula for Paneitz operator from conformal geometry and it shows that the operator D factorizes through the bundles $\mathcal{E}_A^{A'}$ and $\mathcal{E}^{A'} \otimes \boxplus \mathcal{E}_A[-1]$. Such a formula may be deduced directly from an alternative construction of the operator D, described in the appendix, c.f. proposition B.1.

Let S^1 and S^2 denote the irreducible components of the symmetric Rho-tensor S lying in $\mathcal{E}_{(AB)}^{(A'B')}$ and $\mathcal{E}_{[AB]}[-1]$ respectively. Then the following holds.

Proposition 3.12. *The operator* D*, which is a non-standard invariant operator on functions on torsion-free structures, is given by* $\mathsf{D} = \Pi_{\boxplus} \circ \tilde{\mathsf{D}}$ *where*

$$(\tilde{\mathsf{D}} f)_{abcd} = \nabla_a (\nabla_b \nabla_c - 4\mathsf{S}^1_{bc} + 4\mathsf{S}^2_{bc}) \nabla_d f \tag{3.24}$$

PROOF. We start with the formula (3.21) for D and we show that it is equivalent to (3.24). Let us rewrite the leading term first. By lemma 3.10, it can be written as

$$3\nabla_{(a} \nabla_b \nabla_c \nabla_{d)} f \equiv \nabla_a \nabla_b \nabla_c \nabla_d f + \nabla_a \nabla_c \nabla_b \nabla_d f + \nabla_a \nabla_c \nabla_d \nabla_b f$$

modulo $\mathrm{Ker}(\Pi_{\boxplus})$ for any function $f \in \mathcal{E}$. The last two summands are equal becuse ∇_a is torsion-free. Therefore, we can rewrite the right-hand side as $3\nabla_a\nabla_b\nabla_c\nabla_d f - 2\nabla_a(\nabla_b\nabla_c - \nabla_c\nabla_b)\nabla_d f$ and then the use of Ricci identity yields

$$3\nabla_{(a}\nabla_b\nabla_c\nabla_{d)} f \equiv 3\nabla_a\nabla_b\nabla_c\nabla_d f - 2\nabla_a R_{bcd}{}^e\nabla_e f$$

We know that the only non-zero part of the Weyl curvature $U_{bcd}{}^e = R_{bcd}{}^e + \partial(\mathsf{P})_{bcd}{}^e$ in the torsion-free case is the harmonic curvature. But since the harmonic curvature is symmetric in three unprimed indices, it vanishes under our projection and thus we have $R_{bcd}{}^e\nabla_e f \equiv -\partial(\mathsf{P})_{bcd}{}^e\nabla_e f$, which by 3.15 equals

$$R_{bcd}{}^e\nabla_e f \equiv \mathsf{P}_{bc}\nabla_d f + \mathsf{P}_{cb}\nabla_d f + \mathsf{P}_{cd}\nabla_b f \mod \mathrm{Ker}(\Pi_{\boxplus}).$$

This is obviously equivalent to $2\mathsf{S}_{bc}\nabla_d f + \mathsf{S}_{cd}\nabla_b f$ and so we conclude

$$3\nabla_{(a}\nabla_b\nabla_c\nabla_{d)} f \equiv 3\nabla_a\nabla_b\nabla_c\nabla_d f - 4\nabla_a\mathsf{S}_{bc}\nabla_d f - 2\nabla_a\mathsf{S}_{cd}\nabla_b f.$$

Now we distribute the parenthesis and we add lower-order terms from (3.21). Then by lemma 3.10 we obtain

$$(\tilde{\mathsf{D}}f)_{abcd} \equiv 3\nabla_a\nabla_b\nabla_c\nabla_d f - 4(\nabla_a\mathsf{S}_{bc})\nabla_d f - 2(\nabla_c\mathsf{S}_{ab})\nabla_d f$$
$$- 12\mathsf{S}_{ad}\nabla_b\nabla_c f + 6\mathsf{S}_{cd}\nabla_a\nabla_b f \mod \mathrm{Ker}(\Pi_{\boxplus}),$$

which, dropping the factor 3, is by (3.23) equivalent to

$$(\tilde{\mathsf{D}}f)_{abcd} \equiv \nabla_a\nabla_b\nabla_c\nabla_d f - 2(\nabla_a\mathsf{S}_{bc})\nabla_d f - 4\mathsf{S}_{bc}\nabla_a\nabla_d f + 2\mathsf{S}_{cd}\nabla_a\nabla_e f. \quad (3.25)$$

This formula is the one obtained from the alternative construction described in the appendix and it is easy to see its equivalence to (3.24). Indeed, it may be rewritten as

$$(\tilde{\mathsf{D}}f)_{abcd} \equiv \nabla_a(\nabla_b\nabla_c\nabla_d f - 4\mathsf{S}_{bc}\nabla_d f + 2\mathsf{S}_{cd}\nabla_b f) + 2(\nabla_a\mathsf{S}_{bc} - \nabla_c\mathsf{S}_{ab})\nabla_d f$$

and equation 3.23 says that the second summand on the right-hand side vanishes. Now we replace the tensor S by the sum $\mathsf{S}^1 + \mathsf{S}^2$ of its irreducible components. Since the projection $\Pi_{\boxplus}$ contracts C' and D', it maps the term with S^1_{cd} to zero, and thus we get

$$(\tilde{\mathsf{D}}f)_{abcd} \equiv \nabla_a(\nabla_b\nabla_c\nabla_d f - 4\mathsf{S}^1_{bc}\nabla_d f - 4\mathsf{S}^2_{bc}\nabla_d f + 2\mathsf{S}^2_{cd}\nabla_b f).$$

And since from (5) in lemma 3.10 we conclude $\nabla_a\mathsf{S}^2_{cd}\nabla_b f \equiv 4\nabla_a\mathsf{S}^2_{bc}\nabla_d f$, we end up with formula (3.24). □

The equation (3.24) shows that $\tilde{\mathsf{D}}$ can be written as a composition of three operators: $f \mapsto \mu_a = \nabla_a f$, $\mu_a \mapsto \tilde{\nu}_{abc} = \nabla_c\nabla_a\mu_b + 4(\mathsf{S}^2_{ca} - \mathsf{S}^1_{ca})\mu_b$, and $\tilde{\nu}_{abc} \mapsto \tilde{\rho}_{abcd} = \nabla_a\tilde{\nu}_{bcd} \equiv \nabla_d\tilde{\nu}_{abc}$. The first one is oviously the differential $(df)_a$.

According to the definitions of the projections $\Pi_{\yng(2,1,1)}$ and $\Pi_{\yng(2,2)}$, the composition of the last operator with the projection $\Pi_{\yng(2,2)}$ equals

$$\tilde{\nu}_{abc} \mapsto \frac{1}{2}(\nabla^{B'}_{[D}\Pi_{\yng(2,1,1)}(\tilde{\nu})^{A'}_{|AB|C]} + \nabla^{B'}_{[B}\Pi_{\yng(2,1,1)}(\tilde{\nu})^{A'}_{|CD|A]})\epsilon_{A'B'}.$$

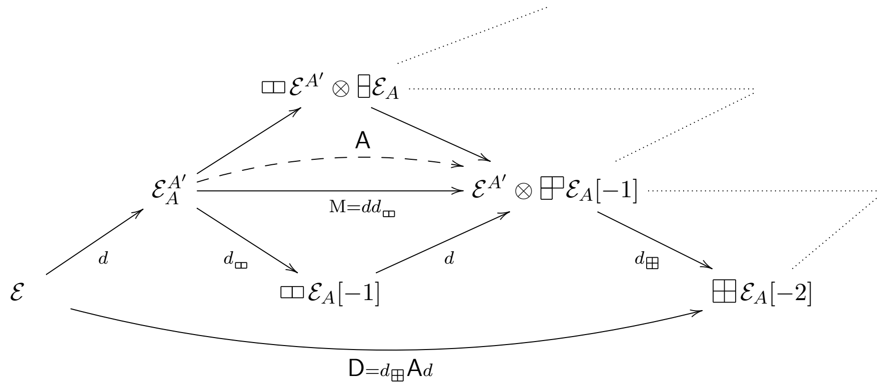
But this is the formula from lemma 2.3 for the action of the exterior derivative $d_{\yng(2,2)}$ on $\Pi_{\yng(2,1,1)}(\tilde{\nu})$, and hence we get the following

Corollary 3.13. *Let* A *be a non-invariant operator* $\mathcal{E}^{A'}_A \to \mathcal{E}^{A'} \otimes \yng(2,1,1)\mathcal{E}_A[-1]$ *given by* $\mathsf{A} = \Pi_{\yng(2,1,1)} \circ \tilde{\mathsf{A}}$, *where*

$$\tilde{\mathsf{A}}(\mu)_{abc} = \nabla_c \nabla_a \mu_b + 4(\mathsf{S}^2_{ca} - \mathsf{S}^1_{ca})\mu_b.$$

Then the invariant operator D *coincides with the (unique) projection of* $d\mathsf{A}d$ *to the bundle* $\yng(2,2)\mathcal{E}_A[-2]$.

Hence we found out that the case of a torsion-free almost Grassmannian geometry is completely parallel to the situation in conformal geometry as it is described in **[6]** and **[12]**. Namely, the standard procedure of the curved Casimir construction yields a vanishing operator $d_{\yng(2,2)}\mathrm{M}d$ (M is an analog of the Maxwell operator from conformal geometry). The construction also gives an invariant operator D in a "non-standard" way which we described above and it may be viewed as an analog of the Paneitz operator from conformal geometry. A formula for D is obtained from $d_{\yng(2,2)}\mathrm{M}d$ by replacing the invariant operator M by the non-invariant operator A, as illustrated in the following figure.



The special form of the operator D in the torsion-free case leads to the definition of an analog of conformal Q-curvature which was intensively studied recently, e.g. **[12]**. It is a polynomial expression in curvature associated to a scale which is not invariant but enjoys certain special properties with respect to changes of scale. Namely, changing the scale the Q-curvature depends linearly on the change and this dependance is described by the operator D. In contrast to the conformal case, the Grassmannian analog

Q of the Q-curvature is not a scalar (for $q > 2$) but it is a section of $\boxplus\mathcal{E}_A[-2] \subset \Lambda^4\mathcal{E}_A^{A'} = \Lambda^4 T^*M$. The precise formulation is given in the following proposition. For a proof see appendix B.

Proposition 3.14. *In the case of a torsion-free almost Grassmannian geometry, there exists a section* $\mathsf{Q}_{ABCD} \in \mathcal{E}_{\boxplus(ABCD)}[-2]$ *which transforms as* $\hat{\mathsf{Q}} = \mathsf{Q} + \mathsf{D}\omega$ *under the change of Weyl structure given by an exact one-form* $\Upsilon = \nabla\omega$. *This section is given by* $\mathsf{Q} = \Pi_{\boxplus}(\tilde{\mathsf{Q}})$, *where*

$$\tilde{\mathsf{Q}}_{abcd} = -\nabla_a\nabla_b\mathsf{P}_{cd} + 2\mathsf{P}_{ac}\mathsf{P}_{bd} - 2\mathsf{P}_{ab}\mathsf{P}_{cd}.$$

3.2. The case of non-vanishing torsion

In the case that the torsion of the Weyl connection ∇ does not vanish, the operator M obtained from the curved Casimir construction does not coincide with the composition $dd_{\boxplus}$ of components of exterior derivatives and so the compositions Md and $d_{\boxplus}$M do not vanish. Indeed, one can make these compositions explicit using the formula (3.18) for M and check that they are actually non-zero invariant operators of order two with the torsion in symbol. This means that the construction of an invariant operator D via the curved Casimir operator, as described in the previous section, fails in the sense that the operator D_σ defined by (3.7) depends on the choice of Weyl structure. Namely, its linearized transformation is non-zero of order two in the initial function and of order one in Υ.

Nevertheless, we prove in this section that there exists a fix which makes D_σ into an invariant operator on structures with arbitrary torsion. A formula for this operator, which we dentote by $\mathsf{D}^{\mathrm{corr}}$ is given in theorem 3.15 below. The proof is via an explicit calculation of its linearized transformation. Using the symmetry of the torsion we show that the invariance is a consequence of the algebraic Bianchi identity. After that we show in proposition 3.21 that there is no correction of the form $d_{\boxplus}(\mathsf{A} + \mathrm{corr.})d$.

3.2.1. Main result. In order to find a fix of D_σ, it is worthwhile to take the simplest formula which defines this operator in the torsion-free case. Hence we consider that D_σ is given by formula (3.24). Since it is of the form $d_{\boxplus}\mathsf{A}d$, it is natural to try to find a correction of the form $d_{\boxplus}(\mathsf{A} + \mathrm{corr.})d$ first. Of course, by a correction we mean a linear combination of lower-order terms involving torsion (reffered as correction terms in sequel). We find out that the correction of this form leads to an operator which is not invariant but has a very simple transformation formula. This formula is quadratic in the torsion, and can be canceled by adding a correction which we denote by C, and which is formed by terms of the form $T^2 \cdot \nabla^2 f$ and $T \cdot \nabla T \cdot \nabla f$. Hence we get an invariant extension of D to structures with non-vanishing torsion. Precisely, using the projection $\Pi_{\boxplus}$ defined in the previous section by equation (3.20) we conclude the following.

Theorem 3.15. *There exists an extension* D^{corr} *of the non-standard operator* $\mathcal{E} \to \boxplus\mathcal{E}_A[-2]$ *which is invariant on almost Grassmannian structures of type* $(2, q)$ *with an arbitrary torsion. It is defined by applying the projection* $\Pi_{\boxplus}$ *to*

$$\tilde{\mathsf{D}}^{corr}_{abcd} = \nabla_a(\nabla_b\nabla_c - 4\mathsf{S}^1_{bc} + 4\mathsf{S}^2_{bc} - 4T_{bc}{}^e\nabla_e - \frac{6}{q+2}\nabla_e T_{bc}{}^e)\nabla_d + \tilde{\mathsf{C}}_{abcd},$$

where the correction $\tilde{\mathsf{C}}$ *is equal to*

$$\begin{aligned} \tilde{\mathsf{C}}^{A'B'C'D'}_{A\,B\,C\,D} &= -2T^{B'E'F}_{B\,C\,F'}T^{A'F'E}_{A\,F\,E'}\nabla^{C'}_{E}\nabla^{D'}_{D} + \tfrac{4}{q-2}T^{B'E'F}_{B\,C\,F'}(\nabla^{C'}_{E}T^{A'F'E}_{A\,F\,E'})\nabla^{D'}_{D} \\ &\quad -\tfrac{12}{q^2-4}T^{B'C'F}_{B\,C\,F'}(\nabla^{E'}_{E}T^{A'F'E}_{A\,F\,E'})\nabla^{D'}_{D}. \end{aligned}$$

Remark 3.1. It follows from the proof of proposition 3.21 below that the correction appearing in the first part of operator $\mathsf{D}^{\mathrm{corr}}$ is the unique correction of terms in linearized transformation of D_σ, which are linear in the torsion. This correction can be obtained by a modification of the construction of D described in appendix B. On the other hand, it turns out that the form of correction C is not unique due to the existence of second-order invariant operators with square of torsion in their leading parts. We can even write down immediately one of these operators. Denoting by # the algebraic action of the torsion, it is given by a projection of $dT\#T\#d$ to $\boxplus\mathcal{E}_A[-2]$.

Remark 3.2. The formula for $\mathsf{D}^{\mathrm{corr}}$ in theorem 3.15 is written in terms of an exact Weyl connection, and is invariant with respect to exact changes. As explained in remark 3.3, the formula for $\mathsf{D}^{\mathrm{corr}}$ can be easily modified to a formula invariant to all changes of Weyl structure by adding some other correction terms, c.f. formula (3.37) .

PROOF OF THE THEOREM 3.15. The idea of the proof is very simple. We compute the linearized transformation of $\tilde{\mathsf{D}}^{\mathrm{corr}}$ and we show that it vanishes under the projection $\Pi_{\boxplus}$. We proceed in three steps. First we use basic transformation laws for transformations of covariant derivatives of tensors and properties of the projection $\Pi_{\boxplus}$ to make explicit the variation of

$$\tilde{\mathsf{D}}^1_{abcd} := \nabla_a(\nabla_b\nabla_c - 4\mathsf{S}^1_{bc} + 4\mathsf{S}^2_{bc} - 4T_{bc}{}^e\nabla_e - \frac{6}{q+2}\nabla_e T_{bc}{}^e)\nabla_d. \qquad (3.26)$$

In the second step, we use the algebraic Binchi identity to express this variation exclusively in terms of torsion. Then we determine the variation of correction $\tilde{\mathsf{C}}$ and we show $\delta(\tilde{\mathsf{D}}^1 + \tilde{\mathsf{C}}) \equiv 0 \mod \mathrm{Ker}(\Pi_{\boxplus})$.

3.2.2. Transformation of D^1. First recall that the outer covariant derivatives define invariant operators d and $d_{\boxplus}$. Hence equation (3.26) defining D^1 has the form $\mathsf{D}^1 = d_{\boxplus}\mathsf{A}^1 d$, where A^1 is a (non-invariant) operator which is given by applying the projection $\Pi_{\boxplus}$ defined by (3.14) to

$$(\tilde{\mathsf{A}}^1\mu)_{abc} := \nabla_c\nabla_a\mu_b - 4\mathsf{S}^1_{ca}\mu_b + 4\mathsf{S}^2_{ca}\mu_b - 4T_{ca}{}^e\nabla_e\mu_b - \frac{6}{q+2}(\nabla_e T_{ca}{}^e)\mu_b.$$

It is easy to see from this equation that A^1 is a torsion extension of the (non-invariant) operator A from corollary 3.13, and so D^1 is of the form $d_{\boxplus}(\mathsf{A} + \text{corr.})d$. Thus a computation of the linearized transformation of D^1 boils down to a computation of the transformation of A^1. Namely, the variation satisfies $\delta(\mathsf{D}^1 f) = d_{\boxplus}\delta(\mathsf{A}^1)(df)$. The variation of A^1 is obtained easily by applying basic rules for transformations of covariant derivatives of tensors and and some properties of the projection $\Pi_{\boxplus}$.

Lemma 3.16. *For an one-form μ the variation $\delta(\mathsf{A}^1)(\mu)$ is given by applying the projection $\Pi_{\boxplus}$ to*

$$\delta(\tilde{\mathsf{A}}^1)(\mu)_{abc} \equiv 4\Upsilon_{[a}\nabla_{c]}\mu_b + 4\mu_{[c}\nabla_{|b|}\Upsilon_{a]} + 2\Upsilon_e T_{ac}{}^e\mu_b - 4\Upsilon_b T_{ac}{}^e\mu_e.$$

PROOF. Since δ satisfies a Leibnitz formula, the variation of A^1 is equal to the projection $\Pi_{\boxplus}$ of

$$\begin{aligned}\delta(\tilde{\mathsf{A}}^1)(\mu)_{abc} &= \delta(\nabla_c)(\nabla_a\mu_b) + \nabla_c\delta(\nabla_a)(\mu_b) - 4\delta(\mathsf{S}^1_{ca})\mu_b + 4\delta(\mathsf{S}^2_{ca})\mu_b \\ &\quad -4T_{ca}{}^e\delta(\nabla_e\mu_b) - \tfrac{6}{q+2}\delta(\nabla_e)(T_{ca}{}^e)\mu_b.\end{aligned}$$

Now we analyze these summands one by one. Let us start with the first line of the previous equation. According to the rule for the transformation of the covariant derivative of a two-tensor described in section 2.2.3, we have

$$\delta(\nabla^{C'}_C)(\nabla^{A'}_A\mu^{B'}_B) = -\Upsilon^{A'}_C\nabla^{C'}_A\mu^{B'}_B - \Upsilon^{B'}_C\nabla^{A'}_A\mu^{C'}_B - \Upsilon^{C'}_A\nabla^{A'}_C\mu^{B'}_B - \Upsilon^{C'}_B\nabla^{A'}_A\mu^{B'}_C.$$

Since alternation over three primed indices vanishes, after contracting A' and B' we may replace the first two terms by $-\Upsilon^{C'}_C\nabla^{A'}_A\mu^{B'}_B\epsilon_{A'B'}$. After skewing in A and B we may replace the second two summands by $-\Upsilon^{C'}_C\nabla^{A'}_{[A}\mu^{B'}_{B]} + 3\Upsilon^{C'}_{[C}\nabla^{A'}_A\mu^{B'}_{B]}$, and thus we get

$$\delta(\nabla^{C'}_C)(\nabla^{A'}_{[A}\mu^{B'}_{B]}\epsilon_{A'B'}) = (-2\Upsilon^{C'}_C\nabla^{A'}_{[A}\mu^{B'}_{B]} + 3\Upsilon^{C'}_{[C}\nabla^{A'}_A\mu^{B'}_{B]})\epsilon_{A'B'}.$$

By lemma 3.10, the alternation over three unprimed indices vanishes under $\Pi_{\boxplus}$ and so the previous equation implies

$$\delta(\nabla_c)(\nabla_a\mu_b) \equiv -2\Upsilon_c\nabla_a\mu_b \quad \text{mod Ker}(\Pi_{\boxplus}). \tag{3.27}$$

Applying the transformation law for the covariant derivative of a one-form given in section 2.2.3 to the second summand, i.e. $\nabla_c\delta(\nabla_a)(\mu_b)$, we get

$$\nabla^{C'}_C\delta(\nabla^{A'}_A)(\mu^{B'}_B) = \nabla^{C'}_C(-\Upsilon^{B'}_A\mu^{A'}_B - \Upsilon^{A'}_B\mu^{B'}_A).$$

And since the projection $\Pi_{\boxplus}$ by definition alternates the indices A, B and contracts A', B' with a two-form ϵ, this equation yields

$$\nabla_c\delta(\nabla_a)(\mu_b) \equiv 2\nabla_c\Upsilon_a\mu_b = 2(\nabla_c\Upsilon_a)\mu_b + 2\Upsilon_a\nabla_c\mu_b. \tag{3.28}$$

For next terms, i.e. $-4\delta(\mathsf{S}^1_{ca})\mu_b$ and $4\delta(\mathsf{S}^2_{ca})\mu_b$, by (2.11) we get

$$-4\delta(\mathsf{S}^1)_{C\ A}^{C'A'}\mu_B^{B'} = -4\mu_B^{B'}\nabla_{(C}^{(C'}\Upsilon_{A)}^{A')}$$

$$4\delta(\mathsf{S}^2)_{C\ A}^{C'A'}\mu_B^{B'} = 4\mu_B^{B'}\nabla_{[C}^{[C'}\Upsilon_{A]}^{A']}.$$

Now since we compute

$$\epsilon_{A'B'}\mathrm{Alt}_{AB}(4\mu_B^{B'}\nabla_{[C}^{[C'}\Upsilon_{A]}^{A']}) = \epsilon_{A'B'}(\mu_C^{C'}\nabla_{[A}^{A'}\Upsilon_{B]}^{B'} - \mu_{[C}^{C'}\nabla_A^{A'}\Upsilon_{B]}^{B'}),$$

we get $4\mu_B^{B'}\nabla_{[C}^{[C'}\Upsilon_{A]}^{A']} \equiv \mu_C^{C'}\nabla_A^{A'}\Upsilon_B^{B'}$ modulo $\mathrm{Ker}(\Pi_{\boxplus})$ by definition of the action of $\Pi_{\boxplus}$ on $\mu_b\nabla_c\Upsilon_a$. For the other term we use the equation

$$\nabla_{(C}^{(C'}\Upsilon_{A)}^{A')} = \frac{1}{2}\nabla_C^{C'}\Upsilon_A^{A'} + \frac{1}{2}\nabla_A^{A'}\Upsilon_C^{C'} - \nabla_{[C}^{[C'}\Upsilon_{A]}^{A']},$$

and so modulo $\mathrm{Ker}(\Pi_{\boxplus})$ we obtain

$$-4\delta(\mathsf{S}^1_{ca})\mu_b + 4\delta(\mathsf{S}^2_{ca})\mu_b \equiv -4\mu_b\nabla_{(c}\Upsilon_{a)} + 2\mu_c\nabla_a\Upsilon_b. \tag{3.29}$$

Now summing the equations (3.27), (3.28) and (3.29) together, we get

$$\delta(\tilde{\mathsf{A}}\mu)_{abc} \equiv -2\Upsilon_c\nabla_a\mu_b + 2\mu_b\nabla_c\Upsilon_a + 2\Upsilon_a\nabla_c\mu_b - 4\mu_b\nabla_{(c}\Upsilon_{a)} + 2\mu_c\nabla_a\Upsilon_b,$$

and since by definition $\Pi_{\boxplus}$ factorizes through $\mathcal{E}_{(ab)c}$, we may rewrite the right-hand side as follows,

$$\delta(\tilde{\mathsf{A}}\mu)_{abc} \equiv 4\Upsilon_{[a}\nabla_{c]}\mu_b + 4\mu_{[c}\nabla_{|b|}\Upsilon_{a]} \quad \mathrm{mod}\ \mathrm{Ker}(\Pi_{\boxplus}). \tag{3.30}$$

Now let us continue with correction terms of A^1. The first term, $T_{ca}{}^e\delta(\nabla_e\mu_b)$ is computed according to the rule for the transformation of covariant derivative of an one-form. Namely, we have

$$\delta(\nabla_E^{E'}\mu_B^{B'}) = -\Upsilon_E^{B'}\mu_B^{E'} - \Upsilon_B^{E'}\mu_E^{B'} = -\Upsilon_B^{B'}\mu_E^{E'} - \Upsilon_E^{E'}\mu_B^{B'} + 4\Upsilon_{[B}^{[B'}\mu_{E]}^{E']}.$$

Since by definition $\Pi_{\boxplus}$ contracts A' and B', and since the torsion is trace-free, we get

$$4T_{C\ A\ E'}^{C'A'E}\Upsilon_{[B}^{[B'}\mu_{E]}^{E']} \equiv 2T_{C\ A\ E'}^{C'E'E}\Upsilon_{[B}^{B'}\mu_{E]}^{A'} = 0 \quad \mathrm{mod}\ \mathrm{Ker}(\Pi_{\boxplus}).$$

From here and the eqaution above we conclude

$$T_{ca}{}^e\delta(\nabla_e\mu_b) \equiv -2T_{ca}{}^e\Upsilon_{(e}\mu_{b)} \quad \mathrm{mod}\ \mathrm{Ker}(\Pi_{\boxplus}) \tag{3.31}$$

Concerning the second correction term, i.e. $(\nabla_eT_{ca}{}^e)\mu_b$, the transformation law (2.10) and the trace-freeness of the torsion directly yields

$$\delta(\nabla_e)(T_{ca}{}^e) = (q+2)(\Upsilon_eT_{ca}{}^e).$$

By this equation and equations (3.30) and (3.31), the variation $\delta(\tilde{\mathsf{A}}^1\mu)_{abc} = \delta(\tilde{\mathsf{A}}\mu)_{abc} - 4T_{ca}{}^e\delta(\nabla_e\mu_b) - \frac{6}{q+2}\delta(\nabla_e)(T_{ca}{}^e)\mu_b$ is equivalent modulo $\mathrm{Ker}(\Pi_\boxplus)$ to

$$\delta(\tilde{\mathsf{A}}^1\mu)_{abc} \equiv 4\Upsilon_{[a}\nabla_{c]}\mu_b + 4\mu_{[c}\nabla_{|b|}\Upsilon_{a]} + 8T_{ca}{}^e\Upsilon_{(e}\mu_{b)} - 6(\Upsilon_e T_{ca}{}^e)\mu_b,$$

and the result follows by rewritting the last two terms on the right. $\square$

Having an expression for the transformation of the operator A^1, we can now easily express the variation $\delta(\mathsf{D}^1 f) = d_\boxplus\delta(\mathsf{A}^1)(df)$. According to the definition of $d_\boxplus$, its action on $\delta(\mathsf{A})(df)$ is obtained as the image of $\nabla_d\delta(\tilde{\mathsf{A}})(df)_{abc}$ under the projection $\Pi_\boxplus$. Hence the previous lemma for $\mu_a = \nabla_a f$ immediately implies that

$$\begin{aligned}\delta(\tilde{\mathsf{D}}^1 f)_{abcd} \quad &\equiv \quad \nabla_d(4\Upsilon_{[a}\nabla_{c]}\nabla_b f + 4(\nabla_b\Upsilon_{[a})\nabla_{c]}f \\ &\qquad + 2\Upsilon_e T_{ac}{}^e\nabla_b f - 4\Upsilon_b T_{ac}{}^e\nabla_e f) \bmod \mathrm{Ker}(\Pi_\boxplus)\end{aligned} \tag{3.32}$$

Now we use the Ricci identity and properties of the projection $\Pi_\boxplus$ summarized in lemma 3.10 to put this variation into the following simple form.

Lemma 3.17. *For a function $f \in \mathcal{E}$ the variation $\delta(\mathsf{D}^1 f)$ is given by applying the projection $\Pi_\boxplus$ to*

$$\delta(\tilde{\mathsf{D}}^1 f)_{abcd} \equiv 4(R_{cab}{}^e - \nabla_a T_{bc}{}^e + T_{bc}{}^f T_{af}{}^e)\Upsilon_{[e}\nabla_{d]}f \tag{3.33}$$

Proof. Let us treat the terms in (3.32) which does not contain torsion first. Acting with the outer covariant derivative we get

$$\begin{aligned}\nabla_d(4\Upsilon_{[a}\nabla_{c]}\nabla_b f + 4(\nabla_b\Upsilon_{[a})\nabla_{c]}f) &\equiv 4(\nabla_d\Upsilon_{[a})\nabla_{c]}\nabla_b f + 4\Upsilon_{[a}\nabla_{|d|}\nabla_{c]}\nabla_b f \\ &\quad + 4(\nabla_d\nabla_b\Upsilon_{[a})\nabla_{c]}f + 4(\nabla_b\Upsilon_{[a})\nabla_{|d|}\nabla_{c]}f.\end{aligned}$$

Now let us consider the two summands which are of order one in Υ. By (2) in lemma 3.10, their sum is equivalent to

$$4(\nabla_d\Upsilon_{[b})(\nabla_{c]}\nabla_a f - \nabla_{|a|}\nabla_{c]}f) = -4(\nabla_d\Upsilon_{[b})T_{c]a}{}^e\nabla_e f.$$

and this obviously is equivalent to $2(\nabla_d\Upsilon_b)T_{ac}{}^e\nabla_e f$ modulo $\mathrm{Ker}(\Pi_\boxplus)$ since $(\nabla_d\Upsilon_c)T_{ba}{}^e\nabla_e f \equiv 0$ due to the skew-symmetry of the torsion and by the definition of $\Pi_\boxplus$. Concerning the term of order three in f, commuting the derivatives of f we get

$$4\Upsilon_{[a}\nabla_{|d|}\nabla_{c]}\nabla_b f = 4\Upsilon_{[a}\nabla_{|d}\nabla_{b|}\nabla_{c]}f - 4\Upsilon_{[a}\nabla_{|d|}(T_{c]b}{}^e\nabla_e f)$$

Expanding the alternation in the second term on the right we conclude that this term is equal to $2\Upsilon_b\nabla_d(T_{ac}{}^e\nabla_e f)$ modulo $\mathrm{Ker}(\Pi_\boxplus)$. Thus we get

$$\begin{aligned}\nabla_d(4\Upsilon_{[a}\nabla_{c]}\nabla_b f + 4(\nabla_b\Upsilon_{[a})\nabla_{c]}f) &\equiv 4\Upsilon_{[a}\nabla_{|d}\nabla_{b|}\nabla_{c]}f + 4(\nabla_d\nabla_b\Upsilon_{[a})\nabla_{c]}f \\ &\quad + 2\Upsilon_b\nabla_d(T_{ac}{}^e\nabla_e f) + 2(\nabla_d\Upsilon_b)T_{ac}{}^e\nabla_e f,\end{aligned}$$

and the last two terms on the right obviously sum up to $2\nabla_d(\Upsilon_b T_{ac}{}^e \nabla_e f)$. Now adding the terms in (3.32) containing torsion we obtain

$$\begin{aligned}\delta(\tilde{\mathsf{D}}^1 f)_{abcd} \equiv {} & 4\Upsilon_{[a}\nabla_{|d}\nabla_{b|}\nabla_{c]}f + 4(\nabla_d\nabla_b\Upsilon_{[a})\nabla_{c]}f \\ & + 2\nabla_d(\Upsilon_e T_{ac}{}^e\nabla_b f - \Upsilon_b T_{ac}{}^e\nabla_e f) \mod \operatorname{Ker}(\Pi_{\boxplus}).\end{aligned}$$

According to the property (3) in lemma 3.10, we may write

$$\begin{aligned}\delta(\tilde{\mathsf{D}}^1 f)_{abcd} \equiv {} & 4\Upsilon_a\nabla_{[d}\nabla_{b]}\nabla_c f + 4(\nabla_{[d}\nabla_{b]}\Upsilon_a)\nabla_c f \\ & + 2\nabla_d(\Upsilon_e T_{ac}{}^e\nabla_b f - \Upsilon_b T_{ac}{}^e\nabla_e f).\end{aligned}$$

Now applying Ricci identity to both the terms in the first line and removing the bracket in the second line yields

$$\begin{aligned}\delta(\tilde{\mathsf{D}}^1 f)_{abcd} \equiv {} & 2\Upsilon_a(R_{dbc}{}^e\nabla_e f - T_{db}{}^e\nabla_e\nabla_c f) + 2(R_{dba}{}^e\Upsilon_e f - T_{db}{}^e\nabla_e\Upsilon_a)\nabla_c f \\ & + 2(\nabla_d\Upsilon_e)T_{ac}{}^e\nabla_b f + 2\Upsilon_e(\nabla_d T_{ac}{}^e)\nabla_b f + 2\Upsilon_e T_{ac}{}^e\nabla_d\nabla_b f \\ & - 2(\nabla_d\Upsilon_b)T_{ac}{}^e\nabla_e f - 2\Upsilon_b(\nabla_d T_{ac}{}^e)\nabla_e f - 2\Upsilon_b T_{ac}{}^e\nabla_d\nabla_e f,\end{aligned}$$

and this obviously is equivalent to

$$\begin{aligned}\delta(\tilde{\mathsf{D}}^1 f)_{abcd} \equiv {} & 4R_{cab}{}^e\Upsilon_{[e}\nabla_{d]}f - 4(\nabla_a T_{bc}{}^e)\Upsilon_{[e}\nabla_{d]}f + 4\Upsilon_a T_{bc}{}^e\nabla_{[e}\nabla_{d]}f \\ & + 4(\nabla_{[e}\Upsilon_{b]})T_{ac}{}^e\nabla_d f + 2\Upsilon_e T_{ac}{}^e\nabla_d\nabla_b f - 2(\nabla_d\Upsilon_b)T_{ac}{}^e\nabla_e f.\end{aligned}$$

modulo $\operatorname{Ker}(\Pi_{\boxplus})$. The first two terms on the right are those appearing in the formula (3.33). The sum of the next two terms is by Ricci identity and by exactness of the Weyl structure equal to

$$4\Upsilon_a T_{bc}{}^e\nabla_{[e}\nabla_{d]}f + 4(\nabla_{[e}\Upsilon_{b]})T_{ac}{}^e\nabla_d f = 2\Upsilon_a T_{bc}{}^e T_{de}{}^f\nabla_f f + 2T_{be}{}^f\Upsilon_f T_{ac}{}^e\nabla_d f,$$

and this is evidently equivalent to $4T_{bc}{}^f T_{af}{}^e\Upsilon_{[e}\nabla_{d]}f$ modulo $\operatorname{Ker}(\Pi_{\boxplus})$. The result then follows from the fact that the last two terms in the previous formula for $\delta(\tilde{\mathsf{D}}^1 f)$ cancel. This can be deduced as follows. Since the torsion is skew-symmetric, we may apply again the property (3) in lemma 3.10 to get

$$2\Upsilon_e T_{ac}{}^e\nabla_d\nabla_b f - 2(\nabla_d\Upsilon_b)T_{ac}{}^e\nabla_e f \equiv 2\Upsilon_e T_{ac}{}^e\nabla_{[d}\nabla_{b]}f - 2(\nabla_{[d}\Upsilon_{b]})T_{ac}{}^e\nabla_e f.$$

This is by Ricci identity and exactness of Υ equivalent to $2T_{ac}{}^e T_{bd}{}^f\Upsilon_{[e}\nabla_{f]}f$ modulo $\operatorname{Ker}(\Pi_{\boxplus})$, and by (2) in 3.10 we obtain

$$2T_{ac}{}^e T_{bd}{}^f\Upsilon_{[e}\nabla_{f]}f \equiv 2T_{ac}{}^{(e}T_{bd}{}^{f)}\Upsilon_{[e}\nabla_{f]}f = 0.$$

□

3.2.3. Use of the Bianchi identity. Although the formula (3.33) for the transformation of D^1 has a nice form, it turns out that there is no simple correction, which would cancel the curvature term appearing in the formula. On the other hand, the formula reminds the algebraic Bianchi identity, c.f. equation (1) in proposition 1.11, and so it is natural to use this identity to get a simplification. Indeed, the following lemma shows, that the curvature term in (3.33) can be replaced by terms involving torsion, which then leads to a very simple form of this formula.

Namely, let ϑ_{ab} be a two-form and let $\vartheta^1_{ab}, \vartheta^2_{ab}$ denote its irreducible components in subbundles isomorphic to $\mathcal{E}^{(A'B')}_{[AB]}$ and $\mathcal{E}_{(AB)}[-1]$ respectively. Then the action of the curvature on ϑ_{ab} satisfies the following.

Lemma 3.18. *For any two-form $\vartheta_{ab} = \vartheta^1_{ab} + \vartheta^2_{ab}$, we have*

$$R_{cab}{}^e\vartheta_{ed} \equiv (\nabla_a T_{bc}{}^e)\vartheta_{ed} - T_{bc}{}^f T_{af}{}^e(\vartheta^1_{ed} - \vartheta^2_{ed}) \quad \operatorname{mod} \operatorname{Ker}(\Pi_\boxplus).$$

Proof. First we prove that the curvature $R_{cab}{}^e$ can be replaced by the Weyl curvature $U_{cab}{}^e$ by showing $(\partial\mathsf{P})_{cab}{}^e\vartheta_{ed} \equiv 0$ modulo $\operatorname{Ker}(\Pi_\boxplus)$ for any two-form ϑ_{ed}. According to formula (2.6) for the action of the differential on the Rho-tensor, we have

$$(\partial\mathsf{P})_{cab}{}^e\vartheta_{ed} = -\mathsf{P}^{A'B'}_{A\,C}\vartheta^{C'D'}_{B\,D} + \mathsf{P}^{C'B'}_{C\,A}\vartheta^{A'D'}_{B\,D} - \mathsf{P}^{A'C'}_{A\,B}\vartheta^{B'D'}_{C\,D} + \mathsf{P}^{C'A'}_{C\,B}\vartheta^{B'D'}_{A\,D}.$$

Since the projection $\Pi_\boxplus$ contracts A', B' and C', D' with two-form ϵ and since it alternates the indices A, B and C, D by definition, we get

$$(\partial\mathsf{P})_{cab}{}^e\vartheta_{ed} \equiv -2\mathsf{P}^{C'A'}_{C\,A}\vartheta^{B'D'}_{B\,D} - \mathsf{P}^{A'B'}_{A\,C}(\vartheta^2)^{C'D'}_{B\,D} - \mathsf{P}^{A'C'}_{A\,B}(\vartheta^1)^{B'D'}_{C\,D}$$

modulo $\operatorname{Ker}(\Pi_\boxplus)$. The skew-symmetry of ϑ^1 in the unprimed indices and the skew-symmetry of ϑ^2 in the primed indices by (5) in lemma 3.10 yields the following two equations:

$$2\mathsf{P}^{A'C'}_{A\,C}(\vartheta^1)^{B'D'}_{B\,D} \equiv \mathsf{P}^{A'C'}_{A\,B}(\vartheta^1)^{B'D'}_{C\,D} \quad \operatorname{mod} \operatorname{Ker}(\Pi_\boxplus),$$
$$2\mathsf{P}^{A'C'}_{A\,C}(\vartheta^2)^{B'D'}_{B\,D} \equiv \mathsf{P}^{A'B'}_{A\,C}(\vartheta^2)^{C'D'}_{B\,D} \quad \operatorname{mod} \operatorname{Ker}(\Pi_\boxplus).$$

Hence replacing the corresponding terms in the equation above, we get $(\partial\mathsf{P})_{cab}{}^e\vartheta_{ed} \equiv -4\mathsf{P}_{(ca)}\vartheta_{bd}$, and this vanishes under the projection $\Pi_\boxplus$ by property (3) in lemma 3.10 since it is symmetric in two indices and skewsymmetric in the other two. Hence we conclude $R_{cab}{}^e\vartheta_{ed} \equiv U_{cab}{}^e\vartheta_{ed}$.

Now we apply lemma A.2 from appendix to express the Weyl curvature in $U_{cab}{}^e\vartheta_{ed}$ in terms of torsion modulo terms in $\operatorname{Ker}(\Pi_\boxplus)$. In order to do so, we first write $U_{cab}{}^e$ according to (2.9) as

$$U_{cab}{}^e = U^{C'A'B'}_{C\,A\quad E'}\delta^E_B - U^{C'A'E}_{C\,A\quad B}\delta^{B'}_{E'}.$$

Then by the definition of the projection $\Pi_{\boxplus}$, we get

$$U_{acb}{}^{e}\vartheta_{ed} \equiv U_{C\ A\ \ \ \ E'}^{C'[A'B']}\vartheta_{B\ D}^{E'D'} - U_{C[A\ \ \ B]}^{C'A'E}\vartheta_{E\ D}^{B'D'} \mod \mathrm{Ker}(\Pi_{\boxplus}). \tag{3.34}$$

We see that the components of the curvature tensor U we need to express in terms of the torsion are $U_{C[A\ \ \ B]}^{(C'A')E}$, $U_{C[A\ \ \ B]}^{[C'A']E}$, $U_{(C\ A)\ \ \ E'}^{C'[A'B']}$ and $U_{[C\ A]\ \ \ E'}^{C'[A'B']}$. The latter three components are obtained directly from lemma A.2. The first component is not listed in this lemma but it can be easily expressed by means of the other components:

$$U_{C[A\ \ \ B]}^{(C'A')E} = \frac{1}{3}U_{B(C\ \ \ A)}^{(C'A')E} - \frac{1}{3}U_{A(C\ \ \ B)}^{(C'A')E} + U_{[CA\ \ \ B]}^{(C'A')E} \tag{3.35}$$

Before replacing the components of U by the expressions from the appendix, we make one more simplification yet. If we go through the list of the irreducible components of the curvature tensor U which is displayed in (2.7), then we see that only four of them can contribute to the projection $\Pi_{\boxplus}$. Namely, the target bundle $\mathcal{E}_{\boxplus(ABCD)}[-2]$ shows up only in the tensor product of the component $\mathcal{E}_{[AB]}^{(A'B')}$ of $\Lambda^2(\mathcal{E}_A^{A'})$ with the component of U lying in $\mathcal{E}_{[CD]}^{(C'D')}$ or $\mathcal{E}_{\boxplus(CDE)_0}^{(C'D')F}$ and in the tensor product of the component $\mathcal{E}_{(AB)}[-1]$ of $\Lambda^2(\mathcal{E}_A^{A'})$ with the component of U lying in $\mathcal{E}_{(CD)}[-1]$ or $\mathcal{E}_{\boxplus(CDE)_0}^{F}[-1]$. This implies that the other components of U can be neglected. Then equation (A.24) from the appendix simplifies as follows.

$$U_{B(C\ \ \ A)}^{(C'A')E} = -\nabla_{(C}^{I'}T_{A)B\ I'}^{C'A'E} + T_{B\ (C|F'|}^{I'(C'|F|}T_{A\ )F\ I'}^{A')F'E} - \tfrac{1}{q+2}\nabla_{I}^{I'}T_{B(C|I|}^{C'A'I}\delta_{A)}^{E} \mod \text{terms in } \mathcal{E}_{(AB)}^{(A'B')}.$$

Now we use this equation to replace the corresponding terms in equation (3.35). We obtain a formula which can be simplified by use of the fact that the torsion is skew-symmetric in the two lower unprimed indices and that for any such tensor $v_{CAB} \in \mathcal{E}_{C[AB]}$, we have

$$\frac{1}{3}v_{(CA)B} - \frac{1}{3}v_{(CB)A} = \frac{1}{3}v_{CAB} + \frac{1}{3}v_{[A|C|B]} = v_{[A|C|B]} - v_{[ACB]}.$$

Using this we conclude that the first component is equal to

$$U_{C[A\ \ \ B]}^{(C'A')E} = -\nabla_{[B}^{I'}T_{A]C\ I'}^{A'C'E} + T_{[B|C\ F'|}^{I'(C'|F|}T_{A\ ]F\ I'}^{A')F'E} - \tfrac{1}{q+2}\nabla_{I}^{I'}T_{C[A|I|}^{C'A'I}\delta_{B]}^{E} \mod \text{terms in } \mathcal{E}_{(AB)}^{(A'B')}, \mathcal{E}_{[ABC]_0}^{(A'B')D}.$$

The second component is obtained directly from equation (A.24), which simplifies to

$$U_{C[A\ \ \ B]}^{[C'A']E} = -T_{[B|C\ F'|}^{I'[C'|F|}T_{A\ ]F\ I'}^{A']F'E} \mod \text{terms in } \mathcal{E}_{(AB)}^{(A'B')}, \mathcal{E}_{[ABC]_0}^{(A'B')D}.$$

Next, by (A.21) and by (A.23) we get

$$U_{(C\ A)\ \ \ E'}^{C'[A'B']} = 0 \mod \text{terms in } \mathcal{E}_{(AB)}^{(A'B')},$$

respectively

$$U_{[C\ A]\quad E'}^{\ C'[A'B']} = -\tfrac{1}{q+2}\nabla_I^{I'} T_{C\ A\ I'}^{C'[A'|I|}\delta_{E'}^{B']} \quad \text{mod terms in } \mathcal{E}_{[AB]}[-1].$$

Now we have collected all components we need and we substitute the corresponding terms in (3.34). We obtain

$$U_{cab}{}^e\vartheta_{ed} \equiv (\nabla_B^{I'} T_{A\ C\ I'}^{A'C'E} - T_{BC\ F'}^{I'(C'|F|} T_{A\ F\ I'}^{A')F'E} + T_{B\ C\ \ F'}^{I'[C'|F|} T_{A\ F\ I'}^{A']F'E})\vartheta_{E\ D}^{B'D'}.$$

modulo $\operatorname{Ker}(\Pi_\boxplus)$. Since by definition $\Pi_\boxplus$ acts by a complete contraction of primed indices and there is no non-zero complete contraction of $\mathcal{E}^{(A'C')[B'D']}$ and $\mathcal{E}^{[A'C'](B'D')}$, we may write

$$U_{cab}{}^e\vartheta_{ed} \equiv (\nabla_B^{I'} T_{A\ C\ I'}^{A'C'E})\vartheta_{E\ D}^{B'D'} - T_{BC\ F'}^{I'C'F} T_{A\ F\ I'}^{A'F'E}(\vartheta_{\ E\ D}^{(B'D')} - \vartheta_{\ E\ D}^{[B'D']})$$

modulo $\operatorname{Ker}(\Pi_\boxplus)$. The symmetry and trace-freeness of the torsion implies that the right-hand side allows a unique complete contraction of primed indices. This implies in turn that we can swap the indices B' and the upper I' in the last expression. Then the desired statement is obtained by renaming the summing index I' to E' and rewriting in the tensor indices. □

Now we set $\vartheta_{ab} := \Upsilon_{[a}\nabla_{b]}f$ and apply this lemma to the formula (3.33) for the variation of the operator D^1. We immediately get

$$\delta(\tilde{\mathsf{D}}^1 f)_{abcd} \equiv 8T_{bc}{}^f T_{af}{}^e \vartheta^2_{ed} \quad \operatorname{mod}\operatorname{Ker}(\Pi_\boxplus) \tag{3.36}$$

where ϑ^2_{ed} is the component of $\Upsilon_{[e}\nabla_{d]}f$ in $\mathcal{E}_{(AB)}[-1]$. Hence we found out that the operator D^1 has very simple transformation law. In particular, we have proved that D^1 is almost invariant in the sense that it corrects all terms which are linear in torsion.

3.2.4. Transformation of C. To finish the proof of theorem 3.15, we need to show that the variation of the correction C cancels the variation of D^1. In order to do so, we make the variations of individual terms which constitue C explicit by applying basic rules for transformations of covariant derivatives of tensors and by applying properties of $\Pi_\boxplus$ and properties of the torsion.

Lemma 3.19. *The following equations hold modulo* $\operatorname{Ker}(\Pi_\boxplus)$*:*

$$\begin{aligned}
&(a)\quad \delta(T_{B\ C\ F'}^{B'E'F} T_{A\ F\ E'}^{A'F'E}\nabla_E^{C'}\nabla_D^{D'}f) &&\equiv 2T_{B\ C\ F'}^{B'E'F} T_{A\ F\ E'}^{A'F'E}\Upsilon_{[E}^{\ C'}\nabla_{D]}^{D'}f,\\
&(b)\quad \delta(T_{B\ C\ F'}^{B'E'F}(\nabla_E^{C'} T_{A\ F\ E'}^{A'F'E})\nabla_D^{D'}f) &&\equiv (q-2)T_{B\ C\ F'}^{B'E'F} T_{A\ F\ E'}^{A'F'E}\Upsilon_E^{C'}\nabla_D^{D'}f\\
& && \quad + 3T_{B\ C\ F'}^{B'C'F} T_{A\ F\ E'}^{A'F'E}\Upsilon_E^{E'}\nabla_D^{D'}f,\\
&(c)\quad \delta(T_{B\ C\ F'}^{B'C'F}(\nabla_E^{E'} T_{A\ F\ E'}^{A'F'E})\nabla_D^{D'}f) &&\equiv (q+2)T_{B\ C\ F'}^{B'C'F} T_{A\ F\ E'}^{A'F'E}\Upsilon_E^{E'}\nabla_D^{D'}f,
\end{aligned}$$

PROOF. (a): Since the torsion is invariant, it suffices to determine the variation of $\nabla^{C'}_{E}\nabla^{D'}_{D} f$. Applying the rule for the transformation of a one-form from 2.2.3 we have

$$\delta(\nabla^{C'}_{E}\nabla^{D'}_{D} f) = -\Upsilon^{D'}_{E}\nabla^{C'}_{D} f - \Upsilon^{C'}_{D}\nabla^{D'}_{E} f,$$

and after contracting the indices C' and D' we obtain $2\epsilon_{C'D'}\Upsilon^{C'}_{[E}\nabla^{D'}_{D]} f$, and the result follows (by definition of $\Pi_{\boxplus}$).
(b): Due to the invariance of torsion, it is sufficient to determine the variation of $\nabla^{C'}_{E} T^{A'F'E}_{A\,F\,E'}$. Due to the trace-freeness of torsion, the general formula (2.10) for the variation of covariant derivative of torsion in our case simplifies to

$$\delta(\nabla^{C'}_{E})(T^{A'F'E}_{A\,F\,E'}) = q\Upsilon^{C'}_{E} T^{A'F'E}_{A\,F\,E'} - \Upsilon^{A'}_{E} T^{C'F'E}_{A\,F\,E'} - \Upsilon^{F'}_{E} T^{A'C'E}_{A\,F\,E'} + \Upsilon^{I'}_{E} T^{A'F'E}_{A\,F\,I'}\delta^{C'}_{E'}.$$

This can be also written as

$$\begin{aligned}\delta(\nabla^{C'}_{E})(T^{A'F'E}_{A\,F\,E'}) = (q-2)\Upsilon^{C'}_{E} T^{A'F'E}_{A\,F\,E'} + 2\Upsilon^{[C'}_{E} T^{A']F'E}_{A\;\,F\,E'} + 2\Upsilon^{[C'}_{E} T^{F']A'E}_{A\;\,F\,E'}\\ + \Upsilon^{I'}_{E} T^{A'F'E}_{A\,F\,I'}\delta^{C'}_{E'}.\end{aligned}$$

Since primed indices are two-dimensional and the torsion is symmetric in primed indices, the second term on the right equals

$$2\Upsilon^{[C'}_{E} T^{A']F'E}_{A\;\,F\,E'} = \Upsilon^{I'}_{E} T_{I'A\;F\,E'}^{\;\;\;\;\,F'E}\epsilon^{C'A'}$$

and the same holds for the third term up to a permutation of A' and F'. Therefore, variation (b) satisfies

$$\begin{aligned}\delta(T^{B'E'F}_{B\,C\,F'}(\nabla^{C'}_{E} T^{A'F'E}_{A\,F\,E'})\nabla^{D'}_{D} f) = (q-2) T^{B'E'F}_{B\,C\,F'} T^{A'F'E}_{A\,F\,E'}\Upsilon^{C'}_{E}\nabla^{D'}_{D} f\\ + T^{B'E'F}_{B\,C\,F'}(\Upsilon^{I'}_{E} T_{I'A\;F\,E'}^{\;\;\;\;\,F'E}\epsilon^{C'A'} + \Upsilon^{I'}_{E} T_{I'A\;F\,E'}^{\;\;\;\;\,A'E}\epsilon^{C'F'} + \Upsilon^{I'}_{E} T^{A'F'E}_{A\,F\,I'}\delta^{C'}_{E'})\nabla^{D'}_{D} f\end{aligned}$$

and it is sufficient to show that contraction with $\epsilon_{A'B'}\epsilon_{C'D'}$ of terms in the second line leads to identical terms, which sum up to the contraction of $3\Upsilon_e T_{bc}{}^f T_{af}{}^e \nabla_d f$. Since there is no permutation on unprimed indices, we supress them for a moment. Then the contraction of the second line may be displayed as

$$\epsilon_{A'B'}\epsilon_{C'D'} T^{B'E'}{}_{F'}(\Upsilon^{I'} T_{E'}{}^{F'}{}_{I'}\epsilon^{C'A'} + \Upsilon^{I'} T_{E'}{}^{A'}{}_{I'}\epsilon^{C'F'} + \Upsilon^{I'} T^{A'F'}{}_{I'}\delta^{C'}_{E'})\nabla^{D'} f$$

and since the torsion is symmetric, it is easy to see that all terms are equal to $\Upsilon^{I'} T^{J'E'F'} T_{E'F'I'}\nabla_{J'} f$ and thus their sum is equal to

$$3\Upsilon^{I'} T^{J'E'F'} T_{E'F'I'}\nabla_{J'} f = 3 T^{B'C'F}_{B\,C\,F'} T^{A'F'E}_{A\,F\,E'}\Upsilon^{E'}_{E}\nabla^{D'}_{D} f\epsilon_{A'B'}\epsilon_{C'D'}.$$

(c): Analogously to the previous case, it suffices to compute the variation of $\nabla^{E'}_{E} T^{A'F'E}_{A\,F\,E'}$. Since the torsion is trace-free, we get directly from (2.10) that it is equal to $(q+2)\Upsilon^{E'}_{E} T^{A'F'E}_{A\,F\,E'}$ and hence the result follows. $\square$

Now it is an easy observation that C corresponds to a linear combination of terms from the previous lemma written schematically as $-2(a)+\frac{4}{q-2}(b)-\frac{12}{q^2-4}(c)$ and thus the variation of C is equivalent to

$$\begin{aligned}\delta(\tilde{\mathsf{C}}f)^{A'B'C'D'}_{A\,B\,C\,D} &\equiv -4T^{B'E'F}_{B\,C\,F'}T^{A'F'E}_{A\,F\,E'}\Upsilon^{\;C'}_{[E}\nabla^{D'}_{D]}f + 4T^{B'E'F}_{B\,C\,F'}T^{A'F'E}_{A\,F\,E'}\Upsilon^{C'}_{E}\nabla^{D'}_{D}f \\ &= -4T^{B'E'F}_{B\,C\,F'}T^{A'F'E}_{A\,F\,E'}\Upsilon^{\;C'}_{(E}\nabla^{D'}_{D)}f.\end{aligned}$$

And since evidently

$$T^{B'E'F}_{B\,C\,F'}T^{A'F'E}_{A\,F\,E'}\Upsilon^{\;C'}_{(E}\nabla^{D'}_{D)}f \equiv 2T^{B'C'F}_{B\,C\,F'}T^{A'F'E}_{A\,F\,E'}\Upsilon^{[E'}_{(E}\nabla^{D']}_{D)}f$$

modulo $\mathrm{Ker}(\Pi_{\boxplus})$, we have

$$\delta(\tilde{\mathsf{C}}f)_{abcd} \equiv -8T_{bc}{}^{f}T_{af}{}^{e}\vartheta^2_{ed} \mod \mathrm{Ker}(\Pi_{\boxplus}),$$

where ϑ^2_{ed} is the component of $\Upsilon_{[e}\nabla_{d]}f$ in $\mathcal{E}_{(ED)}[-1]$. Visibly this is up to the sign the same formula as formula (3.33) for $\delta(\tilde{\mathsf{D}}^1 f)_{abcd}$. Hence the sum of these two variations vanishes under the projection $\Pi_{\boxplus}$, and so we obtain

$$\delta(\mathsf{D}^{\mathrm{corr}}) = \delta(\mathsf{D}^1) + \delta(\mathsf{C}) = 0,$$

which shows that the operator $\mathsf{D}^{\mathrm{corr}}$ is invariant. □

Remark 3.3. Let us now explain how to get a formula for $\mathsf{D}^{\mathrm{corr}}$ which is invariant to all changes of Weyl structures. If we trace back where we used the exactness of the Weyl connection, we find that it was only at the end of the section 3.2.2. Concretely, there we used twice that the one-form Υ describing the change of Weyl structure is closed, i.e.

$$(d\Upsilon)_{ab} = 2\nabla_{[a}\Upsilon_{b]} + T_{ab}{}^{e}\Upsilon_e = 0.$$

In the case of a change described by an one-form Υ which is not closed, the equation (3.33) has the form

$$\begin{aligned}\delta(\tilde{\mathsf{D}}^1 f)_{abcd} \equiv\, & 4(R_{cab}{}^{e} - \nabla_a T_{bc}{}^{e} + T_{bc}{}^{f}T_{af}{}^{e})\Upsilon_{[e}\nabla_{d]}f \\ & + 2(d\Upsilon)_{eb}T_{ac}{}^{e}\nabla_d f - (d\Upsilon)_{db}T_{ac}{}^{e}\nabla_e f.\end{aligned}$$

We did not use exactness in sections 3.2.3 and 3.2.4, and so we deduce

$$\delta(\tilde{\mathsf{D}}^{\mathrm{corr}} f)_{abcd} \equiv 2(d\Upsilon)_{eb}T_{ac}{}^{e}\nabla_d f - (d\Upsilon)_{db}T_{ac}{}^{e}\nabla_e f.$$

Now we easily find a correction which leads to an operator invariant to arbitrary change of the Weyl structure since there is an obvious correction which cancels $(d\Upsilon)_{ab}$. Namely, by (2.11) we have $\delta(\mathsf{P}_{[ab]}) = \nabla_{[a}\Upsilon_{b]}$, and from (2.10) trace-freeness of the torsion we conclude $\delta(\nabla_e T_{ab}{}^{e}) = (q+2)\Upsilon_e T_{ab}{}^{e}$. Hence

$$\delta(2\mathsf{P}_{[ab]} + \frac{1}{(q+2)}\nabla_e T_{ab}{}^{e}) = (d\Upsilon)_{ab},$$

and so the operator invariant to arbitrary change is given by applying the projection $\Pi_{\boxplus}$ to

$$\tilde{\mathsf{D}}^{\mathrm{corr}}_{abcd} - 2(2\mathsf{P}_{[eb]} + \frac{1}{(q+2)}\nabla_f T_{eb}{}^f)T_{ac}{}^e\nabla_d + (2\mathsf{P}_{[db]} + \frac{1}{(q+2)}\nabla_f T_{db}{}^f)T_{ac}{}^e\nabla_e. \tag{3.37}$$

3.2.5. Non-existence of factorization. Now we prove that there exists no correction of the form $d_{\boxplus}(\mathsf{A}+\mathrm{corr.})d$, which would lead to an invariant operator. Indeed, there is not much feedom in varying the second order operator A. This is reflected in the following lemma.

Lemma 3.20. *Any torsion-extension of* $A : \mathcal{E}^{A'}_A \to \mathcal{E}^{A'} \otimes \begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}\mathcal{E}_A[-1]$ *has the form* $A^{tor} = \Pi_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}} \circ \tilde{A}^{tor}$ *where*

$$(\tilde{A}^{tor}\mu)_{abc} = (\tilde{A}\mu)_{abc} + \alpha T_{ca}{}^e\nabla_e\mu_b + \beta(\nabla_e T_{ca}{}^e)\mu_b$$

for some numerical coefficients α, β.

Proof. Since $\mu \mapsto T^2 \cdot \mu$ is invariant, it is easy to see that potential correction terms of $\mathsf{A}\mu$ are of the form $T \cdot \nabla\mu$ or $(\nabla T) \cdot \mu$. As we described in the previous chapter, the torsion lies in

$$T^{A'B'C}_{A\ B\ C'} \in (\mathcal{E}^{(A'B')}{}_{C'})_0 \otimes (\mathcal{E}_{[AB]}{}^C)_0 \cong \square\square\square\,\mathcal{E}^{A'} \otimes {}_{q-1}\begin{smallmatrix}\square&\square\\ \vdots&\\ \square&\end{smallmatrix}\mathcal{E}_A[2].$$

Therefore, both correction terms $T \cdot \nabla\mu$ and $(\nabla T) \cdot \mu$ lie in

$$(\otimes^2\mathcal{E}^{A'} \otimes \square\square\square\,\mathcal{E}^{A'}) \otimes (\otimes^2\mathcal{E}_A \otimes {}_{q-1}\begin{smallmatrix}\square&\square\\ \vdots&\\ \square&\end{smallmatrix}\mathcal{E}_A)[2].$$

Since there is a unique way of the contraction of primed indices and there are two projections of the tensor product of $\mathcal{E}_A$'s to $\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}\mathcal{E}_A[-1]$, the multiplicity of the bundle $\mathcal{E}^{A'} \otimes \begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}\mathcal{E}_A[-1]$ in this tensor product is two. It means that we have four potential correction terms - two for each of the two types. Using the projection $\Pi_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}}$, the representatives may be written as $\Pi_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}}(\nu)$ where $\nu^{A'B'C'}_{A\ B\ C}$ equals one of the following four expressions

$$T^{C'A'E}_{C\ A\ E'}\nabla^{E'}_E\mu^{B'}_B\,, T^{C'A'E}_{C\ A\ E'}\nabla^{E'}_B\mu^{B'}_E\,, (\nabla^{E'}_E T^{C'A'E}_{C\ A\ E'})\mu^{B'}_B\,, (\nabla^{E'}_B T^{C'A'E}_{C\ A\ E'})\mu^{B'}_E$$

The first two terms differ by $T^{C'A'E}_{C\ A\ E'}\nabla^{E'}_{[E}\mu^{B'}_{B]}$ and this is equivalent to the invariant expression $T^{C'A'E}_{C\ A\ E'}\nabla^{(E'}_{[E}\mu^{B')}_{B]}$ modulo $\mathrm{Ker}(\Pi_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}})$. Therefore, the first two terms transform in the same way. We can easily prove (e.g. by a direct computation) that, up to the sign, the fourth term also transforms in the same way as the first two terms do. This implies that each torsion-correction of the "middle" operator A may be written as projection $\Pi_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}}$ of a linear combination of the first and the third term, i.e. a linear combination of $T_{bc}{}^e\nabla_e\mu_d$ and $(\nabla_e T_{bc}{}^e)\mu_d$. $\square$

Now we observe that the correction terms from this lemma are those appearing in the formula for the operator D^1 given by the equation (3.26). Hence an immediate consequence of this lemma is that any torsion-extension of operator D of a form $d_{\boxplus}(A + \text{corr.})d$ is given by a projection under $\Pi_{\boxplus}$ of

$$(\tilde{\mathsf{D}}^{tor} f)_{abcd} := (\tilde{\mathsf{D}}^1 f)_{abcd} + \alpha \nabla_a (T_{bc}{}^e \nabla_e \nabla_d f) + \beta \nabla_a ((\nabla_e T_{bc}{}^e) \nabla_d f) \quad (3.38)$$

for some new coefficients α, β. We prove that demanding the invariance in some special cases yields $\alpha = \beta = 0$. In other words, if D^{tor} is invariant, then it coincides with D^1, which is a contradiction. Thus we conclude

Proposition 3.21. *There is no extension of the operator* D *to an operator of the form* $d(A + corr.)d$ *which is invariant on almost Grassmannian structures with non-vanishing torsion.*

PROOF. Assume that such an invariant operator exists. Then we know that it is given by equation (3.38) for some scalars α, β. By assumption, the variation of D^{tor} is zero. On the other hand, we deduce from equation (3.31) in the proof of theorem 3.15 that variations of the correction terms are

$$\delta^1_{abcd} := \delta(\nabla_a (T_{bc}{}^e \nabla_e \nabla_d f)) \equiv -2 \nabla_d (T_{ca}{}^e \Upsilon_{(e} \nabla_{b)} f) \quad (3.39)$$

$$\delta^2_{abcd} := \delta(\nabla_a ((\nabla_e T_{bc}{}^e) \nabla_d f)) \equiv (q+2) \nabla_d (T_{ca}{}^e \Upsilon_e \nabla_b f) \quad (3.40)$$

modulo $\mathrm{Ker}(\Pi_{\boxplus})$. Now we assume an action of D^{tor} on a function f which satisfies $\nabla_a f(0) = 0$. Then from (3.36), we get $\delta(\mathsf{D}^1 f) = 0$ at the origin and from (3.39) and (3.40), we conclude that at 0 we have

$$\delta^1_{abcd} \equiv -T_{ca}{}^e \Upsilon_e \nabla_d \nabla_b f - T_{ca}{}^e \Upsilon_b \nabla_d \nabla_e f$$

$$\delta^2_{abcd} \equiv (q+2) T_{ca}{}^e \Upsilon_e \nabla_d \nabla_b f$$

modulo $\mathrm{Ker}(\Pi_{\boxplus})$. According to property (3) in lemma 3.10, we get

$$T_{ca}{}^e \Upsilon_e \nabla_d \nabla_b f \equiv T_{ca}{}^e \Upsilon_e \nabla_{[d} \nabla_{b]} f \mod \mathrm{Ker}(\Pi_{\boxplus})$$

and this vanishes at the origin since the alternation of the second covariant derivative of f is given by an action of the torsion on the first derivative. Thus at the origin, we obtain

$$\delta^1_{abcd} \equiv -T_{ca}{}^e \Upsilon_b \nabla_{(d} \nabla_{e)} f \quad \text{and} \quad \delta^2_{abcd} \equiv 0 \mod \mathrm{Ker}(\Pi_{\boxplus}).$$

Hence demanding invariance of D^{tor} yields equation $\alpha T_{ca}{}^e \Upsilon_b \nabla_{(d} \nabla_{e)} f \equiv 0$, which is in the case of non-vanishing torsion satisfied only for $\alpha = 0$ since the value of the symmetrized second derivative of f and the form Υ_a at the origin can be chosen freely.

Next assume a change of Weyl structures described by a one-form Υ_a which satisfies $\Upsilon_a(0) = 0$. In such a case, we also get $\delta(\mathsf{D}^1) = 0$ at the origin according to (3.36) and for exact Weyl structures, equations (3.39) and (3.40) simplify as follows

$$\delta^1_{abcd} \equiv -(\nabla_{(d}\Upsilon_{e)})T_{ca}{}^e\nabla_b f,$$

$$\delta^2_{abcd} \equiv (q+2)(\nabla_{(d}\Upsilon_{e)})T_{ca}{}^e\nabla_b f \quad \operatorname{mod}\operatorname{Ker}(\Pi_{\boxplus}).$$

Thus demanding invariance of D^{tor} in this case yields equation

$$(-\alpha + (q+2)\beta)(\nabla_{(e}\Upsilon_{a)})T_{bc}{}^e\nabla_d f \equiv 0$$

Since the derivative of f and the derivative of Υ_a can be chosen freely at the origin and since α is fixed to 0, this equation is satisfied only for $\beta = 0$. Hence we see that D^{tor} must be equal to D^1, which is a contradiction since D^1 is not invariant. Indeed, (3.36) shows that its linearized transformation contains a non-zero term quadratic in torsion. □

Remark 3.4. It follows from the proof of this proposition that any invariant torsion-extension of the non-standard invariant operator on functions must be of a form D^1+ terms quadratic in torsion, i.e. terms of the form $T^2\cdot\nabla^2$, $T\cdot(\nabla T)\cdot\nabla$. These terms cancel the variation of D^1 given by equation (3.36), which has the form $\delta(\mathsf{D}^1) = \Upsilon\cdot T^2\cdot\nabla$. In a sense, the invariance of the torsion can be viewed as the obstruction to existence of an invariant extension of D in a form $d_{\boxplus}(\mathsf{A}+\text{corr.})d$ similarly as the invariance of the Bach tensor can be thought of as the obstruction to the existence of the curved analogue of the cube of the Laplacian in conformal geometry in the dimension 4, cf. [**14**].

APPENDIX A

Bianchi identity

Irreducible components of curvature. Let us consider a linear connection $\nabla_A^{A'}$ on the tangent bundle $TM = \mathcal{E}_{A'}^A$ induced by a Weyl connection $\sigma^*\omega_0$. According to the description of its curvature in section 2.2.2, we have

$$R_{A\ B\ \ D}^{A'B'C} = U_{A\ B\ \ D}^{A'B'C} - \mathsf{P}_{A\ D}^{A'B'}\delta_B^C + \mathsf{P}_{B\ D}^{B'A'}\delta_A^C$$
$$R_{A\ B\ \ C'}^{A'B'D'} = U_{A\ B\ \ C'}^{A'B'D'} + \mathsf{P}_{A\ B}^{A'D'}\delta_{C'}^{B'} - \mathsf{P}_{B\ A}^{B'D'}\delta_{C'}^{A'},$$

for the corresponding curvatures of $\nabla_A^{A'}$ on $\mathcal{E}^A$ and $\mathcal{E}^{A'}$ respectively. Since $R_{A\ B\ \ I}^{A'B'I} = -R_{A\ B\ \ I'}^{A'B'I'}$, taking traces over C and D and over C' and D' in the previous equations yields $U_{A\ B\ \ I}^{A'B'I} = -U_{A\ B\ \ I'}^{A'B'I'}$. Moreover, for exact Weyl connections these traces satisfy

$$U_{ab}{}^{I'}{}_{I'} = -U_{ab}{}^{I}{}_{I} = 2\mathsf{P}_{[ab]}. \tag{A.1}$$

Since the components of Weyl curvature $U_{A\ B\ \ D}^{A'B'C}$ and $U_{A\ B\ \ C'}^{A'B'D'}$ are sections of bundles $\mathcal{E}_A^{A'} \wedge \mathcal{E}_B^{B'} \otimes \mathcal{E}^C{}_D$ respectively $\mathcal{E}_A^{A'} \wedge \mathcal{E}_B^{B'} \otimes \mathcal{E}^{D'}{}_{C'}$, they decompose according to the obvious decompositions $\mathcal{E}_A^{A'} \wedge \mathcal{E}_B^{B'} = \mathcal{E}_{[AB]}^{(A'B')} \oplus \mathcal{E}_{(AB)}[-1]$ and $\mathcal{E}^C{}_D = (\mathcal{E}^C{}_D)_0 \oplus \mathcal{E}$, respectively $\mathcal{E}^{D'}{}_{C'} = (\mathcal{E}^{D'}{}_{C'})_0 \oplus \mathcal{E}$. The last two decompositions are decompositions into a trace-free part and a trace part. And since we have shown above that these traces are the same up to the sign, we may write the decompositions as

$$\begin{aligned} U_{A\ B\ \ C}^{A'B'E} &= \epsilon^{A'B'}\varphi_{AB}{}^E{}_C + V_{A\ B}^{A'B'E}{}_C - 2(\epsilon^{A'B'}\varphi_{AB} + V_{A\ B}^{A'B'})\delta_C^E \\ U_{A\ B\ \ E'}^{A'B'C'} &= \epsilon^{A'B'}\varphi_{AB}{}^{C'}{}_{E'} + V_{A\ B}^{A'B'C'}{}_{E'} + q(\epsilon^{A'B'}\varphi_{AB} + V_{A\ B}^{A'B'})\delta_{E'}^{C'}, \end{aligned} \tag{A.2}$$

for objets $V_{A\ B}^{A'B'}$, $V_{A\ B}^{A'B'C'}{}_{E'}$, $V_{A\ B}^{A'B'E}{}_C$ symmetric in A', B', skewsymmetric in A, B and satisfying $V_{AB}^{A'B'I'}{}_{I'} = V_{AB}^{A'B'I}{}_I = 0$, and for objects φ_{AB}, $\varphi_{AB}{}^{C'}{}_{E'}$, $\varphi_{AB}{}^E{}_C$ symmetric in A, B and satisfying $\varphi_{AB}{}^{I'}{}_{I'} = \varphi_{AB}{}^I{}_I = 0$.

In terms of V's and φ's, we can easily write down the corresponding representatives of the irreducible representations occuring in the decomposition

2.7 of the Weyl curvature. Namely, for the traces of U we have

$$\begin{aligned}
2\cdot\mathcal{E}_{(AB)}^{(A'B')} &\rightsquigarrow \epsilon^{I'(A'}\varphi_{AB}{}^{B')}{}_{I'}, V_{I\ (A}^{A'B'I}{}_{B)} \\
2\cdot\mathcal{E}_{[AB]}[-1] &\rightsquigarrow \varphi_{I[A}{}^{I}{}_{B]}, V_{AB}^{I'[A'B']}{}_{I'}\epsilon_{A'B'} \\
2\cdot\mathcal{E}_{(AB)}[-1] &\rightsquigarrow \varphi_{AB}, \varphi_{I(A}{}^{I}{}_{B)} \\
3\cdot\mathcal{E}_{[AB]}^{(A'B')} &\rightsquigarrow V_{A\ B}^{A'B'}, V_{A\ B}^{I'(A'B')}{}_{I'}, V_{I\ [B}^{A'B'I}{}_{C]},
\end{aligned}$$

and modulo these trace-terms, the representatives of the other irreducible components are

$$\begin{aligned}
\mathcal{E}_{(ABC)_0}{}^{D}[-1] &\rightsquigarrow \varphi_{(AB}{}^{D}{}_{C)} \\
\mathcal{E}_{[AB]}^{(A'B'C'D')}[1] &\rightsquigarrow V_{A\ B}^{(A'B'C')}{}_{E'}\epsilon^{E'D'} \\
\mathcal{E}_{\boxplus(ABC)_0}^{(A'B')D} &\rightsquigarrow V_{A\ (B}^{A'B'D}{}_{C)} \\
\mathcal{E}_{\boxplus(ABC)}^{D}[-1] &\rightsquigarrow \varphi_{A[B}{}^{D}{}_{C]} \\
\mathcal{E}_{[ABC]_0}^{(A'B')D} &\rightsquigarrow V_{[A\ B}^{A'B'D}{}_{C]}
\end{aligned}$$

Now the ∂^*-closedness of U gives relations between some of these representatives. To make these relations precise, we substitute the components of U into equation (2.8) describing ∂^*-closedness of U by V's and φ's defined by (A.2). We obtain

$$\begin{aligned}
\epsilon^{A'B'}\varphi_{AI}{}^{I}{}_{B} + V_{A\ I}^{A'B'I}{}_{B} - \epsilon^{A'I'}\varphi_{AB}{}^{B'}{}_{I'} - V_{A\ B}^{A'I'B'}{}_{I'} \\
- (q+2)(\epsilon^{A'B'}\varphi_{AB} + V_{A\ B}^{A'B'}) = 0.
\end{aligned}$$

The term $\varphi_{AB}{}^{B'A'} = -\epsilon^{A'I'}\varphi_{AB}{}^{B'}{}_{I'}$ is symmetric in A' and B' due to the definition of $\varphi_{AB}{}^{C'}{}_{D'}$ and the fact that the fibre dimension of $\mathcal{E}^{A'}$ equals two. Thus taking symmetrizations and alternations in A, B and A', B' of the equation above yields the following

Lemma A.1. *The ∂^*-closedness of U is equivalent to the following system of equations*

$$V_{I(A}^{A'B'I}{}_{B)} = \varphi_{AB}{}^{A'B'} \tag{A.3}$$

$$V_{A\ B}^{I'[A'\ B']}{}_{I'} = \epsilon^{A'B'}\varphi_{I[A}{}^{I}{}_{B]} \tag{A.4}$$

$$\varphi_{AB} = \frac{1}{q+2}\varphi_{I(A}{}^{I}{}_{B)} \tag{A.5}$$

$$V_{A\ B}^{A'B'} = -\frac{1}{q+2}(V_{I[A}^{A'B'I}{}_{B]} + V_{A\ B}^{I'(A'\ B')}{}_{I'}) \tag{A.6}$$

Algebraic Bianchi identity. It follows from the description of the harmonic curvature in section 2.1.6 that its homogeneous component of degree two is the irreducible component of the Weyl curvature which lies in the trace-free part of $\mathcal{E}_{(ABC)}{}^{D}$, i.e. it is the trace-free part of $\varphi_{(AB}{}^{D}{}_{C)}$. In the case that the harmonic curvature of homogeneity one, i.e. the torsion vanishes, this is the only non-zero part of the Weyl curvature. It means in turn that the all other irreducible components of U can be expressed in terms of torsion. To find these formulae, we apply the algebraic Bianchi identiy which by 1.11 has the form

$$U_{[ab}{}^{e}{}_{c]} = \nabla_{[a}T_{bc]}{}^{e} - T_{[ab}{}^{f}T_{c]f}{}^{e}. \tag{A.7}$$

In order to express irreducible components of U, one should consider various traces and symmetrizations of this equation. First, let us consider the trace over c and e. Expanding the alternations, we get

$$U_{ab}{}^{i}{}_{i}+U_{ia}{}^{i}{}_{b}+U_{bi}{}^{i}{}_{a} = \nabla_{a}T_{bi}{}^{i}+\nabla_{i}T_{ab}{}^{i}+\nabla_{b}T_{ia}{}^{i}-T_{ab}{}^{j}T_{ij}{}^{i}-T_{ia}{}^{j}T_{bj}{}^{i}-T_{bi}{}^{j}T_{aj}{}^{i}$$

Now observe that equation (2.8) describing the ∂^{*}-closedness of U is equivalent to $U_{ia}{}^{i}{}_{b}=0$. Together with the trace-freeness of the torsion, we get

$$U_{ia}{}^{i}{}_{b} = U_{bi}{}^{i}{}_{a} = \nabla_{a}T_{bi}{}^{i} = \nabla_{b}T_{ia}{}^{i} = T_{ab}{}^{j}T_{ij}{}^{i} = 0,$$

and since evidently $-T_{ia}{}^{j}T_{bj}{}^{i}-T_{bi}{}^{j}T_{aj}{}^{i}=0$, the contracted Bianchi identity simplifies to $U_{ab}{}^{i}{}_{i}=\nabla_{i}T_{ab}{}^{i}$. Now when we substitute U by expression (A.2), we get

$$-2q(q+2)(\epsilon^{A'B'}\varphi_{AB}+V^{A'B'}_{A\ B}) = \nabla^{I'}_{I}T^{A'B'I}_{A\ B\ I'}.$$

And since $T^{A'B'I}_{A\ B\ I'} = T^{(A'B')I}_{[A\ B]\ I'}$ according to (2.4), taking the parts which are symmetric respectively skew-symmetric in A and B gives the following two equations

$$\varphi_{AB} = 0 \tag{A.8}$$

$$V^{A'B'}_{A\ B} = -\frac{1}{2q(q+2)}\nabla^{I'}_{I}T^{A'B'I}_{A\ B\ I'}. \tag{A.9}$$

Then identities (A.8) and (A.5) immediately imply

$$\varphi_{I(A}{}^{I}{}_{B)} = 0. \tag{A.10}$$

In order to express the other irreducible components of U, it is convenient to consider the trace of equation (A.7) over C' and E'. Expanding the alternations therein and replacing the tensor U by its components according to (2.9), we get

$$2U^{A'B'E}_{A\ B\ C} - U^{A'B'I'}_{A\ B\ I'}\delta^{E}_{C} + U^{B'A'E}_{C\ A\ B} - U^{I'A'B'}_{B\ A\ I'}\delta^{E}_{C} + U^{B'A'E}_{B\ C\ A} - U^{B'I'A'}_{B\ A\ I'}\delta^{E}_{C}$$
$$= \nabla^{I'}_{C}T^{A'B'E}_{A\ B\ I'} - T^{I'A'J}_{C\ A\ J'}T^{B'J'E}_{B\ J\ I'} - T^{B'I'J}_{B\ C\ J'}T^{A'J'E}_{A\ J\ I'}.$$

Now we substitute the Weyl curvature U according to (A.2):

$$
\begin{aligned}
&2\epsilon^{A'B'}\varphi_{AB}{}^{E}{}_{C} + 2V_{A\ B}^{A'B'E}{}_{C} - 2(q+2)V_{A\ B}^{A'B'}\delta_C^E + \epsilon^{B'A'}\varphi_{CA}{}^{E}{}_{B} + V_{C\ A}^{B'A'E}{}_{B} \\
&-\varphi_{CA}{}^{A'B'}\delta_B^E - V_{C\ A}^{I'A'B'}{}_{I'}\delta_B^E - (q+2)V_{C\ A}^{B'A'}\delta_B^E + \epsilon^{B'A'}\varphi_{BC}{}^{E}{}_{A} \\
&+V_{B\ C}^{B'A'E}{}_{A} - \varphi_{BC}{}^{A'B'}\delta_A^E - V_{B\ C}^{B'I'A'}{}_{I'}\delta_A^E - (q+2)V_{B\ C}^{B'A'}\delta_A^E \\
&= \nabla_C^{I'}T_{A\ B\ I'}^{A'B'E} - T_{C\ A\ J'}^{I'A'J}T_{B\ J\ I'}^{B'J'E} - T_{B\ C\ J'}^{B'I'J}T_{A\ J\ I'}^{A'J'E}
\end{aligned}
\tag{A.11}
$$

In order to obtain trace-terms first, let us now compute the trace over A and E of this equation.

$$
\begin{aligned}
&2\epsilon^{A'B'}\varphi_{IB}{}^{I}{}_{C} - \epsilon^{A'B'}\varphi_{IC}{}^{I}{}_{B} + 2V_{I\ B}^{A'B'I}{}_{C} - V_{I\ C}^{A'B'I}{}_{B} \\
&-\varphi_{CB}{}^{AB'} + q\varphi_{BC}{}^{A'B'} + V_{B\ C}^{I'A'B'}{}_{I'} - qV_{B\ C}^{I'B'A'}{}_{I'} \\
&-(q-3)(q+2)V_{B\ C}^{A'B'} = -T_{C\ I\ J'}^{I'A'J}T_{B\ J\ I'}^{B'J'I}.
\end{aligned}
\tag{A.12}
$$

The symmetrization of this equation in A' and B' yields

$$
\begin{aligned}
&2V_{I\ B\ C}^{A'B'I} - V_{I\ C\ B}^{A'B'I} + (q-1)\varphi_{BC}{}^{A'B'} - (q-1)V_{B\ C}^{I'(A'B')}{}_{I'} \\
&-(q-3)(q+2)V_{B\ C}^{A'B'} = -T_{C\ I\ J'}^{I'(A'|J|}T_{B\ J\ I'}^{B')J'I}
\end{aligned}
$$

Now we are going to symmetrize and alternate in the indices B and C in order to get equations for V's and φ's appearing in this equation. The alternation yields

$$
3V_{I\ [B\ C]}^{A'B'I} - (q-1)V_{B\ C}^{I'(A'B')}{}_{I'} - (q-3)(q+2)V_{B\ C}^{A'B'} = T_{I\ [C\ |J'|}^{I'(A'|J|}T_{B]\ J\ I'}^{B')J'I} = 0,
$$

which, combined with equations (A.6) and (A.9), implies

$$
V_{I\ [B\ C]}^{A'B'I} = -2V_{B\ C}^{A'B'} = \frac{1}{q(q+2)}\nabla_I^{I'}T_{B\ C\ I'}^{A'B'I} \tag{A.13}
$$

$$
V_{B\ C}^{I'(A'B')}{}_{I'} = -qV_{B\ C}^{A'B'} = \frac{1}{2(q+2)}\nabla_I^{I'}T_{B\ C\ I'}^{A'B'I}. \tag{A.14}
$$

On the other hand, the symmetrization yields

$$
V_{I\ (B\ C)}^{A'B'I} + (q-1)\varphi_{BC}{}^{A'B'} = T_{I\ (B\ |J'|}^{I'(A'|J|}T_{C)\ J\ I'}^{B')J'I}.
$$

From here and (A.3) we get

$$
\varphi_{BC}{}^{A'B'} = V_{I\ (B\ C)}^{A'B'I} = \frac{1}{q}T_{I\ (B\ |J'|}^{I'(A'|J|}T_{C)\ J\ I'}^{B')J'I}. \tag{A.15}
$$

Taking the alternation over indices A' and B' in (A.12) leads to

$$
2\epsilon^{A'B'}\varphi_{IB}{}^{I}{}_{C} - \epsilon^{A'B'}\varphi_{IC}{}^{I}{}_{B} + (q+1)V_{B\ C}^{I'[A'B']}{}_{I'} = -T_{C\ I\ J'}^{I'[A'|J|}T_{B\ J\ I'}^{B']J'I}.
$$

It follows from (A.10) that all terms are actually skewsymmetric in B and C and thus this equation is equivalent to

$$3\epsilon^{A'B'}\varphi_{I[B}{}^{I}{}_{C]} + (q+1)V^{I'[A'B']}_{B\,C\quad\;I'} = T^{I'[A'|J|}_{I\;[C\;|J'|}T^{B']J'I}_{B]\;J\;I'}.$$

This together with the equation (A.4) yields

$$V^{I'[A'B']}_{B\,C\quad\;I'} = \epsilon^{A'B'}\varphi_{I[B}{}^{I}{}_{C]} = \frac{1}{q+4}T^{I'[A'|J|}_{I\;[C\;|J'|}T^{B']J'I}_{B]\;J\;I'}. \tag{A.16}$$

The equations (A.8), (A.9), (A.10), (A.13), (A.14), (A.15), (A.16) form a complete list of expressions for traces of V's and φ's. Now we come back to the equation (A.11) in order to express also the trace-free parts. Taking the alternation of A' and B' in (A.11) yields

$$\begin{aligned}&2\epsilon^{A'B'}\varphi_{AB}{}^{E}{}_{C} - \epsilon^{A'B'}\varphi_{CA}{}^{E}{}_{B} - \epsilon^{A'B'}\varphi_{BC}{}^{E}{}_{A} - V^{I'[A'B']}_{C\,A}{}_{I'}\delta^{E}_{B} - V^{I'[B'A']}_{C\,B}{}_{I'}\delta^{E}_{A}\\ &= -T^{I'[A'J}_{C\;A\;J'}T^{B']J'E}_{B\;J\;I'} - T^{I'[A'J}_{C\;B\;J'}T^{B']J'E}_{A\;J\;I'}.\end{aligned}$$

The term $\varphi_{AB}{}^{E}{}_{C}$ has two trace-free components. The first one is the harmonic part of the Weyl curvature which lies in $\mathcal{E}^{E}_{(ABC)_0}$ and which we obviously cannot express from the previous equation. In order to express the second one, which is the component in $\mathcal{E}^{E}_{\boxplus(ABC)_0}$, we alternate the previous equation in B and C. We get

$$\begin{aligned}3\epsilon^{A'B'}\varphi_{A[B}{}^{E}{}_{C]} = \;& -T^{I'[A'|J|}_{[C|A\;J'|}T^{B']J'E}_{B]\;J\;I'} - T^{I'[A'|J|}_{C\;B\;J'}T^{B']J'E}_{A\;J\;I'}\\ &+V^{I'[A'B']}_{[C|A\qquad I'|}\delta^{E}_{B]} + V^{I'[B'A']}_{B\,C\qquad I'}\delta^{E}_{A}.\end{aligned}$$

This equation can be writte also in the form

$$\begin{aligned}\epsilon^{A'B'}\varphi_{A[B}{}^{E}{}_{C]} = \;& T^{I'[A'J}_{A\;[C|J'|}T^{B']J'E}_{B]\;J\;I'} - T^{I'[A'J}_{[A\;C|J'|}T^{B']J'E}_{B]\;J\;I'}\\ &-V^{I'[A'B']}_{A[C\qquad |I'|}\delta^{E}_{B]} + V^{I'[A'B']}_{[A\,C\qquad |I'|}\delta^{E}_{B]}.\end{aligned}$$

Then the application of (A.16) gives

$$\begin{aligned}\epsilon^{A'B'}\varphi_{A[B}{}^{E}{}_{C]} = \;& T^{I'[A'J}_{A\;[C|J'|}T^{B']J'E}_{B]\;J\;I'} - T^{I'[A'J}_{[A\;C|J'|}T^{B']J'E}_{B]\;J\;I'}\\ &-\tfrac{1}{q+4}T^{I'[A'|J|}_{I\;A\;J'}T^{B']J'I}_{[B\;J\;I'}\delta^{E}_{C]} + \tfrac{1}{q+4}T^{I'[A'|J|}_{I\;[A|J'|}T^{B']J'I}_{B\;|J\;I'|}\delta^{E}_{C]}.\end{aligned} \tag{A.17}$$

Now we take the symmetrization of (A.11) in A' and B'. We obtain

$$\begin{aligned}&2V^{A'B'E}_{A\,B}{}_{C} - 2(q+2)V^{A'B'}_{A\,B}\delta^{E}_{C} + V^{A'B'E}_{C\,A}{}_{B} - \varphi_{CA}{}^{A'B'}\delta^{E}_{B}\\ &-V^{I'(A'\;B')}_{C\,A}{}_{I'}\delta^{E}_{B} - (q+2)V^{A'B'}_{C\,A}\delta^{E}_{B} + V^{A'B'E}_{B\,C}{}_{A} + \varphi_{BC}{}^{A'B'}\delta^{E}_{A}\\ &-V^{I'(B'\;A')}_{B\,C}{}_{I'}\delta^{E}_{A} - (q+2)V^{A'B'}_{B\,C}\delta^{E}_{A}\\ &= \nabla^{I'}_{C}T^{A'B'E}_{A\;B\;I'} - T^{I'(A'J}_{C\;A\;J'}T^{B')J'E}_{B\;J\;I'} - T^{I'(A'J}_{B\;C\;J'}T^{B')J'E}_{A\;J\;I'},\end{aligned}$$

and this can be simplified with the help of (A.14) to

$$\begin{aligned}&2V_{A\ B}^{A'B'E}{}_{C}+V_{C\ A}^{A'B'E}{}_{B}+V_{B\ C}^{A'B'E}{}_{A}-\varphi_{CA}{}^{A'B'}\delta_{B}^{E}\\&+\varphi_{BC}{}^{A'B'}\delta_{A}^{E}-2V_{C\ A}^{A'B'}\delta_{B}^{E}-2V_{B\ C}^{A'B'}\delta_{A}^{E}-2(q+2)V_{A\ B}^{A'B'}\delta_{C}^{E}\\&=\nabla_{C}^{I'}T_{A\ B\ I'}^{A'B'E}-T_{C\ A\ J'}^{I'(A'J}T_{B\ J\ I'}^{B')J'E}-T_{B\ C\ J'}^{I'(A'J}T_{A\ J\ I'}^{B')J'E}.\end{aligned}$$

Alternating over A, B and C and applying (A.9) then yields

$$\begin{aligned}V_{[A\ B\ \ C]}^{A'B'E} &= \tfrac{1}{4}(\nabla_{[C}^{I'}T_{A\ B]\ I'}^{A'B'E}-2T_{[C\ A\ J'}^{I'(A'J}T_{B]\ J\ I'}^{B')J'E}\\&\quad-\tfrac{q+4}{q(q+2)}\nabla_{I}^{I'}T_{[A\ B|I'|}^{A'B'I}\delta_{C]}^{E}),\end{aligned}\tag{A.18}$$

while the symmetrization in B and C gives

$$\begin{aligned}&V_{A\ (B\ \ C)}^{A'B'E}-\varphi_{A(C}{}^{A'B'}\delta_{B)}^{E}+\varphi_{BC}{}^{A'B'}\delta_{A}^{E}-2(q+1)V_{A\ (B}^{A'B'}\delta_{C)}^{E}\\&=\nabla_{(C}^{I'}T_{|A|B)\ I'}^{A'B'E}+T_{A\ (C\ J'}^{I'(A'J}T_{B)\ J\ I'}^{B')J'E}.\end{aligned}$$

Then by equations (A.15) and (A.9) we have

$$\begin{aligned}V_{A\ (B\ \ C)}^{A'B'E}=&-\nabla_{(C}^{I'}T_{B)A\ I'}^{A'B'E}+T_{A\ (C\ J'}^{I'(A'J}T_{B)\ J\ I'}^{B')J'E}\\&+\tfrac{1}{q}T_{I\ A\ |J'|}^{I'(A'|J|}T_{(B\ J\ I'}^{B')J'I}\delta_{C)}^{E}+\tfrac{1}{q}T_{I\ \ B\ \ J'}^{I'(A'|J|}T_{C\ J\ I'}^{B')J'I}\delta_{A}^{E}\\&-\tfrac{q+1}{q(q+2)}\nabla_{I}^{I'}T_{A(B|I'|}^{A'B'I}\delta_{C)}^{E}.\end{aligned}\tag{A.19}$$

Now the only bit missing is $V_{A\ B\ \ \ E'}^{(A'B'C')}$. In order to obtain a formula for this term, we come back and consider a trace of the Bianchi identity (A.7) over indices C and E. It has the form

$$\begin{aligned}&U_{A\ B\ \ I}^{A'B'I}-qU_{A\ B\ \ E'}^{A'B'C'}+U_{I\ A\ \ B}^{C'A'I}\delta_{E'}^{B'}-U_{B\ A\ \ E'}^{C'A'B'}+U_{B\ I\ \ A}^{B'C'I}\delta_{E'}^{A'}-U_{B\ A\ \ E'}^{B'C'A'}\\&=\nabla_{I}^{C'}T_{A\ B\ E'}^{A'B'I}-T_{I\ A\ J'}^{C'A'J}T_{B\ J\ E'}^{B'J'I}-T_{B\ I\ J'}^{B'C'J}T_{A\ J\ E'}^{A'J'I},\end{aligned}$$

and replacing U by φ's and V's according to (A.2) we get

$$\begin{aligned}&-q\epsilon^{A'B'}\varphi_{AB}{}^{C'}{}_{E'}-qV_{A\ B}^{A'B'C'}{}_{E'}-q(q+2)V_{A\ B}^{A'B'}\delta_{E'}^{C'}\\&+\epsilon^{C'A'}\varphi_{IA}{}^{I}{}_{B}\delta_{E'}^{B'}+V_{I\ A}^{C'A'I}{}_{B}\delta_{E'}^{B'}-\epsilon^{C'A'}\varphi_{BA}{}^{B'}{}_{E'}-V_{B\ A}^{C'A'B'}{}_{E'}\\&-(q+2)V_{B\ A}^{C'A'}\delta_{E'}^{B'}+\epsilon^{B'C'}\varphi_{IB}{}^{I}{}_{A}\delta_{E'}^{A'}-V_{I\ B}^{B'C'I}{}_{A}\delta_{E'}^{A'}\\&-\epsilon^{B'C'}\varphi_{BA}{}^{A'}{}_{E'}-V_{B\ A}^{B'C'A'}{}_{E'}-(q+2)V_{B\ A}^{B'C'}\delta_{E'}^{A'}\\&=\nabla_{I}^{C'}T_{A\ B\ E'}^{A'B'I}-T_{I\ A\ J'}^{C'A'J}T_{B\ J\ E'}^{B'J'I}-T_{B\ I\ J'}^{B'C'J}T_{A\ J\ E'}^{A'J'I}.\end{aligned}$$

Its symmetrization over A', B' and C' gives

$$\begin{aligned}&(2-q)V^{(A'B'C')}_{A\,B\quad\;\;E'}+2V^{(C'A'|I|}_{I\,[A\quad\;\;B]}\delta^{B')}_{E'}+(q+2)(2-q)V^{(A'B'}_{A\,B}\delta^{C')}_{E'}\\&=\nabla^{(C'}_{I}T^{A'B')I}_{A\,B\,E'}-2T^{(C'A'|J|}_{I\,[A\,J'}T^{B')J'I}_{B]\,J\,E'}.\end{aligned}$$

From here and by (A.13) and (A.9), we get

$$\begin{aligned}V^{(A'B'C')}_{A\,B\quad\;\;E'}=&\ \tfrac{1}{2-q}(\nabla^{(C'}_{I}T^{A'B')I}_{A\,B\,E'}-2T^{(C'A'|J|}_{I\,[A\,|J'|}T^{B')J'I}_{B]\,J\,E'}\\&-\tfrac{q}{2(q+2)}\nabla^{I'}_{I}T^{(A'B'|I|}_{A\,B\;\;I'}\delta^{C')}_{E'}).\end{aligned}\tag{A.20}$$

Now we have expressions in terms of torsion for all irreducible components of the Weyl curvature (up to its harmonic part). We can omit the use of V's and φ's and rewrite these results using only the components ${U_{ab}}^{E}{}_{C}$ and ${U_{ab}}^{C'}{}_{E'}$ and algebraic operations applied to its indices.

Lemma A.2. *The components of the Weyl curvature* ${U_{ab}}^{e}{}_{c}={U_{ab}}^{E}{}_{C}\delta^{C'}_{E'}-{U_{ab}}^{C'}{}_{E'}\delta^{E}_{C}$ *satisfy the following equations.*

$$U^{A'B'I}_{A\,B\quad I}=-U^{A'B'I'}_{A\,B\quad I'}=\frac{1}{q+2}\nabla^{I'}_{I}T^{A'B'I}_{A\,B\,I'}$$

$$U^{\;I'[A'B']}_{(B\,C)\quad I'}=U^{\;I'(A'B')}_{[B\,C]\quad I'}=U^{[A'B']I}_{I\,(B\quad C)}=U^{(A'B')I}_{I\,[B\quad C]}=0$$

$$U^{\;I'(A'B')}_{(B\,C)\quad I'}=U^{(A'B')I}_{I\,(B\quad C)}=\frac{1}{q}T^{I'(A'|J|}_{I\,(B\,|J'|}T^{B')J'I}_{C)\,J\,I'}$$

$$U^{\;I'[A'B']}_{[B\,C]\quad I'}=U^{[A'B']I'}_{I\,[B\quad C]}=\frac{1}{q+4}T^{I'[A'|J|}_{I\,[C\,|J'|}T^{B']J'I}_{B]\,J\,I'}$$

$$\begin{aligned}U^{(A'B'C')}_{[A\,B]\quad E'}=&-\frac{1}{q-2}\nabla^{(C'}_{I}T^{A'B')I}_{A\,B\,E'}+\frac{2}{q-2}T^{(C'A'|J|}_{I\,[A\,|J'|}T^{B')J'I}_{B]\,J\,E'}\\&+\frac{1}{q^2-4}\nabla^{I'}_{I}T^{(A'B'|I|}_{A\,B\;\;I'}\delta^{C')}_{E'}\end{aligned}$$

$$U^{\;A'B'C'}_{(A\,B)\quad E'}=-\frac{2}{q}T^{I'C'|J|}_{I\,(A\,|J'|}T^{[A'|J'I|}_{B)\,J\,I'}\delta^{B']}_{E'}\tag{A.21}$$

$$U^{[A'B']E}_{A\,[B\quad C]}=T^{I'[A'J}_{A\,[C|J'|}T^{B']J'E}_{B]\,J\,I'}-T^{I'[A'J}_{[A\,C|J'|}T^{B']J'E}_{B]\,J\,I'}\tag{A.22}$$

$$\begin{aligned}U^{A'[B'C']}_{[A\,B]\quad E'}=&-\frac{1}{q+2}\nabla^{I'}_{I}T^{A'[B'|I|}_{A\;\;B\;\;I'}\delta^{C']}_{E'}+\frac{1}{q+4}T^{I'A'J}_{I[C|J'|}T^{[B'|J'I|}_{B]\,J\,I'}\delta^{C']}_{E'}\\&-\frac{1}{q+4}T^{I'[A'|J|}_{I\;\;A\;\;J'}T^{B']J'I}_{[B\,J\,I'}\delta^{E}_{C]}+\frac{1}{q+4}T^{I'[A'|J|}_{I\,[A\,|J'|}T^{B']J'I}_{B\,|J\,I'|}\delta^{E}_{C]}\end{aligned}\tag{A.23}$$

$$
\begin{aligned}
U_{[A\;B\;\;\;\;C]}^{(A'B')E} &= \frac{1}{4}\nabla_{[C}^{I'}T_{A\;B]\;I'}^{A'B'E} - \frac{1}{2}T_{[C\;A\;J'}^{I'(A'J}T_{B]\;J\;I'}^{B')J'E} \\
&+ \frac{1}{4(q+2)}\nabla_{I}^{I'}T_{[A\;B|I'|}^{A'B'I}\delta_{C]}^{E} \\
U_{A(B\;\;\;\;C)}^{(A'B')E} &= -\nabla_{(C}^{I'}T_{B)A\;I'}^{A'B'E} + T_{A\;(C\;J'}^{I'(A'J}T_{B)\;J\;I'}^{B')J'E} \\
&+ \frac{1}{q}T_{I\;A\;|J'|}^{I'(A'|J|}T_{(B\;J\;I'}^{B')J'I}\delta_{C)}^{E} + \frac{1}{q}T_{I\;\;B\;\;J'}^{I'(A'|J|}T_{C\;J\;I'}^{B')J'I}\delta_{A}^{E} \\
&- \frac{1}{q+2}\nabla_{I}^{I'}T_{A(B|I'|}^{A'B'I}\delta_{C)}^{E}
\end{aligned}
\tag{A.24}
$$

PROOF. Taking various traces in the defining equations (A.2) for φ's and V's yields

$$
\begin{aligned}
U_{A\;B\;\;\;I}^{A'B'I} &= -U_{A\;B\;\;\;I'}^{A'B'I'} = -2qV_{A\;B}^{A'B'} \\
U_{B\;C\;\;\;I'}^{I'A'B'} &= \varphi_{BC}{}^{A'B'} + V_{B\;C\;\;\;I'}^{I'A'B'} + qV_{B\;C}^{A'B'} \\
U_{I\;B\;\;\;C}^{A'B'I} &= \epsilon^{A'B'}\varphi_{IB}{}^{I}{}_{C} + V_{I\;\;B\;\;C}^{A'B'I} - 2V_{C\;B}^{A'B'}
\end{aligned}
$$

Here we used that $\varphi_{AB} = 0$ according to (A.8). From the second equality and (A.14), we conclude

$$
\begin{aligned}
&U_{(B\;C)\;\;\;I'}^{\;I'[A'B']} = U_{[B\;C]\;\;\;I'}^{\;I'(A'B')} = 0, \\
&U_{(B\;C)\;\;\;I'}^{\;I'(A'B')} = \varphi_{BC}{}^{A'B'}, \\
&U_{[B\;C]\;\;\;I'}^{\;I'[A'B']} = V_{B\;C\;\;\;\;I'}^{I'[A'B']}
\end{aligned}
$$

and the third one together with equations (A.10) and (A.13) implies

$$
\begin{aligned}
&U_{I\;(B\;\;\;\;C)}^{[A'B']I} = U_{I\;[B\;\;\;\;C]}^{(A'B')I} = 0, \\
&U_{I\;(B\;\;\;\;C)}^{(A'B')I} = V_{I\;(B\;\;C)}^{A'B'I}, \\
&U_{I\;[B\;\;\;\;C]}^{[A'B']I'} = \epsilon^{A'B'}\varphi_{I[B}{}^{I}{}_{C]}
\end{aligned}
$$

Taking suitable algebraic operations in (A.2) gives the following expressions for the non-trace terms.

$$
\begin{aligned}
&U_{[A\;B]\;\;\;E'}^{(A'B'C')} = V_{A\;B\;\;\;\;E'}^{(A'B'C')} + qV_{A\;B}^{(A'B'}\delta_{E'}^{C')}, \\
&U_{A\;[B\;\;\;\;C]}^{[A'B']E} = \epsilon^{A'B'}\varphi_{A[B}{}^{E}{}_{C]}, \\
&U_{[A\;B\;\;\;\;C]}^{(A'B')E} = V_{[A\;B\;\;C]}^{A'B'E} - 2V_{[A\;B}^{A'B'}\delta_{C]}^{E} \\
&U_{A(B\;\;\;\;C)}^{(A'B')E} = V_{A(B\;\;C)}^{A'B'E} - 2V_{A(B}^{A'B'}\delta_{C)}^{E}
\end{aligned}
$$

For the remaining components of the Weyl curvature, we have

$$
U_{(A\;B)\;\;\;E'}^{\;A'B'C'} = \epsilon^{A'B'}\varphi_{AB}{}^{C'}{}_{E'} = -2\varphi_{AB}{}^{C'[A'}\delta_{E'}^{B']}
$$

and

$$U_{[A\ B]\quad E'}^{A'[B'C']} = V_{A\ B\quad E'}^{A'[B'C']} + qV_{A\ B}^{A'[B'}\delta_{E'}^{C']} = -V_{A\ B\quad I'}^{I'A'[B'}\delta_{E'}^{C']} + qV_{A\ B}^{A'[B'}\delta_{E'}^{C']}$$

The rest is a direct consequence of equations (A.9), (A.15), (A.16), (A.17), (A.18), (A.19) and (A.20). □

Corollary A.3. *In the torsion-free case, the Weyl curvature* $U_{ab}{}^{e}{}_{c}$ *satisfies*

$$U_{ab}{}^{e}{}_{c} = W_{AB}{}^{E}{}_{C}\epsilon^{A'B'}\delta_{E'}^{C'},$$

for some $W_{AB}{}^{E}{}_{C} \in \mathcal{E}_{(ABC)_0}^{E}[-1]$.

By (A.1) and by equation (1) in the previous lemma, we have

Corollary A.4. *The skew-symmetric part of the Rho-tensor associated to an exact Weyl connection satisfies*

$$-2(q+2)\mathsf{P}_{[ab]} = \nabla_i T_{ab}{}^{i}. \tag{A.25}$$

In particular, it vanishes in the torsion-free case.

Differential Bianchi identity. According to proposition 1.11, the differential Bianchi identity has the form

$$\nabla_{[a}R_{bc]}{}^{e}{}_{d} - T_{[ab}{}^{f}R_{c]f}{}^{e}{}_{d} = 0.$$

We assume in this paragraph that the Weyl connection ∇_a is torsion-free. Then the differential Bianchi identity simplifies to $\nabla_{[a}R_{bc]}{}^{e}{}_{d} = 0$. Now we rewrite this equation using the structure of the Riemannian curvature $R_{bc}{}^{e}{}_{d}$ described in 2.2.2. Namely, we replace $R_{bc}{}^{e}{}_{d}$ by $U_{bc}{}^{e}{}_{d} - \partial(\mathsf{P})_{bc}{}^{e}{}_{d}$ which yields

$$\nabla_{[a}U_{bc]}{}^{e}{}_{d} = \nabla_{[a}\partial(\mathsf{P})_{bc]}{}^{e}{}_{d} \tag{A.26}$$

This equivalent reformulation of the differential Bianchi identity can be further simplified because the vanishing of the torsion implies that by A.3 the only non-zero part of $U_{bc}{}^{e}{}_{d}$ is its harmonic part $W_{AB}{}^{E}{}_{C} \in \mathcal{E}_{(ABC)_0}^{E}[-1]$. Hence if we set

$$Q_{abc} := 2\nabla_{[a}\mathsf{P}_{b]c},$$

we conclude from (A.26) the following.

Lemma A.5. *In the case of a Grassmannian geometry (i.e. the torsion-free case), the differential Bianchi identity implies*

$$Q_{A\ B\ C}^{A'B'C'} = \frac{1}{1-q}\epsilon^{A'B'}\nabla_{E}^{C'}W_{A\ B}{}^{E}{}_{C},$$

where $W_{AB}{}^{E}{}_{C}$ *is the harmonic part of the curvature lying in* $\mathcal{E}_{(ABC)_0}^{E}[-1]$.

PROOF. It follows from the previous discussion that the left-hand side of (A.26) equals

$$\frac{1}{3}\sum_{\text{cycl}(abc)} \epsilon^{B'C'}\nabla^{A'}_{A} W_{B\ C}{}^{E}{}_{D}\delta^{D'}_{E'}$$

while the right-hand side equals

$$\frac{1}{3}\sum_{\text{cycl}(abc)} (-\mathsf{P}^{B'D'}_{B\ C}\delta^{C'}_{E'}\delta^{E}_{D} + \mathsf{P}^{C'D'}_{C\ B}\delta^{B'}_{E'}\delta^{E}_{D} - \mathsf{P}^{B'C'}_{B\ D}\delta^{E}_{C}\delta^{D'}_{E'} + \mathsf{P}^{C'B'}_{C\ D}\delta^{E}_{B}\delta^{D'}_{E'})$$

according to equation (2.6) for $\partial(\mathsf{P})$. Now it is useful to use the tensor Q_{abc} in order to expand this sum. Then we get that equation (A.26) is equivalent to the equation

$$\begin{aligned}
&\epsilon^{B'C'}\nabla^{A'}_{A} W_{B\ C}{}^{E}{}_{D}\delta^{D'}_{E'} + \epsilon^{C'A'}\nabla^{B'}_{B} W_{C\ A}{}^{E}{}_{D}\delta^{D'}_{E'} + \epsilon^{A'B'}\nabla^{C'}_{C} W_{A\ B}{}^{E}{}_{D}\delta^{D'}_{E'}\\
&= -Q^{A'B'D'}_{A\ B\ C}\delta^{C'}_{E'}\delta^{E}_{D} - Q^{A'B'C'}_{A\ B\ D}\delta^{D'}_{E'}\delta^{E}_{C} + Q^{A'C'D'}_{A\ C\ B}\delta^{B'}_{E'}\delta^{E}_{D}\\
&+Q^{A'C'B'}_{A\ C\ D}\delta^{D'}_{E'}\delta^{E}_{B} - Q^{B'C'D'}_{B\ C\ A}\delta^{A'}_{E'}\delta^{E}_{D} - Q^{B'C'A'}_{B\ C\ D}\delta^{D'}_{E'}\delta^{E}_{A}
\end{aligned}$$

Now the claim of the lemma is obtained from here by taking suitable traces and algebraic operations. Namely, taking the trace over D' and E' yields

$$\begin{aligned}
&2\epsilon^{B'C'}\nabla^{A'}_{A} W_{B\ C}{}^{E}{}_{D} + 2\epsilon^{C'A'}\nabla^{B'}_{B} W_{C\ A}{}^{E}{}_{D} + 2\epsilon^{A'B'}\nabla^{C'}_{C} W_{A\ B}{}^{E}{}_{D}\\
&= -Q^{A'B'C'}_{A\ B\ C}\delta^{E}_{D} - 2Q^{A'B'C'}_{A\ B\ D}\delta^{E}_{C} + Q^{A'C'B'}_{A\ C\ B}\delta^{E}_{D}\\
&+2Q^{A'C'B'}_{A\ C\ D}\delta^{E}_{B} - Q^{B'C'A'}_{B\ C\ A}\delta^{E}_{D} - 2Q^{B'C'A'}_{B\ C\ D}\delta^{E}_{A}
\end{aligned}$$

and since

$$-Q_{abc} + Q_{acb} - Q_{bca} = -Q_{[abc]} = -\nabla_{[a}\mathsf{P}_{bc]} = 0$$

in the case of vanishing torsion, the previous equation is equivalent to

$$\begin{aligned}
&\epsilon^{B'C'}\nabla^{A'}_{A} W_{B\ C}{}^{E}{}_{D} + \epsilon^{C'A'}\nabla^{B'}_{B} W_{C\ A}{}^{E}{}_{D} + \epsilon^{A'B'}\nabla^{C'}_{C} W_{A\ B}{}^{E}{}_{D}\\
&= -Q^{A'B'C'}_{A\ B\ D}\delta^{E}_{C} + Q^{A'C'B'}_{A\ C\ D}\delta^{E}_{B} - Q^{B'C'A'}_{B\ C\ D}\delta^{E}_{A}
\end{aligned}$$

Now the trace over C and E yields

$$\epsilon^{A'B'}\nabla^{C'}_{E} W_{A\ B}{}^{E}{}_{D} = -qQ^{A'B'C'}_{A\ B\ D} + Q^{A'C'B'}_{A\ B\ D} - Q^{B'C'A'}_{B\ A\ D} \tag{A.27}$$

since the tensor W is trece-free. The tensor Q_{abc} is skewsymmetric in the first two enteries and so it splitts as $Q = Q^1 + Q^2$, where $(Q^1)^{A'B'C'}_{A\ B\ C} = Q^{(A'B')C'}_{[A\ B]\ D}$ and $(Q^2)^{A'B'C'}_{A\ B\ C} = Q^{[A'B']C'}_{(A\ B)\ D}$. Then the symmetrization of (A.27) over A', B' and C' implies

$$(Q^1)^{(A'B'C')}_{A\ B\ D} = 0$$

and the alternation of (A.27) in A' and B' gives

$$\epsilon^{A'B'}\nabla^{C'}_{E} W_{A\ B}{}^{E}{}_{D} = -qQ^{[A'B']C'}_{A\ B\ D} + 2Q^{[A'|C'|B']}_{(A\ B)\ D} = (1-q)(Q^2)^{A'B'C'}_{A\ B\ D}$$

since $2(Q^2)^{[A'|C'|B']}_{A\;B\;\;D} = (Q^2)^{A'B'C'}_{A\;B\;D}$. The alternation of (A.27) in B' and C' leads to

$$\epsilon^{A'[B'}\nabla^{C']}_{E}W_{A\;B}{}^{E}{}_{D} = -(q+1)Q^{A'[B'C']}_{A\;B\;D} - Q^{[B'C']A'}_{B\;A\;D}$$

which is equivalent to

$$-\frac{1}{2}\epsilon^{B'C'}\nabla^{A'}_{E}W_{A\;B}{}^{E}{}_{D} = -(q+1)(Q^1)^{A'[B'C']}_{A\;B\;D} + \frac{q-1}{2}(Q^2)^{B'C'A'}_{A\;B\;D}.$$

Inserting the previous equation for Q^2 into this equation yields

$$(Q^1)^{A'[B'C']}_{A\;B\;D} = 0.$$

All together, we have got

$$Q^{A'B'C'}_{A\;B\;C} = (Q^2)^{A'B'C'}_{A\;B\;C} = \frac{1}{1-q}\epsilon^{A'B'}\nabla^{C'}_{E}W_{A\;B}{}^{E}{}_{C}.$$

□

APPENDIX B

An analogue of Q-curvature

In the first part we give an alternative construction of the invariant operator $\mathsf{D} : \mathcal{E} \to \boxplus \mathcal{E}_A[-2]$ in the torsion-free case. In the second one we use this construction to prove proposition 3.14, which shows the existence of an Grassmannian analogue of the conformal Q-curvature.

An alternative contruction of D. We will proceed along the lines of [**12**] where the second power of conformal Laplacian is constructed since this is the conformal analogue of our operator D. In contrast with the previous construction, where we used an operator for splitting the tractor bundle $\boxplus \mathcal{E}_\alpha$ induced by curved Casimir operators, we will use an operator for splitting the tractor bundle $\Lambda^2 \mathcal{E}_\alpha$ and then we act by the analogue of conformal Laplacian.

So let us start with the tractor bundle $\Lambda^2 \mathcal{E}_\alpha$ instead of $\boxplus \mathcal{E}_\alpha$, and let us compute the action of $T^*M = \mathcal{E}_A^{A'}$ and $TM = \mathcal{E}_{A'}^A$ on it. It follows from the elementary action of $\Upsilon \in T^*M$ and $\xi \in TM$ on the costandard tractor bundle $\mathcal{E}_\alpha = \mathcal{E}_{A'} \oplus \mathcal{E}_A$ that the action on

$$\Lambda^2 \mathcal{E}_\alpha = \mathcal{E}[1] \oplus \mathcal{E}_A^{A'}[1] \oplus \mathcal{E}_{[AB]}$$

is

$$\Upsilon \bullet \begin{pmatrix} f \\ \mu_A^{A'} \\ \rho_{AB} \end{pmatrix} = \begin{pmatrix} 0 \\ -\Upsilon_A^{A'} f \\ 2\Upsilon_{[A}^{A'} \mu_{B]}^{B'} \epsilon_{A'B'} \end{pmatrix}, \quad \xi \bullet \begin{pmatrix} f \\ \mu_A^{A'} \\ \rho_{AB} \end{pmatrix} = \begin{pmatrix} -\xi_{A'}^A \mu_A^{A'} \\ -\xi_{B'}^B \rho_{AB} \epsilon^{A'B'} \\ 0 \end{pmatrix}$$

Now it is easy to express the action of the tractor connection, fundamental derivative and the differential part of the curved Casimir operator on $\Lambda^2 \mathcal{E}_\alpha$. The tractor connection is defined by $\nabla_\psi^{\mathcal{T}} t = \nabla_\psi t + \mathsf{P}(\psi) \bullet t + \xi \bullet t$ which yields

$$(\nabla^{\mathcal{T}})_A^{A'} \begin{pmatrix} f \\ \mu_B^{B'} \\ \rho_{BC} \end{pmatrix} = \begin{pmatrix} \nabla_A^{A'} f - \mu_A^{A'} \\ \nabla_A^{A'} \mu_B^{B'} - \mathsf{P}_{A\ B}^{A'B'} f - \epsilon^{A'B'} \rho_{AB} \\ \nabla_A^{A'} \rho_{BC} + 2\mathsf{P}_{A[B}^{A'B'} \mu_{C]}^{C'} \epsilon_{B'C'} \end{pmatrix} \tag{B.1}$$

The differential part of the curved Casimir operator $\mathcal{C}$ equals $-2\sum_l \varphi^l \bullet (\nabla_{\psi_l} t + \mathsf{P}(\psi_l) \bullet t)$ for a local basis φ^l of T^*M and its dual basis ψ^l. Therefore, the action of $\mathcal{C}$ on $\Lambda^2\mathcal{E}_\alpha$ can be written as

$$\mathcal{C}\begin{pmatrix} f \\ \mu_A^{A'} \\ \rho_{AB} \end{pmatrix} = \begin{pmatrix} c_0 f \\ c_1\mu_A^{A'} + 2\nabla_A^{A'} f \\ c_2\rho_{AB} - 4\nabla_{[A}^{A'}\mu_{B]}^{B'}\epsilon_{A'B'} + 4\mathsf{P}_{[AB]}^{A'B'} f\epsilon_{A'B'} \end{pmatrix}$$

where c_0, c_1, c_2 are Casimir eigenvalues on the irreducible bundles $\mathcal{E}[1]$, $\mathcal{E}_A^{A'}[1]$ and $\Lambda^2\mathcal{E}_A$ respectively. Obviously, the same formula holds for the action of $\mathcal{C}$ on $\Lambda^2\mathcal{E}_\alpha[w]$ but the eigenvalues become polynomial in the weight w. It follows from the computation of Casimir eigenvalues in the previous section that the differences to the first eigenvalue are

$$\begin{pmatrix} 0 \\ -2w-2 \\ -4w-8 \end{pmatrix}$$

and so there are two degenerate cases – the first one is $w = -1$ and the second one is $w = -2$.

In a non-degenerate case, the operator $(\mathcal{C}-c_1)(\mathcal{C}-c_2)$ induces an invariant splitting $S_w : \mathcal{E}[w+1] \to \Lambda^2\mathcal{E}_\alpha[w]$ which is an analog of the conformal tractor-D operator. Using the formula for the curved Casimir displayed above, we get

$$S_w(f) = \begin{pmatrix} (w+1)(w+2)f \\ (w+2)\nabla_A^{A'} f \\ -\nabla_{[A}^{A'}\nabla_{B]}^{B'} f\epsilon_{A'B'} + (w+1)\mathsf{P}_{A\ B}^{A'B'} f\epsilon_{A'B'} \end{pmatrix}$$

In the case of $w = -1$, we have $c_0 = 0$ (the corresponding bundle is the bundle of functions on M in such a case), $c_1 = 0$ and $c_2 = -4$. Therefore, $\mathcal{C}$ induces an invariant operator $\mathcal{E} \to \mathcal{E}_A^{A'}$ which is the exterior derivative of course. The splitting operator $S_{-1} : \mathcal{E} \to \Lambda^2\mathcal{E}_\alpha[-1]$ is induced by $\mathcal{C}(\mathcal{C}+4)$ and has the form

$$S_{-1}(f) = \begin{pmatrix} 0 \\ \nabla_A^{A'} f \\ -\nabla_{[A}^{A'}\nabla_{B]}^{B'} f\epsilon_{A'B'} \end{pmatrix}$$

We observe that the operator S_{-1} factorizes through $\mathcal{E}_A^{A'}$ and so we can view it as an operator $\mathcal{E}_A^{A'} \to \Lambda^2\mathcal{E}_\alpha[-1]$ acting on $\nabla_A^{A'} f$.

Since we have $c_0 = c_2$ and $c_1 = c_0 + 2$ in the second degenerate case $w = -2$, the operator $(\mathcal{C} - c_0)(\mathcal{C} - c_0 - 2)$ induces an invariant operator

$\Box_{AB} : \mathcal{E}[-1] \to \Lambda^2\mathcal{E}_A[-2]$. It follows form the formula for curved Casimir that

$$\Box_{AB} = \epsilon_{A'B'}(\nabla^{A'}_{[A}\nabla^{B'}_{B]} + \mathsf{P}^{A'B'}_{[AB]})$$

This is a strongly invariant operator and so we can replace the connection $\nabla^{A'}_A$ by the tractor connection $(\nabla^{\mathcal{T}})^{A'}_A$. We obtain an invariant operator $\Box^{\mathcal{T}}_{AB}$ acting on weighted tractor bundles of the weight -1. Therefore, we can construct the invariant operator

$$\Box^{\mathcal{T}}_{AB} \circ S_{-1} : \mathcal{E} \to \Lambda^2\mathcal{E}_A \otimes \Lambda^2\mathcal{E}_\alpha[-2]$$

The target space has the following composition series.

$$\Lambda^2\mathcal{E}_\alpha \otimes \Lambda^2\mathcal{E}_A[-2] = \Lambda^2\mathcal{E}_A[-1] \oplus \begin{array}{c} \mathcal{E}^{A'}\mathcal{E}_A\ydiagram{2,1}\,\mathcal{E}_A[-1] \\ \oplus \\ \mathcal{E}^{A'}\Lambda^3\mathcal{E}_A[-1] \end{array} \oplus \begin{array}{c} \boxplus\,\mathcal{E}_A[-2] \\ \oplus \\ \ydiagram{2,1,1}\,\mathcal{E}_A[-2] \\ \oplus \\ \Lambda^4\mathcal{E}_A[-2] \end{array}$$

Using the formula for the tractor connection given above, we make the formula for $\Box^{\mathcal{T}}_{AB} \circ S_{-1}$ explicit. It is easy to show that we get zero in the projecting slot $\Lambda^2\mathcal{E}_A[-1]$. In the two slots in the middle, we get an expression which also vanishes in the torsion-free case. Therefore, we can project $\Box^{\mathcal{T}}_{AB} \circ S_{-1}$ to $\boxplus\mathcal{E}_A[-2]$ which yields an invariant operator $\mathcal{E} \to \boxplus\mathcal{E}_A[-2]$. It turns out that this operator coincides with the operator D and so we obtain an alternative construction of D which yields also a new formula for D.

Proposition B.1. *The action of the operator $\Box^{\mathcal{T}}_{AB} \circ S_{-1}$ on $f \in \mathcal{E}$ is given by*

$$\Box^{\mathcal{T}}_{AB} \circ S_{-1}(f) = \begin{pmatrix} 0 \\ 0 \mid 0 \\ \mathsf{D}(f) \mid * \mid * \end{pmatrix}$$

Proof. We compute the action of $\Box^{\mathcal{T}}_{AB}$ on $S_{-1}(f)$ explicitely using the expression for the action of the tractor connection. Then we show that only the slots in the bottom are non-zero and that we obtain the formula defining the operator D in the slot corresponding to $\boxplus\mathcal{E}_A[-2]$.

The action of

$$\Box^{\mathcal{T}}_{AB} = \epsilon_{A'B'}(\nabla^{\mathcal{T}})^{A'}_{[A}(\nabla^{\mathcal{T}})^{B'}_{B]} + \epsilon_{A'B'}\mathsf{P}^{A'B'}_{[AB]}$$

on the section

$$S_{-1}(f) = \begin{pmatrix} 0 \\ \nabla^{C'}_{C} f \\ -\nabla^{C'}_{[C}\nabla^{D'}_{D]} f \epsilon_{C'D'} \end{pmatrix}$$

of $\Lambda^2\mathcal{E}_\alpha[-1]$ is obtained by a double use of the expression (B.1) for the action of tractor connection on $\Lambda^2\mathcal{E}_\alpha$. In the first slot, we get

$$\epsilon_{A'B'}(-\nabla^{A'}_{[A}\nabla^{B'}_{B]} - \nabla^{B'}_{[B}\nabla^{A'}_{A]} - \epsilon^{A'B'}\epsilon_{C'D'}\nabla^{C'}_{[A}\nabla^{D'}_{B]}) = 0$$

In the second slot, we end up with the expression

$$\begin{aligned} &\epsilon_{A'B'}(\nabla^{A'}_{[A}\nabla^{B'}_{B]}\nabla^{C'}_{C} f - \nabla^{C'}_{C}\nabla^{A'}_{[A}\nabla^{B'}_{B]} f + 3\nabla^{C'}_{[C}\nabla^{A'}_{A}\nabla^{B'}_{B]} f \\ &\mathsf{P}_{[A|C|}{}^{A'C'}\nabla^{B'}_{B]} f + \mathsf{P}^{C'A'}_{C[A}\nabla^{B'}_{B]} f + \mathsf{P}_{[A\ B]}^{A'B'}\nabla^{C'}_{C} f - 3\mathsf{P}_{[C\ A}^{C'A'}\nabla^{B'}_{B]} f) \end{aligned} \tag{B.2}$$

which defines an invariant operator $\mathcal{E} \to \mathcal{E}^{C'}_{[AB]C}$. Now we split this formula into the irreducible parts according to $\mathcal{E}^{C'}_{[AB]C} = \mathcal{E}^{C'}_{[ABC]} \oplus \mathcal{E}^{C'}_{\boxplus(ABC)}$. The component in $\mathcal{E}^{C'}_{\boxplus(ABC)}$ can be written with the help of the projection $\Pi_{\boxplus}$ defined in (3.14) as $\Pi_{\boxplus}$ applied to

$$\nabla_a\nabla_b\nabla_c f - \nabla_c\nabla_a\nabla_b f + \mathsf{P}_{ac}\nabla_b f + \mathsf{P}_{ca}\nabla_b f + \mathsf{P}_{ab}\nabla_c f.$$

It follows from the proof of lemma 3.5 that the part with the Rho-tensor equals $\partial(\mathsf{P})_{acb}{}^e\nabla_e f$. Moreover, the first two terms can be rewritten by applying twice Ricci identity, and so the last expression may be written as $\Pi_{\boxplus}$ applied to

$$-\nabla_a(T_{bc}{}^e\nabla_e f) - T_{ac}{}^e\nabla_e\nabla_b f + R_{acb}{}^e\nabla_e f + \partial(\mathsf{P})_{acb}{}^e\nabla_e f.$$

Now we observe that in the torsion-free case the first two terms vanish and the second two sum up to $U_{acb}{}^e\nabla_e f$. The Weyl curvature is by A.3 equal to $U_{ac}{}^e{}_b = W_{AC}{}^E{}_B\epsilon^{A'C'}\delta^{B'}_{E'}$ for some $W_{AC}{}^E{}_B \in \mathcal{E}^E_{(ACB)_0}[-1]$ and thus the term $U_{acb}{}^e\nabla_e f$ is maped to zero under $\Pi_{\boxplus}$ by the definition of $\Pi_{\boxplus}$. Hence we conclude that the projection of (B.2) to $\mathcal{E}^{C'}_{\boxplus(ABC)}$ vanish. Similarly, the projection of (B.2) to the slot corresponding to $\mathcal{E}^{C'}_{[ABC]}$ yields

$$\epsilon_{A'B'}(-U_{[A\ C\ B]}^{A'C'B'E}{}_{E'}\nabla^{E'}_{E} f + T^{A'C'E}_{[A\ C|E'}\nabla^{E'}_{E|}\nabla^{B'}_{B]} f)$$

which also vanishes in the case of vanishing torsion, and so the whole formula (B.2) appearing in the second level of $\square^{\mathcal{T}}_{AB} \circ S_{-1}(f)$ vanishes. In the bottom level, we get

$$\begin{aligned} \epsilon_{A'B'}\epsilon_{C'D'}\mathrm{Alt}_{AB} \circ \mathrm{Alt}_{CD}(&-\nabla^{A'}_{A}\nabla^{B'}_{B}\nabla^{C'}_{C}\nabla^{D'}_{D} f + 4\mathsf{P}_{B\ C}^{B'C'}\nabla^{A'}_{A}\nabla^{D'}_{D} f \\ &- 2\mathsf{P}_{A\ C}^{A'B'}\nabla^{C'}_{[B}\nabla^{D'}_{D]} f - \mathsf{P}_{A\ B}^{A'B'}\nabla^{C'}_{C}\nabla^{D'}_{D} f + 2\nabla^{A'}_{A}\mathsf{P}_{B\ C}^{B'C'}\nabla^{D'}_{D} f) \end{aligned}$$

Since

$$\mathrm{Alt}_{CD}(\mathsf{P}^{A'B'}_{A\ C}\nabla^{C'}_{[B}\nabla^{D'}_{D]}f) = \frac{1}{2}\mathsf{P}^{A'B'}_{A\ B}\nabla^{C'}_{[C}\nabla^{D'}_{D]}f - \frac{3}{2}\mathsf{P}^{A'B'}_{A[B}\nabla^{C'}_{C}\nabla^{D'}_{D]}f,$$

the projection of the bottom slot to $\mathcal{E}_{\boxplus(ABCD)}[-2]$ can be obtained by applying the projection $\Pi_{\boxplus}$ defined in (3.20) to

$$-\nabla_a\nabla_b\nabla_c\nabla_d f + 4\mathsf{P}_{bc}\nabla_a\nabla_d f - 2\mathsf{P}_{ab}\nabla_c\nabla_d f + 2(\nabla_a\mathsf{P}_{bc})\nabla_d f.$$

But up to the sign this is formula (3.25) for the operator D. $\square$

Proof of proposition 3.14. Now we use the alternative construction of the operator D from proposition B.1 to construct an analogue Q of the Q-curvature in such way that the form of its transformation becomes obvious. At first, let us choose a scale and consider the following non-invariant section of the tractor bundle $\Lambda^2\mathcal{E}_\alpha$

$$I^\sigma = \begin{pmatrix} -1 \\ 0 \\ -\mathsf{P}^{C'D'}_{[C\ D]}\epsilon_{C'D'} \end{pmatrix}.$$

The reason why we assume such section is that it has a special transformation. Namely, its linearized transformation under the change of Weyl structure described by a one-form $\Upsilon^{A'}_A$ is

$$\delta(I^\sigma) = \begin{pmatrix} 0 \\ \Upsilon^{C'}_C \\ -\nabla^{C'}_{[C}\Upsilon^{D'}_{D]}\epsilon_{C'D'} \end{pmatrix}$$

From here we see that if $\Upsilon^{A'}_A$ is an exact form given by $\nabla^{A'}_A\omega$, we get

$$\delta(I^\sigma) = S_{-1}(\omega)$$

Now we form $\square^{\mathcal{T}}_{AB}I^\sigma$ which is a section of $\Lambda^2\mathcal{E}_A \otimes \Lambda^2\mathcal{E}_\alpha[-2]$. Then the previous equation together with the invariance of $\square^{\mathcal{T}}_{AB}$ imply

$$\delta(\square^{\mathcal{T}}_{AB}I^\sigma) = \square^{\mathcal{T}}_{AB}S_{-1}(\omega) \tag{B.3}$$

In the perspective of proposition B.1, this equation says that $\square^{\mathcal{T}}_{AB}I^\sigma$ transforms in the bottom-slot of $\Lambda^2\mathcal{E}_A \otimes \Lambda^2\mathcal{E}_\alpha[-2]$ only and, moreover, the piece which corresponds to $\boxplus\mathcal{E}_A[-2]$ transforms essentially via $\mathsf{D}(\omega)$. In order to state this properly, we compute $\square^{\mathcal{T}}_{AB}I^\sigma$ explicitely.

From the definition of $\square^{\mathcal{T}}_{AB}$, we get

$$\square^{\mathcal{T}}_{AB} I^\sigma = \left(\epsilon_{A'B'}(\nabla^{\mathcal{T}})^{A'}_{[A}(\nabla^{\mathcal{T}})^{B'}_{B]} + \epsilon_{A'B'}\mathsf{P}^{A'B'}_{[AB]}\right)\begin{pmatrix} -1 \\ 0 \\ -\mathsf{P}^{C'D'}_{[CD]}\epsilon_{C'D'}\end{pmatrix}$$

Let us compute the leading term first. From equation (B.1) for the action of the tractor curvature, we get

$$(\nabla^{\mathcal{T}})^{B'}_{B}\begin{pmatrix} -1 \\ 0 \\ -\mathsf{P}^{C'D'}_{[CD]}\epsilon_{C'D'}\end{pmatrix} = \begin{pmatrix} 0 \\ \mathsf{P}^{B'C'}_{B\ C} + \epsilon^{B'C'}\mathsf{P}^{E'F'}_{[BC]}\epsilon_{E'F'} \\ -\nabla^{B'}_{B}\mathsf{P}^{C'D'}_{[CD]}\epsilon_{C'D'}\end{pmatrix}$$

and the action of $(\nabla^{\mathcal{T}})^{A'}_{A}$ on the resulting section yields

$$\begin{pmatrix} -\mathsf{P}^{B'A'}_{B\ A} - \epsilon^{B'A'}\mathsf{P}^{E'F'}_{[BA]}\epsilon_{E'F'} \\ \nabla^{A'}_{A}\mathsf{P}^{B'C'}_{B\ C} + \epsilon^{B'C'}\nabla^{A'}_{A}\mathsf{P}^{E'F'}_{[BC]}\epsilon_{E'F'} + \epsilon^{A'C'}\nabla^{B'}_{B}\mathsf{P}^{E'F'}_{[A\ C]}\epsilon_{E'F'} \\ (-\nabla^{A'}_{A}\nabla^{B'}_{B}\mathsf{P}^{C'D'}_{[CD]} + 2\mathsf{P}^{A'C'}_{A[C}\mathsf{P}^{B'D'}_{|B|D]} - \mathsf{P}^{A'B'}_{A[C}\mathsf{P}^{C'D'}_{|B|D]} + \mathsf{P}^{A'B'}_{A[C}\mathsf{P}^{C'D'}_{D]B})\epsilon_{C'D'}\end{pmatrix}$$

We see from here that the first slot of $\square^{\mathcal{T}}_{AB} I^\sigma$ equals

$$\epsilon_{A'B'}(-\mathsf{P}^{B'A'}_{[BA]} - \epsilon^{B'A'}\mathsf{P}^{E'F'}_{[BA]}\epsilon_{E'F'}) - \epsilon_{A'B'}\mathsf{P}^{A'B'}_{[AB]} = 0$$

while the second one equals

$$\begin{aligned}&\epsilon_{A'B'}(\nabla^{A'}_{[A}\mathsf{P}^{B'C'}_{B]C} + \nabla^{C'}_{[A}\mathsf{P}^{A'B'}_{B]C} - \nabla^{C'}_{[A}\mathsf{P}^{A'B'}_{|C|B]}) \\ &= \epsilon_{A'B'}(\nabla^{A'}_{[A}\mathsf{P}^{B'C'}_{B]C} - \nabla^{C'}_{C}\mathsf{P}^{A'B'}_{[AB]} + 3\nabla^{C'}_{[C}\mathsf{P}^{A'B'}_{A\ B]})\end{aligned}$$

This slot corresponds to the bundle $\mathcal{E}_{[AB]} \otimes \mathcal{E}^{C'}_{C}[-2]$ which splits into two irreducible parts. The first one is $\mathcal{E}^{C'}_{[ABC]}[-2]$ and we find easily that the projection of the formula to this part equals

$$2\nabla^{(C'}_{[C}\mathsf{P}^{A')B'}_{A\ B]}\epsilon_{A'B'}.$$

And since we consider the torsion-free case, this is by A.5 equal to

$$\frac{1}{1-q}\epsilon^{C'A'}\nabla^{B'}_{E}W_{[C\ A}{}^{E}{}_{B]} = 0.$$

The other part of the second slot lies in $\mathcal{E}^{C'}_{\boxplus(ABC)}[-2]$, and it can be obviously written as $\Pi_{\boxplus}$ applied to $\nabla_a\mathsf{P}_{bc} - \nabla_c\mathsf{P}_{ab}$. Since we consider a torsion-free case, the Rho-tensor P_{ab} coincides with its symmetric part S_{ab} and so lemma 3.23 (which is also a consequence of A.5) shows that $\nabla_a\mathsf{P}_{bc} - \nabla_c\mathsf{P}_{ab} \equiv 0$ mod $\mathrm{Ker}(\Pi_{\boxplus})$. Thus we conclude from the differential Bianchi identity that the whole second slot vanishes. The third slot of $\Box^{\mathcal{T}}_{AB} I^\sigma$ lies in $\mathcal{E}_{[AB]} \otimes \mathcal{E}_{[CD]}[-2]$ and equals

$$\begin{aligned}
&\mathrm{Alt}_{AB} \circ \mathrm{Alt}_{CD}(-\nabla^{A'}_{A} \nabla^{B'}_{B} \mathsf{P}^{C'D'}_{C\ D} + 2\mathsf{P}^{A'C'}_{A\ C} \mathsf{P}^{B'D'}_{B\ D} - \mathsf{P}^{C'D'}_{A\ C} \mathsf{P}^{A'B'}_{B\ D} \\
&\qquad + \mathsf{P}^{C'D'}_{A\ C} \mathsf{P}^{A'B'}_{D\ B} - \mathsf{P}^{A'B'}_{A\ B} \mathsf{P}^{C'D'}_{C\ D})\epsilon_{A'B'}\epsilon_{C'D'} \\
&= \mathrm{Alt}_{AB} \circ \mathrm{Alt}_{CD}(-\nabla^{A'}_{A} \nabla^{B'}_{B} \mathsf{P}^{C'D'}_{C\ D} + 2\mathsf{P}^{A'C'}_{A\ C} \mathsf{P}^{B'D'}_{B\ D} - 2\mathsf{P}^{A'B'}_{A\ B} \mathsf{P}^{C'D'}_{C\ D} \\
&\qquad + 3\mathsf{P}^{A'B'}_{A[B} \mathsf{P}^{C'D'}_{C\ D]})\epsilon_{A'B'}\epsilon_{C'D'}.
\end{aligned}$$

We conclude from this formula that its projection to $\mathcal{E}_{\boxplus(ABCD)}[-2]$ can be written as $\Pi_{\boxplus}(\tilde{\mathsf{Q}})_{ABCD}$ where

$$\tilde{\mathsf{Q}}_{abcd} = -\nabla_a \nabla_b \mathsf{P}_{cd} + 2\mathsf{P}_{ac}\mathsf{P}_{bd} - 2\mathsf{P}_{ab}\mathsf{P}_{cd}.$$

Since we have just proved

$$\Box^{\mathcal{T}}_{AB} I^\sigma = \begin{pmatrix} 0 \\ 0 \mid 0 \\ \mathsf{Q}_{ABCD} \mid * \mid * \end{pmatrix},$$

and from B.1 we know

$$\Box^{\mathcal{T}}_{AB} S_{-1}(\omega) = \begin{pmatrix} 0 \\ 0 \mid 0 \\ \mathsf{D}(\omega) \mid * \mid * \end{pmatrix},$$

it follows from (B.3) that

$$\delta(Q) = \mathsf{D}(\omega).$$

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