DISSERTATION

Titel der Dissertation

"Lefschetz Numbers of Involutions on Arithmetic Subgroups of Inner Forms of the Special Linear Group"

Verfasser

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Für meine Eltern

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Contents

Introduction

On the whole this thesis shall be a contribution to the theory of arithmetic groups. An arithmetic subgroup Γ of a linear algebraic $\mathbb{Q}$-group G is, in vague terms, a subgroup of $G(\mathbb{Q})$ commensurable with the group of $\mathbb{Z}$-points of G. For instance classical groups of matrices with integral entries, such as $\mathrm{GL}_n(\mathbb{Z})$, are arithmetic groups.

Arithmetic groups can be considered from various different perspectives, and each point of view is interesting in itself. From the viewpoint of group theory arithmetic groups are families of examples of infinite groups which combine many nice properties. For instance they are finitely presented, residually finite and virtually torsion-free. Moreover, arithmetic groups arise naturally in number theory and their properties are closely connected to important arithmetic objects, such as quadratic forms over the integers or automorphic forms. Finally one can also consider arithmetic groups as geometric objects, which is the standpoint taken in this thesis. From the outset geometric methods have played a crucial role in the theory of arithmetic groups as developed by Borel, Harder, Harish-Chandra, Raghunathan, Serre and many others.

Let Γ be an arithmetic subgroup of a linear algebraic group G defined over the field of rational numbers $\mathbb{Q}$. The group $G(\mathbb{R})$ of real points of G is a real Lie group and one can consider Γ as a discrete subgroup of $G(\mathbb{R})$. For every maximal compact subgroup $K \subseteq G(\mathbb{R})$ the homogeneous space $X := K\backslash G(\mathbb{R})$ is diffeomorphic to a Euclidean space. The arithmetic group Γ acts properly on X and the geometry of X/Γ reflects the properties of Γ. In particular, geometric invariants of Γ, such as the cohomology, can be studied via the space X/Γ. To make this precise, assume that Γ is torsion-free. If E is a Γ-module, then there is a locally constant sheaf $\widetilde{E}$ on the space X/Γ and a natural isomorphism

$$H^\bullet(\Gamma, E) \xrightarrow{\simeq} H^\bullet(X/\Gamma, \widetilde{E}),$$

where the left hand side denotes the abstract group cohomology of the group Γ and the right hand side denotes the cohomology of the space X/Γ with values in the sheaf $\widetilde{E}$.

The cohomology of arithmetic groups has been extensively studied, nevertheless it is still a rather mysterious invariant of arithmetic groups. For instance, in general it is a very involved task to compute, or estimate, a single Betti number. Luckily, the Euler characteristic of arithmetic groups is understood rather well. Harder's Gauß-Bonnet Theorem makes it possible to give explicit formulas for the Euler characteristics of arithmetic subgroups of semisimple groups. In turn one can get an impression of the size of the Betti numbers if the Euler characteristic is non-zero. Furthermore, the question whether or not the Euler characteristic vanishes, merely depends on the surrounding Lie group $G(\mathbb{R})$. For example, an arithmetically defined lattice in the special linear group $\mathrm{SL}_n(\mathbb{R})$ always has Euler characteristic zero, provided $n \geq 3$. In this case the Euler characteristic does not reveal any information on the cohomology of the arithmetic group.

Let $\tau : G \to G$ be a $\mathbb{Q}$-rational automorphism of finite order. Suppose that the torsion-free arithmetic group Γ is τ-stable, i.e. $\tau(\Gamma) = \Gamma$, then we can define the Lefschetz number of τ by

$$\mathcal{L}(\tau, \Gamma, \mathbb{C}) = \sum_{i=0}^{\infty} (-1)^i \operatorname{Tr}\big(\tau_i : H^i(\Gamma, \mathbb{C}) \to H^i(\Gamma, \mathbb{C})\big),$$

where τ_i denotes the map induced by τ in the cohomology of degree i. Note that this is indeed a finite sum since arithmetic groups have finite cohomological dimension. Analogously one can also define Lefschetz numbers for cohomology with non-trivial coefficients, provided τ acts on the coefficients in a suitable way. The Euler characteristic is the Lefschetz number of the identity, so Lefschetz numbers are a generalisation of the Euler characteristic. The idea to study Lefschetz numbers in the context of arithmetic groups goes back to Harder [**Har75**].

In topology there is a general Lefschetz fixed point principle which roughly says that the Lefschetz number of an automorphism of finite order of a topological space is the Euler characteristic of the space of points fixed by the automorphism. Rohlfs used this principle to develop a general method for the computation of Lefschetz numbers in the cohomology of arithmetic groups, see [**Roh78**] or [**Roh90**]. Let $\tau : G \to G$ be a finite order automorphism and let Γ be a torsion-free, τ-stable arithmetic group. There is a τ-stable maximal compact subgroup $K \subseteq G(\mathbb{R})$ and thus τ naturally acts on X/Γ. The decisive observation is that the space $(X/\Gamma)^\tau$ of τ-fixed points decomposes into several connected components,

$$(X/\Gamma)^\tau = \bigsqcup_{\eta \in H^1(\tau, \Gamma)} \mathfrak{F}(\eta),$$

which can be parametrised by the first non-abelian Galois cohomology set $H^1(\tau, \Gamma)$. The components $\mathfrak{F}(\eta)$ are homeomorphic to spaces of the form $X(\eta)/\Gamma(\eta)$, for certain arithmetic groups $\Gamma(\eta)$. This reduces the computation of Lefschetz numbers to the computation of Euler characteristics of arithmetic groups. However, it can be very difficult to determine the cohomology set $H^1(\tau, \Gamma)$. We emphasise that this thesis substantially relies on Rohlfs' ideas.

Rohlfs' method has been applied to several different groups. Rohlfs computed the Lefschetz number of the Cartan involution on principal congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$ in [**Roh81**]. Moreover, he used the Lefschetz number of the Galois automorphism on Bianchi groups to obtain non-vanishing statements for the cuspidal cohomology, see [**Roh85**]. Further, Lai [**Lai91**] obtained some results on the Lefschetz number of the Cartan involution on congruence subgroups of $\mathrm{SL}_n(\mathbb{Z}[i])$. The Lefschetz number of the Cartan involution on principal congruence subgroups of $\mathrm{SO}(n,1)(\mathbb{Z})$ was computed by Rohlfs-Speh in [**RS87**]. In [**RS98**] Rohlfs and Schwermer treated the Lefschetz number of the Cartan involution on the symplectic group over totally real number fields. Finally, we want to mention the article [**RS93**] of Rohlfs and Speh, where they enhance the theory of Lefschetz numbers to obtain a general formula which relates Lefschetz numbers with orbital integrals. For their applications it is admissible to shrink the involved arithmetic groups in order to avoid a bad behaviour at certain places. On the contrary, it is a focal point of this thesis to compute explicit Lefschetz numbers for arithmetic congruence groups without passing on to subgroups of finite index.

Even though Rohlfs' method works for an arbitrary automorphism of finite order, in most cases only the Lefschetz number of the Cartan involution (or some Galois action) on some (mostly split) group has been explicitly calculated. This thesis has the ambition to cover new classes of examples where concise Lefschetz numbers can be computed. In particular, we focus on inner forms of the special linear group.

Let F be an algebraic number field and let A be a central simple F-algebra. The associated (reduced) norm-one group $G = \mathrm{SL}_A$ is a semisimple, simply connected algebraic group defined over F. Moreover, G is an inner form of the special linear group. Recall that an involution $\tau : A \to A$ of the first kind is an F-linear endomorphism of order two such that $\tau(xy) = \tau(y)\tau(x)$ for all $x, y \in A$. If A has an involution τ of the first kind then G has an automorphism $\tau^* : G \to G$ of order two, defined as the composition of τ with the group inversion. In this thesis we develop a set of tools which, in principle, makes it possible to study the Lefschetz numbers of automorphisms of the form τ^*. The methods work well if τ is an involution of symplectic type (see Def. III.4.4), which yields the main theorem of this thesis. Under some restrictions the approach of this thesis can also be used to treat involutions of the second kind, this is involutions that are not F-linear but which act with a quadratic Galois automorphism on the centre. This will be explained in an application to Bianchi congruence groups. Involutions of orthogonal type, however, are more difficult to handle and the methods in this thesis are not sufficient to treat them.

If A has an involution of the first kind, then A is isomorphic to a matrix algebra $M_n(D)$ where D is a quaternion algebra. So we may restrict to algebras of this form. Moreover, let D be a quaternion algebra, there is a canonical involution of symplectic type $\tau_c : D \to D$. The involution τ_c induces an involution of symplectic type τ on $M_n(D)$, defined by

$$\tau(x) := \tau_c(x)^T$$

for every $x \in M_n(D)$. This involution is, in a way, the prototypical example of an involution of symplectic type since all other symplectic involutions on $M_n(D)$ are of the form $x \mapsto h\tau(x)h^{-1}$ for some $h \in \mathrm{GL}_n(D)$ satisfying $\tau(h) = h$. We only consider this specific involution τ.

Let $\mathcal{O}$ be the ring of integers of the number field F and let $\Lambda_D \subseteq D$ be a maximal $\mathcal{O}$-order. We fix an integer $n \geq 1$ and we consider the F-algebra $A = M_n(D)$ with the associated norm-one group $G = \mathrm{SL}_A$. Let $\mathfrak{a} \subseteq \mathcal{O}$ be a non-trivial ideal and consider the principal congruence subgroup

$$\Gamma(\mathfrak{a}) := \{\, g \in M_n(\Lambda_D) \mid \mathrm{nrd}_A(g) = 1 \text{ and } g \equiv 1 \mod \mathfrak{a} \,\}.$$

These subgroups are τ^*-stable. Let $\rho : G \to \mathrm{GL}(W)$ be a rational representation of G (defined over the algebraic closure of F) on a finite dimensional vector space. If W is equipped with a compatible τ^*-action, then we can define the Lefschetz number $\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W)$ of τ^* in the group cohomology $H^\bullet(\Gamma(\mathfrak{a}), W)$.

Theorem. *Assume that $\Gamma(\mathfrak{a})$ is torsion-free. If D is totally definite, we assume further that $n \geq 2$. The Lefschetz number $\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W)$ is zero if F is not totally real.*

If F is totally real, the Lefschetz number is given by the following formula

$$\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W) \;=\; 2^{-r}\, \mathrm{N}(\mathfrak{a})^{n(2n+1)} \Delta_{rd}(D)^{n(n+1)/2}\, \mathrm{Tr}(\tau^*|W) \prod_{j=1}^{n} M(j, \mathfrak{a}, D).$$

Here $\Delta_{rd}(D)$ denotes the signed reduced discriminant of D, r denotes the number of real places of F ramified in D, and

$$M(j, \mathfrak{a}, D) := \zeta_F(1-2j) \prod_{\mathfrak{p}|\mathfrak{a}} \bigl(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^{2j}}\bigr) \prod_{\substack{\mathfrak{p} \in \mathrm{Ram}_f(D) \\ \mathfrak{p} \nmid \mathfrak{a}}} \bigl(1 + (\frac{-1}{\mathrm{N}(\mathfrak{p})})^j\bigr)$$

where $\mathrm{Ram}_f(D)$ denotes the set of finite places of F where D ramifies.

Note that, if F is totally real, then the Lefschetz number vanishes if and only if the trace $\mathrm{Tr}(\tau^*|W)$ is zero. This makes it possible to use the Lefschetz number for applications like the estimation of Betti numbers.

For involutions of the second kind the developed methods are explained by means of the example of Bianchi congruence groups. Let E be an imaginary quadratic number field and let $\mathcal{O}$ be its ring of integers. We consider the linear algebraic $\mathbb{Q}$-group G obtained by restriction of scalars $E/\mathbb{Q}$ from the special linear group SL_2 over E. On the central simple E-algebra $M_2(E)$ we consider the involution of the second kind $\tau : M_2(E) \to M_2(E)$ defined by $x \mapsto \sigma(x)^T$, where σ denotes the non-trivial Galois automorphism of the extension $E/\mathbb{Q}$. As before, τ induces an automorphism of order two $\tau^* : G \to G$ which is defined over $\mathbb{Q}$. Let $m \geq 3$ be an integer. The principal congruence subgroups

$$\Gamma(m) := \{\, g \in \mathrm{SL}_2(\mathcal{O}) \mid g \equiv 1 \mod m \,\}$$

are τ^*-stable. For every rational representation $\rho : G \to \mathrm{GL}(W)$ on some finite dimensional vector space W, which is equipped with a compatible τ^*-action, we define the Lefschetz number $\mathcal{L}(\tau^*, \Gamma(m), W)$. The result is the following theorem.

Theorem. *Let $E = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field with $d \equiv 1 \mod 4$. For every integer $m \geq 3$ the Lefschetz number $\mathcal{L}(\tau^*, \Gamma(m), W)$ in the group cohomology $H^\bullet(\Gamma(m), W)$ is*

$$\mathcal{L}(\tau^*, \Gamma(m), W) = \frac{m^3|d|}{12}\,\mathrm{Tr}(\tau^*|W) \prod_{p|m}\Big(1 - \frac{1}{p^2}\Big) \prod_{\substack{p|d\\ p\nmid m}}\Big(1 + \frac{(-1)^{(p-1)/2}}{p}\Big).$$

The Lefschetz number is non-zero precisely when $\mathrm{Tr}(\tau^*|W)$ does not vanish.

A guide for the reader. This thesis is divided into five chapters complemented by three appendices which contain additional material included for reasons of completeness. Every paragraph is numbered consecutively in the form chapter.section.paragraph. For instance V.4.2 denotes the second paragraph in the fourth section of chapter V. Theorems, proposition, corollaries, etc. are mostly paragraphs in their own right and hence are part of the consecutive numeration. Sometimes a lemma or proposition belongs to a paragraph an has no number. These are results that will be needed only in the context of the surrounding paragraph.

The first chapter can be considered as a preface in sheaf cohomology. The main result is a generalisation of the fixed point principle mentioned above: The Lefschetz number of an automorphism of finite order is the Euler characteristic of its set of fixed points. We establish a general version of this principle based on the approach of Verdier [**Ver73**]. More precisely, we work with cohomology of equivariant sheaves and arbitrary systems of supports. The reader who is only interested in the main theorem of this thesis will primarily need Corollary I.7.14.

The second chapter begins with a small recapitulation of the cohomology theory of groups and its connections to the cohomology theory of sheaves. Afterwards we give an extensive account of Rohlfs' decomposition of the set of fixed points of finite group actions on the space X/Γ. We intended to distinguish between topological, differential and arithmetic properties. This means that we study abstract groups acting on a topological space first. Then we turn to discrete subgroups of Lie groups and only finally we specialise to arithmetic groups. In the last section we treat an adelic version of the Rohlfs decomposition. The adelic viewpoint is the most important for all applications in this thesis.

Smooth group schemes over Dedekind rings play a decisive role in this thesis. The reason is that the computation of the Euler characteristic of an arithmetic group Γ in a linear algebraic $\mathbb{Q}$-group G is much easier if Γ is a congruence group with respect to some smooth $\mathbb{Z}$-model of G. Further, smoothness of the underlying $\mathbb{Z}$-model simplifies the computation of the involved non-abelian Galois cohomology sets. In fact, keeping track of the smoothness properties of group schemes is our way to avoid dealing with "bad primes".

In Chapter III we introduce and analyse the group schemes occurring in this thesis. In particular, let A be a central simple algebra over some number field F with ring of integers $\mathcal{O}$. Given an $\mathcal{O}$-order $\Lambda \subseteq A$, we define the associated general linear group GL_Λ, and the associated special linear group SL_Λ, as group schemes over $\mathcal{O}$. Whereas the general linear group is always smooth, the special linear group need not be smooth over $\mathcal{O}$. We establish a criterion for the smoothness of SL_Λ, in particular, the special linear group over a maximal order is smooth. Furthermore, we consider involutions and their associated fixed point groups (resp. group schemes). We shall establish a criterion for their smoothness as well. Finally we consider the associated non-abelian Galois cohomology sets. The pfaffian for involutions of symplectic type turns out to be a useful tool analysing non-abelian cohomology. The theory of hermitian forms is another valuable device in this context, and it provides the powerful theorem of Fainsilber-Morales III.5.9. We conclude the third chapter with a short summary of integration on smooth schemes over complete discrete valuation rings.

The fourth chapter is devoted to Harder's Gauß-Bonnet Theorem. We give an adelic reformulation which is based on the notion of smooth group scheme. It was our intention to state a precise version which is suitable for direct applications. In order to do this, we recall some facts on the Gauß-Bonnet form and the Euler-Poincaré measure. We conclude the fourth chapter demonstrating Harder's Theorem by means of the well-known example SL_2.

Chapter V contains the proof of the main theorem and applications. We proceed in two big steps. The first step is the calculation of the non-abelian cohomology sets which describe the fixed point decomposition. This is done in Section V.2. The utilities for this step were developed in the third chapter. The second step is the computation of the Euler characteristics of the fixed point components. The necessary results are taken from chapter four. Eventually, we combine the two steps using the adelic decomposition obtained in the second chapter. We give applications of the main theorem concerning the growth of Betti numbers and the cohomology of arithmetically defined Fuchsian groups. In the added Section V.6 we proceed in a similar fashion to obtain the result for Bianchi congruence groups.

Appendix A is a short, but rather self-contained, summary of the theory of maximal compact subgroups of Lie groups. This theory is essential in many points of this thesis and we felt it would be worth writing down a suitable compilation. All results are well-know, however they are spread over the literature or are at least folklore.

The second appendix is essentially a collection of some technical results on projective modules, schemes, norms, traces and symmetric algebras. There should be no need to read this appendix on its own. It should only be contacted if the reader is interested in details which did not become clear.

Appendix C is a short account of the theory of, as we call them, *degenerate quaternion algebras.* These algebras arise naturally as quotients of maximal orders in quaternion division algebras over p-adic fields. The entire chapter is rather elementary and its main purpose is to calculate the orders of certain finite groups occurring in the computation of the Euler characteristics. This subject did not really fit into one of the five chapters, nevertheless the results are indispensable. We encourage the reader interested in finite dimensional algebras to take a look at it.

Permanent Notation and Conventions

Throughout this thesis every ring has a unit element and every module is unital, i.e. the unit of the ring acts as the identity. A central simple algebra A over a field k is by definition a *finite dimensional* k-algebra with centre k, such that A and (0) are the only two-sided ideals of A. A quaternion algebra is a central simple algebra of dimension 4. Throughout every smooth manifold has finite dimension. Similarly, all schemes are affine and of finite type.

List of permanent notation.

$\mathbb{Z}$	ring of integers
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	the fields of rational, real, complex numbers respectively
$\mathbb{H}$	Hamilton's division quaternion $\mathbb{R}$-algebra
$\mathrm{char}(k)$	characteristic of the field k
$\lvert X\rvert$	cardinality of the set X
$\emptyset$	the empty set
$R^\times$	group of units of the ring R
$\mathrm{Quot}(R)$	quotient field of the integral domain R
$M_n(R)$	ring of $n \times n$ matrices with entries in the ring R
$\dim(V)$	dimension of the vector space V
$\mathrm{Tr}(\varphi\lvert V)$	trace of the endomorphism φ on V
$\det(\varphi)$	determinant of the endomorphism φ
x^T	transpose of the matrix or vector x
$\mathrm{diag}(x_1, \dots, x_n)$	diagonal matrix with entries $x_1, \dots, x_n$
$I_{p,q}$	$(p+q) \times (p+q)$ diagonal matrix $\mathrm{diag}(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q})$
nrd_A	reduced norm on the central simple algebra A
trd_A	reduced trace on A
δ_{ij}	Kronecker delta, 1 if $i = j$, zero otherwise
$\mathrm{Hom}_R(M, N)$	set of R-linear homomorphisms $M \to N$
$\mathrm{End}_R(M)$	set of R-linear endomorphism of M
$S_R(M)$	symmetric R-algebra over the R-module M
$\mathfrak{X} \times_A B$	base change of the A-scheme $\mathfrak{X}$ to B
$\mathrm{Res}_{B/A}(\mathfrak{X})$	Weil restriction of scalars from B to A
$[G : H]$	index of the subgroup $H \subseteq G$
$\mathrm{Gal}(E/F)$	Galois group of the Galois extension E/F
$\overline{F}$	algebraic closure of the field F

CHAPTER I

A Lefschetz Fixed Point Formula

I.1. Introduction

I.1.1. The objective of this chapter is to prove a useful theorem relating Lefschetz numbers of automorphisms of finite order to the Euler characteristic of the set of fixed points of the automorphism. A simple version of this theorem could be formulated as follows: Let X be a topological space satisfying certain cohomological finiteness conditions and let $\tau : X \to X$ be an automorphism of finite prime order. If the space X^τ, consisting of all fixed points of τ, satisfies certain cohomological finiteness conditions, then the Lefschetz number of τ equals the Euler characteristic of X^τ. This kind of result is well-known and appears in various disguises in the literature, for example in [**Ver73**], [**Die87**, p. 225], and [**Bro82b**] combined with [**Zar69**]. However, these results are not applicable, at least not without changes, to the situations we have in mind.

I.1.2. The version of the theorem proved here, as well as the proof, is a generalisation of Verdier's account [**Ver73**]. Whereas Verdier works with cohomology with compact supports and constant coefficients, our intention was to obtain a similar result for different families of supports and a wider class of coefficient systems. Indeed, the Lefschetz fixed point principle obtained in this chapter holds for cohomology with values in flat equivariant sheaves and for paracompactifying invariant systems of supports, see I.7.12.

The chapter is organised as follows. We first recall some facts from the theory of derived categories and derived functors. In Section I.3 we prove a simple version of the finite tor amplitude criterion. Afterwards, in Section I.4, we recall some facts from the cohomology of sheaves, in particular concerning the cohomological dimension of topological spaces. After having established the necessary projection formula in Section I.5, we study actions of finite groups on topological spaces in Section I.6. In particular, we consider the cohomology of equivariant sheaves. The statements and proofs of the main results are given in the last section.

I.2. Derived Categories and Functors

I.2.1. We will assume some familiarity with the theory of triangulated categories and derived categories, as can be found e.g. in [**GM96**] and [**Har66**]. All functors occurring in this section are covariant, unless they are explicitly defined to be contravariant. In this section let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ denote abelian categories. The categories we have in mind are: the category of (left or right) modules over a ring and the category of sheaves of such modules on some topological space. By a *complex* $X^\bullet$ of objects in $\mathcal{A}$ we mean a collection of objects $(X^n)_{n\in\mathbb{Z}}$ of $\mathcal{A}$ together with a family of morphisms $d^n : X^n \to X^{n+1}$, called differentials, such that $d^{n+1} \circ d^n = 0$ for all integers n. Together with the obvious notion of morphism we get the category of $\mathcal{A}$-complexes, which we denote by $\mathrm{Kom}(\mathcal{A})$. A complex $X^\bullet$ is said to be bounded below (resp. above) if there is some integer n_0 such that $X^n = 0$ for all $n < n_0$ (resp. for all $n > n_0$). A complex is bounded if it is bounded below and bounded above. The full subcategory of bounded below (resp. bounded above, bounded) complexes will be denoted by $\mathrm{Kom}^+(\mathcal{A})$ (resp. $\mathrm{Kom}^-(\mathcal{A})$, $\mathrm{Kom}^b(\mathcal{A})$). We consider the corresponding homotopy category $\mathrm{K}(\mathcal{A})$ whose objects are complexes and the morphisms are *homotopy classes* of morphisms of complexes. Recall that $\mathrm{K}(\mathcal{A})$ is a triangulated category. As before we obtain the full triangulated subcategories $\mathrm{K}^+(\mathcal{A})$, $\mathrm{K}^-(\mathcal{A})$ and $\mathrm{K}^b(\mathcal{A})$.

I.2.2. Recall that for every integer n there is a cohomology functor

$$H^n : \mathrm{Kom}(\mathcal{A}) \to \mathcal{A}.$$

Given a complex $X^\bullet$ of objects of $\mathcal{A}$ the n-th cohomology $H^n(X)$ is defined as the quotient $\ker(d^n)/\operatorname{im}(d^{n-1})$. A morphism of $\mathcal{A}$-complexes $f : X^\bullet \to Y^\bullet$ is called a *quasi-isomorphism* if $H^n(f)$ is an isomorphism for every integer n. By localisation of $\mathrm{K}(\mathcal{A})$ with respect to the multiplicative system of quasi-isomorphisms we get the *derived category* of $\mathcal{A}$, denoted by $\mathrm{D}(\mathcal{A})$. The derived category is again a triangulated category. The localisation functor $\mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ will be denoted by Q. As before, we obtain full subcategories $\mathrm{D}^+(\mathcal{A})$, $\mathrm{D}^-(\mathcal{A})$ and $\mathrm{D}^b(\mathcal{A})$. The functor $\mathcal{A} \to \mathrm{D}(\mathcal{A})$ which maps an object $X \in \mathrm{Ob}(\mathcal{A})$ to the complex $X^\bullet$ defined by $X^0 = X$ and $X^n = 0$ for all $n \neq 0$ is fully faithful. Sometimes we will not distinguish the object X and the associated complex $X^\bullet$.

I.2.3. We remind the reader of the notion of derived functor. Recall that a functor F between triangulated categories is called *exact* (or a ∂-functor) if it commutes with the translations and maps distinguished triangles to distinguished triangles. Note that every additive functor $F : \mathcal{A} \to \mathcal{B}$ induces an exact functor $\widetilde{F} : \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$. Clearly, $\widetilde{F}$ restricted to $\mathrm{K}^+, \mathrm{K}^-, \mathrm{K}^b$ is still an exact functor.

Definition. Let $F : \mathrm{K}^-(\mathcal{A}) \to \mathrm{K}^-(\mathcal{B})$ be an exact functor. A *left derived* functor for F is a pair $(\underline{\mathrm{L}}F, \xi)$, where

$$\underline{\mathrm{L}}F : D^-(\mathcal{A}) \to D^-(\mathcal{B})$$

is an exact functor and

$$\xi : \underline{\mathrm{L}}F \circ Q \to Q \circ F$$

is a natural transformation, satisfying the following *universal property*: Whenever a functor $G : \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$ and a natural transformation $\zeta : G \circ Q \to Q \circ F$ are given, there is a unique transformation $\eta : G \to \underline{\mathrm{L}}F$ such that $\zeta = \xi \circ (\eta \circ Q)$.

If a left derived functor exists, it is unique up to unique transformation. There is an analogous definition of *right derived functor* by reversing the directions of the involved natural transformations, see [**Har66**, p. 50].

I.2.4. *Adapted classes.* In order to answer the question of existence of derived functors we introduce the notion of *adapted classes* as defined in [**GM96**, p. 187]. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, we shall denote the induced exact functor $\mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$ by F as well.

Definition. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. A class $\mathcal{L} \subseteq \mathrm{Ob}(\mathcal{A})$ of objects of $\mathcal{A}$ is said to be *adapted* to F if the three following conditions are fulfilled:

(i) $\mathcal{L}$ is closed under finite direct sums.
(ii) F maps acyclic bounded above complexes of objects in $\mathcal{L}$ to acyclic complexes.
(iii) Any object of $\mathcal{A}$ is the quotient of an object in $\mathcal{L}$.

There is also a definition of adapted class for left exact functors. We only have to change “bounded above“ to “bounded below“ in (ii) and replace (iii) by: Any object of $\mathcal{A}$ can be embedded into an object in $\mathcal{L}$.

Theorem. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor, which admits an adapted class $\mathcal{L} \subseteq \mathrm{Ob}(\mathcal{A})$. Then F has a left derived functor $\underline{\mathrm{L}}F : \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$. Moreover, for every complex $X^\bullet$ of objects in $\mathcal{L}$ there is an isomorphism

$$\underline{\mathrm{L}}F(X^\bullet) \xrightarrow{\simeq} F(X^\bullet)$$

in $\mathrm{D}^-(\mathcal{B})$.

A proof can be found in [**GM96**, Thm. 8, p. 189] or [**Har66**, Thm. 5.1, p. 53]. Of course, there is an analogous theorem stating that left exact functors with adapted class have right derived functors.

I.2.5. *Example.* Let R be some ring and fix some right module M over R. Tensoring with M defines a right exact functor

$$M \otimes_R \,\cdot\, : \mathsf{Mod}_l(R) \to \mathsf{Ab}$$

from the category $\mathsf{Mod}_l(R)$ of left R-modules to the category of abelian groups. The class of flat (left) R-modules is an adapted class for the functor $M \otimes_R \,\cdot\,$, which has therefore a left derived functor, usually denoted by $M \otimes^{\mathrm{L}}_R \,\cdot\,$.

I.2.6. *Acyclic objects.* Let $\mathcal{A}$ be an abelian category and assume that $\mathcal{A}$ has enough injectives, then the class of injective objects is adapted to any left exact functor $F : \mathcal{A} \to \mathcal{B}$ (cf. [**GM96**, Thm. 12 p. 194]). Let $X \in \mathrm{Ob}(\mathcal{A})$ and define $\mathrm{R}^p F(X) := H^p(\underline{\mathrm{R}}F(X))$ to obtain the classical right derived functors of F. An object $X \in \mathrm{Ob}(\mathcal{A})$ is said to be F-*acyclic* if $\mathrm{R}^p F(X) = 0$ for all $p \geq 1$. Equivalently, this means that $\underline{\mathrm{R}}F(X)$ is isomorphic to $F(X)$ in the derived category.

I.2.7. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor which admits an adapted class. Sometimes one would like to extend the derived functor $\underline{\mathrm{R}}F : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ to the entire derived category $\mathrm{D}(\mathcal{A})$. This means, is there a functor $\underline{\mathrm{R}}F : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ satisfying the suitable universal property?

Proposition. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories, such that $\mathcal{A}$ has enough injectives. If there is some integer n such that $\mathrm{R}^p F = 0$ for all $p > n$, then F has a right derived functor

$$\underline{\mathrm{R}}F : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B}).$$

Moreover, for any complex $X^\bullet$ consisting of F-acyclic objects there is an isomorphism

$$\underline{\mathrm{R}}F(X^\bullet) \xrightarrow{\simeq} F(X^\bullet)$$

in the derived category of $\mathcal{B}$.

Proof. By Thm. 7.6 in [**Ive86**, p. 55] we know:

(i) Every complex $X^\bullet \in \mathrm{Ob}(\mathrm{Kom}(\mathcal{A}))$ admits a quasi-isomorphism into a complex consisting of F-acyclic objects.
(ii) F takes acyclic complexes of F-acyclic objects to acyclic complexes.

Now we only need to apply Thm. 5.1 of [**Har66**, p. 52] using the class of complexes of F-acyclic objects as adapted. □

I.2.8. *Example.* Let X be a topological space and let Φ be a paracompactifying family of supports (see I.4.1 or [**God58**, p. 150]). Moreover, let $\mathcal{G}$ be a sheaf of abelian groups on X. By $\Gamma_\Phi(X, \mathcal{G})$ we denote the group of sections of $\mathcal{G}$ over X with support in Φ. We get a left exact functor

$$\Gamma_\Phi(X, \,\cdot\,) : \mathsf{Sh}_X(\mathsf{Ab}) \to \mathsf{Ab}$$

from the category $\mathsf{Sh}_X(\mathsf{Ab})$ of sheaves of abelian groups on X to the category Ab of abelian groups. This functor admits an adapted class, namely the class of Φ-soft sheaves on X. This follows directly from II. Thm. 3.5.4 in [**God58**]. We conclude that $\Gamma_\Phi(X, \cdot)$ has a right derived functor

$$\underline{\mathrm{R}}\Gamma_\Phi(X,\, \cdot\,) : \mathrm{D}^+(\mathsf{Sh}_X(\mathsf{Ab})) \to \mathrm{D}^+(\mathsf{Ab})$$

and we know how to compute this functor on Φ-soft sheaves. One could also use the fact that $\mathsf{Sh}_X(\mathsf{Ab})$ has enough injectives to verify the existence of the derived functor. If in addition X has finite Φ-dimension, say $\dim_\Phi X = n$, then $\mathrm{R}^p\,\Gamma_\Phi = 0$ for all $p > n$ and by means of Prop. I.2.7 we deduce that the derived functor can be extended to

$$\underline{\mathrm{R}}\Gamma_\Phi(X,\, \cdot\,) : \mathrm{D}(\mathsf{Sh}_X(\mathsf{Ab})) \to \mathrm{D}(\mathsf{Ab}).$$

Since Φ-soft sheaves are acyclic with respect to Γ_Φ, we know how the derived functor behaves on arbitrary complexes of Φ-soft sheaves. Of course, everything in this paragraph similarly applies to sheaves of R-modules for any ring R.

I.2.9. *Bi-functors.* Having the tensor product example in mind, it seems plausible that one can also derive bi-functors. We want to describe the appropriate setting. Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be an additive bi-functor between abelian categories. We say that F is right exact if it is right exact in both arguments. This means, whenever we fix an object $X \in \mathrm{Ob}(\mathcal{A})$ (resp. $Y \in \mathrm{Ob}(\mathcal{B})$) the functor $F(X, \cdot)$ (resp. $F(\cdot\,, Y)$) is right exact. Similarly, a bi-functor between triangulated categories is called *exact* if it is exact with respect to both arguments. The notion of left derived functor of an exact bi-functor $F : \mathrm{K}^-(\mathcal{A}) \times \mathrm{K}^-(\mathcal{B}) \to \mathrm{K}^-(\mathcal{C})$ is defined along the lines of Definition I.2.3.

I.2.10. An additive functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces an exact functor

$$\widetilde{F} : \mathrm{K}^-(\mathcal{A}) \times \mathrm{K}^-(\mathcal{B}) \to \mathrm{K}^-(\mathcal{C}).$$

In this case a bit of caution is needed to define $\widetilde{F}$. For complexes $X^\bullet \in \mathrm{Ob}(\mathrm{K}^-(\mathcal{A}))$ and $Y^\bullet \in \mathrm{Ob}(\mathrm{K}^-(\mathcal{B}))$ we set

$$\widetilde{F}(X^\bullet, Y^\bullet)^n := \bigoplus_{p+q=n} F(X^p, Y^q).$$

Notice that for every integer n the right hand side is a *finite* direct sum. For all pairs of integers p and q we have a canonical injection $i_{p,q} : F(X^p, Y^q) \to \widetilde{F}(X^\bullet, Y^\bullet)^{p+q}$. Let d_X (resp. d_Y) denote the differential on the complex $X^\bullet$ (resp. $Y^\bullet$). The differential d on $\widetilde{F}(X^\bullet, Y^\bullet)^{p+q}$ is defined by

$$d \circ i_{p,q} := F(d_X^p, \mathrm{Id}) + (-1)^p F(\mathrm{Id}, d_Y^q).$$

One can check that this defines an exact functor of triangulated categories. Let F be a right exact bi-functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and let $\mathcal{L}_1 \subseteq \mathrm{Ob}(\mathcal{A})$ and $\mathcal{L}_2 \subseteq \mathrm{Ob}(\mathcal{B})$ be classes of objects. We say that the pair $(\mathcal{L}_1, \mathcal{L}_2)$ is an adapted class for F if the following two conditions hold:

- For every object $X \in \mathrm{Ob}(\mathcal{A})$ the class $\mathcal{L}_2$ is adapted for $F(X,\, \cdot\,)$,
- For every object $Y \in \mathrm{Ob}(\mathcal{B})$ the class $\mathcal{L}_1$ is adapted for $F(\,\cdot\,, Y)$.

In this case $\widetilde{F}$ admits a left derived functor $\underline{\mathrm{L}}F$ and there is an isomorphism

$$\underline{\mathrm{L}}F(X^\bullet, Y^\bullet) \xrightarrow{\simeq} \widetilde{F}(X^\bullet, Y^\bullet)$$

in the category $\mathrm{D}^-(\mathcal{C})$, whenever $X^\bullet$ is a complex of $\mathcal{L}_1$-objects or $Y^\bullet$ is a complex of objects in $\mathcal{L}_2$. This can be proven using the same arguments as for functors in one variable.

I.2.11. *Example.* Let R be a ring and let $\mathsf{Mod}_l(R)$ (resp. $\mathsf{Mod}_r(R)$) denote the category of left (resp. right) R-modules. The tensor product is a right exact bi-functor

$$\cdot \otimes_R \cdot : \mathsf{Mod}_r(R) \times \mathsf{Mod}_l(R) \to \mathsf{Ab}.$$

The classes of flat right and flat left modules are adapted, and we deduce the existence of a left derived functor

$$\cdot \otimes^{\mathrm{L}}_R \cdot : \mathrm{D}^-(\mathsf{Mod}_r(R)) \times \mathrm{D}^-(\mathsf{Mod}_l(R)) \to \mathrm{D}^-(\mathsf{Ab}).$$

I.2.12. *Example.* Let R be a ring and let X be a topological space. For any sheaf $\mathcal{G}$ of left R-modules and any right R-module M we can define a sheaf $M \otimes_R \mathcal{G}$ of abelian groups, obtained by sheafification of the presheaf $U \mapsto M \otimes_R \mathcal{G}(U)$. Since the tensor product commutes with direct limits, it is easy to see that the stalk of $M \otimes_R \mathcal{G}$ at $x \in X$ is isomorphic to $M \otimes_R \mathcal{G}_x$. Thus we have a right exact bi-functor

$$\cdot \otimes_R \cdot : \mathsf{Mod}_r(R) \times \mathsf{Sh}_X(\mathsf{Mod}_l(R)) \to \mathsf{Sh}_X(\mathsf{Ab}),$$

where $\mathsf{Sh}_X(\mathsf{Mod}_l(R))$ (resp. $\mathsf{Sh}_X(\mathsf{Ab})$) denotes the category sheaves of left R-modules (resp. abelian groups) on X. We show that the tensor product functor has a left derived functor. It follows directly from the description of the stalks that for any sheaf $\mathcal{G}$ of left R-modules the class of *flat* right R-modules is adapted for the functor $M \mapsto M \otimes_R \mathcal{G}$. Next, we have to find an adapted class for the first argument. Recall that a sheaf $\mathcal{G}$ of left R-modules is called *flat* if the stalk $\mathcal{G}_x$ is a flat module for any $x \in X$. It is known that every sheaf of R-modules is a quotient of a flat sheaf (see [**B$^+$84**, p. 52, 1.11(a)] or [**GM96**, Prop. 5, p. 222]). We deduce that the class of flat sheaves of left R-modules is adapted for the tensor product. Thus, there is a left derived functor

$$\cdot \otimes^{\mathrm{L}}_R \cdot : \mathrm{D}^-(\mathsf{Mod}_r(R)) \times \mathrm{D}^-(\mathsf{Sh}_X(\mathsf{Mod}_l(R))) \to \mathrm{D}^-(\mathsf{Sh}_X(\mathsf{Ab})),$$

Furthermore, given a bounded above complex $\mathcal{G}^\bullet$ of left R-modules and a (bounded above) complex $M^\bullet$ of right R-modules, we have an isomorphism

$$M^\bullet \otimes^{\mathrm{L}}_R \mathcal{G}^\bullet \xrightarrow{\simeq} M^\bullet \widetilde{\otimes}_R \mathcal{G}^\bullet,$$

whenever one of the complexes consists entirely of flat objects. Here, we denote by $M^\bullet \widetilde{\otimes}_R \mathcal{G}^\bullet$ the totalised tensor product functor as in I.2.10. For simplicity, we shall omit the tilde from now on and write $M^\bullet \otimes_R \mathcal{G}^\bullet$ for the totalised complex.

I.3. The Finite Tor-Amplitude Criterion

I.3.1. In this section R always denotes a ring. Moreover, the following definition is essential.

Definition. A complex of left R-modules $E^\bullet$ is said to have *finite tor-amplitude* if it is isomorphic in $\mathrm{D}(\mathsf{Mod}_l(R))$ to a bounded complex consisting entirely of flat modules.

Our objective is to prove a criterion that allows to determine whether an object in $D^-(\mathsf{Mod}_l(R))$ has finite tor-amplitude. Even better, in some cases we will be able to replace flat by projective modules. Probably one can attribute this criterion to Grothendieck. At least, it seems to have emerged from SGA 6 [**BGI71**, I, Prop. 5.1]. Another source for the finite tor-amplitude criterion is [**Har66**, II.4.2]. The criterion we are going to prove is simpler, nevertheless it appears to be inaccessible in the literature. Before we state and prove the criterion, we will recall some useful results on modules.

I.3.2. *Pure exact sequences.* A short exact sequence of left R-modules

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$$

is called *pure* if for any right R-module M the induced sequence

$$0 \longrightarrow M \otimes L' \longrightarrow M \otimes L \longrightarrow M \otimes L'' \longrightarrow 0$$

is exact. In this case, we say that L' is a *pure* submodule of L.

I.3.3. Lemma. *Let*

$$0 \longrightarrow L' \xrightarrow{\alpha} L \longrightarrow L'' \longrightarrow 0 \tag{1}$$

be a short exact sequence of left R-modules and suppose that L is flat. Then the following are equivalent:

(i) The sequence is pure exact.
(ii) L'' is flat.
(iii) For any finitely generated *right module M the induced sequence*

$$0 \longrightarrow M \otimes L' \longrightarrow M \otimes L \longrightarrow M \otimes L'' \longrightarrow 0$$

is exact.

Proof. The equivalence of (i) and (ii) is the statement of Corollary (4.86) in [**Lam99**]. It is obvious that (i) implies (iii). For the last implication, assume that the sequence (1) stays exact when we tensorise with finitely generated modules. Let M be any left R-module, we have to show that the morphism

$$\mathrm{Id}_M \otimes \alpha : M \otimes L' \to M \otimes L$$

is injective. Pick any $x \in \ker(\mathrm{Id}_M \otimes \alpha)$ and write it as

$$x = \sum_{i=1}^{k} m_i \otimes a_i$$

with some $a_i \in L'$ and $m_i \in M$ for $1 \le i \le k$. Define M' to be the R-submodule of M generated by $m_1, \ldots, m_k$. Let $\iota : M' \to M$ be the canonical injection. We get a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & M \otimes L' \arrow[r, "\mathrm{Id}_M \otimes \alpha"] & M \otimes L \\
0 \arrow[r] & M' \otimes L' \arrow[r] \arrow[u, "\iota\otimes \mathrm{Id}_{L'}"] & M' \otimes L \arrow[u] \\
 & & 0 \arrow[u]
\end{tikzcd}$$

Exactness of the last column follows from the flatness of L. We see that $\iota \otimes \mathrm{Id}_{L'}$ is injective and we deduce $x = 0$. $\square$

I.3.4. Theorem (Finite Tor-Amplitude Criterion). *Let $E^\bullet$ be a bounded above complex of left R-modules. Then $E^\bullet$ has finite tor-amplitude if and only if there is some integer $n \in \mathbb{Z}$ such that*

$$H^p(M \otimes^{\mathrm{L}}_R E^\bullet) \;=\; 0$$

for any finitely generated right R-module M and any integer $p < n$.

Proof. Assume that $E^\bullet$ has finite tor-amplitude, then (by definition) $E^\bullet$ is isomorphic in $\mathrm{D}^-(\mathsf{Mod}_l(R))$ to a bounded complex $F^\bullet$ where each F^p is flat. Let M be any right R-module. Since the class of flat modules is adapted to the tensor product, we get isomorphisms

$$M \otimes_R^{\mathrm{L}} E^\bullet \;\cong\; M \otimes_R^{\mathrm{L}} F^\bullet \;\cong\; (M \otimes_R F^\bullet)$$

in the derived category $\mathrm{D}^-(\mathsf{Ab})$. Take $n \in \mathbb{Z}$ such that $F^p = 0$ for all $p < n$, then clearly

$$H^p(M \otimes_R^{\mathrm{L}} E^\bullet) = H^p(M \otimes_R F^\bullet) = 0$$

for all $p < n$.

Conversely, assume that there is an integer n such that $H^p(M \otimes_R^{\mathrm{L}} E^\bullet) = 0$ for all $p < n$ and any finitely generated module M. We may assume that $E^\bullet$ is a complex of free modules and that $n = 0$. We denote by $d^i : E^i \to E^{i+1}$ the i-th differential of $E^\bullet$. For any finitely generated right R-module M we have $M \otimes_R^{\mathrm{L}} E^\bullet \cong M \otimes_R E^\bullet$ in $\mathrm{D}^-(\mathsf{Ab})$, thus

$$\ker(\mathrm{Id}_M \otimes d^i) = \mathrm{im}(\mathrm{Id}_M \otimes d^{i-1})$$

for all $i < 0$. We obtain a commutative diagram

$$\begin{tikzcd}
\ker(\mathrm{Id}_M \otimes d^i) \arrow[r, equal] & \mathrm{im}(\mathrm{Id}_M \otimes d^{i-1}) \\
M \otimes \ker(d^i) \arrow[u, "\alpha^i"] \arrow[r, equal] & M \otimes \mathrm{im}(d^{i-1}) \arrow[u, "\beta^{i-1}"']
\end{tikzcd} \tag{2}$$

for all $i < 0$, where α and β denote the canonical maps. Since β^i is onto for all i, we see that α^i is onto for all negative integers i. Via diagram chasing in

$$\begin{tikzcd}
 & M \otimes \ker(d^i) \arrow[r] \arrow[d, "\alpha^i"'] & M \otimes E^i \arrow[r] \arrow[d, equal] & M \otimes \mathrm{im}(d^i) \arrow[r] \arrow[d, "\beta^i"] & 0 \\
0 \arrow[r] & \ker(\mathrm{Id}_M \otimes d^i) \arrow[r] & M \otimes E^i \arrow[r] & \mathrm{im}(\mathrm{Id}_M \otimes d^i) \arrow[r] & 0
\end{tikzcd}$$

one can show that β^i is an isomorphism for all $i < 0$. Finally, looking at the diagram (2) we conclude that α^i is an isomorphism for all $i < 0$ as well. Hence the short sequence

$$0 \longrightarrow M \otimes \ker(d^i) \longrightarrow M \otimes E^i \longrightarrow M \otimes \mathrm{im}(d^i) \longrightarrow 0$$

is exact for any finitely generated right R-module M. Therefore, by Lemma I.3.3, the R-module $\mathrm{im}(d^i)$ is flat for all negative integers i. Define a complex $F^\bullet$ by

$$F^i \;=\; \begin{cases} E^i & \text{if } i \geq 0 \\ \mathrm{im}(d^{-1}) & \text{if } i = -1 \\ 0 & \text{if } i < 0. \end{cases}$$

The canonical map $E^\bullet \to F^\bullet$ is a quasi-isomorphism. □

I.3.5. Corollary. *Let R be a ring such that every flat left R-module has finite projective dimension*[1]*. A bounded above complex $X^\bullet$ of left R-modules has finite tor-amplitude if and only if it is isomorphic in $\mathrm{D}^-(\mathsf{Mod}_l(R))$ to a bounded complex of projective modules.*

Proof. In the proof of the finite tor-amplitude criterion I.3.4 we can add another step. Find a finite projective resolution of $\mathrm{im}(d^{-1})$:

$$0 \to P^k \to P^{k-1} \to \cdots \to P^0 \to \mathrm{im}(d^{-1}) \to 0.$$

[1]A module has finite projective dimension if it has a projective resolution of finite length.

Then we are able to define a bounded complex $\widetilde{F}^\bullet$ as follows

$$\widetilde{F}^i = \begin{cases} E^i & \text{if } i \geq 0 \\ P^{-i-1} & \text{if } -(k+1) \leq i \leq -1 \\ 0 & \text{else.} \end{cases}$$

Again, the canonical map $\widetilde{F}^\bullet \to F^\bullet$ is a quasi-isomorphism. □

I.3.6. *Remark.* Let R be a left noetherian ring. Then a bounded complex $P^\bullet$ of projective left R-modules is homotopy equivalent to a bounded complex of *finitely generated* projective R-modules if and only if $H^i(P^\bullet)$ is a finitely generated R-module for every integer i. This follows from a useful lemma of Brown (cf. [**Bro74**, p. 231]).

I.4. Cohomology of Sheaves and Cohomological Dimension

I.4.1. In the entire section X and Y will denote topological Hausdorff spaces. Our objective is to establish some lemmata on cohomological dimension.

Recall that a family of supports on X is a collection Φ of closed subspaces, which is closed under finite unions and which satisfies: whenever $S \in \Phi$ and $A \subset S$ is a closed subset, then $A \in \Phi$. A family Φ of supports is said to be *paracompactifying* if all elements of Φ are paracompact and every $S \in \Phi$ has a neighbourhood which belongs to Φ. For more details see [**God58**, p. 150] or [**Kul70**, 12.5]. The most prominent examples are the family of all closed sets in a paracompact space and the family of compact sets in a locally compact space. For any subspace $Z \subset X$ we obtain a family of supports $\Phi_{|Z} := \{ S \in \Phi \mid S \subseteq Z \}$ on Z. If Φ is paracompactifying and Z is locally closed, then $\Phi_{|Z}$ is paracompactifying as well (cf. [**Kul70**, 13.8]). Given a sheaf $\mathcal{G}$ on X, we denote the set of sections of $\mathcal{G}$ with support in Φ by $\Gamma_\Phi(X, \mathcal{G})$. We write $\Gamma(U, \mathcal{G})$ as well as $\mathcal{G}(U)$ for the set of all sections of $\mathcal{G}$ over some subset $U \subseteq X$.

I.4.2. Let Φ be a paracompactifying family of supports on X. We say that X has finite Φ-dimension if there is some natural number n such that $H^i_\Phi(X, \mathcal{G}) = 0$ for any sheaf of abelian groups $\mathcal{G}$ and any $i > n$. The smallest possible n is called the Φ-dimension of X, denoted $\dim_\Phi X$. If X is paracompact and Φ is the family of all closed subspaces, then the Φ-dimension is usually called the *cohomological dimension* of X and is simply denoted by $\dim X$. Moreover, for any paracompactifying family Φ of supports we have

$$(3) \qquad \dim_\Phi X \leq n \;\Leftrightarrow\; \dim S \leq n \;\text{ for all } S \in \Phi$$

(cf. [**God58**, II.4.14.1]). Finite cohomological dimension is a local property:

Theorem. A paracompact space X has cohomological dimension $\dim X \leq n$ if and only if every point has a closed neighbourhood V of cohomological dimension $\dim V \leq n$.

(cf. [**God58**, II.4.14.1] and [**Kul70**, 24.5]). Another fact we will freely use, can be found in [**God58**, II.4.9.1]: Let Φ be a family of supports on X and assume $\dim_\Phi X \leq n$. Then $\dim_{\Phi|A} A \leq n$ for any closed subspace A of X. If Φ is paracompactifying, this follows also from (3) for any locally closed $A \subseteq X$.

I.4.3. Lemma. *Let $f : X \to Y$ be a surjective local homeomorphism between paracompact Hausdorff spaces. Then $\dim X \leq n$ if and only if $\dim Y \leq n$.*

Proof. Since X and Y are paracompact Hausdorff spaces, every point admits a neighbourhood basis of closed sets. Thus, every point $x \in X$ has a closed neighbourhood homeomorphic to a closed neighbourhood of $f(x) \in Y$. Conversely, using surjectivity of f, every point $y \in Y$ has a closed neighbourhood homeomorphic to a closed neighbourhood of some x in X. Now the result follows readily from I.4.2. $\square$

I.4.4. Corollary. *Let X, Y be topological Hausdorff spaces and Φ (resp. Ψ) a paracompactifying family of supports on X (resp. Y). Suppose there is a surjective local homeomorphism $f : X \to Y$ such that $f^{-1}(S) \in \Phi$ for all $S \in \Psi$, then*

$$\dim_\Phi X \leq n \implies \dim_\Psi Y \leq n.$$

Proof. Let $S \in \Psi$ be a support in Y, then $S' := f^{-1}(S) \in \Phi$ and $f_{|S'} : S' \to S$ is a surjective local homeomorphism between paracompact spaces. $\square$

I.4.5. *Remark.* Note that the property $f^{-1}(S) \in \Phi$ for all $S \in \Psi$ can be a strong condition. In the case of locally compact spaces and compact supports this means that the map f is proper. By the way, one can also prove the converse implication, but then the additional assumption $f(\Phi) \subseteq \Psi$ is necessary.

I.4.6. Definition. Let X be a Hausdorff space, Φ a paracompactifying family of supports and $\mathcal{G}$ a sheaf of left R-modules on X. We say that the triple $(X, \Phi, \mathcal{G})$ is of *finite type* (with respect to R) if for every integer n the cohomology group $H^n_\Phi(X,\ \mathcal{G})$ is a finitely generated R-module.

I.4.7. *Remark.* Let X, Φ and $\mathcal{G}$ be as in I.4.6 and assume the ring R to be left noetherian. Let $A \subset X$ be a closed subspace and let $U = X \setminus A$ be the complement. If two of the three triples $(X, \Phi, \mathcal{G})$, $(A, \Phi_{|A}, \mathcal{G}_{|A})$, $(U, \Phi_{|U}, \mathcal{G}_{|U})$ are of finite type, then the third is of finite type as well. This follows from the long exact sequence

$$\cdots \to H^n_{\Phi|U}(U, \mathcal{G}_{|U}) \longrightarrow H^n_\Phi(X, \mathcal{G}) \longrightarrow H^n_{\Phi|A}(A, \mathcal{G}_{|A}) \longrightarrow H^{n+1}_{\Phi|U}(U, \mathcal{G}_{|U}) \to \cdots$$

associated with a closed subspace (see [**God58**], Thm. II.4.10.1).

I.5. A Projection Formula

I.5.1. In this section we are going to explain a specific projection formula. Let X be a topological Hausdorff space and Φ a paracompactifying family of supports. We shall always assume that $n = \dim_\Phi X$ is finite. The aim is to show the identity

$$\underline{\mathrm{R}}\Gamma_\Phi(X, M \otimes^{\mathrm{L}}_R \mathcal{G}) \cong M \otimes^{\mathrm{L}}_R \underline{\mathrm{R}}\Gamma_\Phi(X, \mathcal{G})$$

for any right noetherian ring R, any sheaf $\mathcal{G}$ of left R-modules and any finitely generated right R-module M.

I.5.2. Let R be a ring, $\mathcal{G}$ a sheaf of left R-modules and M a right R-module. In general, the presheaf defined by $U \mapsto M \otimes_R \mathcal{G}(U)$ is not a sheaf. In particular, we should not expect equality of $\Gamma_\Phi(X, M \otimes \mathcal{G})$ and $M \otimes \Gamma_\Phi(X, \mathcal{G})$. However, the theory of derived categories will be able to fix this defect if we can show this equality for "sufficiently many" modules M.

Lemma. Let P be a finitely generated projective right R-module. For any sheaf $\mathcal{G}$ of left modules the presheaf $U \mapsto P \otimes_R \mathcal{G}(U)$ is indeed a sheaf. This means, for any open $U \subseteq X$ we have

$$\Gamma(U, P \otimes_R \mathcal{G}) \ = \ P \otimes_R \Gamma(U, \mathcal{G}).$$

The identity $\Gamma_\Phi(X, P \otimes_R \mathcal{G}) \ = \ P \otimes_R \Gamma_\Phi(X, \mathcal{G})$ holds for any family of supports Φ.

PROOF. Observation: If $\mathcal{F}_1$ and $\mathcal{F}_2$ are presheaves of abelian groups, then $\mathcal{F}_1 \oplus \mathcal{F}_2$ is a sheaf if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are sheaves. Therefore, we may assume that P is a finitely generated *free* R-module. We choose a basis $e_1, \ldots, e_m$ of P and obtain an isomorphism between the presheaf $U \mapsto P \otimes \mathcal{G}(U)$ and the sheaf $\bigoplus_{i=1}^m \mathcal{G}e_i$.

The identity $\Gamma_\Phi(X, P \otimes_R \mathcal{G}) = P \otimes_R \Gamma_\Phi(X, \mathcal{G})$ is obvious if P is free and follows for projective modules via a direct sum argument. $\square$

I.5.3. *Remark.* Let X be a locally compact space and consider cohomology with compact supports. There is a canonical isomorphism

$$\Gamma_c(X, M \otimes_R \mathcal{G}) = M \otimes_R \Gamma_c(X, \mathcal{G})$$

for any flat R-module M (cf. Lemma 2.5.12 in [**KS90**]).

I.5.4. Corollary. *Let $\mathcal{G}$ be a sheaf of left R-modules on X, P a finitely generated projective right module and Φ a family of supports on X. If $\mathcal{G}$ is flabby (soft, Φ-soft), then $P \otimes_R \mathcal{G}$ is flabby (soft, Φ-soft).*

PROOF. We only have to observe that even for closed sets $A \subseteq X$ the identity $\Gamma(A, P \otimes \mathcal{G}) = P \otimes \Gamma(A, \mathcal{G})$ holds. $\square$

I.5.5. Theorem (Projection Formula)**.** *Let R be a right noetherian ring, X a topological Hausdorff space with a paracompactifying family of supports Φ. Assume X has finite Φ-dimension. Then we have an isomorphism*

$$\underline{\mathrm{R}}\Gamma_\Phi(X, M \otimes_R^{\mathrm{L}} \mathcal{G}) \cong M \otimes_R^{\mathrm{L}} \underline{\mathrm{R}}\Gamma_\Phi(X, \mathcal{G})$$

in the derived category $\mathrm{D}(\mathsf{Ab})$, *for any sheaf of left R-modules $\mathcal{G}$ and any finitely generated right R-module M.*

PROOF. Take a resolution $P^\bullet \to M$ of M by finitely generated projective modules. Here we need the assumption that R is right noetherian. Furthermore, there is a finite resolution $\mathcal{G} \to \mathcal{F}^\bullet$ of $\mathcal{G}$ by Φ-soft sheaves. Here we use that X has finite Φ-dimension to apply Satz 24.4 in [**Kul70**]. Therefore, we have an isomorphism

$$M \otimes_R^{\mathrm{L}} \mathcal{G} \xrightarrow{\simeq} P^\bullet \otimes_R^{\mathrm{L}} \mathcal{F}^\bullet$$

in the derived category $\mathrm{D}^-(\mathsf{Sh}_X(\mathsf{Ab}))$. Since projective modules are flat, and flat modules are adapted to the tensor product (see I.2.12), we see that the right hand side is isomorphic in the derived category to the totalised complex $P^\bullet \otimes_R \mathcal{F}^\bullet$ defined by

$$(P^\bullet \otimes_R \mathcal{F}^\bullet)^m = \bigoplus_{p+q=m} P^p \otimes \mathcal{F}^q$$

as in I.2.10. Using Corollary I.5.4 we see that this is a complex of Φ-soft sheaves. Since Φ-soft sheaves are Γ_Φ-acyclic, the complex is adapted to the (extended) derived functor $\underline{\mathrm{R}}\Gamma_\Phi$. Thus, by I.2.8 and I.5.2 we obtain

$$\underline{\mathrm{R}}\Gamma_\Phi(X, P^\bullet \otimes_R^{\mathrm{L}} \mathcal{F}^\bullet) \cong P^\bullet \otimes_R \Gamma_\Phi(X, \mathcal{F}^\bullet).$$

Finally, we use once more that projective modules are adapted to the tensor product to obtain the result:

$$\underline{\mathrm{R}}\Gamma_\Phi(X, M \otimes_R^{\mathrm{L}} \mathcal{G}) \cong P^\bullet \otimes_R^{\mathrm{L}} \Gamma_\Phi(X, \mathcal{F}^\bullet) \cong M \otimes_R^{\mathrm{L}} \underline{\mathrm{R}}\Gamma_\Phi(X, \mathcal{G}).$$

$\square$

I.5.6. Corollary. *Let R be a right noetherian ring, X a Hausdorff space and Φ a paracompactifying family of supports. Assume X has finite Φ-dimension. For any flat sheaf $\mathcal{G}$ of left R-modules the complex $\underline{\mathrm{R}}\Gamma_{\Phi}(X,\mathcal{G})$ has finite tor-amplitude.*

PROOF. Let M be any finitely generated right R-module. Note that $M \otimes_R^{\mathrm{L}} \mathcal{G}$ is isomorphic to $M \otimes_R \mathcal{G}$ in the derived category because $\mathcal{G}$ is flat. By the projection formula we get

$$M \otimes_R^{\mathrm{L}} \underline{\mathrm{R}}\Gamma_{\Phi}(X,\mathcal{G}) \cong \underline{\mathrm{R}}\Gamma_{\Phi}(X, M \otimes_R \mathcal{G}).$$

Now, we only have to apply the finite tor-amplitude criterion I.3.4 using the assumption that X has finite Φ-dimension. $\square$

I.6. Actions of Finite Groups

I.6.1. Let X be a Hausdorff space. Let G be a finite group acting on X from the left by homeomorphisms:

$$\rho : G \times X \longrightarrow X \quad \text{with} \ \ \rho(g,x) =: gx.$$

Even though all groups act from the left we will always denote quotient spaces in this section by right quotients, e.g. X/G. We introduce a bunch of notation which was intentionally chosen to resemble Verdier's notation (cf. [**Ver73**]). We denote the stabiliser of a point $x \in X$ by $\mathrm{Stab}(x)$. For a subgroup H of G we put

$$\begin{aligned} X^H &:= \{\, x \in X \mid hx = x \ \text{ for all } h \in H \,\} \\ X_H &:= \{\, x \in X \mid \mathrm{Stab}(x) = H \,\}. \end{aligned}$$

Moreover, let $\mathrm{Cl}(G)$ denote the set of conjugacy classes of subgroups of G. For a class $\mathcal{C} \in \mathrm{Cl}(G)$ we define

$$\begin{aligned} X^{\mathcal{C}} &= \bigcup_{H \in \mathcal{C}} X^H \\ X_{\mathcal{C}} &= \bigcup_{H \in \mathcal{C}} X_H \;=\; \{\, x \in X \mid \mathrm{Stab}(x) \in \mathcal{C} \,\}. \end{aligned}$$

Note that X^H and $X^{\mathcal{C}}$ are closed subsets of X, whereas the sets X_H and $X_{\mathcal{C}}$ are locally closed in X.

I.6.2. *The action of G on $X_{\mathcal{C}}$.* Let $\mathcal{C} \in \mathrm{Cl}(G)$ be a conjugacy class. The space $X_{\mathcal{C}}$ is G-invariant since

$$\mathrm{Stab}(gx) \;=\; g\,\mathrm{Stab}(x)g^{-1}$$

for every $x \in X$ and $g \in G$. Let $N_G(H)$ denote the normaliser of a subgroup H in G.

Lemma. The space $X_{\mathcal{C}}$ is topologically the disjoint union of the X_H with $H \in \mathcal{C}$. Moreover, for every $H \in \mathcal{C}$ the inclusion $j : X_H \to X_{\mathcal{C}}$ induces a homeomorphism

$$X_H/N_G(H) \xrightarrow{\simeq} X_{\mathcal{C}}/G.$$

PROOF. Let H be an element of $\mathcal{C}$. Since $X^H \cap X_{\mathcal{C}} = X_H$, we see that X_H is closed in $X_{\mathcal{C}}$. Furthermore, $U := X \setminus \bigcup_{K \not\leq H} X^K$ is open and again $U \cap X_{\mathcal{C}} = X_H$. We conclude that the inclusion $j : X_H \to X_{\mathcal{C}}$ is open and closed, and obviously factors to a homeomorphism $X_H/N_G(H) \xrightarrow{\simeq} X_{\mathcal{C}}/G$. $\square$

I.6.3. *A technical Lemma.* This paragraph is devoted to the proof of a technical lemma. Note that if H and K are subgroups of G, then $X^H \cap X^K = X^{\langle H,K\rangle}$ where $\langle H, K\rangle$ denotes the subgroup generated by H and K. Thus, the finite collection of closed sets $S := \{\, X^H \mid H \leq G \,\}$ is closed under intersections. This is the family one should have in mind while reading the next lemma.

Lemma. Let X be a Hausdorff space, Φ a paracompactifying family of supports, and R a left noetherian ring. Moreover, let $\mathcal{E}$ be a sheaf of left R-modules on X. Let S be a finite collection of closed subsets of X such that $A, B \in S$ implies $A \cap B \in S$. For subsets T, Z of S we define locally closed subspaces

$$V(Z,T) := \bigcup_{B\in Z} B \setminus \bigcup_{A\in T} A.$$

Then the following hold

(i) If $\dim_\Phi X$ is finite, then for all subsets $Z, T \subseteq S$ the dimension of $V(Z,T)$ with respect to $\Phi_{|V(Z,T)}$ is finite.
(ii) Suppose that for all $A \in S$ the triple $(A, \Phi_{|A}, \mathcal{E}_{|A})$ is of finite type (w.r.t. R). Then for all subsets $Z, T \subseteq S$ the triple $(V(Z,T), \Phi_{|V(Z,T)}, \mathcal{E}_{|V(Z,T)})$ is of finite type.

Proof. The first claim follows from the last remark in I.4.2. We will prove the second statement by double induction on the cardinalities of Z and T. For the induction basis observe that the claim holds for $T = \emptyset$ and $|Z| = 1$ by assumption.

Let $1 \leq t \leq |S|$ be an integer, and assume that the claim holds for any $Z = \{B\}$ and any $T \subseteq S$ with $|T| < t$. Take a subset T of S which has exactly t elements. Then choose some $A \in T$ and put $T' = T \setminus \{A\}$. Observe the following two relations

$$\begin{aligned} V(\{B\},T') \cap A \;&=\; V(\{B \cap A\},T'), \\ V(\{B\},T') \setminus V(\{B\cap A\},T') \;&=\; V(\{B\},T') \setminus A \;=\; V(\{B\},T). \end{aligned}$$

This means that the set $V(\{B\cap A\},T')$ is closed in $V(\{B\},T')$ and its complement is $V(\{B\},T)$. The induction hypothesis for $V(\{B\},T')$ and $V(\{B\cap A\},T')$ and the long exact sequence associated with a closed subspace (cf. I.4.7) yield the claim for T.

Let $2 \leq z \leq |S|$ be an integer. For the second induction we may assume that the claim holds for any T and all $Z \subseteq S$ with $|Z| < z$. Take $Z \subseteq S$ with exactly z elements. Choose any $B \in Z$ and put $Z' := Z \setminus \{B\}$. The following relations hold for any subset $T \subseteq S$

$$\begin{aligned} V(Z,T) \cap B \;&=\; V(\{B\},T), \\ V(Z,T) \setminus V(\{B\},T) \;&=\; V(Z,T) \setminus B \;=\; V(Z', T \cup \{B\}). \end{aligned}$$

Once again the statement follows from the induction hypothesis and the argument of Remark I.4.7. □

I.6.4. Lemma. *Let X be a Hausdorff space and G a finite group acting on X. If X is paracompact, then the quotient space X/G is paracompact.*

Proof. Let $\pi : X \to X/G$ denote the canonical projection. It is continuous, open, closed, and surjective. Let $(U_i)_{i\in I}$ be any open cover of X/G, so we have to find a locally finite refinement. We set $V_i := \pi^{-1}(U_i)$ and obtain an open cover of X. Since X is assumed to be paracompact, we can find a locally finite refinement $(V'_j)_{j\in J}$. We construct another open cover $(W_j)_{j\in J}$ of X as

$$W_j := GV'_j.$$

Since all the V_i are G-invariant, the collection $(W_j)_{j\in J}$ is still a refinement of $(V_i)_{i\in I}$. We claim that $(W_j)_J$ is still locally finite. Pick $x \in X$; for any $g \in G$ we find a neighbourhood $N(g)$ of gx which intersects only finitely many of the V'_j. Define a neighbourhood N of x by

$$N := \bigcap_{g\in G} g^{-1}N(g).$$

If N meets W_j for some $j \in J$, then $gV'_j \cap N \neq \emptyset$ for some $g \in G$. We see that $N(g^{-1})\cap V'_j$ is not empty and deduce that there is only a finite number of such j. Moreover, there is a refinement of the cover $(U_i)_{i\in I}$ given by the sets $U'_j := \pi(W_j)$. This refinement is locally finite since $\pi(N)$ is an open neighbourhood of $\pi(x)$ intersecting only a finite number of the sets U'_j. □

I.6.5. *Supports.* A family of supports Φ on X is said to be *G-invariant* if for every $S \in \Phi$ and every $g \in G$ also gS belongs to Φ. Let $\pi : X \to X/G$ be the canonical quotient map. From an invariant family of supports we obtain a family of supports $\pi(\Phi)$ on X/G and the assumption that Φ is G-invariant implies that $S \in \pi(\Phi)$ if and only if $\pi^{-1}(S) \in \Phi$. Moreover, Lemma I.6.4 implies that $\pi(\Phi)$ is paracompactifying if Φ is paracompactifying.

I.6.6. *Direct image sheaves.* Let X be a Hausdorff space with a finite transformation group G. We study the direct image functor associated with the canonical projection $\pi : X \to X/G$. Recall that if $\mathcal{G}$ is a sheaf on X, the direct image sheaf $\pi_*(\mathcal{G})$ is the sheaf on X/G which is defined by $\pi_*(\mathcal{G})(U) = \mathcal{G}(\pi^{-1}(U))$.

Lemma. Let $\mathcal{G}$ be a sheaf on X and let $\bar{x}$ be a point in X/G. Then we can describe the stalk of $\pi_*(\mathcal{G})$ at $\bar{x}$ by

$$\pi_*(\mathcal{G})_{\bar{x}} \;=\; \prod_{x\in\pi^{-1}(\bar{x})} \mathcal{G}_x \;=\; \mathcal{G}(\pi^{-1}(\bar{x})).$$

PROOF. We begin with the second equality. Let $p : \widetilde{\mathcal{G}} \to X$ be the *étalé* space associated with the sheaf $\mathcal{G}$. Since X is Hausdorff, $\pi^{-1}(\bar{x})$ is discrete and we have

$$\mathcal{G}(\pi^{-1}(\bar{x})) = \{\, \sigma : \pi^{-1}(\bar{x}) \to \widetilde{\mathcal{G}} \mid p(\sigma(x)) = x \text{ for all } x \,\} = \prod_{x\in\pi^{-1}(\bar{x})} \mathcal{G}_x$$

Consider the first equality. Given an open set $U \subseteq X$ containing the fibre over $\bar{x}$, we find an open neighbourhood W of $\bar{x}$ such that $\pi^{-1}(W)$ is contained in U. More precisely, put $W := \pi(\bigcap_{g\in G} gU)$. Eventually, it is easily seen that

$$\pi_*(\mathcal{G})_{\bar{x}} = \varinjlim_{\bar{x}\in W} \pi_*(\mathcal{G})(W) = \varinjlim_{\pi^{-1}(\bar{x})\subset U} \mathcal{G}(U) = \prod_{x\in\pi^{-1}(\bar{x})} \mathcal{G}_x$$ □

Theorem. The functor $\pi_* : \mathsf{Sh}_X(\mathsf{Ab}) \to \mathsf{Sh}_{X/G}(\mathsf{Ab})$ is exact.

PROOF. Let $\mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G}$ be an exact sequence of sheaves of abelian groups on the space X. This means, by definition, that the sequence of stalks

$$\mathcal{E}_x \xrightarrow{f_x} \mathcal{F}_x \xrightarrow{g_x} \mathcal{G}_x$$

is exact for every $x \in X$. Moreover, the sequence

$$\bigoplus_{x\in\pi^{-1}(\bar{x})} \mathcal{E}_x \xrightarrow{\oplus f} \bigoplus_{x\in\pi^{-1}(\bar{x})} \mathcal{F}_x \xrightarrow{\oplus g} \bigoplus_{x\in\pi^{-1}(\bar{x})} \mathcal{G}_x$$

has to be exact for every $\bar{x} \in X/G$. However, by the lemma this is just the induced sequence in the stalks of the direct image sheaves at $\bar{x}$. □

I.6.7. Theorem. *Let Φ be a G-invariant system of supports on X. For any sheaf $\mathcal{G}$ of abelian groups on X, there is an isomorphism*

$$H^\bullet_\Phi(X,\ \mathcal{G}) \xrightarrow{\simeq} H^\bullet_{\pi(\Phi)}(X/G,\ \pi_*(\mathcal{G})).$$

If $\mathcal{G}$ is a sheaf of R-modules, then this is an isomorphism of R-modules. In particular, the triple $(X, \Phi, \mathcal{G})$ is of finite type with respect to R (see I.4.6) exactly if $(X/G, \pi(\Phi), \pi_*(\mathcal{G}))$ is of finite type.

Proof. Note that by definition $\mathcal{G}(X) = \pi_*(\mathcal{G})(X/G)$. Take s in $\mathcal{G}(X)$ and let S be its support. As an element of $\pi_*(\mathcal{G})(X/G)$ the section s has support $\pi(S)$. Using that Φ is G-invariant, we see that s has support in Φ if and only if it has support in $\pi(\Phi)$ when considered as a section of $\pi_*(\mathcal{G})$.

It is well known that the direct image of a flabby sheaf is flabby (cf. [**God58**, Thm. 3.1.1]). Now, by the Theorem in I.6.6 the functor π_* is exact, therefore it takes flabby resolutions to flabby resolutions. Using the observation above, it follows that $H^\bullet_\Phi(X,\ \mathcal{G})$ and $H^\bullet_{\pi(\Phi)}(X/G,\ \pi_*(\mathcal{G}))$ can be computed by the same complex. □

I.6.8. *G-sheaves.* Let X be a Hausdorff space with a left action of a finite group G. We recall Grothendieck's concept of a *G-sheaf*, or G-equivariant sheaf, as defined in [**Gro57**, Chap. V].

Definition. A sheaf $\mathcal{E}$ on X together with a continuous left G-action on the étalé space $\widetilde{\mathcal{E}}$ is called a *G-sheaf* if the projection $p : \widetilde{\mathcal{E}} \to X$ is G-equivariant.

Usually we will simply say that $\mathcal{E}$ is a G-sheaf without giving a name to the action on the étalé space. Whenever we have a section $\sigma : U \to \widetilde{\mathcal{E}}$ of a G-sheaf and an element $g \in G$, we get a section ${}^g\sigma : gU \to \widetilde{\mathcal{E}}$ defined by

$${}^g\sigma(x) := g\sigma(g^{-1}x).$$

Expressed differently, for every $g \in G$ and any subspace $U \subseteq X$ we get a bijection

$$\kappa(g, U) : \Gamma(U, \mathcal{E}) \to \Gamma(gU, \mathcal{E})$$

sending σ to ${}^g\sigma$. Moreover, we have the relation $\kappa(h, gU) \circ \kappa(g, U) = \kappa(hg, U)$ for all $h, g \in G$ and the maps $\kappa(g, U)$ are compatible with restriction. These two properties can be used to give an alternative (but equivalent) definition of G-sheaves. In particular, we get an isomorphism of sheaves $\kappa(g) : g_*(\mathcal{E}) \to \mathcal{E}$.

Let R be a ring. We say that a G-sheaf $\mathcal{E}$ is a *G-sheaf of R-modules* if $\mathcal{E}$ is a sheaf of R-modules and the maps $\kappa(g, U)$ are homomorphisms of R-modules.

I.6.9. *Cohomology of G-sheaves.* If $\mathcal{E}$ is a G-sheaf of R-modules and Φ a G-invariant system of supports, then the cohomology $H^\bullet_\Phi(X,\ \mathcal{E})$ is equipped with an $R[G]$-module structure. One can show that the category of G-sheaves of left R-modules is an abelian category with enough injectives (cf. Prop. 5.1.2 in [**Gro57**]). For a G-sheaf $\mathcal{E}$ the global sections $\Gamma_\Phi(X, \mathcal{E})$ with support in Φ form an $R[G]$-module, so Γ_Φ is a left exact functor from the category of G-sheaves of R-modules into the category of $R[G]$-modules. Since the category of G-sheaves has enough injectives, we can derive this functor and obtain the appropriate definition of *cohomology of G-sheaves*.

If $U \subseteq X$ is an open G-invariant subset, then the R-module $\Gamma(U, \mathcal{E})$ is even a left $R[G]$-module. Therefore, the direct image sheaf $\pi_*(\mathcal{E})$ is a sheaf of left $R[G]$-modules on X/G. The isomorphism of Theorem I.6.7 is indeed an isomorphism of left $R[G]$-modules.

I.6.10. *Example.* Let M be any left R-module. The constant sheaf $\mathcal{E} = M_X$ is an equivariant sheaf in a natural way. Recall that for any open set U in X we can identify the sections of $\mathcal{E}$ over U with

$$\mathcal{E}(U) \;=\; \{\, f : U \to M \mid f \text{ locally constant} \,\}.$$

For any $g \in G$ there is a natural map $\kappa(g,U) : \mathcal{E}(U) \to \mathcal{E}(gU)$ defined by

$$\kappa(g,U)(f)(x) = f(g^{-1}x).$$

It is easy to see that this furnishes $\mathcal{E}$ with the structure of a G-sheaf of left R-modules.

I.6.11. *Twisted G-sheaves.* Let $\mathcal{E}$ be a G-sheaf of R-modules on a space X. A *character* of G with values in R is a homomorphism of groups $\psi : G \to R^\times$ from G into the unit group of R. Such a character ψ can be used to construct a new G-sheaf $\mathcal{E} \otimes \psi$ from $\mathcal{E}$, which we will call the *ψ-twisted G-sheaf.* We define for any open set $U \subseteq X$ the sections over U by $(\mathcal{E} \otimes \psi)(U) := \mathcal{E}(U)$ as R-modules, but we change the action of G. Let κ denote the G-sheaf structure of $\mathcal{E}$. We define the ψ-twisted structure κ^ψ by

$$\kappa^\psi(g,U)(s) = \psi(g)\kappa(g,U)(s)$$

for any open set U and any section $s \in \mathcal{E}(U) = (\mathcal{E} \otimes \psi)(U)$.

It is easy to see that twisting with a character leads to the same effect in the cohomology. More precisely, if Φ is a G-invariant system of supports, we have an isomorphism of $R[G]$-modules

$$H^\bullet_\Phi(X, \mathcal{E} \otimes \psi) \cong H^\bullet_\Phi(X, \mathcal{E}) \otimes \psi.$$

I.6.12. Lemma. *Let X be a Hausdorff space with a finite transformation group G, let Φ be a G-invariant system of supports and let $\mathcal{E}$ be a G-sheaf of left R-modules. Suppose X is a topologically disjoint union*

$$X \;=\; \coprod_{j=1}^{r} g_j Y$$

where $g_1, \ldots, g_r$ is a set of representatives of left cosets of $H := \{\, g \in G \mid gY = Y \,\}$. Then there is an isomorphism of $R[G]$-modules

$$\operatorname{ind}_H^G(H^\bullet_{\Phi|Y}(Y, \mathcal{E}_{|Y})) \;\xrightarrow{\;\simeq\;}\; H^\bullet_\Phi(X, \mathcal{E}),$$

where $\operatorname{ind}_H^G(H^\bullet_{\Phi|Y}(Y, \mathcal{E}_{|Y})) \;:=\; R[G] \otimes_{R[H]} H^\bullet_{\Phi|Y}(Y, \mathcal{E}_{|Y})$ denotes the induced representation.

Proof. There is a direct sum decomposition

$$H^\bullet_\Phi(X, \mathcal{E}) \;=\; \bigoplus_{j=1}^{r} H^\bullet_{\Phi|g_jY}(g_jY, \mathcal{E}_{|g_jY})$$

of $H^\bullet_\Phi(X, \mathcal{E})$ as R-module. One checks that the map induced by g_j maps the summand $H^\bullet_{\Phi|Y}(Y, \mathcal{E}_{|Y})$ isomorphically to $H^\bullet_{\Phi|g_jY}(g_jY, \mathcal{E}_{|g_jY})$. Note that $g\Phi_{|Y} = \Phi_{|gY}$ for all $g \in G$. Therefore, there is an $R[H]$-bilinear map

$$R[G] \times H^\bullet_{\Phi|Y}(Y, \mathcal{E}_{|Y}) \;\to\; H^\bullet_\Phi(X, \mathcal{E}),$$

which induces an isomorphism

$$R[G] \otimes_{R[H]} H^\bullet_{\Phi|Y}(Y, \mathcal{E}_{|Y}) \;\to\; H^\bullet_\Phi(X, \mathcal{E}). \qquad \square$$

I.6.13. *Sheaves of $R[G]$-modules.* Let R be a commutative integral domain with quotient field F and let G be a finite group. Moreover, let $\mathcal{E}$ be a sheaf of left $R[G]$-modules on some space X. Given a character $\psi : G \to R^\times$, we define the *ψ-eigensheaf* $\mathcal{E}_\psi$ by

$$\mathcal{E}_\psi(U) \;:=\; \{\, s \in \mathcal{E}(U) \mid \forall\, g \in G \quad gs = \psi(g)s \,\}$$

for all open $U \subset X$. It is easy to see that this is indeed a sheaf of R-modules on X. It is worth noting, that the stalk of $\mathcal{E}_\psi$ at x is the ψ-eigenspace of $\mathcal{E}_x$. If ψ is the trivial character, then we get the sheaf of fixed elements, which we denote by $\mathcal{E}^G$.

I.6.14. Lemma. *Let R be a commutative domain of characteristic zero, F its field of fractions and Φ any family of supports. Let $\mathcal{E}$ be a sheaf of $R[G]$-modules on X and assume that the stalks are without R-torsion. Then the canonical map*

$$\mathcal{E}^G \;\longrightarrow\; R \otimes_{R[G]} \mathcal{E}$$

is injective and induces an isomorphism

$$F \otimes_R H^\bullet_\Phi(X, \mathcal{E}^G) \xrightarrow{\;\simeq\;} F \otimes_R H^\bullet_\Phi(X, R \otimes_{R[G]} \mathcal{E}).$$

Proof. It suffices to check the injectivity on the stalks. Let E be an $R[G]$-module which is torsion-free as R-module. We show that the canonical map $\alpha : E^G \to R \otimes_{R[G]} E$ is injective. Consider the map $E \to E^G$ given by $v \mapsto \sum_{g\in G} gv$. We get an induced map $\beta : R \otimes_{R[G]} E \to E^G$. The composition $\beta \circ \alpha$ is the multiplication with the order $|G|$ of G. Since E contains no R-torsion and R of characteristic 0, we obtain that α is injective. Furthermore, we see that $|G|(R \otimes_{R[G]} E)$ lies in the image of α. Summing up, there is a short exact sequence of sheaves of R-modules

$$0 \longrightarrow \mathcal{E}^G \longrightarrow R \otimes_{R[G]} \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is a sheaf of R-modules with the property that all the stalks of $\mathcal{G}$ consist entirely of $|G|$-torsion elements. Hence the cohomology $H^\bullet_\Phi(X, \mathcal{G})$ consists of $|G|$-torsion elements. Tensoring with F in the long exact sequence yields the claim. $\square$

I.6.15. *Decomposition of sheaves of $R[G]$-modules.* As before, let R be a commutative domain of characteristic zero and let F be its quotient field. Let G be a finite *abelian* group. Given a sheaf of $R[G]$-modules $\mathcal{E}$ on a space X, we can decompose $\mathcal{E}$ into eigensheaves. Assume that R contains all the $|G|$-th roots of unity and let $\widehat{G}$ denote the group of characters $G \to R^\times$.

Lemma. Let E be an $R[G]$-module which is torsion-free as R-module. For $\psi \in \widehat{G}$ we denote by E_ψ the ψ-eigenspace. The sum of all eigenspaces is direct and the quotient $E/(\bigoplus_{\psi\in\widehat{G}} E_\psi)$ consists of $|G|$-torsion elements.

Proof. Note that all characters $G \to \overline{F}$ are already defined as characters into R. So we may use orthogonality relations to see that for $e \in E$ we have

$$|G|e = \sum_{g\in G} \sum_{\psi\in\widehat{G}} \psi(g)ge = \sum_{\psi\in\widehat{G}} \sum_{g\in G} \psi(g)ge.$$

It is easy to verify that $\sum_{g\in G} \psi(g)ge$ is in the ψ^{-1}-eigenspace. Moreover, one can use orthogonality relations to show that the sum is direct. $\square$

Lemma. Let G be a finite abelian group and let $\mathcal{E}$ be a sheaf of $R[G]$-modules. If all the stalks are torsion-free as R-modules, then the canonical map

$$\bigoplus_{\psi\in\widehat{G}} \mathcal{E}_\psi \longrightarrow \mathcal{E}$$

is injective and induces an isomorphism

$$\bigoplus_{\psi \in \widehat{G}} F \otimes_R H^\bullet_\Phi(X, \mathcal{E}_\psi) \xrightarrow{\simeq} F \otimes_R H^\bullet_\Phi(X, \mathcal{E}).$$

for all systems of supports Φ.

PROOF. Use the previous lemma to check injectivity on the stalks. The second assertion follows as in the proof of Lemma I.6.14. □

I.6.16. *Remark.* Let R be a commutative domain of characteristic zero, let G be a finite group and let $\psi : G \to R^\times$ be a character. Suppose that G acts on the space X and let $\mathcal{E}$ be a G-sheaf of left R-modules. We assume that a normal subgroup $N \trianglelefteq G$ acts trivially on X. In this case $\mathcal{E}$ is also a sheaf of $R[N]$-modules and we can form the eigensheaf $\mathcal{E}_{\psi|N}$ for the restricted character $\psi_{|N} : N \to R^\times$. Then $\mathcal{E}_{\psi|N}$ is again a G-sheaf. This follows from the assumptions that N is normal and that ψ is a character on the group G.

If G is an abelian group and R contains all $|G|$-th roots of unity, then every character of N extends to a character on G. Hence for every character ψ of N the eigensheaf $\mathcal{E}_\psi$ is a G-sheaf and the isomorphism

$$\bigoplus_{\psi \in \widehat{N}} F \otimes_R H^\bullet_\Phi(X, \mathcal{E}_\psi) \xrightarrow{\simeq} F \otimes_R H^\bullet_\Phi(X, \mathcal{E}).$$

is indeed an isomorphism of $F[G]$-modules.

I.7. The Fixed Point Formula

I.7.1. *Grothendieck groups.* Let R be a left noetherian ring. We denote by $\mathsf{G}_0(R)$ the Grothendieck group of finitely generated left R-modules modulo short exact sequences. Recall the construction: Take the set[2] I of isomorphism classes of finitely generated left R-modules. The isomorphism class of some module M will be denoted by $[M]$ or $[M]_R$. Take the free abelian group over I and factor out the relations $[M] = [M'] + [M'']$ for any short exact sequence

$$0 \to M' \longrightarrow M \longrightarrow M'' \to 0$$

of finitely generated left R-modules.

Moreover, let $\mathsf{K}_0(R)$ denote the Grothendieck group of finitely generated projective R-modules. The image of the Cartan homomorphism $\mathsf{K}_0(R) \to \mathsf{G}_0(R)$ will be denoted by $\mathsf{G}_0^{\mathrm{proj}}(R)$. In other words, $\mathsf{G}_0^{\mathrm{proj}}(R)$ is the subgroup of $\mathsf{G}_0(R)$ generated by the isomorphism classes of the finitely generated projective modules.

I.7.2. *Base change and restriction.* Let S be another left noetherian ring and let $\iota\colon R \to S$ be a morphism of rings. If S is a *flat* right R-module via ι, then there is a *base change* homomorphism

$$\mathsf{bc}_R^S : \mathsf{G}_0(R) \to \mathsf{G}_0(S) \text{ defined by } [M]_R \mapsto [S \otimes_R M]_S.$$

It is easy to see that bc_R^S maps $\mathsf{G}_0^{\mathrm{proj}}(R)$ to $\mathsf{G}_0^{\mathrm{proj}}(S)$. Moreover, if $S \to T$ is another homomorphism of rings such that T is a flat S-module, then $\mathsf{bc}_S^T \circ \mathsf{bc}_R^S = \mathsf{bc}_R^T$.

[2]One should check that this is indeed a set.

If S is finitely generated as left R-module, then every finitely generated left module over S can be made a finitely generated left R-module by restriction of scalars to R. We obtain the *restriction* homomorphism

$$\mathsf{res}_S^R : \mathsf{G}_0(S) \to \mathsf{G}_0(R) \text{ defined by } [M]_S \mapsto [M]_R.$$

Again we have $\mathsf{res}_S^R \circ \mathsf{res}_T^S = \mathsf{res}_T^R$ whenever these maps are defined.

I.7.3. *Euler characteristic.* Let R denote a left noetherian ring. Let X be a Hausdorff space, Φ a paracompactifying family of supports and $\mathcal{E}$ a sheaf of left R-modules. If $\dim_\Phi X = n$ is finite and the triple $(X, \Phi, \mathcal{E})$ is of finite type, then we can define the Euler characteristic

$$\chi_\Phi(X, \mathcal{E}; R) \;:=\; \sum_{i=0}^{n} (-1)^i [H_\Phi^i(X, \mathcal{E})]_R \;\in \mathsf{G}_0(R).$$

Furthermore, let G be a finite group acting on X from the left. Note that the group ring $R[G]$ is again left noetherian. If Φ is G-invariant and $\mathcal{E}$ is a G-sheaf of R-modules (cf. I.6.8, I.6.5), then we can define a Euler characteristic with respect to $R[G]$:

$$\chi_\Phi(X, \mathcal{E}; R[G]) \;:=\; \sum_{i=0}^{n} (-1)^i [H_\Phi^i(X, \mathcal{E})]_{R[G]} \;\in \mathsf{G}_0(R[G]).$$

Using I.6.7 we see that

$$\chi_\Phi(X, \mathcal{E}; R[G]) \;=\; \chi_{\pi(\Phi)}(X/G, \pi_*(\mathcal{E}); R[G])$$

whenever both sides are defined.

In the special case where R is a commutative integral domain and F is its field of fractions, we introduce the short notation

$$\chi_\Phi(X, \mathcal{E})_F \;:=\; \dim\big(\mathsf{bc}_R^F(\chi_\Phi(X, \mathcal{E}; R))\big) \;\in \mathbb{Z}.$$

Here $\dim : \mathsf{G}_0(F) \to \mathbb{Z}$ denotes the dimension isomorphism.

I.7.4. *Small rings.* We say that a ring R is *small* if its cardinality is at most $\aleph_n$ for some natural number n. This is useful due to the following result: In a ring of cardinality at most $\aleph_n$ every flat module has projective dimension less or equal than $n+1$ (see [**Sim74**]). Given a small ring R and a finite group G, the group ring $R[G]$ is still a small ring.

I.7.5. Theorem. *Let R be a small noetherian*[3] *ring. Let X be a Hausdorff space, Φ a paracompactifying family of supports and $\mathcal{E}$ a* flat *sheaf of left R-modules. Suppose X has finite Φ-dimension and the triple $(X, \Phi, \mathcal{E})$ is of finite type, then*

$$\chi_\Phi(X, \mathcal{E}; R) \;\in\; \mathsf{G}_0^{proj}(R).$$

Proof. By Corollary I.5.6 the cohomology $H_\Phi^\bullet(X, \mathcal{E})$ can be computed by a complex $Z^\bullet$ of finite tor-amplitude. Using the assumptions that R is small and that $(X, \Phi, \mathcal{E})$ is of finite type, we may apply I.3.5 and I.3.6 to find a bounded complex of finitely generated projective R-modules isomorphic to $Z^\bullet$ in $\mathrm{D}^b(\mathsf{Mod}_l(R))$. □

[3] A ring is noetherian if it is left and right noetherian.

I.7.6. This inconspicuous theorem has remarkable consequences. As a first step we describe how to apply it to G-sheaves. We use the notation and assumptions of the previous theorem. Furthermore, we assume that a finite group G operates from the left on the space X and that Φ is G-invariant.

Lemma. Suppose G acts freely on X. If $\mathcal{E}$ is a flat G-sheaf of left R-modules, then $\pi_*(\mathcal{E})$ is a flat sheaf of $R[G]$-modules.

PROOF. Use the lemma in paragraph I.6.6 to deduce that for any point $\bar{x} = \pi(x)$ in the quotient space X/G, there is an isomorphism

$$\pi_*(\mathcal{E})_{\bar{x}} \xrightarrow{\simeq} \mathcal{E}_x \otimes_R R[G].$$

Hence the stalks of $\pi_*(\mathcal{E})$ are flat as $R[G]$-modules. □

Corollary. Let $\mathcal{E}$ be a flat G-sheaf of left R-modules. If G acts freely on X, then

$$\chi_\Phi(X, \mathcal{E}; R[G]) \in \mathsf{G}_0^{\mathrm{proj}}(R[G]).$$

PROOF. First we apply Corollary I.4.4 to deduce that the space X/G has finite $\pi(\Phi)$-dimension. By Theorem I.6.7 the triple $(X/G, \pi(\Phi), \pi_*(\mathcal{E}))$ is of finite type with respect to R, thus *a fortiori* of finite type with respect to $R[G]$. Now, the claim follows from the lemma and Theorem I.7.5. □

I.7.7. *Swan's theorem.* Let R be the ring of integers of an algebraic number field F and let G be a finite group. A useful theorem of Swan states: If P is a projective $R[G]$-module, then $F[G] \otimes_{R[G]} P$ is a *free* $F[G]$-module (see [**CR88**, Thm. 78.3]). We can rephrase this using the language of Grothendieck groups introduced in I.7.1:

$$\mathsf{bc}_{R[G]}^{F[G]}(\mathsf{G}_0^{\mathrm{proj}}(R[G])) = \mathbb{Z} \cdot \mathrm{Reg}_F(G) \subset \mathsf{G}_0(F[G]),$$

where $\mathrm{Reg}_F(G)$ denotes the class of the regular representation $F[G]$ in $\mathsf{G}_0(F[G])$. Consequently, applying this to the setting of paragraph I.7.6, there is some integer $m \in \mathbb{Z}$ depending on X, $\mathcal{E}$ and Φ such that

$$\mathsf{bc}_{R[G]}^{F[G]}(\chi_\Phi(X, \mathcal{E}; R[G])) = m \,\mathrm{Reg}_F(G).$$

We compute the integer m via the base change map $\mathsf{bc}_{F[G]}^{F} : \mathsf{G}_0(F[G]) \to \mathsf{G}_0(F)$.

I.7.8. Proposition. *Let R be the ring of integers of an algebraic number field F. Let X be a Hausdorff space with a left action of a finite group G. Assume that there is a normal subgroup $N \trianglelefteq G$ such that N acts trivial and G/N acts freely on X. Moreover, let Φ be a G-invariant paracompactifying family of supports and let $\mathcal{E}$ be a flat G-sheaf of R-modules. Suppose there is a character $\psi : G \to R^\times$ such that $\mathcal{E} = \mathcal{E}_{\psi|N}$ (cf. I.6.13).*

If $\dim_\Phi X$ is finite and the triple $(X, \Phi, \mathcal{E})$ is of finite type with respect to R, then

$$\mathsf{bc}_{R[G]}^{F[G]}(\chi_\Phi(X, \mathcal{E}; R[G])) = \chi_{\pi(\Phi)}(X/G, \pi_*(\mathcal{E})_\psi)_F \; [\mathrm{ind}_N^G(\psi_{|N})]_{F[G]}.$$

PROOF. We twist the G-sheaf with ψ^{-1} (see I.6.11). The normal subgroup N acts trivially on the twisted sheaf $\mathcal{E} \otimes \psi^{-1}$, so we may consider it as a G/N-sheaf. From I.7.7 and the assumption that G/N acts freely, we see that

$$\mathsf{bc}_{R[G/N]}^{F[G/N]}(\chi_\Phi(X, \mathcal{E} \otimes \psi^{-1}; R[G/N])) = m \,\mathrm{Reg}_F(G/N) \tag{4}$$

for some integer $m \in \mathbb{Z}$. Now, we apply $\mathsf{res}_{F[G/N]}^{F[G]}$ and use the effect of twisting in the cohomology (cf. I.6.11) to obtain

$$\mathsf{bc}_{R[G]}^{F[G]}(\chi_\Phi(X, \mathcal{E}; R[G])) = m[\mathrm{ind}_N^G(\psi_{|N})]_{F[G]}.$$

We are left with the task of computing the number m. For simplicity we put $H := G/N$. Apply the base change $\mathsf{bc}^F_{F[H]}$ to equation (4). One should notice that F with the trivial H-action is a flat $R[H]$-module. Hence we can compute $F \otimes_{R[H]} H^\bullet_\Phi(X, \mathcal{E} \otimes \psi^{-1})$ with the complex

$$F \otimes^{\mathrm{L}}_{R[H]} \underline{\mathrm{R}}\Gamma_{\pi(\Phi)}(X/G, \pi_*(\mathcal{E}) \otimes \psi^{-1}).$$

Using the composition theorem for derived functors (see [**KS90**, Prop. 1.8.7] or [**GM96**, p. 200]) and the projection formula I.5.5, we obtain an isomorphism

$$F \otimes^{\mathrm{L}}_{R[H]} \underline{\mathrm{R}}\Gamma_{\pi(\Phi)}(X/G, \pi_*(\mathcal{E}) \otimes \psi^{-1}) \;\cong\; F \otimes^{\mathrm{L}}_{R} \underline{\mathrm{R}}\Gamma_{\pi(\Phi)}(X/G, R \otimes_{R[H]} \pi_*(\mathcal{E} \otimes \psi^{-1}))$$

in the derived category. Finally, Lemma I.6.14 implies that we may replace the sheaf $R \otimes_{R[H]} \pi_*(\mathcal{E} \otimes \psi^{-1})$ by $\pi_*(\mathcal{E} \otimes \psi^{-1})^H$. The latter is isomorphic to $\pi_*(\mathcal{E})_\psi$ and the claim follows. □

I.7.9. *Remark.* Suppose $\mathcal{E} = R_X$ is the constant sheaf. Under the assumptions of I.7.8 where $N = 1$ and ψ is the trivial character we obtain

$$\chi_{\pi(\Phi)}(X/G, \pi_*(\mathcal{E})^G)_F \;=\; \chi_{\pi(\Phi)}(X/G, R)_F,$$

because is not difficult to check that $\pi_*(R_X)^G = R_{X/G}$.

Let X be a locally compact space and let Φ be the family of compact supports. Let $\mathcal{E} = \mathbb{Z}_X$ be the constant sheaf on X. It is a G-sheaf in a natural way for any finite group G acting on X (see I.6.10). Specialising Proposition I.7.8 to this case one obtains Verdier's Lemma [**Ver73**]. In order to generalise Verdier's Theorem one needs the following technical lemma (which Verdier only states as a fact).

I.7.10. *A technical lemma.* Let R be any noetherian ring and let X be a Hausdorff space with a left action of a finite group G. Moreover, let Φ be a paracompactifying G-invariant system of supports and let $\mathcal{E}$ be a G-sheaf of left R-modules. Suppose that $\dim_\Phi X$ is finite and that for any subgroup $H \leq G$ the triple $(X^H, \Phi_{|X^H}, \mathcal{E}_{|X^H})$ is of finite type w.r.t. R.

Then for any conjugacy class of subgroups $C \in \mathrm{Cl}(G)$ the dimension $\dim_{\Phi|X_C} X_C$ is finite and the triple $(X_C, \Phi_{|X_C}, \mathcal{E}_{|X_C})$ is of finite type. Furthermore, we show that

$$\chi_\Phi(X, \mathcal{E}; R[G]) \;=\; \sum_{C \in \mathrm{Cl}(G)} \chi_{\Phi|X_C}(X_C, \mathcal{E}_{|X_C}; R[G]). \tag{5}$$

The first claim follows directly from paragraph I.6.3. It remains show formula (5).

Put a total order $\prec$ on the set $\mathrm{Cl}(G)$ of conjugacy classes of subgroups of G, in such a way that it extends the partial order induced by the cardinality of a representative. This means, given two classes $C, C' \in \mathrm{Cl}(G)$ and representatives $H \in C$ and $H' \in C'$, whenever $|H| < |H'|$ then $C \prec C'$. For any space Y on which G acts, we define $\mathrm{maxFix}(Y)$ to be the maximal class $C \in \mathrm{Cl}(G)$ satisfying $X_C \neq \emptyset$. We prove the claim by induction on $\mathrm{maxFix}(X)$. If $\mathrm{maxFix}(X) = \{\{1\}\}$, then the group G acts freely and there is nothing to prove. Now assume that the result holds for all spaces Y with $\mathrm{maxFix}(Y) \prec C$ and suppose $\mathrm{maxFix}(X) = C$. Then $X_C = X^C$ is closed and we can use the long exact sequence associated with this subspace (cf. [**God58**, p. 190]) to obtain

$$\chi_\Phi(X, \mathcal{E}; R[G]) \;=\; \chi_{\Phi|X_C}(X_C, \mathcal{E}_{|X_C}; R[G]) + \chi_{\Phi|X \setminus X_C}(X \setminus X_C, \mathcal{E}_{|X \setminus X_C}; R[G]).$$

Here one needs to check that the long exact sequence is a sequence of $R[G]$-modules. Finally, we apply the induction hypothesis to the space $X \setminus X_C$.

I.7.11. Theorem (Fixed Point Formula I). *Let X be a Hausdorff space equipped with a finite transformation group G and a G-invariant paracompactifying family of supports Φ. Suppose that $\dim_\Phi X$ is finite and that for any subgroup $H \leq G$ the triple $(X^H, \Phi_{|X^H}, \mathbb{Z}_{X^H})$ is of finite type.*

Then

$$\mathsf{bc}_{\mathbb{Z}[G]}^{\mathbb{Q}[G]}\big(\chi_\Phi(X, \mathbb{Z}; \mathbb{Z}[G])\big) \;=\; \sum_{C \in \mathrm{Cl}(G)} \chi_{\pi(\Phi|X_C)}(X_C/G, \mathbb{Z})_{\mathbb{Q}} \,\cdot\, \mathrm{ind}_C^G,$$

where $\mathrm{ind}_C^G := [\mathrm{ind}_H^G(1)]$ *denotes the isomorphism class of the representation induced by the trivial representation of some element $H \in C$.*

Proof. Note that by I.7.10 (resp. I.6.3) the dimension $\dim_{\Phi|X_C} X_C$ is finite and the triple $(X_C, \Phi_{|X_C}, \mathcal{E}_{|X_C})$ is of finite type for any conjugacy class $C \in \mathrm{Cl}(G)$. We determine the terms in formula (5).

Let $C \in \mathrm{Cl}(G)$ be a conjugacy class, recall that by Lemma I.6.2

$$X_C = \coprod_{H \in C} X_H.$$

Therefore, we may use Lemma I.6.12 to see that

$$\chi_{\Phi|X_C}(X_C, \mathbb{Z}_{X_C}; \mathbb{Z}[G]) \;=\; \mathsf{bc}_{\mathbb{Z}[N(H)]}^{\mathbb{Z}[G]}\big(\chi_{\Phi|X_H}(X_H, \mathbb{Z}_{X_H}; \mathbb{Z}[N(H)])\big),$$

where $N(H)$ denotes the normaliser of H in G. Note that on the space X_H the finite group $N(H)/H$ acts freely and we can apply Proposition I.7.8 and Remark I.7.9 to deduce

$$\mathsf{bc}_{\mathbb{Z}[N(H)]}^{\mathbb{Q}[N(H)]}\big(\chi_{\Phi|X_H}(X_H, \mathbb{Z}_{X_H}; \mathbb{Z}[N(H)])\big) = \chi_{\pi(\Phi|X_H)}(X_H/N(H), \mathbb{Z})_{\mathbb{Q}}[\mathrm{ind}_H^{N(H)}(1)].$$

To obtain the result one has to apply the base change $\mathsf{bc}_{\mathbb{Q}[N(H)]}^{\mathbb{Q}[G]}$ and use the homeomorphism $X_C/G \cong X_H/N(H)$. □

I.7.12. Theorem (Fixed Point Formula II). *Let X be a Hausdorff space with the action of a finite* abelian *group G. Let R be the ring of integers of an algebraic number field F and assume that R contains all $|G|$-th roots of unity. Let Φ be a paracompactifying G-invariant system of supports and let $\mathcal{E}$ be a flat G-sheaf of R-modules. Suppose that $\dim_\Phi X$ is finite and that for every subgroup $H \leq G$ the triple $(X^H, \Phi_{|X^H}, \mathcal{E}_{|X^H})$ is of finite type for R. Then*

$$\mathsf{bc}_{R[G]}^{F[G]}\chi_\Phi(X, \mathcal{E}; R[G]) \;=\; \sum_{H \leq G} \sum_{\psi \in \widehat{H}} \chi_{\pi(\Phi|X_H)}(X_H/G, \pi_*(\mathcal{E}_{|X_H})_{\widetilde{\psi}})_F \, [\mathrm{ind}_H^G(\psi)]$$

where $\widetilde{\psi}$ denotes any chosen extension of ψ to G.

Proof. As in I.7.10, we have

$$\chi_\Phi(X, \mathcal{E}; R[G]) = \sum_{H \leq G} \chi_{\Phi|X_H}(X_H, \mathcal{E}_{|X_H}; R[G]).$$

Every subgroup H acts trivially on X_H and G/H acts freely on X_H. From the eigensheaf decomposition I.6.15 together with Remark I.6.16 we conclude

$$\chi_{\Phi|X_H}(X_H, \mathcal{E}_{|X_H}; R[G]) = \sum_{\psi \in \widehat{H}} \chi_{\Phi|X_H}(X_H, (\mathcal{E}_{|X_H})_\psi; R[G]).$$

Finally, we merely apply Proposition I.7.8. □

I.7.13. *Lefschetz numbers.* Let F be a field and let G be a finite group. For every $g \in G$ the trace induces a homomorphism of abelian groups

$$\mathrm{Tr}_g : \mathsf{G}_0(F[G]) \to F,$$

defined by $[V] \mapsto \mathrm{Tr}(g|V)$.

Return to the setting of paragraph I.7.10. We assume R to be a noetherian domain with quotient field F. We define the *Lefschetz number* of an element $g \in G$ by

$$\mathcal{L}_\Phi(g, X, \mathcal{E}) \;:=\; \mathrm{Tr}_g\, \mathsf{bc}_{R[G]}^{F[G]}\big(\chi_\Phi(X, \mathcal{E}; R[G])\big).$$

It is obvious that this definition does only depend on g and not on the choice of the group G. By (5) the following holds:

$$\mathcal{L}_\Phi(g, X, \mathcal{E}) \;=\; \sum_{C \in \mathrm{Cl}(G)} \mathcal{L}_{\Phi|X_C}(g, X_C, \mathcal{E}_{|X_C}).$$

This is particularly useful in combination with Theorem I.7.12.

I.7.14. Corollary. *Suppose $g : X \to X$ is an automorphism of finite order and G the cyclic group generated by g. Under the assumptions of Theorem I.7.12, we obtain the identity*

$$\mathcal{L}_\Phi(g, X, \mathcal{E}) \;=\; \mathcal{L}_{\Phi|X^g}(g, X^g, \mathcal{E}_{|X^g}) \;=\; \sum_{\psi \in \widehat{G}} \chi_{\Phi|X^g}(X^g, (\mathcal{E}_{|X^g})_\psi)_F\; \psi(g).$$

If $\mathcal{E} = \mathbb{Z}_X$ is the constant sheaf, then

$$\mathcal{L}_\Phi(g, X, \mathbb{Z}) \;=\; \chi_{\Phi|X^g}(X^g, \mathbb{Z})_{\mathbb{Q}}.$$

Proof. Since the trace of g vanishes on every representation which is induced from a proper subgroup of G, the assertions follow directly from Thm. I.7.12 and Thm. I.7.11. $\square$

CHAPTER II

Lefschetz Numbers in the Cohomology of Groups

II.1. Introduction

II.1.1. In this chapter we describe how the results of the first chapter can be applied to study Lefschetz numbers of finite order automorphisms in the cohomology of arithmetic groups. In order to do so, we identify the cohomology of the arithmetic group with the sheaf cohomology of a suitably chosen topological space and transform the automorphism to a finite order transformation of this space. As seen in the first chapter, we have to understand the fixed points of this transformation. The important ingredient to manage this task is Rohlfs' decomposition of the set of fixed points into several components, which themselves can be seen to be classifying spaces for arithmetic subgroups in groups of smaller rank. In order to verify the finiteness conditions needed to apply the theorems of chapter I, we will use the fact that torsion-free arithmetic groups are of type (FL).

II.1.2. We proceed as follows: In section II.2 we summarise some facts on groups of type (FL) and we use these in order to understand basic properties of Lefschetz numbers in the cohomology of such groups. In section II.3 we give a purely topological account of the Rohlfs decomposition, to stress that some properties of the fixed point components do not rely on any hypothesis. We put this together for arithmetic groups in section II.5. Finally, we give an adelic version of the decomposition.

II.2. Groups of Type (FL)

II.2.1. In this section we consider the cohomology of groups as defined, for instance, in [**Ser71**] or [**Bro82a**]. Familiarity with the basic concepts is assumed. We will only focus on the groups of so-called type (FL) and most results are very elementary. We will conclude this section relating the introduced notions to their analogues in sheaf cohomology.

II.2.2. *Type (FL).* Let Γ be a group. Following Serre [**Ser71**] we say that Γ is of type *(FL)* if the trivial Γ-module $\mathbb{Z}$ has a resolution of finite length consisting of free left $\mathbb{Z}[\Gamma]$-modules of finite rank. This means, there is an exact sequence of $\mathbb{Z}[\Gamma]$-modules

$$0 \to L_n \to L_{n-1} \to \dots \to L_0 \to \mathbb{Z} \to 0,$$

such that the L_i are free of finite rank. In this case we define the Euler characteristic $\chi(\Gamma)$ as

$$\chi(\Gamma) := \sum_{i=0}^{n} (-1)^i \operatorname{rank}(L_i).$$

The Euler characteristic is independent of the chosen resolution $(L_i)_i$ (see [**Ser71**, p. 82]).

Definition. The group Γ is said to be of type *(WFL)*, if Γ is virtually torsion-free and every torsion-free subgroup of finite index is of type (FL) (cf. [**Ser71**, p. 101]).

II.2.3. *Remark.* Let G be a linear algebraic $\mathbb{Q}$-group and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup of $G(\mathbb{Q})$, then it can be shown, as a corollary to the compactification theorem of Borel and Serre, that Γ is of type (WFL) (see [**BS73**, Thm. 11.4.4]). In particular, every torsion-free, arithmetically defined group is of type (FL). These are the only examples of groups of type (FL) that we will consider.

II.2.4. *Equivariant cohomology.* Let Γ be a group and let Ξ be a group which acts from the left on Γ by automorphisms. We will denote this action by left exponents

$$\begin{aligned} \Xi \times \Gamma &\to \Gamma \\ (s, \gamma) &\mapsto {}^{s}\gamma. \end{aligned}$$

Let R be any ring and consider the semidirect product group $\Gamma \rtimes \Xi$. Given a left $R[\Gamma \rtimes \Xi]$-module E, the collection E^{Γ} of Γ-fixed elements is an $R[\Xi]$-module. This can be easily seen since for $e \in E^{\Gamma}$, $s \in \Xi$ and all $\gamma \in \Gamma$ we have

$$\gamma({}^{s}e) = {}^{s}\big(({}^{s^{-1}}\gamma)e\big) = {}^{s}e.$$

In other words, taking the Γ-fixed points is a left exact covariant functor from the category of left $R[\Gamma \rtimes \Xi]$-modules to the category of $R[\Xi]$-modules

$$\mathrm{Fix}_{\Gamma} : \mathsf{Mod}_l(R[\Gamma \rtimes \Xi]) \to \mathsf{Mod}_l(R[\Xi]).$$

The category $\mathsf{Mod}_l(R[\Gamma \rtimes \Xi])$ has enough injectives, so there is a right derived functor

$$\underline{\mathrm{R}}\,\mathrm{Fix}_{\Gamma} : \mathrm{D}^{+}(\mathsf{Mod}_l(R[\Gamma \rtimes \Xi])) \to \mathrm{D}^{+}(\mathsf{Mod}_l(R[\Xi])).$$

For an $R[\Gamma \rtimes \Xi]$-module E the associated cohomology groups $H^{\bullet}\underline{\mathrm{R}}\,\mathrm{Fix}_{\Gamma}(E)$ are hence $R[\Xi]$-modules. It is easily checked that $H^{\bullet}\underline{\mathrm{R}}\,\mathrm{Fix}_{\Gamma}(E)$ considered as R-module is isomorphic to the usual group cohomology $H^{\bullet}(\Gamma, E)$. So, whenever E is an $R[\Gamma \rtimes \Xi]$-module we will consider the cohomology modules $H^{\bullet}(\Gamma, E)$ as left $R[\Xi]$-modules.

II.2.5. *Euler characteristic.* Let Γ be a group of type (FL). Let R be a left noetherian ring and let E be an $R[\Gamma]$-module. The cohomology groups $H^{\bullet}(\Gamma, E)$ are in a natural way equipped with a left R-module structure. Moreover, if E is finitely generated as R-module, then the assumption that Γ is of type (FL) implies that $H^{q}(\Gamma, E)$ is a finitely generated R-module for any degree q. Note further that $H^{q}(\Gamma, E)$ is trivial for all sufficiently large q. Thus we can define the Euler characteristic

$$\chi(\Gamma, E; R) := \sum_{q=0}^{\infty} (-1)^{q} [H^{q}(\Gamma, E)]_{R} \in \mathsf{G}_0(R),$$

where $\mathsf{G}_0(R)$ denotes the Grothendieck group of finitely generated left R-modules (see I.7.1). We introduce a notation similar to the one in I.7.3: If R is a commutative noetherian domain with quotient field F, we put

$$\chi(\Gamma, E)_F := \dim\big(\mathsf{bc}_R^F(\chi(\Gamma, E; R))\big).$$

Suppose there is a finite group Ξ acting on Γ by automorphisms as in paragraph II.2.4. Since Ξ is finite, the group ring $R[\Xi]$ is again a left noetherian ring. Let E be an $R[\Gamma \rtimes \Xi]$-module and recall that $H^{q}(\Gamma, E)$ is a left $R[\Xi]$-module. If we assume, as before, that Γ is of type (FL) and that E is finitely generated as R-module, we see that $H^q(\Gamma, E)$ is finitely generated over $R[\Xi]$. We define the Euler characteristic

$$\chi(\Gamma, E; R[\Xi]) \;:=\; \sum_{q=0}^{\infty}(-1)^q[H^q(\Gamma, E)]_{R[\Xi]} \;\in \mathsf{G}_0(R[\Xi]),$$

with values in the Grothendieck group $\mathsf{G}_0(R[\Xi])$ of finitely generated left $R[\Xi]$-modules.

II.2.6. Lemma. *Let Γ be a group of type (FL). Let R be a left noetherian ring and M a flat right R-module. For any $R[\Gamma]$-module E and for all natural numbers q there is an isomorphism of abelian groups*

$$H^q(\Gamma, M \otimes_R E) \cong M \otimes_R H^q(\Gamma, E).$$

Furthermore, if E is finitely generated over R, then

$$\chi(\Gamma, E; R) \;=\; \chi(\Gamma)[E]_R$$

Proof. Take a free finite rank resolution of finite length

$$0 \to L_n \to L_{n-1} \to \ldots \to L_0 \to \mathbb{Z} \to 0,$$

of $\mathbb{Z}$ as $\mathbb{Z}[\Gamma]$-module. This is possible because Γ is assumed to be of type (FL). Using that L_i is free of finite rank, it is easy to see that $\mathrm{Hom}_\Gamma(L_i, M \otimes_R E)$ is canonically isomorphic (as abelian group) to $M \otimes_R \mathrm{Hom}_\Gamma(L_i, E)$ for all i. Now, use that M is a flat R-module to obtain the first claim.

Moreover, if E is finitely generated over R, we obtain

$$\begin{aligned}\chi(\Gamma, E; R) &= \sum_{i=0}^{n}(-1)^i[H^i(\Gamma, E)]_R \\ &= \sum_{i=0}^{n}(-1)^i[\mathrm{Hom}_\Gamma(L_i, E)]_R \\ &= \sum_{i=0}^{n}(-1)^i \operatorname{rank}(L_i)[E]_R \;=\; \chi(\Gamma)[E]_R.\end{aligned}$$

□

II.2.7. Corollary. *Let R be a commutative noetherian domain with quotient field F. If E is finitely generated as R-module, then*

$$\chi(\Gamma, E)_F \;=\; \chi(\Gamma)\dim_F(F \otimes_R E).$$

II.2.8. *Lefschetz numbers.* In the spirit of paragraph I.7.13 we define Lefschetz numbers. Let R be a commutative noetherian domain and let F be its quotient field. Let Ξ be a finite group, then the group ring $R[\Xi]$ is a noetherian ring. Furthermore, let Γ be a group of type (FL) and suppose that Ξ acts by automorphisms on Γ.

Let E be a left $R[\Gamma \rtimes \Xi]$-module and we assume that E is finitely generated over R (or equivalently over $R[\Xi]$). For $\xi \in \Xi$ we define the *Lefschetz number* of ξ

$$\mathcal{L}(\xi, \Gamma, E) \;:=\; \mathrm{Tr}_\xi\big(\mathsf{bc}_{R[\Xi]}^{F[\Xi]}(\chi(\Gamma, E; R[\Xi]))\big).$$

If Ξ acts trivially on Γ, then $R[\Gamma \rtimes \Xi] \cong R[\Xi][\Gamma]$. In this case we can apply Lemma II.2.6 to see that

$$\mathcal{L}(\xi, \Gamma, E) \;=\; \chi(\Gamma)\,\mathrm{Tr}(\xi|F \otimes_R E).$$

II.2.9. *Cohomology of sheaves.* In this section we were, up to now, concerned with the cohomology of groups. Now, we want to recall the connection between the cohomology of groups and the cohomology theory of sheaves. Let Γ be a (discrete) group and let X be some locally compact Hausdorff space on which Γ acts (continuously) from the right. We denote the canonical quotient map onto the orbit space by $\pi : X \to X/\Gamma$.

Let R be a ring. With every $R[\Gamma]$-module E we can associate a sheaf $\widetilde{E}$ of left R-modules on X/Γ, by defining the sections of $\widetilde{E}$ over an open set $U \subseteq X/\Gamma$ to be the locally constant functions $f : \pi^{-1}(U) \to E$ such that $f(x\gamma) = \gamma^{-1} f(x)$ for all $\gamma \in \Gamma$ and $x \in \pi^{-1}(U)$. To be concise, this means

$$\widetilde{E}(U) = \{\, f : \pi^{-1}(U) \to E \text{ loc. const.} \mid \big(\forall\, \gamma \in \Gamma, x \in \pi^{-1}(U)\big)\; f(x\gamma) = \gamma^{-1} f(x) \,\}.$$

Suppose that Γ acts freely and properly on X, then we can also describe the associated étalé space. Take the product space $X \times E$ with the product topology induced by the discrete topology on E. The group Γ acts from the right on $X \times E$ by $(x, e)\gamma := (x\gamma, \gamma^{-1}e)$. Denote the quotient space under this action by $X \times_\Gamma E$; it is equipped with a natural local homeomorphism q onto X/Γ. This is precisely the étalé space associated with $\widetilde{E}$. One can check that the stalk at any $x \in X/\Gamma$ is isomorphic to E as R-module. Consequently, if E is an $R[\Gamma]$-module which is flat as R-module, then $\widetilde{E}$ is a flat sheaf of left R-modules.

Note that a morphism $\varphi : E \to F$ of $R[\Gamma]$-modules induces a morphism of sheaves $\widetilde{\varphi} : \widetilde{E} \to \widetilde{F}$, where $\widetilde{E}$ and $\widetilde{F}$ denote the associated sheaves. In other words, we defined a functor

$$\widetilde{\cdot} : \mathsf{Mod}_l(R[\Gamma]) \to \mathsf{Sh}_{X/\Gamma}(\mathsf{Mod}_l(R))$$

from the category of left $R[\Gamma]$-modules to the category of sheaves of left R-modules on X/Γ. Note that if Γ acts freely and properly, then this functor is exact.

II.2.10. *Automorphisms and Ξ-sheaves.* We keep the notation and assumptions of paragraph II.2.9. Let Ξ be a finite group which acts (from the left) by automorphisms on Γ and on X. We denote both actions by left exponents and we assume that they are compatible, i.e.

$${}^{s}(x\gamma) = {}^{s}x\, {}^{s}\gamma$$

for all $s \in \Xi$, $x \in X$ and $\gamma \in \Gamma$. Under these assumptions the group Ξ acts naturally on the quotient space X/Γ.

For every left $R[\Gamma \rtimes \Xi]$-module E, the associated sheaf $\widetilde{E}$ is in a natural way a Ξ-sheaf (cf. I.6.8). This can be seen as follows: for an open set $U \subseteq X/\Gamma$, a section $f \in \widetilde{E}(U)$, and an element $s \in \Xi$ we define $\kappa(s, U)(f) \in \widetilde{E}({}^{s}U)$ by

$$\kappa(s, U)(f)(x) := {}^{s}f({}^{s^{-1}}x).$$

This is well-defined since

$$\kappa(s, U)(f)(x\gamma) = {}^{s}f({}^{s^{-1}}(x\gamma)) = {}^{s}f({}^{s^{-1}}x\, {}^{s^{-1}}\gamma) = \gamma^{-1}\kappa(s, U)(f)(x).$$

II.2.11. *Essentially acyclic spaces.* Let X be a Hausdorff topological space. Given an abelian group A, we denote the constant sheaf with stalk A over X by A_0. We say that the space X is *essentially acyclic* if for every abelian group A and every integer $q \geq 1$ the sheaf cohomology group $H^q(X, A_0)$ is trivial.

If X is paracompact, locally contractible and contractible, then X is essentially acyclic. This can be see as follows: For paracompact, locally contractible spaces sheaf cohomology agrees with singular cohomology (cf. [**Kul70**, p. 116]). Further, a contractible space is acyclic for singular cohomology, thus such a space X is essentially acyclic. In particular every Euclidean space is essentially acyclic.

II.2.12. Proposition. *Let X be a connected, locally compact, and essentially acyclic Hausdorff space. Let Γ be a group which acts freely and properly on X and let Ξ be some finite group which acts compatibly on Γ and X. Let R be any ring and let E be an $R[\Gamma \rtimes \Xi]$-module. There is a natural isomorphism of $R[\Xi]$-modules*

$$H^\bullet(\Gamma, E) \xrightarrow{\simeq} H^\bullet(X/\Gamma, \widetilde{E}).$$

PROOF. We only sketch the proof, the technical details are left to the reader.

Consider the two left exact functors $\mathrm{Fix}_\Gamma, T : \mathsf{Mod}_l(R[\Gamma \rtimes \Xi]) \to \mathsf{Mod}_l(R[\Xi])$, where Fix_Γ is the fixed point functor $E \mapsto E^\Gamma$ and T is the functor $E \mapsto \widetilde{E}(X/\Gamma)$. Since X is connected, $\widetilde{E}(X/\Gamma)$ is the space of constant functions $X \to E^\Gamma$, i.e. T is naturally isomorphic to Fix_Γ.

The functor T is the composition of the functor $\widetilde{\cdot}$ and the global section functor $\Gamma(X/\Gamma, \cdot)$. By assumption Γ acts freely and properly and so the functor

$$\widetilde{\cdot} : \mathsf{Mod}_l(R[\Gamma \times \Xi]) \to \Xi\text{-}\mathsf{Sh}_{X/\Gamma}(\mathsf{Mod}_l(R))$$

is exact.

Using that X is essentially acyclic, one can verify that this functor maps injective $R[\Gamma \rtimes \Xi]$-modules to acyclic sheaves. Now the claim follows from the composition theorem for derived functors (cf. III.§7 in [**GM96**]). $\square$

II.2.13. *Groups of type (FL) and sheaf cohomology.* Let Γ be a group of type (FL) which acts freely and properly on a Euclidean space X. Moreover, let R be some ring. Given any $R[\Gamma]$-module E, which is finitely generated as R-module, we consider the associated sheaf $\widetilde{E}$ on the orbit space X/Γ. Now, using Proposition II.2.12, we see that the triple $(X/\Gamma, \Phi, \widetilde{E})$ is of finite type over R (cf. I.4.6), where Φ denotes the system of all supports on X/Γ.

II.3. Rohlfs Decomposition in an Abstract Setting

II.3.1. In the entire section X denotes a locally compact Hausdorff space and Γ a (discrete) group acting continuously, freely and properly on X from the right. For this action we will always use the notation

$$\begin{aligned} X \times \Gamma &\to X \\ (x, \gamma) &\mapsto x\gamma. \end{aligned}$$

Moreover, we assume that there is some (discrete) group Ξ which acts continuously on X from the left. This action will be denoted by left exponents

$$\begin{aligned} \Xi \times X &\to X \\ (s, x) &\mapsto {}^s x. \end{aligned}$$

Furthermore, we assume that Ξ acts from the left on Γ by group homomorphisms and this action will be denoted by left exponents as well

$$\begin{aligned} \Xi \times \Gamma &\to \Gamma \\ (s, \gamma) &\mapsto {}^s \gamma. \end{aligned}$$

Finally, we make the assumption that these actions are *compatible*, this means

$${}^s(x\gamma) = {}^s x\, {}^s \gamma$$

for all $s \in \Xi$, $x \in X$ and $\gamma \in \Gamma$.

Let $\pi : X \to X/\Gamma$ be the canonical projection. Since Γ acts properly, the quotient space X/Γ is a locally compact Hausdorff space. Note that the group Ξ acts continuously on the orbit space X/Γ. In this abstract setting we describe the *Rohlfs decomposition* of the set of fixed points $(X/\Gamma)^\Xi$. We deal with this abstract situation in order to point out that some aspects of Rohlfs' decomposition do not rely on any assumption. Originally Rohlfs developed this decomposition in the setting of arithmetic groups (see [**Roh78, Roh81**]). We will consider this more specific situation in Section II.5.

II.3.2. *Non-abelian Galois cohomology.* We will use the language of non-abelian Galois cohomology, for details we refer to [**Ser64**], [**Ser79**, p. 123-126] or [**KMRT98**, Ch. VII]. A 1-cocycle (or simply a cocycle) for Ξ with values in Γ is a function $a : \Xi \to \Gamma$ such that

$$a_{st} = a_s\, {}^s a_t$$

for all $s, t \in \Xi$, where a_s denotes the value of a at $s \in \Xi$. The constant function onto the neutral element in Γ is called the trivial cocycle. The set of all 1-cocycles with values in Γ will be denoted by $Z^1(\Xi, \Gamma)$. Two cocycles $a, b \in Z^1(\Xi, \Gamma)$ are said to be equivalent (or *cohomologous*), written $a \sim b$, if there is some $\gamma \in \Gamma$ such that

$$\gamma^{-1} a_s\, {}^s\gamma = b_s$$

for all $s \in \Xi$. The first non-abelian Galois cohomology set $H^1(\Xi, \Gamma)$ is defined to be $Z^1(\Xi, \Gamma)$ modulo the equivalence relation $\sim$. It is a pointed set; the distinguished element is the class of the trivial cocycle.

The reader should be aware of the fact that the group Ξ is some abstract group and not a profinite group. So, we deviate slightly from the classical setting in which non-abelian Galois cohomology has been defined originally.

II.3.3. *Twisting.* Given a cocycle $b \in Z^1(\Xi, \Gamma)$, we get b-twisted actions of Ξ on X and on Γ. They are given by

$$(s, x) \mapsto {}^s x b_s^{-1} =: {}^{s|b}x$$

and

$$(s, \gamma) \mapsto b_s\, {}^s\gamma b_s^{-1} =: {}^{s|b}\gamma.$$

for all $s \in \Xi$, $x \in X$ and $\gamma \in \Gamma$. We observe that the twisted actions are again compatible, i.e.

$${}^{s|b}(x\gamma) = {}^s(x\gamma) b_s^{-1} = {}^{s|b}x\, {}^{s|b}\gamma.$$

We introduce the following notation for the sets of fixed points of the b-twisted actions on X and Γ:

$$X(b) := \{\, x \in X \mid {}^{s|b}x = x \ \text{ for all } s \in \Xi \,\},$$
$$\Gamma(b) := \{\, \gamma \in \Gamma \mid {}^{s|b}\gamma = \gamma \ \text{ for all } s \in \Xi \,\}.$$

Obviously, $X(b)$ is a closed subset of X and $\Gamma(b)$ is a subgroup of Γ. Also, using that Γ acts freely on X, we see that $X(b)$ and $X(c)$ are disjoint, if $b, c \in Z^1(\Xi, \Gamma)$ are distinct cocycles.

II.3.4. Lemma. *Let $b, c \in Z^1(\Xi, \Gamma)$ be two cocycles. The sets $\pi(X(b))$ and $\pi(X(c))$ are either disjoint or equal. More precisely, they agree if and only if $b \sim c$ and in this case*

$$X(b)\gamma = X(c),$$

where $\gamma \in \Gamma$ is such that $c_s = \gamma^{-1} b_s\, {}^s\gamma$ for all $s \in \Xi$. Moreover,

$$\pi^{-1}\big((X/\Gamma)^\Xi\big) = \bigsqcup_{b \in Z^1(\Xi, \Gamma)} X(b).$$

Proof. Assume first that b and c are cohomologous. There is $\gamma \in \Gamma$ such that $c_s = \gamma^{-1} b_s\, {}^s\gamma$ for all $s \in \Xi$. Take $x \in X(b)$, then we have

$$^{s|c}(x\gamma) = {}^s x\, {}^s\gamma c_s^{-1} = x b_s\, {}^s\gamma c_s^{-1} = x\gamma.$$

In particular $X(b)\gamma = X(c)$ and $\pi(X(b)) = \pi(X(c))$. Suppose now that

$$\pi(X(b)) \cap \pi(X(c)) \neq \emptyset,$$

then we find $x \in X(b)$, $y \in X(c)$ and $\gamma \in \Gamma$ such that $x\gamma = y$. For all $s \in \Xi$ we see that

$$x\gamma c_s = y c_s = {}^s y = {}^s x\, {}^s\gamma = x b_s\, {}^s\gamma.$$

Since the action of Γ is free, we obtain $c_s = \gamma^{-1} b_s\, {}^s\gamma$.

To prove the last claim, we pick some $x \in X$ such that $\pi(x)$ is a fixed point for Ξ. Consequently, for all $s \in \Xi$ there is some $b_s \in \Gamma$ satisfying ${}^s x = x b_s$. We only have to check that $b = (b_s)_{s \in \Xi}$ is a cocycle. This follows from the assumption that Γ acts freely together with the following calculation for $s, t \in \Xi$:

$$x b_{st} = {}^{st}x = {}^s(x b_t) = x b_s\, {}^s b_t. \qquad \square$$

II.3.5. Corollary. *For all cohomology classes $\eta \in H^1(\Xi, \Gamma)$ we can define a set $\mathfrak{F}(\eta)$ of Ξ-fixed points in X/Γ by*

$$\mathfrak{F}(\eta) = \pi(X(b)),$$

where $b \in Z^1(\Xi, \Gamma)$ is any representative of the class η. The sets $\mathfrak{F}(\eta)$ are well-defined and the set of all fixed points in X/Γ is the disjoint union of all the $\mathfrak{F}(\eta)$, i.e.

$$(X/\Gamma)^\Xi = \bigsqcup_{\eta \in H^1(\Xi,\Gamma)} \mathfrak{F}(\eta). \tag{6}$$

II.3.6. *Remark.* So far we only know that (6) holds set-theoretically, this means that the $\mathfrak{F}(\eta)$ are disjoint sets. Though we would like to have this result topologically, meaning that the spaces $\mathfrak{F}(\eta)$ are open and closed in $(X/\Gamma)^\Xi$. This can indeed be achieved if we are willing to assume that Ξ is a *finitely generated* group. For our purposes this is no restriction since in all applications the group Ξ will be finite.

II.3.7. Lemma. *Assume that Ξ is a finitely generated group. For every cocycle $b \in Z^1(\Xi, \Gamma)$ there is an open set $U(b) \subseteq X$ which contains $X(b)$ but which does not meet any $X(c)$ with $c \neq b$.*

Proof. Let $b \in Z^1(\Xi, \Gamma)$ be a cocycle. Let $t_0, t_1, \ldots, t_n$ be a finite list of generators for the group Ξ such that t_0 is the neutral element in Ξ. We define $X^{n+1} := \prod_{i=0}^n X$ and consider the continuous map

$$\Psi : X \to X^{n+1},$$
$$x \mapsto ({}^{t_i|b}x)_{i=0,\ldots,n}.$$

Let $\Delta \subseteq X^{n+1}$ denote the diagonal. Then $X(b) = \Psi^{-1}(\Delta)$ because $x \in \Psi^{-1}(\Delta)$ if and only if the stabiliser of x under the b-twisted Ξ-action contains all the generators $t_0, \ldots, t_n$. Since Γ acts properly and freely, we find an open neighbourhood V_x for any $x \in X$ such that $V_x \cap V_x\gamma = \emptyset$ for all $\gamma \in \Gamma$ with $\gamma \neq 1$. Define the open neighbourhood $W := \bigcup_{x \in X} V_x \times \cdots \times V_x \subset X^{n+1}$ of the diagonal Δ. The set $U(b) := \Psi^{-1}(W)$ is an open neighbourhood of $X(b)$.

Let $c \in Z^1(\Xi, \Gamma)$. We have to check that $U(b)$ does not meet $X(c)$ whenever $c \neq b$. Suppose we find $x \in X(c) \cap U(b)$. This means $\Psi(x) \in W$, so we find a $y \in X$ such that ${}^{t_i|b}x \in V_y$ for all $i = 0, \ldots, n$. In particular $x \in V_y$. Eventually, we see that

$x = {}^{t_i|c}x = {}^{t_i}xc_{t_i}^{-1} \in V_y$. We conclude that $V_y b_{t_i} \cap V_y c_{t_i} \neq \emptyset$ and consequently $b_{t_i} = c_{t_i}$ for all i. Thus, using that the t_i generate Ξ, we get $b = c$. □

II.3.8. Corollary. *Assume that Ξ is a finitely generated group. For every class $\eta \in H^1(\Xi, \Gamma)$ the fixed point set $\mathfrak{F}(\eta)$ is open and closed in $(X/\Gamma)^{\Xi}$. In particular, $\mathfrak{F}(\eta)$ is a closed subspace of X/Γ.*

Proof. Let $b \in \eta$ be a representing cocycle. We claim that

$$\pi(U(b)) \cap (X/\Gamma)^{\Xi} = \mathfrak{F}(\eta).$$

Let $c \in Z^1(\Xi, \Gamma)$ be a cocycle and suppose that $\pi(X(c)) \cap \pi(U(b)) \neq \emptyset$. Thus, there exist $x \in X(c)$ and $\gamma \in \Gamma$ such that $x\gamma \in U(b)$. However, by Lemma II.3.4, the set $X(c)\gamma$ equals $X(c')$ where $c'_s = \gamma^{-1} c_s\, {}^s\gamma$. By the choice of the set $U(b)$ we deduce $b = c'$ and therefore also $\pi(X(c)) = \mathfrak{F}(\eta)$. This proves the claim.

Finally, we deduce from the claim that the sets $\mathfrak{F}(\eta)$ are open in $(X/\Gamma)^{\Xi}$. Moreover, $(X/\Gamma)^{\Xi}$ is the disjoint union of those sets, thus they are also closed. □

II.3.9. Theorem. *Assume that Ξ is a finitely generated group. Let $b \in Z^1(\Xi, \Gamma)$ be such that $X(b) \neq \emptyset$ and let $\eta \in H^1(\Xi, \Gamma)$ be the class of b . Then $\Gamma(b)$ acts on $X(b)$ and there is a canonical homeomorphism*

$$X(b)/\Gamma(b) \xrightarrow{\simeq} \mathfrak{F}(\eta).$$

In particular the canonical map $X(b)/\Gamma(b) \longrightarrow X/\Gamma$ is a closed embedding.

Proof. Since the b-twisted actions on X and Γ are compatible, it is clear that $\Gamma(b)$ acts on $X(b)$. Moreover, if $x \in X(b)$ and $\gamma \in \Gamma$ are such that $x\gamma \in X(b)$ then we see that $\gamma \in \Gamma(b)$ since the action of Γ is free. So, we obtain a continuous and bijective map

$$f : X(b)/\Gamma(b) \to \mathfrak{F}(\eta).$$

It remains to show that f is also open. Take any open set $V \subseteq X(b)$. Hence there is an open set W in X such that $W \cap X(b) = V$. We may assume that $W \subseteq U(b)$. Eventually we find $\pi(V) = \pi(W) \cap \mathfrak{F}(\eta)$, which proves the theorem. □

II.3.10. Let R be any ring and let E be a left $R[\Gamma \rtimes \Xi]$-module. We consider the associated Ξ-sheaf $\widetilde{E}$ on X/Γ. Let $\eta \in H^1(\Xi, \Gamma)$ be such that $\mathfrak{F}(\eta) \neq \emptyset$. We want to understand the restriction of $\widetilde{E}$ to the closed subspace $\mathfrak{F}(\eta)$. One should be aware of the fact that this is a sheaf of $R[\Xi]$-modules over $\mathfrak{F}(\eta)$.

For this purpose we introduce some notation. For a cocycle $b \in Z^1(\Xi, \Gamma)$ let $E(b)$ denote the $R[\Xi][\Gamma(b)]$-module obtained from E by restricting the Γ-action to $\Gamma(b)$ and twisting the Ξ-action with b. This means that $s \in \Xi$ acts on $e \in E$ via

$${}^{s|b}e := b_s\, {}^s e.$$

Note that the actions of $\Gamma(b)$ and Ξ indeed commute since

$${}^{s|b}(\gamma e) = b_s\, {}^s\gamma\, {}^s e = \gamma b_s\, {}^s e = \gamma\, {}^{s|b}e$$

for all $\gamma \in \Gamma(b)$. Furthermore, we can associate the sheaf $\widetilde{E(b)}$ of $R[\Xi]$-modules over $X(b)/\Gamma(b)$ with the module $E(b)$.

II.3.11. Proposition. *Let $\eta \in H^1(\Xi, \Gamma)$ be a class such that $\mathfrak{F}(\eta) \neq \emptyset$ and choose a representative $b \in \eta$. Over the homeomorphism $f : X(b)/\Gamma(b) \to \mathfrak{F}(\eta)$ there is an isomorphism of sheaves of $R[\Xi]$-modules*

$$\varphi : \widetilde{E(b)} \xrightarrow{\simeq} \widetilde{E}_{|\mathfrak{F}(\eta)}.$$

PROOF. We will work with the associated étalé spaces. For the time being ignore the Ξ-actions. Let $q : X \times_\Gamma E \to X/\Gamma$ denote the associated étalé space for $\widetilde{E}$ and let $q' : X(b) \times_{\Gamma(b)} E \to X(b)/\Gamma(b)$ be the associated space for $\widetilde{E(b)}$. Define the continuous map $\varphi : X(b) \times_{\Gamma(b)} E \to X \times_\Gamma E$ sending $(x, e)\Gamma(b)$ to $(x, e)\Gamma$. The image of φ lies in $q^{-1}(\mathfrak{F}(\eta))$. The situation can be understood by peering at the following commutative diagram.

$$\begin{tikzcd}
X(b) \times E \arrow[r, "\text{open, closed}"] \arrow[d, "\text{loc. homeo.}"'] & \coprod_{c\in\eta} X(c) \times E \arrow[r] \arrow[d, "\text{loc. homeo.}"] & X \times E \arrow[d, "\text{loc. homeo.}"] \\
X(b) \times_{\Gamma(b)} E \arrow[r, "\varphi"] \arrow[d, "q'"'] & q^{-1}(\mathfrak{F}(\eta)) \arrow[r, "\text{closed}"] \arrow[d] & X \times_\Gamma E \arrow[d, "q"] \\
X(b)/\Gamma(b) \arrow[r, "\simeq", "f"'] & \mathfrak{F}(\eta) \arrow[r, "\text{closed}"] & X/\Gamma
\end{tikzcd}$$

To show that φ is a homeomorphism one simply checks that it is injective and surjective. The upper left square of the diagram can be used to show that φ is open. Finally, we have to show that the Ξ-action on the stalks agree. Take $x \in X(b)$, $e \in E$ and some $s \in \Xi$, then

$${}^s(x, e)\Gamma = ({}^s x, {}^s e)\Gamma = (x b_s, {}^s e)\Gamma = (x, b_s\, {}^s e)\Gamma.$$

We see that we obtain the b-twisted Ξ-action on E. □

II.4. Rohlfs Decomposition and Discrete Subgroups of Lie Groups

II.4.1. In this section G always denotes a real Lie group with a finite number of connected components. We will make frequent use of an important result of Mostow which generalises the Cartan decomposition for semisimple groups ([**Mos55b**, Thm. 3.1, 3.2]). The reader can find the statement and proof of this theorem in Appendix A (Theorem A.3.4). This result is known under various combinations of the names Cartan, Iwasawa, Malcev and Mostow, and it appears in various versions in the literature (e.g. [**HT94**] or [**Pla66**]). We will refer to Theorem A.3.4 briefly as *Mostow's Theorem.*

II.4.2. We denote by Ξ a finite group of automorphisms of G. By Proposition A.4.2 there is a maximal compact subgroup K of G that is stable under the action of Ξ. We define the smooth manifold X as the homogeneous space

$$X := K\backslash G.$$

By Mostow's Theorem the manifold X is diffeomorphic to a Euclidean space. Moreover, since K is Ξ-stable, the finite group Ξ is a transformation group of X.

Let Γ be a discrete torsion-free subgroup of G. In this case the group Γ acts freely and properly on X from the left. Observe that we can use II.2.12 to study the cohomology of Γ via the cohomology of the space X/Γ.

Finally, we make the assumption that the group Γ is stable under the action of the finite group Ξ. Under these assumptions we are in the situation of section II.3 where Ξ and Γ act compatibly on the space X. We are going to study the structure of the fixed point sets $X(b)$ for $b \in Z^1(\Xi, G)$. In particular we will show that they are non-empty and are diffeomorphic to a Euclidean space.

II.4.3. Compared with the purely topological setting in II.3, we can now take advantage of the inclusion map $\iota : \Gamma \to G$ and the induced map

$$\iota_* : H^1(\Xi, \Gamma) \to H^1(\Xi, G)$$

to analyse the structure of the fixed point components $\mathfrak{F}(\eta)$ for classes $\eta \in H^1(\Xi, \Gamma)$. Let $b, c \in Z^1(\Xi, G)$ be cohomologous cocycles, this means, there is $g \in G$ such that $g^{-1} b_s\, {}^s g = c_s$ for all $s \in \Xi$. Just like in Lemma II.3.4 one sees that

$$X(b)g \;=\; X(c). \tag{7}$$

It is important to note that the spaces $X(b)$ are indeed submanifolds of X. This follows from the fact that they are sets of points fixed by the action of a finite group[1]. Hence, we conclude that $X(b)$ and $X(c)$ are diffeomorphic whenever $b \sim c$.

II.4.4. Theorem. *Let Ξ be a finite group acting on G and let K be a Ξ-stable maximal compact subgroup of G. If $j : K \to G$ denotes the canonical inclusion, then the induced map*

$$j_* :\; H^1(\Xi, K) \xrightarrow{\;\simeq\;} H^1(\Xi, G)$$

is bijective.

II.4.5. *Remark.* If G is a connected semisimple group with finite centre, the theorem is a result due to J. Rohlfs (see [**Roh81**, Lem. 1.4]). His argument can be generalised to give the injectivity of the map j_*, as we will see in the proof. Another special case of this theorem was treated already by Borel and Serre [**BS64**, 6.8]. The general version of this theorem is due to An and Wang [**AW08**, Thm. 3.1]. We use their elegant argument for the surjectivity of the canonical map j_*.

Proof of Theorem II.4.4. We first show that the map j_* is injective. Let $b, c \in Z^1(\Xi, K)$ be two cocycles with values in K. Suppose they are equivalent as cocycles with values in G, i.e. there is some $g \in G$ such that $g^{-1} b_s\, {}^s g = c_s$ for all $s \in \Xi$. Let $P \subseteq G$ be a Euclidean complement to K (which exists by Mostow's Theorem). Moreover, we can assume that P is stable under conjugation with K and stable under the action of Ξ (cf. A.4.2). We may write $g = kp$ with unique $k \in K$ and $p \in P$. We obtain

$$p^{-1} k^{-1} b_s\, {}^s k\, {}^s p = c_s$$

for all $s \in \Xi$, and so

$$k^{-1} b_s\, {}^s k\, {}^s p = c_s (c_s^{-1} p c_s).$$

Using that P is stable under conjugation with elements in K and stable under Ξ, we deduce, from the uniqueness of Mostow's decomposition, that $k^{-1} b_s\, {}^s k = c_s$. This means that b and c define the same class in $H^1(\Xi, K)$.

Now we show that j_* is indeed surjective. Let $(b_s)_{s \in \Xi}$ be a cocycle in $Z^1(\Xi, G)$. Consider the semidirect product group $G \rtimes \Xi$. Note that this is a Lie group with a finite number of connected components. The finite subset

$$H := \{(b_s, s) \in G \rtimes \Xi \mid s \in \Xi\}$$

[1] One could proceed as follows to verify that $X(b)$ is a submanifold: Put a Riemannian metric on X which is invariant under the b-twisted Ξ-action. This is possible, because Ξ is finite. Then use, for instance, Lemma 3.5 in [**Kio10**].

is a subgroup of $G \rtimes \Xi$. By Mostow's Theorem we can conjugate the subgroup H into the maximal compact subgroup $K \rtimes \Xi$ by an element in G. This means, there is $g \in G$ such that $g^{-1} b_s\, {}^s g \in K$ for all $s \in \Xi$. $\square$

II.4.6. Lemma. *Let $b \in Z^1(\Xi, G)$ be a cocycle. The fixed point set $X(b)$ of the b-twisted Ξ-action is non-empty.*

PROOF. If we assume $b \in Z^1(\Xi, K)$, there is an obvious fixed point $o = K \in X$. Let $b \in Z^1(\Xi, G)$ be an arbitrary cocycle. The surjectivity of the canonical map $j_* \colon H^1(\Xi, K) \to H^1(\Xi, G)$ implies that there is $g \in G$ such that $c_s := g^{-1} b_s\, {}^s g$ lies in the maximal compact subgroup K. Finally the assertion follows from equation (7) in II.4.3 which states $X(b)g = X(c)$. $\square$

II.4.7. Let $b \in Z^1(\Xi, G)$ be a cocycle for Ξ with values in G. We define the b-twisted action of Ξ on G by

$$ {}^{s|b}g := b_s\, {}^s g b_s^{-1} $$

for all $s \in \Xi$. The set of fixed points of the b-twisted action on G will be denoted by

$$ G(b) \;=\; \{\, g \in G \mid {}^{s|b}g = g \;\text{ for all } s \in \Xi \,\}. $$

Furthermore, if b is a cocycle with values in K, then K is stable under the b-twisted action and we define

$$ K(b) = G(b) \cap K. $$

Note that we can apply Lemma A.4.4 to the b-twisted action to see that $K(b)$ is a maximal compact subgroup of $G(b)$. We deduce that $K(b)\backslash G(b)$ is diffeomorphic to a Euclidean space.

Corollary. Let $b \in Z^1(\Xi, K)$, then $X(b)$ is diffeomorphic to $K(b)\backslash G(b)$. In particular, for every cocycle $b \in Z^1(\Xi, G)$ the space $X(b)$ is Euclidean.

PROOF. Via twisting we can assume that $b = 1$. The long exact sequence

$$ 1 \longrightarrow K^{\Xi} \longrightarrow G^{\Xi} \longrightarrow X^{\Xi} \longrightarrow H^1(\Xi, K) \xrightarrow{j_*} H^1(\Xi, G) $$

and the injectivity of j_* yield that G^{Ξ} acts transitively on X^{Ξ}. For more details the reader is referred to [**Kio10**, 3.20].

The second assertion follows immediately from Theorem II.4.4 and equation (7). $\square$

II.5. Application to Arithmetic Groups

II.5.1. We fix some notation for this section. Let G be a linear algebraic group defined over some number field F. Let V_∞ denote the set of archimedean places of F. Given a place $v \in V_\infty$ we denote the completion of F with respect to v by F_v.

Let τ be an automorphism of finite order of G which is defined over F. We write $\Xi := \langle \tau \rangle$ for the finite cyclic group generated by the automorphism τ. Also, Γ always denotes a torsion-free, τ-stable arithmetic subgroup of $G(F)$.

Consider the real Lie group G_∞ associated with G, which is defined as

$$ G_\infty := \prod_{v \in V_\infty} G(F_v). $$

We fix some τ-stable, maximal compact subgroup $K \subseteq G_\infty$ and define

$$ X := K\backslash G_\infty $$

which is diffeomorphic to a Euclidean space. The group Γ acts freely and properly on X from the right. Furthermore, τ induces automorphisms of finite order on X and on the quotient X/Γ. To keep the notation simple we will denote these automorphisms again by the symbol τ.

Note that such a group Γ is of type (FL), so we can define Lefschetz numbers in the cohomology of Γ (see II.2.8). Using the results we already established, it is quite easy to get nice theorems relating those Lefschetz numbers to the developed fixed point theory (see Theorems II.5.5 and II.5.9). Originally, in the case of semisimple algebraic groups, these results are due to Rohlfs (cf. [**Roh81**], [**Roh90**], [**RS93**]). We felt it might be beneficial to extend Rohlfs' result to arbitrary linear algebraic F-groups. We try to give a detailed account in this section. Nevertheless, the case of semisimple groups is still the most interesting one. We start by giving a theorem for cohomology with constant coefficients (see II.5.5). Afterwards we will consider cohomology with non-trivial coefficients. However, in order to do this we will need some easy results on rational representations of G.

II.5.2. *Remark.* Given G and τ as above, it is easy to see that torsion-free, τ-stable arithmetic subgroups indeed exist. Take any torsion-free arithmetic subgroup $\Gamma \subseteq G(F)$ of G. Recall that such a group exists since finitely generated linear groups (in characteristic zero) are virtually torsion-free (cf. 4.8. in [**Weh73**], see also Thm. III.2.3). Moreover, since τ is defined over F, we know that $\tau^m(\Gamma)$ is an arithmetic subgroup for all powers m of τ. Let n be the order of τ, then

$$\Gamma' := \bigcap_{m=0}^{n-1} \tau^m(\Gamma)$$

is a τ-stable, torsion-free arithmetic subgroup of G.

II.5.3. We combine the results of the previous sections. Recall that we have a concise description of the set $(X/\Gamma)^\Xi$ of fixed points of Ξ in X/Γ, more precisely we have

$$(X/\Gamma)^\Xi = \bigsqcup_{\eta\in H^1(\Xi,\Gamma)} \mathfrak{F}(\eta)$$

(see Cor. II.3.5, equation (6)). This is a disjoint union in the sense of topology (II.3.8). For a class $\eta \in H^1(\Xi,\Gamma)$ the set $\mathfrak{F}(\eta)$ is a non-empty submanifold of X/Γ. If $b \in \eta$ is a representing cocycle, we have a diffeomorphism

$$X(b)/\Gamma(b) \xrightarrow{\simeq} \mathfrak{F}(\eta) \tag{8}$$

(cf. II.3.9). Moreover, the space $X(b)$ is Euclidean (see II.4.7) and, in particular, it is an essentially acyclic space (see II.2.11). An important further comment is that the non-abelian cohomology set $H^1(\Xi,\Gamma)$ is a finite set (see Prop. 3.8 in [**BS64**]). It is worth noting that the groups $\Gamma(b)$ for $b \in Z^1(\Xi,\Gamma)$ are of type (FL). This can be easily seen since $\Gamma(b)$ is a torsion-free arithmetic subgroup of the linear algebraic F-group $G(b)$ of fixed points under the b-twisted Ξ-action (cf. Prop. 3.6 in [**BS64**]).

II.5.4. Let Φ denote the system of all supports in X/Γ, i.e. the collection of all closed sets. Since Γ is of type (FL), we see (using II.2.12) that the triple $(X/\Gamma,\Phi,R)$ is of finite type over R for any ring R. By the same argument the triple $(\mathfrak{F}(\eta),\Phi_{|\mathfrak{F}(\eta)},R)$ is of finite type for any $\eta \in H^1(\Xi,\Gamma)$. We define the Euler characteristic

$$\chi(\mathfrak{F}(\eta)) := \chi_{\Phi|\mathfrak{F}(\eta)}(\mathfrak{F}(\eta),\mathbb{Q})_{\mathbb{Q}}$$

using the notation from paragraph I.7.3. Take a representative $b \in Z^1(\Xi,\Gamma)$. Looking at (8) we observe that

$$\chi(\mathfrak{F}(\eta)) = \chi_{\Phi|\mathfrak{F}(\eta)}(\mathfrak{F}(\eta),\mathbb{Q})_{\mathbb{Q}} = \chi(\Gamma(b),\mathbb{Q})_{\mathbb{Q}} = \chi(\Gamma(b)).$$

It is worth noting that $H^\bullet(\Gamma(b),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}\cong H^\bullet(\Gamma(b),\mathbb{Q})$ which follows again from the fact that $\Gamma(b)$ is of type (FL). Consequently,

$$\chi(\mathfrak{F}(\eta)) = \chi(\Gamma(b),\mathbb{Z})_{\mathbb{Q}} = \chi_{\Phi|\mathfrak{F}(\eta)}(\mathfrak{F}(\eta),\mathbb{Z})_{\mathbb{Q}}.$$

Now we are able to state and prove a slightly generalised version of Rohlfs' fixed point theorem.

II.5.5. Theorem. *Let Γ be a torsion-free, τ-stable arithmetic subgroup of G. Then the Lefschetz number $\mathcal{L}(\tau,\Gamma,\mathbb{Q})$ satisfies*

$$\mathcal{L}(\tau,\Gamma,\mathbb{Q}) \;=\; \sum_{\eta\in H^1(\Xi,\Gamma)} \chi(\mathfrak{F}(\eta)).$$

Moreover, for every representative $b\in\eta$ we have the identity

$$\chi(\mathfrak{F}(\eta)) \;=\; \chi(\Gamma(b)).$$

Proof. The second assertion was explained in the previous paragraph. To prove the first statement, we observe that (by II.2.12)

$$\mathcal{L}(\tau,\Gamma,\mathbb{Q}) = \mathcal{L}(\tau,\Gamma,\mathbb{Z}) = \mathcal{L}_\Phi(\tau,X/\Gamma,\mathbb{Z}),$$

where Φ denotes the system of all supports. So, we only have to apply Corollary I.7.14 and use that the Euler characteristic is additive for topologically disjoint unions.

In order to do this we have to check that the finiteness assumptions are fulfilled. First note, that Φ is clearly Ξ-invariant and paracompactifying since X/Γ is a paracompact space. Furthermore, X/Γ is a finite dimensional manifold and thus it has finite cohomological dimension (cf. [**Deo76**]). Finally, the triples $(\mathfrak{F}(\eta),\Phi_{|\mathfrak{F}(\eta)},\mathbb{Z})$ are of finite type for any η and since $H^1(\Xi,\Gamma)$ is finite we get that $((X/\Gamma)^\Xi,\Phi_{|(X/\Gamma)^\Xi},\mathbb{Z})$ is of finite type. Analogously this follows for all subgroups of Ξ and we can apply Cor. I.7.14. $\square$

II.5.6. *Example.* Let $G:=\mathbb{G}_a^n$ be the n-fold direct product of the additive group defined over $\mathbb{Q}$. This is an abelian group hence inversion defines a $\mathbb{Q}$-automorphism $\tau: G\to G$ of order two. The lattice $\mathbb{Z}^n\subseteq\mathbb{Q}^n = G(\mathbb{Q})$ is a torsion-free, τ-stable arithmetic subgroup. Note that $G(\mathbb{R})=\mathbb{R}^n$ and $K=\{0\}$ is the unique maximal compact subgroup. Hence $X/\Gamma\cong\mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional torus. It is easy to see that $H^1(\langle\tau\rangle,\mathbb{Z}^n)\cong(\mathbb{Z}/2\mathbb{Z})^n$. Moreover, since G is abelian there are no twisted τ-actions, this means $\Gamma(b)=\Gamma^\tau=\{0\}$ for all $b\in Z^1(\langle\tau\rangle,\mathbb{Z}^n)$. We deduce $\chi(\Gamma(b))=1$ for all b. Now we apply the theorem to obtain

$$\mathcal{L}(\tau,\mathbb{Z}^n,\mathbb{Q}) = 2^n.$$

II.5.7. Theorem. *Let Γ be a torsion-free, τ-stable arithmetic subgroup of G. Let k be a number field and let R be its ring of integers. Further, let E be an $R[\Gamma\rtimes\Xi]$-module which is finitely generated and projective as R-module. Then the Lefschetz number $\mathcal{L}(\tau,\Gamma,E)$ satisfies*

$$\mathcal{L}(\tau,\Gamma,E) \;=\; \sum_{\eta\in H^1(\Xi,\Gamma)} \chi(\mathfrak{F}(\eta))\,\mathrm{Tr}(\tau|k\otimes_R E(b_\eta)),$$

where b_η is any representative of the class η.

Proof. The Lefschetz number does not change if we replace k by a larger number field k'. So we may assume that R contains all n-th roots of unity, where n is the order of the automorphism τ. We apply Corollary I.7.14 after checking the finiteness conditions just as in the proof of Thm. II.5.5. Here we need that E is a finitely generated and projective R-module, which also implies that the sheaf $\widetilde{E}$ is flat. Eventually Prop. II.3.11 tells us that $\widetilde{E}_{|\mathfrak{F}(\eta)}$ is essentially $\widetilde{E(b_\eta)}$, and so Prop. II.2.12 implies $\mathcal{L}_{\Phi|\mathfrak{F}(\eta)}(\tau, \mathfrak{F}(\eta), \widetilde{E}_{|\mathfrak{F}(\eta)}) = \mathcal{L}(\tau, \Gamma(b_\eta), E(b_\eta))$. Hence the claim follows from the observations made in II.2.8. □

II.5.8. *Rational representations.* Usually, we want to use rational representations of the group G as coefficient systems. In order to do this we have to introduce some notations and prove an auxiliary result.

Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation defined over the algebraic closure $\overline{F}$ of F. Here V is a finite dimensional $\overline{F}$-vector space with an F-structure V_F. For an extension field k of F we write V_k for the k-rational points of V. As before we write Ξ for the group generated by the automorphism τ. Suppose we have an action of Ξ on $V = V_{\overline{F}}$ (denoted by left exponents). We say that this action is *compatible* with ρ, if

$$ {}^s(\rho(g)v) \;=\; \rho({}^s g)\, {}^s v $$

for all $v \in V$, $s \in \Xi$ and $g \in G(\overline{F})$.

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation defined over $\overline{F}$ with a compatible action of Ξ.

(i) There is a finite extension k/F such that ρ and the action of Ξ are defined over k.

(ii) Let $\Gamma \subseteq G(F)$ be a τ-stable arithmetic subgroup and let R be the ring of integers of a number field k as in (i). Then there is a $(\Gamma \rtimes \Xi)$-stable R-lattice[2] Λ in V_k.

Proof. The first assertion is clear. To verify the second we choose an R-arithmetic subgroup $\Gamma' \subseteq G(k)$, i.e. an arithmetic subgroup of the algebraic k-group $G \times_F k$, such that Γ' contains Γ. Since the representation ρ is defined over k, there is a Γ'-stable R-lattice $\Lambda' \subset V_k$ (cf. 7.13 in [**Bor69**]). We define another R-lattice

$$ \Lambda := \sum_{s \in \Xi} {}^s\Lambda'. $$

Since the action of Ξ is defined over k, this is indeed an R-lattice in V_k. The group Γ is Ξ-stable and it follows from the compatibility of the action of Ξ with ρ that the lattice Λ is stable under Γ and stable under the action of Ξ. □

II.5.9. Theorem. *Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation defined over $\overline{F}$ together with a compatible Ξ-action. Let Γ be a torsion-free, τ-stable arithmetic subgroup of G. Then the Lefschetz number $\mathcal{L}(\tau, \Gamma, V)$ satisfies*

$$ \mathcal{L}(\tau, \Gamma, V) \;=\; \sum_{\eta \in H^1(\Xi, \Gamma)} \chi(\mathfrak{F}(\eta))\, \mathrm{Tr}(\tau | V(b_\eta)), $$

where b_η is any cocycle in the class η.

Proof. Choose any F-structure V_F on V. We fix an extension field k of F and an R-lattice Λ as in Lemma II.5.8. Apply Theorem II.5.7 to Λ. Finally, choose an embedding of k into $\overline{F}$ and note that $\overline{F} \otimes_R H^\bullet(\Gamma, \Lambda) \cong H^\bullet(\Gamma, V)$ as $\overline{F}[\Xi]$-modules. □

[2] An R-lattice in V_k is a finitely generated R-submodule of V_k which spans V_k over k.

II.5.10. *Remark.* Note that the trace $\mathrm{Tr}(\tau|V(b_\eta))$ does not depend on the chosen representative $b_\eta \in \eta$. This follows from Lemma II.5.11 below, which will further simplify the application of Theorem II.5.9. Though usually the cohomology set $H^1(\Xi, \Gamma)$ is very large, the image of the canonical map $H^1(\Xi,\Gamma) \to H^1(\Xi, G(\overline{F}))$ typically contains only a small number of elements.

II.5.11. Lemma. *Let $b, c \in Z^1(\Xi, G(F))$ be cocycles. If b and c are cohomologous in $H^1(\Xi, G(\overline{F}))$, then the traces of the twisted actions of τ of V agree, i.e.*

$$\mathrm{Tr}(\tau|V(b)) = \mathrm{Tr}(\tau|V(c)).$$

Proof. Suppose b and c are cohomologous over $\overline{F}$, then there is $g \in G(\overline{F})$ such that $g^{-1} b_s\, {}^s g = c_s$ holds for all $s \in \Xi$. The c-twisted action of Ξ on V is given by

$${}^{s|c}v = c_s\, {}^s v = g^{-1} b_s\, {}^s(gv).$$

Thus $\tau|c$ and $\tau|b$ are conjugate in $\mathrm{GL}(V)$ and therefore they have equal trace. □

II.6. The Adelic Decomposition

II.6.1. There is also an adelic version of Rohlfs' decomposition which can be found in [**Roh90**]. We will describe it in this section. It turns out to be very powerful since it arranges the terms in the Lefschetz number formula II.5.9 in the appropriate way.

II.6.2. Let F be an algebraic number field. Let $\mathbb{A}$ denote the ring of adeles of F and let $\mathbb{A}_f$ denote the ring of finite adeles. A linear algebraic group G over F has *strong approximation* if $G(F)$ is dense in $G(\mathbb{A}_f)$ with respect to the adelic topology. For example, unipotent groups and F-simple, simply connected groups with a non-compact associated Lie group G_∞ have strong approximation (see p. 427 in [**PR94**]). Moreover, all semidirect products of such groups have strong approximation.

Let G be a linear algebraic F-group with strong approximation. Choose a maximal compact subgroup $K_\infty \subseteq G_\infty$ and set $X := K_\infty\backslash G_\infty$. Furthermore, let $K_f \subseteq G(\mathbb{A}_f)$ be an open compact subgroup and let $\Gamma := G(F) \cap K_f$ be the arithmetic group defined by this open compact subgroup. There is a homeomorphism

$$X/\Gamma \xrightarrow{\simeq} K_\infty K_f\backslash G(\mathbb{A})/G(F).$$

To see this, consider with the inclusion $G_\infty \to G(\mathbb{A})$ and use strong approximation to observe that it factors to such a homeomorphism.

II.6.3. Lemma. *Let G be an algebraic F-group with strong approximation. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let $\Gamma := G(F) \cap K_f$ be the associated arithmetic group. Then Γ is torsion-free if and only if $G(F)$ acts freely on $K_\infty K_f\backslash G(\mathbb{A})$.*

Proof. This is a special case of Lemma IV.6.4 below. □

II.6.4. *Fixed points.* Let G be an algebraic group defined over F and let Ξ be a finite group of F-rational automorphisms of G. We fix a Ξ-stable maximal compact subgroup $K_\infty \subseteq G_\infty$ (cf. A.4.2). If $K_f \subset G(\mathbb{A}_f)$ is a Ξ-stable open compact subgroup of $G(\mathbb{A}_f)$, then we obtain an action of Ξ on the double coset space

$$S(K_f) := K_\infty K_f\backslash G(\mathbb{A})/G(F).$$

We are going to describe the set of Ξ-fixed points $S(K_f)^\Xi$ following Rohlfs [**Roh90**] under the assumption that $G(F)$ acts freely on $K_\infty K_f\backslash G(\mathbb{A})$.

Let $a \in G(\mathbb{A})$ be a representative of a Ξ-fixed point in $S(K_f)$. Then for every $s \in \Xi$ there are $k_s \in K_\infty K_f$ and $g_s \in G(F)$ such that

$$ {}^s a \;=\; k_s^{-1} a g_s. \tag{9}$$

Since we assumed the action of $G(F)$ on $K_\infty K_f \backslash G(\mathbb{A})$ to be free, we see that k_s and g_s are unique. Moreover, for all $s, t \in \Xi$ we obtain

$$k_{st}^{-1} a g_{st} = {}^{st}a = {}^s k_t^{-1} k_s^{-1} a g_s \, {}^s g_t,$$

and we may conclude that $(k_s)_{s\in\Xi}$ (resp. $(g_s)_{s\in\Xi}$) is a 1-cocycle for Ξ in $K_\infty K_f$ (resp. $G(F)$). Additionally, equation (9) yields that these cocycles become cohomologous when considered as cocycles with values in $G(\mathbb{A})$. It is easily checked that we obtain a pair of equivalent cocycles if we replace a by another representative of the same fixed point.

Summing up, we constructed a map

$$\vartheta : S(K_f)^\Xi \;\longrightarrow\; \mathcal{H}^1(\Xi)$$

where $\mathcal{H}^1(\Xi)$ denotes the fibred product

$$\mathcal{H}^1(\Xi) := H^1(\Xi, K_\infty K_f) \underset{H^1(\Xi, G(\mathbb{A}))}{\times} H^1(\Xi, G(F)).$$

Clearly, the map ϑ is surjective and we can decompose $S(K_f)^\Xi$ into the fibres

$$S(K_f)^\Xi \;=\; \bigsqcup_{\eta\in\mathcal{H}^1(\Xi)} \vartheta^{-1}(\eta). \tag{10}$$

II.6.5. *Fixed points and strong approximation.* We keep the setting of the previous paragraph. Assume that G has strong approximation and define $\Gamma = G(F) \cap K_f$. By Lemma II.6.3 this is a torsion-free arithmetic group. Recall that we have a homeomorphism $X/\Gamma \cong S(K_f)$. We want to compare the adelic decomposition of the fixed point space in (10) with the decomposition obtained in paragraph II.5.3.

Let η be a class in $H^1(\Xi, \Gamma)$. Let $i : \Gamma \to G(F)$ be the inclusion, then we get a class $i_*(\eta) \in H^1(\Xi, G(F))$. Similarly, if $\ell : \Gamma \to K_f$ denotes the (diagonal) inclusion, then we get a class $\ell_*(\eta) \in H^1(\Xi, K_f)$. Moreover, we write $j : K_\infty \to G_\infty$ and $h : \Gamma \to G_\infty$ for the natural inclusion maps. Recall that $j_* : H^1(\Xi, K_\infty) \to H^1(\Xi, G_\infty)$ is a bijection (see Theorem II.4.4). So, we also get a class $j_*^{-1} h_*(\eta) \in H^1(\Xi, K_\infty)$. In total we constructed a mapping

$$\alpha : H^1(\Xi, \Gamma) \longrightarrow \mathcal{H}^1(\Xi),$$

given by $\eta \mapsto \big((j_*^{-1}h_*(\eta), \ell_*(\eta)),\; i_*(\eta)\big)$.

Lemma. The map α is surjective.

Proof. Let $\eta \in \mathcal{H}^1(\Xi)$. We find cocycles $(k_s) \in Z^1(\Xi, K_\infty K_f)$ and (g_s) in $Z^1(\Xi, G(F))$ as well as some $a \in G(\mathbb{A})$ such that ${}^s a = k_s^{-1} a g_s$ holds for all $s \in \Xi$, and such that η is represented by $((k_s)_s, (g_s)_s)$. Using strong approximation we can assume that $a = a_\infty \in G_\infty$. However with this assumption it follows immediately that $g_s \in \Gamma$ for all s, and it is easy to check that η is the image of the class of $(g_s)_s$ in $H^1(\Xi, \Gamma)$ under the map α. $\square$

In particular, this implies that the set $\mathcal{H}^1(\Xi)$ is finite (cf. II.5.3). The following diagram is commutative

$$\begin{tikzcd}
X/\Gamma^\Xi \arrow[r, "\simeq"] \arrow[d] & S(K_f)^\Xi \arrow[d, "\vartheta"] \\
H^1(\Xi, \Gamma) \arrow[r, "\alpha"'] & \mathcal{H}^1(\Xi)
\end{tikzcd}$$

and we can read off that the fibres of ϑ are unions of fixed point components obtained in II.5.3. More precisely, let $\eta \in \mathcal{H}^1(\Xi)$, then

$$\vartheta^{-1}(\eta) = \bigsqcup_{\mu \in \alpha^{-1}(\eta)} \mathfrak{F}(\mu).$$

It follows directly that the fibres of ϑ are open and closed in $S(K_f)^\Xi$. One could as well say that ϑ is continuous with respect to the discrete topology on $\mathcal{H}^1(\Xi)$.

II.6.6. *Fixed point groups.* For G and Ξ as above, let $\gamma \in Z^1(\Xi, G(F))$ be a cocycle. The γ-twisted Ξ-action (see II.4.7) on G is defined over F and the group of fixed points is a linear algebraic group which will be denoted $G(\gamma)$. Similarly, given a cocycle $k \in Z^1(\Xi, K_\infty K_f)$ we define the k-twisted action of Ξ on $K_\infty K_f$ by ${}^{s|k}g := k_s\, {}^s g k_s^{-1}$. The corresponding group of fixed points under this action will be written $(K_\infty K_f)^{\Xi|k}$.

II.6.7. Lemma (Rohlfs, see 3.5 in [**Roh90**])**.** *We use the notation from paragraph II.6.4. Let G be an algebraic F-group and Ξ a finite group of automorphisms of G defined over F. Let $K_f \subset G(\mathbb{A}_f)$ be a Ξ-stable open compact subgroup such that $G(F)$ acts freely on $K_\infty K_f \backslash G(\mathbb{A})$.*

Let $\eta \in \mathcal{H}^1(\Xi)$ be a class represented by a pair of cocycles $k = (k_s)_s \in Z^1(\Xi, K_\infty K_f)$ and $\gamma = (\gamma_s)_s \in Z^1(\Xi, G(F))$. Take $a \in G(\mathbb{A})$ such that ${}^s a = k_s^{-1} a \gamma_s$ for all $s \in \Xi$. There is a homeomorphism

$$a^{-1}(K_\infty K_f)^{\Xi|k} a \backslash G(\gamma)(\mathbb{A})/G(\gamma)(F) \xrightarrow{\simeq} \vartheta^{-1}(\eta).$$

Proof. Let $\pi : G(\mathbb{A}) \to S(K_f)$ denote the canonical projection. Let $x \in G(\gamma)(\mathbb{A})$, then $\pi(ax) \in \vartheta^{-1}(\eta)$ because

$${}^s(ax) = k_s^{-1} a \gamma_s\, {}^s x = k_s^{-1} a x \gamma_s$$

for all $s \in \Xi$. We get a continuous map $\rho : G(\gamma)(\mathbb{A}) \to \vartheta^{-1}(\eta)$ defined by $\rho(x) = \pi(ax)$. We show that the map ρ is surjective. Take $y \in \vartheta^{-1}(\eta)$, then (as one can check) we can find a representative $b \in G(\mathbb{A})$ with $\pi(b) = y$ and such that ${}^s b = k_s^{-1} b \gamma_s$ for all $s \in \Xi$. Now, $a^{-1}b$ is obviously in $G(\gamma)(\mathbb{A})$ and $\rho(a^{-1}b) = y$.

Further, it is obvious that for $x, y \in G(\gamma)(\mathbb{A})$ which represent the same double coset in $a^{-1}(K_\infty K_f)^{\Xi|k} a \backslash G(\gamma)(\mathbb{A})/G(\gamma)(F)$ one has $\rho(x) = \rho(y)$. Conversely, assume $\rho(x) = \rho(y)$. We find $c \in K_\infty K_f$ and $q \in G(F)$ such that

$$caxq = ay.$$

For all $s \in \Xi$, we observe that

$$k_s^{-1} caxq\gamma_s = k_s^{-1} ay\gamma_s = {}^s(ay) = {}^s c k_s^{-1} ax\gamma_s\, {}^s q.$$

By assumption $G(F)$ acts freely on $K_\infty K_f \backslash G(\mathbb{A})$ and we obtain $c \in (K_\infty K_f)^{\Xi|k}$ and $q \in G(\gamma)(F)$. This means that x and y define the same double coset in

$$a^{-1}(K_\infty K_f)^{\Xi|k} a \backslash G(\gamma)(\mathbb{A})/G(\gamma)(F).$$

We conclude that ρ factors to a continuous bijection

$$\rho' : a^{-1}(K_\infty K_f)^{\Xi|k} a \backslash G(\gamma)(\mathbb{A})/G(\gamma)(F) \longrightarrow \vartheta^{-1}(\eta)$$

and we have to verify that it is indeed a homeomorphism. Apply Theorem II.3.9, with $X = G(\mathbb{A})$ and the γ-twisted Ξ-action, to see that the canonical map

$$G(\gamma)(\mathbb{A})/G(\gamma)(F) \to G(\mathbb{A})/G(F)$$

is a closed embedding. Moreover, the projection $p : G(\mathbb{A})/G(F) \to S(K_f)$ is proper since $K_\infty K_f$ is compact. Using this it follows immediately, that ρ' is closed and is hence a homeomorphism. $\square$

II.6.8. Theorem (cf. [**Roh90**]). *Let G be an algebraic F-group with strong approximation and let τ be an automorphism of finite order defined over F. We write Ξ for the finite cyclic group generated by τ. Let $K_f \subset G(\mathbb{A}_f)$ be a τ-stable open compact subgroup such that $\Gamma := G(F) \cap K_f$ is torsion-free. Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation defined over $\overline{F}$ with a compatible Ξ-action. Then we have*

$$\mathcal{L}(\tau, \Gamma, V) \;=\; \sum_{\eta \in \mathcal{H}^1(\Xi)} \chi(\vartheta^{-1}(\eta)) \,\mathrm{Tr}(\tau | V(b_\eta)),$$

where $b_\eta \in G(F)$ is any representative of the $H^1(\Xi, G(F))$ component of η.

Proof. Use Theorem II.5.9 with the result that $\vartheta^{-1}(\eta)$ is the union of some of the sets $\mathfrak{F}(\mu)$ (see II.6.5). Since all of the classes $\mu \in \alpha^{-1}(\eta)$ define the same class in $H^1(\Xi, G(F))$, the traces of all the twisted actions of such representatives agree (cf. Lem. II.5.11). $\square$

II.6.9. *Remark.* At first sight it seems that Theorem II.6.8 has no advantage over the non-adelic Theorem II.5.9. Indeed, we simply arranged the fixed point components $\mathfrak{F}(\mu)$ in groups denoted $\vartheta^{-1}(\eta)$. Nevertheless, the description of the adelic components $\vartheta^{-1}(\eta)$ in Lemma II.6.7 makes it easier to actually compute to Euler characteristic (cf. Thm. IV.6.12). Even better, the index set $\mathcal{H}^1(\Xi)$ can, in general, be easily determined, whereas computing $H^1(\Xi, \Gamma)$ is a more intricate task.

CHAPTER III

Smooth Group Schemes and Orders in Central Simple Algebras

III.1. Introduction

III.1.1. In this chapter we collect some basic results on groups schemes defined over rings, in particular over Dedekind domains. This is necessary for two reasons. The first is that we will give a reformulation of Harder's Gauß-Bonnet Theorem in the next chapter which exploits smoothness of the underlying groups schemes, and this chapter provides the basic facts.

The second reason is that the matter of interest in Chapter V is the so-called *special linear group* over an order in a central simple algebra. In our approach it is important to regard these groups as group schemes over a Dedekind ring. Even more, not only the special linear group but also the groups of fixed points associated with involutions will occur in the final chapter, where it will be crucial to understand when these group schemes are smooth. It seems that the basic properties of group schemes associated with orders in central simple algebras are not accessible in the literature and for this reason we tried to be precise about the subtle points.

III.1.2. This chapter is organised as follows. We start with some basic properties of group schemes in Section III.2, in particular we define smoothness and we prove a smoothness criterion for kernels of homomorphisms. In the next section we study the group scheme of units associated with an order in a central simple algebra. We also define the *special linear group* over an order (see III.3.8) and we give a simple criterion for the smoothness of special linear groups. In Section III.4 we study involutions on central simple algebras and we define the associated fixed point group schemes. A criterion for the smoothness of these groups is given by Theorem III.4.15. Further, we construct the pfaffian for involutions of symplectic type (see III.4.9). We also consider the non-abelian Galois cohomology sets corresponding to the action of an involution in Section III.5. We will relate these sets with hermitian forms on the underlying order, which leads to the main result of this section: the theorem of Fainsilber-Morales III.5.9. We conclude this chapter in Section III.6 with a short excursion on integration with the canonical measure on a smooth scheme over a complete discrete valuation ring.

III.2. Group Schemes and Smoothness

III.2.1. Let R be a commutative ring with unit. Whenever we speak of a group scheme G over R, we will always mean an *affine* group scheme of *finite type* over R. The associated Hopf algebra will be denoted by $R[G]$. The assumption that G is of finite type, means that $R[G]$ is finitely generated as R-algebra. Moreover, if A is a commutative R-algebra, then $G(A)$ denotes the group of A-rational points of the group scheme G.

III.2.2. *Principal congruence subgroups.* Let G be a group scheme over R. Given an ideal $\mathfrak{a} \subseteq R$, we define the *principal congruence subgroup* $\Gamma(\mathfrak{a}, G)$ of level $\mathfrak{a}$ by

$$\Gamma(\mathfrak{a}, G) := \ker\big(G(R) \to G(R/\mathfrak{a})\big).$$

We usually write $\Gamma(\mathfrak{a})$ for $\Gamma(\mathfrak{a}, G)$ whenever the surrounding group scheme G is clear from the context. If R is a Dedekind ring these congruence groups are torsion-free for most choices of $\mathfrak{a}$. To see this we formulate an old theorem of Minkowski in the language of group schemes (the proof is a Hopf algebra reformulation of the proof of Satz K.16 in [**JS06**]).

III.2.3. Theorem. *Let R be a Dedekind domain of characteristic zero and let G be a group scheme over R. Given a non-trivial ideal $\mathfrak{a} \subseteq R$ such that for every prime number p the ideal $\mathfrak{a}^{p-1}$ does not divide the principal ideal pR, then the congruence subgroup $\Gamma(\mathfrak{a}, G)$ is torsion-free.*

Proof. Suppose $\Gamma(\mathfrak{a}, G)$ has torsion elements. Then we can find an element $g \in \Gamma(\mathfrak{a})$ which has finite order different from one. We may assume, by taking powers, that g has order p, where p is a prime number in $\mathbb{Z}$. In terms of Hopf algebras, g is a morphism of R-algebras $g : R[G] \to R$. Further, $g \in \Gamma(\mathfrak{a})$ means that the diagram

$$\begin{array}{ccc} R[G] & \xrightarrow{\;g\;} & R \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow \\ R & \longrightarrow & R/\mathfrak{a} \end{array}$$

commutes. Here $\varepsilon : R[G] \to R$ denotes the unit of G (or counit of $R[G]$). We set $h := g - \varepsilon$, this is an R-linear map $h : R[G] \to R$ and its image $\mathfrak{b} = h(R[G])$ is an ideal in R which is contained in $\mathfrak{a}$. Since we assumed $g \neq \varepsilon$ the ideal $\mathfrak{b}$ is non-zero. We have $(0) \subsetneq \mathfrak{b} \subseteq \mathfrak{a} \subsetneq R$ and so $\mathfrak{b}$ is a proper ideal in R. Let $\mu : R[G] \to R[G] \otimes_R R[G]$ denote the comultiplication. For any natural number m we can take m-fold products in G, and so we get the m-fold comultiplication

$$\mu^m : R[G] \to \underbrace{R[G] \otimes_R \cdots \otimes_R R[G]}_{m \text{ times}}.$$

Note that $\mu^1 = \mathrm{Id}_{R[G]}$ and $\mu = \mu^2$. By assumption we have $(g, \ldots, g) \circ \mu^p = \varepsilon$. Now, we write $g = \varepsilon + h$ and we obtain

$$\varepsilon \;=\; \varepsilon + ph + \sum_{k=2}^{p} \binom{p}{k} \underbrace{(h, \ldots, h)}_{k} \circ \mu^k. \tag{11}$$

Note that p divides $\binom{p}{k}$ whenever $1 \leq k < p$. Let $j \geq 0$ be the maximal integer such that $\mathfrak{b}^j$ divides pR. By assumption $j < p - 1$. From equation (11) we conclude that $p\mathfrak{b}$ is contained in $\mathfrak{b}^{j+2} + \mathfrak{b}^p \subseteq \mathfrak{b}^{j+2}$. Since $\mathfrak{b}$ is a proper ideal, we can deduce that $\mathfrak{b}^{j+1}$ divides pR. However, this is a contradiction since j was chosen maximal. ☐

III.2.4. *Remark.* For $R = \mathbb{Z}$ the condition in the theorem holds for any ideal $m\mathbb{Z}$ in $\mathbb{Z}$ with $m \geq 3$. Moreover, if R is the ring of integers of an algebraic number field F, then all prime factors of the ideal pR occur with a power less or equal than the degree $[F : \mathbb{Q}]$ of the extension $F/\mathbb{Q}$. Therefore, the condition in the theorem only has to be checked for all primes $p \leq [F : \mathbb{Q}] + 1$.

III.2.5. Definition. Let G be a group scheme over R. The scheme G is said to be *smooth* over R, if for any commutative R-algebra C with nilpotent ideal $I \subseteq C$ the induced homomorphism $G(C) \to G(C/I)$ is surjective.

III.2.6. *Remark.* First note that it suffices to check the smoothness condition for every commutative R-algebra C with an ideal $I \subseteq C$ satisfying $I^2 = 0$ (cf. 19.4.3 in EGA IV, [**Gro64**]).

Further, let G be a smooth group scheme over R, then in certain situations one can lift points also over ideals which are not nilpotent. A useful result for our purposes is the following: Let C be a complete local noetherian ring and let $I \subseteq C$ be a proper ideal (i.e. $I \neq C$), then the homomorphism $G(C) \to G(C/I)$ is surjective (cf. Corollaire 19.3.11, EGA IV, [**Gro64**]).

III.2.7. *The cotanget space.* Let G be a group scheme over R and let $\varepsilon : R[G] \to R$ denote the unit of G. The kernel $I_G := \ker(\varepsilon)$ is called the augmentation ideal of $R[G]$. We define the cotanget module ω_G of G by $\omega_G := I_G/I_G^2$. Recall that for every commutative R-algebra C, the C-points $\mathrm{Lie}(G)(C)$ of the Lie algebra of G and $\mathrm{Hom}_R(\omega_G, C)$ are canonically isomorphic as R-modules (cf. Cor. 3.6 in [**DG70**, p. 208]).

Assume now that R is noetherian and G is smooth. Then ω_G is a finitely generated projective R-module (see II.§4, 4.8 in [**DG70**, p. 215]). Furthermore, the *Theorem of infinitesimal points* (cf. [**DG70**, p. 208]) can be stated as follows: For every commutative R-algebra C with an ideal $J \subseteq C$ satisfying $J^2 = 0$ there is a short exact sequence of groups

$$1 \longrightarrow \mathrm{Lie}(G)(R) \otimes_R J \longrightarrow G(C) \longrightarrow G(C/J) \longrightarrow 1.$$

This sequence is functorial in G and C. Note that $\mathrm{Lie}(G)(R) \otimes_R J$ is isomorphic to $\mathrm{Hom}_R(\omega_G, J)$ since ω_G is finitely generated and projective (use B.1.2).

Smoothness is an infinitesimal condition and a useful criterion for smoothness is usually one that can be formulated as a condition on the Lie algebra level. We will prove such a criterion for kernels of homomorphisms.

III.2.8. Proposition (Smoothness of kernels)**.** *Let R be a commutative noetherian ring and let $f : G \to H$ be morphism between two smooth group schemes over R. If the derivative* $\mathrm{d}(f) : \mathrm{Lie}(G)(R) \to \mathrm{Lie}(H)(R)$ *is surjective, then the group scheme* $K := \ker(f)$ *is smooth over R.*

PROOF. Let C be an R-algebra with an ideal $J \subseteq C$ satisfying $J^2 = 0$. We have to show that the homomorphism $p_K : K(C) \to K(C/J)$ is surjective. For simplicity we write $\mathfrak{g} := \mathrm{Lie}(G)(R)$ and $\mathfrak{h} := \mathrm{Lie}(H)(R)$. There is a commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathfrak{g}\otimes J \arrow[r, "\mathrm{d}(f)\otimes \mathrm{Id}_J"] \arrow[d] & \mathfrak{h}\otimes J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K(C) \arrow[r] \arrow[d, "p_K"'] & G(C) \arrow[r, "f_C"] \arrow[d, "p_G"'] & H(C) \arrow[d, "p_H"'] & \\
0 \arrow[r] & K(C/J) \arrow[r] & G(C/J) \arrow[r, "f_{C/J}"] \arrow[d] & H(C/J) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Take $x \in K(C/J)$ and consider it as an element in $G(C/J)$. Since G is smooth, we find $y \in G(C)$ such that $p_G(y) = x$. The element $f_C(y)$ lies in the kernel of p_H, thus there is $z \in \mathfrak{h} \otimes J$ mapping onto $f_C(y)$. By assumption the map $\mathrm{d}(f)$ is surjective, hence we can find $v \in \mathfrak{g} \otimes J$ with $(\mathrm{d}(f) \otimes \mathrm{Id}_J)(v) = z$. Sending this element to $G(C)$ we obtain $v_C \in G(C)$. By construction $f_C(yv_C^{-1}) = 1_H$ and the element yv_C^{-1} is in the kernel $K(C)$. Clearly, $p_K(yv_C^{-1}) = x$. $\square$

III.3. Groups Associated with Orders in Central Simple Algebras

III.3.1. In this section we will define smooth group schemes associated with units of orders in central simple algebras. In particular, we define the associated *special linear group* of elements having reduced norm one. These groups, especially the cohomology of their congruence subgroups, will be the topic of Chapter V. We will also need the notations defined in this section later on. For some technical background on determinants and traces we will often refer to Appendix B.

III.3.2. Lemma. *Let R be a ring and let $\mathbb{A}^1$ denote the affine line over R. Let $\mathfrak{X}$ be an affine scheme over R with a morphism $f : \mathfrak{X} \to \mathbb{A}^1$. The subfunctor $\mathfrak{Y}$ (from the category of commutative R-algebras to the category of sets) defined by*

$$A \mapsto \{\, y \in \mathfrak{X}(A) \mid f(y) \in A^\times \}$$

is an affine scheme and the natural transformation $\mathfrak{Y} \to \mathfrak{X}$ is a morphism of schemes. If $\mathfrak{X}$ is of finite type (resp. smooth), then $\mathfrak{Y}$ has the same property.

Proof. Let $R[\mathfrak{X}]$ be the coordinate ring of $\mathfrak{X}$ and let $P \in R[\mathfrak{X}]$ be the polynomial defining f. Note that $\mathfrak{Y}$ is canonically isomorphic to the functor

$$A \mapsto \{\, (y, z) \in \mathfrak{X}(A) \times A \mid f(y)z = 1 \,\}.$$

Using this it is easily checked that the R-algebra $S := R[\mathfrak{X}] \otimes_R R[T]/(P \otimes T - 1)$ represents $\mathfrak{Y}$. Clearly, S is of finite type if $R[\mathfrak{X}]$ is of finite type.

It remains to show that $\mathfrak{Y}$ is smooth, whenever $\mathfrak{X}$ is smooth. Assume $\mathfrak{X}$ to be smooth and take a commutative R-algebra C with an ideal J such that $J^2 = 0$. By assumption $\mathfrak{X}(C) \to \mathfrak{X}(C/J)$ is surjective, so given $y \in \mathfrak{Y}(C/J)$ we find $x \in \mathfrak{X}(C)$ projecting onto y. By assumption $f(x) + J$ is a unit in C/J. In particular, we find $z \in C$ with $f(x)z \in 1+J$. However, $1+J$ consists entirely of units and thus $f(x) \in C^\times$. We deduce that $\mathfrak{Y}$ is smooth. $\square$

III.3.3. *Unit groups.* Let R be a commutative ring and let Λ be a (possibly non-commutative) R-algebra. We assume that Λ is finitely generated and projective considered as an R-module. The associated ring scheme Λ_a, defined by $\Lambda_a(A) = \Lambda \otimes_R A$ is smooth, affine, and of finite type (cf. B.1.7 and 19.3.2 in EGA IV [**Gro64**]). The norm of the algebra defines a morphism of schemes

$$\mathrm{N}_{\Lambda/R} : \Lambda_a \to \mathbb{A}^1$$

to the affine line (this is explained in B.2.11). Moreover, $x \in \Lambda \otimes_R A$ is a unit if and only if $\mathrm{N}_{\Lambda/R}(x) \in A^\times$ (see Prop. B.2.10). Therefore Lemma III.3.2 implies that the associated unit group functor $\mathrm{GL}_\Lambda : A \mapsto (\Lambda \otimes_R A)^\times$ is a smooth, affine scheme of finite type. To stress this once more: in this thesis GL_Λ is always a functor and not a group. Given a commutative R-algebra A, the group of A-rational points $\mathrm{GL}_\Lambda(A)$ is defined by $\mathrm{GL}_\Lambda(A) := (\Lambda \otimes A)^\times$. Sometimes we write $\mathrm{GL}(\Lambda) := \mathrm{GL}_\Lambda(R)$ for the group of R-rational points. If we take $\Lambda = R$ then we call GL_Λ the *multiplicative group* (or multiplicative group scheme) defined over R, and we denote it by $\mathbb{G}_m$.

Let Λ be arbitrary again, then the norm map defines a morphism of R-group schemes

$$\mathrm{N}_{\Lambda/R} : \mathrm{GL}_\Lambda \to \mathbb{G}_m.$$

Usually we shall call a morphism into the multiplicative group scheme a *character*.

III.3.4. *The Lie algebra of a unit group.* We keep the notation of the previous paragraph III.3.3. We will briefly describe the Lie algebra of GL_Λ. Everything in this paragraph is straightforward and we avoid giving details. All the basic notions can be found in II.§4 in [**DG70**].

There is an obvious isomorphism of group schemes $\Lambda_a \to \mathrm{Lie}(\mathrm{GL}_\Lambda)$ sending an element $x \in \Lambda_a(A) = \Lambda \otimes_R A$ to $1 + \varepsilon x \in \mathrm{Lie}(\mathrm{GL}_\Lambda)(A)$ (with $\varepsilon^2 = 0$ as in [**DG70**]). We will use this isomorphism to identify Λ_a and $\mathrm{Lie}(\mathrm{GL}_\Lambda)$. Under this identification the adjoint action Ad of GL_Λ on Λ_a is given by conjugation and the Lie bracket is given by the usual law $[x, y] = xy - yx$ for all $x, y \in \Lambda_a(A)$. In particular, we identify the Lie algebra of the multiplicative group $\mathbb{G}_m$ over R with the affine line $\mathbb{A}^1$ equipped with the trivial Lie bracket. Using this identification we can describe the derivative of the norm (see also B.2.11).

Proposition. The derivative of the norm character $\mathrm{N}_{\Lambda/R} : \mathrm{GL}_\Lambda \to \mathbb{G}_m$ is the trace $\mathrm{Tr}_{\Lambda/R} : \Lambda_a \to \mathbb{A}^1$.

Proof. We freely use the notation from Appendix B, paragraph B.2.11. For any commutative R-algebra A, and any $x \in \Lambda_a(A)$. One easily verifies

$$\mathrm{d}(\mathrm{N}_{\Lambda/R})(x) = \mathrm{N}_{\Lambda/R}(1 + \varepsilon x) = \det(l_{1+\varepsilon x}) = \det(\mathrm{Id}_\Lambda + \varepsilon l_x) \overset{\star}{=} 1 + \varepsilon \,\mathrm{Tr}_{\Lambda/R}(x),$$

where $\varepsilon^2 = 0$. The equality $(\star)$ is direct if Λ is a free module and can be verified by a direct sum argument otherwise. $\square$

III.3.5. *Central simple algebras.* Let k be a field. Further, let S be a central simple k-algebra[1]. We recall the definitions of the reduced norm and trace. Let E be a splitting field of S, that is, an extension field of k such that there is an isomorphism of E-algebras

$$\alpha : S \otimes_k E \xrightarrow{\simeq} M_n(E).$$

[1]Recall: In this thesis every central simple algebra is assumed to be finite dimensional.

Given such a splitting α, one defines the reduced norm $\mathrm{nrd}_{S/k}$ and reduced trace $\mathrm{trd}_{S/k}$ of an element $x \in S$ by

$$\mathrm{nrd}_{S/k}(x) := \det(\alpha(x \otimes 1)),$$
$$\mathrm{trd}_{S/k}(x) := \mathrm{Tr}(\alpha(x \otimes 1)).$$

One can check that these definitions are independent of the choice of E and the choice of the splitting α. Moreover, both maps, reduced norm and reduced trace, take values in k (cf. (9.3) in [**Rei03**]). Finally, one can show that they are polynomial functions on S, i.e. we can realise them as elements in the symmetric algebra $S_k(S^*)$ (see IX, §2, Prop. 6 in [**Wei95**]). More concisely there is a unique element $\mathrm{nrd}_{S/k} \in S_k(S^*)$ such that for every splitting field E and every splitting α the induced map between the symmetric algebras $S(\alpha^*) : S_E(M_n(E)^*) \to S_k(S^*) \otimes_k E$ maps the determinant polynomial to $\mathrm{nrd}_{S/k} \otimes_k 1$. Analogously, one has such a result for the reduced trace. Note that in this formulation the assumption that k is not a finite field (as made by Weil in [**Wei95**, IX, §2]) is not necessary. We deduce that there are morphisms of k-schemes

$$\mathrm{nrd}_{S/k}, \mathrm{trd}_{S/k} : S_a \to \mathbb{A}^1,$$

where $\mathbb{A}^1$ denotes affine line defined over k. As expected, the reduced norm is multiplicative and the reduced trace is k-linear.

III.3.6. *Orders in central simple algebras.* Let R be a Dedekind ring and let k be the quotient field of R. Let S be a central simple k-algebra and let $\Lambda \subset S$ be an R-order. Since Λ is a full R-lattice in S, this is, a finitely generated R-submodule which spans S over k, we see that Λ is a finitely generated, projective R-module (here we use that R is a Dedekind ring, cf. (4.13) in [**Rei03**]). In particular, we obtain the unit group GL_Λ, which is a smooth group scheme defined over R (see III.3.3).

We show that the reduced norm and reduced trace are defined over R. For the reduced trace this is an easy observation. The reduced trace is linear, so we have $\mathrm{trd}_{S/k} \in S^* := \mathrm{Hom}_k(S, k)$. Moreover, the reduced trace is R-valued on the order Λ, thus we can interpret the reduced trace as an element $\mathrm{trd}_{\Lambda/R} \in \Lambda^* := \mathrm{Hom}_R(\Lambda, R)$. This means that the reduced trace is a morphism of schemes $\mathrm{trd}_{\Lambda/R} : \Lambda_a \to \mathbb{A}^1$ defined over R.

We are going to prove an analogous result for the reduced norm, however, this is a little bit more tedious. We identify the symmetric algebra $S_R(\Lambda^*)$ with an R-subalgebra of $S_k(S^*)$ via the canonical injection $S_R(\Lambda^*) \to S_k(S^*)$ (cf. Lemma B.3.3). The reduced norm is related to the norm via

$$N_{S/k} = \mathrm{nrd}_{S/k}^n \in S_k(S^*),$$

where n^2 is the dimension of the central simple algebra S (cf. (9.7) in [**Rei03**]). Since $S_R(\Lambda^*)$ is integrally closed in $S_k(S^*)$ (see B.3.3), we see that $\mathrm{nrd}_{S/k} \in S_R(\Lambda^*)$, and thus we have a morphism of schemes

$$\mathrm{nrd}_{\Lambda/R} : \Lambda_a \to \mathbb{A}^1.$$

III.3.7. Lemma. *Let R, S and Λ be as above. The reduced norm induces a morphism of R-group schemes*

$$\mathrm{nrd}_{\Lambda/R} : \mathrm{GL}_\Lambda \to \mathbb{G}_m$$

into the multiplicative group defined over R. The derivative $\mathrm{d}(\mathrm{nrd}_{\Lambda/R}) : \Lambda_a \to \mathbb{A}^1$ is given by the reduced trace.

Proof. As a first step one has to verify that the reduced norm is multiplicative, then it is clear that we obtain a morphism of R-group schemes $\mathrm{nrd} : \mathrm{GL}_\Lambda \to \mathbb{G}_m$. It suffices to verify the multiplicativity over fields (due to B.3.5). However, it is well-known that the reduced norm is multiplicative on central simple algebras.

Finally, we have to calculate the derivative. Let E be any extension field of k. The derivative $\mathrm{d}(\mathrm{nrd}_{\Lambda/R})$ is uniquely determined by the E-linear map $\mathrm{d}(\mathrm{nrd}_{\Lambda/R})_E$ from $S \otimes_k E$ to E. Choose E to be a splitting field of S and the claim follows from the fact that the derivative of the determinant is the trace. □

III.3.8. Definition. Let R be a Dedekind ring with quotient field k. Let Λ be an R-order in a central simple k-algebra. The R-group scheme defined as the kernel of the reduced norm $\mathrm{nrd}_{\Lambda/R} : \mathrm{GL}_\Lambda \to \mathbb{G}_m$ is called the *special linear group* over Λ and is denoted by SL_Λ.

III.3.9. *Remark.* We say that an R-order Λ in a central simple algebra is *smooth*, if the reduced trace $\mathrm{trd}_{\Lambda/R} : \Lambda \to R$ is surjective. It follows from Lemma III.3.7 and the smoothness criterion for kernels (Proposition III.2.8) that SL_Λ is smooth if the order Λ is smooth.

We will conclude this section with some results on smooth orders. In particular we want to show that maximal orders are always smooth.

III.3.10. Lemma. *Let R be a Dedekind ring with quotient field k and let S be a central simple k-algebra. For an R-order $\Lambda \subseteq S$ the following are equivalent:*

(i) The order Λ is smooth.

(ii) For every prime ideal $\mathfrak{p} \subseteq R$, the localisation $\Lambda_\mathfrak{p}$ is a smooth $R_\mathfrak{p}$-order in the k-algebra S.

(iii) For every prime ideal $\mathfrak{p} \subseteq R$, the completion $\hat{\Lambda}_\mathfrak{p}$ is a smooth $\hat{R}_\mathfrak{p}$-order in the $\hat{k}_\mathfrak{p}$-algebra $\hat{S} = S \otimes_k \hat{k}_\mathfrak{p}$.

Here $\hat{k}_\mathfrak{p}$ denotes the quotient field of the completion $\hat{R}_\mathfrak{p}$ of $R_\mathfrak{p}$.

Proof. The equivalence of (i) and (ii) is immediate since the map $\mathrm{trd}_{\Lambda/R} : \Lambda \to R$ is onto if and only if for every prime ideal $\mathfrak{p} \subseteq R$ the induced map $\mathrm{trd}_{\Lambda_\mathfrak{p}/R_\mathfrak{p}} : \Lambda_\mathfrak{p} \to R_\mathfrak{p}$ is onto (cf. (3.16) in [**Rei03**]). To prove the equivalence of (ii) and (iii) we may assume that R is a local ring with maximal ideal $\mathfrak{m}$. However, a linear map $\alpha : \Lambda \to R$ is not surjective if and only if $\alpha(\Lambda) \subseteq \mathfrak{m}$. Since the maximal ideal of $\hat{R}$ is $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$ (cf. III. §3.4, Prop. 8 [**Bou72**]), we see that $\hat{\alpha}(\hat{\Lambda}) = \alpha(\Lambda)\hat{R}$ is contained in $\hat{m}$ if and only if $\alpha(\Lambda)$ is contained in $\mathfrak{m}$. □

III.3.11. Lemma. *Let R be a Dedekind ring with quotient field k and assume that the residue field $R/\mathfrak{p}$ is finite for every prime ideal $\mathfrak{p}$ of R. Then every maximal R-order in a central simple k-algebra is smooth.*

Proof. Since an order Λ is maximal in S if and only if all $\mathfrak{p}$-adic completions are maximal orders (see (11.6) in [**Rei03**]), we can use Lemma III.3.10 to assume that R is a complete discrete valuation ring. Recall that S is isomorphic to a matrix algebra $M_r(D)$ over a central division algebra D. Moreover, D has a unique maximal R-order $\Delta \subseteq D$ and Λ is (up to conjugation) the maximal order $M_r(\Delta)$ in S (see (17.3) in [**Rei03**]). It is known that the reduced trace of a matrix $x = (x_{ij})_{i,j=1}^r \in M_r(D)$ is given by

$$\mathrm{trd}_{S/k}(x) = \sum_{i=1}^{r} \mathrm{trd}_{D/k}(x_{ii})$$

(cf. Cor. 2, IX.§2 in [**Wei95**]). Hence we may assume that $S = D$ is a division algebra and $\Lambda = \Delta$ is the unique maximal order. Let $\dim_k D = n^2$ and let ℓ/k be the unique unramified extension of k of degree $[\ell : k] = n$. The field ℓ embeds into D as a maximal subfield and the reduced trace $\mathrm{trd}_{D/k}$ on the elements of ℓ agrees with the field trace $\mathrm{Tr}_{\ell/k}$ (cf. proof of (14.9) in [**Rei03**]). Let o_ℓ denote the valuation ring of ℓ. The image of o_ℓ under the embedding $\ell \to D$ lies in the maximal order Δ. Finally the surjectivity of $\mathrm{trd}_{D/k} : \Delta \to R$ follows from the well-known surjectivity of the field trace $\mathrm{Tr}_{\ell/k} : o_\ell \to R$. □

III.4. Involutions and their Fixed Point Groups over Orders

III.4.1. In this section we are going to define and study the group of fixed points associated with an involution on a central simple algebra. Our main examples are involutions of symplectic type, the associated groups of fixed points are called symplectic groups. We recall the definition of an involution of *symplectic type* in III.4.4. Moreover, we will construct the appropriate *pfaffian* associated with an involution of symplectic type.

We will further consider orders stable under an involution. We shall see that the fixed point group associated with an involution is a smooth group scheme over the underlying Dedekind ring if the order satisfies a simple property – which will be called smoothness with respect to the involution (cf. III.4.13).

Throughout the section k denotes a field of characteristic $\mathrm{char}(k) \neq 2$.

III.4.2. Definition. Let A be a k-algebra. An *involution* τ on A is a k-linear mapping $\tau : A \to A$ of order two such that $\tau(xy) = \tau(y)\tau(x)$ for all $x, y \in A$.

III.4.3. *Remark.* Let A be a central simple k-algebra. By our definition an involution τ is the identity restricted to the centre k. This means, following the notation of [**KMRT98**], that τ is an involution of the *first kind*. If we want to consider involutions of the *second kind*, this is, involutions which do not induce the identity on the centre, we can consider A as an ℓ-algebra, where $\ell \subset k$ is a subfield of index two. We will mostly be interested in involutions of the first kind, note however that for most results we do not assume that k is the centre of A and so they are valid for involutions of the second kind as well.

III.4.4. Definition. Let A be a central simple k-algebra equipped with an involution $\tau : A \to A$. We say that the involution τ is of *symplectic type*, if there is a splitting field ℓ of the algebra A, a splitting

$$\varphi : A \otimes_k \ell \xrightarrow{\simeq} M_n(\ell)$$

and a skew symmetric matrix $a \in M_n(\ell)$ satisfying $\varphi(\tau(x)) = a\varphi(x)^T a^{-1}$ for all elements $x \in A \otimes_k \ell$. If this is the case, then every splitting (over any splitting field) has this property.

III.4.5. Definition. Let τ be an involution on the k-algebra A and let Λ be a τ-stable R-order in A. For every R-module C, we define the C-module of symmetric elements

$$\mathrm{Sym}(\Lambda \otimes_R C, \tau) := \{\, x \in \Lambda \otimes_R C \mid (\tau \otimes \mathrm{Id}_C)(x) = x \,\}.$$

III.4.6. Lemma. *Let R be a Dedekind ring with quotient field k and let Λ be a τ-stable order in a finite dimensional k-algebra A. The module $E := \mathrm{Sym}(\Lambda, \tau)$ is a direct summand of Λ.*

Proof. Consider the short exact sequence of R-modules

$$0 \longrightarrow E \longrightarrow \Lambda \longrightarrow \Lambda/E \longrightarrow 0.$$

It suffices to show that Λ/E is projective since this implies that the short exact sequence splits. The module Λ is finitely generated over R, hence the quotient Λ/E is finitely generated as well. Further, Λ/E is R torsion-free. To see this, we take $x \in \Lambda$ and $r \in R \setminus \{0\}$ with $rx \in E$ and we see that

$$r\tau(x) = \tau(rx) = rx.$$

Since Λ is R torsion-free, we find $\tau(x) = x$ and so $x \in E$. The assumption that R is a Dedekind ring yields that Λ/E is projective (see (4.13) in [**Rei03**]). □

III.4.7. *Remark.* Since E is a direct summand of Λ (see Lemma III.4.6) we can think of $E \otimes_R C$ as a C-submodule of $\Lambda \otimes_R C$ and clearly $E \otimes_R C$ is contained in $\mathrm{Sym}(\Lambda \otimes_R C, \tau)$. However, the module $E \otimes_R C$ can be strictly smaller than $\mathrm{Sym}(\Lambda \otimes_R C, \tau)$.

III.4.8. Corollary. *If C is a flat R-module, then* $\mathrm{Sym}(\Lambda \otimes_R C, \tau) = E \otimes_R C$.

Proof. The kernel of the R-linear map $\varphi : \Lambda \to \Lambda$, defined by $x \mapsto x - \tau(x)$, is $E = \mathrm{Sym}(\Lambda, \tau)$. Similarly, the kernel of $\varphi \otimes \mathrm{Id}_C$ is $\mathrm{Sym}(\Lambda \otimes C, \tau)$. Consider the short exact sequence of R-modules

$$0 \longrightarrow E \longrightarrow \Lambda \xrightarrow{\varphi} \mathrm{im}(\varphi) \longrightarrow 0.$$

The flatness of C implies that the sequence

$$0 \longrightarrow E \otimes_R C \longrightarrow \Lambda \otimes_R C \xrightarrow{\varphi \otimes \mathrm{Id}_C} \mathrm{im}(\varphi) \otimes_R C \longrightarrow 0$$

is exact. Using flatness of C once again we obtain further that the canonical map $\mathrm{im}(\varphi) \otimes C \to \Lambda \otimes C$ is injective. Consequently, $E \otimes_R C$ is the kernel of $\varphi \otimes \mathrm{Id}_C$. □

III.4.9. *Construction of the pfaffian.* Let R be a Dedekind ring with quotient field k. Let A be a central simple k-algebra with an involution $\tau : A \to A$ of symplectic type. Moreover, let $\Lambda \subset A$ be an R-order which is stable under the involution τ. Consider the R-module E of τ-symmetric elements, this is

$$E := \mathrm{Sym}(\Lambda, \tau) = \{\, x \in \Lambda \mid x = \tau(x) \,\}.$$

Note that E is finitely generated and projective since R is a Dedekind ring and Λ is projective (cf. [**Lam99**, 2.26]).

The inclusion $\iota : E \to \Lambda$ induces a morphism of R-algebras

$$S(\iota^*) : S_R(\Lambda^*) \to S_R(E^*).$$

In III.3.6 we showed that the reduced norm $\mathrm{nrd}_{\Lambda/R}$ is actually a morphism of schemes $\Lambda_a \to \mathbb{A}^1$ defined over R. This means, the reduced norm is given by a polynomial function $\mathrm{nrd}_{\Lambda/R} \in S_R(\Lambda^*)$. We define $\mathrm{nrd}_{|E} := S(\iota^*)(\mathrm{nrd}_{\Lambda/R}) \in S_R(E^*)$. We are going to show that there is a *pfaffian*, i.e. a polynomial $\mathrm{pf}_\tau \in S_R(E^*)$ such that $\mathrm{nrd}_{|E} = \mathrm{pf}_\tau^2$. We base the construction on the construction of the pfaffian in [**KMRT98**, (2.9)], however we further show that the pfaffian is defined over R.

Let L/k be any field extension. It follows from (2.9) in [**KMRT98**] that for every $x \in E \otimes_R L$ the reduced norm $\mathrm{nrd}_{|E}(x)$ is a square in L. Therefore, we may apply Corollary B.3.5 to deduce that there is a polynomial $f \in S_R(E^*)$ such that

$$f^2 = \mathrm{nrd}_{|E}\,.$$

We normalise this polynomial $\mathrm{pf}_\tau := \pm f$ such that $\mathrm{pf}_\tau(1) = 1$ and we call pf_τ the *pfaffian* with respect to τ.

III.4.10. Lemma. *In the setting of III.4.9, let $S(\tau^*)$ denote the automorphism of the symmetric R-algebra $S_R(\Lambda^*)$ which is induced by τ. The following assertions hold:*

(i) $S(\tau^*)(\mathrm{nrd}_{\Lambda/R}) = \mathrm{nrd}_{\Lambda/R}$, *and*

(ii) for any commutative R-algebra C, any $y \in \Lambda \otimes_R C$ and all $x \in E \otimes_R C$, we have $\tau(y)xy \in E \otimes_R C$. Moreover, the following identity holds:

$$\mathrm{pf}_\tau(\tau(y)xy) = \mathrm{nrd}_{\Lambda/R}(y)\,\mathrm{pf}_\tau(x).$$

In particular, $\mathrm{nrd}_{\Lambda/R}(\tau(x)) = \mathrm{nrd}_{\Lambda/R}(x)$ *for all* $x \in \Lambda$.

Proof. To prove the first claim we may work over fields, due to Corollary B.3.5. However, over fields this is the well-known statement (2.2) in [**KMRT98**].

The same proof works for the second statement, as soon as we have verified that $\tau(y)xy \in E \otimes_R C$. Both sides are polynomial functions on $\Lambda \times E$. If they agree over all fields then they agree as polynomials (Cor. B.3.5). However, over fields this is the result (2.13) in [**KMRT98**].

It remains to show that $\tau(y)xy \in E \otimes_R C$. We can write $y = \sum_i u_i \otimes c_i$ for certain $u_i \in \Lambda$ and $c_i \in C$. The claim is linear in x, hence we may assume $x = e \otimes c$ with $e \in E$ and $c \in C$. We calculate

$$\begin{aligned}\tau(y)xy &= \sum_{i,j} \tau(u_i)eu_j \otimes cc_ic_j \\ &= \sum_i \tau(u_i)eu_i \otimes cc_i^2 + \sum_{i<j} (\tau(u_i)eu_j + \tau(u_j)eu_i) \otimes cc_ic_j,\end{aligned}$$

and we see that $\tau(y)xy$ is in $E \otimes_R C$ since $\tau(u_i)eu_i$ and $\tau(u_i)eu_j + \tau(u_j)eu_i$ are elements of E.

□

III.4.11. *Fixed point groups of involutions.* Let R be a Dedekind ring with quotient field k. Moreover, let A be a finite dimensional k-algebra with involution τ. Recall that we always assume an involution τ to be k-linear. We choose a τ-stable R-order Λ in A. Note that Λ is a finitely generated and projective R-module.

We consider the unit group scheme GL_Λ as defined in III.3.3. The involution τ induces an automorphism τ^* of the group GL_Λ, defined by

$$\tau^* := \mathrm{inv} \circ \tau_{|\,\mathrm{GL}_\Lambda}$$

where $\mathrm{inv} : \mathrm{GL}_\Lambda \to \mathrm{GL}_\Lambda$ denotes the inversion. Here we use that τ is k-linear, otherwise $\tau : \Lambda_a \to \Lambda_a$ is no morphism of schemes. Consider the group scheme $G := \mathrm{GL}_\Lambda^{\tau^*}$ of τ^*-fixed points, this is

$$G(C) := \{\, x \in (\Lambda \otimes_R C)^\times \mid \tau^*(x) = x \,\} = \{\, x \in (\Lambda \otimes_R C)^\times \mid \tau(x)x = 1 \,\}$$

for any commutative R-algebra C. We usually write $G = G(\Lambda, \tau)$ for the group scheme of fixed points. We end this section establishing a criterion for the smoothness of these group schemes.

III.4.12. *The Lie algebra of the fixed point group.* Keep the notation of the previous paragraph III.4.11. We give a description of the Lie algebra of the fixed point group $G := G(\Lambda, \tau)$. Let C be a commutative R-algebra. The C-points of the Lie algebra of G can be described as

$$\mathrm{Lie}(G)(C) = \{\, x \in \Lambda \otimes_R C \mid \tau(x) = -x \,\}.$$

This claim can be easily verified using the same identification as in III.3.4. Note that the derivative of $\tau^* : \mathrm{GL}_\Lambda \to \mathrm{GL}_\Lambda$ is $x \mapsto -\tau(x)$.

III.4.13. Definition. Let $E := \mathrm{Sym}(\Lambda, \tau)$ be the R-module of τ-symmetric elements in Λ. We say that the order Λ is *τ-smooth* if the R-linear map $s : \Lambda \to E$ which maps x to $\tau(x) + x$ is surjective.

III.4.14. *Remark.* Note that τ-smoothness is a local property. Let $\mathfrak{p} \subseteq R$ be a prime ideal and let $R_\mathfrak{p}$ be the localisation of R with respect to $\mathfrak{p}$. Moreover, let $\hat{R}_\mathfrak{p}$ denote the $\mathfrak{p}$-adic completion of $R_\mathfrak{p}$, which is a flat R-module. Corollary III.4.8 shows that $\mathrm{Sym}(\Lambda \otimes_R \hat{R}_\mathfrak{p}, \tau) = \mathrm{Sym}(\Lambda, \tau) \otimes_R \hat{R}_\mathfrak{p}$. Therefore the map $s : \Lambda \to \mathrm{Sym}(\Lambda, \tau)$ is surjective if and only if for every prime ideal $\mathfrak{p}$ the local maps $\hat{s}_\mathfrak{p} : \Lambda \otimes \hat{R}_\mathfrak{p} \to \mathrm{Sym}(\Lambda \otimes \hat{R}_\mathfrak{p}, \tau)$ are surjective (use, for instance, (5.2) in [**Rei03**]).

III.4.15. Theorem. *Let R be a Dedekind ring with quotient field k, let A be a finite dimensional k-algebra with involution τ, and let $\Lambda \subset A$ be a τ-stable R-order. If Λ is τ-smooth then the group scheme $G(\Lambda, \tau)$ is smooth over R.*

Proof. We set $G := G(\Lambda, \tau)$. Let C be a commutative R-algebra with an ideal $I \subseteq C$ such that $I^2 = 0$. We have to show that the canonical map $G(C) \to G(C/I)$ is surjective. Take $\bar{y} \in G(C/I)$. Since the unit group scheme GL_Λ is smooth (see III.3.3), we find $y \in \mathrm{GL}_\Lambda(C) = (\Lambda \otimes C)^\times$ which maps to $\bar{y}$ modulo I. Since $\bar{y}$ is in the fixed point group of τ^*, this implies that

$$\tau(y)y = 1 + \rho$$

with some $\rho \in \Lambda \otimes I$.

We consider $E := \mathrm{Sym}(\Lambda, \tau)$ and we obtain $\tau(y)y \in E \otimes_R C$ by Lemma III.4.10. Consequently, there is $u \in \Lambda \otimes_R C$ such that $\tau(u) + u = y$. Moreover, we have $1 \in E$, thus there is some $v \in \Lambda \otimes_R C$ with $\tau(v) + v = 1$. We deduce that $\rho = \tau(u-v) + (u-v)$ is an element in $E \otimes_R C$, and thus

$$\rho \in E \otimes_R C \cap \Lambda \otimes_R I = E \otimes_R I$$

(use Lemma B.1.9 combined with Lemma III.4.6).

As last step we use once again that Λ is τ-smooth and deduce that there is some $w \in \Lambda \otimes I$ with $\rho = \tau(w) + w$. We put $y' := y(1 - w)$ which is congruent $\bar{y}$ modulo I and satisfies

$$\tau(y')y' = (1 - \tau(w))\tau(y)y(1 - w) = (1 - \tau(w))(1 + \rho)(1 - w) = 1 + \rho - \tau(w) - w = 1.$$

Therefore $y' \in G(C)$ and y' maps to $\bar{y} \in G(C/I)$ under the canonical map. $\square$

III.4.16. *Example.* We consider an example of a τ-smooth order, which will be treated in more detail in Lemma V.1.8. Let D be a quaternion algebra defined over the field of rational numbers $\mathbb{Q}$. There is a canonical involution of symplectic type $\tau_c : D \to D$, which is usually called the conjugation. Let $\Lambda \subseteq D$ be a $\mathbb{Z}$-order and note that Λ is indeed τ_c-stable. Since $\tau_c(x) + x = \mathrm{trd}_D(x)$, we see that Λ is smooth if and only if it is τ_c-smooth. Suppose Λ is a maximal order, then we conclude with III.3.11 that Λ

is smooth and τ_c-smooth. The associated fixed point group is the special linear group over Λ.

III.5. Involutions and Non-abelian Galois Cohomology

III.5.1. Let R be a Dedekind ring with quotient field k, as usual we assume that the characteristic of k is not 2. Let A be a finite dimensional k-algebra with an involution τ. We consider a τ-stable R-order Λ in A and the induced automorphism τ^* on the group schemes GL_Λ and SL_Λ defined over R (cf. III.3.3 and III.3.8).

The purpose of this section is to study the non-abelian Galois cohomology sets $H^1(\tau^*, \mathrm{SL}_\Lambda(C))$ and $H^1(\tau^*, \mathrm{GL}_\Lambda(C))$ for commutative R-algebras C. Two tools will enable us to say something about the cohomology sets in question. The first is the pfaffian which turns out to be essentially a map on the cohomology (cf. III.5.3). The second tool is the theory of hermitian forms which works nicely if we assume the order to be τ-smooth.

III.5.2. *Fixed point groups and symplectic involutions.* Let A be a central simple k-algebra and let τ be an involution of symplectic type. We call the fixed point group scheme $G = G(\Lambda, \tau)$ the *symplectic group* associated with τ.

Let $x \in G(C)$ for any commutative R-algebra C. We see from $\tau(x)x = 1$ and Lemma III.4.10 that

$$\mathrm{nrd}_{\Lambda/R}(x) = \mathrm{pf}_\tau(\tau(x)x) = \mathrm{pf}_\tau(1) = 1.$$

This means that the reduced norm restricts to the trivial character on the fixed point group.

III.5.3. *The cohomological pfaffian.* Assume that A is a central simple k-algebra and that τ in an involution of symplectic type. Let C be a commutative R-algebra and assume that C is flat as R-module.

A cocycle $b \in Z^1(\tau^*, \mathrm{GL}_\Lambda(C))$ is an element of $(\Lambda \otimes C)^\times$ which satisfies $b\,{}^{\tau^*}b = 1$, or equivalently $b = {}^{\tau}b$. In other words $Z^1(\tau^*, \mathrm{GL}_\Lambda(C)) = \mathrm{Sym}(\Lambda \otimes_R C, \tau) \cap \mathrm{GL}_\Lambda(C)$. The assumption that C is flat combined with Corollary III.4.8 yields that

$$\mathrm{Sym}(\Lambda \otimes_R C, \tau) = E \otimes_R C.$$

Therefore we can apply the pfaffian associated with τ to cocycles in $Z^1(\tau^*, \mathrm{GL}_\Lambda(C))$. Moreover, if b and c are cohomologous cocycles, there is $y \in \mathrm{GL}_\Lambda(C)$ such that

$$b = {}^{\tau}ycy.$$

It follows from Lemma III.4.10 that $\mathrm{pf}_\tau(b) = \mathrm{nrd}_{\Lambda/R}(y)\,\mathrm{pf}_\tau(c)$. Therefore the pfaffian defines a morphism of pointed sets

$$\mathrm{pf}_\tau : H^1(\tau^*, \mathrm{GL}_\Lambda(C)) \to C^\times / \mathrm{nrd}_{\Lambda/R}(\mathrm{GL}_\Lambda(C)).$$

By the same reasoning we obtain a morphism of pointed sets

$$\mathrm{pf}_\tau : H^1(\tau^*, \mathrm{SL}_\Lambda(C)) \to \{\, x \in C^\times \mid x^2 = 1 \,\}.$$

For simplicity we introduce the notation $C^{(2)} := \{\, x \in C^\times \mid x^2 = 1 \,\}$ and we define $C_\Lambda^\times := \mathrm{nrd}_{\Lambda/R}(\mathrm{GL}_\Lambda(C))$.

III.5.4. Proposition (The cohomological diagram for symplectic involutions). *Let A be a central simple k-algebra and let τ be an involution of symplectic type on A. For every commutative R-algebra C which is flat as R-module, there is a commutative diagram of pointed sets with exact rows.*

$$\begin{tikzcd} C^{(2)} \cap C_\Lambda^\times \arrow[r, "\delta"] \arrow[d, equal] & H^1(\tau^*, \mathrm{SL}_\Lambda(C)) \arrow[r, "j_*"] \arrow[d, "\mathrm{pf}_\tau"'] & H^1(\tau^*, \mathrm{GL}_\Lambda(C)) \arrow[r, "\mathrm{nrd}"] \arrow[d, "\mathrm{pf}_\tau"'] & C^\times/(C_\Lambda^\times)^2 \arrow[d, equal] \\ C^{(2)} \cap C_\Lambda^\times \arrow[r] & C^{(2)} \arrow[r] & C^\times/C_\Lambda^\times \arrow[r, ".^2"] & C^\times/(C_\Lambda^\times)^2 \end{tikzcd}$$

The map δ is injective and the lower row is an exact sequence of groups. Here j_* denotes the map induced by the inclusion $j : \mathrm{SL}_\Lambda(C) \to \mathrm{GL}_\Lambda(C)$.

Proof. The short exact sequence of groups

$$1 \longrightarrow \mathrm{SL}_\Lambda(C) \xrightarrow{j} \mathrm{GL}_\Lambda(C) \xrightarrow{\mathrm{nrd}} C_\Lambda^\times \longrightarrow 1$$

is even an exact sequence of groups with τ^*-action, where τ^* acts on $C_\Lambda^\times$ by inversion. Consider the initial segment of the associated long exact sequence in the cohomology (see Prop. I.38 [**Ser64**]):

$$1 \longrightarrow \mathrm{SL}_\Lambda(C)^{\tau^*} \xrightarrow{j} \mathrm{GL}_\Lambda(C)^{\tau^*} \xrightarrow{\mathrm{nrd}} C_\Lambda^\times \cap C^{(2)} \longrightarrow \dots$$

It follows directly from III.5.2 that $\mathrm{SL}_\Lambda(C)^{\tau^*} \xrightarrow{j} \mathrm{GL}_\Lambda(C)^{\tau^*}$ is bijective. Thus the long exact sequence takes the form

$$1 \longrightarrow C_\Lambda^\times \cap C^{(2)} \xrightarrow{\delta} H^1(\tau^*, \mathrm{SL}_\Lambda(C)) \longrightarrow H^1(\tau^*, \mathrm{GL}_\Lambda(C)) \longrightarrow H^1(\tau^*, C_\Lambda^\times).$$

It is easy to see that $H^1(\tau^*, C_\Lambda^\times)$ is $C_\Lambda^\times/(C_\Lambda^\times)^2$ which is a subgroup of $C^\times/(C_\Lambda^\times)^2$. Hence we simply replace the last term by $C^\times/(C_\Lambda^\times)^2$. This yields the upper row of the diagram. It is an easy exercise to verify that the lower row is an exact sequence of groups.

It remains to verify the commutativity of the rectangles. The middle one is obviously commutative by definition of the pfaffian in the cohomology. For the last rectangle we simply use that $\mathrm{pf}_\tau(g)^2 = \mathrm{nrd}(g)$ for all $g \in Z^1(\tau^*, \mathrm{GL}_\Lambda(C))$ by the construction of the pfaffian.

Consider the first rectangle. We recall the definition of the connecting morphism δ: Given $c \in C_\Lambda^\times \cap C^{(2)}$, we can find an element $g \in \mathrm{GL}_\Lambda(C)$ such that $\mathrm{nrd}_{\Lambda/R}(g) = c$, then $\delta(c)$ is defined to be the class of $g^{-1}\,{}^{\tau^*}\!g$. The pfaffian of $g^{-1}\,{}^{\tau^*}\!g$ is

$$\mathrm{pf}_\tau(g^{-1}\,{}^{\tau^*}\!g) = \mathrm{nrd}(g)^{-1} = c^{-1} = c$$

(see Lemma III.4.10). This proves the commutativity of the first rectangle.

Finally, note that δ is injective since $\mathrm{pf}_\tau \circ \delta$ is injective. □

III.5.5. Corollary. *In the setting of the previous proposition: $x \in H^1(\tau^*, \mathrm{GL}_\Lambda(C))$ lies in the image of j_* if and only if $\mathrm{pf}_\tau(x)$ lies in the image of the canonical map $C^{(2)} \to C^\times/C_\Lambda^\times$.*

Proof. Let $\alpha : C^{(2)} \to C^\times/C_\Lambda^\times$ denote the canonical map. Suppose the class $x \in H^1(\tau^*, \mathrm{GL}_\Lambda(C))$ is in the image of j_*, then we obtain immediately that $\mathrm{pf}_\tau(x)$ lies in the image of α.

Conversely, suppose $\mathrm{pf}_\tau(x) = \alpha(u)$ for some $u \in C^{(2)}$. Then the diagram shows that $\mathrm{nrd}_{\Lambda/R}(x) = 1$ in $C^\times/(C_\Lambda^\times)^2$ and therefore x lies in the image of j_*. □

III.5.6. *Twisting involutions.* Let A be a finite dimensional k-algebra with an involution τ and let Λ be a τ-stable R-order.

Given an element $b \in \mathrm{Sym}(\Lambda, \tau) \cap \Lambda^\times$, we can twist the involution τ with b. More precisely, we define $\tau|b : A \to A$ by $x \mapsto b\,{}^\tau x b^{-1}$. It is easily verified that this is again an involution on A, and since $b \in \Lambda^\times$, the order Λ is $\tau|b$-stable.

Suppose Λ is τ-smooth, we claim that Λ is $\tau|b$-smooth as well. Take an element $y \in \mathrm{Sym}(\Lambda, \tau|b)$, this is $y = b\,{}^\tau y b^{-1}$. Consequently, $yb \in \mathrm{Sym}(\Lambda, \tau)$ and by τ-smoothness there is an element $z \in \Lambda$ which satisfies ${}^\tau z + z = yb$. The element b is a unit in Λ, hence we may write $z = wb$ for $w = zb^{-1} \in \Lambda$ and it follows that ${}^{\tau|b}w + w = y$. We have shown that Λ is $\tau|b$-smooth.

Two further comments are useful. First, note that if A is a central simple k-algebra and τ is of symplectic type, then $\tau|b$ is again of symplectic type. Second, for all $b \in \mathrm{Sym}(\Lambda, \tau) \cap \Lambda^\times$ we have $(\tau|b)^* = \mathrm{int}(b) \circ \tau^*$ on the group scheme GL_Λ. Since $b = {}^\tau b$ is equivalent to $b\,{}^{\tau^*}b = 1$, such an element b is a cocycle for $H^1(\tau^*, \Lambda^\times)$. If we now twist τ^* with the cocycle b (cf. II.3.3), we obtain

$$\tau^*|b := \mathrm{int}(b) \circ \tau^* = (\tau|b)^*.$$

III.5.7. *Hermitian forms and τ-smoothness.* The notion of τ-smoothness is related to the theory of even hermitian forms. Let Λ be a τ-stable order and let M be a finitely generated and projective right Λ-module. A hermitian form h (or more precisely a 1-hermitian form) with respect to τ on M is said to be *even* if there is a τ-sesquilinear form $s : M \times M \to \Lambda$ such that $h = s + s^*$. Here s^* is the sesquilinear form defined by

$$s^*(x, y) := {}^\tau s(y, x).$$

Now, it follows immediately that Λ is τ-smooth if and only if every hermitian form on Λ (considered as right Λ-module) is even. This is useful since even hermitian forms can be handled easier than arbitrary hermitian forms.

III.5.8. *Hermitian forms and non-abelian Galois cohomology.* Recall that we defined the automorphism

$$\tau^* : \mathrm{GL}_\Lambda \to \mathrm{GL}_\Lambda \quad \text{by} \quad \tau^* = \mathrm{inv} \circ \tau.$$

(cf. III.4.11). Let C be a commutative R-algebra, we consider the non-abelian Galois cohomology set $H^1(\tau^*, \mathrm{GL}_\Lambda(C))$. A cocycle is an element $a \in (\Lambda \otimes_R C)^\times$ which satisfies $a\,{}^{\tau^*}a = 1$, this is $a = {}^\tau a$. This means that we have

$$Z^1(\tau^*, \mathrm{GL}_\Lambda(C)) = (\Lambda \otimes_R C)^\times \cap \mathrm{Sym}(\Lambda \otimes_R C, \tau).$$

We may consider this set as the set of regular hermitian forms with respect to τ on the right $\Lambda \otimes_R C$-module $\Lambda \otimes_R C$. Moreover, two cocycles a and b are cohomologous if and only if there is $u \in (\Lambda \otimes_R C)^\times$ such that ${}^\tau uau = b$. We conclude that $H^1(\tau^*, \mathrm{GL}_\Lambda(C))$ classifies the regular hermitian forms on $\Lambda \otimes_R C$ with respect to the involution τ.

Suppose that R is a complete discrete valuation ring, then we can use the observation above to prove (a slight variation of) a theorem due to Fainsilber and Morales. The proof is essentially the one given by Fainsilber and Morales in [**FM99**], however we assume that the order is τ-smooth to eliminate the restriction on the residual characteristic of R.

III.5.9. Theorem (Fainsilber, Morales [**FM99**]). *Let k be a field which is complete for a discrete valuation and let R be its valuation ring. Let A be a central simple k-algebra with involution[2] τ. Suppose Λ is a τ-stable maximal R-order in A. If Λ is τ-smooth, then the canonical map*

$$j_* : H^1(\tau^*, \Lambda^\times) \to H^1(\tau^*, A^\times)$$

is injective.

Proof. Suppose we have cocycles $a, b \in Z^1(\tau^*, \Lambda^\times)$ which are equivalent over $A^\times$, i.e. there is $u \in A^\times$ such that ${}^\tau uau = b$. Consider the right Λ-module Λ^2 equipped with the hermitian form $h = \langle a, -b \rangle$, this means $h((x,y),(x',y')) = {}^\tau xax' - {}^\tau yby'$. Note that this is an even hermitian form since we assumed Λ to be τ-smooth. Define the Λ-linear map $f : \Lambda^2 \to A$ by $f(x,y) := x - uy$. The image of f is isomorphic to a right ideal in Λ (multiply with suitable $m \in R \setminus \{0\}$) and since maximal orders are hereditary (cf. [**Rei03**, (17.3)]), we see that the image of f is a projective Λ-module. Therefore the kernel M of f is a direct summand of Λ^2. There is a canonical isomorphism of right Λ-modules $M \cong \Lambda \cap u\Lambda = \{ x \in \Lambda \mid u^{-1}x \in \Lambda \}$. The latter is a right ideal in Λ and hence we have $\Lambda \cap u\Lambda = \rho\Lambda$ for some $\rho \in \Lambda$ (see (17.3) in [**Rei03**]). Take $m \in R \setminus \{0\}$ such that $mu^{-1} \in \Lambda$, then $m\Lambda \subseteq \rho\Lambda$. We conclude that $\rho \in A^\times$ and therefore $M \cong \Lambda$ as right Λ-modules.

We claim that $M = M^\perp$ in (Λ^2, h). To verify this we take $(x,y),(x',y') \in M$, this means $x = uy$ and $x' = uy'$, and we find

$$h((x,y),(x',y')) = {}^\tau xax' - {}^\tau yby' = {}^\tau y\, {}^\tau uauy' - {}^\tau yby' = 0.$$

Thus $M \subseteq M^\perp$. Conversely, if we have $(x',y') \in M^\perp$, then

$$h((\rho, u^{-1}\rho),(x',y')) = {}^\tau \rho ax' - {}^\tau \rho\, {}^\tau u^{-1}by' = {}^\tau \rho a(x' - uy').$$

We deduce $(x',y') \in M$ since ρ is a unit in A.

By Knebusch's Lemma (Ch. 7, Lemma 3.7 (ii) in [**Sch85**]) we can conclude that (Λ^2, h) is isometric to the hyperbolic space $\mathcal{H}(M) \cong \mathcal{H}(\Lambda)$. Here we use that the form h is even. By the same argument again we see that $(\Lambda^2, \langle a, -a \rangle) \cong \mathcal{H}(\Lambda)$. So we have $(\Lambda^2, \langle a, -a \rangle) \cong (\Lambda^2, \langle a, -b \rangle)$ and the theorem would follow from Witt cancellation. Indeed, Λ is a semilocal ring (see [**Rei03**, (17.5)]) and Λ has f-rank 1 as a Λ-module. Finally all involved hermitian form are even and thus Witt cancellation is possible (see VI. Cor. (5.7.4) in [**Knu91**]). $\square$

III.6. Integration: A Theorem of Weil

III.6.1. Let R be a complete discrete valuation ring and let F be its field of fractions. The unique maximal ideal of R will be denoted $\mathfrak{p}$, moreover we fix a uniformizer $\pi \in \mathfrak{p}$. We assume that the residue class field $k = R/\mathfrak{p}$ is a finite field with q elements. Furthermore, we obtain the corresponding valuation $|\cdot|$ on F normalised such that $|\pi| = q^{-1}$.

Let $\mathfrak{X}$ be a smooth affine scheme of finite type over R. André Weil observed that integration of a gauge form over the R-rational points $\mathfrak{X}(R)$ is essentially the number of k-rational points of the scheme $\mathfrak{X}$ (cf. Thm. 2.2.5 [**Wei82**]). We will give a short account based on the treatment of Oesterlé [**Oes84**]. Another source for Weil's theorem written in modern terminology is [**Bat99**].

[2] The theorem and its proof are correct for involutions of the second kind as well!

III.6.2. *Canonical measure.* Let $\mathfrak{X}$ be a smooth affine scheme of finite type over R and assume that the fibres are of dimension d. The set $\mathfrak{X}(F)$ of F-rational points of $\mathfrak{X}$ is an F-analytic manifold of dimension d and $\mathfrak{X}(R)$ is an open and compact submanifold. There is a unique Radon measure ν on $\mathfrak{X}(R)$ such that the fibres of the canonical map $\mathfrak{X}(R) \to \mathfrak{X}(R/\mathfrak{p}^n)$ have measure q^{-dn} (cf. 2.2 in [**Oes84**]). This measure is called the canonical measure on $\mathfrak{X}(R)$ and we have

$$\int_{\mathfrak{X}(R)} \nu = \frac{|\mathfrak{X}(k)|}{q^d}.$$

III.6.3. *Volume density.* Let G be a smooth R-group scheme of dimension d. We write $\mathfrak{g} = \mathrm{Lie}(G)(F)$ for the Lie algebra of G over F. Let $B : \mathfrak{g} \times \mathfrak{g} \to F$ be a non-degenerate F-bilinear form. The form B defines a left $G(F)$-invariant density $|\mathrm{vol}_B|$ on the analytic manifold $G(F)$ which we call the volume density. The volume density is uniquely determined by the relation $|\mathrm{vol}_B|\,(e_1 \wedge \cdots \wedge e_d) = |\det(B(e_i, e_j))|^{1/2}$ for all $e_1, \ldots, e_d \in \mathfrak{g}$.

We fix the unique additive Haar measure on F such that R has measure 1. Via this measure the volume density on $G(F)$ induces a left Haar measure on the locally compact group $G(F)$ and we will denote this measure still with $|\mathrm{vol}_B|$. As a next step we want to compare this measure with the canonical measure ν on $G(R)$.

III.6.4. *The modulus factor.* Let $B : \mathfrak{g} \times \mathfrak{g} \to F$ be a non-degenerate F-bilinear form. We write $\mathfrak{g}_R = \mathrm{Lie}(G)(R)$ for the R-points of the Lie algebra, this is a full R-lattice in $\mathfrak{g}$. Since R is a principal ideal domain the module $\mathfrak{g}_R$ has an R-basis, say $e_1, \ldots, e_d$. We define the *modulus factor* $m(B)$ of B by

$$m(B) := |\det(B(e_i, e_j))|^{1/2}.$$

This definition is independent of the choice of the basis. This can be verified as follows: take another R-basis $e'_1, \ldots, e'_d$ of $\mathfrak{g}_R$, then $\det(B(e'_i, e'_j)) = u^2 \det(B(e_i, e_j))$ for a unit $u \in R^\times$. The claim is immediate since $|u| = 1$.

Suppose that B restricts to a form $\mathfrak{g}_R \times \mathfrak{g}_R \to R$ and suppose further that this form induces an isomorphism of R-modules $\mathfrak{g}_R \to \mathrm{Hom}_R(\mathfrak{g}_R, R)$, then the modulus factor is one, i.e. $m(B) = 1$.

III.6.5. Theorem. *Let G be a smooth group scheme defined over R and of dimension d. Further, let $B : \mathfrak{g} \times \mathfrak{g} \to F$ be a non-degenerate bilinear form. The following identity of measures on $G(R)$ holds:*

$$|\mathrm{vol}_B| = m(B)\,\nu.$$

In particular,

$$\int_{G(R)} |\mathrm{vol}_B| = m(B) \frac{|G(k)|}{q^d}.$$

Proof. Take a basis $e_1, \ldots, e_d$ of $\mathfrak{g}_R$. It follows from Proposition 2.5 in [**Oes84**] that the measure μ induced by the left invariant differential form $\omega_e = e_1 \wedge \cdots \wedge e_d$ agrees with the canonical measure on $G(R)$. By definition of the volume density we have $|\mathrm{vol}_B| = m(B)\mu$ and the claim follows. $\square$

CHAPTER IV

A Reformulation of Harder's Gauß-Bonnet Theorem

IV.1. Introduction

IV.1.1. In order to use the machinery developed in Chapter II there is an important ingredient missing. No matter which Lefschetz number we try to compute, we will finally be faced with the problem of computing the Euler characteristic of an arithmetic group. In general this can be a difficult task. Luckily, there is a method to calculate the Euler characteristic of a torsion-free arithmetic subgroup in a semi-simple group: Harder's *Gauß-Bonnet Formula* (see [**Har71**]). In this chapter we will give a reformulation of Harder's theorem which hinges on the notion of smooth group scheme. Using Grothendieck's concept of smoothness combined with Weil's theorem (cf. Section III.6), we will obtain a quite explicit formula for the Euler characteristic of a congruence group in a smooth arithmetic group scheme.

IV.1.2. *Assumption.* Throughout the chapter all Lie groups are assumed to have only a finite number of connected components.

IV.2. The Gauß-Bonnet Form

IV.2.1. The purpose of this section is to introduce some notation from differential geometry. In particular, we give a concise definition of the Gauß-Bonnet form; the motivation is to stress that it is uniquely determined from the Riemann metric and the chosen orientation. This is necessary since almost every book on differential geometry uses different sign conventions for the curvature tensor and forms. We decided to use most conventions from Helgason's book [**Hel78**] and we complement them with notation from Hicks [**Hic64**].

IV.2.2. *Notation.* Let M be a smooth real manifold of dimension n. The $\mathbb{R}$-algebra of smooth real valued functions on M is denoted by $C^\infty(M)$. The tangent space at a point $p \in M$ will be denoted by T_pM. We write $\mathfrak{X}(M)$ for the $C^\infty(M)$-module of smooth vector fields on M. For $1 \leq k \leq n$ the space of differential k-forms on M is denoted by $\Omega^k(M)$. The $C^\infty(M)$-module of smooth sections of the density bundle is denoted by $\mathrm{Vol}(M)$.

Suppose that M is orientable and let $\omega \in \Omega^n(M)$ be an orientation form, that is, a top-degree form without zeros. Then we can associate with ω a density $|\omega|$, and the map $f\omega \mapsto f|\omega|$ defines an isomorphism of $C^\infty(M)$-modules $\Omega^n(M) \to \mathrm{Vol}(M)$. This isomorphism is independent of ω up to orientation. More precisely, if $\omega' = f\omega$ with $f > 0$ on M, then $|\omega'| = f|\omega|$ and we get the same isomorphism. In other words, an orientation $\mathfrak{o}$ of M induces a uniquely determined isomorphism $j_{\mathfrak{o}} : \Omega^n(M) \to \mathrm{Vol}(M)$ of $C^\infty(M)$-modules. Note further that $j_{-\mathfrak{o}} = -j_{\mathfrak{o}}$.

IV.2.3. Let (M, g) be a pseudo-Riemannian manifold of dimension n. Let $|\mathrm{vol}_g|$ denote the volume density associated with the metric g. This means, if $p \in M$ is a point and

$e_1, \dots, e_n$ is a basis of the tangent space T_pM, then

$$|\mathrm{vol}_g|\,(e_1 \wedge \cdots \wedge e_n)_p = |\det(g_p(e_i, e_j))|^{1/2}.$$

Suppose that (M, g) is an oriented Riemannian manifold. The volume form of M associated with g will be denoted by $\mathrm{vol}_g \in \Omega^n(M)$. Clearly, the volume density $|\mathrm{vol}_g|$ is the density associated with vol_g.

IV.2.4. *Curvature.* Let (M, g) be a pseudo-Riemannian manifold. We use the standard notation $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ for the affine connection induced by the metric g. The affine connection yields the curvature tensor

$$\begin{aligned} R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \quad &\text{defined by} \\ R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,& \end{aligned}$$

where we use the sign convention of Helgason [**Hel78**, Ch. I, §8] and Hicks [**Hic64**, Ch. 5].

IV.2.5. Lemma. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a local isometry of pseudo Riemann manifolds. Let ∇^i (resp. R^i) denote the connection (resp. curvature) on M_i. Then the pull-backs are natural:*

(i) $f^*(|\mathrm{vol}_{g_2}|) = |\mathrm{vol}_{g_1}|$,
(ii) $f^*(R^2) = R^1$, *and*
(iii) $\nabla^1_{f^*(X)} f^*(Y) = f^*(\nabla^2_X Y)$ *for all vector fields* $X, Y \in \mathfrak{X}(M_2)$.

Proof. Volume density, affine connection and curvature are local objects, so we may assume that f is an isometry. Since all three objects are uniquely determined by the metric, the claim is obvious. $\square$

IV.2.6. *The Gauß-Bonnet form.* Let (M, g) be an oriented Riemann manifold of dimension n. We will define the Gauß-Bonnet form Ω_M on M. If the dimension n is odd, we define $\Omega_M := 0$. Now assume that the dimension $\dim(M) = n = 2p$ is even. Let R be the curvature tensor on M and let $e_1, \dots, e_n$ be a local orthonormal frame of positive orientation on some open set $U \subset M$. We define the associated curvature forms $R_{ij} \in \Omega^2(U)$ for all $i, j \le n$. Let X, Y be vector fields on M and write $R(X, Y)e_j$ with respect to the chosen frame

$$R(X, Y)e_j = \sum_{i=1}^{n} R_{ij}(X, Y)e_i,$$

this determines the smooth forms $R_{ij} \in \Omega^2(U)$ uniquely. Moreover, we have the relation $R_{ij} = -R_{ji}$ for all i, j (cf. [**Hic64**, p. 72]). We define the *Gauß-Bonnet* form on U by

$$\begin{aligned} \Omega_U :=& \frac{1}{(2\pi)^p} \operatorname{Pf}(R_{ij}) \\ =& \frac{1}{(4\pi)^p p!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) R_{\sigma(1),\sigma(2)} \wedge R_{\sigma(3),\sigma(4)} \wedge \cdots \wedge R_{\sigma(n-1),\sigma(n)}. \end{aligned}$$

Here Pf denotes the pfaffian of skew-symmetric matrices. One can show that this definition is independent of the local frame (as long as one does not change the orientation) and hence this defines a global form Ω_M of top-degree on M (cf. [**Hic64**, p. 115]). Note that the Gauß-Bonnet form depends on the chosen orientation, more precisely, the opposite orientation yields the negative Gauß-Bonnet form.

IV.2.7. Lemma. *Let (M_1, g_1) and (M_2, g_2) be oriented Riemann manifolds and let $f : M_1 \to M_2$ be an orientation preserving, local isometry. Then $f^*(\mathrm{vol}_{g_2}) = \mathrm{vol}_{g_1}$ and $f^*(\Omega_{M_2}) = \Omega_{M_1}$.*

Proof. Both forms are local objects and uniquely determined by the orientation and the metric. □

IV.2.8. *The Gauß-Bonnet density.* Let (M, g) be an orientable Riemann manifold of dimension n. The choice of an orientation $\mathfrak{o}$ on M induces an isomorphism $j_{\mathfrak{o}}$ between the $C^\infty(M)$-module of n-forms $\Omega^n(M)$ and the module of densities $\mathrm{Vol}(M)$ (see IV.2.2). The image of the Gauß-Bonnet form Ω_M under this isomorphism is called the *Gauß-Bonnet density*, and is denoted by Ω_M^+. The Gauß-Bonnet density is independent of the chosen orientation. This can be seen as follows: The Gauß-Bonnet form changes the sign, when we change the orientation of the manifold. Similarly, we have $j_{-\mathfrak{o}} = -j_{\mathfrak{o}}$ and the claim follows.

Having this in mind, it is natural to ask for a definition of the Gauß-Bonnet density on an unoriented Riemannian manifold (M, g). One way to achieve this is to work locally since locally every manifold is orientable. As before one obtains the following:

Lemma. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a local isometry of Riemannian manifolds, then $f^*(\Omega_{M_2}^+) = \Omega_{M_1}^+$.

Another way to define the Gauß-Bonnet density is to work with the orientable double cover instead. Consider the orientable double cover $\pi : \widetilde{M} \to M$ with the Riemann metric $\tilde{g} := \pi^*(g)$. Let $\tau : \widetilde{M} \to \widetilde{M}$ be the non-trivial deck transformation, this means, τ acts transitively on the fibres of π. Then $\tilde{g}$ is τ-invariant and so are the Gauß-Bonnet form and density on $\widetilde{M}$. Hence there is a unique density Ω_M^+ on M, such that $\pi^*(\Omega_M^+) = \Omega_{\widetilde{M}}^+$. This density is the Gauß-Bonnet density on M.

IV.3. Symmetric Spaces

IV.3.1. In this section we recall some basic facts about homogeneous spaces, in particular about Riemannian globally symmetric spaces. Recall that all Lie groups are assumed to have a finite number of connected components. Given a real Lie group G, we denote the (real) Lie algebra of G by $\mathfrak{g}$.

IV.3.2. Lemma. *Let G be a real Lie group and let $H \subseteq G$ be a closed Lie subgroup with Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$. Consider the homogeneous space $X = H\backslash G$ and the natural submersion $\pi : G \to X$; define $o := \pi(e)$. Suppose that $\mathfrak{p} \subseteq \mathfrak{g}$ is an $\mathrm{Ad}(H)$-stable complement of $\mathfrak{h}$ in $\mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Assume further, that B is an $\mathrm{Ad}(H)$-invariant non-degenerate bilinear form on $\mathfrak{p}$. Then there is a unique G-invariant pseudo Riemann metric Q on X, such that the tanget map $(T_e\pi)_{|\mathfrak{p}} : \mathfrak{p} \to T_oX$ is an isometry with respect to B and Q_o*

Proof. First note that Q is uniquely determined by Q_o since G acts transitively on X. Let α denote the inverse of $(T_e\pi)_{|\mathfrak{p}}$. Let $x = og^{-1} \in X$ for some $g \in G$, and take tanget vectors $v, w \in T_xX$. Then one defines

$$Q_x(v, w) := B(\alpha(T_x r_g v), \alpha(T_x r_g w)),$$

where r_g denotes the right multiplication with g on X. It is an easy exercise to check that this gives a well-defined G-invariant pseudo Riemann metric on X. □

IV.3.3. *Remark.* Suppose that X is a homogeneous G-space equipped with an invariant Riemann metric Q. Then it follows from Lemma IV.2.5 that the volume density $|\mathrm{vol}_Q|$ is G-invariant. Moreover, Lemma IV.2.8 implies that the Gauß-Bonnet density is G-invariant. We deduce that there is a constant factor $c_X \in \mathbb{R}$ such that

$$\Omega_X^+ = c_X \, |\mathrm{vol}_Q| \, .$$

IV.3.4. *Symmetric spaces.* We briefly recall some facts on symmetric spaces following Helgason [**Hel78**, Ch. IV]. A pair $(\mathfrak{g}, s)$ consisting of a real Lie algebra $\mathfrak{g}$ and an involutive automorphism s is said to be an *orthogonal symmetric Lie algebra* if the subalgebra $\mathfrak{k} := \mathfrak{g}^s$ of fixed points is compactly imbedded in $\mathfrak{g}$ (cf. [**Hel78**, IV.§3]). We usually write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the eigenspace for the eigenvalue -1 of the involution s.

Let (G, K) be a pair consisting of a *connected* real Lie group G with Lie algebra $\mathfrak{g}$ and a closed connected subgroup K. We say that (G, K) is associated with the orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ if $\mathfrak{k} = \mathfrak{g}^s$ is the Lie algebra of K. In this case, the homogeneous space $X = K\backslash G$ is a locally Riemannian symmetric space with respect to any G-invariant Riemann metric on X (cf. [**Hel78**, IV. Prop. 3.6]). Moreover, the affine connection (and hence the curvature tensor) on X is independent of the chosen G-invariant metric (cf. [**Hel78**, IV. Cor. 4.3]). In particular, we see that the Gauß-Bonnet density is independent of the chosen G-invariant metric.

Let $\pi : G \to X$ be the canonical submersion and put $o := \pi(e)$. The tangent map $T_e\pi$ induces an isomorphism $\mathfrak{p} \to T_oX$, and we will always identify $\mathfrak{p}$ with the tangent space T_oX. The curvature tensor R on X at the point o is given by

$$R_o(u, v)w = -[[u, v], w] \tag{12}$$

for all $u, v, w \in \mathfrak{p}$ (see [**Hel78**, IV. Thm. 4.2]).

IV.3.5. *Semisimple Lie groups and symmetric spaces of non-compact type.* Let G be a semisimple real Lie group having a finite centre. We write G^0 for the connected component of the identity. Let $K \subset G$ be a maximal compact subgroup, then $KG^0 = G$ and $K \cap G^0 = K^0$ (see A.3.4). Hence the embedding $G^0 \to G$ induces a diffeomorphism $K^0\backslash G^0 \to K\backslash G$, this means, we may assume that G is connected whenever we study the homogeneous space $K\backslash G$.

Let $\mathfrak{k}$ be the Lie algebra of K. We consider a Cartan decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{p}$ is the -1 eigenspace with respect to a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{k} = \mathfrak{g}^\theta$. Then $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie algebra and the pair (G^0, K^0) is associated. If G is not compact, the homogeneous space $K\backslash G$ is a Riemannian globally symmetric space of non-compact type with respect to any G-invariant Riemann metric (see [**Hel78**, VI. Thm. 1.1]). Moreover, the exponential map yields a diffeomorphism $\mathfrak{p} \to K\backslash G$. In general we do not exclude the case where G is compact, however, this is not the case of interest since the space $K\backslash G$ consists of a single point.

IV.3.6. *Complexifications.* Let G be a real Lie group. A *complexification* of G is a complex Lie group $G_{\mathbb{C}}$ together with an embedding of G as a closed subgroup such that the Lie algebra of $G_{\mathbb{C}}$ is the complexification of the Lie algebra of G. Note that not every Lie group has a complexification and a complexification is in general *not* unique up to isomorphism.

IV.3.7. *The compact dual group.* Let G be a connected real semisimple Lie group with finite centre which has a complexification $G_{\mathbb{C}}$. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}_{\mathbb{C}}$ be the complexification, i.e. the Lie algebra of $G_{\mathbb{C}}$. Moreover, let K be a maximal compact subgroup and consider the associated Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

The real vector space $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}$ is a real Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and is even a compact real form of $\mathfrak{g}_{\mathbb{C}}$ (cf. [**Kna02**, p. 360]). Let G_u be the unique connected (a priori virtual) Lie subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{u}$. Since the real semisimple Lie algebra $\mathfrak{u}$ is a compact form, the Lie group G_u is compact and thus closed in $G_{\mathbb{C}}$ (see IV, Thm. 4.69 in [**Kna02**]). Moreover, since K is connected, we see that $K \subset G_u$ is a subgroup.

We say that G_u is the *compact dual group* of G containing K with respect to the complexification $G_{\mathbb{C}}$. Note that the compact dual group G_u really depends on the chosen complexification $G_{\mathbb{C}}$.

IV.3.8. *The compact dual symmetric space.* We define a compact dual symmetric space in the setting of the previous paragraph IV.3.7. We choose any $\mathrm{Ad}_G(K)$-invariant, positive definite bilinear form B on $\mathfrak{p}$ and we consider the Riemannian symmetric space $X = K\backslash G$ with the G-invariant metric induced by B (see Lemma IV.3.2).

As a next step, we transfer the form B to the space $i\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$ in such a way that multiplication with i is an isometry. We can use this to define a G_u-invariant Riemann metric on the compact homogeneous space $X_u := K\backslash G_u$. Note that the pair $(\mathfrak{u}, \theta_u)$, where θ_u acts trivial on $\mathfrak{k}$ and acts with -1 on $i\mathfrak{p}$, is an orthogonal symmetric Lie algebra. Moreover, the pair (G_u, K) is associated with $(\mathfrak{u}, \theta_u)$ and hence the quotient $X_u := K\backslash G_u$ is a compact Riemannian globally[1] symmetric space.

We say that X_u is the *compact dual symmetric space* of X with respect to $G_{\mathbb{C}}$. We stress that the compact dual symmetric space X_u is in general not unique and depends on the chosen complexification $G_{\mathbb{C}}$.

IV.3.9. Definition. The *rank* $\mathrm{rk}(K)$ of a compact real Lie group K is the dimension of any maximal torus of K. The *rank* $\mathrm{rk}(\mathfrak{g})$ of a complex reductive Lie algebra $\mathfrak{g}$ is the dimension of any Cartan subalgebra. If K is a compact Lie group with Lie algebra $\mathfrak{k}$, then $\mathrm{rk}(K) = \mathrm{rk}(\mathfrak{k}_{\mathbb{C}})$ (cf. [**Kna02**, p. 254]).

IV.3.10. *Remark.* Recall the following theorem: Let U be a compact connected semisimple Lie group with Lie algebra $\mathfrak{u}$ and let T be a maximal torus in U with Lie algebra $\mathfrak{t} \subset \mathfrak{u}$. Let H be any real Lie group with Lie algebra $\mathfrak{h}$. A morphism of Lie algebras $\alpha : \mathfrak{u} \to \mathfrak{h}$ is the derivative of a homomorphism of Lie groups $U \to H$ if and only if $\alpha_{|\mathfrak{t}}$ is the derivative of a homomorphism $T \to H$.

We apply this in the following situation: Let G be a connected semisimple Lie group with finite centre and let K be a maximal compact subgroup. We assume that G has a complexification $G_{\mathbb{C}}$ and further that $\mathrm{rk}(K) = \mathrm{rk}(\mathfrak{g}_{\mathbb{C}})$. Thus a maximal torus T in K is also maximal in G_u. Let G'_u be another compact connected Lie group with Lie algebra $\mathfrak{u}$ and assume G'_u contains K. Then T is again a maximal torus in G'_u and hence $G_u \cong G'_u$ by the theorem mentioned above. This means that the compact dual group is unique up to isomorphism whenever $\mathrm{rk}(K) = \mathrm{rk}(\mathfrak{g}_{\mathbb{C}})$.

[1]Strictly speaking, we only know that it is a locally symmetric space. Since K is connected, one can use Ex. 10, p. 349 in [**Hel78**] to show that it is indeed globally symmetric.

IV.4. Harder's Gauß-Bonnet Theorem

IV.4.1. In this section we briefly recall Harder's Gauß-Bonnet Theorem. Before we do this, we remind the reader of the classical Gauß-Bonnet Theorem. Since we use the (non-standard) notion of Gauß-Bonnet density, we give a short proof connecting this notion with the classical Gauß-Bonnet form.

IV.4.2. Theorem (The Gauß-Bonnet Theorem). *Let M be a compact Riemann manifold and let Ω_M^+ be the Gauß-Bonnet density (IV.2.8), then*

$$\int_M \Omega_M^+ = \chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M.

Proof. If M is of odd dimension, then $\chi(M) = 0$ and the theorem follows since we defined $\Omega_M^+ = 0$. Moreover, if M is of even dimension and orientable, this is the classical Theorem due to Allendoerfer, Chern and Weil (see for instance [**Che44**] or [**Hic64**, p. 115]). Finally, if M is of even dimension but not orientable, then we take the orientable double cover $\pi : \widetilde{M} \to M$. Since $\Omega_{\widetilde{M}}^+ = \pi^*(\Omega_M^+)$, we obtain

$$2\chi(M) = \chi(\widetilde{M}) = \int_{\widetilde{M}} \Omega_{\widetilde{M}}^+ = 2\int_M \Omega_M^+.$$

The first equality is a special case of Theorem I.7.11. □

IV.4.3. Let G be a semisimple algebraic group defined over $\mathbb{Q}$ and let $\Gamma \subset G(\mathbb{Q})$ be a torsion-free arithmetic subgroup. Fix a maximal compact subgroup $K \subset G(\mathbb{R})$ and consider the symmetric space $X = K\backslash G(\mathbb{R})$. The space X is Euclidean and Γ acts properly and freely on X, thus $H^\bullet(\Gamma, \mathbb{R}) \cong H^\bullet(X/\Gamma, \mathbb{R})$ (see II.2.12).

Whenever X/Γ is compact, we can use the Gauß-Bonnet Theorem to calculate the Euler characteristic of Γ. Although the space X/Γ is in general not compact, it always has finite invariant volume, i.e. finite volume with respect to G-invariant measures on X (see [**BHC62**, §9]). Thus it is natural to ask whether we can still calculate the Euler characteristic of Γ by integration over the Gauß-Bonnet density. The answer is provided by Harder's Gauß-Bonnet Theorem.

IV.4.4. Theorem (Harder). *If G is a semisimple algebraic $\mathbb{Q}$-group and $\Gamma \subset G(\mathbb{Q})$ a torsion-free arithmetic subgroup, then*

$$\chi(\Gamma) = \int_{X/\Gamma} \Omega_{X/\Gamma}^+.$$

Proof. See [**Har71**]. □

IV.4.5. *Remark.* Let G be a semisimple algebraic group defined over some number field F. If $\Gamma \subset G(F)$ is a torsion-free arithmetic group, then Harder's formula can also be used to calculate the Euler characteristic of Γ. Let $G' = \mathrm{Res}_{F/\mathbb{Q}}\, G$ be the Weil restriction of scalars, then $G(F) = G'(\mathbb{Q})$ and Γ is also an arithmetic subgroup of the $\mathbb{Q}$-group G'.

IV.5. The Euler-Poincaré Measure

IV.5.1. Let G be a unimodular real Lie group. Recall that an *Euler-Poincaré measure* μ_χ on G (in the sense of Serre [**Ser71**]) is a Haar measure on G such that for every torsion-free cocompact subgroup $\Gamma \subseteq G$ one has $\mu_\chi(G/\Gamma) = \chi(\Gamma)$. Note that every such torsion-free cocompact subgroup Γ is of type (FL) (see Prop. 18 in [**Ser71**]). Further, if G has such a subgroup, then there is at most one Euler-Poincaré measure. The purpose of this section is to determine the Euler-Poincaré measure on a semisimple real Lie group.

IV.5.2. Throughout the section G denotes a real semisimple Lie group with finite centre. Moreover, we assume that G has a complexification $G_\mathbb{C}$ (see IV.3.6). Fix a maximal compact subgroup K of G. We write G_u for the compact dual group of G^0 containing K^0 with respect to the complexification $G_\mathbb{C}$ (IV.3.7).

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to K, as before we get the compact real form $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. We fix an $\mathrm{Ad}(K)$-invariant, positive definite bilinear form $B_\mathfrak{p}$ on $\mathfrak{p}$ and we transfer it to $i\mathfrak{p}$ such that multiplication with i is an isogeny – we denote this form $B_{i\mathfrak{p}}$. Further we choose any non-degenerate bilinear form $B_\mathfrak{k}$ on $\mathfrak{k}$. We define the non-degenerate form B to be the orthogonal sum $B = B_\mathfrak{k} \perp B_\mathfrak{p}$. For later reference we will say that such a bilinear form B is *nice* with respect to the Cartan decomposition. By abuse of notation we also denote the non-degenerate form $B_\mathfrak{k} \perp B_{i\mathfrak{p}}$ on $\mathfrak{u}$ by B.

We define $X := K\backslash G$ and we equip it with the G-invariant Riemann metric induced by $B_\mathfrak{p}$ (see Lemma IV.3.2). We also define the compact dual $X_u := K^0\backslash G_u$ and we consider the G_u-invariant metric on X_u induced by $B_{i\mathfrak{p}}$.

IV.5.3. *Basic observation.* The volume density $\left|\mathrm{vol}_{B_\mathfrak{p}}\right|$ for $B_\mathfrak{p}$ and the Gauß-Bonnet density on X are left G-invariant (cf. IV.3.3). Therefore there is some real number $c_X \in \mathbb{R}$ such that

$$\Omega_X^+ = c_X \left|\mathrm{vol}_{B_\mathfrak{p}}\right|.$$

Moreover, if $\Gamma \subseteq G$ is a discrete, torsion-free subgroup, then the canonical map quotient $\pi_\Gamma : X \to X/\Gamma$ is a local isometry and thus we find

$$\pi_\Gamma^*(\Omega_{X/\Gamma}^+) = \Omega_X^+ = c_X \left|\mathrm{vol}_{B_\mathfrak{p}}\right| = c_X \pi_\Gamma^*(\left|\mathrm{vol}_{X/\Gamma}\right|),$$

where we write $\left|\mathrm{vol}_{X/\Gamma}\right|$ for the volume density on X/Γ induced from $B_\mathfrak{p}$. Therefore we find an Euler-Poincaré measure μ_χ on G, namely

$$\mu_\chi := c_X \,\mathrm{vol}_{B_\mathfrak{k}}(K)^{-1} \left|\mathrm{vol}_B\right|. \tag{13}$$

We will determine the factor c_X using *Hirzebruch's proportionality principle* following the approach of Harder [**Har71**] and Serre [**Ser71**].

IV.5.4. Theorem. *Let G be a semisimple real Lie group with finite centre and assume there is a complexification $G_\mathbb{C}$. If $\dim(X)$ is odd or if $\mathrm{rk}(\mathfrak{g}_\mathbb{C}) < \mathrm{rk}(\mathfrak{k}_\mathbb{C})$, then $\mu_\chi = 0$ is the Euler-Poincaré measure. Otherwise, if $\dim(X) = 2p$ is even and $\mathrm{rk}(\mathfrak{g}_\mathbb{C}) = \mathrm{rk}(\mathfrak{k}_\mathbb{C})$, then*

$$\mu_\chi := \frac{(-1)^p \left|W(\mathfrak{g}_\mathbb{C})\right|}{\left|\pi_0(G)\right| \left|W(\mathfrak{k}_\mathbb{C})\right|} \mathrm{vol}_B(G_u)^{-1} \left|\mathrm{vol}_B\right|$$

is the Euler-Poincaré measure on G. Here $\pi_0(G) = G/G^0$, $W(\mathfrak{g}_\mathbb{C})$ (resp. $W(\mathfrak{k}_\mathbb{C})$) denotes the Weyl group of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ (resp. $\mathfrak{k}_\mathbb{C}$), and G_u denotes the compact dual of G^0 containing K^0.

Proof. We have to find the factor c_X in equation (13). If $\dim(X)$ is odd, then $\Omega_X^+ = 0$ and we have $c_X = 0$. So, we can assume that $\dim(X) = 2p$ is even.

We write $\pi : G \to X = K\backslash G$ and $\pi_u : G_u \to X_u = K^0\backslash G_u$ for the canonical projections. Moreover, we identify the tanget space at $o = \pi(e)$ (resp. $o = \pi_u(e)$) with $\mathfrak{p}$ (resp. $i\mathfrak{p}$). Let R (resp. R^u) denote the curvature tensor on X (resp. X_u) – say for the metric induced from $B_\mathfrak{p}$ (resp. $B_{i\mathfrak{p}}$). We know that

$$R_o(u,v)w = -[[u,v],w] \text{ and } R^u_o(iu,iv)iw = i[[u,v],w]$$

for all $u,v,w \in \mathfrak{p}$ (cf. equation (12) in IV.3.4). So, up to multiplication with the imaginary unit i, the curvature tensor R^u is the negative of the tensor R at the chosen points o. Further, by definition of $B_{i\mathfrak{p}}$, multiplication with i is an isogeny between $B_\mathfrak{p}$ and $B_{i\mathfrak{p}}$. Using that the Gauß-Bonnet form is a homogeneous polynomial of degree p in terms of the curvature (see IV.2.6), we see that

$$\Omega^+_{X_u} = (-1)^p c_X \left|\mathrm{vol}_{B_{i\mathfrak{p}}}\right|. \tag{14}$$

We will use this relation to determine c_X. By a well-known result of Hopf and Samelson [**HS41**] the Euler characteristic of a homogeneous space $X_u = K^0\backslash G_u$ is zero if and only if $\mathrm{rk}(K^0) < \mathrm{rk}(G_u)$. Moreover, if $\mathrm{rk}(K^0) = \mathrm{rk}(G_u)$, then $\chi(X_u) = |W(G_u)|\left|W(K^0)\right|^{-1}$, where $W(G_u)$ and $W(K^0)$ denote the Weyl groups with respect to some maximal torus.

For a connected compact Lie group H we have $\mathrm{rk}(H) = \mathrm{rk}(\mathfrak{h}_\mathbb{C})$ (cf. IV.3.9) and there is an isomorphism between the Weyl groups $W(H) \to W(\mathfrak{h}_\mathbb{C})$ (see Thm. 4.54 in [**Kna02**]). Thus integration of equation (14) yields

$$c_X = \begin{cases} 0 & \text{if } \mathrm{rk}(\mathfrak{k}_\mathbb{C}) < \mathrm{rk}(\mathfrak{g}_\mathbb{C}), \\ (-1)^p\,|W(\mathfrak{g}_\mathbb{C})|\,|W(\mathfrak{k}_\mathbb{C})|^{-1}\,\mathrm{vol}_B(G_u)^{-1}\,\mathrm{vol}_{B_\mathfrak{k}}(K^0) & \text{if } \mathrm{rk}(\mathfrak{k}_\mathbb{C}) = \mathrm{rk}(\mathfrak{g}_\mathbb{C}). \end{cases}$$

Finally, we use that $\pi_0(G) \cong \pi_0(K)$ (cf. A.3.4) and the claim follows from (13). □

IV.5.5. *Remark.* In the proof of Theorem IV.5.4 we assumed the form B to be *nice* in the sense of paragraph IV.5.2. However, once the result is established it is easy to see that the result holds for any non-degenerate form $B : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$.

Let B be a nice form (as in the theorem) and let $C : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ be any non-degenerate form. We transfer C to a form C_u on $\mathfrak{u}$ such that the isomorphism of real vector spaces $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p} \to \mathfrak{k}\oplus i\mathfrak{p} = \mathfrak{u}$, defined by $v + w \mapsto v + iw$ for all $v \in \mathfrak{k}$ and $w \in \mathfrak{p}$, is an isometry. By abuse of notation we often write C instead of C_u for the form on $\mathfrak{u}$.

Take any basis $e_1,\dots,e_d$ of $\mathfrak{g}$. Then $|\mathrm{vol}_B| = \lambda\,|\mathrm{vol}_C|$ on G, where

$$\lambda = |\det(B(e_i,e_j))\det(C(e_i,e_j))^{-1}|^{1/2}.$$

However, we also have $\mathrm{vol}_B(G_u) = \lambda\,\mathrm{vol}_C(G_u)$ with the same factor λ and we deduce that the theorem holds for any non-degenerate bilinear form on $\mathfrak{g}$.

IV.5.6. *The volume of G_u.* Suppose the form $B : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ is such that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal. In this case we may modify the form B_u on $\mathfrak{u}$ slightly: We can define $B_{i\mathfrak{p}}$ on $i\mathfrak{p}$ such that $B_{i\mathfrak{p}}(iu,iv) := -B_\mathfrak{p}(u,v)$. This does not change the volume density $|\mathrm{vol}_B|$, and hence it does not change the volume of G_u. Nevertheless, the form obtained in this way is often more convenient since it agrees with the $\mathbb{C}$-linear extension of B to $\mathfrak{g}_\mathbb{C}$.

IV.6. Adelic Formulation and Smoothness

IV.6.1. Let F be an algebraic number field and let $\mathcal{O} \subset F$ be the ring of algebraic integers in F. Let V be the set of places of F. We have $V = V_\infty \cup V_f$ where V_∞ (resp. V_f) denotes the set of archimedean (resp. finite) places. Given a place $v \in V$, we denote the completion of F with respect to v by F_v. If v is a finite place, $\mathcal{O}_v$ denotes the valuation ring in F_v. We denote the ring of adeles of F by $\mathbb{A}$ and the ring of finite adeles is denoted by $\mathbb{A}_f$.

IV.6.2. Let G be a linear algebraic group defined over F. We define

$$G_\infty := G(F \otimes_{\mathbb{Q}} \mathbb{R}) = \prod_{v \in V_\infty} G(F_v)$$

and we fix for every place $v \in V_\infty$ a maximal compact subgroup $K_v \subseteq G(F_v)$. The group $\mathrm{K}_\infty := \prod_{v \in V_\infty} K_v$ is a maximal compact subgroup of G_∞.

Let $K_f \subseteq G(\mathbb{A}_f)$ be an open compact subgroup of the locally compact group $G(\mathbb{A}_f)$. Borel showed that $G(\mathbb{A})$ is the disjoint union of a finite number m of double cosets, i.e.

$$G(\mathbb{A}) = \bigsqcup_{i=1}^{m} G_\infty K_f x_i G(F)$$

for some representatives $x_1, \ldots, x_m \in G(\mathbb{A}_f)$ (see Thm. 5.1 in [**Bor63**]). For every $i = 1, \ldots, m$ we obtain arithmetic subgroups $\Gamma_i \subseteq G(F)$ defined by

$$\Gamma_i := G(F) \cap x_i^{-1} K_f x_i.$$

IV.6.3. Proposition. *There is a G_∞-equivariant homeomorphism*

$$K_f \backslash G(\mathbb{A})/G(F) \xrightarrow{\simeq} \bigsqcup_{i=1}^{m} G_\infty/\Gamma_i.$$

Here the right hand side denotes the topologically disjoint union.

Proof. For every $i = 1, \ldots, m$ we have a continuous mapping

$$f_i : G_\infty \to K_f \backslash G(\mathbb{A})/G(F)$$

defined by $g \mapsto K_f x_i g G(F)$. Clearly, the images of f_i and f_j are disjoint for $i \neq j$ since x_i and x_j represent distinct cosets. Suppose that $f_i(g) = f_i(g')$ for some $g, g' \in G_\infty$. Then there are $k \in K_f$ and $\gamma \in G(F)$ such that $x_i g = k x_i g' \gamma$. We conclude $g = g'\gamma$ and $\gamma = x_i^{-1} k^{-1} x_i$. We see that $\gamma \in \Gamma_i$ and further that g and g' represent the same element in G_∞/Γ_i. Conversely, let $g \in G_\infty$ and take $\gamma \in \Gamma_i$ then one easily checks that $f_i(g\gamma) = f_i(g)$.

This means we found a bijective and continuous mapping

$$f = \bigsqcup f_i : \bigsqcup_{i=1}^{m} G_\infty/\Gamma_i \to K_f \backslash G(\mathbb{A})/G(F).$$

Since $G_\infty K_f x_i G(F)$ is open and closed in $G(\mathbb{A})$, we can deduce that $f_i(G_\infty)$ is an open and closed subset of $K_f \backslash G(\mathbb{A})/G(F)$. The maps f_i are obviously open, thus f is a homeomorphism. It is easily verified that f is G_∞-equivariant using that $x_i \in G(\mathbb{A}_f)$ commutes with G_∞. $\square$

IV.6.4. Lemma. *The group $G(F)$ acts freely on the quotient $K_\infty K_f \backslash G(\mathbb{A})$ if and only if the groups Γ_i are torsion-free for all $i = 1, \ldots, m$.*

Proof. Assume that all the groups Γ_i are torsion-free and suppose that the action of $G(F)$ on $K_\infty K_f \backslash G(\mathbb{A})$ is not free. Then there is some $i \in \{1, \ldots, m\}$ so that

$$g_\infty x_i \gamma = k_\infty k_f g_\infty x_i$$

holds for some $g_\infty \in G_\infty$, $k_\infty \in K_\infty$, $k_f \in K_f$ and $\gamma \in G(F)$ with $\gamma \neq 1$. We deduce that $x_i\gamma = k_f x_i$, this means $\gamma \in \Gamma_i$. Moreover, we see that $g_\infty\gamma = k_\infty g_\infty$ and conclude that $\gamma \in g_\infty^{-1}K_\infty g_\infty \cap \Gamma_i = \{1\}$. This contradicts the assumption $\gamma \neq 1$.

Conversely, assume that $G(F)$ acts freely on $K_\infty K_f\backslash G(\mathbb{A})$. Suppose that $\gamma \in \Gamma_i$ is a torsion element. Such an element is contained in a maximal compact subgroup of G_∞, thus there is $g_\infty \in G_\infty$ with $\gamma \in g_\infty^{-1}K_\infty g_\infty$ and we have $g_\infty\gamma = k_\infty g_\infty$ for some $k_\infty \in K_\infty$. We write $\gamma = x_i^{-1}k_f x_i$ with some $k_f \in G(\mathbb{A}_f)$ and we find

$$g_\infty x_i\gamma = k_\infty k_f g_\infty x_i.$$

Since $G(F)$ acts freely, we get $\gamma = 1$, i.e. every torsion element is trivial. □

IV.6.5. *The Tamagawa measure.* Let G be a connected semisimple linear algebraic F-group of dimension d. Let $\mathfrak{g} = \mathrm{Lie}(G)(F)$ be the Lie algebra of G over F. For every place $v \in V$ there is a canonical isomorphism of Lie algebras

$$\mathfrak{g}_v := \mathrm{Lie}(G(F_v)) \cong \mathfrak{g} \otimes_F F_v.$$

Let $\omega : \bigwedge^d \mathfrak{g} \to F$ be a non-trivial differential form of highest degree on $\mathfrak{g}$. By F_v-linear continuation we get a top-degree form on $\mathfrak{g}_v$ and therefore a left invariant density $|\omega|_v$ on the F_v-analytic Lie group $G(F_v)$. Since G is semisimple, $G(F_v)$ is unimodular and we conclude that $|\omega|_v$ is right invariant as well.

We fix Haar measures μ_v on F_v for every place v such that

- $\mu_v(\mathcal{O}_v) = 1$ if $v \in V_f$ is a finite place,
- $\mu_v([0,1]) = 1$ if v is a real place, and
- $\mu_v([0,1] + [0,1]i) = 2$ if v is a complex place.

With these measures the density $|\omega|_v$ induces a Haar measure on $G(F_v)$ which will be denoted by the same symbol.

Let G_0 be a smooth model of G defined over $\mathcal{O}[1/m]$ for some $m \in \mathbb{Z}$, this is, G_0 is a smooth group scheme over the ring $\mathcal{O}[1/m]$ and $G_0 \times F \cong G$. Let S be the set of all finite places $v \in V_f$ that do not divide m. For all $v \in S$ the group $G_0(\mathcal{O}_v)$ is defined and it is an open compact subgroup of $G(F_v)$. For $v \in S$ we define $\nu_v := \int_{G_0(\mathcal{O}_v)} |\omega|_v$. One can show (using Weil's theorem on integration cf. III.6) that the product $\prod_{v\in S}\nu_v$ is absolutely convergent (see §1 in [**Ono66a**]). Therefore the product measure $\prod_{v\in V}|\omega|_v$ is well-defined on the adele group $G(\mathbb{A})$. The measure

$$\tau := |d_F|^{-d/2}\prod_{v\in V}|\omega|_v$$

is called the *Tamagawa measure* on $G(\mathbb{A})$. It follows directly from the product formula that the Tamagawa measure is independent of the chosen form ω. We equip the discrete group $G(F)$ with the counting measure and define

$$\tau(G) := \int_{G(\mathbb{A})/G(F)} \tau.$$

The number $\tau(G)$ is called the *Tamagawa number* of G.

Recall the following theorem: If G is simply connected, then $\tau(G) = 1$. This was conjectured by A. Weil and he gave a proof for some classical groups [**Wei82**]. A general proof was given later by Kottwitz [**Kot88**].

IV.6.6. Lemma. *Let G be a d-dimensional semisimple connected linear algebraic group defined over F. Fix a non-degenerate F-bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to F$ on the Lie algebra. For every place $v \in V$ we have the left invariant volume density $|\mathrm{vol}_B|_v$ on*

$G(F_v)$ associated with B (see III.6.3). Then the Tamagawa measure on $G(\mathbb{A})$ is given by

$$\tau = |d_F|^{-d/2} \prod_{v\in V} |\mathrm{vol}_B|_v \,.$$

PROOF. Let $e_1, \dots, e_d$ be a basis of $\mathfrak{g}$ over F and take the dual basis $\varepsilon_1, \dots, \varepsilon_d$ of $\mathrm{Hom}_F(\mathfrak{g}, F)$. We define the non-trivial form of highest degree $\omega = \varepsilon_1 \wedge \cdots \wedge \varepsilon_d$ on $\mathfrak{g}$. By definition of the volume density we have

$$|\mathrm{vol}_B|_v = |\det(B(e_i, e_j))|_v^{1/2}\, |\omega|_v \,.$$

By the product formula we know that $|\det(B(e_i, e_j))|_v = 1$ for almost all places v and further that $\prod_{v\in V} |\det(B(e_i, e_j))|_v = 1$. □

IV.6.7. *The modulus factors.* Let G be a smooth group scheme defined over $\mathcal{O}$. For any commutative $\mathcal{O}$-algebra R we write $\mathfrak{g}_R := \mathrm{Lie}(G)(R)$ to denote the R-points of the Lie algebra of G. Let $B : \mathfrak{g}_F \times \mathfrak{g}_F \to F$ be a non-degenerate F-bilinear form. For every finite place $v \in V_f$ we define the *modulus factor* $m(B)_v$ as in III.6.4. This means, take an $\mathcal{O}_v$-basis $e_1, \dots, e_n$ of the free $\mathcal{O}_v$-module $\mathfrak{g}_v := \mathfrak{g}_{\mathcal{O}_v}$, and define

$$m(B)_v := |\det(B(e_i, e_j))|_v^{1/2} \,.$$

For almost all finite places $v \in V_f$ we have $m(B)_v = 1$. To see this, take an F-basis of $\mathfrak{g}_F$ and note that it is an $\mathcal{O}_v$-basis of $\mathfrak{g}_v$ for almost all finite places v. This allows us to define the *global modulus factor* $m(B) := \prod_{v\in V_f} m(B)_v$.

IV.6.8. We introduce some notation concerning the infinite places. Let G be a connected semisimple F-group. Recall that

$$G_\infty := \prod_{v\in V_\infty} G(F_v)$$

and we choose a maximal compact subgroup

$$K_\infty := \prod_{v\in V_\infty} K_v$$

with maximal compact subgroups $K_v \subseteq G(F_v)$. We write $\mathfrak{g}_v$ for the Lie algebra of $G(F_v)$. If $v \in V_\infty$ is an archimedean place, then $\mathfrak{g}_{v,\mathbb{C}}$ denotes the complexification of $\mathfrak{g}_v$, where $\mathfrak{g}_v$ is considered as a real Lie algebra regardless of whether v is a real or complex place. Let $\mathfrak{g}_\infty$ (resp. $\mathfrak{k}$) be the real Lie algebra of G_∞ (resp. K_∞) and let $\mathfrak{g}_{\infty,\mathbb{C}}$ (resp. $\mathfrak{k}_\mathbb{C}$) denote the complexification. As before we define the symmetric space $X := K_\infty \backslash G_\infty$.

IV.6.9. Theorem (Adelic Euler characteristic formula - Part 1)**.** *Let G be a connected semisimple algebraic group defined over F. Let $K_f \subseteq G(\mathbb{A}_f)$ be an open compact subgroup such that $G(F)$ acts freely on $K_\infty K_f \backslash G(\mathbb{A})$. If $\dim(X)$ is odd or if $\mathrm{rk}(\mathfrak{k}_\mathbb{C}) < \mathrm{rk}(\mathfrak{g}_{\infty,\mathbb{C}})$, then*

$$\chi(K_\infty K_f \backslash G(\mathbb{A})/G(F)) = 0.$$

PROOF. By Proposition IV.6.3 there is a homeomorphism

$$K_\infty K_f \backslash G(\mathbb{A})/G(F) \xrightarrow{\simeq} \bigsqcup_{i=1}^{m} X/\Gamma_i$$

with certain torsion-free (see Lemma IV.6.4) arithmetic groups $\Gamma_i \subseteq G_\infty$. It follows from Harder's Gauß-Bonnet Theorem and the vanishing of the Euler-Poincaré measure on G_∞ (Thm. IV.5.4) that $\chi(\Gamma_i) = \chi(X/\Gamma_i) = 0$. Finally the claim follows from the fact that the Euler characteristic is additive for topologically disjoint unions. □

IV.6.10. *Remark.* Note that $\mathfrak{g}_{\infty,\mathbb{C}} = \bigoplus_{v\in V_\infty} \mathfrak{g}_{v,\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}} = \bigoplus_{v\in V_\infty} \mathfrak{k}_{v,\mathbb{C}}$. In particular, we obtain that $\mathrm{rk}(\mathfrak{k}_{\mathbb{C}}) = \mathrm{rk}(\mathfrak{g}_{\infty,\mathbb{C}})$ if and only if $\mathrm{rk}(\mathfrak{k}_{v,\mathbb{C}}) = \mathrm{rk}(\mathfrak{g}_{v,\mathbb{C}})$ for all archimedean places v.

Suppose that F has a complex place w. We claim that $\mathrm{rk}(\mathfrak{k}_{w,\mathbb{C}}) < \mathrm{rk}(\mathfrak{g}_{w,\mathbb{C}})$. To verify this, one has to realise that $\mathfrak{k}_w$ is a compact real form of the complex Lie algebra $\mathfrak{g}_w$ (cf. [**Kna02**, Cor. 6.22]). In particular, we see that $\mathfrak{k}_{w,\mathbb{C}} \cong \mathfrak{g}_w$. Furthermore, it is easy to check that $\mathfrak{g}_{w,\mathbb{C}}$ is isomorphic to $\mathfrak{g}_w \oplus \mathfrak{g}_w$ as a complex Lie algebra. Since $\mathfrak{g}_w$ is a complex semisimple (thus non-trivial) Lie algebra, we have $\mathrm{rk}(\mathfrak{g}_w) \geq 1$. We conclude that $\mathrm{rk}(\mathfrak{g}_{w,\mathbb{C}}) = 2\,\mathrm{rk}(\mathfrak{g}_w) > \mathrm{rk}(\mathfrak{g}_w) = \mathrm{rk}(\mathfrak{k}_{w,\mathbb{C}})$. Consequently, if we want to study spaces $K_\infty K_f\backslash G(\mathbb{A})/G(F)$ with non-vanishing Euler characteristic, then we have to assume that F is a totally real number field.

IV.6.11. *Congruence groups.* In the adelic formulation of Harder's Gauß-Bonnet Theorem we want to focus on congruence groups which are given by a specific description. Let G be a smooth $\mathcal{O}$-group scheme. For every finite place $v \in V_f$, let $\alpha_v \geq 1$ be a natural number and we assume that $\alpha_v = 1$ for almost all $v \in V_f$. Let v be a finite place and let $\mathfrak{p}_v \subseteq \mathcal{O}_v$ be the unique prime ideal in $\mathcal{O}_v$. We define π_v to be the reduction morphism

$$\pi_v : G(\mathcal{O}_v) \to G(\mathcal{O}_v/\mathfrak{p}_v^{\alpha_v}).$$

Further, we assume that we are given a subgroup U_v of the finite group $G(\mathcal{O}_v/\mathfrak{p}_v^{\alpha_v})$ for every place $v \in V_f$. For a place $v \in V_f$ the group $K_v(U) := \pi_v^{-1}(U_v)$ is an open compact subgroup of $G(\mathcal{O}_v)$. If we additionally impose the assumption that $U_v = G(\mathcal{O}_v/\mathfrak{p}_v^{\alpha_v})$ for almost all v, then the group

$$K(U) := \prod_{v\in V_f} K_v(U)$$

is an open compact subgroup of the group $G(\mathbb{A}_f)$ of points in the finite adeles. We say that $K(U)$ is the congruence group associated with the *local datum*

$$U = (U_v)_v = (U_v, \alpha_v)_v$$

(usually the numbers α_v are considered to be implicitly a part of the datum U).

IV.6.12. Theorem (Adelic Euler characteristic formula - Part 2)**.** *Let F be a totally real algebraic number field and let $\mathcal{O}$ be its ring of integers. Let G be a smooth group scheme over $\mathcal{O}$ such that $G \times_{\mathcal{O}} F$ is a connected semisimple group of dimension d. We fix any non-degenerate F-bilinear form $B : \mathfrak{g}_F \times \mathfrak{g}_F \to F$. Furthermore, let $K_f = K(U)$ be a congruence subgroup of $G(\mathbb{A}_f)$ given by a local datum (U, α) such that $G(F)$ acts freely on $K_\infty K_f\backslash G(\mathbb{A})$.*

If $\dim(X) = 2p$ is even and $\mathrm{rk}(\mathfrak{k}_{\mathbb{C}}) = \mathrm{rk}(\mathfrak{g}_{\infty,\mathbb{C}})$, then the Euler characteristic of $K_\infty K_f\backslash G(\mathbb{A})/G(F)$ is given by

$$\chi(K_\infty K_f\backslash G(\mathbb{A})/G(F)) \\ = (-1)^p\,|d_F|^{d/2}\,\frac{|W(\mathfrak{g}_{\infty,\mathbb{C}})|\ \tau(G)}{|\pi_0(G_\infty)|\,|W(\mathfrak{k}_{\mathbb{C}})|}\,\mathrm{vol}_B(G_u)^{-1} m(B)^{-1} \prod_{v\in V_f} \frac{\mathrm{N}(\mathfrak{p}_v)^{d\alpha_v}}{|U_v|}.$$

Here $\tau(G)$ is the Tamagawa number of G, $\mathrm{N}(\mathfrak{p}_v)$ denotes the cardinality of the residue class field $\mathcal{O}_v/\mathfrak{p}_v$, and G_u denotes the compact dual group of G^0_∞ (the remaining notation is as in Theorem IV.5.4).

PROOF. Let $x_1, \ldots, x_m \in G(\mathbb{A}_f)$ be representatives, as in IV.6.2, of the finitely many double cosets in $G_\infty K_f\backslash G(\mathbb{A})/G(F)$. We consider the torsion-free arithmetic groups $\Gamma_i := G(F) \cap x_i^{-1} K_f x_i$. Let $\mathcal{F}_i$ be a Borel measurable fundamental domain for the right action of Γ_i on G_∞. Here we mean a fundamental domain in the strict sense, i.e. $\mathcal{F}_i$ is a set of representatives for G_∞/Γ_i (for the existence of measurable fundamental domains see Bourbaki – Intégration, VII.§2 Ex. 12 [**Bou63**]). The set $\mathcal{F}$ defined as the union $\bigsqcup_{i=1}^m \mathcal{F}_i K_f x_i \subseteq G(\mathbb{A})$ is a Borel measurable fundamental domain for the right action of $G(F)$ on $G(\mathbb{A})$. For simplicity we write $|\mathrm{vol}_B|_\infty = \prod_{v\in V_\infty} |\mathrm{vol}_B|_v$ and $|\mathrm{vol}_B|_f := \prod_{v\in V_f} |\mathrm{vol}_B|_v$. Due to Theorem IV.5.4 we have

$$\chi(K_\infty K_f\backslash G(\mathbb{A})/G(F)) = \sum_{i=1}^m \chi(X/\Gamma_i) = \lambda \sum_{i=1}^m \int_{\mathcal{F}_i} |\mathrm{vol}_B|_\infty$$

where $\lambda = (-1)^p \frac{|W(\mathfrak{g}_{\infty,\mathbb{C}})|}{|\pi_0(G_\infty)||W(\mathfrak{k}_\mathbb{C})|} \mathrm{vol}_B(G_u)^{-1}$. By multiplication with the volume of K_f, which is simply $\mathrm{vol}_B(K_f) = \int_{K_f} |\mathrm{vol}_B|_f$, and by Lemma IV.6.6, we obtain

$$\sum_{i=1}^m \int_{\mathcal{F}_i} |\mathrm{vol}_B|_\infty \mathrm{vol}_B(K_f) = \int_{\mathcal{F}} \prod_{v\in V} |\mathrm{vol}_B|_v = |d_F|^{d/2} \int_{\mathcal{F}} \tau = |d_F|^{d/2} \tau(G).$$

This means, we have

$$\chi(K_\infty K_f\backslash G(\mathbb{A})/G(F)) = \lambda\, |d_F|^{d/2}\, \tau(G)\, \mathrm{vol}_B(K_f)^{-1}.$$

Finally we are left with the task of determining $\mathrm{vol}_B(K_f)$. We shall exploit that K_f is given by the local datum (U,α). Since $\mathrm{vol}_B(K_f) = \prod_{v\in V_f} \mathrm{vol}_B(K_v(U))$ and the scheme G is smooth, we can apply Weil's theorem (see III.6.5) in every finite place to deduce

$$\mathrm{vol}_B(K_f) = \prod_{v\in V_f} m(B)_v \frac{|U_v|}{\mathrm{N}(\mathfrak{p}_v)^{d\alpha_v}}.$$

Now the claim follows readily. □

IV.6.13. *Remark.* What is the idea behind the form B? The reason we introduced the form B is that usually there is a natural choice for a Riemannian metric on G_u and with respect to this metric the volume is well-known. This means that the task of defining B is a question of reverse engineering: We know which form we want to have on G_u and we have to transport it to the Lie algebra. For instance, the volume of the compact dual group G_u with respect to the metric induced by the Killing form has been computed by Ono, see 3.4 in [**Ono66b**]. Another reference for the volumes of compact Lie groups is [**Mac80**].

IV.6.14. *An example:* SL_2. We illustrate the adelic formula with the well-known example SL_2. Let F be a totally real algebraic number field of degree $r = [F:\mathbb{Q}]$ and let $\mathcal{O}$ be its ring of integers. Let $G = \mathrm{SL}_2$ be the special linear group defined over $\mathcal{O}$. The trace $\mathrm{Tr} : M_2(\mathcal{O}) \to \mathcal{O}$ is onto and it follows from Remark III.3.9 that G is a smooth scheme over $\mathcal{O}$.

Let $\mathfrak{a} \subset \mathcal{O}$ be a non-trivial proper ideal. We consider the principal congruence group of level $\mathfrak{a}$, this is $\Gamma(\mathfrak{a}) := \ker(\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/\mathfrak{a}))$. We will always assume that $\mathfrak{a}$ is chosen such that $\Gamma(\mathfrak{a})$ is torsion-free (this is possible, see Thm. III.2.3).

The group $\Gamma(\mathfrak{a})$ is given by a local datum (U,α). For a prime ideal $\mathfrak{p} \subseteq \mathcal{O}$ which divides $\mathfrak{a}$ we define $\alpha_\mathfrak{p} := \nu_\mathfrak{p}(\mathfrak{a})$, where $\nu_\mathfrak{p}(\mathfrak{a})$ denotes the maximal positive integer e with $\mathfrak{p}^e|\mathfrak{a}$. Moreover, we set $U_\mathfrak{p} = \{1\} \subseteq \mathrm{SL}_2(\mathcal{O}/\mathfrak{p}^{\alpha_\mathfrak{p}})$. Otherwise, i.e. if $\mathfrak{p}$ does not divide $\mathfrak{a}$, we set $\alpha_\mathfrak{p} = 1$ and $U_\mathfrak{p} = \mathrm{SL}_2(\mathcal{O}/\mathfrak{p})$. Let $K_f := K(U) \subseteq \mathrm{SL}_2(\mathbb{A}_f)$ be the associated open compact subgroup, then $\Gamma(\mathfrak{a}) = \mathrm{SL}_2(F) \cap K_f$.

It is obvious that

$$G_\infty = \prod_{v\in V_\infty} \mathrm{SL}_2(F_v) \cong \mathrm{SL}_2(\mathbb{R})^r,$$

and we fix the maximal compact subgroup

$$K_\infty = \mathrm{SO}(2)^r.$$

The symmetric space $X = K_\infty \backslash G_\infty$ is a product of r copies of the Poincaré upper half-plane. We conclude that $p = r$ and $\pi_0(G_\infty) = \{1\}$. Since SL_2 is a semisimple and simply connected group, the Tamagawa number $\tau(G)$ is equal to 1 (see [**Kot88**]). Further, SL_2 has strong approximation (cf. II.6.2), in particular we obtain a homeomorphism $X/\Gamma(\mathfrak{a}) \cong K_f K_\infty \backslash \mathrm{SL}_2(\mathbb{A}) / \mathrm{SL}_2(F)$, and consequently

$$\chi(K_f K_\infty \backslash \mathrm{SL}_2(\mathbb{A}) / \mathrm{SL}_2(F)) = \chi(\Gamma(\mathfrak{a})).$$

Clearly, we have $\mathfrak{g}_{\infty,\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})^r$, thus $\mathrm{rk}(\mathfrak{g}_{\infty,\mathbb{C}}) = r$ and $|W(\mathfrak{g}_{\infty,\mathbb{C}})| = 2^r$. The Lie algebra $\mathfrak{k}_\mathbb{C} \cong \mathfrak{so}_2^r \otimes_\mathbb{R} \mathbb{C}$ is abelian of dimension r, hence the rank $\mathrm{rk}(\mathfrak{k}_\mathbb{C})$ equals r and $|W(\mathfrak{k}_\mathbb{C})| = 1$. It is well-known that the compact dual of $\mathrm{SL}_2(\mathbb{R})$ is the special unitary group $\mathrm{SU}(2)$, this implies $G_u \cong \mathrm{SU}(2)^r$. It remains to choose some non-degenerate bilinear form $B : \mathfrak{sl}_2(F) \times \mathfrak{sl}_2(F) \to F$, and to calculate the volume $\mathrm{vol}_B(G_u)$ and the associated modulus factor $m(B)$.

Consider the Lie algebra $\mathfrak{g}_F = \mathfrak{sl}_2(F) = \{\, Y \in M_2(F) \mid \mathrm{Tr}(Y) = 0 \,\}$. We consider the F-basis given by

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \;\text{ and } u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\iota : F \to \mathbb{R}$ be an embedding, then the image of w spans the Lie algebra $\mathfrak{so}_2$ of the maximal compact subgroup $\mathrm{SO}(2)$. We denote the span of h and u in $\mathfrak{sl}_2(\mathbb{R})$ by $\mathfrak{p}$, then

$$\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{so}_2 \oplus \mathfrak{p}$$

is a Cartan decomposition of $\mathfrak{sl}_2(\mathbb{R})$. We define the form $B : \mathfrak{sl}_2(F) \times \mathfrak{sl}_2(F) \to F$ to be the unique bilinear form having w, h, u as an orthonormal basis. Let $\mathfrak{su}_2$ be the Lie algebra of the compact dual group $SU(2)$. Since $\mathfrak{su}_2 = \mathfrak{so}_2 \oplus i\mathfrak{p}$, the vectors w, ih, iu form an $\mathbb{R}$-basis of $\mathfrak{su}_2$ and, by definition, the form B on $\mathfrak{su}_2$ is such that w, ih, iu is an orthonormal basis. Furthermore, there is a well-known diffeomorphism $\psi : \mathrm{SU}(2) \to S^3$ between the special unitary group and the 3-sphere. More precisely, ψ is defined by $\psi : g \mapsto ge_1$ where e_1 is the first vector in the standard basis of $\mathbb{C}^2$. We transfer the $\mathrm{SU}(2)$-invariant metric induced by B to the sphere S^3. It is easy to check that this defines the standard Riemann metric on S^3 considered as a submanifold of $\mathbb{C}^2$. We conclude $\mathrm{vol}_B(\mathrm{SU}(2)) = \mathrm{vol}(S^3) = 2\pi^2$ and therefore

$$\mathrm{vol}_B(G_u) = \mathrm{vol}_B(\mathrm{SU}(2))^r = 2^r \pi^{2r}.$$

Finally we determine the modulus factor. It follows from Lemma III.3.7 that the $\mathcal{O}$-points of the Lie algebra of G are given by

$$\mathfrak{g}_\mathcal{O} = \mathrm{Lie}(G)(\mathcal{O}) = \{\, Y \in M_2(\mathcal{O}) \mid \mathrm{Tr}(Y) = 0 \,\}.$$

This is a free $\mathcal{O}$-module with basis e, f, h where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;\text{ and }\; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that w, h, u is *not* an $\mathcal{O}$-basis of $\mathfrak{g}_\mathcal{O}$. We have $e = \frac{1}{2}u - \frac{1}{2}w$ and $f = \frac{1}{2}u + \frac{1}{2}w$. Evaluating the form B on this basis (in the order e,f,h), we get the Gramian matrix

$$[B] = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This shows that the modulus factor $m(B)$ is given by

$$m(B) = \prod_{v \in V_f} |\det([B])|_v^{1/2} = \prod_{v \in V_\infty} |\det([B])|_v^{-1/2} = 2^r.$$

Notice that we use the product formula to replace the product over the finite places by the product over the archimedean places.

Substituting these results into the adelic formula IV.6.12 we obtain

$$\chi(\Gamma(\mathfrak{a})) = (-1)^r \, |d_F|^{3/2} \, 2^r (2^r \pi^{2r})^{-1} 2^{-r} \prod_{v \in V_f} \frac{\mathrm{N}(\mathfrak{p}_v)^{3\alpha_v}}{|U_v|}$$

For a non-trivial ideal $\mathfrak{b} \subseteq \mathcal{O}$ we define the norm $\mathrm{N}(\mathfrak{b}) := |\mathcal{O}/\mathfrak{b}|$. Let v be a finite place and let $\mathfrak{p}$ be the associated prime ideal in $\mathcal{O}$, then $\mathrm{N}(\mathfrak{p}) = \mathrm{N}(\mathfrak{p}_v)$. Moreover, it is well-known that $|\mathrm{SL}_2(\mathcal{O}/\mathfrak{p})| = \mathrm{N}(\mathfrak{p})(\mathrm{N}(\mathfrak{p})^2 - 1)$ (see 3.3.1 in [**Wil09**]). In total we obtain the identity

$$\chi(\Gamma(\mathfrak{a})) = (-1)^r \, |d_F|^{3/2} \, (2^r \pi^{2r})^{-1} \, \mathrm{N}(\mathfrak{a})^3 \zeta_F(2) \prod_{\mathfrak{p} | \mathfrak{a}} (1 - \mathrm{N}(\mathfrak{p})^{-2}).$$

Here ζ_F denotes the zeta function of the number field F. As a last step we apply the functional equation of the zeta function (cf. VII-§6, Thm. 3 in [**Wei95**]), which implies

$$\zeta_F(2) \, |d_F|^{3/2} \left(2\pi^2\right)^{-r} = (-1)^r \zeta_F(-1).$$

Theorem. Let $\mathfrak{a} \subseteq \mathcal{O}$ be a non-trivial proper ideal, such that the principal congruence group $\Gamma(\mathfrak{a})$ is torsion-free. Then the Euler characteristic of $\Gamma(\mathfrak{a})$ is given by the following formula:

$$\chi(\Gamma(\mathfrak{a})) = \zeta_F(-1) \, \mathrm{N}(\mathfrak{a})^3 \prod_{\mathfrak{p} | \mathfrak{a}} (1 - \mathrm{N}(\mathfrak{p})^{-2}).$$

This is a special case of Harder's results for Chevalley groups [**Har71**].

CHAPTER V

Lefschetz Numbers of Involutions of Symplectic Type

V.1. Introduction

V.1.1. In this section we compute the Lefschetz number of an involution of symplectic type on principal congruence subgroups of inner forms of the special linear group. For this purpose we combine the tools developed in the previous chapters. Remark V.1.2 explains why we may restrict our investigations to matrix algebras over quaternion algebras.

One should keep in mind that the central result is the adelic Lefschetz number formula II.6.8. Whenever we want to apply this theorem, there are two important steps to do. First step: understand the involved first non-abelian Galois cohomology sets. Second step: compute the Euler characteristic of the fixed point components. In the second step we use the adelic formula IV.6.12 obtained from Harder's Gauß-Bonnet Theorem.

In this section we introduce the employed notions and fix some notation. The aim of the second section is to determine various non-abelian cohomology sets. In the third section we describe the fixed point groups and we compute their Euler characteristics. The main theorem V.4.2 and some applications are given in Section V.4. We also give an application of the main theorem to arithmetically defined fuchsian groups in Section V.5. Finally, in Section V.6, we compute a Lefschetz number on Bianchi congruence groups to show that the developed tools can also be used to treat involutions of the second kind.

V.1.2. *Remark.* Let F be a number field and let A be a central simple F-algebra, which has an involution of the first kind $\sigma : A \to A$. Then A is isomorphic to the opposed F-algebra A^{op}. This means that the class of A has order two in the Brauer group of F. It follows from (32.19) in [**Rei03**] that A is isomorphic to a matrix algebra $M_n(D)$ over a quaternion algebra D.

Assume $A = M_n(D)$. Every quaternion algebra is equipped with a unique involution of symplectic type $\tau_c : D \to D$ which induces an involution of symplectic type $\tau : A \to A$ (see V.1.5). If, moreover, σ is of symplectic type, then $\sigma = \mathrm{int}(g) \circ \tau$ for an element $g \in \mathrm{GL}_n(D)$ with $\tau(g) = g$.

Without loss of generality we merely consider central simple algebras of the form $M_n(D)$. Due to the above observation it is only a minor restriction if we focus on the involution τ induced by the conjugation on the quaternion algebra.

V.1.3. *Notation.* The following notations and conventions will be used throughout the chapter (except for the last section V.6). Let F denote an algebraic number field and let $\mathcal{O}$ be its ring of integers. We denote the set of places of F by V, we further write V_∞ (resp. V_f) for the subset of archimedean (resp. non-archimedean) places. We will not distinguish a finite place from the associated prime ideal. Let $v \in V$ be a place of F, then we denote the completion of F at v by F_v. The valuation ring of F_v is denoted by $\mathcal{O}_v$ and the prime ideal in $\mathcal{O}_v$ is denoted by $\mathfrak{p}_v$. For a non-zero ideal $\mathfrak{a} \subseteq \mathcal{O}$ the ideal norm is defined by $\mathrm{N}(\mathfrak{a}) := |\mathcal{O}/\mathfrak{a}|$. As usual $\mathbb{A}$ denotes the ring of adeles of F and $\mathbb{A}_f$ is the ring of finite adeles.

Let D be a quaternion algebra over F, i.e. a central simple F-algebra of degree four. Note that, even though we use the symbol D, the quaternion algebra D may or may not be a division algebra. Given a place v, we define $D_v := D \otimes_F F_v$. If D_v is isomorphic to $M_2(F_v)$, we say that D *splits* at the place v. Otherwise D_v is a division algebra and we say that D is *ramified* at v. Let $\mathrm{Ram}(D) \subset V$ be the finite set places where D ramifies, and let $\mathrm{Ram}_f(D)$ (resp. $\mathrm{Ram}_\infty(D)$) denote the subset of finite (resp. archimedean) places.

V.1.4. Definition. The *signed reduced discriminant* $\Delta_{rd}(D)$ of D is the integer

$$\Delta_{rd}(D) := (-1)^r \prod_{\mathfrak{p} \in \mathrm{Ram}_f(D)} \mathrm{N}(\mathfrak{p}).$$

where $r = |\,\mathrm{Ram}_\infty(D)|$.

V.1.5. *The canonical involution.* On the quaternion algebra D we have the *canonical involution*, sometimes called *conjugation*,

$$\tau_c : D \to D \;\text{ denoted by }\; x \mapsto \overline{x}.$$

Given a description of D as $D = Q(a, b|F)$ with $a, b \in F^\times$, this is, there is a basis $1, i, j, ij$ of D with $i^2 = a$, $j^2 = b$ and $ij = -ji$, then the conjugation is defined by

$$\tau_c : x_0 + x_1 i + x_2 j + x_3 ij \mapsto x_0 - x_1 i - x_2 j - x_3 ij.$$

Note that the conjugation is F-linear, i.e. it is an involution of the first kind on D. Moreover, τ_c is an involution of symplectic type (cf. III.4.4).

The elements fixed by conjugation are precisely the elements of F. The conjugation is related to the reduced norm and trace of D by

$$\mathrm{trd}_D(x) = x + \overline{x},$$
$$\mathrm{nrd}_D(x) = x\overline{x} = \overline{x}x$$

for all x in D.

V.1.6. *Orders.* Let Λ_D be an $\mathcal{O}$-order in D. Then Λ_D is τ_c-stable, as can be seen as follows: Let $x \in \Lambda_D$, then

$$\overline{x} = x + \overline{x} - x = \mathrm{trd}_D(x) - x.$$

Recall that $\mathrm{trd}_D(x) \in \mathcal{O}$ because x is integral. Since $\mathcal{O} \subseteq \Lambda_D$, we obtain $\overline{x} \in \Lambda_D$. Moreover, it follows directly from the definitions that Λ_D is smooth if and only if Λ_D is τ_c-smooth (see III.3.9 and III.4.13 for the definitions).

We will assume from now on that Λ_D is a maximal $\mathcal{O}$-order in D. In particular, it is a smooth and τ_c-smooth order (see Lemma III.3.11).

V.1.7. Let n be a positive integer. Consider the central simple F-algebra

$$A := M_n(D)$$

of $n \times n$-matrices with entries in the quaternion algebra D. The canonical involution on D induces an involution τ on A defined by

$$\tau(x) := {}^{\tau}x := \overline{x}^T,$$

i.e. conjugate every entry in the matrix x and then transpose the matrix. It is easily checked that this defines an involution of symplectic type on A (cf. Lem. 1.18 in [**Kio10**] or (2.23) in [**KMRT98**]).

V.1.8. Lemma. *Let $\Lambda_D \subseteq D$ be a maximal $\mathcal{O}$-order. The $\mathcal{O}$-order $\Lambda = M_n(\Lambda_D)$ in A is maximal, τ-stable, smooth and τ-smooth.*

PROOF. Since Λ_D is stable under conjugation, it is obvious that Λ is τ-stable. Moreover, it follows from (21.6) in [**Rei03**] that Λ is a maximal $\mathcal{O}$-order. In turn Lemma III.3.11 shows that Λ is also a smooth order.

Finally we need to check that Λ is τ-smooth. Let $x \in \mathrm{Sym}(\Lambda, \tau)$ be an element which is fixed by τ. This means that $x = (x_{ij})$ satisfies

$$\begin{aligned} &x_{ij} = \overline{x_{ji}} \ \text{ for all } \ i \neq j, \ \text{ and} \\ &x_{ii} \in \mathcal{O}. \end{aligned}$$

The order Λ_D is smooth, therefore there is, for every $i = 1, \ldots, n$, an element $z_i \in \Lambda_D$ with $\mathrm{trd}_D(z_i) = z_i + \overline{z_i} = x_{ii}$. Now we define the upper triangular element $y \in \Lambda$ by

$$y_{ij} := \begin{cases} 0 & \text{if } i > j \\ x_{ij} & \text{if } i < j \\ z_i & \text{if } i = j, \end{cases}$$

and it is easy to see that $y + {}^{\tau}y = x$. We deduce that Λ is τ-smooth. □

V.1.9. We define $G := \mathrm{SL}_\Lambda$ to be the special linear group over the order Λ (see Def. III.3.8). From the previous lemma and III.3.9 we deduce that G is a smooth group scheme over $\mathcal{O}$. Moreover, the involution τ induces an automorphism τ^* of GL_Λ where $\tau^* = \mathrm{inv} \circ \tau$ (cf. III.4.11). Clearly τ^* has order two and it restricts to an automorphism of $G = \mathrm{SL}_\Lambda$.

The real Lie group G_∞ associated with G is

$$G_\infty := \prod_{v \in V_\infty} G(F_v) \ \cong \ \mathrm{SL}_{2n}(\mathbb{R})^s \times \mathrm{SL}_n(\mathbb{H})^r \times \mathrm{SL}_{2n}(\mathbb{C})^t.$$

Here s denotes the number real places of F where D splits, r is the number of real places where D ramifies, and t is the number of complex places of F. The symbol $\mathbb{H}$ is used for Hamilton's quaternion division algebra and $\mathrm{SL}_n(\mathbb{H})$ is the group of elements with reduced norm one in the central simple $\mathbb{R}$-algebra $M_n(\mathbb{H})$. Note that $[F : \mathbb{Q}] = r+s+t$. For every archimedean place v we fix a τ^*-stable maximal compact subgroup $K_v \subseteq G(F_v)$, then the group $K_\infty := \prod_{v \in V_\infty} K_v$ is a τ^*-stable maximal compact subgroup of G_∞.

The objective of this chapter is to study the cohomology of congruence subgroups, arising from the group scheme SL_Λ, using Lefschetz numbers. Let $\mathfrak{a} \subseteq \mathcal{O}$ be a proper ideal, we define the principal congruence subgroup

$$\Gamma(\mathfrak{a}) := \ker\big(G(\mathcal{O}) \to G(\mathcal{O}/\mathfrak{a})\big)$$

of level $\mathfrak{a}$. We shall always assume that $\Gamma(\mathfrak{a})$ is torsion-free (see Theorem III.2.3 for a sufficient condition on $\mathfrak{a}$). Note that the groups $\Gamma(\mathfrak{a})$ are always τ^*-stable.

These groups can be described by local data. Let $\mathfrak{p} \subseteq \mathcal{O}$ be a prime ideal of $\mathcal{O}$ and let v be the associated finite place. Let $\nu_\mathfrak{p}(\mathfrak{a})$ be the maximal exponent e such that $\mathfrak{p}^e$ divides $\mathfrak{a}$, then $\mathfrak{a}\mathcal{O}_v = \mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}\mathcal{O}_v$. We obtain an open and compact subgroup $K_v \subseteq G(\mathcal{O}_v)$ defined as

$$K_v := \ker\big(G(\mathcal{O}_v) \longrightarrow G(\mathcal{O}_v/\mathfrak{a}\mathcal{O}_v)\big).$$

We form the direct product $K_f := \prod_{v \in V_f} K_v$, this is an open and compact subgroup of the locally compact group $G(\mathbb{A}_f)$. Clearly, $\Gamma(\mathfrak{a}) = G(F) \cap K_f$.

V.1.10. *Objective.* Given a rational representation $\rho : G\times_{\mathcal{O}}\overline{F} \to \mathrm{GL}(V)$ defined over $\overline{F}$ with a compatible τ^*-action (see II.5.8), we consider the group cohomology $H^\bullet(\Gamma(\mathfrak{a}), V)$. Question: What is the Lefschetz number $\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), V)$? In this chapter we are going to answer this question.

As indicated above, the basic result is Rohlfs' Theorem in the adelic formulation II.6.8. In order to apply this theorem we determine the non-abelian Galois cohomology set $\mathcal{H}^1(\tau^*)$. Afterwards we analyse the fixed point components and we calculate their Euler characteristics. The main result can be found in Theorem V.4.2.

V.1.11. *Assumptions.* We keep the notation introduced in this section. Moreover, we always assume that

(i) The order Λ_D is a maximal order in D, and
(ii) the ideal $\mathfrak{a} \subseteq \mathcal{O}$ is non-trivial and chosen such that $\Gamma(\mathfrak{a})$ is torsion-free.

V.2. Hermitian Forms and Galois Cohomology

V.2.1. In this section we determine the non-abelian Galois cohomology set $\mathcal{H}^1(\tau^*)$. Recall that $\mathcal{H}^1(\tau^*)$ is the fibred product

$$\mathcal{H}^1(\tau^*) := H^1(\tau^*, K_\infty K_f) \underset{H^1(\tau^*,G(\mathbb{A}))}{\times} H^1(\tau^*, G(F))$$

defined in II.6.4. In order to determine this set we need to calculate local and global cohomology sets. The global problem is to determine $H^1(\tau^*, G(F))$, whereas locally we have to calculate $H^1(\tau^*, G(F_v))$ and $H^1(\tau^*, K_v)$ for every place v. This task amounts to the classification of certain hermitian forms over quaternion algebras, which is basically well-known (see for instance [**Shi63**, §2] or [**Sch85**, Ch. 10]). We try to present the theory in a way that fits best to the applications we have in mind.

V.2.2. Lemma (cf. Prop. 2.1 in [**Shi63**]). *Let k be a field (*$\mathrm{char}(k) \neq 2$*) and let H be a quaternion k-algebra with canonical involution τ_c. Let E be a free right H-module of finite rank equipped with a regular hermitian form $b : E \times E \to H$. Then E has an H-basis $e_1, \dots, e_m$ such that*

$$b(e_i, e_j) = 0 \quad \textit{for all } i \neq j$$

and $b(e_i, e_i) \in k^\times$. Moreover, given a set of representatives $R \subseteq k^\times$ for the classes $k^\times / \mathrm{nrd}_H(H^\times)$, then the basis can be chosen such that $b(e_i, e_i) \in R$ for every i.

Proof. Claim: There is $x \in E$ such that $b(x, x) \in k^\times$. Suppose $b(x, x) = 0$ for all $x \in E$, then

$$0 = b(x + y, x + y) - b(x, x) - b(y, y) = \mathrm{trd}_H(b(x, y))$$

for all $x, y \in E$. This contradicts the assumption that b is regular.

Take $e_1 \in E$ with $b(e_1, e_1) \in K^\times$. The space $V = xH$ with $b_{|V}$ is regular, thus E is a direct sum $E = V \oplus V^\perp$ (see Ch. 7, Thm. 1.4 in [**Sch85**]). The k-algebra H is simple and has precisely one isomorphism class of simple modules. Thus, for dimension reasons, $V^\perp$ is a free H-module of smaller rank and the assertion follows by induction on the rank.

Note further that replacing e_1 by $e_1\lambda$ for some $\lambda \in H^\times$ yields

$$b(e_1\lambda, e_1\lambda) = \mathrm{nrd}_H(\lambda) b(e_1, e_1). \qquad \square$$

V.2.3. The previous lemma is already the key step towards the classification of quaternion hermitian forms since the set of classes $k^\times / \mathrm{nrd}_H(H^\times)$ is mostly pretty small.

We introduce the following notation: Given two integers $p, q \geq 0$ with $p+q = n$ we define the diagonal matrix

$$I_{p,q} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q})$$

in $M_n(\Lambda_D)$.

V.2.4. Corollary. *Let $v \in V$ be a place of F. If v is a real place where D is ramified, then*

$$H^1(\tau^*, \mathrm{GL}_\Lambda(F_v)) \cong \{\, I_{p,q} \mid p, q \geq 0 \text{ with } p+q = n \,\}.$$

This means that the matrices $I_{p,q}$ are a system of representatives for the cohomology classes. The cohomology is trivial for all places $v \in V \setminus \mathrm{Ram}_\infty(D)$, i.e.

$$H^1(\tau^*, \mathrm{GL}_\Lambda(F_v)) = \{1\}.$$

Proof. Let $b \in Z^1(\tau^*, \mathrm{GL}_\Lambda(F_v))$ be a cocycle, this is, b is an element of $\mathrm{GL}_n(D_v)$ satisfying $b = {}^\tau b$. Such a matrix b defines a regular hermitian form on the free right D_v-module D_v^n. By Lemma V.2.2 there is $g \in \mathrm{GL}_n(D_v)$ such that $gb\,{}^\tau g$ is diagonal and we can achieve that the entries are contained in a set of representatives of the classes in $F_v^\times / \mathrm{nrd}_{D_v}(D_v^\times)$. For all $v \in V \setminus \mathrm{Ram}_\infty(D)$, i.e. v is not a real ramified place, the reduced norm $D_v^\times \to F_v^\times$ is surjective (see (33.4) in [**Rei03**]). The second assertion follows immediately.

Let $v \in \mathrm{Ram}_\infty(D)$, then $D_v \cong \mathbb{H}$. In this case the image of the reduced norm is the group of positive real numbers, i.e. $\mathrm{nrd}_{D_v}(D_v^\times) = \mathbb{R}^\times_{>0}$. Therefore the set $\{\, I_{p,q} \mid p, q \geq 0 \text{ with } p+q = n \,\}$ clearly contains a representative of every cohomology class.

We need to check that $I_{p,q}$ is cohomologous to $I_{p',q'}$ only if $p = p'$ and $q = q'$. However, the notion of *positive definite* space is well defined for hermitian forms over $\mathbb{H}$, and p is the $\mathbb{H}$-dimension of a maximal positive definite subspace for the form defined by $I_{p,q}$. This determines p uniquely and the claim follows (more details on definite spaces can be found in [**Kio10**, 1.34]). □

V.2.5. Definition. Let $v \in \mathrm{Ram}_\infty(D)$. For a cocycle $b \in Z^1(\tau^*, \mathrm{GL}_\Lambda(F_v))$ which is cohomologous to $I_{p,q}$ we say that the *signature* of b is the pair (p,q).

V.2.6. Corollary. *Let $v \in V_f$ be a finite place, then $H^1(\tau^*, \mathrm{GL}_\Lambda(\mathcal{O}_v)) = \{1\}$.*

Proof. The $\mathcal{O}$-order Λ is maximal and τ-smooth (see V.1.8) and the same holds for the $\mathcal{O}_v$-order $\Lambda \otimes \mathcal{O}_v$ (cf. (11.6) in [**Rei03**] and Remark III.4.14). By the theorem of Fainsilber-Morales (Thm. III.5.9) the canonical map

$$H^1(\tau^*, \mathrm{GL}_\Lambda(\mathcal{O}_v)) \to H^1(\tau^*, \mathrm{GL}_\Lambda(F_v))$$

is injective, and hence the assertion follows immediately from Corollary V.2.4. □

V.2.7. Now we start to attack the global problem and analyse $H^1(\tau^*, \mathrm{GL}_\Lambda(F))$. For a place $v \in \mathrm{Ram}_\infty(D)$ let $\iota_v : F \to F_v$ denote the associated embedding. Consider

$$F_D^\times := \{\, x \in F^\times \mid \iota_v(x) > 0 \text{ for all } v \in \mathrm{Ram}_\infty(D) \,\}.$$

Due to a theorem of Hasse-Schilling-Maass $F_D^\times$ is the image of the reduced norm $\mathrm{nrd}_D \colon D^\times \to F^\times$ (see [**Rei03**, (33.15)]). Moreover, F is a dense subset embedded diagonally in $\prod_{v\in\mathrm{Ram}_\infty(D)} F_v$, thus

$$F^\times/F_D^\times \cong \prod_{v\in\mathrm{Ram}_\infty(D)} \mathbb{R}^\times/\mathbb{R}^\times_{>0} \cong \{\pm 1\}^r.$$

We need a variation of Lemma V.2.2.

V.2.8. Lemma. *Assume D is a quaternion division algebra and let $R \subset F^\times$ be a set of representatives of the classes in $F^\times/F_D^\times$. Let E be a finite dimensional right D-vector space with a regular hermitian form $b : E \times E \to D$. There is a D-basis $e_1, \dots, e_m$ of E such that*

(i) $b(e_i, e_j) = 0$ for all $i \neq j$,
(ii) $b(e_i, e_i) \in R$ for all $i = 1, \dots, m$, and
(iii) for all $v \in \mathrm{Ram}_\infty(D)$ and every $i = 1, \dots, m-1$:

$$\iota_v(b(e_i, e_i)) < 0 \implies \iota_v(b(e_{i+1}, e_{i+1})) < 0.$$

Proof. We proceed by induction on the dimension m of E. For $m = 1$ there is nothing to show. We assume $m > 1$. As in the proof of Lemma V.2.2 we see that there is $x \in E$ with $b(x, x) \neq 0$. For every such x we define a set of places

$$S(x) := \{\, v \in \mathrm{Ram}_\infty(D) \mid \iota_v(b(x, x)) > 0 \,\}.$$

Choose $e_1 \in E$ such that $b(e_1, e_1) \neq 0$ and such that $|S(e_1)|$ is maximal among all such elements. After multiplication with an element in $D^\times$ we can assume $b(e_1, e_1) \in R$. Note that this does not change the set $S(e_1)$, i.e. $S(e_1) = S(e_1\mu)$ for all $\mu \in D^\times$. The one-dimensional space $V := e_1 D$ is regular for the form b and we get an orthogonal decomposition $E = V \oplus V^\perp$ (cf. [**Sch85**, 7.1.4]). By the induction hypothesis there is a basis $e_2, \dots, e_m$ of $V^\perp$ which satisfies the claimed properties. It is clear that $e_1, \dots, e_m$ is a basis of E which meets the first two conditions.

It remains to show that $v \notin S(e_1)$ implies $v \notin S(e_2)$ for every real place v. Suppose that there is a place $v \in \mathrm{Ram}_\infty(D)$ which is in $S(e_2)$ but not in $S(e_1)$. Choose $\mu \in F^\times$ such that

- $\iota_v(\mu)^2 > \frac{|\iota_v(b(e_1,e_1))|}{\iota_v(b(e_2,e_2))}$, and
- $\iota_w(\mu)^2 < \frac{\iota_w(b(e_1,e_1))}{|\iota_w(b(e_2,e_2))|}$ for all $w \in S(e_1)$.

Consider $x := e_1 + e_2\mu$. This element satisfies

$$\iota_v(b(x, x)) = \iota_v(b(e_1, e_1)) + \iota_v(\mu)^2 \iota_v(b(e_2, e_2)) > 0.$$

Moreover, for any place $w \in S(e_1)$ we obtain

$$\iota_w(b(x, x)) = \iota_w(b(e_1, e_1)) + \iota_w(\mu)^2 \iota_w(b(e_2, e_2)) > 0.$$

We have shown that $|S(x)| > |S(e_1)|$, which contradicts the choice of e_1, and we conclude that there is no such place v. $\square$

V.2.9. Proposition (Hasse principle). *The canonical map*

$$f : H^1(\tau^*, \mathrm{GL}_\Lambda(F)) \longrightarrow \prod_{v\in\mathrm{Ram}_\infty(D)} H^1(\tau^*, \mathrm{GL}_\Lambda(F_v))$$

induced by the inclusions is bijective. This means a class in $H^1(\tau^*, \mathrm{GL}_\Lambda(F))$ is uniquely determined by its signatures at the real ramified places.

Proof. If the set $\mathrm{Ram}_\infty(D)$ is empty, then $H^1(\tau^*, \mathrm{GL}_\Lambda(F)) = \{1\}$ due to Lemma V.2.2 and the fact that $\mathrm{nrd}_D(D^\times) = F^\times$. Hence there is nothing to show.

Assume that $\mathrm{Ram}_\infty(D) \neq \emptyset$, in particular D is a division algebra. The map f is surjective since $F^\times$ is dense in $\prod_{v\in\mathrm{Ram}_\infty(D)} F_v$. More precisely, we have elements in $F^\times$ which embed with arbitrary combination of signs in $\prod_{v\in\mathrm{Ram}_\infty(D)} F_v$. The diagonal matrices having these elements as entries yield every local combination of signatures.

Let $R \subseteq F^\times$ be a set of representatives for the classes in $F^\times/F_D^\times$. We apply Lemma V.2.8 to see that every class in $H^1(\tau^*, \mathrm{GL}_\Lambda(F))$ can be represented by a diagonal matrix b with entries $b_i \in R$ such that

$$\iota_v(b_i) < 0 \implies \iota_v(b_{i+1}) < 0$$

for every place $v \in \mathrm{Ram}_\infty(D)$ and every index i. Since there are precisely $(n+1)^r$ many such matrices, we can deduce that the map f is bijective. □

V.2.10. *The pfaffian of diagonal matrices.* We want to explain how to compute the pfaffian associated with τ (see III.4.9) for diagonal matrices. Let k be any extension field of F, for example a local completion. Given a diagonal matrix $x = \mathrm{diag}(x_1, \dots, x_n)$ with entries in k, we can consider x as a τ-fixed matrix in $A \otimes_F k = M_n(D \otimes_F k)$.

Lemma. For $x = \mathrm{diag}(x_1, \dots, x_n)$ with entries in some extension field k of F, the Pfaffian of x is the product of all entries, i.e.

$$\mathrm{pf}_\tau(x) = x_1 x_2 \cdots x_n.$$

Proof. We can assume without loss of generality that k is algebraically closed. In this case $D \otimes_F k \cong M_2(k)$ and the reduced norm $\mathrm{nrd}_D : D \otimes_F k \to k$ agrees with the determinant, in particular it is surjective. This means, for given $i \in \{1, \dots, n\}$, we can write $x_i = \mathrm{nrd}_D(y_i) = \overline{y_i} y_i$ for some $y_i \in D \otimes_F k$. Consider the matrix $y = \mathrm{diag}(y_1, \dots, y_n) \in M_n(D \otimes_F k)$, this matrix satisfies $\tau(y)y = x$. By Lemma III.4.10 we obtain

$$\mathrm{pf}_\tau(x) = \mathrm{nrd}_A(y) = \prod_{i=1}^n \mathrm{nrd}_D(y_i) = \prod_{i=1}^n x_i.$$

Here we used that the reduced norm of a diagonal matrix in $M_n(D \otimes_F k)$ is the product of the reduced norms of the entries (see IX, §2, Cor. 2 in [**Wei95**]). □

Corollary. Let C be a commutative $\mathcal{O}$-algebra which is an integral domain of characteristic 0, then the pfaffian $\mathrm{pf}_\tau : \mathrm{Sym}(\Lambda, \tau) \otimes_{\mathcal{O}} C \to C$ is surjective.

Proof. Let k be the quotient field of C. By assumption $\mathcal{O}$ injects into C, thus k is an extension field of F. Note that the canonical map

$$\mathrm{Sym}(\Lambda, \tau) \otimes_{\mathcal{O}} C \to \mathrm{Sym}(\Lambda, \tau) \otimes_{\mathcal{O}} k$$

is injective since $\mathrm{Sym}(\Lambda, \tau)$ is a projective $\mathcal{O}$-module.

Let $c \in C$ be given. The matrix $x = \mathrm{diag}(c, 1, \dots, 1) \in \Lambda \otimes_{\mathcal{O}} C$ actually lies in $\mathrm{Sym}(\Lambda, \tau) \otimes_{\mathcal{O}} C$. The lemma, applied with the field k, yields $\mathrm{pf}_\tau(x) = c$. □

V.2.11. The final step in this section is to transfer the results on non-abelian Galois cohomology with values in GL_Λ to the group $G = \mathrm{SL}_\Lambda$. Our main tool is the cohomological diagram for symplectic involutions III.5.4.

V.2.12. Lemma. *Let k be an extension field of F such that $\mathrm{nrd}_\Lambda : \mathrm{GL}_\Lambda(k) \to k^\times$ is onto. Then the Pfaffian induces a bijection*

$$\mathrm{pf}_\tau : H^1(\tau^*, \mathrm{SL}_\Lambda(k)) \xrightarrow{\simeq} \{\pm 1\}.$$

This holds in particular for $k = F_v$ where $v \in V \setminus \mathrm{Ram}_\infty(D)$ is a place.

Proof. As in Corollary V.2.4 one can show that the first non-abelian cohomology set $H^1(\tau^*, \mathrm{GL}_\Lambda(k))$ is trivial. The cohomological diagram for symplectic involutions (see III.5.4) collapses to

$$\begin{array}{ccccc} \{\pm 1\} & \xrightarrow{\delta} & H^1(\tau^*, \mathrm{SL}_\Lambda(k)) & \longrightarrow & 1 \\ \Vert & & \downarrow{\scriptstyle \mathrm{pf}_\tau} & & \\ \{\pm 1\} & = & \{\pm 1\}. & & \end{array}$$

Here we used that $\mathrm{nrd}_\Lambda : \mathrm{GL}_\Lambda(k) \to k^\times$ is surjective . By III.5.4 the morphism δ is injective, and thus bijective.

Recall that the reduced norm $\mathrm{nrd}_\Lambda : \mathrm{GL}_\Lambda(F_v) \to F_v^\times$ is surjective for any place $v \in V$ which is not a real place ramified in D (see (33.4) in [**Rei03**]). □

V.2.13. Lemma. *Let $v \in \mathrm{Ram}_\infty(D)$. The canonical map*

$$j_* : H^1(\tau^*, \mathrm{SL}_\Lambda(F_v)) \to H^1(\tau^*, \mathrm{GL}_\Lambda(F_v))$$

is bijective.

Proof. In this case the reduced norm takes only positive values in $F_v \cong \mathbb{R}$. Therefore the cohomological diagram for symplectic involutions III.5.4 yields

$$\begin{array}{ccccc} 1 & \longrightarrow & H^1(\tau^*, \mathrm{SL}_\Lambda(F_v)) & \xrightarrow{j_*} & H^1(\tau^*, \mathrm{GL}_\Lambda(F_v)) \\ & & \downarrow{\scriptstyle \mathrm{pf}_\tau} & & \downarrow{\scriptstyle \mathrm{pf}_\tau} \\ 1 & \longrightarrow & \{\pm 1\} & \xrightarrow{\simeq} & \mathbb{R}^\times/\mathbb{R}^\times_{>0} \end{array}$$

It follows directly from Corollary III.5.5 that j_* is surjective. Moreover, twisting the upper row with cocycles for $H^1(\tau^*, \mathrm{SL}_\Lambda(F_v))$ shows that j_* is indeed injective. For more details on twisting in non-abelian cohomology the reader may consult [**Ser64**, I,5.4]. Note that twisting an involution of symplectic type gives an involution of symplectic type (see III.5.6). □

V.2.14. Lemma. *Let v be a finite place and let $\mathfrak{p}_v \subseteq \mathcal{O}_v$ be the prime ideal. For an integer $m \geq 0$ we define $K_v(m) := \ker(G(\mathcal{O}_v) \to G(\mathcal{O}_v/\mathfrak{p}_v^m))$. Then the pfaffian induces a bijection*

$$\mathrm{pf}_\tau : H^1(\tau^*, K_v(m)) \xrightarrow{\simeq} \begin{cases} \{\pm 1\} & \text{if } -1 \equiv 1 \mod \mathfrak{p}_v^m, \\ \{1\} & \text{otherwise}. \end{cases}$$

Proof. We start with the special case $m = 0$, this is $K_v(m) = \mathrm{SL}_\Lambda(\mathcal{O}_v)$. Here the claim follows just as in the proof of Lemma V.2.12 from the cohomological diagram for symplectic involutions III.5.4, Corollary V.2.6, and the fact that the reduced norm $\mathrm{nrd}_\Lambda : \mathrm{GL}_\Lambda(\mathcal{O}_v) \to \mathcal{O}_v^\times$ is onto (combine (14.1) and Ex. 5, p. 152 in [**Rei03**]).

For $m \geq 1$ consider the short exact sequence of groups

$$1 \longrightarrow K_v(m) \longrightarrow \mathrm{SL}_\Lambda(\mathcal{O}_v) \longrightarrow \mathrm{SL}_\Lambda(\mathcal{O}_v/\mathfrak{p}_v^m) \longrightarrow 1.$$

Note that this sequence uses that the order Λ, and hence the group scheme SL_Λ, is smooth (cf. V.1.8). We obtain a long exact sequence of pointed sets

$$G^{\tau^*}(\mathcal{O}_v) \xrightarrow{\pi} G^{\tau^*}(\mathcal{O}_v/\mathfrak{p}_v^m) \xrightarrow{\delta} H^1(\tau^*, K_v(m)) \xrightarrow{j_m} H^1(\tau^*, G(\mathcal{O}_v)).$$

It follows from Remark III.5.2 that the fixed point group G^{τ^*} is just the group scheme $G(\Lambda, \tau)$ defined in III.4.11. Since the group scheme $G(\Lambda, \tau)$ is smooth (Thm. III.4.15), the canonical map π is surjective, and so δ is trivial. Via twisting (cf. III.5.6) we obtain that j_m is injective.

We use that the pfaffian is a morphism of schemes defined over $\mathcal{O}$ (cf. III.4.9): Given a cocycle $b \in Z^1(\tau^*, K_v(m))$, we have $\mathrm{pf}_\tau(b) \equiv 1 \mod \mathfrak{p}_v^m$. Consequently, if 1 and -1 are not congruent modulo $\mathfrak{p}_v^m$, then $H^1(\tau^*, K_v(m)) = \{1\}$ and the claim follows.

Assume now that $-1 \equiv 1 \mod \mathfrak{p}_v^m$, then the matrix $\mathrm{diag}(-1, 1, \ldots, 1)$ lies in $K_v(m)$ and has pfaffian -1 (cf. V.2.10). $\square$

V.2.15. Lemma. *Assume that* $\mathrm{Ram}_\infty(D)$ *is not empty. Then the canonical morphism of pointed sets*

$$j_* : H^1(\tau^*, \mathrm{SL}_\Lambda(F)) \longrightarrow H^1(\tau^*, \mathrm{GL}_\Lambda(F))$$

is injective. The image consists of precisely those classes $x \in H^1(\tau^*, \mathrm{GL}_\Lambda(F))$ *which satisfy* $\mathrm{pf}_\tau(x) = \pm 1 \cdot F_D^\times$.

If otherwise D *splits at every real place, then the Pfaffian induces a bijection*

$$\mathrm{pf}_\tau : H^1(\tau^*, \mathrm{SL}_\Lambda(F)) \xrightarrow{\simeq} \{\pm 1\}.$$

Proof. Assume that $\mathrm{Ram}_\infty(D)$ is empty. By the Hasse-Schilling-Maass Theorem (cf. (33.15) in [**Rei03**]) the reduced norm $\mathrm{GL}_\Lambda(F) \to F^\times$ is surjective. Hence the second assertion follows from Lemma V.2.12.

Now we assume that $\mathrm{Ram}_\infty(D)$ is not empty. The image of the reduced norm $\mathrm{nrd}_A : A^\times \to F^\times$ is $F_D^\times$ which can not contain the element -1 since $\mathrm{Ram}_\infty(D)$ is not empty. Look at the cohomological diagram for symplectic involutions III.5.4

$$\begin{array}{ccccc}
1 & \longrightarrow & H^1(\tau^*, \mathrm{SL}_\Lambda(F)) & \xrightarrow{j_*} & H^1(\tau^*, \mathrm{GL}_\Lambda(F)) \\
 & & \big\downarrow \mathrm{pf}_\tau & & \big\downarrow \mathrm{pf}_\tau \\
1 & \longrightarrow & \{\pm 1\} & \longrightarrow & F^\times / F_D^\times
\end{array}.$$

Twisting shows that the map j_* is injective. The assertion about the image of j_* follows immediately from Corollary III.5.5. $\square$

V.2.16. *Remark.* Assume that $\mathrm{Ram}_\infty(D)$ is not empty. Let $x \in H^1(\tau^*, \mathrm{GL}_\Lambda(F))$ be a cohomology class. For every place $v \in \mathrm{Ram}_\infty(D)$ the class x considered as a class in $H^1(\tau^*, \mathrm{GL}_\Lambda(F_v))$ has a local signature (p_v, q_v). Then according to Lemma V.2.15 the class x lies in the image of j_* if and only if

$$q_v \equiv q_w \mod 2$$

for every pair of places $v, w \in \mathrm{Ram}_\infty(D)$. This means that either all q_v are even or all q_v are odd.

V.2.17. Theorem. *Let* $K_f = \prod_{v \in V_f} K_v \subseteq G(\mathbb{A}_f)$ *be the open compact subgroup associated with the congruence subgroup* $\Gamma(\mathfrak{a})$ *(cf. V.1.9). Consider the set* $\mathcal{H}^1(\tau^*)$ *defined in V.2.1. The projection* $\pi : \mathcal{H}^1(\tau^*) \to H^1(\tau^*, G(F))$ *is injective and there is a short exact sequence of pointed sets*

$$1 \longrightarrow \mathcal{H}^1(\tau^*) \xrightarrow{\pi} H^1(\tau^*, G(F)) \xrightarrow{\mathrm{pf}_\tau} \{\pm 1\} \longrightarrow 1.$$

Proof. Consider the cohomology set $H^1(\tau^*, K_\infty K_f)$, which is canonically isomorphic to the direct product $H^1(\tau^*, K_\infty) \times H^1(\tau^*, K_f)$. Recall that the canonical map $H^1(\tau^*, K_\infty) \to H^1(\tau^*, G_\infty)$ is bijective (see II.4.4). Moreover, for every finite place $v \in V_f$ the group K_v is of the form

$$K_v(m) = \ker\bigl(G(\mathcal{O}_v) \to G(\mathcal{O}_v/\mathfrak{p}_v^m)\bigr)$$

for some integer m. It follows from Lemma V.2.14 and Lemma V.2.12 that the inclusion $K_v \to G(F_v)$ induces an injection $H^1(\tau^*, K_v) \to H^1(\tau^*, G(F_v))$. Therefore the canonical map $H^1(\tau^*, K_\infty K_f) \to H^1(\tau^*, G(\mathbb{A}))$ is injective and we conclude that the projection $\pi : \mathcal{H}^1(\tau^*) \to H^1(\tau^*, G(F))$ is injective.

Moreover, we know from the considerations on diagonal matrices in V.2.10 that the pfaffian $\mathrm{pf}_\tau : H^1(\tau^*, G(F)) \to \{\pm 1\}$ is surjective.

It remains to understand the image of π. Since $\Gamma(\mathfrak{a})$ is (by assumption) torsion-free, we know that -1 is not congruent 1 modulo $\mathfrak{a}$. In particular there is a prime ideal $\mathfrak{p}$ which divides $\mathfrak{a}$, say $e = \nu_\mathfrak{p}(\mathfrak{a})$, such that 1 and -1 are not congruent modulo $\mathfrak{p}^e$. Let $v \in V_f$ be the finite place associated with $\mathfrak{p}$, then $K_v = K_v(e)$ and $H^1(\tau^*, K_v) = \{1\}$ by Lemma V.2.14. Let $\gamma \in H^1(\tau^*, G(F))$ be in the image of π, say (x, γ) is the inverse image in $\mathcal{H}^1(\tau^*)$. Let x_v be the projection of the class x to $H^1(\tau^*, K_v)$. Since x and γ have the same image in $H^1(\tau^*, G(\mathbb{A}))$, we can deduce that $\mathrm{pf}_\tau(\gamma) = \mathrm{pf}_\tau(x_v) = 1$.

Conversely, given $\gamma \in H^1(\tau^*, G(F))$ in the kernel of the pfaffian, then γ lies in the image of π. Let $c_\infty \in H^1(\tau^*, K_\infty)$ be a cohomology class such that c_∞ and γ define the same class in $H^1(\tau^*, G_\infty)$. Let 1_f denote the trivial class in $H^1(\tau^*, K_f)$, then the triple $(c_\infty, 1_f, \gamma)$ is a class in $\mathcal{H}^1(\tau^*)$ which is mapped to γ by π. □

V.3. The Fixed Point Groups

V.3.1. *Assumption.* Up to paragraph V.3.24 the number field F is *totally real.*

V.3.2. Definition. Let R be a commutative $\mathcal{O}$-algebra (e.g. $\mathcal{O}_v$ or F_v). For a cocycle $\gamma \in Z^1(\tau^*, G(R))$ the R-group scheme $G(\gamma)$ of $\tau^*|\gamma$-fixed points is defined by

$$G(\gamma)(C) := \{\, g \in G(C) \mid g = {}^{\tau^*|\gamma}g \,\}$$

for any commutative R-algebra C. Recall that the γ-twisted τ^*-action is given by ${}^{\tau^*|\gamma}g = \gamma\, {}^{\tau^*}g \gamma^{-1}$.

V.3.3. *Notation.* We define the symplectic group Sp_n over $\mathbb{Z}$ by

$$\mathrm{Sp}_n(R) := \{\, g \in \mathrm{GL}_{2n}(R) \mid g^T J g = J \,\},$$

for every commutative ring R, where J is the standard symplectic matrix

$$J = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$$

Note that in this notation Sp_n is of rank n, but consists of matrices of size $2n \times 2n$.

V.3.4. Given a cocycle $\gamma \in Z^1(\tau^*, G(\mathcal{O}))$ we want to understand the associated group scheme $G(\gamma)$. In particular we want to calculate the Euler characteristic of congruence subgroups of this group. We start with some basic observations and afterwards we collect all the ingredients necessary for an application of the adelic Euler characteristic formula IV.6.12.

For simplicity we define $H := G(1)$, which is a group scheme over $\mathcal{O}$.

V.3.5. *Remark.* Note that the definition of the groups $G(\gamma)$ in paragraph II.6.6 is in some sense a special case of the above definition. The notation is a little bit sloppy since the underlying ring R is not mentioned explicitly. However, we try to point out the domain of definition whenever this seems to be important.

Notice further that, if $\gamma \in Z^1(\tau^*, G(\mathcal{O}))$, then $G(\gamma) = G(\Lambda, \tau|\gamma)$ in the notation of paragraph III.4.11. The reason for this identity is that $G(\Lambda, \tau|\gamma)$ is always a closed subscheme of SL_Λ, i.e. all elements have reduced norm one (see III.5.2). Here $\tau|\gamma$ is the γ-twisted involution on A (cf. III.5.6). Recall that $\tau|\gamma$ is of symplectic type, and that twisting and the operation $*$ commute, i.e. $(\tau|\gamma)^* = \tau^*|\gamma$.

V.3.6. Lemma. *For every $\gamma \in Z^1(\tau^*, G(\mathcal{O}))$ the group scheme $G(\gamma)$ is smooth.*

Proof. By Lemma V.1.8 the order Λ is τ-smooth. Remark III.5.6 implies further that Λ is $(\tau|\gamma)$-smooth as well, and thus Theorem III.4.15 yields that $G(\gamma) = G(\Lambda, \tau|\gamma)$ is smooth. □

V.3.7. Lemma. *Let R be a commutative $\mathcal{O}$-algebra. Suppose the two cocycles γ, γ' in $Z^1(\tau^*, G(R))$ define the same class in $H^1(\tau^*, \mathrm{GL}_\Lambda(R))$, then $G(\gamma)$ and $G(\gamma')$ are isomorphic as group schemes over R.*

Proof. There is $c \in \mathrm{GL}_\Lambda(R)$ which satisfies $\gamma' = c\gamma\, {}^\tau c$. We define a morphism of group schemes $f : G(\gamma) \to G(\gamma')$ by

$$f_C : g \mapsto cgc^{-1}$$

for every commutative R-algebra C and all $g \in G(\gamma)(C)$. This map is well-defined:

$${}^{\tau^*|\gamma'}(cgc^{-1}) = \gamma'\, {}^{\tau^*}c\, {}^{\tau^*}g\, {}^{\tau^*}c^{-1} \gamma'^{-1} = c\gamma\, {}^{\tau^*}g \gamma^{-1} c^{-1} = cgc^{-1}.$$

The inverse map of f is obviously given by $g \mapsto c^{-1}gc$, thus f is an isomorphism. □

V.3.8. Corollary. *Let $\gamma \in Z^1(\tau^*, G(\mathcal{O}))$ be a cocycle. Given a commutative $\mathcal{O}$-algebra R with $H^1(\tau^*, \mathrm{GL}_\Lambda(R)) = \{1\}$, there is an isomorphism of R-group schemes*

$$G(\gamma) \times_{\mathcal{O}} R \xrightarrow{\simeq} G(1) \times_{\mathcal{O}} R.$$

In particular, this holds if $R = \mathcal{O}_v$ for $v \in V_f$ or if $R = k$ is an extension field of F such that the reduced norm $\mathrm{nrd}_\Lambda : \mathrm{GL}_\Lambda(k) \to k^\times$ is surjective (see V.2.6 and V.2.2).

Moreover, if k is a splitting field of D, then $G(\gamma) \times_{\mathcal{O}} k$ is isomorphic to the symplectic group $\mathrm{Sp}_n \times_{\mathbb{Z}} k$ defined over k.

Proof. The first part follows immediately from Lemma V.3.7. For the second assertion note that we can choose a splitting $\varphi : A \otimes k \to M_{2n}(k)$ such that $\varphi(\tau(x))$ equals $J\varphi(x)^T J^{-1}$, where J denotes the standard symplectic matrix (see V.3.3). □

V.3.9. Corollary. *Let $\gamma \in Z^1(\tau^*, G(\mathcal{O}))$ be a cocycle and let $v \in \mathrm{Ram}_\infty(D)$ be a real ramified place. If the class of γ in $H^1(\tau^*, G(F_v))$ has signature (p, q), then there is an isomorphism of real Lie groups*

$$G(\gamma)(F_v) \xrightarrow{\simeq} \mathrm{Sp}(p, q)$$

defined over $F_v \cong \mathbb{R}$. Here $\mathrm{Sp}(p, q)$ is the real Lie group defined by

$$\mathrm{Sp}(p, q) := \{\, g \in \mathrm{GL}_n(\mathbb{H}) \mid \overline{g}^T I_{p,q} g = I_{p,q} \,\}.$$

Proof. This follows from Lemma V.3.7 and the description of the cohomology set $H^1(\tau^*, \mathrm{GL}_\Lambda(F_v))$ in Lemma V.2.4. □

V.3.10. Let $\gamma \in Z^1(\tau^*, G(\mathcal{O}))$ be a cocycle. Consider the real Lie group

$$G(\gamma)_\infty = \prod_{v \in V_\infty} G(\gamma)(F_v)$$

associated with the group $G(\gamma)$. For a real ramified place $v \in \mathrm{Ram}_\infty(D)$ let (p_v, q_v) denote the local signature of the cohomology class of γ in $H^1(\tau^*, G(F_v))$. It follows readily from the Corollaries V.3.8 and V.3.9 that there is an isomorphism of real Lie groups

$$G(\gamma)_\infty \xrightarrow{\simeq} \mathrm{Sp}_n(\mathbb{R})^s \times \prod_{v \in \mathrm{Ram}_\infty(D)} \mathrm{Sp}(p_v, q_v).$$

Here s denotes the number of real places of F which split D. Note that $G(\gamma)_\infty$ is connected and semisimple. The real Lie algebra $\mathfrak{g}(\gamma)_\infty$ of $G(\gamma)_\infty$ is isomorphic to

$$\mathfrak{g}(\gamma)_\infty \cong \mathfrak{sp}(n, \mathbb{R})^s \oplus \bigoplus_{v \in \mathrm{Ram}_\infty(D)} \mathfrak{sp}(p_v, q_v).$$

V.3.11. *Maximal compact subgroups.* Recall that every maximal compact subgroup of the real Lie group $\mathrm{Sp}_n(\mathbb{R})$ is isomorphic to the unitary group $\mathrm{U}(n)$ (see for instance Prop. 2.22 in [**MS98**]).

Consider the group $\mathrm{Sp}(n) := \mathrm{Sp}(n, 0)$. One can check that this is a compact connected semisimple real Lie group (see [**Kna02**, p. 111]). Moreover, it is a maximal compact subgroup of the special linear group $\mathrm{SL}_n(\mathbb{H})$.

Let $p, q \geq 0$ be integers with $p + q = n$. The Lie group $\mathrm{Sp}(p, q)$ is connected and semisimple [**Kna02**, Prop. 1.145]. The compact group $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$ is, in an obvious way, a subgroup of $\mathrm{Sp}(p, q)$.

Lemma. The group $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$ is a maximal compact subgroup of $\mathrm{Sp}(p, q)$.

PROOF. Consider the automorphism $\sigma := \tau^* | I_{p,q}$ of $\mathrm{SL}_n(\mathbb{H})$. The maximal compact subgroup $\mathrm{Sp}(n)$ is σ-stable since we have

$$I_{p,q} x^{-1} I_{p,q} I_{p,q} (\overline{x}^T)^{-1} I_{p,q} = I_{p,q} x^{-1} (\overline{x}^T)^{-1} I_{p,q} = 1$$

for all $x \in \mathrm{Sp}(n)$. The group $\mathrm{Sp}(p, q)$ is the fixed point group of σ and by Lemma A.4.4 the group $\mathrm{Sp}(p, q) \cap \mathrm{Sp}(n)$ is maximal compact in $\mathrm{Sp}(p, q)$. It is readily checked that $\mathrm{Sp}(p, q) \cap \mathrm{Sp}(n) = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$. □

Given any maximal compact subgroup $K(\gamma)_\infty \subseteq G(\gamma)_\infty$, we obtain an isomorphism of Lie groups

$$K(\gamma)_\infty \xrightarrow{\simeq} \mathrm{U}(n)^s \times \prod_{v \in \mathrm{Ram}_\infty(D)} \mathrm{Sp}(p_v) \times \mathrm{Sp}(q_v).$$

V.3.12. *The symmetric space.* Consider the associated Riemannian symmetric space $X(\gamma) := K(\gamma)_\infty \backslash G(\gamma)_\infty$ (cf. IV.3.5). We have $\dim G(\gamma) = n(2n + 1)$ and thus

$$\dim G(\gamma)_\infty = n(2n + 1)[F : \mathbb{Q}].$$

The dimension of the unitary group $\mathrm{U}(n)$ is n^2 and consequently

$$\dim K(\gamma)_\infty = sn^2 + \sum_{v \in \mathrm{Ram}_\infty(D)} p_v(2p_v + 1) + q_v(2q_v + 1).$$

Subtraction of both dimensions yields

$$\dim X(\gamma) = sn(n + 1) + \sum_{v \in \mathrm{Ram}_\infty(D)} 4 p_v q_v,$$

which is obviously an *even* number.

V.3.13. *Lie algebras and complexifications.* We complexify the Lie algebra $\mathfrak{g}(\gamma)_\infty$ and we obtain an isomorphism

$$\mathfrak{g}(\gamma)_\infty \otimes_\mathbb{R} \mathbb{C} \cong \mathfrak{sp}(n, \mathbb{C})^{[F:\mathbb{Q}]}.$$

The rank of this complex semisimple Lie algebra is $\mathrm{rk}(\mathfrak{g}(\gamma)_{\infty,\mathbb{C}}) = n[F : \mathbb{Q}]$. Let $\mathfrak{k}(\gamma)_\infty$ denote the Lie algebra of the maximal compact subgroup $K(\gamma)_\infty$. The complexification of this Lie algebra is isomorphic to

$$\mathfrak{k}(\gamma)_\infty \otimes_\mathbb{R} \mathbb{C} \cong \mathfrak{gl}(n, \mathbb{C})^s \oplus \bigoplus_{v \in \mathrm{Ram}_\infty(D)} \mathfrak{sp}(p_v, \mathbb{C}) \oplus \mathfrak{sp}(q_v, \mathbb{C}).$$

The rank of $\mathfrak{k}(\gamma)_{\infty,\mathbb{C}}$ is $sn + \sum_{v\in\mathrm{Ram}_\infty(D)} p_v + q_v = n[F : \mathbb{Q}]$. Thus the complexified Lie algebras $\mathfrak{g}(\gamma)_{\infty,\mathbb{C}}$ and $\mathfrak{k}(\gamma)_{\infty,\mathbb{C}}$ have equal rank. The Weyl groups of these complex reductive Lie algebras are well-known, in particular we get

$$|W(\mathfrak{g}(\gamma)_{\infty,\mathbb{C}})| = (2^n n!)^{[F:\mathbb{Q}]}, \text{ and}$$
$$|W(\mathfrak{k}(\gamma)_{\infty,\mathbb{C}})| = (n!)^s \prod_{v\in\mathrm{Ram}_\infty(D)} 2^{p_v} p_v! \cdot 2^{q_v} q_v!$$

as can be found in [**Hum72**, p. 66]. The quotient of the cardinalities of the two Weyl groups is given by

$$\frac{|W(\mathfrak{g}(\gamma)_{\infty,\mathbb{C}})|}{|W(\mathfrak{k}(\gamma)_{\infty,\mathbb{C}})|} = 2^{ns} \prod_{v\in\mathrm{Ram}_\infty(D)} \binom{n}{p_v}.$$

V.3.14. *Remark.* For this paragraph we fix an embedding $\iota : F \to \mathbb{C}$. The linear algebraic F-group $G(\gamma) \times_\mathcal{O} F$ is an inner form of the symplectic group Sp_n, in particular this means $G(\gamma) \times_\mathcal{O} \mathbb{C} \cong \mathrm{Sp}_n \times_\mathbb{Z} \mathbb{C}$ (see V.3.8).

Lemma. $G(\gamma) \times_\mathcal{O} F$ is a semisimple and simply connected F-group.

Proof. Consider the Lie algebra $\mathrm{Lie}(G(\gamma))(\mathbb{C}) \cong \mathfrak{sp}(n, \mathbb{C})$. It is well-known that this is a complex semisimple Lie algebra. Since F is of characteristic zero, we can deduce that $G(\gamma)$ is semisimple. Moreover, one can show that the group Sp_n is simply connected (cf. §2.3.2 in [**PR94**]). □

In particular, the Tamagawa number $\tau(G(\gamma))$ is equal to one [**Kot88**].

V.3.15. *The metric form B.* Recall that the Lie algebra of $G(\gamma)$ is a functor $\mathrm{Lie}(G(\gamma))$ which assigns to a commutative $\mathcal{O}$-algebra C the C-Lie algebra

$$\mathrm{Lie}(G(\gamma))(C) = \{\, x \in (\Lambda \otimes_\mathcal{O} C)^\times \mid (\tau|\gamma)(x) = -x \,\}$$

(see III.4.12). For simplicity we write $\mathfrak{g}(\gamma)_C$ instead of $\mathrm{Lie}(G(\gamma))(C)$.

Consider the non-degenerate bilinear form $B : \mathfrak{g}(\gamma)_F \times \mathfrak{g}(\gamma)_F \to F$ defined by $B(x, y) := -\frac{1}{2}\,\mathrm{trd}_A(xy)$. Let $\iota : F \to \mathbb{C}$ be an embedding of F into the field of complex numbers. The central simple algebra $A = M_n(D)$ splits over $\mathbb{C}$ and we can choose a splitting $A \to M_{2n}(\mathbb{C})$ such that $\tau|\gamma$ is the standard symplectic involution. Via this splitting the Lie algebra $\mathfrak{g}(\gamma)_\mathbb{C}$ is isomorphic to the complex semisimple Lie algebra $\mathfrak{sp}(n, \mathbb{C})$. The form B, extended $\mathbb{C}$-linear to $\mathfrak{sp}(n, \mathbb{C})$, is given by $B(x, y) = -\frac{1}{2}\,\mathrm{Tr}(xy)$. Recall that the Killing form β on $\mathfrak{sp}(n, \mathbb{C})$ is the form $\beta(x, y) = (2n+2)\,\mathrm{Tr}(xy)$ (cf. III, §8 in [**Hel78**]) and hence B is a multiple of the Killing form.

V.3.16. Proposition. *Consider the compact Lie group* $\mathrm{Sp}(n)$ *and its Lie algebra*

$$\mathfrak{sp}(n) := \{ x \in M_n(\mathbb{H}) \mid \overline{x}^T + x = 0 \}.$$

Let B *be the positive definite* $\mathbb{R}$*-bilinear form* $B : \mathfrak{sp}(n) \times \mathfrak{sp}(n) \to \mathbb{R}$ *defined by* $B(x,y) := -\frac{1}{2}\operatorname{trd}(xy)$. *With respect to the right invariant Riemann metric induced by* B*, the group* $\mathrm{Sp}(n)$ *has the volume*

$$\operatorname{vol}_B(\mathrm{Sp}(n)) = \prod_{j=1}^{n} \frac{(2\pi)^{2j}}{2 \cdot (2j-1)!}.$$

Proof. Note that the right invariant metric induced by B is also left invariant since B is a multiple of the Killing form and thus is $\mathrm{Ad}(\mathrm{Sp}(n))$-invariant.

We proceed by induction on n. Consider the action of $\mathrm{Sp}(n)$ on $\mathbb{H}^n$ given by $(g,v) \mapsto gv$. This action restricts to a transitive action on the unit sphere $S^{4n-1} \subseteq \mathbb{H}^n$, i.e. there is an action

$$\mathrm{Sp}(n) \times S^{4n-1} \to S^{4n-1}.$$

Consider the last standard basis vector $e_n := (0,\ldots,0,1)^T \in \mathbb{H}^n$, which is an element of S^{4n-1}. The stabiliser of e_n is the group $\mathrm{Sp}(n-1)$ embedded into $\mathrm{Sp}(n)$ as a block in the upper left corner, i.e.

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1. \end{pmatrix}$$

We get a direct sum decomposition of the Lie algebra $\mathfrak{sp}(n) = \mathfrak{sp}(n-1) \oplus \mathfrak{t}$ as $\mathbb{R}$-vector space where

$$\mathfrak{t} := \{ x \in M_n(\mathbb{H}) \mid \overline{x}^T + x = 0 \text{ and } x_{kl} = 0 \text{ for all } k,l < n \}.$$

Note that $\mathfrak{t}$ is orthogonal to $\mathfrak{sp}(n-1)$ with respect to the form B. Let f be the surjective orbit map $\mathrm{Sp}(n) \to S^{4n-1}$ defined by $g \mapsto ge_n$. This makes $\mathrm{Sp}(n)$ into a fibre bundle over S^{4n-1} with fibre $\mathrm{Sp}(n-1)$. The derivative of f at 1 yields an isomorphism of $\mathbb{R}$-vector spaces

$$\mathrm{d}(f) : \mathfrak{t} \xrightarrow{\ \simeq\ } T_{e_n}S^{4n-1} = \{ v \in \mathbb{H}^n \mid v \perp e_n \}.$$

We denote the standard $\mathbb{R}$-basis of $\mathbb{H}$ by $1, i, j, ij$, this is $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$, with $i^2 = -1$, $j^2 = -1$ and $ij = -ji$. Consider the $\mathbb{R}$-basis of $\mathfrak{t}$ which consists of the following vectors:

$$\begin{aligned} \alpha E_{k,n} - \overline{\alpha} E_{n,k} &\text{ for } k = 1,\ldots,n-1 \text{ and } \alpha \in \{1,i,j,ij\}, \text{ and} \\ \alpha E_{n,n} &\text{ with } \alpha \in \{i,j,ij\}. \end{aligned}$$

This basis is mapped to the standard basis of the tangent space $T_{e_n}S^{4n-1}$ via $\mathrm{d}(f)$. Transfer the form $B_{|\mathfrak{t}}$ to the tangent space $T_{e_n}S^{4n-1}$, we calculate the values on the standard basis in order to understand the induced metric on S^{4n-1}. For $k,l < n$ and $\alpha,\beta \in \{1,i,j,ij\}$ we have

$$\begin{aligned} B(\alpha E_{k,n} - \overline{\alpha} E_{n,k}, \beta E_{l,n} - \overline{\beta} E_{n,l}) &= -\frac{1}{2}\operatorname{trd}(-\alpha\overline{\beta} E_{k,l} - \delta_{k,l}\overline{\alpha}\beta E_{n,n}) \\ &= \frac{1}{2}\operatorname{trd}(\alpha\overline{\beta} + \overline{\alpha}\beta)\delta_{k,l} = 2\delta_{\alpha,\beta}\delta_{k,l}. \end{aligned}$$

Here $\delta_{\alpha,\beta}$ equals one if $\alpha = \beta$ and zero otherwise. For $k < n$, $\alpha \in \{1,i,j,ij\}$ and $\beta \in \{i,j,ij\}$ we have

$$B(\alpha E_{k,n} - \overline{\alpha} E_{n,k}, \beta E_{n,n}) = 0.$$

And finally for $\alpha, \beta \in \{i,j,ij\}$ we obtain

$$B(\alpha E_{n,n}, \beta E_{n,n}) = -\frac{1}{2}\operatorname{trd}(\alpha\beta E_{n,n}) = \delta_{\alpha\beta}.$$

Let μ denote the canonical volume density on S^{4n-1}. The left $\mathrm{Sp}(n)$-invariant form (for simplicity denoted by B) on S^{4n-1} which is induced by $B_{|\mathfrak{t}}$, has the volume density

$$|\mathrm{vol}_B| = 4^{n-1}\mu.$$

Consequently, $\mathrm{vol}_B(S^{4n-1}) = 4^{n-1}\frac{2\pi^{2n}}{(2n-1)!} = 2^{2n-1}\frac{\pi^{2n}}{(2n-1)!}$. This yields the induction start: Using $\mathrm{Sp}(1) \cong S^3$ we obtain

$$\mathrm{vol}_B(\mathrm{Sp}(1)) = \frac{(2\pi)^2}{2}.$$

Let $n \geq 2$ and assume the claim holds for $\mathrm{Sp}(n-1)$. By the above considerations we may conclude

$$\mathrm{vol}_B(\mathrm{Sp}(n)) = \mathrm{vol}_B(S^{4n-1})\,\mathrm{vol}_B(\mathrm{Sp}(n-1)) = \frac{(2\pi)^{2n}}{2(2n-1)!}\prod_{j=1}^{n-1}\frac{(2\pi)^{2j}}{2(2j-1)!}.$$

□

V.3.17. *The modulus factor.* Consider the F-bilinear form $B : \mathfrak{g}(\gamma)_F \times \mathfrak{g}(\gamma)_F \to F$ defined by $B(x,y) := -\frac{1}{2}\,\mathrm{trd}_A(xy)$. We want to calculate the global modulus factor $m(B) = \prod_{v \in V_f} m(B)_v$ (cf. IV.6.7). Note that Λ is in general not a free $\mathcal{O}$-module, this is the reason why we have to work locally.

We will start with the finite places $v \in V_f$ where D splits. The main observation is: We can assume the $\Lambda \otimes_{\mathcal{O}} \mathcal{O}_v = M_{2n}(\mathcal{O}_v)$ and $\tau|\gamma$ is the standard symplectic involution. This will be shown in the following two lemmata.

V.3.18. Lemma. *Let R be a complete discrete valuation ring with quotient field k and let $\pi \in R$ be a prime element. Further, let $m \geq 1$ be an integer. Every element $g \in \mathrm{GL}_m(k)$ which satisfies $gM_m(R)g^{-1} = M_m(R)$ has the form $g = \pi^e u$ with $e \in \mathbb{Z}$ and $u \in \mathrm{GL}_m(R)$.*

PROOF. Let $g \in \mathrm{GL}_m(k)$ with $gM_m(R)g^{-1} = M_m(R)$. Consider the R-lattice $gM_m(R) = M_m(R)g \subseteq M_m(k)$. Let c be an integer such that $\pi^c gM_m(R)$ is contained in $M_m(R)$. Thus $\pi^c gM_m(R) = M_m(R)\pi^c g$ is a two-sided ideal in $M_m(R)$. Using Theorem (17.3) in [**Rei03**], we conclude that $\pi^c gM_m(R) = \pi^b M_m(R)$ for some integer $b \geq 0$. We obtain that $\pi^c g = \pi^b u$ for a unit $u \in \mathrm{GL}_m(R)$. □

V.3.19. Lemma. *Let R be a complete discrete valuation ring with quotient field k of characteristic* $\mathrm{char}(k) \neq 2$. *Let σ be an involution of symplectic type on $M_{2n}(k)$ and let $\Lambda \subseteq M_{2n}(k)$ be a maximal R-order which is σ-stable.*

There is an element $g \in \mathrm{GL}_{2n}(k)$ such that

(i) $g\Lambda g^{-1} = M_{2n}(R)$, and

(ii) $g\sigma(x)g^{-1} = J(gxg^{-1})^T J^{-1}$, where J denotes the standard symplectic matrix (cf. V.3.3).

PROOF. It follows from Theorem (17.3) in [**Rei03**] that there is an invertible matrix $a \in \mathrm{GL}_{2n}(k)$ such that $a\Lambda a^{-1} = M_{2n}(R)$. Moreover, σ is an involution of symplectic type and we can consider $\mathrm{int}(a) : M_{2n}(k) \to M_{2n}(k)$ as a splitting of the central simple k-algebra $M_{2n}(k)$. There is a matrix $h \in \mathrm{GL}_{2n}(k)$ such that $h^T = -h$ and $\mathrm{int}(a)(\sigma(x)) = h(\mathrm{int}(a)(x))^T h^{-1}$ for every $x \in M_{2n}(k)$.

Using that Λ is σ-stable, we see that $hM_{2n}(R)h^{-1} = M_{2n}(R)$. By Lemma V.3.18 we can choose h to be an element in $\mathrm{GL}_{2n}(R)$. On a free module over a complete discrete valuation ring, there is only one regular symplectic form up to isogeny ($\mathrm{char}(k) \neq 2!$), this means that there is $b \in \mathrm{GL}_{2n}(R)$ such that $bhb^T = J$. Finally, we define $g := ba$ and observe

$$g\sigma(x)g^{-1} = bh(axa^{-1})^T h^{-1}b^{-1} = J(b^{-1})^T(axa^{-1})^T b^T J^{-1} = J(gxg^{-1})^T J^{-1}$$

for every $x \in M_{2n}(k)$. □

V.3.20. Corollary. *Let $\gamma \in Z^1(\tau^*, G(\mathcal{O}))$ be a cocycle and let $v \in V_f$ be a finite place of F which splits D. There is an isomorphism of group schemes over $\mathcal{O}_v$:*

$$G(\gamma) \times_{\mathcal{O}} \mathcal{O}_v \xrightarrow{\simeq} \mathrm{Sp}_n \times_{\mathbb{Z}} \mathcal{O}_v.$$

Proof. This follows directly from the previous lemma since $\tau|\gamma$ is an involution of symplectic type. □

V.3.21. Proposition. *Let $v \in V_f$ be a finite place which splits D. Consider the bilinear form $B : \mathfrak{g}(\gamma)_F \times \mathfrak{g}(\gamma)_F \to F$ defined by $B(x,y) = -\frac{1}{2}\,\mathrm{trd}(xy)$. The local modulus factor (see IV.6.7) is*

$$m(B)_v = |2|_v^{-n}.$$

Proof. By Corollary V.3.20 we can assume $G(\gamma) = \mathrm{Sp}_n$ over $\mathcal{O}_v$. This means

$$\mathfrak{g}(\gamma)_{\mathcal{O}_v} = \mathfrak{sp}(n, \mathcal{O}_v) = \{\, x \in M_{2n}(\mathcal{O}_v) \mid x^T J + Jx = 0 \,\}.$$

Note that the form B is given by the analogous formula $B(x,y) = -\frac{1}{2}\,\mathrm{Tr}(xy)$. Recall that the elements of $\mathfrak{sp}(n, \mathcal{O}_v)$ are matrices of the form

$$\begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$$

with $a, b, c \in M_n(\mathcal{O}_v)$ where b and c are symmetric matrices. Let $E_{s,t}$ denote the elementary $2n \times 2n$ matrix with exactly one entry 1 in position (s,t). We choose an $\mathcal{O}_v$-basis of $\mathfrak{sp}(n, \mathcal{O}_v)$ which is made up of the following elements:

(1) $a_{i,j} := E_{i,j} - E_{j+n,i+n}$ for all $i, j \in \{1, \ldots, n\}$,
(2) $b_{i,j} := E_{i,j+n} + E_{j,i+n}$ for all $1 \le i < j \le n$,
(3) $c_{i,j} := E_{i+n,j} + E_{j+n,i}$ for all $1 \le i < j \le n$, and
(4) $b_i := E_{i,i+n}$ and $c_i := E_{i+n,i}$ for all $i \in \{1, \ldots, n\}$.

We evaluate the form B on all the basis vectors.

It is an easy observation that

$$\begin{aligned} 0 &= B(a_{i,j}, c_{k,l}) = B(a_{i,j}, b_{k,l}) = B(a_{i,j}, b_k) = B(a_{i,j}, c_k) \\ &= B(c_{i,j}, c_{k,l}) = B(b_{i,j}, b_{k,l}) = B(c_i, c_j) = B(b_i, b_j). \end{aligned}$$

for all i, j, k, l. Moreover, one readily verifies that $B(b_{i,j}, c_k) = B(c_{i,j}, b_k) = 0$ for all i, j, k. Consider the remaining cases.

Case 1: For all $i, j, k, l \in \{1, \ldots, n\}$ we have

$$\begin{aligned} B(a_{i,j}, a_{k,l}) &= -\frac{1}{2}\,\mathrm{Tr}\big((E_{i,j} - E_{j+n,i+n})(E_{k,l} - E_{l+n,k+n})\big) \\ &= -\frac{1}{2}\,\mathrm{Tr}\big(\delta_{j,k}E_{i,l} + \delta_{i,l}E_{j+n,k+n}\big) = -\delta_{j,k}\delta_{i,l} \end{aligned}$$

Case 2: For all $i < j \le n$ and $k < l \le n$ one obtains

$$\begin{aligned} B(b_{i,j}, c_{k,l}) &= -\frac{1}{2}\,\mathrm{Tr}\big((E_{i,j+n} + E_{j,i+n})(E_{k+n,l} + E_{l+n,k})\big) \\ &= -\frac{1}{2}\,\mathrm{Tr}\big(E_{i,l}\delta_{j,k} + E_{j,l}\delta_{k,i} + E_{i,k}\delta_{j,l} + E_{j,k}\delta_{l,i}\big) \\ &= -\delta_{i,k}\delta_{j,l} \end{aligned}$$

Case 3: For all $i, j \in \{1, \dots, n\}$ we observe further that

$$B(b_i, c_j) = -\frac{1}{2}\operatorname{Tr}(E_{i,i+n}E_{j+n,j}) = -\frac{1}{2}\delta_{i,j}.$$

Using these results, we are able to calculate the modulus factor and obtain

$$m(B)_v = \left|\det\begin{pmatrix} 0 & -1/2 \\ -1/2 & 0\end{pmatrix}\right|_v^{n/2} = |2|_v^{-n}. \qquad \square$$

V.3.22. Proposition. *Let $v \in \mathrm{Ram}_f(D)$ be a finite ramified place and let $\mathfrak{p} \subset \mathcal{O}$ be the associated prime ideal. The local modulus factor for the group $G(\gamma)$ and the form B defined in V.3.15 is*

$$m(B)_v = |2|_v^{-n}\,\mathrm{N}(\mathfrak{p})^{-n(n+1)/2}.$$

Proof. The F_v-algebra $D_v := D \otimes_F F_v$ is the unique quaternion division algebra over F_v and $\Delta := \Lambda_D \otimes_{\mathcal{O}} \mathcal{O}_v$ is the unique maximal order in D_v.

Due to Corollary V.3.8 we can assume that $\gamma = 1$, i.e. $G(\gamma) \times \mathcal{O}_v$ is isomorphic to $H := G(1) \times \mathcal{O}_v$. We define

$$\mathfrak{h} := \mathrm{Lie}(H)(\mathcal{O}_v) = \{\, x \in M_n(\Delta) \mid \tau(x) = -x \,\}.$$

Recall that $\tau(x) = \overline{x}^T$.

Take an $\mathcal{O}_v$-basis v_0, v_1, v_2, v_3 of Δ such that $\mathrm{trd}_D(v_0) = 1$ and $\mathrm{trd}_D(v_i) = 0$ for $i = 1, 2, 3$. Such a basis exists since $\mathrm{trd}_D : \Delta \to \mathcal{O}_v$ is surjective (maximal orders are smooth, see III.3.11). We construct an $\mathcal{O}_v$-basis of the Lie algebra $\mathfrak{h}$, which consists of the following elements

(1) $a_{s,i} := v_s E_{i,i}$ for all $s \in \{1, 2, 3\}$ and $i \in \{1, \dots, n\}$, and
(2) $b_{s,i,j} := v_s E_{i,j} - \overline{v_s} E_{j,i}$ for all $s \in \{0, 1, 2, 3\}$ and $i, j \in \{1, \dots, n\}$ with $i < j$.

We calculate the form B on all basis vectors. Observe that $B(a_{s,i}, b_{t,k,l}) = 0$ for all s, t, i, k, l. Moreover, for $s, t \in \{1, 2, 3\}$ and $i, j \in \{1, \dots, n\}$ we find

$$B(a_{s,i}, a_{t,j}) = -\frac{1}{2}\operatorname{trd}\big(v_s E_{i,i} v_t E_{j,j}\big) = -\frac{1}{2}\delta_{i,j}\operatorname{trd}_D(v_s v_t).$$

Finally, let $s, t \in \{0, 1, 2, 3\}$ and let $i, j, k, l \in \{1, \dots, n\}$ with $i < j$ and $k < l$. We determine the form B on the associated basis vectors and obtain

$$\begin{aligned} B(b_{s,i,j}, b_{t,k,l}) &= -\frac{1}{2}\operatorname{trd}\big((v_s E_{i,j} - \overline{v_s} E_{j,i})(v_t E_{k,l} - \overline{v_t} E_{l,k})\big) \\ &= -\frac{1}{2}\operatorname{trd}\big(v_s v_t E_{i,l}\delta_{j,k} - \overline{v_s} v_t E_{j,l}\delta_{k,i} - v_s \overline{v_t} E_{i,k}\delta_{j,l} + \overline{v_t}\,\overline{v_s} E_{j,k}\delta_{i,l}\big) \\ &= \frac{1}{2}\operatorname{trd}_D(\overline{v_s} v_t + v_s \overline{v_t})\delta_{i,k}\delta_{j,l} = \operatorname{trd}_D(v_s \overline{v_t})\delta_{i,k}\delta_{j,l}. \end{aligned}$$

Summing up we obtain a formula for the modulus factor

$$m(B)_v^2 = \Big|\frac{1}{8}\det(\operatorname{trd}(v_s v_t))_{s,t=1,2,3}\Big|_v^n \cdot \Big|\det(\operatorname{trd}(v_s \overline{v_t}))_{s,t=0,1,2,3}\Big|_v^{n(n-1)/2} \tag{15}$$

Since the elements $\overline{v_0}, \overline{v_1}, \overline{v_2}, \overline{v_3}$ form an $\mathcal{O}_v$-basis of Δ as well, we see that the second term $\big|\det(\operatorname{trd}(v_s\overline{v_t}))_{s,t=0,1,2,3}\big|_v$ is the valuation of the discriminant of Δ. It is known that the discriminant of Δ is $\mathfrak{p}_v^2$ (see (14.9) in [**Rei03**]).

To calculate the first term in equation (15) we consider $w_0 := 1$ and we define $w_s = v_s$ for $s = 1, 2, 3$. Note that w_0, w_1, w_2, w_3 is in general not an $\mathcal{O}_v$-basis of Δ since $\mathrm{trd}_D(1) = 2$ need not be a unit in $\mathcal{O}_v$. We can write

$$w_0 = 1 = r_0 v_0 + r_1 v_1 + r_2 v_2 + r_3 v_3$$

for certain r_0, r_1, r_2, r_3 in $\mathcal{O}_v$. Applying the reduced trace we get $2 = \mathrm{trd}_D(1) = r_0$. Furthermore, this implies that the matrix $\big(\mathrm{trd}(w_s w_t)\big)_{s,t=0,1,2,3}$ can be written as a product of matrices

$$\begin{pmatrix} 2 & r_1 & r_2 & r_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \big(\mathrm{trd}(v_i v_j)\big)_{i,j=0,1,2,3} \begin{pmatrix} 2 & 0 & 0 & 0 \\ r_1 & 1 & 0 & 0 \\ r_2 & 0 & 1 & 0 \\ r_3 & 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$\big(\mathrm{trd}(w_s w_t)\big)_{s,t=0,1,2,3} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \mathrm{trd}(v_1 v_1) & \mathrm{trd}(v_1 v_2) & \mathrm{trd}(v_1 v_3) \\ 0 & \mathrm{trd}(v_2 v_1) & \mathrm{trd}(v_2 v_2) & \mathrm{trd}(v_2 v_3) \\ 0 & \mathrm{trd}(v_3 v_1) & \mathrm{trd}(v_3 v_2) & \mathrm{trd}(v_3 v_3) \end{pmatrix}.$$

We deduce that $\big|\det(\mathrm{trd}(v_s v_t))_{s,t=1,2,3}\big|_v = |2|_v \, \mathrm{N}(\mathfrak{p})^{-2}$. In total the local modulus factor is

$$m(B)_v = |2|_v^{-n} \, \mathrm{N}(\mathfrak{p})^{-n-n(n-1)/2} = |2|_v^{-n} \, \mathrm{N}(\mathfrak{p})^{-n(n+1)/2}. \qquad \square$$

V.3.23. Corollary. *Let $\gamma \in Z^1(\tau^*, G(\mathcal{O}))$ be a cocycle. The global modulus factor $m(B)$ for the group $G(\gamma)$ with respect to the form B defined in V.3.15 is*

$$m(B) = 2^{n[F:\mathbb{Q}]} \prod_{\mathfrak{p} \in \mathrm{Ram}_f(D)} \mathrm{N}(\mathfrak{p})^{-n(n+1)/2} = 2^{n[F:\mathbb{Q}]} (-1)^{rn(n+1)/2} \Delta_{rd}(D)^{-n(n+1)/2},$$

where $\Delta_{rd}(D)$ denotes the signed reduced discriminant of D (see V.1.4).

Proof. By Prop. V.3.21, Prop. V.3.22, and an application of the product formula we obtain

$$m(B) = \prod_{v \in V_f} |2|_v^{-n} \prod_{\mathfrak{p} \in \mathrm{Ram}_f(D)} \mathrm{N}(\mathfrak{p})^{-n(n+1)/2} = 2^{n[F:\mathbb{Q}]} \prod_{\mathfrak{p} \in \mathrm{Ram}_f(D)} \mathrm{N}(\mathfrak{p})^{-n(n+1)/2}. \quad \square$$

V.3.24. Let $\gamma \in Z^1(\tau^*, G(\mathcal{O}))$ be a cocycle. We are now able to compute the Euler characteristic of torsion-free arithmetic subgroups of $G(\gamma)$. In the next theorem we will give a precise formula for principal congruence subgroups. More general subgroups can be treated analogously.

We fix the following notation. Let F be a number field (which need not be totally real) and let $\mathcal{O}$ denote its ring of integers. Let $\mathfrak{a} \subset \mathcal{O}$ be a proper ideal. For a finite place $v \in V_f$ we define $K_v(\gamma, \mathfrak{a}) := \ker\big(G(\gamma)(\mathcal{O}_v) \to G(\gamma)(\mathcal{O}_v/\mathfrak{a}\mathcal{O}_v)\big)$. Note that $K_v(\gamma, \mathfrak{a}) = G(\gamma)(\mathcal{O}_v)$ for almost all places. The group

$$K_f(\gamma, \mathfrak{a}) := \prod_{v \in V_f} K_v(\gamma, \mathfrak{a})$$

is an open compact subgroup of the locally compact group $G(\gamma)(\mathbb{A}_f)$. This subgroup is given by a local datum (U, α) (cf. IV.6.11). Let $v \in V_f$ be a finite place and let $\mathfrak{p}$ be the associated prime ideal. Let $e = \nu_{\mathfrak{p}}(\mathfrak{a})$ be the exponent of $\mathfrak{p}$ in $\mathfrak{a}$. Then $\alpha_v = 1$ and $U_v = G(\gamma)(\mathcal{O}/\mathfrak{p})$ if is $e = 0$, otherwise $\alpha_v = e$ and $U_v = \{1\} \subseteq G(\gamma)(\mathcal{O}/\mathfrak{p}^e)$.

Let $G(\gamma)_\infty = \prod_{v \in V_\infty} G(\gamma)(F_v)$ and let $K(\gamma)_\infty \subseteq G(\gamma)_\infty$ be a maximal compact subgroup.

For every real ramified place $v \in \mathrm{Ram}_\infty(D)$ we denote the local signature of the class of γ in $H^1(\tau^*, G(F_v))$ by (p_v, q_v) (cf. V.2.5).

V.3.25. Theorem. *In the setting of V.3.24 assume that $G(\gamma)(F)$ acts freely on $K(\gamma)_\infty K_f(\gamma,\mathfrak{a})\backslash G(\gamma)(\mathbb{A})$. The Euler characteristic of the double quotient space*

$$S(\mathfrak{a}) := K(\gamma)_\infty K_f(\gamma,\mathfrak{a})\backslash G(\gamma)(\mathbb{A})/G(\gamma)(F)$$

is non-zero if and only if F is totally real. In this case the following formula holds

$$\chi\big(S(\mathfrak{a})\big) = 2^{-nr}\,\mathrm{N}(\mathfrak{a})^{n(2n+1)}\Delta_{rd}(D)^{n(n+1)/2}\prod_{v\in\mathrm{Ram}_\infty(D)}\binom{n}{p_v}\prod_{j=1}^{n}M(j,\mathfrak{a},D),$$

where $M(j,\mathfrak{a},D)$ is defined as

$$M(j,\mathfrak{a},D) := \zeta_F(1-2j)\prod_{\mathfrak{p}\mid\mathfrak{a}}\big(1-\frac{1}{\mathrm{N}(\mathfrak{p})^{2j}}\big)\prod_{\substack{\mathfrak{p}\in\mathrm{Ram}_f(D)\\ \mathfrak{p}\nmid\mathfrak{a}}}\big(1+(\frac{-1}{\mathrm{N}(\mathfrak{p})})^{j}\big).$$

Here r is the number of real places of F where D is ramified and $\Delta_{rd}(D)$ denotes the signed reduced discriminant of D (cf. V.1.4). The sign of $\chi(S(\mathfrak{a}))$ is $(-1)^{sn(n+1)/2}$, where s denotes the number of real places where D splits.

Proof. It follows from Remark IV.6.10 that the Euler characteristic vanishes whenever F has a complex place. Therefore we may assume that F is totally real. We want to apply the adelic Euler characteristic formula (Theorem IV.6.12). We know that $G(\gamma)$ is a smooth group scheme over $\mathcal{O}$ (see Lemma V.3.6). Further $G\times_{\mathcal{O}}F$ is an inner form of the symplectic group, and is thus a semisimple and simply connected algebraic group of dimension $d=n(2n+1)$ (cf. V.3.14). Note further, that by assumption $G(\gamma)(F)$ acts freely on $K(\gamma)_\infty K_f(\gamma,\mathfrak{a})\backslash G(\gamma)(\mathbb{A})$.

Moreover, we observe that $\dim X(\gamma)$ is even (cf. V.3.12) and that the complexified Lie algebras $\mathfrak{k}(\gamma)_\infty\otimes\mathbb{C}$ and $\mathfrak{g}(\gamma)_{\infty,\mathbb{C}}$ have equal rank (cf.V.3.13). We conclude that the Euler characteristic does not vanish and that we can apply the adelic formula IV.6.12.

We fix the non-degenerate bilinear form $B:\mathfrak{g}(\gamma)_F\times\mathfrak{g}(\gamma)_F\to F$ defined by $B(x,y) := -\frac{1}{2}\operatorname{trd}_A(xy)$. We want to calculate the volume of the compact dual with respect to B. It is easy to see that the compact dual group $G(\gamma)_u$ of $G(\gamma)_\infty$ is isomorphic to $\mathrm{Sp}(n)^{[F:\mathbb{Q}]}$. Recall that B is a multiple of the Killing form (see V.3.15). Let $v\in V_\infty$ be a real place and consider a Cartan decomposition

$$\mathfrak{g}(\gamma)_{F_v} = \mathfrak{k}(\gamma)_v\oplus\mathfrak{m}(\gamma)_v$$

of the Lie algebra. Since B is a multiple of the Killing form, the spaces $\mathfrak{k}(\gamma)_v$ and $\mathfrak{m}(\gamma)_v$ are orthogonal with respect to B (cf. (6.25) in [**Kna02**]). By IV.5.6 we may simply extend the form B to a $\mathbb{C}$-linear form on $\mathfrak{g}(\gamma)_{F_v}\otimes\mathbb{C}$ and restrict it to the compact real form. This yields the same multiple of the Killing form and hence B is given by the same formula on the compact dual. Therefore the volume is

$$\mathrm{vol}_B(G(\gamma)_u) = \Big(\prod_{j=1}^{n}\frac{(2\pi)^{2j}}{2\cdot(2j-1)!}\Big)^{[F:\mathbb{Q}]}$$

according to Prop. V.3.16. Using the global modulus factor, calculated in Cor. V.3.23, and the quotient of the orders of the involved Weyl groups, derived in V.3.13, the adelic formula yields

$$\begin{aligned}\chi(S(\mathfrak{a})) =&(-1)^{[F:\mathbb{Q}]n(n+1)/2}\,|d_F|^{d/2}\,2^{ns}\prod_{v\in\mathrm{Ram}_\infty(D)}\binom{n}{p_v}\\ &\cdot\Big(\prod_{j=1}^{n}\frac{2\cdot(2j-1)!}{(2\pi)^{2j}}\Big)^{[F:\mathbb{Q}]}2^{-n[F:\mathbb{Q}]}\Delta_{rd}(D)^{n(n+1)/2}\prod_{\mathfrak{p}\in V_f}\frac{\mathrm{N}(\mathfrak{p})^{d\alpha_{\mathfrak{p}}}}{|U_{\mathfrak{p}}|}.\end{aligned}\tag{16}$$

Here s denotes the number of real places of F which split D. The only terms that can be negative are $(-1)^{[F:\mathbb{Q}]n(n+1)/2}$ and the signed reduced discriminant. Consequently, the sign of the Euler characteristic is $(-1)^{sn(n+1)/2}$.

Let $v \in V_f$ be a finite place with associated prime ideal $\mathfrak{p}$ and consider the term $\frac{\mathrm{N}(\mathfrak{p})^{d\alpha_\mathfrak{p}}}{|U_\mathfrak{p}|}$. We have to distinguish three cases:

Case (1): D splits at v and $\mathfrak{p}$ does not divide $\mathfrak{a}$. In this case $\alpha_\mathfrak{p} = 1$ and $U_\mathfrak{p} = G(\gamma)(\mathcal{O}/\mathfrak{p})$. Since $G(\gamma)$ is isomorphic to Sp_n over $\mathcal{O}_v$ (see Corollary V.3.20), there is an isomorphism of finite groups $G(\gamma)(\mathcal{O}/\mathfrak{p}) \cong \mathrm{Sp}_n(\mathcal{O}/\mathfrak{p})$. From 3.5 in [**Wil09**] we deduce that

$$\big|G(\gamma)(\mathcal{O}/\mathfrak{p})\big| = \mathrm{N}(\mathfrak{p})^d \prod_{j=1}^n \big(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^{2j}}\big),$$

and moreover that

$$\frac{\mathrm{N}(\mathfrak{p})^{d\alpha_\mathfrak{p}}}{|U_\mathfrak{p}|} = \prod_{j=1}^n \big(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^{2j}}\big)^{-1}.$$

Case (2): D is ramified at v and $\mathfrak{p}$ does not divide $\mathfrak{a}$. In this situation we have $\alpha_\mathfrak{p} = 1$ and $U_\mathfrak{p} = G(\gamma)(\mathcal{O}/\mathfrak{p})$. The theory of degenerate quaternion algebras, as can be found in Appendix C, makes it possible to determine the order of this finite group. A combination of Corollary C.4.4 and Corollary C.3.4 shows that

$$\big|G(\gamma)(\mathcal{O}/\mathfrak{p})\big| = \mathrm{N}(\mathfrak{p})^d \prod_{j=1}^n \big(1 - \frac{(-1)^j}{\mathrm{N}(\mathfrak{p})^{j}}\big).$$

So, we may conclude

$$\frac{\mathrm{N}(\mathfrak{p})^{d\alpha_\mathfrak{p}}}{|U_\mathfrak{p}|} = \prod_{j=1}^n \big(1 - \frac{(-1)^j}{\mathrm{N}(\mathfrak{p})^{j}}\big)^{-1}.$$

Case (3): $\mathfrak{p}$ divides $\mathfrak{a}$. In this case $\alpha_v = \nu_\mathfrak{p}(\mathfrak{a})$ and $|U_\mathfrak{p}| = 1$. Consequently,

$$\frac{\mathrm{N}(\mathfrak{p})^{d\alpha_\mathfrak{p}}}{|U_\mathfrak{p}|} = N(\mathfrak{p})^{d\nu_\mathfrak{p}(\mathfrak{a})}.$$

The product of these terms is

$$\prod_{\mathfrak{p}\in V_f} \frac{\mathrm{N}(\mathfrak{p})^{d\alpha_\mathfrak{p}}}{|U_\mathfrak{p}|} = \mathrm{N}(\mathfrak{a})^d \prod_{j=1}^n \Big(\zeta_F(2j) \prod_{\mathfrak{p}|\mathfrak{a}} \big(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^{2j}}\big) \prod_{\substack{\mathfrak{p}\in \mathrm{Ram}_f(D) \\ \mathfrak{p}\nmid\mathfrak{a}}} \big(1 + (\frac{-1}{\mathrm{N}(\mathfrak{p})})^j\big)\Big).$$

Here ζ_F denotes the zeta function of the number field F.

Note that $d = n(2n+1) = \sum_{j=1}^n 4j - 1$ and so $|d_F|^{d/2} = \prod_{j=1}^n |d_F|^{(4j-1)/2}$. The functional equation of the zeta function of the totally real number field F (see VII.§6, Thm. 3 in [**Wei95**]) yields

$$\zeta_F(2j)\, |d_F|^{(4j-1)/2} \Big(\frac{2\cdot(2j-1)!}{(2\pi)^{2j}}\Big)^{[F:\mathbb{Q}]} = (-1)^{j[F:\mathbb{Q}]} \zeta_F(1-2j)$$

for every integer $j \geq 1$. Using this we see that

$$|d_F|^{d/2} \Big(\prod_{j=1}^n \frac{2\cdot(2j-1)!}{(2\pi)^{2j}}\Big)^{[F:\mathbb{Q}]} \prod_{j=1}^n \zeta_F(2j) = (-1)^{[F:\mathbb{Q}]n(n+1)/2} \prod_{j=1}^n \zeta_F(1-2j).$$

Substitute this into equation (16), then a simple calculation proves the claim. $\square$

V.3.26. Corollary (cf. Harder [**Har71**]). *Let F be a totally real algebraic number field and let $\mathcal{O}$ denote its ring of integers. Given a proper ideal $\mathfrak{a} \subseteq \mathcal{O}$ such that the principal congruence subgroup*

$$\Gamma(\mathrm{Sp}_n, \mathfrak{a}) := \ker\bigl(\mathrm{Sp}_n(\mathcal{O}) \to \mathrm{Sp}_n(\mathcal{O}/\mathfrak{a})\bigr)$$

is torsion-free. Then the Euler characteristic of $\Gamma(\mathrm{Sp}_n, \mathfrak{a})$ is

$$\chi\bigl(\Gamma(\mathrm{Sp}_n, \mathfrak{a})\bigr) = N(\mathfrak{a})^{n(2n+1)} \prod_{j=1}^{n} \Bigl(\zeta_F(1-2j) \prod_{\mathfrak{p}|\mathfrak{a}} (1 - \mathrm{N}(\mathfrak{p})^{-2j})\Bigr).$$

Proof. Apply Theorem V.3.25 with $D = M_2(F)$, $\Lambda_D = M_2(\mathcal{O})$ and $\gamma = 1$. In this case $G(\gamma)$ is isomorphic over $\mathcal{O}$ to the symplectic group $\mathrm{Sp}_n \times_{\mathbb{Z}} \mathcal{O}$. Note that, by IV.6.4, $\mathrm{Sp}_n(F)$ acts freely on $K(1)_\infty K_f(1, \mathfrak{a}) \backslash \mathrm{Sp}_n(\mathbb{A})$ since $\Gamma(\mathrm{Sp}_n, \mathfrak{a})$ is torsion-free. Further, the generic fibre of $\mathrm{Sp}_n \times_{\mathbb{Z}} \mathcal{O}$ is F-simple, simply connected and for every place $v \in V_\infty$ the real Lie group $\mathrm{Sp}_n(F_v) \cong \mathrm{Sp}_n(\mathbb{R})$ is not compact. Thus Sp_n has strong approximation (see Thm. 7.12 in [**PR94**]) and consequently

$$\chi\bigl(\Gamma(\mathrm{Sp}_n, \mathfrak{a})\bigr) = \chi\bigl(K(1)_\infty K_f(1, \mathfrak{a}) \backslash \mathrm{Sp}_n(\mathbb{A}) / \mathrm{Sp}_n(F)\bigr)$$

(this follows from II.6.2). □

V.4. The Main Theorem and Applications

V.4.1. The notation and assumptions are those of the introduction V.1. We recall some notation. As usual F denotes an algebraic number field and $\mathcal{O}$ denotes its ring of integers. Let D be a quaternion algebra defined over F and let $\Lambda_D \subseteq D$ be a maximal $\mathcal{O}$-order. Let $n \geq 1$ be an integer, we consider the central simple F-algebra $A = M_n(D)$ and the maximal $\mathcal{O}$-order $\Lambda = M_n(\Lambda_D)$.

The conjugation on the quaternion algebra $\tau_c : D \to D$ induces an involution of symplectic type $\tau : A \to A$, defined by $\tau : x \mapsto \tau_c(x)^T$. Further $G := \mathrm{SL}_\Lambda$ is the smooth $\mathcal{O}$-group scheme defined as the kernel of the reduced norm over the order Λ (cf. III.3.8). The involution τ composed with the inversion induces an automorphism τ^* of order two of G.

We say that the quaternion algebra D over F is *totally definite*, if F is totally real and D ramifies at every real place of F.

V.4.2. Main Theorem. *Let $\mathfrak{a} \subseteq \mathcal{O}$ be a proper ideal such that the principal congruence group*

$$\Gamma(\mathfrak{a}) := \ker\bigl(\mathrm{SL}_\Lambda(\mathcal{O}) \to \mathrm{SL}_\Lambda(\mathcal{O}/\mathfrak{a})\bigr)$$

is torsion-free. Let $\rho : \mathrm{SL}_\Lambda \times_{\mathcal{O}} \overline{F} \to \mathrm{GL}(W)$ be a rational representation of $\mathrm{SL}_\Lambda \times_{\mathcal{O}} \overline{F}$ on a finite dimensional $\overline{F}$-vector space W equipped with a compatible $\langle \tau^* \rangle$-action (as defined in II.5.8). If D is totally definite, we assume additionally that $n \geq 2$.

The Lefschetz number of τ^* in the cohomology $H^\bullet(\Gamma(\mathfrak{a}), W)$ is zero if F is not totally real. Whereas if F is totally real, the Lefschetz number is given by the following formula:

$$\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W) \;=\; 2^{-r}\, \mathrm{N}(\mathfrak{a})^{n(2n+1)} \Delta_{rd}(D)^{n(n+1)/2}\, \mathrm{Tr}(\tau^*|W) \prod_{j=1}^{n} M(j, \mathfrak{a}, D).$$

Here $\Delta_{rd}(D)$ denotes the signed reduced discriminant of D (cf. V.1.4), r denotes the number of real places ramified in D, and

$$M(j,\mathfrak{a},D) := \zeta_F(1-2j)\prod_{\mathfrak{p}|\mathfrak{a}}\big(1-\frac{1}{\mathrm{N}(\mathfrak{p})^{2j}}\big)\prod_{\substack{\mathfrak{p}\in\mathrm{Ram}_f(D)\\ \mathfrak{p}\nmid\mathfrak{a}}}\big(1+(\frac{-1}{\mathrm{N}(\mathfrak{p})})^j\big).$$

Assume that F is totally real. The Lefschetz number vanishes precisely if $\mathrm{Tr}(\tau^*|W)=0$. If the Lefschetz number is non-zero, then the sign of the Lefschetz number is the sign of

$$(-1)^{sn(n+1)/2}\,\mathrm{Tr}(\tau^*|W),$$

where s denotes the number of real places where D splits.

Proof. The algebraic group $G\times_{\mathcal{O}}F$ has strong approximation since it is an F-simple, simply connected group and $G_\infty\cong \mathrm{SL}_{2n}(\mathbb{R})^s\times\mathrm{SL}_n(\mathbb{H})^r\times\mathrm{SL}_{2n}(\mathbb{C})^t$ is not compact. Since the group $\mathrm{SL}_1(\mathbb{H})$ is compact, we need the assumption that $n\geq 2$ if D is totally definite.

Let $K_\infty\subseteq G_\infty$ be a τ^*-stable maximal compact subgroup. Further, let K_f be the open compact subgroup of $G(\mathbb{A}_f)$, which satisfies $\Gamma(\mathfrak{a})=K_f\cap G(F)$ (see V.1.9). Since $\Gamma(\mathfrak{a})$ is torsion-free and τ^*-stable, we can apply Theorem II.6.8 and we obtain

$$\mathcal{L}(\tau^*,\Gamma(\mathfrak{a}),W) \;=\; \sum_{\eta\in\mathcal{H}^1(\tau^*)}\chi\big(\vartheta^{-1}(\eta)\big)\,\mathrm{Tr}(\tau^*|W(\gamma_\eta)). \tag{17}$$

Here γ_η is any representative of the $H^1(\tau^*,G(F))$ component of η and

$$\vartheta:\big(K_\infty K_f\backslash G(\mathbb{A})/G(F)\big)^{\tau^*}\to\mathcal{H}^1(\tau^*)$$

is the surjective continuous map defined in II.6.4.

By Theorem V.2.17 the projection $\pi:\mathcal{H}^1(\tau^*)\to H^1(\tau^*,G(F))$ is injective and there is an exact sequence of pointed sets

$$1\longrightarrow\mathcal{H}^1(\tau^*)\xrightarrow{\;\pi\;}H^1(\tau^*,G(F))\xrightarrow{\mathrm{pf}_\tau}\{\pm1\}\longrightarrow 1. \tag{18}$$

We deduce that, given a class $\eta\in\mathcal{H}^1(\tau^*)$, every representative $\gamma_\eta\in\pi(\eta)$ has pfaffian one, and hence they all describe the same class in $H^1(\tau^*,G(\overline{F}))$ (see Lemma V.2.12). An application of Lemma II.5.11 shows that $\mathrm{Tr}(\tau^*|W(\gamma_\eta))=\mathrm{Tr}(\tau^*|W)$ for every class $\eta\in\mathcal{H}^1(\tau^*)$.

As a next step we describe the fixed point components. Let $\eta\in\mathcal{H}^1(\tau^*)$, it follows from paragraph II.6.5 that we can choose representing cocycles k_η in $Z^1(\tau^*,K_\infty K_f)$ and γ_η in $Z^1(\tau^*,\Gamma(\mathfrak{a}))$, and an element $a_\infty\in G_\infty$ such that

$$\eta=([k_\eta],[\gamma_\eta])\quad\text{ and }\quad {}^{\tau^*}a_\infty=k_\eta^{-1}a_\infty\gamma_\eta.$$

We write $k_\eta=k_\infty k_0$ with $k_\infty\in K_\infty$ and $k_0\in K_f$. Note that $k_0=\gamma_\eta$ considered as elements in $G(\mathbb{A}_f)$. By Lemma II.6.7 there is a homeomorphism

$$\vartheta^{-1}(\eta)\xrightarrow{\;\simeq\;}(a_\infty^{-1}K_\infty^{\tau^*|k_\infty}a_\infty)K_f(\gamma_\eta,\mathfrak{a})\backslash G(\gamma_\eta)(\mathbb{A})/G(\gamma_\eta)(F).$$

The maximal compact subgroup $a_\infty^{-1}K_\infty a_\infty\subseteq G_\infty$ is stable under $\tau^*|\gamma_\eta$. Therefore the group of fixed points $(a_\infty^{-1}K_\infty a_\infty)^{\tau^*|\gamma_\eta}=(a_\infty^{-1}K_\infty^{\tau^*|k_\infty}a_\infty)$ is a maximal compact subgroup of $G(\gamma_\eta)_\infty$ (cf. A.4.4).

Let $v\in\mathrm{Ram}_\infty(D)$ and let (p_v,q_v) denote the local signature of γ_η at v. By Theorem V.3.25 the Euler characteristic of the fixed point component is zero if F has a complex place. If F is totally real, which we assume from now on, then

$$\chi(\vartheta^{-1}(\eta))=2^{-nr}\,\mathrm{N}(\mathfrak{a})^{n(2n+1)}\Delta_{rd}(D)^{n(n+1)/2}\prod_{v\in\mathrm{Ram}_\infty(D)}\binom{n}{p_v}\prod_{j=1}^{n}M(j,\mathfrak{a},D).$$

The short exact sequence (18), in combination with the Hasse principle V.2.9 and Lemma V.2.15, shows that the map which takes all the local signatures at the real ramified places induces a bijection

$$\mathcal{H}^1(\tau^*) \xrightarrow{\simeq} \prod_{v \in \mathrm{Ram}_\infty(D)} \{\, (p_v, q_v) \mid p_v + q_v = n \text{ and } q_v \text{ even} \,\}.$$

The following identity can be easily verified

$$\sum_{\eta \in \mathcal{H}^1(\tau^*)} \prod_{v \in \mathrm{Ram}_\infty(D)} \binom{n}{q_v} = \sum_{u_1,\dots,u_r=0}^{[\frac{n}{2}]} \prod_{i=1}^{r} \binom{n}{2u_i} = 2^{r(n-1)}.$$

As a final step we substitute all results in formula (17) and we observe:

$$\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W) \;=\; 2^{-r}\, \mathrm{N}(\mathfrak{a})^{n(2n+1)} \Delta_{rd}(D)^{n(n+1)/2}\, \mathrm{Tr}(\tau^*|W) \prod_{j=1}^{n} M(j, \mathfrak{a}, D).$$

Note that the Lefschetz number is non-zero precisely when F is totally real and $\mathrm{Tr}(\tau^*|W)$ does not vanish. ☐

V.4.3. *Remark on the rationality of zeta-values.* The Lefschetz number $\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W)$ is an integer. This follows from the fact that τ^* has order two. In particular, if F is a totally real number field, $D = M_2(F)$, and W is the trivial representation, then one can use the main theorem to show by induction on n that the values of the zeta function ζ_F at odd negative integers are non-zero rational numbers. This is a classical theorem due to Siegel [**Sie69**] and Klingen [**Kli62**].

V.4.4. *Assumption.* We have seen in the proof of the main theorem that the group scheme G has strong approximation if and only if (a) $n = 1$ and D is not totally definite, or (b) $n \geq 2$.

For the rest of this section we keep the setting of the main theorem, in particular we assume that G has strong approximation. We will moreover assume that F is a totally real number field. We conclude this section with an application of the main theorem to asymptotic results on the total Betti number.

V.4.5. Definition. Given a torsion-free arithmetic group $\Gamma \subseteq G(F)$, we define the *total Betti number* $B(\Gamma, W)$ of Γ with respect to the representation (W, ρ) by

$$B(\Gamma, W) := \sum_{i=0}^{\infty} b_i(\Gamma, W),$$

where $b_i(\Gamma, W) = \dim H^i(\Gamma, W)$ denotes the i-th Betti number.

V.4.6. Corollary. *The absolute value of the Lefschetz number yields a lower bound for the total Betti number $B(\Gamma(\mathfrak{a}), W)$, more precisely*

$$B(\Gamma(\mathfrak{a}), W) \geq 2^{-r}\, \mathrm{N}(\mathfrak{a})^{n(2n+1)} \big| \Delta_{rd}(D)^{n(n+1)/2}\, \mathrm{Tr}(\tau^*|W) \prod_{j=1}^{n} M(j, \mathfrak{a}, D) \big|.$$

PROOF. The automorphism τ^* is of order two, therefore the trace of τ^* on the finite dimensional vector space $H^i(\Gamma(\mathfrak{a}), W)$ has an absolute value less or equal than $b_i(\Gamma(\mathfrak{a}), W) = \dim H^i(\Gamma(\mathfrak{a}), W)$. The total Betti number is the sum of all Betti numbers

$b_i(\Gamma(\mathfrak{a}), W)$ and the Lefschetz number is the alternating sum of the traces, so

$$B(\Gamma(\mathfrak{a}), W) = \sum_i b_i(\Gamma(\mathfrak{a}), W) \geq \sum_i |\operatorname{Tr}(\tau^*|H^i(\Gamma(\mathfrak{a}), W))| \geq |\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W)|. \qquad \square$$

V.4.7. Lemma. *Since G has strong approximation, the index of $\Gamma(\mathfrak{a})$ in $G(\mathcal{O})$ is*

$$\mathrm{N}(\mathfrak{a})^{4n^2-1} \prod_{\substack{\mathfrak{p}|\mathfrak{a} \\ \mathfrak{p}\notin \mathrm{Ram}_f(D)}} \Big(\prod_{j=2}^{2n}(1-\frac{1}{\mathrm{N}(\mathfrak{p})^j})\Big) \prod_{\substack{\mathfrak{p}|\mathfrak{a} \\ \mathfrak{p}\in \mathrm{Ram}_f(D)}} \Big((1+\frac{1}{\mathrm{N}(\mathfrak{p})})\prod_{j=2}^{n}(1-\frac{1}{\mathrm{N}(\mathfrak{p})^{2j}})\Big).$$

In particular, the term $[G(\mathcal{O}) : \Gamma(\mathfrak{a})]N(\mathfrak{a})^{-4n^2+1}$ is bounded from above and from below independent of $\mathfrak{a}$,

$$\prod_{j=2}^{2n} \zeta_F(j)^{-1} \leq [G(\mathcal{O}) : \Gamma(\mathfrak{a})]N(\mathfrak{a})^{-4n^2+1} \leq \prod_{\mathfrak{p}\in \mathrm{Ram}_f(D)} (1+\frac{1}{\mathrm{N}(\mathfrak{p})}).$$

Proof. Using the smoothness of the group scheme combined with strong approximation, there is a short exact sequence of groups

$$1 \longrightarrow \Gamma(\mathfrak{a}) \longrightarrow G(\mathcal{O}) \longrightarrow G(\mathcal{O}/\mathfrak{a}) \longrightarrow 1,$$

from which we deduce

$$[G(\mathcal{O}) : \Gamma(\mathfrak{a})] = \prod_{\mathfrak{p}|\mathfrak{a}} |G(\mathcal{O}_\mathfrak{p}/\mathfrak{a}\mathcal{O}_\mathfrak{p})|.$$

Let $\mathfrak{p}$ be a prime ideal which divides $\mathfrak{a}$, say $\nu_\mathfrak{p}(\mathfrak{a}) = e \geq 1$. Then $\mathcal{O}_\mathfrak{p}/\mathfrak{a}\mathcal{O}_\mathfrak{p} \cong \mathcal{O}_\mathfrak{p}/\mathfrak{p}^e\mathcal{O}_\mathfrak{p}$ and it follows from the smoothness of G that

$$|G(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^e\mathcal{O}_\mathfrak{p})| = \mathrm{N}(\mathfrak{p})^{(e-1)d}|G(\mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p})|$$

where d is the dimension of the group $G \times_\mathcal{O} F$ (use III.2.7 or [**Oes84**, 2.1]). The dimension of G is $d = 4n^2 - 1$.

If $\mathfrak{p} \in \mathrm{Ram}_f(D)$, then the theory of degenerate quaternion algebras implies that

$$|G(\mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p})| = \mathrm{N}(\mathfrak{p})^{4n^2-1}(1+\mathrm{N}(\mathfrak{p})^{-1})\prod_{j=2}^{n}(1-\mathrm{N}(\mathfrak{p})^{-2j})$$

(combine of Corollary C.2.6 with Corollary C.4.4).

If otherwise $\mathfrak{p} \notin \mathrm{Ram}_f(D)$, then $G \times_\mathcal{O} \mathcal{O}_\mathfrak{p}$ is isomorphic to the special linear group SL_{2n}. We deduce that

$$|G(\mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p})| = \mathrm{N}(\mathfrak{p})^{4n^2-1}\prod_{j=2}^{2n}(1-\mathrm{N}(\mathfrak{p})^{-j})$$

due to 3.3.1 in [**Wil09**]. Now the assertions can be readily verified. $\square$

V.4.8. Corollary (Growth of the Lefschetz number)**.** *We assume that $\operatorname{Tr}(\tau^*|W)$ does not vanish. There are positive real numbers a and b, depending only on F, D, W and n, such that*

$$a[G(\mathcal{O}) : \Gamma(\mathfrak{a})]^{\frac{n(2n+1)}{4n^2-1}} \leq |\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W)| \leq b[G(\mathcal{O}) : \Gamma(\mathfrak{a})]^{\frac{n(2n+1)}{4n^2-1}}$$

holds for every ideal $\mathfrak{a}$ so that $\Gamma(\mathfrak{a})$ is torsion-free.

Proof. It follows directly from the main theorem, that there are two positive real numbers $a' < b'$, depending on F, D, W and n, such that

$$a' \,\mathrm{N}(\mathfrak{a})^{n(2n+1)} \leq |\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W)| \leq b' \,\mathrm{N}(\mathfrak{a})^{n(2n+1)}.$$

Now the claim is immediate from Lemma V.4.7. $\square$

V.4.9. Corollary (Growth of the total Betti number). *Let $\Gamma_0 \subset G(F)$ be an arithmetic subgroup. For any ideal $\mathfrak{a} \subset \mathcal{O}$ we define $\Gamma_0(\mathfrak{a}) := \Gamma_0 \cap \Gamma(\mathfrak{a})$. There is a positive real number $\kappa > 0$, depending on F, D, Γ_0, W and n, so that*

$$B(\Gamma_0(\mathfrak{a}), W) \geq \kappa[\Gamma_0 : \Gamma_0(\mathfrak{a})]^{\frac{n(2n+1)}{4n^2-1}}$$

holds for every ideal $\mathfrak{a}$ such that $\Gamma(\mathfrak{a})$ is torsion-free.

Proof. Since $\Gamma_0(\mathfrak{a})$ is a subgroup of finite index in $\Gamma(\mathfrak{a})$, we obtain (from VII, Prop. 6 in [**Ser79**]) that $b_i(\Gamma(\mathfrak{a}), W) \leq b_i(\Gamma_0(\mathfrak{a}), W)$. By Cor. V.4.6 and the growth of the Lefschetz number, there is a positive real number a, depending on F, D, W and n, satisfying

$$B(\Gamma(\mathfrak{a}), W) \geq a[G(\mathcal{O}) : \Gamma(\mathfrak{a})]^{\frac{n(2n+1)}{4n^2-1}}$$

for every proper ideal $\mathfrak{a} \subseteq \mathcal{O}$ such that $\Gamma(\mathfrak{a})$ is torsion-free. We obtain

$$\begin{aligned} B(\Gamma_0(\mathfrak{a}), W) &= \sum_i b_i(\Gamma_0(\mathfrak{a}), W) \geq \sum_i b_i(\Gamma(\mathfrak{a}), W) \\ &\geq a[G(\mathcal{O}) : \Gamma(\mathfrak{a})]^{\frac{n(2n+1)}{4n^2-1}} \geq a[G(\mathcal{O}) \cap \Gamma_0 : \Gamma_0(\mathfrak{a})]^{\frac{n(2n+1)}{4n^2-1}} \\ &= a\big([\Gamma_0 : G(\mathcal{O}) \cap \Gamma_0]^{-1}[\Gamma_0 : \Gamma_0(\mathfrak{a})]\big)^{\frac{n(2n+1)}{4n^2-1}}. \end{aligned}$$

We define $\kappa = a[\Gamma_0 : G(\mathcal{O}) \cap \Gamma_0]^{-\frac{n(2n+1)}{4n^2-1}}$. $\square$

V.5. Application: The Cohomology of Arithmetic Fuchsian Groups

V.5.1. In this section we consider the special case $n = 1$ in the main theorem. Further, we will assume that the quaternion algebra D is split at precisely on real place. In this situation the arithmetic groups considered are Fuchsian groups. The main theorem makes it possible to compute the cohomology of those groups and the genus of the associated Riemann surface. As an application we are able to give explicit dimension formulas for the spaces of modular forms of these groups.

V.5.2. *Assumptions.* Let F be a totally real algebraic number field and let D be a division quaternion algebra over F such that $\mathrm{Ram}_\infty(D) = V_\infty \setminus \{v_0\}$ for some place $v_0 \in V_\infty$. This means, there is precisely one real place (namely v_0) where D is split, and so $r = [F : \mathbb{Q}] - 1$.

Let $\Lambda = \Lambda_D$ be a maximal $\mathcal{O}$-order in D. We consider the group scheme $G = \mathrm{SL}_\Lambda$ defined over $\mathcal{O}$. The associated real Lie group is

$$G_\infty \cong \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_1(\mathbb{H})^r.$$

Note that the group $\mathrm{SL}_1(\mathbb{H})$ is compact and so the projection $p_1 : G_\infty \to \mathrm{SL}_2(\mathbb{R})$ onto the first factor is a proper and open homomorphism of Lie groups. In particular, every discrete torsion-free subgroup $\Gamma \subseteq G_\infty$ maps via p_1 isomorphically to a discrete subgroup in $\mathrm{SL}_2(\mathbb{R})$.

Let $\mathfrak{a} \subseteq \mathcal{O}$ be a proper ideal such that $\Gamma(\mathfrak{a}) := \ker\big(G(\mathcal{O}) \to G(\mathcal{O}/\mathfrak{a})\big)$ is torsion-free (see criterion III.2.3 for the existence of such ideals). We will interpret $\Gamma(\mathfrak{a})$ as a subgroup of $\mathrm{SL}_2(\mathbb{R})$. Note that, since we assumed D to be a division algebra, the group $\Gamma(\mathfrak{a})$ is a cocompact Fuchsian group (see Thm. 5.4.1 in [**Kat92**]).

Let $\mathfrak{h} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ be the Poincaré upper half-plane. To be consistent with the classical setting, throughout this section the group $\Gamma(\mathfrak{a})$ acts from the left on $\mathfrak{h}$. However, we still denote the quotient space $\mathfrak{h}/\Gamma(\mathfrak{a})$ by a right quotient.

V.5.3. Proposition. *The compact Riemann surface $\mathfrak{h}/\Gamma(\mathfrak{a})$ has genus*

$$g = 1 + 2^{-[F:\mathbb{Q}]}\,\mathrm{N}(\mathfrak{a})^3\,|\Delta_{rd}(D)\zeta_F(-1)| \prod_{\mathfrak{p}|\mathfrak{a}}\bigl(1-\mathrm{N}(\mathfrak{p})^{-2}\bigr) \prod_{\substack{\mathfrak{p}\in \mathrm{Ram}_f(D)\\ \mathfrak{p}\nmid\mathfrak{a}}} \bigl(1-\mathrm{N}(\mathfrak{p})^{-1}\bigr)$$

This implies an explicit formula for the first Betti number $b_1(\Gamma(\mathfrak{a}))$ since

$$b_1(\Gamma(\mathfrak{a})) = \dim H^1(\Gamma(\mathfrak{a}),\mathbb{C}) = 2g.$$

Proof. Consider the main theorem for $n = 1$. Note that for $n = 1$ the automorphism τ^* is actually the identity. This means that, using the main theorem with the trivial representation, we obtain

$$\mathcal{L}(\tau^*,\Gamma(\mathfrak{a}),\mathbb{C}) = \chi(\Gamma(\mathfrak{a})) = \chi(\mathfrak{h}/\Gamma(\mathfrak{a})).$$

Note that the sign of the Lefschetz number is -1. Since $\chi(\mathfrak{h}/\Gamma(\mathfrak{a})) = 2-2g$, the claim follows immediately. □

V.5.4. Definition. Let $k \geq 2$ be an integer. A holomorphic modular form of weight k for the group $\Gamma(\mathfrak{a})$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that

$$f(\gamma\cdot z) = (cz+d)^k f(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma(\mathfrak{a})$ and $z \in \mathfrak{h}$. Let $S_k(\Gamma(\mathfrak{a}))$ denote the $\mathbb{C}$-vector space of holomorphic weight k modular forms for $\Gamma(\mathfrak{a})$.

V.5.5. Corollary. *Let $k \geq 2$ be an integer. The dimension of the space $S_k(\Gamma(\mathfrak{a}))$ of holomorphic weight k modular forms for $\Gamma(\mathfrak{a})$ is*

$$\varepsilon + (k-1)2^{-[F:\mathbb{Q}]}\,\mathrm{N}(\mathfrak{a})^3\,|\Delta_{rd}(D)\zeta_F(-1)| \prod_{\mathfrak{p}|\mathfrak{a}}\bigl(1-\mathrm{N}(\mathfrak{p})^{-2}\bigr) \prod_{\substack{\mathfrak{p}\in \mathrm{Ram}_f(D)\\ \mathfrak{p}\nmid\mathfrak{a}}} \bigl(1-\mathrm{N}(\mathfrak{p})^{-1}\bigr),$$

where $\varepsilon = 1$ if $k = 2$, and $\varepsilon = 0$ otherwise.

Proof. Recall that $\Gamma(\mathfrak{a})$ is assumed to be torsion-free, thus $\Gamma(\mathfrak{a})$ contains no elliptic elements. Moreover, $\Gamma(\mathfrak{a})$ is cocompact and so there are no cusps. Shimura obtained formulas for the dimension of the space $S_k(\Gamma(\mathfrak{a}))$ depending on the genus. More precisely, in this special situation, we have

$$\dim(S_k(\Gamma(\mathfrak{a})) = \varepsilon + (k-1)(g-1),$$

see Theorems 2.24 and 2.25 in [**Shi71**]. The assertion follows immediately by substitution of the genus of $\mathfrak{h}/\Gamma(\mathfrak{a})$ into Shimura's formula. □

V.5.6. *Remark.* Note that the method used here is applicable to all cocompact arithmetically defined torsion-free Fuchsian groups Γ. More precisely, one needs to find a division quaternion algebra D, and a maximal order such that Γ is commensurable with $G(\mathcal{O})$. As a next step, one chooses some ideal $\mathfrak{a}$ and calculates the indices $[\Gamma : \Gamma(\mathfrak{a})\cap\Gamma]$ and $[\Gamma(\mathfrak{a}) : \Gamma(\mathfrak{a})\cap\Gamma]$. Then the Euler characteristic of Γ is $\chi(\Gamma) = [\Gamma : \Gamma(\mathfrak{a})\cap\Gamma]^{-1}[\Gamma(\mathfrak{a}) : \Gamma(\mathfrak{a})\cap\Gamma]\chi(\Gamma(\mathfrak{a}))$ (apply Prop. 5(c) in [**Ser71**]).

V.6. Addendum: A Lefschetz Number on Bianchi Groups

V.6.1. The main theorem V.4.2 has an obvious weak point: If the underlying number field has a complex place, then the Lefschetz number vanishes. Henceforth, in this situation, we are not able to extract information on the on the Betti numbers.

The reason for the vanishing of the Lefschetz number was already pointed out in Remark IV.6.10. More precisely, the Euler-Poincaré measure on complex semisimple Lie groups is zero, due to the involved rank conditions. We conclude that, in order to find non-vanishing Lefschetz numbers on arithmetic groups, the fixed point components may not be quotients of symmetric spaces associated with complex semisimple Lie groups.

In other words, if the underlying number field has a complex place, then we have to compute Lefschetz numbers of *involutions of the second kind*. Conversely, the Lefschetz number of an involution of the first kind is always zero. The machinery developed in this thesis, in particular in Chapter III, enables us to calculate Lefschetz numbers of involutions of the second kind. The procedure is very similar to the treatment of involutions of symplectic type. In this section we indicate how this can be done, using the example of Bianchi modular groups.

V.6.2. Let $E = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field, where $d \leq -1$ is a square-free negative integer. Let $\mathcal{O}$ denote the ring of integers of E. The non-trivial Galois automorphism of the extension $E/\mathbb{Q}$ will be denoted by σ. For simplicity we write $\overline{x} := \sigma(x)$ for all $x \in E$.

Consider the central simple E-algebra $A := M_2(E)$ equipped with the involution $\tau : A \to A$ defined by $x \mapsto \overline{x}^T$. This means we transpose the matrix x and apply the Galois automorphism σ to every entry. The involution τ is an involution of the second kind on A, i.e. τ is $\mathbb{Q}$-linear but not E-linear. We study the maximal $\mathcal{O}$-order $\Lambda := M_2(\mathcal{O})$ in A.

V.6.3. Lemma. *The order $\Lambda = M_2(\mathcal{O})$ is τ-smooth if and only if $d \equiv 1 \mod 4$.*

Proof. Assume Λ is τ-smooth. Then the matrix $x := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is in $\mathrm{Sym}(\Lambda, \tau)$ and hence, by τ-smoothness we find $y \in \Lambda$ such that $y + \overline{y}^T = x$. We observe

$$1 = y_{11} + \overline{y_{11}} = \mathrm{Tr}_{E/\mathbb{Q}}(y_{11}).$$

Such an element $y_{11} \in \mathcal{O}$ exists if and only if $d \equiv 1 \mod 4$.

Conversely, assume $d \equiv 1 \mod 4$. Let $x \in \mathrm{Sym}(\Lambda, \tau)$, then $x = \left(\begin{smallmatrix} a & z \\ \overline{z} & b \end{smallmatrix}\right)$ with $a, b \in \mathbb{Z}$ and $z \in \mathcal{O}$. Take $\omega \in \mathcal{O}$ with $\mathrm{Tr}_{E/\mathbb{Q}}(\omega) = 1$ and define $y = \left(\begin{smallmatrix} a\omega & z \\ 0 & b\omega \end{smallmatrix}\right)$. Clearly, $y + \overline{y}^T = x$. $\square$

V.6.4. *Assumption.* We shall assume from now on that $d \equiv 1 \mod 4$. In particular, the order $M_2(\mathcal{O})$ is τ-smooth and the trace $\mathrm{Tr}_{E/\mathbb{Q}} : \mathcal{O} \to \mathbb{Z}$ is surjective.

Recall that the ring of integers is $\mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}\omega$ where $\omega = \frac{1+\sqrt{d}}{2}$ (see X. Bsp. 1.9 in [**JS06**]).

V.6.5. Consider the smooth $\mathbb{Z}$-group scheme SL_2. We want to study the group scheme $\mathrm{SL}_2 \times_{\mathbb{Z}} \mathcal{O}$ over $\mathcal{O}$. However, the involution τ is not $\mathcal{O}$-linear, so it does not induce an automorphism τ^* on $\mathrm{SL}_2 \times_{\mathbb{Z}} \mathcal{O}$. Therefore we consider the restriction of scalars $G := \mathrm{Res}_{\mathcal{O}/\mathbb{Z}}(\mathrm{SL}_2 \times_{\mathbb{Z}} \mathcal{O})$. This is a smooth $\mathcal{O}$-group scheme and τ induces an automorphism τ^* of order two of G, defined as $\tau^* := \mathrm{inv} \circ \tau$.

The associated real Lie group G_∞ is

$$G_\infty = G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{C}).$$

The automorphism τ^* is a Cartan involution and $K_\infty := \mathrm{SU}(2)$ is the group of fixed points, which is automatically a τ^*-stable, maximal compact subgroup.

Let $m \geq 3$ be an integer. We consider the principal congruence subgroup

$$\Gamma(m) := \ker\big(\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/m\mathcal{O})\big).$$

Note that the assumption $m \geq 3$ implies that $\Gamma(m)$ is torsion-free (cf. III.2.3). This group is given by a local datum: for a prime number $p \in \mathbb{Z}$, we define

$$K_p := \ker\big(G(\mathbb{Z}_p) \to G(\mathbb{Z}_p/m\mathbb{Z}_p)\big).$$

The product $K_f := \prod_p K_p$ is an open compact subgroup of the locally compact group $G(\mathbb{A}_f)$. Here $\mathbb{A}$ denotes the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ denotes the ring of finite adeles of $\mathbb{Q}$. We obtain $\Gamma(m) = G(\mathbb{Q}) \cap K_f$.

V.6.6. *Non-abelian Galois cohomology.* We determine the non-abelian Galois cohomology set $H^1(\tau^*, G(R))$ for the rings $R = \mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{R}$ and $\mathbb{A}$.

Observe that a cocycle $b \in Z^1(\tau^*, G(R))$ is an element $b \in \mathrm{SL}_2(\mathcal{O} \otimes_{\mathbb{Z}} R)$ which satisfies $\bar{b}^T = b$, this is, b is a hermitian matrix with determinant 1. Further, two cocycles b and c are cohomologous if there is an element $g \in \mathrm{SL}_2(\mathcal{O} \otimes_{\mathbb{Z}} R)$ such that $b = \bar{g}^T cg$. Thus $H^1(\tau^*, G(R))$ classifies unimodular hermitian forms.

V.6.7. Proposition. *(a) For every prime number $p \in \mathbb{Z}$ the cohomology sets $H^1(\tau^*, G(\mathbb{Q}_p))$ and $H^1(\tau^*, G(\mathbb{Z}_p))$ are trivial.*

(b) The cohomology set $H^1(\tau^*, G(\mathbb{R}))$ consists of the two classes ± 1.

(c) The canonical inclusion $\mathbb{Q} \to \mathbb{R}$ induces a bijection of pointed sets

$$H^1(\tau^*, G(\mathbb{Q})) \xrightarrow{\simeq} H^1(\tau^*, G(\mathbb{R})).$$

(d) The diagonal embedding $\mathbb{Q} \to \mathbb{A}$ induces a bijection of the first non-abelian cohomology sets

$$H^1(\tau^*, G(\mathbb{Q})) \xrightarrow{\simeq} H^1(\tau^*, G(\mathbb{A})).$$

Proof. Ad (a): Let p be a prime number. We have to treat different cases depending on the splitting behaviour of p in $\mathcal{O}$.

Suppose that p splits in $\mathcal{O}$. Then there is an isomorphism of rings

$$f : \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$$

such that $f \circ (\sigma \otimes \mathrm{Id}) = \mathrm{swap} \circ f$, where "swap" denotes the automorphism which swaps the two components. Therefore $G(\mathbb{Z}_p) \cong \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p)$ and the Galois automorphism acts by swapping the two components. Now a direct calculation yields $H^1(\tau^*, G(\mathbb{Z}_p)) = \{1\}$. The same argument works for $\mathbb{Q}_p$.

Suppose that p is inert or ramified in $\mathcal{O}$. In this case $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}_p = E \otimes_{\mathbb{Q}} \mathbb{Q}_p =: E_p$ is a quadratic extension field of $\mathbb{Q}_p$. It is known that hermitian forms for $E_p/\mathbb{Q}_p$ are classified by dimension and determinant (cf. Chap. 10, 1.6 (ii) in [**Sch85**]). Therefore, given $b \in Z^1(\tau^*, \mathrm{SL}_2(E_p))$, we find $g \in \mathrm{GL}_2(E_p)$ such that $\bar{g}^T bg = 1$. However, since $\overline{\det(g)}\det(g) = 1$, we may alter g such that $\det(g) = 1$. We conclude that $H^1(\tau^*, G(\mathbb{Q}_p)) = \{1\}$.

Note that $\mathcal{O} \otimes \mathbb{Z}_p = \mathcal{O}_p$ is the valuation ring of E_p. Similarly, given a cocycle $b \in Z^1(\tau^*, G(\mathbb{Z}_p))$, the class of b goes to the trivial class in $H^1(\tau^*, \mathrm{GL}_2(E_p))$. We can use the theorem of Fainsilber and Morales (cf. III.5.9) to deduce that the class of b in $H^1(\tau^*, \mathrm{GL}_2(\mathcal{O}_p))$ is trivial. Applying the same argument as for $\mathbb{Q}_p$, we deduce that b represents the trivial class in $H^1(\tau^*, G(\mathbb{Z}_p))$.

Ad (b): This follows from the well-known classification of hermitian forms over $\mathbb{C}$ by their signature. Therefore, given $b \in Z^1(\tau^*, \mathrm{SL}_2(\mathbb{C}))$, we find $g \in \mathrm{GL}_2(\mathbb{C})$ such that $\overline{g}^T bg = \pm 1$. Altering g we can achieve $\det(g) = 1$.

Ad (c): This follows from the fact that the $E/\mathbb{Q}$ hermitian forms are classified by dimension, determinant and signature at the complex place (cf. Chap. 10, Ex. 1.6 (iv) in [**Sch85**]). Use the same argument as in the proof of (b).

Ad (d): This is immediate from the combination of (a) and (c). $\square$

V.6.8. *The fixed point groups.* It follows from the previous proposition that it is sufficient to study the fixed point groups $G^{\tau^*} = G(1)$ and $G(-1)$. Since -1 is central, twisting with -1 has no effect. This means, we merely need to analyse $H := G(1) = G^{\tau^*}$. In particular we need the smoothness of the group scheme H over $\mathbb{Z}$.

V.6.9. Lemma. *The group scheme of τ^*-fixed points $H = G^{\tau^*}$ is smooth over the ring of integers $\mathbb{Z}$.*

Proof. We assumed $d \equiv 1 \mod 4$, thus by Lemma V.6.3 the order $\Lambda := M_2(\mathcal{O})$ is τ-smooth. Consider the $\mathbb{Z}$-group scheme $G(\Lambda, \tau)$ defined in III.4.11. Recall that for a commutative ring C, the group of C-rational points is

$$G(\Lambda, \tau)(C) = \{ g \in \mathrm{GL}_2(\mathcal{O} \otimes_{\mathbb{Z}} C) \mid \overline{g}^T g = 1 \}.$$

In other words $G(\Lambda, \tau) = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}}(\mathrm{GL}_2 \times_{\mathbb{Z}} \mathcal{O})^{\tau^*}$ and by Theorem III.4.15 this group scheme is smooth over $\mathbb{Z}$.

Let $\mathbb{G}_m$ be the multiplicative group defined over $\mathbb{Z}$. The norm $\mathrm{N}_{E/\mathbb{Q}}$ defines a morphism of $\mathbb{Z}$-group schemes

$$\mathrm{N}_{E/\mathbb{Q}} : \mathrm{Res}_{\mathcal{O}/\mathbb{Z}}(\mathbb{G}_m \times_{\mathbb{Z}} \mathcal{O}) \to \mathbb{G}_m,$$

and the kernel of this homomorphism will be denoted by $\mathbb{G}_m^{(1)}$. The derivative of the norm is the trace, and by assumption the trace $\mathrm{Tr}_{E/\mathbb{Q}} : \mathcal{O} \to \mathbb{Z}$ is surjective. The smoothness criterion for kernels (Prop. III.2.8) yields that $\mathbb{G}_m^{(1)}$ is smooth over $\mathbb{Z}$.

The determinant defines a homomorphism $\det : G(\Lambda, \tau) \to \mathbb{G}_m^{(1)}$ of smooth group schemes over $\mathbb{Z}$. Recall that the Lie algebra of $G(\Lambda, \tau)$ is

$$\mathrm{Lie}(G(\Lambda, \tau))(\mathbb{Z}) = \{ x \in M_2(\mathcal{O}) \mid \overline{x}^T + x = 0 \}$$

(see III.4.12). Moreover, $\mathrm{Lie}(\mathbb{G}_m^{(1)})(\mathbb{Z}) = \{ a \in \mathcal{O} \mid \mathrm{Tr}_{E/\mathbb{Q}}(a) = 0 \}$. The derivative of the determinant is the trace, which is clearly surjective as a map

$$\mathrm{Tr} : \mathrm{Lie}(G(\Lambda, \tau))(\mathbb{Z}) \to \mathrm{Lie}(\mathbb{G}_m^{(1)})(\mathbb{Z}).$$

The smoothness criterion for kernels III.2.8 yields the claim since

$$H = \ker\bigl(\det : G(\Lambda, \tau) \to \mathbb{G}_m^{(1)}\bigr). \qquad \square$$

V.6.10. Corollary. *Let $p \in \mathbb{Z}$ be a prime number and let $e \geq 1$ be some integer. Define $K_p(e) := \ker\bigl(G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^e\mathbb{Z})\bigr)$. The group $K_p(e)$ is τ^*-stable and*

$$H^1(\tau^*, K_p(e)) = \{1\}.$$

Proof. Since G is smooth over $\mathbb{Z}$, there is a short exact sequence of groups with τ^*-action:

$$1 \longrightarrow K_p(e) \longrightarrow G(\mathbb{Z}_p) \longrightarrow G(\mathbb{Z}/p^e\mathbb{Z}) \longrightarrow 1.$$

We obtain a long exact sequence in Galois cohomology

$$H(\mathbb{Z}_p) \xrightarrow{\pi} H(\mathbb{Z}/p^e\mathbb{Z}) \longrightarrow H^1(\tau^*, K_p(e)) \longrightarrow H^1(\tau^*, G(\mathbb{Z}_p)).$$

The last term in the sequence is trivial due to Proposition V.6.7 (a). Moreover, the map π is surjective since H is a smooth group scheme over $\mathbb{Z}$. The claim follows immediately. $\square$

V.6.11. Corollary. *Consider the fibred product*

$$\mathcal{H}^1(\tau^*) := H^1(\tau^*, K_\infty K_f) \underset{H^1(\tau^*, G(\mathbb{A}))}{\times} H^1(\tau^*, G(\mathbb{Q})).$$

The projection $\pi : \mathcal{H}^1(\tau^*) \to H^1(\tau^*, G(\mathbb{Q}))$ is a bijection.

Proof. This follows from Prop. V.6.7 (d), the previous corollary, and Thm. II.4.4. $\square$

V.6.12. *Local volumes.* The real Lie group associated with H is the special unitary group

$$H_\infty = H(\mathbb{R}) = \mathrm{SU}(2).$$

It is a compact Lie group with Lie algebra

$$\mathfrak{h}_\infty = \mathfrak{su}_2 = \{\, X \in M_2(\mathbb{C}) \mid \overline{X}^T + X = 0 \text{ and } \mathrm{Tr}(X) = 0 \,\}.$$

We consider the Lie algebra

$$\mathfrak{h} = \mathrm{Lie}(H)(\mathbb{Q}) = \{\, X \in M_2(E) \mid \overline{X}^T + X = 0 \text{ and } \mathrm{Tr}(X) = 0 \,\}$$

with the non-degenerate $\mathbb{Q}$-bilinear form $B(x,y) := -\frac{1}{2}\mathrm{Tr}(xy)$. Note that B takes only values in $\mathbb{Q}$. Moreover, since SU(2) is compact, it is automatically its compact dual group.

V.6.13. Lemma. *The volume of* SU(2) *with respect to the metric form B is*

$$\mathrm{vol}_B(\mathrm{SU}(2)) = 2\pi^2.$$

Proof. Let e_1 denote the first basis vector in $\mathbb{C}^2$. The map $\varphi : \mathrm{SU}(2) \to S^3$ defined by $g \mapsto ge_1$ is a left SU(2)-equivariant diffeomorphism. We transfer the form B to S^3. The tanget map $\mathrm{d}(\varphi) : \mathfrak{su}_2 \to T_{e_1}S^3$ is given by $x \mapsto xe_1$. Every element $x \in \mathfrak{su}_2$ has the form $x = \left(\begin{smallmatrix} ia & -\overline{z} \\ z & -ia \end{smallmatrix}\right)$ for some $a \in \mathbb{R}$ and $z \in \mathbb{C}$. Thus the matrix x is mapped to the vector $(ia, z)^T$ under $\mathrm{d}(\varphi)$. We obtain

$$B((ia,z),(ib,w)) = -\frac{1}{2}\mathrm{Tr}\Big(\left(\begin{smallmatrix} ia & -\overline{z} \\ z & -ia \end{smallmatrix}\right)\left(\begin{smallmatrix} ib & -\overline{w} \\ w & -ib \end{smallmatrix}\right)\Big) = ab + \frac{1}{2}(\overline{z}w + z\overline{w}).$$

This is the standard Riemann metric on S^3 and hence

$$\mathrm{vol}_B(SU(2)) = \mathrm{vol}(S^3) = 2\pi^2. \qquad \square$$

V.6.14. *The modulus factor.* We determine the modulus factor $m(B)$ defined in paragraph IV.6.7. Consider the Lie algebra

$$\mathfrak{h}_\mathbb{Z} = \mathrm{Lie}(H)(\mathbb{Z}) = \{\, x \in M_2(\mathcal{O}) \mid \overline{x}^T + x = 0 \text{ and } \mathrm{Tr}(x) = 0 \,\}.$$

We find a $\mathbb{Z}$-basis of $\mathrm{Lie}(H)(\mathbb{Z})$ consisting of the elements

$$h = \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix}, \quad e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & -\overline{\omega} \\ \omega & 0 \end{pmatrix},$$

where $\omega = \frac{1+\sqrt{d}}{2}$. We evaluate B on this basis and find

$$B(h,h) = -d,\ B(e,e) = 1,\ B(e,f) = \frac{1}{2}, \text{ and } B(f,f) = \frac{1-d}{4},$$

and clearly $B(e,h) = B(f,h) = 0$. Summing up, the product formula yields that

$$m(B) = \prod_p m(B)_p = m(B)_\infty^{-1} = \frac{2}{|d|}.$$

V.6.15. Theorem. *Let $m \geq 3$ be an integer and define*

$$K_{H,p} := K_p \cap H(\mathbb{Z}_p) = \ker\big(H(\mathbb{Z}_p) \to H(\mathbb{Z}_p/m\mathbb{Z}_p)\big),$$

as well as the open compact subgroup $K_{H,f} := \prod_p K_{H,p} \subset H(\mathbb{A}_f)$. The group $H(\mathbb{Q})$ acts freely on $K_{H,f}\backslash H(\mathbb{A}_f)$ and the Euler characteristic of the double quotient space $K_{H,f}\backslash H(\mathbb{A}_f)/H(\mathbb{Q})$ is

$$\chi(K_{H,f}\backslash H(\mathbb{A}_f)/H(\mathbb{Q})) = \frac{m^3\,|d|}{24} \prod_{p|m}(1 - \frac{1}{p^2}) \prod_{\substack{p|d \\ p\nmid m}}(1 + \frac{(-1)^{(p-1)/2}}{p})$$

Proof. Since $m \geq 3$, the group $\Gamma(m) \subseteq G(\mathbb{Q})$ is torsion-free and thus, since G has strong approximation, $G(\mathbb{Q})$ acts freely on $K_\infty K_f\backslash G(\mathbb{A})$ (cf. Lemma IV.6.4). The canonical map $H(\mathbb{A}_f) \to G(\mathbb{A}_f)$ induces a closed, $H(\mathbb{Q})$-equivariant embedding

$$K_{H,f}\backslash H(\mathbb{A}_f) \to K_\infty K_f\backslash G(\mathbb{A}),$$

henceforth the action of $H(\mathbb{Q})$ on $K_{H,f}\backslash H(\mathbb{A}_f)$ is free.

Now we apply the adelic Euler characteristic formula IV.6.12. Note that $\mathrm{SU}(2)$ is connected and that $H \times_\mathbb{Z} \mathbb{Q}$ is a $\mathbb{Q}$-form of SL_2. Therefore $H \times_\mathbb{Z} \mathbb{Q}$ is semisimple and simply connected, and has Tamagawa number 1 due to [**Kot88**]. We obtain

$$\chi(K_{H,f}\backslash H(\mathbb{A}_f)/H(\mathbb{Q})) = \frac{|d|}{4\pi^2} \prod_p \frac{p^{3\alpha_p}}{|U_p|},$$

where (U,α) is the local datum describing $\Gamma(m)$ (cf. IV.6.11). This means, we have $\alpha_p = \nu_p(m)$ and $|U_p| = 1$ for every prime number p dividing m. For all prime numbers which do not divide m the local datum is given by $\alpha_p = 1$ and $U_p = H(\mathbb{Z}/p\mathbb{Z})$.

Let p be a prime number, we determine the finite groups $H(\mathbb{Z}/p\mathbb{Z})$. For simplicity we define $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$.

Case 1: Suppose p splits in E, then it follows from the Chinese remainder theorem that $\mathcal{O}\otimes_\mathbb{Z}\mathbb{F}_p \cong \mathbb{F}_p \times \mathbb{F}_p$. Further, the Galois automorphism acts by swapping the two components. So $G(\mathbb{F}_p) = \mathrm{SL}_2(\mathbb{F}_p \times \mathbb{F}_p)$ and $H(\mathbb{F}_p) \cong \mathrm{SL}_2(\mathbb{F}_p)$. The cardinality of this group is $|H(\mathbb{F}_p)| = p^3(1 - p^{-2})$.

Case 2: Suppose p is inert in E. In this case $\ell := \mathcal{O} \otimes_\mathbb{Z} \mathbb{F}_p$ is the unique finite field with p^2 elements and $H(\mathbb{F}_p)$ is the special unitary group SU_2 of the field extension $\ell/\mathbb{F}_p$. We find $|H(\mathbb{F}_p)| = p^3(1 - p^{-2})$, see 3.6 in [**Wil09**].

Case 3: Suppose p is ramified in E, i.e. p divides d. There is an isomorphism of rings $\mathcal{O}\otimes_\mathbb{Z}\mathbb{F}_p \cong \mathbb{F}_p \oplus \mathbb{F}_p\mu$ with $\mu^2 = 0$. The Galois automorphism acts with -1 on μ. The projection morphism $q : \mathbb{F}_p \oplus \mathbb{F}_p\mu \to \mathbb{F}_p$, which sends $x + y\mu$ to x, induces a homomorphism of finite groups $q_* : H(\mathbb{F}_p) \to \mathrm{SO}_2(\mathbb{F}_p)$. Here the group SO_2 is defined by $\mathrm{SO}_2(\mathbb{F}_p) = \{\, x \in \mathrm{SL}_2(\mathbb{F}_p) \mid x^T x = 1 \,\}$. We obtain a short exact sequence of finite groups

$$1 \longrightarrow \mathrm{Sym}_2(\mathbb{F}_p)_0 \longrightarrow H(\mathbb{F}_p) \xrightarrow{q_*} \mathrm{SO}_2(\mathbb{F}_p) \longrightarrow 1.$$

Here $\mathrm{Sym}_2(\mathbb{F}_p)_0$ denotes the set of symmetric 2×2 matrices with entries in $\mathbb{F}_p$ which have vanishing trace. Summing up, we obtain

$$|H(\mathbb{F}_p)| = p^3(1 - (-1)^{(p-1)/2}p^{-1})$$

using § 3.7.2 in [**Wil09**]. Note that it depends on the behaviour of p modulo 4 whether the orthogonal group is of plus or of minus type.

We multiply all the terms and obtain

$$\chi(K_{H,f}\backslash H(\mathbb{A}_f)/H(\mathbb{Q})) = \frac{|d|}{4\pi^2} m^3 \zeta(2) \prod_{p|m} \bigl(1-p^{-2}\bigr) \prod_{\substack{p|d \\ p\nmid m}} \bigl(1+(-1)^{(p-1)/2}p^{-1}\bigr)$$

The claim follows from the fact that $\zeta(2) = \frac{\pi^2}{6}$. □

V.6.16. Theorem. *Let $E = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field with $d \equiv 1 \mod 4$ and let $\mathcal{O}$ be its ring of integers. For an integer $m \geq 3$ consider the congruence subgroup*

$$\Gamma(m) := \ker\bigl(\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/m\mathcal{O})\bigr).$$

Let $G = \mathrm{Res}_{E/\mathbb{Q}}(\mathrm{SL}_2 \times E)$ be the algebraic $\mathbb{Q}$-group obtained by restriction of scalars and let $\tau^* : G \to G$ denote the automorphism of order two defined by $x \mapsto (\overline{x}^T)^{-1}$.

Let $\rho : G \times_{\mathbb{Q}} \overline{\mathbb{Q}} \to \mathrm{GL}(W)$ be a rational representation on a finite dimensional $\overline{\mathbb{Q}}$-vector space W equipped with a compatible τ^*-action.

The Lefschetz number $\mathcal{L}(\tau^*, \Gamma(m), W)$ of τ^* in the cohomology $H^\bullet(\Gamma(m), W)$ is

$$\mathcal{L}(\tau^*, \Gamma(m), W) = \frac{m^3|d|}{12} \mathrm{Tr}(\tau^*|W) \prod_{p|m} \bigl(1-p^{-2}\bigr) \prod_{\substack{p|d \\ p\nmid m}} \bigl(1+(-1)^{(p-1)/2}p^{-1}\bigr).$$

The Lefschetz number is non-zero precisely when $\mathrm{Tr}(\tau^*|W)$ *does not vanish, in this case the sign of $\mathcal{L}(\tau^*, \Gamma(m), W)$ is the sign of* $\mathrm{Tr}(\tau^*|W)$.

Proof. The group G has strong approximation, so we can apply the adelic Lefschetz number formula II.6.8. We obtain

$$\mathcal{L}(\tau^*, \Gamma(m), W) \;=\; \sum_{\eta \in \mathcal{H}^1(\tau^*)} \chi(\vartheta^{-1}(\eta))\, \mathrm{Tr}(\tau^*|W(b_\eta)),$$

where b_η is a representative of the $\mathbb{Q}$-component of η. We know that $\mathcal{H}^1(\tau^*)$ consists of precisely two elements, represented by the cocycles $\pm 1 \in Z^1(\tau^*, G(\mathbb{Q}))$ (see Corollary V.6.11 and Proposition V.6.7). Note that $G(\overline{\mathbb{Q}}) = \mathrm{SL}_2(\overline{\mathbb{Q}}) \times \mathrm{SL}_2(\overline{\mathbb{Q}})$ and the non-trivial Galois automorphism of $E/\mathbb{Q}$ acts by swapping the two components. Hence we find $H^1(\tau^*, G(\overline{\mathbb{Q}})) = \{1\}$ and so $\mathrm{Tr}(\tau^*|W) = \mathrm{Tr}(\tau^*|W(-1))$ (see Lemma II.5.11).

Consider the class $\eta \in \mathcal{H}^1(\tau^*)$ which maps to the trivial class in $H^1(\tau^*, G(\mathbb{Q}))$. By Lemma II.6.7 the fixed point component $\vartheta^{-1}(\eta)$ is homeomorphic to the double coset space $K_{H,f}\backslash H(\mathbb{A}_f)/H(\mathbb{Q})$.

Consider the class $\eta' \in \mathcal{H}^1(\tau^*)$ which is represented by $-1 \in H^1(\tau^*, G(\mathbb{Q}))$. Since -1 is central, twisting has no effect. More precisely, let p be a prime number and let $a_p \in G(\mathbb{Z}_p)$ be such that $a_p(-1)\,{}^{\tau}a_p = 1$ (such an element exists by V.6.7). Set $a_\infty := 1$ and consider the adelic point $a = (a_\infty, (a_p)_p) \in G(\mathbb{A})$. For every prime number p we put $k_p := 1$ and we set $k_\infty := -1$. Then we have ${}^{\tau^*}a = k^{-1}a(-1)$ and by Lemma II.6.7 we obtain

$$\vartheta^{-1}(\eta') \cong a_f^{-1} K_f^{\tau^*} a_f \backslash H(\mathbb{A}_f)/H(\mathbb{Q}),$$

where $a_f = (a_p)_p \in G(\mathbb{A}_f)$. Note that $K_f^{\tau^*} = K_{H,f}$ in the notation of Theorem V.6.15. For every prime number p the identity ${}^{\tau^*}a_p = -a_p$ implies that $a_p^{-1}K_{H,p}a_p = K_{H,p}$, i.e.

$$\vartheta^{-1}(\eta') \cong K_{H,f}\backslash H(\mathbb{A}_f)/H(\mathbb{Q}).$$

Now, the assertion follows directly from Theorem V.6.15. □

V.6.17. *Remark.* Theorem V.6.16 can be used, just like the main theorem, to deduce lower bounds for the total Betti number.

Let $m \geq 3$ and consider the trivial representation. The associated group of real points $G(\mathbb{R})$ is isomorphic to $\mathrm{SL}_2(\mathbb{C})$ and the maximal compact subgroup is $\mathrm{SU}(2)$. The associated symmetric space $\mathrm{SU}(2)\backslash \mathrm{SL}_2(\mathbb{C})$ has dimension 3 and is usually called the hyperbolic 3-space. The $\mathbb{Q}$-rank of G is one and hence the cohomological dimension of $\Gamma(m)$ is $2 = 3-1$ (this follows from the Borel-Serre compactification [**BS73**, 11.4.4]). The Betti number $b_0(\Gamma(m))$ of $\Gamma(m)$ is one, and we deduce that

$$1 + b_1(\Gamma(m)) + b_2(\Gamma(m)) \geq \frac{m^3|d|}{12} \prod_{p|m} \big(1-p^{-2}\big) \prod_{\substack{p|d\\ p\nmid m}} \big(1+(-1)^{(p-1)/2}p^{-1}\big).$$

The index $[\mathrm{SL}_2(\mathcal{O}) : \Gamma(m)]$ is less than m^6, thus we obtain

$$b_1(\Gamma(m)) + b_2(\Gamma(m)) \gg [\mathrm{SL}_2(\mathcal{O}) : \Gamma(m)]^{1/2}$$

as m goes to infinity.

However, this order of growth is too small. Considering cohomology classes induced from the boundary of the Borel-Serre compactification one can show that the first Betti number grows at least as fast as $[\mathrm{SL}_2(\mathcal{O}) : \Gamma(m)]^{2/3}$ (see 6.1 in [**KS12**]).

V.6.18. *Lefschetz numbers on Bianchi groups.* Bianchi groups and their principal congruence subgroups are the most prominent examples of arithmetic groups for which Lefschetz numbers have been computed. Rohlfs calculated the Lefschetz number of the Galois automorphism on the full Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$ (see [**Roh85**], and cf. §4 in [**Roh78**]), here $\mathcal{O}_d$ denotes the ring of integers of the imaginary quadratic number field $E = \mathbb{Q}(\sqrt{d})$. For principal congruence subgroups the Lefschetz number of the Galois automorphism can be found in [**KS12**, Cor. 6.3] and also in [**ST12**]. Krämer [**Krä85**] further considered the Lefschetz number of the Galois involution twisted by the conjugation with the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The main application of the Lefschetz numbers obtained in [**Roh85**], [**Krä85**] and [**ST12**] is to establish lower bounds for the dimension of the cuspidal cohomology of Bianchi groups.

V.6.19. *Remark.* The results of this section can be generalised to treat principal congruence subgroups in $\mathrm{SL}_n(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of an imaginary quadratic number field $E = \mathbb{Q}(\sqrt{d})$. In particular, if $d \equiv 1 \mod 4$, the tools developed in Chapter III work as in the example treated above. For $E = \mathbb{Q}(i)$ and $\mathcal{O} = \mathbb{Z}[i]$ the Lefschetz number of the automorphism $x \mapsto (\overline{x}^T)^{-1}$ has already been considered by Lai (see [**Lai91**]).

APPENDIX A

Lie Groups and Maximal Compact Subgroups

A.1. Introduction

A.1.1. In this Appendix we summarise some important results concerning the structure of real Lie groups and their maximal compact subgroups. These results are spread all over the literature and it seemed more convenient to assemble them here stated in the way we need them.

Section A.2 contains a small recapitulation of continuous cohomology of compact topological groups. In Section A.3 we give a detailed proof of *Mostow's Theorem*. Finally, we apply Mostow's Theorem to obtain some results concerning actions of finite groups on Lie groups (see Section A.4).

A.2. Continuous Cohomology

A.2.1. Let V be a finite dimensional real or complex vector space and let G be a locally compact Hausdorff[1] topological group. We consider V as a topological vector space equipped with the Euclidean topology. Furthermore, let π denote a continuous representation of G on V. This means that π is a homomorphism of groups $\pi : G \to \mathrm{GL}(V)$ which induces a continuous map $G \times V \to V$. We are going to define continuous cohomology of G with values in V.

Define $C^q(G, V) := \{ f : G^q \to V \mid f \text{ continuous} \}$ for all integers $q \geq 1$, and set $C^0(G, V) := V$. We define the differential $d_q : C^q(G, V) \to C^{q+1}(G, V)$ by the formula

$$\begin{aligned} d_q(f)(x_0, \ldots, x_q) &= x_0 \cdot f(x_1, \ldots, x_q) \\ &\quad + \sum_{i=0}^{q-1} (-1)^{i+1} f(x_0, \ldots, x_i x_{i+1}, \ldots, x_q) + (-1)^{q+1} f(x_0, \ldots, x_{q-1}). \end{aligned}$$

The cohomology $H^\bullet_{\mathrm{ct}}(G, V)$ of this complex is called the *continuous cohomology* of G with values in V. It is a continuous analog of abstract group cohomology. More details on continuous cohomology can be found in [**BW00**, IX].

We are only interested in the case when G is a compact group. Note that all the results especially apply to finite groups G.

A.2.2. Theorem (cf. IX Prop. 1.12 in [**BW00**]). *If G is compact, then*

$$H^q_{ct}(G, V) = 0$$

for all $q > 0$.

Proof. Let $q \geq 0$ be given. Take a $(q+1)$-cocycle $f \in C^{q+1}(G, V)$, this is, f satisfies $d_{q+1}(f) = 0$. We show that f is a coboundary. Fix a Haar measure for G such that $\mathrm{vol}(G) = 1$. Recall that G is compact and thus G is a unimodular group. We

[1] In this appendix, we will always assume all topological groups to be Hausdorff.

define $\alpha \in C^q(G, V)$ by averaging, i.e.

$$\alpha(x_0, \dots, x_{q-1}) := \int_G f(x_0, \dots, x_{q-1}, y) dy$$

We claim that $d_q(\alpha) = \pm f$. This can be seen by a simple calculation. First recall that

$$\begin{aligned} d_q(\alpha)(x_0, \dots, x_q) =& x_0 \cdot \alpha(x_1, \dots, x_q) \\ &+ \sum_{i=0}^{q-1} (-1)^{i+1} \alpha(x_0, \dots, x_i x_{i+1}, \dots, x_q) \\ &+ (-1)^{q+1} \alpha(x_0, \dots, x_{q-1}). \end{aligned}$$

We consider the first summand and we observe

$$\begin{aligned} x_0 \cdot \alpha(x_1, \dots, x_q) &= \int_G x_0 \cdot f(x_1, \dots, x_q, x_{q+1}) dx_{q+1} \\ &= \int_G \sum_{i=0}^{q} (-1)^i f(x_0, \dots, x_i x_{i+1}, \dots, x_{q+1}) + (-1)^{q+1} f(x_0, \dots, x_q) dx_{q+1} \\ &= \sum_{i=0}^{q-1} (-1)^i \alpha(x_0, \dots, x_i x_{i+1}, \dots, x_q) \\ &\quad + (-1)^q \alpha(x_0, \dots, x_{q-1}) + (-1)^{q+1} f(x_0, \dots, x_q) \end{aligned}$$

We see that $d_q(\alpha) = (-1)^{q+1} f$ and the theorem follows immediately. $\square$

A.2.3. Using the above vanishing theorem for H^1_{ct} and H^2_{ct}, we can deduce some powerful structural theorems for topological groups. More precisely, we can use it to study topological groups which are extensions of compact groups by vector groups.

A.2.4. Corollary. *Let G be a compact topological group and let V be a real finite dimensional continuous G-module. The topological semidirect product group*

$$L := V \rtimes G$$

has the following properties.

(i) G is a maximal compact subgroup.
(ii) Every compact subgroup of L is conjugate (under V) to a subgroup of G.
(iii) All maximal compact subgroups in L are conjugate.
(iv) A maximal compact subgroup of L touches every connected component.

Proof. The statement (i) follows from the fact that V has no compact subgroups except $\{0\}$. So, if K is a compact subgroup of L containing G, then $K \cap V = \{0\}$. We may deduce $K = G$.

Let $K \subset L$ be any compact subgroup. Let $p_G : L \to G$ denote the projection onto G and let $p_V : L \to V$ denote the projection onto V. Note that p_G is a group homomorphism, whereas p_V is in general no homomorphism. The image $K^* := p_G(K)$ of K under p_G is a subgroup of G. As before $K \cap V = \{0\}$ and therefore p_G restricts to an isomorphism of topological[2] groups $K \to K^*$. Let $\phi : K^* \to K$ denote the inverse isomorphism. Furthermore, we obtain a continuous map $f := p_V \circ \phi$ from K^* to V. For any $g \in K^*$ the element $k := (f(g), g)$ is, by construction, the unique member of K with $p_G(k) = g$. Moreover, the map f is a 1-cocycle for K^* with values in V since

$$(f(gh), gh) = (f(g), g)(f(h), h) = (f(g) + (g \cdot f(h)), gh)$$

[2]This map is a homeomorphism since K and K^* are compact and Hausdorff.

for all $g, h \in K^*$. Using the vanishing theorem for H^1_{ct}, we see that there is some vector $v \in V$ such that $f(g) = g \cdot v - v$ for all $g \in K^*$. Consequently,

$$(v, 1)(f(g), g)(-v, 1) = (f(g) + v - g \cdot v, g) = (0, g).$$

Hence the element $(v, 1) \in L$ takes K into G. This proves assertion (ii). The remaining statements follow readily. $\square$

A.2.5. Corollary. *Let G be a compact topological group and let V be a finite dimensional real vector space. Moreover, let L be some locally compact extension group of G by V. This means, there is a short exact sequence of groups*

$$1 \longrightarrow V \xrightarrow{\iota} L \xrightarrow{\pi} G \longrightarrow 1 \tag{19}$$

with continuous homomorphisms ι and π. We make the additional assumption that there is some continuous function $\sigma : G \to L$ s.t. $\pi \circ \sigma = \mathrm{Id}$. Then $L \cong V \rtimes G$ with a suitable continuous action of G on V.

Proof. Note that V can be identified with a closed subgroup of L using an appropriate open mapping theorem. The compact group G acts on V via $g \cdot v := \sigma(g) v \sigma(g)^{-1}$. This is a continuous action by group homomorphisms, hence by real linear maps. Clearly, the induced map $G \times V \to V$ is continuous.

We define a continuous function $f : G \times G \to V$ by $f(x, y) := \sigma(x)\sigma(y)\sigma(xy)^{-1}$ for all $x, y \in G$. We show that f is a 2-cocycle for $H^2_{\mathrm{ct}}(G, V)$. This is immediate from the following calculation for $x, y, z \in G$:

$$\begin{aligned} f(x, y) + f(xy, z) =& \sigma(x)\sigma(y)\sigma(xy)^{-1}\sigma(xy)\sigma(z)\sigma(xyz)^{-1} \\ =& \sigma(x)\sigma(y)\sigma(z)\sigma(yz)^{-1}\sigma(x)^{-1}\sigma(x)\sigma(yz)\sigma(xyz)^{-1} \\ =& x \cdot f(y, z) + f(x, yz). \end{aligned}$$

By the vanishing theorem for H^2_{ct} we obtain a continuous function $\alpha : G \to V$ such that

$$f(x, y) = \alpha(x) + x \cdot \alpha(y) - \alpha(xy)$$

for all $x, y \in G$. Furthermore, we define the continuous function $\tau : G \to L$ by $\tau(g) = \alpha(g)^{-1}\sigma(g)$. Then τ is a section of π, i.e. a continuous group homomorphism $G \to L$ right inverse to π. This can be seen as follows

$$\tau(xy) = \alpha(xy)^{-1}\sigma(xy) = (\sigma(x)\alpha(y)\sigma(x)^{-1}\alpha(x))^{-1}\sigma(x)\sigma(y) = \tau(x)\tau(y).$$

Since G is compact the image of τ is closed in L and so τ is an embedding. We conclude that

$$\iota \times \tau : V \rtimes G \to L$$

is an isomorphism of topological groups (here we apply an open mapping theorem). $\square$

A.2.6. *Lie Groups.* We conclude this section with a remark on the additional assumption in Corollary A.2.5, i.e. the existence of the map σ. We will see that σ always exists if G and L are finite dimensional Lie groups and π is a surjective submersion.

Lemma. Let V be a real finite dimensional vector group, G a compact Lie group and L some extension Lie group. Hence, we have a short exact sequence

$$1 \longrightarrow V \xrightarrow{\iota} L \xrightarrow{\pi} G \longrightarrow 1, \tag{20}$$

where we assume π to be a surjective submersion, i.e. G is a quotient group of L by a normal vector subgroup. Then there is a continuous function $\sigma : G \to L$ right inverse to π. Consequently, by A.2.5, there is even a smooth section homomorphism.

PROOF. Since π is a surjective submersion, we can find smooth local sections of π. Therefore, for every $x \in G$ there is a compact neighbourhood U_x of x and a continuous function $\sigma_x : U_x \to L$ right inverse to π. Compactness of G yields a finite number of such x, say $x_1, \ldots, x_n$, such that the interiors of $U_{x_1}, \ldots, U_{x_n}$ cover G.

It suffices to prove the following assertion: if we have U_1, U_2 and σ_1, σ_2 as above, then we can construct a section $\sigma : U_1 \cup U_2 \to L$. First note that we have a continuous function ρ on $W := U_1 \cap U_2$ to V defined by $\rho(x) := \sigma_1(x)\sigma_2(x)^{-1}$. As U_2 is a compact T_2-space and V a real vector space we can use Tietze's theorem to extend ρ to a continuous function $\rho^* : U_2 \to V$. Define a new section $\sigma_2^* : U_2 \to L$ by $\sigma_2^*(x) = \rho^*(x)\sigma_2(x)$. Observe that $\sigma_2^* = \sigma_1$ on $U_1 \cap U_2$ and thus the claim follows. $\square$

A.3. Maximal Compact Subgroups of fcc Groups

A.3.1. We shall call a real Lie group with a finite number of connected components an *fcc group*. This notion was introduced by Mostow in [**Mos55b**]. The aim of this section is to prove the main structural theorem for maximal compact subgroups of fcc groups (see Thm. A.3.4). This theorem is a generalisation of the *Cartan decomposition* for connected semisimple groups (cf. [**Kna02**, VI.3]). Many people contributed to this theorem, in particular Cartan, Iwasawa, Malcev and Mostow, however we will briefly call it *Mostow's Theorem*. We will follow the account of Mostow given in [**Mos55b**]. Moreover, we use the notion of an *exp-complement* inspired by Mostow's definition of *exp-set* (cf. [**Mos55a**]). It should be pointed out that parts of this theorem can be generalised to certain topological groups (cf. [**HT94**], [**Pla66**]).

A.3.2. *exp-complements.* Let G be a real Lie group and let K be a closed Lie subgroup. Let $\mathfrak{g}$ denote the Lie algebra of G and let $\mathfrak{k} \subseteq \mathfrak{g}$ denote the Lie algebra of K. An *exp-complement* E of K in G is a finite collection of linear $\mathrm{Ad}(K)$-stable subspaces $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \subseteq \mathfrak{g}$ of the Lie algebra such that

- $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m$ and
- the map $\mu_E : K \times \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m \to G$ given by

  $$\mu_E(k, X_1, \ldots, X_m) = k \exp(X_1) \ldots \exp(X_m)$$

  is a diffeomorphism.

If $K = G$, the empty collection of subspaces is an exp-complement. Most subgroups of G will not have an exp-complement, but we will see that maximal compact subgroups of fcc groups admit exp-complements. Note further that if K has an exp-complement, then $K\backslash G$ is diffeomorphic to a Euclidean space.

A.3.3. Lemma. *Let K be a compact Lie group and let V, W be two finite dimensional real K-representations. Suppose there is a surjective K-homomorphism $\pi : V \to W$ and moreover, that W is the direct sum*

$$W = \bigoplus_{i=1}^{m} W_i$$

of K-stable subspaces W_i. Then there are K-stable subspaces $V_1, \ldots, V_m$ of V such that

$$V = \ker(\pi) \oplus \bigoplus_{i=1}^{m} V_i$$

and such that $\pi(V_i) = W_i$ for every i.

Proof. This follows from the fact that finite dimensional representations of compact groups are completely reducible (see [**Kna02**, Cor. 4.7]). More precisely, choose for every i a K-stable complement V_i of $\ker(\pi)$ in the inverse image $\pi^{-1}(W_i)$. $\square$

A.3.4. Theorem (cf. Thm. 3.1 [**Mos55b**]). *Let G be a Lie group with finitely many connected components. Then the following hold*

- *(A) Every compact subgroup of G is contained in a maximal one. In particular, G has maximal compact subgroups.*
- *(B) All maximal compact subgroups are conjugate under elements in G^0.*
- *(C) If K is maximal compact, then $G = KG^0$*
- *(D) Every maximal compact subgroup of G admits an exp-complement.*

A.3.5. *Remark.* The proof of Mostow's Theorem is rather difficult, but the effort will pay off. The tedious discussion of many different cases in the proof should not conceal the fact that the two basic cases one has to treat are: 1. extensions of compact groups by vector groups and 2. semisimple groups. The first case was treated in A.2 via the theory of continuous cohomology. The case of semisimple groups is the classical theory of Cartan, which is well-known and accessible in the literature.

Proof of Mostow's Theorem. We prove the theorem (all statements simultaneously) by induction on the dimension of G. If $\dim G = 0$, then G is a finite group and the theorem holds trivially.

Let $\mathfrak{g}$ denote the (real) Lie algebra of G. We assume $\dim G = n$ under the induction hypothesis that the theorem holds for all fcc groups of smaller dimension. We distinguish two cases: (i) $\mathfrak{g}$ has a radical and (ii) $\mathfrak{g}$ is semisimple.

Case (i): Let $\mathfrak{r} \subset \mathfrak{g}$ denote the radical of $\mathfrak{g}$ and let $\mathfrak{a}$ denote the last non-zero factor of the derived series of $\mathfrak{r}$. Then $\mathfrak{a}$ is an abelian and characteristic ideal in $\mathfrak{g}$. Let A denote the unique virtual connected Lie subgroup of G with Lie algebra $\mathfrak{a}$. It is a normal abelian subgroup and the same holds for its closure $\bar{A}$. There is an isomorphism $\bar{A} \cong T \times V$ where T is a torus and V is a vector group. We have to discuss three cases:

Case (i)-a: $T \neq \{1\}$.
Then T is a compact (closed) normal Lie subgroup of G and so $H = G/T$ is of smaller dimension than G. Let $\pi : G \to H$ denote the canonical projection. Let $L \subset H$ be a maximal compact subgroup of H, then $\pi^{-1}(L)$ is maximal compact in G. Moreover, if $K \subset G$ is any compact subgroup of G, then $\pi(K)$ is compact, hence contained in some maximal compact subgroup L of H. Therefore the group K lies in the maximal compact subgroup $\pi^{-1}(L)$. This proves (A).

Fix a maximal compact subgroup L in H and let K be any maximal compact subgroup of G. There is some $g \in G^0$ (note $H^0 = \pi(G^0)$) such that

$$\pi(g)\pi(K)\pi(g)^{-1} = L,$$

thus $gKg^{-1} \subset \pi^{-1}(L)$. The equality $gKg^{-1} = \pi^{-1}(L)$ follows from the maximality of K. We deduce (B). Statement (C) follows from a simple observation: If $K = \pi^{-1}(L)$ is maximal compact, then $LH^0 = H$ implies $KG^0 = G$.

Finally, let $K = \pi^{-1}(L)$ be a maximal compact subgroup. We need to find an exp-complement of K. By the induction hypothesis an exp-complement $E' = (\mathfrak{p}_1', \ldots, \mathfrak{p}_m')$ to L in H exists. Here the $\mathfrak{p}_i'$ are $\mathrm{Ad}(L)$-stable subspaces of the Lie algebra $\mathfrak{h}$ of H. Using Lemma A.3.3 we obtain $\mathrm{Ad}(K)$-stable subspaces $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \bigoplus_{i=1}^{m} \mathfrak{p}_i$$

and moreover $\mathrm{d}\,\pi(\mathfrak{p}_i) = \mathfrak{p}_i'$ for all i. One has to check that $\mu_E : K \times \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m \to G$ is a diffeomorphism. This can be done using the following commutative diagram.

$$\begin{array}{ccc} K \times \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m & \xrightarrow{\mu_E} & G \\ {\scriptstyle \pi\times\mathrm{d}\,\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ L \times \mathfrak{p}_1' \oplus \cdots \oplus \mathfrak{p}_m' & \xrightarrow[\mu_{E'}]{\simeq} & H \end{array}$$

Case (i)-b: T is trivial and G/V is compact.
In this case G is an extension of a compact group by a vector group and the result follows from the Corollaries A.2.4 and A.2.5. To get an exp-complement we simply take $\mathfrak{p}$ to be the Lie algebra of V in $\mathfrak{g}$.

Case (i)-c: T is trivial and G/V is not compact.
Let $\pi : G \to G/V$ denote the canonical projection and let $L \subset G/V$ be a maximal compact subgroup of G/V. Set $L^* = \pi^{-1}(L)$. Since π restricts to a surjective submersion $L^* \to L$ (use that π is a submersion), we get that L^* is an extension of L by V, hence $L^* \cong V \rtimes L$ by A.2.5. Since by assumption G/V is not compact, we get that L^* is a closed subgroup of G having smaller dimension. Use the induction hypothesis on L^* and take $K \subset L^*$ to be maximal compact. Then $\pi(K) = L$ since we have $L^* \cong V \rtimes L$. We claim that K is maximal compact in G. To verify this take any compact subgroup C of G containing K. Since K is maximal compact in L^*, we must have $C \cap L^* = K$. Now, using that $\pi(C) \supset \pi(K) = L$ is a compact subgroup together with the maximality of L, we obtain $\pi(C) = L$ and consequently $C = K$. Note that $K(L^*)^0 = L^*$ implies $KG^0 = G$, this proves claim (C).

Now, let C be any compact subgroup in G, we show that we can conjugate C under G^0 into K. From this (A) and (B) will follow immediately. Consider $\pi(C)$ in G/V. We find, using the induction hypothesis, an element $g \in G^0$, such that $\pi(g)\pi(C)\pi(g)^{-1} \subset L$. This means that gCg^{-1} lies in L^*. Using the induction hypothesis once again for L^* proves the claim.

As a last step we have to construct an exp-complement to K. We take an exp-complement $E' = (\mathfrak{p}_1', \ldots, \mathfrak{p}_m')$ of L in G/V. Since $\pi(K) = L$ and by Lemma A.3.3, we can find $\mathrm{Ad}(K)$-stable subspaces $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of $\mathfrak{g}$ satisfying

$$\mathfrak{g} = \mathfrak{k} \oplus \mathrm{Lie}(V) \oplus \bigoplus_{i=1}^{m} \mathfrak{p}_i$$

and $\mathrm{d}\,\pi(\mathfrak{p}_i) = \mathfrak{p}_i'$. Note that $\mathfrak{v} := \mathrm{Lie}(V)$ is an $\mathrm{Ad}(K)$-stable subspace of $\mathfrak{g}$. We claim that $E = (\mathfrak{v}, \mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ is an exp-complement to K in G. We can use the commutative diagram

$$\begin{array}{ccc} K \times \mathfrak{v} \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m & \xrightarrow{\mu_E} & G \\ {\scriptstyle \pi\times\mathrm{d}\,\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ L \times \mathfrak{p}_1' \oplus \cdots \oplus \mathfrak{p}_m' & \xrightarrow[\mu_{E'}]{\simeq} & G/V \end{array}$$

to detect that μ_E is a diffeomorphism.

So finally, we consider case (ii) and assume that $\mathfrak{g}$ is semisimple. Then G is a semisimple group with finitely many connected components. Consider the identity component G^0, this is a connected semisimple group. In this situation there is the well-known Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

(see [**Kna02**, Chap. VI]). Let K^0 denote the fixed point group of the corresponding global Cartan involution on G^0. An important point is that K^0 is maximal compact only if G^0 has finite centre. Once again we have distinguish two cases.

Case (ii)-a: $K^0 = G^0$.
This means that K^0 is a semisimple group. Moreover, since $\mathrm{Ad}(K^0)$ is compact, we may use Weyl's theorem (cf. Thm. 4.69 [**Kna02**]) to deduce that K^0 (and hence G) is compact. In this situation all claims hold trivially.

Case (ii)-b: K^0 is strictly smaller than G^0 (i.e. K^0 has lower dimension).
In this case K^0 is connected and it is known that it agrees with its normaliser $N_{G^0}(K^0)$. Define $K = N_G(K^0)$, it is clear that K^0 is indeed the identity component of K since $K^0 = K \cap G^0$. Furthermore, we claim $KG^0 = G$. To check this we fix any $g \in G$. The group $\mathrm{Ad}_{G^0}(gK^0g^{-1})$ is compact and thus gK^0g^{-1} has a fixed point when operating on the Riemannian symmetric space of non-compact type G^0/K^0 (see I. Thm. 13.5 in [**Hel78**]). Hence we find some element $x \in G^0$ such that $gkg^{-1}x \in xK^0$ for all $k \in K^0$. This means $x^{-1}g \in K = N_G(K^0)$. We can deduce that K is an fcc group and we can apply the induction hypothesis to K.

Let $L \subset K$ be a maximal compact subgroup. We claim that L is maximal compact in G. To verify this, take any compact subgroup $C \subset G$. It operates with fixed points on the symmetric space $G/K \simeq G^0/K^0$ (cf. I.13.5 in [**Hel78**]). Hence we can find some $g \in G^0$ such that $Cg \subset gK$, i.e. $g^{-1}Cg \subset K$. Eventually, we apply the induction hypothesis to K and conjugate $g^{-1}Cg$ into L. This proves (A) and (B). From $LK^0 = K$ and $KG^0 = G$ we get $LG^0 = G$ and thus (C).

Eventually, we have to find an exp-complement of L in G. We have an exp-complement of L in K and we know that the map $\mu^0 : K^0 \times \mathfrak{p} \to G^0$ defined by $(k, X) \mapsto k\exp(X)$ is a diffeomorphism (see [**Kna02**, Thm. 6.31]). We only need to convince ourselves that the analogous map $\mu : K \times \mathfrak{p} \to G$ is a diffeomorphism. This can be easily checked. □

A.4. Applications and Finite Groups of Automorphisms

A.4.1. In this section we will deduce some useful results from Mostow's Theorem. It is so powerful that most results will follow almost directly. In the entire section G denotes a Lie group with finitely many connected components and Ξ denotes a finite group of automorphisms of G.

A.4.2. Proposition. *Let G be an fcc group and let Ξ be a finite group of automorphisms of G. Then there is a Ξ-stable maximal compact subgroup K of G. Moreover, every such K has an exp-complement $E = (\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ such that all the $\mathfrak{p}_i$ are stable under Ξ.*

Proof. Consider the semidirect product Lie group $G \rtimes \Xi$. Since Ξ is finite this group has finitely many connected components. Let C be a maximal compact subgroup of $G \rtimes \Xi$ which contains the finite (compact!) subgroup $\{1\} \rtimes \Xi$. Let

$$\pi : G \rtimes \Xi \to G$$

denote the projection onto G, then $K = \pi(C)$ is a compact subgroup of G which is stable under Ξ. This follows from $\{1\} \rtimes \Xi \subseteq C$. We check that K is maximal compact: Suppose K^* is a maximal compact subgroup of G which contains K. By Mostow's Theorem we can conjugate $K^* \rtimes \{1\}$ by some element $(g, \tau) \in G \rtimes \Xi$ into C. Now this implies that $g\tau(K^*)g^{-1} \subseteq K$, thus K is maximal compact.

To find an exp-complement for K, we simply take an exp-complement of C in the group $G \rtimes \Xi$. □

A.4.3. Lemma (Antonyan, Cor. 1.4 [**Ant12**]). *Let G be an fcc group. A compact subgroup $K \subseteq G$ is maximal if and only if $K\backslash G$ is a contractible space.*

Proof. It follows from A.3.4 (D) that $K\backslash G$ is contractible if K is maximal compact. Conversely, suppose that $K\backslash G$ is contractible for a compact subgroup $K \subseteq G$. Take a maximal compact subgroup $L \subseteq G$ which contains K. Then, by choosing an exp-complement $E = (\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ of L, we obtain a diffeomorphism

$$K\backslash G \xrightarrow{\simeq} K\backslash L \times \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m.$$

Hence, the contractible space $K\backslash G$ is homotopy equivalent to the compact manifold $K\backslash L$. We can deduce that $K\backslash L$ is a single point and therefore $K = L$. □

A.4.4. Lemma. *The group G^Ξ of Ξ-fixed points is a Lie group with a finite number of connected components. Moreover, if we choose a Ξ-stable maximal compact K of G, then the group K^Ξ of fixed points in K is a maximal compact subgroup of G^Ξ.*

Proof. We choose K to be Ξ-stable and we choose a Ξ-stable exp-complement $E = (\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ as in Proposition A.4.2. We get a diffeomorphism

$$\mu_E : K \times \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m \to G$$

(cf. A.3.2). Note that Ξ acts linearly on $\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m$. The group of fixed points G^Ξ is closed and hence a Lie subgroup. Restricting μ_E we get a diffeomorphism

$$\mu^* : K^\Xi \times \mathfrak{p}_1^\Xi \oplus \cdots \oplus \mathfrak{p}_m^\Xi \to G^\Xi.$$

Since K^Ξ is compact it has a finite number of connected components. Furthermore, $\mathfrak{p}_1^\Xi \oplus \cdots \oplus \mathfrak{p}_m^\Xi$ is Euclidean (hence connected) and so G^Ξ has a finite number of connected components.

Moreover, μ^* factors to a diffeomorphism

$$f : \mathfrak{p}_1^\Xi \oplus \cdots \oplus \mathfrak{p}_m^\Xi \to K^\Xi\backslash G^\Xi$$

and consequently this is a contractible space. Now, it follows from Antonyan's Lemma (A.4.3) that K^Ξ is a maximal compact subgroup of G^Ξ. □

APPENDIX B

Projective Modules, Determinants and Traces

B.1. Projective Modules and Reflexivity

B.1.1. Throughout the section R denotes a commutative ring. We summarise some well-known results on projective modules and reflexivity properties. In particular we will see that, given a finitely generated and projective R-module M, the functor $A \mapsto M \otimes_R A$ from the category of commutative R-algebras to the category of R-modules is representable.

Though all results in this section are well-known, it seemed more convenient to include them explicitly.

B.1.2. Lemma. *Let R be a commutative ring and let M, B and C be R-modules. Assume that M is finitely generated and projective, then the canonical map*

$$\rho_M : \mathrm{Hom}_R(M, B) \otimes_R C \longrightarrow \mathrm{Hom}_R(M, B \otimes_R C)$$

sending $\varphi \otimes c$ to the morphism $m \mapsto \varphi(m) \otimes c$ is an isomorphism of R-modules.

Proof. It is obvious that ρ is R-linear. We start proving the claim under the assumption that M is free of finite rank. Let $x_1, \dots, x_n$ be an R-basis of M. For every R-module N there is an isomorphism of R-modules

$$\beta_N : \mathrm{Hom}_R(M, N) \xrightarrow{\simeq} \bigoplus_{i=1}^{n} N$$

defined by $\varphi \mapsto (\varphi(x_i))_i$. The following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}_R(M,B) \otimes_R C \arrow[r, "\rho_M"] \arrow[d, "\beta_B \otimes \mathrm{Id}"'] & \mathrm{Hom}_R(M, B \otimes_R C) \arrow[d, "\beta_{B\otimes C}"] \\
(\bigoplus_{i=1}^n B) \otimes_R C \arrow[r, "\simeq"] & \bigoplus_{i=1}^n B \otimes_R C
\end{tikzcd}$$

and the claim follows for free modules.

Assume now that M is finitely generated and projective. We find a free module F of finite rank and a projective module Q such that F is the direct sum $F = M \oplus Q$. For every R-module N there is a canonical isomorphism

$$\alpha_N : \mathrm{Hom}_R(M, N) \oplus \mathrm{Hom}_R(Q, N) \longrightarrow \mathrm{Hom}_R(F, N).$$

The following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_R(M, B \otimes_R C) \oplus \mathrm{Hom}_R(Q, B \otimes_R C) \arrow[r, "\alpha_{B\otimes C}"] & \mathrm{Hom}_R(F, B \otimes_R C) \\
\mathrm{Hom}_R(M,B) \otimes_R C \oplus \mathrm{Hom}_R(Q,B) \otimes_R C \arrow[r, "\alpha_B \otimes \mathrm{Id}"] \arrow[u, "\rho_M \oplus \rho_Q"] & \mathrm{Hom}(F,B) \otimes_R C \arrow[u, "\rho_F"']
\end{tikzcd}$$

and the assertion follows, using that $\rho_M \oplus \rho_Q$ is an isomorphism if and only if both maps are isomorphisms. □

B.1.3. Let M be an R-module. We define the *R-dual module* of M to be

$$M^* = \mathrm{Hom}_R(M, R).$$

There is an canonical homomorphism of R-modules

$$j : M \to M^{**}$$

defined by $j(x)(\alpha) = \alpha(x)$ for all $x \in M$ and $\alpha \in M^*$. This map is in general no isomorphism. A module M such that the canonical map j is an isomorphism is said to be *reflexive*. Recall that finitely generated projective modules are reflexive (cf. [**Lam99**, (2.11)]. They even satisfy a stronger property. Let N be any R-module. The map $j : M \to M^{**}$ induces a homomorphism of R-modules

$$j_N : M \otimes_R N \longrightarrow \mathrm{Hom}_R(M^*, N).$$

Explicitly this morphism is given by $j_N(x \otimes n)(\alpha) = \alpha(x)n$ for all $x \in M$, $n \in N$ and $\alpha \in M^*$.

B.1.4. Corollary. *Let M be a finitely generated projective R-module. Then for any R-module N the map j_N is an isomorphism of R-modules.*

Proof. As above, let $\rho_{M^*} : \mathrm{Hom}_R(M^*, R) \otimes_R N \to \mathrm{Hom}_R(M^*, N)$ be the canonical isomorphism. Observe that j_N is the composition $\rho_{M^*} \circ (j \otimes \mathrm{Id}_N)$. □

B.1.5. Corollary. *Let A be a commutative R-algebra, then*

$$j_A : M \otimes_R A \to \mathrm{Hom}_R(M^*, A)$$

is an isomorphism of A-modules.

Proof. Verify that j_A is A-linear. □

B.1.6. Corollary. *Let M be a finitely generated and projective R-module, and let A be any commutative R-algebra. The canonical map $M^* \otimes_R M \to \mathrm{End}_R(M)$ is an isomorphism of R-modules. Further, there is a canonical isomorphism*

$$e_M(A) : \mathrm{End}_R(M) \otimes_R A \xrightarrow{\simeq} \mathrm{End}_A(M \otimes_R A)$$

of A-algebras.

Proof. The first claim follows directly from Lemma B.1.2 with $B = R$ and $C = M$. The second claim follows from Lemma B.1.2, substituting $B = M$ and $C = A$, together with the canonical isomorphism $\mathrm{Hom}_R(M, M \otimes_R A) \to \mathrm{End}_A(M \otimes_R A)$. More precisely we define $e_M(A)$ to be the composition of these two isomorphisms, this is, $e_M(A)$ sends $\psi \otimes a$ to $\psi \otimes (a\, \mathrm{Id}_A)$. It is easy to verify that $e_M(A)$ is a homomorphism of A-algebras. □

B.1.7. Corollary. *Let M be a finitely generated projective R-module. The functor $A \mapsto M \otimes_R A$ from the category of commutative R-algebras to the category of R-modules is represented by the symmetric algebra $S_R(M^*)$.*

Proof. Using the universal property of the symmetric algebra we get natural isomorphisms

$$\mathrm{Hom}_{R\text{-algebra}}(S_R(M^*), A) \cong \mathrm{Hom}_R(M^*, A) \cong M \otimes A. \qquad \square$$

B.1.8. From the last corollary we deduce that the functor $A \mapsto M \otimes_R A$ defines an affine commutative group scheme over R. This functor, as well as the scheme defined by this functor, will be denoted by M_a. This means, we sometimes write $M_a(A)$ instead of $M \otimes_R A$.

Moreover, we can consider the elements of $S_R(M^*)$ as polynomial functions on M. Let $f \in S_R(M^*)$, let A be a commutative R-algebra and let $x \in M \otimes_R A$. We explain how to evaluate f at x. Consider the R-linear map $\mathrm{ev}_x : M^* \to A$ defined by $\mathrm{ev}_x(\alpha) := (\alpha \otimes \mathrm{Id}_A)(x) \in A$. By the universal property of $S_R(M^*)$ we get a unique homomorphism of R-algebras $S(\mathrm{ev}_x) : S_R(M^*) \to A$ which extends ev_x. Finally we define $f(x) := S(\mathrm{ev}_x)(f)$.

B.1.9. Lemma. *Let R be a ring and let P be a projective R-module. Let $E \subseteq P$ be a submodule which is a direct summand of P. Moreover, let M be any R-module and let $I \subset M$ be a submodule. Then $P \otimes I$, $E \otimes M$ and $E \otimes I$ map injectively into $P \otimes M$ and*

$$P \otimes I \cap E \otimes M = E \otimes I.$$

Proof. Consider the inclusion $i : I \to M$. Since P is projective, the map $P \otimes I \to P \otimes M$ is injective and similarly $E \otimes I$ embeds into $E \otimes M$. Let E' be a complement of E in P, i.e. $P = E \oplus E'$. The short exact sequence

$$0 \longrightarrow E \longrightarrow P \xrightarrow{\pi} E' \longrightarrow 0$$

splits and hence the sequence

$$0 \longrightarrow E \otimes M \longrightarrow P \otimes M \longrightarrow E' \otimes M \longrightarrow 0$$

is exact. In particular, $E \otimes M$ injects into $P \otimes M$. Finally, let $x \in P \otimes M$ be such that $x \in P \otimes I \cap E \otimes M$. We consider the R-linear map $f := \pi \otimes \mathrm{Id}_M : P \otimes M \to E' \otimes M$. By assumption $x \in E \otimes M$, thus $f(x) = 0$. We may write

$$x = \sum_i e_i \otimes \lambda_i + \sum_j y_j \otimes \mu_j$$

with $\lambda_i, \mu_j \in I$ and $e_i \in E$ and $y_j \in E'$. So the equality $0 = f(x) = \sum_j y_j \otimes \mu_j$ shows that

$$x = \sum_i e_i \otimes \lambda_i \in E \otimes I. \qquad \square$$

B.2. Trace and Determinant

B.2.1. In this section we define determinant and trace for endomorphisms of finitely generated projective modules. This theory is well-known and we summarise it to be able to prove that trace and determinant are algebraic in the appropriate sense. Moreover we will use it to define the norm and the trace on algebras. It seems that the determinant for endomorphisms of finitely generated projective modules was first defined by Goldman [**Gol61**]. The treatment given here was inspired by the elegant account of Helmstetter-Micali [**HM08**, 3.6].

B.2.2. *Definition of determinant and trace.* Let R be a commutative ring and let M be a finitely generated projective R-module. Let N be a complement of M in a free finite rank R-module F, this means $M \oplus N = F$. Let φ be an endomorphism of M, we define the trace $\mathrm{Tr}_R(\varphi|F)$ and the determinant $\det_R(\varphi|F)$ of φ with respect to F by

$$\mathrm{Tr}_R(\varphi|F) \;=\; \mathrm{Tr}(\varphi \oplus 0_N)$$

and

$$\det_R(\varphi|F) \;=\; \det(\varphi \oplus \mathrm{Id}_N).$$

B.2.3. Proposition. *These definitions are independent of the choice of F and N. The trace is R-linear and the determinant is multiplicative.*

Proof. Let N_2 be another complement such that $M \oplus N_2 = F_2$ is a free module of finite rank. Consider $F \oplus F_2 = M \oplus (M \oplus N \oplus N_2)$. By permutation of the factors we easily see that $\det_R(\varphi|F) = \det_R(\varphi|F \oplus F_2) = \det_R(\varphi|F_2)$. The same argument can be used for the trace. The other two claims are clear since they are well-known for free modules. □

B.2.4. The previous proposition shows that we may speak of *the* trace and *the* determinant. From now on we simplify our notation; we denote the determinant of φ by $\det_R(\varphi)$ and the trace by $\mathrm{Tr}_R(\varphi)$.

B.2.5. Corollary. *An endomorphism φ of a finitely generated projective R-module M is invertible if and only if $\det_R(\varphi)$ is a unit in R.*

Proof. If φ is invertible we can use the multiplicativity of the determinant to see that $\det(\varphi)$ is a unit. Conversely, suppose $\det(\varphi)$ is a unit. The claim is well-known if M is free. Let N be an R-module, such that $M \oplus N$ is free of finite rank. We can deduce from $\det(\varphi) = \det(\varphi \oplus \mathrm{Id}_N) \in R^\times$ that $\varphi \oplus \mathrm{Id}_N$ is invertible. However, this can only be the case if φ is an automorphism of M. □

B.2.6. Lemma. *Trace and determinant are natural in the following sense. Let M be a finitely generated and projective R-module, and let A be a commutative R-algebra via the morphism $\iota\colon R \to A$. An endomorphism $\varphi \in \mathrm{End}_R(M)$ induces an A-linear endomorphism $\varphi \otimes \mathrm{Id}_A \in \mathrm{End}_A(M \otimes_R A)$. The determinant and the trace satisfy $\iota(\det_R(\varphi)) = \det_A(\varphi \otimes \mathrm{Id}_A)$ and $\iota(\mathrm{Tr}_R(\varphi)) = \mathrm{Tr}_A(\varphi \otimes \mathrm{Id}_A)$.*

Proof. The claim is obvious for free modules M if we choose a basis and consider matrices. Now, let $F = M \oplus N$ be a free module of finite rank. By definition we get

$$\iota(\det_R(\varphi)) = \iota(\det(\varphi \oplus \mathrm{Id}_N)) = \det((\varphi \otimes \mathrm{Id}_A) \oplus \mathrm{Id}_{N\otimes A}) = \det_A(\varphi \otimes \mathrm{Id}_A).$$

The same argument applies for the trace. □

B.2.7. *The scheme* $\underline{\mathrm{End}}(M)$. Let M be a finitely generated projective R-module. The R-module $\mathrm{End}_R(M)$ is isomorphic to $M^* \otimes_R M$, and hence is finitely generated and projective. Therefore the functor $A \mapsto \mathrm{End}_R(M) \otimes A$ from the category of commutative R-algebras to the category of rings is representable. The associated group (or ring) scheme over R will be denoted by $\underline{\mathrm{End}}(M)$. Moreover, there is a canonical isomorphism $\underline{\mathrm{End}}(M)(A) \cong \mathrm{End}_A(M \otimes_R A)$. We write $\mathbb{A}^1$ for the affine line over R, this is the functor given by $\mathbb{A}^1(A) = A$ for any commutative R-algebra A. Further, for a commutative R-algebra A, the determinant and the trace define maps $\underline{\mathrm{End}}(M)(A) \to \mathbb{A}^1(A)$.

B.2.8. Proposition. *Let M be a finitely generated and projective R-module. The trace and the determinant define morphisms of schemes over R*

$$\underline{\mathrm{End}}(M) \longrightarrow \mathbb{A}^1.$$

PROOF. Using Yoneda's Lemma we simply have to show that trace and determinant are natural transformations of functors. This can be seen using Lemma B.2.6 as follows. Let $\varphi : A \to B$ be a morphism of commutative R-algebras. The following diagram commutes

$$\begin{array}{ccc} \operatorname{End}_R(M) \otimes_R A & \xrightarrow{\operatorname{Id} \otimes \varphi} & \operatorname{End}_R(M) \otimes_R B \\ {\scriptstyle e_M(A)} \downarrow & & \downarrow {\scriptstyle e_M(B)} \\ \operatorname{End}_A(M \otimes_R A) & \xrightarrow{T} & \operatorname{End}_B(M \otimes_R B) \end{array}$$

where T denotes the map induced from $t : \operatorname{End}_A(M \otimes_R A) \to \operatorname{End}_A((M \otimes_R A) \otimes_A B)$ with $t : \psi \mapsto \psi \otimes \operatorname{Id}_B$, and the canonical identification $(M \otimes_R A) \otimes_A B \cong M \otimes_R B$. $\square$

B.2.9. *Norms and traces on algebras.* As a simple application we are able to define norm and trace on R-algebras. Let Λ be any (possibly non-commutative) R-algebra and assume that Λ is finitely generated and projective as R-module. Every $x \in \Lambda$ induces an R-linear map

$$l_x : \Lambda \to \Lambda \quad \text{defined by } \ z \mapsto xz.$$

We define the norm $\mathrm{N}_{\Lambda/R}(x) = \mathrm{N}(x)$ of x to be the determinant of l_x and similarly we set the trace $\operatorname{Tr}_{\Lambda/R}(x) = \operatorname{Tr}(x)$ of x to be the trace of l_x.

B.2.10. Proposition. *The norm on Λ has the following properties:*
(1) For all $x, y \in \Lambda$ we have $\mathrm{N}(xy) = \mathrm{N}(x)\,\mathrm{N}(y)$
(2) An element $x \in \Lambda$ is a unit in Λ if and only if $\mathrm{N}(x)$ is a unit in R.

PROOF. The first claim follows from the identity $l_x \circ l_y = l_{xy}$ and the multiplicativity of the determinant. The second statement can be seen as follows: Take $x \in \Lambda$, from Corollary B.2.5 we know that l_x is bijective if and only if $\mathrm{N}(x) \in R^\times$. Thus one direction is easy: if x is a unit, then l_x is obviously invertible. Conversely, let x be such that l_x is bijective. We find an element $y \in \Lambda$ such that $l_x(y) = xy = 1 \in \Lambda$. Moreover, this implies $1 = \mathrm{N}(xy) = \mathrm{N}(x)\,\mathrm{N}(y)$ and hence $\mathrm{N}(y)$ is a unit in R. By the same argument we get an element $z \in \Lambda$ with $yz = 1$. We obtain that $x = x(yz) = (xy)z = z$ and thus $xy = yx = 1$, i.e. x is a unit. $\square$

B.2.11. *Schematic approach.* We reformulate this in schematic language. Let Λ be an R-algebra which is finitely generated and projective as R-module. Then we obtain an associated scheme of rings Λ_a as the functor defined by $A \mapsto \Lambda \otimes_R A$ for every commutative R-algebra A. Left multiplication yields a morphism of ring schemes

$$l : \Lambda_a \to \underline{\operatorname{End}}(\Lambda).$$

Furthermore, we obtain the norm and trace as morphisms of schemes defined over R

$$\mathrm{N}_{\Lambda/R} := \det \circ l : \Lambda_a \to \mathbb{A}^1,$$
$$\operatorname{Tr}_{\Lambda/R} := \operatorname{Tr} \circ l : \Lambda_a \to \mathbb{A}^1.$$

B.3. Symmetric Algebras

B.3.1. This section is a repository for some results on symmetric algebras over rings. It mainly serves as a toolbox, for example for the construction of the reduced norm and the pfaffian in Chapter III.

B.3.2. Lemma. *Let R be a commutative ring and let A be a commutative R-algebra. The canonical homomorphism*

$$S_R(M) \otimes_R A \to S_A(M \otimes_R A)$$

is an isomorphism of A-algebras for every R-module M.

Proof. The universal property of the symmetric algebra implies that the R-linear map $M \to M \otimes_R A$ induces a map of R-algebras $S_R(M) \to S_A(M \otimes_R A)$ and thus we obtain a canonical morphism of A-algebras $S_R(M) \otimes_R A \to S_A(M \otimes_R A)$. Conversely, the A-linear map $M \otimes_R A \to S_R(M) \otimes_R A$ induces a morphism of A-algebras $S_A(M \otimes_R A) \to S_R(M) \otimes_R A$ which is obviously inverse to the first map. $\square$

B.3.3. Lemma. *Let R be an integrally closed domain with quotient field k and let P be a finitely generated projective R-module. Then the following assertions hold.*

(i) The canonical map $f_P : S_R(P) \to S_k(P \otimes_R k)$ is injective.
(ii) The symmetric algebra $S_R(P)$ is integrally closed in the symmetric algebra $S_k(P \otimes_R k)$.
(iii) The symmetric algebra $S_R(P)$ is integrally closed in its quotient field.

Proof. If P is a free R-modules of finite rank, then all assertions are well-known statements about polynomial rings. Assume now, that P is finitely generated and projective. We choose a finitely generated complement Q such that $F := P \oplus Q$ is free of finite rank. Recall that there is a canonical isomorphism $S_R(F) \to S_R(P) \otimes_R S_R(Q)$ and consider the following commutative diagram

$$\begin{array}{ccc} S_R(F) & \xrightarrow{\simeq} & S_R(P) \otimes_R S_R(Q) \\ {\scriptstyle f_F}\big\downarrow & & \big\downarrow{\scriptstyle f_P \otimes f_Q} \\ S_k(F \otimes_R k) & \xrightarrow{\simeq} & S_k(P \otimes_R k) \otimes_k S_k(Q \otimes_R k). \end{array}$$

We see that the map $f_P \otimes f_Q$ is injective. There is the direct sum decomposition $S_R(Q) = R \oplus S_R(Q)^+$, where $S_R(Q)^+$ denotes the elements of strictly positive degree. We conclude that $S_R(P)$ injects into $S_R(P) \otimes_R S_R(Q)$. Consequently, $f_P(x) = 0$ if and only if $(f_P \otimes f_Q)(x \otimes 1) = 0$, and so (i) follows.

We can be more precise here: The algebra $S_R(P)$ is $\mathbb{N}$-graded and thus the algebra $S_R(P) \otimes_R S_R(Q)$ has an $\mathbb{N}^2$-grading. Similarly $S_k(P \otimes_R k) \otimes_k S_k(Q \otimes_R k)$ has such a grading and the involved morphisms are morphisms of graded algebras. Using the grading it is easy to see that $S_k(P \otimes_R k) \cap S_R(P) \otimes_R S_R(Q) = S_R(P)$ considered as subsets of $S_k(P \otimes_R k) \otimes_k S_k(Q \otimes_R k)$. Hence, if $y \in S_k(P \otimes_R k)$ is integral over $S_R(P)$, then $y \in S_k(P \otimes_R k) \cap S_R(P) \otimes_R S_R(Q) = S_R(P)$. This proves (ii). Finally assertion (iii) follows since $S_k(P \otimes_R k)$ is integrally closed in the quotient field $\mathrm{Quot}(S_R(P))$. $\square$

B.3.4. *Observation.* Let R be a commutative ring and let P be a finitely generated and projective R-module. It follows from Corollary B.1.6 that the canonical map $\gamma : P \otimes_R P^* \to \mathrm{End}_R(P)$, which sends $p \otimes \alpha$ to the endomorphism $x \mapsto \alpha(x)p$, is an isomorphism of R-modules. The element in $P \otimes_R P^*$ which is mapped to the identity by γ will be denoted by $u_P \in P \otimes P^*$. We can write $u_P = \sum_{i=1}^m e_i \otimes \varepsilon_i$ for some elements $e_i \in P$ and $\varepsilon_i \in P^*$.

Let A be any commutative R-algebra and let $\varphi : P^* \to A$ be a homomorphism of R-modules. This homomorphism induces a unique homomorphism of R-algebras $S(\varphi) : S_R(P^*) \to A$ satisfying $S(\varphi)_{|P^*} = \varphi$. Moreover, we define the element u_A in $P \otimes_R A$ by $u_A := (\mathrm{Id}_P \otimes \varphi)(u_P)$.

Lemma. We have $f(u_A) = S(\varphi)(f)$ for all $f \in S_R(P^*)$.

Proof. One can easily check that $T(f) := f(u_A)$ defines a homomorphism of R-algebras form $S_R(P^*)$ to A (see B.1.8). We verify that $T = S(\varphi)$. Since the elements of P^* generate $S_R(P^*)$, it suffices to check $T_{|P^*} = \varphi$. So, take $f \in P^*$ and observe that

$$T(f) = f(u_A) = \sum_{i=1}^m f(e_i)\varphi(\varepsilon_i) = \varphi(\sum_{i=1}^m f(e_i)\varepsilon_i) = \varphi(f).$$

$\square$

B.3.5. Corollary. *Let R be an integrally closed domain with quotient field k. Let P be a finitely generated projective R-module.*

(i) *Let $f \in S_R(P^*)$ be a polynomial function on P. Suppose there is an integer $n > 1$ such that for every field extension F/k and every $p \in P \otimes_R F$*

$$f(p) \in F^n := \{x^n \mid x \in F\}.$$

Then there is $g \in S_R(P^*)$ with $g^n = f$.

(ii) *Let $f, g \in S_R(P^*)$. Suppose that for every field extension F/k and every $p \in P \otimes_R F$ we have $f(p) = g(p)$, then $f = g$ as elements in $S_R(P^*)$.*

Proof. Let F be the quotient field of $S_R(P^*)$. We define $\varphi : P^* \to F$ to be the canonical embedding, which is clearly an R-linear map. Then $S(\varphi) : S_R(P^*) \to F$ is the canonical embedding of $S_R(P^*)$ into its quotient field and we will identify f with $S(\varphi)(f)$. Let $p = u_F$ be as in paragraph B.3.4 and apply the Lemma, then we obtain $f = S(\varphi)(f) = f(p) \in F^n$. Hence we find an element $g \in F$ with $g^n = f$. By Lemma B.3.3 (iii) the ring $S_R(P^*)$ is integrally closed in F. Since g satisfies the equation $X^n - f = 0$ we deduce $g \in S_R(P^*)$.

Consider the second claim. We proceed similarly and take F to be the quotient field of $S_R(P^*)$. As above we choose $p = u_F$ and we find $f = f(p) = g(p) = g$ as elements in $S_R(P^*) \subseteq F$. $\square$

APPENDIX C

Degenerate Quaternion Algebras

C.1. Definition and Basic Properties

C.1.1. In this appendix we introduce the notion of *degenerate quaternion algebra*. The motivation for considering these algebras is that they arise naturally as quotients of maximal orders in quaternion algebras over p-adic fields. As in Chapter III one can associate unit groups and fixed point groups to these algebras and we are interested in the structure of these groups. In particular, we determine the orders of such groups when the algebra is defined over a finite field.

C.1.2. We begin with the definition and basic properties in Section C.1. In Section C.2 we consider algebras of matrices with entries in degenerate quaternion algebras, further we define and study the associated special linear group. The associated fixed point groups, called unitary groups, are treated in Section C.3. Finally we make the connection with maximal orders in quaternion algebras over p-adic fields explicit in Section C.4.

C.1.3. Let K be a field and let L/K be a quadratic Galois extension. Let σ be the non-trivial Galois automorphism of the extension L/K, we usually write $\sigma(x) = \overline{x}$ for all $x \in L$. We construct a 4-dimensional K-algebra

$$A := Q_d(L/K) := L \oplus L\pi$$

with $\pi^2 = 0$ and $\pi x = \overline{x}\pi$ for all $x \in L$. More precisely, the multiplication is defined as

$$(x + y\pi)(u + v\pi) := xu + (y\overline{u} + xv)\pi. \tag{21}$$

C.1.4. Lemma. *The multiplication defined by* (21) *satisfies the laws of associativity and distributivity and makes A into a* 4*-dimensional K-algebra.*

Proof. Simple calculation. □

C.1.5. Definition. The algebra $A = Q_d(L/K)$ is called the *degenerate quaternion algebra* associated with the extension L/K.

C.1.6. *Remark.* From now on A denotes a degenerate quaternion algebra for some quadratic Galois extension L/K. The multiplicative unit in A is the element $1 := 1 + 0\pi$. The element $0 := 0 + 0\pi$ is the neutral element for addition. We identify K with the subset $K + 0\pi \subset A$.

C.1.7. Lemma. *The centre $Z(A)$ of A is precisely K.*

Proof. Let $z := x + y\pi \in Z(A)$. Then $z\pi = \pi z$, and hence

$$x\pi = (x + y\pi)\pi = \pi(x + y\pi) = \overline{x}\pi.$$

We deduce $x \in K$. Moreover, let $a \in L^\times$, we obtain

$$ax + \overline{a}y\pi = (x + y\pi)a = a(x + y\pi) = ax + ay\pi,$$

and consequently $ay = \overline{a}y$. Choose $a \in L \setminus K$, this is $\overline{a} \neq a$, and deduce $y = 0$. $\square$

C.1.8. *Remark.* The lemma yields that A is a central K-algebra. However, A is not simple since

$$\mathfrak{m} := L\pi$$

is a two-sided ideal in A. Note that $\mathfrak{m}$ is a maximal two-sided ideal since the quotient $A/\mathfrak{m}$ is isomorphic to the field L.

C.1.9. Lemma. *An element $z = x + y\pi \in A$ is a unit if and only if $x \neq 0$.*

Proof. If $x = 0$, then $z = y\pi \in \mathfrak{m}$ and z can not be a unit since $\mathfrak{m}$ is a proper ideal. Conversely, suppose that $x \in L^\times$. An easy calculation shows

$$(x + y\pi)(x^{-1} - x^{-1}y\overline{x}^{-1}\pi) = 1 = (x^{-1} - x^{-1}y\overline{x}^{-1}\pi)(x + y\pi).$$

$\square$

C.1.10. Lemma. *The ideal $\mathfrak{m}$ is the only proper left (resp. right, resp. two-sided) ideal in A.*

Proof. Let $I \subset A$ be a proper left ideal of A. By proper we mean that I is neither $\{0\}$ nor A. Let $z = x + y\pi \in I$ be a non-zero element. If $x \neq 0$, then z is a unit and $I = A$, which contradicts the assumption that I is proper. Hence $z = y\pi$ with $y \neq 0$ and it is clear that $I = \mathfrak{m}$. The same argument works if we assume I to be a right ideal. $\square$

C.1.11. Corollary. *A degenerate quaternion algebra is always a non-commutative local ring.*

C.1.12. *The canonical involution.* As for (non-degenerate) quaternion algebras there is a canonical involution on degenerate quaternion algebras. Define a map $\sigma : A \to A$ by

$$\sigma : x + y\pi \mapsto \overline{x} - y\pi.$$

This is a K-linear map of order two and it is indeed an involution:

$$\sigma\big((x + y\pi)(u + v\pi)\big) = \overline{xu} - (y\overline{u} + xv)\pi = (\overline{u} - v\pi)(\overline{x} - y\pi).$$

C.1.13. Definition. The *norm* on A is the function $\mathrm{N} : A \to K$ defined by

$$\mathrm{N}(x + y\pi) := \mathrm{N}_{L/K}(x) = \overline{x}x.$$

Note that $N(z) = \sigma(z)z$ as in quaternion algebras.

C.1.14. Corollary. *An element $z \in A$ is a unit if and only if $\mathrm{N}(z) \neq 0$.*

C.1.15. *Remark.* We can realise a degenerate quaternion algebra A as a K-algebra of 2×2 matrices with entries in L. Consider the K-algebra

$$B := \Big\{ \begin{pmatrix} x & y \\ 0 & \overline{x} \end{pmatrix} \mid x, y \in L \Big\}.$$

The K-linear map $f : A \to B$ defined by

$$x + y\pi \;\mapsto\; \begin{pmatrix} x & y \\ 0 & \overline{x} \end{pmatrix}$$

is an isomorphism of K-algebras.

C.2. Matrix Algebras over Degenerate Quaternion Algebras

C.2.1. Let $A = Q_d(L/K)$ be a degenerate quaternion algebra over a field K. Let $n \geq 1$ be an integer, we want to study the matrix algebra $M_n(A)$ of $n \times n$-matrices with entries in A.

Note that every element $z \in M_n(A)$ can be written in the form $z = x + y\pi$ with uniquely determined $x, y \in M_n(L)$. Let $x \in M_n(L)$, the matrix obtained by applying the non-trivial Galois automorphism of L/K to every entry of x will be denoted by $\overline{x}$

C.2.2. Lemma. *The K-algebra $M_n(A)$ is central. Moreover, the algebra $M_n(A)$ has a unique proper two-sided ideal $\mathfrak{m}_n := M_n(\mathfrak{m}) = M_n(L)\pi$.*

Proof. The K-algebras A and $M_n(K)$ are central, thus $A \otimes_K M_n(K) \cong M_n(A)$ is central (see IX, Satz 3.2 in [**JS06**]). The K-algebra $M_n(K)$ is central and simple, therefore the two-sided ideals in $A \otimes_K M_n(K) \cong M_n(A)$ correspond bijectively to the two-sided ideals in A (see IX, Satz 4.4 in [**JS06**]). However, by Lemma C.1.10, there is only one proper two-sided ideal in A. $\square$

C.2.3. Lemma. *An element $x + y\pi \in M_n(A)$ is a unit if and only if $x \in \mathrm{GL}_n(L)$.*

Proof. Suppose $x + y\pi$ is a unit and let $u + v\pi$ be its inverse. Then

$$1 = (x + y\pi)(u + v\pi) = xu + (y\overline{u} + xv)\pi,$$

and we conclude $xu = 1$, this is $x \in \mathrm{GL}_n(L)$.

Conversely, let $x \in \mathrm{GL}_n(L)$, then one can verify that $x^{-1} - x^{-1}y\overline{x}^{-1}\pi$ is the inverse of $x + y\pi$. $\square$

C.2.4. Corollary. *There is a short exact sequence of groups*

$$1 \longrightarrow M_n(L) \xrightarrow{r} M_n(A)^\times \xrightarrow{\varphi} \mathrm{GL}_n(L) \longrightarrow 1,$$

where φ denotes the projection $x + y\pi \mapsto x$ and r maps $y \in M_n(L)$ to $1 + y\pi$. The short exact sequence splits and hence $M_n(A)^\times \cong \mathrm{GL}_n(L) \ltimes M_n(L)$.

Proof. By Lemma C.2.3 the map $\varphi : M_n(A)^\times \to \mathrm{GL}_n(L)$ is surjective an has kernel $N := \{\, 1 + y\pi \mid y \in M_n(L) \,\}$. Since $(1 + y\pi)(1 + y'\pi) = 1 + (y + y')\pi$, the kernel is isomorphic to the additive group $M_n(L)$.

The group homomorphism $s : \mathrm{GL}_n(L) \to M_n(A)^\times$ sending x to $x + 0\pi$ splits the above sequence. $\square$

C.2.5. *The reduced norm.* The map $\mathrm{N}_d : M_n(A) \to K$ defined by

$$\mathrm{N}_d(x + y\pi) := \mathrm{N}_{L/K}(\det(x)) = \overline{\det(x)}\det(x)$$

is called the *(reduced) norm* on $M_n(A)$. The reduced norm is multiplicative, i.e. $\mathrm{N}_d(zw) = \mathrm{N}_d(z)\,\mathrm{N}_d(w)$ for all $z, w \in M_n(A)$. Moreover, $z \in M_n(A)$ is a unit if and only if $\mathrm{N}_d(z)$ is a unit in K.

From now on we write $\mathrm{GL}_n(A)$ for the unit group $M_n(A)^\times$ and we denote the kernel of the reduced norm $\mathrm{N}_d : \mathrm{GL}_n(A) \to K^\times$ by $\mathrm{SL}_n(A)$. We say that $\mathrm{SL}_n(A)$ is the special linear group over A.

C.2.6. Corollary. *The short sequence of groups*

$$1 \longrightarrow M_n(L) \longrightarrow \mathrm{SL}_n(A) \xrightarrow{\varphi} \mathrm{GL}_n^{(1)}(L) \longrightarrow 1,$$

is exact. Here $\mathrm{GL}_n^{(1)}$ *is the group defined by*

$$\mathrm{GL}_n^{(1)} := \{\, g \in \mathrm{GL}_n(L) \mid \mathrm{N}_{L/K}(\det(g)) = 1 \,\}.$$

If K is a finite field with q elements, then $\mathrm{SL}_n(A)$ *is a finite group of order*

$$|\,\mathrm{SL}_n(A)| = q^{4n^2-1}(1+q^{-1}) \prod_{j=2}^{n} \big(1 - q^{-2j}\big).$$

Proof. The first assertion follows directly from the exact sequence in Cor. C.2.4 and the description of the reduced norm in C.2.5.

Suppose that K is a finite field with q elements, then it follows from the short exact sequence that

$$|\,\mathrm{SL}_n(A)| = |M_n(L)| \cdot |\,\mathrm{GL}_n^{(1)}(L)| = q^{2n^2} q^{2n^2-1}(1+q^{-1}) \prod_{j=2}^{n} \big(1 - q^{-2j}\big).$$

In this identity we use that the order of $\mathrm{SL}_n(L)$ is $q^{2n^2-2}\prod_{j=2}^{n}\big(1-q^{-2j}\big)$ (see 3.3.1 in [**Wil09**]). Moreover, it is obvious that $|\,\mathrm{GL}_n^{(1)}(L)| = (q+1)|\,\mathrm{SL}_n(L)|$. □

C.3. The Unitary Group

C.3.1. Lemma. *Let A be a degenerate quaternion algebra over K. The K-linear map* $\theta : M_n(A) \to M_n(A)$ *defined by*

$$\theta : x + y\pi \;\mapsto\; \overline{x}^T - y^T\pi$$

is an involution[1].

Proof. Clearly θ is a K-linear map of order two. Let $x, y, u, v \in M_n(L)$, then

$$\begin{aligned}\theta((x+y\pi)(u+v\pi)) &= \theta(xu + (y\overline{u} + xv)\pi) = \overline{u}^T\overline{x}^T - (\overline{u}^T y^T + v^T x^T)\pi \\ &= (\overline{u}^T - v^T\pi)(\overline{x}^T - y^T\pi) = \theta(u+v\pi)\theta(x+y\pi).\end{aligned}$$

We conclude that θ is an involution. □

C.3.2. Definition. Let $A = Q_d(L/K)$ be a degenerate quaternion algebra over some field K. Let $n \geq 1$ be an integer, the group

$$\mathrm{U}_n(A) := \{\, z \in M_n(A) \mid \theta(z)z = 1 \,\}$$

is called the *unitary group* of rank n over A. The unitary group is a subgroup of the unit group $\mathrm{GL}_n(A)$.

C.3.3. Proposition. *Let* $\mathrm{Sym}_n(L)$ *denote the additive group of symmetric $n \times n$ matrices with entries in L. There is a split short exact sequence of groups*

$$1 \longrightarrow \mathrm{Sym}_n(L) \longrightarrow \mathrm{U}_n(A) \longrightarrow \mathrm{U}_n(L/K) \longrightarrow 1,$$

[1] For a definition of involution see III.4.2.

where $\mathrm{U}_n(L/K) := \{\, x \in M_n(L) \,|\, \overline{x}^T x = 1 \,\}$ *denotes the unitary group for the quadratic extension* L/K.

Proof. Let $x + y\pi \in M_n(A)$. We calculate

$$\theta(x+y\pi)\,(x+y\pi) = (\overline{x}^T - y^T\pi)(x+y\pi) = \overline{x}^T x + (\overline{x}^T y - y^T\overline{x})\pi.$$

We recognise that if $x + y\pi$ is in $\mathrm{U}_n(A)$, then $x \in \mathrm{U}_n(L/K)$ and further the projection $x + y\pi \mapsto x$ defines a homomorphism of groups $\mathrm{U}_n(A) \to \mathrm{U}_n(L/K)$. We consider the kernel of this map: $1 + y\pi$ is in $\mathrm{U}_n(A)$ if and only if $y \in \mathrm{Sym}_n(L)$. This gives the claimed short exact sequence and moreover $s : \mathrm{U}_n(L/K) \to \mathrm{U}_n(A)$ with $x \mapsto x + 0\pi$ splits the sequence. □

C.3.4. Corollary. *The unitary group over a degenerate quaternion algebra A is a semidirect product* $\mathrm{U}_n(A) \cong \mathrm{U}_n(L/K) \ltimes \mathrm{Sym}_n(L)$. *If K is a finite field with q elements, then*

$$|\,\mathrm{U}_n(A)| = q^{n(2n+1)} \prod_{j=1}^{n} \big(\, 1 - \frac{(-1)^j}{q^j} \,\big).$$

Proof. The exact sequence in the previous proposition splits and thus $\mathrm{U}_n(A)$ is the claimed semi-direct product.

Suppose K is the finite field with q elements. We obtain the identity

$$|\,\mathrm{U}_n(A)| = |\,\mathrm{U}_n(L/K)| \cdot |\,\mathrm{Sym}_n(L)|.$$

It is known that

$$|\,\mathrm{U}_n(L/K)| = q^{n^2} \prod_{j=1}^{n} \big(1 - \frac{(-1)^j}{q^j}\big)$$

(see section 3.6 in [**Wil09**]). Moreover, $|\,\mathrm{Sym}_n(L)|$ has $q^{n(n+1)}$ elements and the claim follows. □

C.4. Relation to Maximal Orders

C.4.1. Let $\mathcal{O}$ be a complete discrete valuation ring and let K be its quotient field. Let $\mathfrak{p} \subset \mathcal{O}$ be the prime ideal of $\mathcal{O}$ and let $\pi \in \mathfrak{p}$ be a uniformizer. The residue class field $\mathcal{O}/\mathfrak{p}$ will be denoted by k and we assume that k is a finite field.

Let D denote a quaternion division algebra over K. Recall that this algebra is unique up to isomorphism (cf. [**Rei03**, (14.5), (14.6)]). Moreover, there is a unique maximal $\mathcal{O}$-order $\Delta \subseteq D$. The order Δ has a unique maximal two-sided ideal $\pi_D\Delta$, where π_D is a prime element of Δ. Note that $\pi_D\Delta = \Delta\pi_D$. We can assume further that $\pi_D^2 = \pi$ (cf. [**Rei03**, (14.5)]).

Let $L \subset D$ be an inertia field, that is, a maximal subfield of D such that the extension L/K is unramified (cf. [**Rei03**, p. 145]). Let $\mathcal{O}_L$ denote the valuation ring of L and let $\ell := \mathcal{O}_L/\pi\mathcal{O}_L$ denote the residue class field. Since L is an unramified quadratic extension of K, the field ℓ is a quadratic extension field of k.

C.4.2. Lemma. *The k-algebra $\Delta/\pi\Delta$ is isomorphic to the degenerate quaternion algebra $Q_d(\ell/k)$.*

Proof. The maximal order Δ has a direct sum decomposition

$$\Delta = \mathcal{O}_L + \mathcal{O}_L\pi_D$$

(see [**Rei03**, p. 146] or [**Vig80**, II, Cor. 1.7]). Let $\widehat{\pi_D}$ denote the image of π_D in $\Delta/\pi\Delta$. Then the k-algebra $\Delta/\pi\Delta$ has a direct sum decomposition

$$\Delta/\pi\Delta = \ell + \ell\widehat{\pi_D}.$$

The relation $\pi_D^2 = \pi$ implies that $\widehat{\pi_D}^2 = 0$. Let σ denote the non-trivial Galois automorphism of L/K. It is well-known that $\pi_D x = \sigma(x)\pi_D$ for all $x \in L$ (cf. II. Cor. 1.7 in [**Vig80**]). Since $\sigma_{|\mathcal{O}_L}$ induces the non-trivial Galois automorphism on ℓ/k, the claim follows directly form the definition of $Q_d(\ell/k)$ in C.1.5. $\square$

C.4.3. Lemma. *Let $A = \Delta/\pi\Delta$. The norm N_d on $M_n(A)$ in the sense of C.2.5 agrees with the function defined by reduced norm polynomial $\mathrm{nrd}_{\Lambda/\mathcal{O}} \in S_{\mathcal{O}}(\Lambda^*)$ with $\Lambda = M_n(\Delta)$ (cf. III.3.6).*

Proof. We define explicitly a splitting $\varphi : M_n(D) \to M_{2n}(L)$ by

$$\varphi(x + y\pi_D) \;=\; \begin{pmatrix} x & y \\ \pi\sigma(y) & \sigma(x) \end{pmatrix}.$$

for all $x, y \in M_n(L)$ (cf. [**Rei03**, (14.7)]). Note that φ maps $M_n(\Delta)$ into $M_{2n}(\mathcal{O}_L)$. The reduced norm on the central simple K-algebra $M_n(D)$ is defined for all $z \in M_n(D)$ by $\mathrm{nrd}(z) := \det(\varphi(z))$. Let $z \in M_n(A)$ be given. We find elements $x, y \in M_n(\mathcal{O}_L)$ such that $z = x + y\pi_D + M_n(\pi\Delta)$. Since the reduced norm is given by the element $\mathrm{nrd}_{\Lambda/\mathcal{O}} \in S_{\mathcal{O}}(\Lambda^*)$, i.e. it is a polynomial function, we have

$$\mathrm{nrd}(z) \equiv \mathrm{nrd}(x + y\pi_D) \mod \mathfrak{p}.$$

As a next step we calculate $\mathrm{nrd}(x + y\pi_D) = \det(\varphi(x + y\pi_D))$ via the splitting φ. We see immediately, that

$$\det \begin{pmatrix} x & y \\ \pi\sigma(y) & \sigma(x) \end{pmatrix} \equiv \det(x)\det(\sigma(x)) \mod \mathfrak{p}.$$

Consequently, $\mathrm{N}_d(z) = \mathrm{nrd}_{\Lambda/\mathcal{O}}(z) \in k$. $\square$

C.4.4. Corollary. *Let $A = \Delta/\pi\Delta$ be the degenerate quaternion algebra for the extension ℓ/k. Consider the maximal order $\Lambda := M_n(\Delta)$ in $M_n(D)$.*

(a) We have $\mathrm{GL}_\Lambda(k) = \mathrm{GL}_n(A)$ and $\mathrm{SL}_\Lambda(k) = \mathrm{SL}_n(A)$ (in the notation of C.2.5).
(b) Let $G = G(\Lambda, \tau)$, where τ is the canonical symplectic involution on $M_n(D)$ (cf. V.1.7), then $G(k) \cong \mathrm{U}_n(A)$.

Proof. We have $\mathrm{GL}_\Lambda(k) = (\Lambda \otimes_{\mathcal{O}} k)^\times = M_n(\Delta/\pi\Delta)^\times$ and, by definition, this unit group is $\mathrm{GL}_n(A)$. Similarly, $\mathrm{SL}_\Lambda(k) = \ker(\mathrm{nrd}_{\Lambda/\mathcal{O}} : \mathrm{GL}_n(A) \to k^\times)$ and the claim follows directly from the lemma.

Finally we observe that the canonical symplectic involution on $M_n(D)$ induces the involution θ on $M_n(A)$ as defined in C.3.1. Thus we find

$$G(k) = \{\, g \in \mathrm{GL}_n(A) \mid \theta(g)g = 1 \,\} = \mathrm{U}_n(A).$$

$\square$

Abstract / Zusammenfassung

English Abstract

In this thesis we refine a method, developed by J. Rohlfs, for the computation of Lefschetz numbers of automorphisms of finite order in the cohomology of arithmetic groups. We adjust his method in order to apply it to arithmetic subgroups of inner forms of the special linear group.

Let A be a central simple algebra over an algebraic number field F. The associated norm-one group G is a semisimple algebraic F-group and it is further an inner form of the special linear group. An involution τ on A induces an automorphism τ^* of order two on G. In this thesis we study the Lefschetz numbers of automorphisms of the form τ^* in the cohomology of principal congruence subgroups associated with maximal $\mathcal{O}$-orders in A, where $\mathcal{O}$ denotes the ring of integers of F. The approach presented in this thesis exploits the smoothness properties of the underlying group schemes and is particularly suited to treat the case where τ is an involution of symplectic type. The main theorem gives an explicit formula for the Lefschetz number of a specific involution of symplectic type. As an application we obtain statements on the asymptotic behaviour of the total Betti number and we compute the dimensions of certain spaces of modular forms. Further one can apply the tools of this thesis to handle involutions of the second kind. We illustrate this with an explicit formula for a Lefschetz number in the cohomology of Bianchi congruence groups.

Deutsche Zusammenfassung

Diese Arbeit soll einen Beitrag zum Verständnis der Kohomologie von arithmetischen Gruppen leisten. Ist Γ eine arithmetisch definierte Untergruppe einer halbeinfachen algebraischen Gruppe G, sagen wir G sei definiert über dem Körper der rationalen Zahlen $\mathbb{Q}$, so ermöglicht Harders Gauß-Bonnet Satz prinzipiell die Bestimmung der Euler Charakteristik von Γ, siehe [**Har71**]. Die Euler Charakteristik ist eine wichtige Invariante von Γ, denn sie vermittelt einen Einblick in das Verhalten der Betti Zahlen der Gruppe Γ, zumindest falls die Euler Charakteristik nicht verschwindet. Es ist in der Tat leicht festzustellen, ob die Euler Charakteristik einer arithmetischen Gruppe verschwindet, denn dies hängt nur von Eigenschaften der umgebenden Lie-Gruppe $G(\mathbb{R})$ ab. Beispielsweise verschwindet die Euler Charakteristik jeder arithmetisch definierten Untergruppe der speziellen linearen Gruppe $\mathrm{SL}_n(\mathbb{R})$, sofern $n \geq 3$ gilt. In diesen Fällen ist es nötig andere kohomologische Invarianten von arithmetischen Gruppen zu studieren.

Als Verallgemeinerung der Euler Charakteristik kann man Lefschetz-Zahlen von Automorphismen endlicher Ordnung studieren. Die notwendigen Methoden wurden von J. Rohlfs entwickelt und erfolgreich angewendet, siehe z.B. [**Roh78, Roh81, Roh90**]. In dieser Arbeit wird die Methode von Rohlfs aufgegriffen und für die Anwendung auf arithmetische Untergruppen von inneren Formen der speziellen linearen Gruppe adaptiert. Im Folgenden werden die genauen Fragestellungen und Ergebnisse dieser Arbeit zusammengefasst.

Sei F ein algebraischer Zahlkörper. Ist A eine zentral einfache F-Algebra, so erhält man die zugehörige Norm-Eins-Gruppe G. Die Gruppe G ist eine F-Form der speziellen linearen Gruppe. Sei $\tau : A \to A$ eine Involution, sagen wir eine Involution erster Art, so induziert diese einen Automorphismus τ^* der Ordnung zwei auf der Gruppe G. Genauer ist τ^* definiert als die Verknüpfung von τ mit der Inversion auf G. In der vorliegenden Arbeit werden Ansätze zur Berechnung von Lefschetz-Zahlen solcher Automorphismen vorgestellt. Die entwickelten Hilfsmittel eigenen sich vor allem für Involutionen vom symplektischen Typ, was wir im Hauptsatz dieser Arbeit zeigen. Auch Involutionen von zweiter Art können behandelt werden, dies wird anhand des Beispiels von Bianchi-Kongruenzuntergruppen erklärt. Beide Ergebnisse werden wir nun kurz beschreiben.

Besitzt A eine Involution vom symplektischen Typ, so folgt schon aus der Existenz einer solchen Involution, dass A isomorph zu einer Matrix-Algebra $M_n(D)$ ist, wobei D eine F-Quaternionenalgebra bezeichnet. Die Quaternionenalgebra D hat eine kanonische Involution vom symplektischen Typ $\tau_c : D \to D$. Mittels dieser Involution erhält man eine weitere Involution τ vom symplektischen Typ auf $A = M_n(D)$. Genauer definiert man τ durch die Zuordnung $x \mapsto \tau_c(x)^T$. Wir beschränken unsere Betrachtungen auf diese spezielle Involution τ. Sei $\mathcal{O}$ der Ring der ganzen Zahlen zu F und sei $\Lambda_D \subseteq D$ eine maximale $\mathcal{O}$-Ordnung. Die Ordnung $\Lambda := M_n(\Lambda_D)$ ist maximal in A und diese Ordnung ist stabil unter der Involution τ. Sei $\mathfrak{a}$ ein echtes Ideal in $\mathcal{O}$, wir betrachten die zugehörige Hauptkongruenzuntergruppe

$$\Gamma(\mathfrak{a}) := \{\, g \in M_n(\Lambda_D) \mid \mathrm{nrd}_A(g) = 1 \text{ und } g \equiv 1 \mod \mathfrak{a} \,\}.$$

Diese Gruppe ist stabil unter τ^*. Sei nun $\rho : G \to \mathrm{GL}(W)$ eine rationale Darstellung der Norm-Eins-Gruppe G auf einem endlich dimensionalen Vektorraum W. Ist W mit einer verträglichen Wirkung von τ^* ausgestattet, so kann man die Lefschetz-Zahl $\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W)$ von τ^* in der Gruppenkohomologie $H^\bullet(\Gamma(\mathfrak{a}), W)$ definieren. In dieser Arbeit wird das folgende Ergebnis bewiesen.

Satz. *Wir nehmen an $\Gamma(\mathfrak{a})$ sei torsionsfrei. Falls D total definit ist, so sei $n \geq 2$. Die Lefschetz-Zahl $\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W)$ verschwindet, wenn F nicht total reell ist.*

Ist F total reell, so ist die Lefschetz-Zahl durch die folgende Formel gegeben

$$\mathcal{L}(\tau^*, \Gamma(\mathfrak{a}), W) \;=\; 2^{-r}\, \mathrm{N}(\mathfrak{a})^{n(2n+1)} \Delta_{rd}(D)^{n(n+1)/2}\, \mathrm{Tr}(\tau^*|W) \prod_{j=1}^{n} M(j, \mathfrak{a}, D).$$

Hierbei bezeichnet $\Delta_{rd}(D)$ die signierte reduzierte Diskriminante von D, mit r bezeichnen wir die Anzahl der reellen Stellen von F an denen D verzweigt, und es ist

$$M(j, \mathfrak{a}, D) := \zeta_F(1-2j) \prod_{\mathfrak{p}|\mathfrak{a}} \bigl(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^{2j}}\bigr) \prod_{\substack{\mathfrak{p}\in \mathrm{Ram}_f(D) \\ \mathfrak{p}\nmid\mathfrak{a}}} \bigl(1 + (\frac{-1}{\mathrm{N}(\mathfrak{p})})^j\bigr)$$

wobei $\mathrm{Ram}_f(D)$ *die Menge der endlichen, in D verzweigten Stellen von F notiert.*

Wir erhalten ein ähnliches Ergebnis für Hauptkongruenzuntergruppen von Bianchi-Gruppen. Sei E ein imaginär-quadratischer Zahlkörper, d.h. $E := \mathbb{Q}(\sqrt{d})$ für eine quadratfreie negative ganze Zahl d. Sei $\mathcal{O}$ der Ring der ganzen Zahlen zu E. Es sei außerdem σ der nicht-triviale Galoisautomorphismus der Erweiterung $E/\mathbb{Q}$. Auf $\mathrm{SL}_2(E)$ gibt es den Automorphismus τ^* definiert durch $x \mapsto (\sigma(x)^T)^{-1}$. Die Kongruenzuntergruppen

$$\Gamma(m) = \{\, g \in \mathrm{SL}_2(\mathcal{O}) \mid g \equiv 1 \mod m\mathcal{O} \,\}$$

sind torsionsfrei und τ^*-stabil für jede ganze Zahl $m \geq 3$. Ein Ergebnis der vorliegenden Arbeit ist der folgende Satz, den wir aus Gründen der Knappheit hier nur für Kohomologie mit trivialen Koeffizienten angeben.

Satz. *Sei $E = \mathbb{Q}(\sqrt{d})$ ein imaginär-quadratischer Zahlkörper mit $d \equiv 1 \mod 4$. Für alle $m \geq 3$ ist die Lefschetz-Zahl $\mathcal{L}(\tau^*, \Gamma(m), \mathbb{C})$ von τ^* in der Gruppenkohomologie $H^\bullet(\Gamma(m), \mathbb{C})$ gegeben durch*

$$\mathcal{L}(\tau^*, \Gamma(m), \mathbb{C}) = \frac{m^3|d|}{12} \prod_{p|m} (1 - \frac{1}{p^2}) \prod_{\substack{p|d \\ p\nmid m}} (1 + \frac{(-1)^{(p-1)/2}}{p}).$$

Es ist ein Anliegen dieser Arbeit die Grundlagen der verwendeten Methoden möglichst allgemein zu etablieren. Daher sind die genannten Ergebnisse nur ein kleiner Teil dieser Dissertation, der sich aus mehreren allgemeinen Resultaten ergibt. Die übrigen Ergebnisse und die Struktur dieser Arbeit werden nun kurz beschrieben.

Im ersten Kapitel dieser Arbeit verallgemeinern wir das folgende Lefschetz'sche Fixpunkt Prinzip: Die Lefschetz-Zahl eines Automorphismus endlicher Ordnung auf einem topologischen Raum ist die Euler Charakteristik des Raumes der Fixpunkte. Wir betrachten dazu äquivariante Garben-Kohomologie mit beliebigen Trägersystemen.

Im zweiten Kapitel geben wir eine detaillierte Darstellung von Rohlfs' Methode zur Beschreibung der Fixpunkte eines Automorphismus endlicher Ordnung auf dem lokal symmetrischen Raum welcher durch eine arithmetische Gruppe definiert wurde. Wir versuchen möglichst allgemein zu arbeiten, d.h. wir behandeln die Rohlfs'sche Zerlegung zunächst rein topologisch ohne Bezug zu arithmetischen Gruppen. Außerdem beschreiben wir auch die adelische Version der Rohlfs-Zerlegung.

Das dritte Kapitel enthält diverse Resultate zu glatten Gruppenschemata über Dedekindringen. Insbesondere werden Gruppenschemata untersucht die mit Ordnungen in zentral einfachen Algebren assoziiert sind, z.B. die spezielle lineare Gruppe über einer Ordnung oder die Fixpunktgruppe bzgl. einer Involution.

Im vierten Kapitel geben wir eine adelische Umformulierung des Harder'schen Gauß-Bonnet Satzes, die sich auf den Begriff des glatten Gruppenschemas stützt. Das Ziel war es eine Version der Gauß-Bonnet Formel anzugeben, welche leicht anzuwenden ist.

Die oben genannten Hauptresultate werden im fünften Kapitel bewiesen. Dazu finden sich in der Arbeit drei Anhänge. Im ersten wird die Theorie der maximal kompakten Untergruppen von Lie-Gruppen kurz dargestellt. Der zweite Anhang ist lediglich eine Sammlung diverser algebraischer Grundlagen. Im dritten Anhang geben wir eine Einführung in die Theorie der degenerierten Quaternionenalgebren.

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