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„On the existence of shadow price processes in utility maximization“

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Summary

In utility maximization with transaction costs, a shadow price is a process lying within the bid-ask spread such that it leads to the same optimal utility as the original process with transaction costs. This notion allows to reduce the problem of utility maximization under transaction costs to corresponding problems in the well-studied frictionless case. However, it has remained quite elusive under which conditions such a process exists. The aim of this thesis is to find an answer to this question.

To this end, the problem of utility maximization in frictionless markets will be introduced in discrete time and solved by following a dual approach from convex optimization. Then transaction costs are added to the model and it will be elaborated on how this modfication affects the process of solving the utility maximization problem where the emphasis is again placed on duality theory.

After this preliminary work, the notion of shadow prices will be introduced in discrete time. It will be proved that a shadow price always exists in one-period models. As regards multi-period models, two counterexamples show that a shadow price may fail to exist in general markets. However, if the underlying financial market is based on a finite probability space, existence is guaranteed and a proof for this assertion will be given. Furthermore, in order to gain additional insights to the existence of shadow prices, connections to duality theory will be presented.

The final part is devoted to analyzing the existence of shadow price processes in a continuous time setting. After developing the underlying financial market model and introducing the problem of utility maximization in this framework, the definition of shadow price processes in continuous time will be established. Then, two results regarding their existence will be presented. While the first one states necessary and sufficient conditions for the existence of shadow prices in the case of utility functions with constant relative risk aversion, the second result guarantees the existence if short selling is ruled out.

Zusammenfassung

Im Bereich der Nutzenmaximierung mit Transaktionskosten bezeichnet ein Schattenpreis einen Prozess, der Werte innerhalb des Bid-Ask Spreads annimmt und zum selben maximalen Erwartungsnutzen wie der ursprüngliche Preisprozess mit Transaktionskosten führt. Dieser Begriff kann dafür benutzt werden, das Problem der Nutzenmaximierung mit Transaktionskosten auf den ausführlich erforschten Fall ohne Transaktionskosten zu reduzieren. Es ist jedoch noch recht unklar unter welchen Bedingungen ein solcher Prozess existiert. Das Ziel dieser Arbeit liegt darin, eine Antwort für diese Existenzfrage zu finden.

Dazu wird zuerst das Problem der Nutzenmaximierung ohne Transaktionskosten und in diskreter Zeit präsentiert und anhand eines Dualitätsprinzips der konvexen Optimierung gelöst. Dann werden Transaktionskosten zum Modell hinzugefügt und es wird erläutert wie diese Veränderung den Lösungsprozess des Problems der Nutzenmaximierung beeinflusst.

Nach dieser Vorbereitung kann der Begriff des Schattenpreises in diskreter Zeit exakt definiert werden. Es wird bewiesen, dass ein Schattenpreis in ein-periodigen Modellen existiert. In mehr-periodigen Modellen kann die Existenz im Allgemeinen fehlschlagen, wie zwei Gegenbeispiele zeigen. Wenn jedoch der zugrunde liegende Wahrscheinlichkeitsraum endlich ist, ist die Existenz garantiert, wofür ein Beweis präsentiert wird. Darüber hinaus werden Verbindungen zur Dualitätstheorie erarbeitet. Dies ermöglicht weitere Einblicke in die Existenz von Schattenpreisen.

Der letzte Teil widmet sich der Analyse von Schattenpreisen in stetiger Zeit. Dafür wird ein stetiges Finanzmarktmodell entwickelt und das Problem der Nutzenmaximierung in selbigem definiert. Darauf aufbauend kann die Definition der Schattenpreise in stetiger Zeit erläutert werden. Es werden zwei Existenzresultate vorgestellt. Während das erste notwendige und hinreichende Bedingungen für die Existenz liefert, garantiert das zweite, dass ein Schattenpreis immer existiert, wenn das Modell keine Leerverkäufe zulässt.

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Chapter 1

Introduction

A classical problem in mathematical finance is to find the optimal trading strategy of an investor in a market with stocks and a risk-free asset so as to maximize expected utility. This problem was first solved by Merton [Mer71] who found an explicit closed-form solution for the optimal trading strategy. The solution suggests that the fraction of wealth invested in stocks is held constant at all times. In Merton's continuous-time setting, the investor's utility function was assumed to be a power function. He also assumed that the stocks' mean rate of return and volatility are constant and that the risk-free asset has constant interest rate. Since these assumptions are quite restrictive, various extensions of this model have been studied.

Many of these extensions rely on the assumption that financial markets allow for frictionless trading, that is, assets can be purchased and sold for the same price. In real markets, this assumption does not hold true since transaction costs are incurred whenever a portfolio is rebalanced: While one receives the bid price for selling assets, one has to pay the higher ask price for purchasing them. This market imperfection has far-reaching consequences as Magill and Constantinides [MC76] heuristically showed. They argued that the optimal strategy of a utility-maximizing investor changes fundamentally when transaction costs are considered:

There no longer exists a constant optimal fraction of wealth invested in stock. Instead, the optimal portfolio now consists of a whole region of holdings in stock, the so-called no-trade region. In this region, transaction costs exceed the benefits of trading. When the holdings leave this region due to changes in the prices of the assets, the investor should trade in order to reach the no-trade region as soon as possible.

Their results were later made precise by Davis and Norman [DN90] who applied methods from stochastic control to the problem of maximizing utility from consumption. They considered a stock whose price is given by a geometric Brownian motion and proportional transaction costs. Moreover, the investor is assumed to have a logarithmic or power utility function. They rigorously proved that in this framework the above described strategy is indeed optimal, and showed how to compute the borders of the no-trade region. In doing so, they relied on quite restrictive technical conditions which could later be removed by Shreve and Soner [SS94].

Recently, more emphasis has been put on a shadow price approach wich was first introduced in the pioneering work of Jouini and Kallal [JK95]. They show that a market is free of arbitrage if and only if there exists an equivalent martingale measure which transforms a process evolving within the bid-ask spread of the frictious price process into a martingale. This strong connection to the fundamental theorem of asset pricing suggest that it is possible to reduce the problem of utility maximization under transaction costs to corresponding problems in the well-studied frictionless case. In utility maximization, a shadow price is a process lying within the bid-ask spread such that it leads to the same optimal utility as the original process with transaction costs.

This concept has been successfully applied to portfolio optimization problems in various models. For example, Kallsen and Muhle-Karbe [KMK10] solve the above mentioned Merton problem with logarithmic utility and proportional transaction costs by applying the duality theory for frictionless markets to a shadow price. In a similar way, Gerhold et al. [GMKS11a] tackle the Davis and Norman problem for logarithmic utility, while Herczegh and Prokaj [HP11] derive a shadow price for the power utility case. Therefore, the shadow price approach appears to be a useful tool in portfolio optimization under transaction costs. However, it has remained quite elusive under which conditions such a process exists. The aim of this thesis is to find an answer to this question. The roadmap to this end is laid out as follows.

Chapter 2 presents a concise outline of the notions which are required in the subsequent analysis. In particular, a model of a financial market will be developed and basic results that hold in this market will be stated. Furthermore, the problem of utility maximization in frictionless markets will be introduced and solved by following a dual approach from convex optimization. Then transaction costs are added to the model and it will be elaborated on how this modification affects the process of solving the utility maximization problem. As regards this, the emphasis is again placed on duality theory.

In **Chapter 3**, the notion of shadow prices will be introduced in discrete time. It will be proved that a shadow price always exists in one-period models. As regards multi-period models, two counterexamples show that a shadow price may fail to exist in general markets. However, if the underlying financial market is based on a finite probability space, existence is guaranteed and a proof for this assertion will be given. Finally, in order to gain additional insights to the existence of shadow prices, connections to the duality theory that was introduced in the previous section will be presented.

Chapter 4 is devoted to analyzing the existence of shadow price processes in a continuous time setting. After developing the underlying financial market model and introducing the problem of utility maximization in this framework, the definition of shadow price processes in continuous time will be established. Then, two results regarding their existence will be presented. While the first one states necessary and sufficient conditions for the existence of shadow prices in the case of utility functions with constant relative risk aversion, the second result guarantees the existence if short selling is ruled out.

The appendices lay out the technical foundations on which major parts of this thesis are based on. More specifically, **Appendix A** highlights important definitions and results from convex analysis. For better readability of the body of the thesis, essential yet extensive and very technical formulas for the main theorem about the existence of shadow prices in continuous time can be found in **Appendix B**.

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Chapter 2

Utility Maximization in Discrete Time

This chapter presents a concise outline of the notions which are required in the subsequent analysis. In particular, a model of a financial market will be developed and basic results that hold in this market will be stated. Then utility functions and the problem of an investor who wants to maximize utility in this frictionless market will be introduced. For the problem of maximizing expected utility from terminal wealth in a financial market model which is based on a finite probability space, an explicit solution will be developed by following a dual approach from convex optimization. Thereupon, the finiteness assumption will be dropped and it will be examined to what extent this affects the solution to the utility maximization problem.

In the second part of this chapter, transaction costs will be added to the model and it will be elaborated on how this modification alters the process of solving the utility maximization problem. To this end, an explicit solution will again be developed for the case of a finite financial market. Similar to the frictionless part, it will then be pointed out how the solution changes when passing to general financial markets. As regards this, the emphasis is again placed on duality theory.

2.1 The Case without Transaction Costs

2.1 The Financial Market Model

Since many works on utility maximization use different frameworks, the aim of this section is to develop a general model of a financial market in order to cover all the important results in a consistent way. To this end, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space where the time set $\{0, 1, ..., T\}$ is finite. The considered financial market shall consist of one risk-free asset with price process S^0 and d risky assets whose prices are given by the d-dimensional stochastic process $S = (S^1, ..., S^d)$. While the risk-free asset can be regarded as a bank account, the risky assets represent stocks. By choosing the bank account as the numéraire, it follows that $S_t^0 = 1$ for all times $t = 0, 1, ..., T$, and the prices of the risky assets are consequently expressed as multiples of S^0. For $i = 1, ..., d$, S^i denotes a strictly positive, adapted process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$. From an economic point of view, this definition is justified: Since one can observe at time t the price of a stock at this time, its price process must be $\mathcal{F}_t$-measurable and hence adapted. Furthermore, the strict positivity stems from the fact that stocks clearly cannot have negative prices. Derivatives such as futures which may have negative prices will not be included in this financial market.

Now consider an investor in this financial market who may allocate his wealth either in the bank account or in any of the risky assets. Denote by $\varphi_t^0 \in \mathbb{R}$ his holdings in the bank account at time t, and let $\varphi_t^i \in \mathbb{R}$ for $i = 1, ..., d$ describe the number of shares held in asset i after rebalancing the portfolio at time $t-1$. Note that negative holdings resemble the short selling of the corresponding asset. Then a **trading strategy** is defined as an $\mathbb{R}^{d+1}$-valued predictable stochastic process $(\varphi^0, \varphi) = (\varphi^0, (\varphi^1, ..., \varphi^d)) = (\varphi_t^0, (\varphi_t^1, ..., \varphi_t^d))_{t=0}^{T+1}$. Again, the economic interpretation is straightforward as the investor must determine the quantity of holdings in a stock based on the information at time $t-1$ without "looking into the future". Thus, the trading strategy is required to be $\mathcal{F}_{t-1}$-measurable, i.e. predictable. The quantity of assets held at time 0 $(\varphi_0^0, \varphi_0) = (\eta_0, \eta) \in \mathbb{R}_{\geq 0}^{d+1}$ is called **initial endowment**.

The investor is not forced to invest his entire wealth. He may also consume his wealth. Let $c_t \in \mathbb{R}_{\geq 0}$ represent the amount of wealth consumed at time t. Then the nonnegative adapted process $c = (c_t)_{t=0}^T$ is called a **consumption process**. The combined pair $((\varphi^0, \varphi), c)$ is called a **portfolio/consumption pair**.

All of the investor's wealth must be consumed, or invested in any of the $d+1$ assets. Adding or withdrawing funds is not allowed. Therefore, any changes in the value of the holdings in the risky assets or consumption of money must be accounted for in the bank account. A portfolio/consumption pair $((\varphi^0, \varphi), c)$ is called **self-financing**, denoted by $((\varphi^0, \varphi), c) \in \mathcal{A}$, if it satisies this condition, i.e.

$$\Delta\varphi_{t+1}^0 = \sum_{i=1}^{d} S_t^i \Delta\varphi_{t+1}^i - c_t = S_t^\top \Delta\varphi_{t+1} - c_t, \qquad 1 \leq t \leq T. \tag{2.1}$$

For some applications, calculations become much easier if one allows "throwing away money", that is, the condition above transforms to $\Delta\varphi_{t+1}^0 \leq S_t^\top \Delta\varphi_{t+1} - c_t$ for all $1 \leq t \leq T$. This modification clearly does not violate the intuitive idea behind the self-financing property. In the remainder, it will be evident from the context which definition of a self-financing portfolio/consumption pair is chosen. Moreover, a self-financing portfolio/consumption pair $((\varphi^0, \varphi), c)$ is called **admissible** if it satisfies $(\varphi_0^0, \varphi_0) = (\eta_0, \eta)$ and $(\varphi_{T+1}^0, \varphi_{T+1}) = (0, 0)$. This means that the positions in the risky assets have to be liquidated at the terminal time T. In addition, the entire cash endowment in the bank account must be consumed at time T. Therefore, the nonnegativity of the consumption process implies that bankruptcy is not allowed. The set of all admissible strategies starting from a nonnegative initial cash endowment $(x, 0)$ will be denoted by $\mathcal{A}(x)$.

The liquidation value at the terminal time T of a given admissible strategy $((\varphi^0, \varphi), c) \in \mathcal{A}(x)$ can be calculated by using the self-financing condition from equation (2.1):

$$\varphi^0_{T+1} = \varphi^0_0 + \sum_{t=1}^{T+1} \Delta\varphi^0_t = x + \sum_{t=1}^{T+1} (S^\top_{t-1}\Delta\varphi_t - c_{t-1}) \tag{2.2}$$

$$= x + \sum_{t=1}^{T} \Delta S^\top_t \varphi_t - \sum_{t=1}^{T+1} c_{t-1} \tag{2.3}$$

Since $\varphi^0_{T+1} = 0$ by definition of an admissible strategy, it follows that the liquidation value c_T can be written as

$$c_T = x + \sum_{t=1}^{T} \Delta S^\top_t \varphi_t - \sum_{t=1}^{T} c_{t-1} \tag{2.4}$$

$$= x + (\varphi \cdot S)_T - \sum_{t=1}^{T} c_{t-1} \tag{2.5}$$

where $(\varphi \cdot S)_T := \sum_{t=1}^{T} \Delta S^\top_t \varphi_t$ denotes the stochastic integral of φ with respect to S. This expression of the liquidation value will play a pivotal role in the subsequent sections when it comes to maximizing expected utility from terminal wealth. In the remainder of this thesis, the pair (φ^0, φ) will stand for an admissible trading strategy $((\varphi^0, \varphi), c)$ with $c_t = 0$ for $t = 0, 1, ..., T - 1$. It is clear that the liquidation value of these strategies represent the terminal highest pay-off attainable with an admissible strategy starting from initial endowment $(x, 0)$. Motivated by this, the subspace $K(x)$ of $L^0(\Omega, \mathcal{F}, P)$ defined by

$$K(x) = \{x + (\varphi \cdot S)_T \mid (\varphi^0, \varphi) \in \mathcal{A}(x)\} \tag{2.6}$$

is called the set of **contingent claims attainable at price x**. In addition, the convex cone $C(x)$ in $L^\infty(\Omega, \mathcal{F}, P)$ defined by

$$C(x) = \{g \in L^\infty(\Omega, \mathcal{F}, P) \mid \text{there exists } f \in K(x) \text{ with } f \geq g\} \tag{2.7}$$

is said to be the set of **contingent claims superreplicable at price x**. The set of **cosumption processes attainable at price x** will be denoted by $\mathcal{C}(x)$. Explicitly,

$$\mathcal{C}(x) = \{c \mid \text{there exists a trading strategy } (\varphi^0, \varphi) \text{ such that } (\varphi^0, \varphi, c) \in \mathcal{A}(x)\}. \tag{2.8}$$

These definitions evidently build the bridge to the concept of arbitrage. If one were able to end up with a positive liquidation value with positive probability by following a self-financing strategy starting with zero initial investment, the market would obviously admit opportunities of arbitrage. Therefore, a financial market S is said to satisfy the **no-arbitrage condition (NA)** if

$$K(0) \cap L^0_+(\Omega, \mathcal{F}, P) = 0 \tag{2.9}$$

where 0 denotes the function identically equal to zero. Since K dominates C from above, this is equivalent to

$$C(0) \cap L^0_+(\Omega, \mathcal{F}, P) = 0. \tag{2.10}$$

As will turn out in the theorem below, there exists a convenient connection of the no-arbitrage condition with martingale theory. A probability measure Q on $(\Omega, \mathcal{F})$ is called an **equivalent martingale measure** for S if $Q \sim P$ and S is a martingale under Q, i.e. $\mathbb{E}_Q[S_{t+1}|\mathcal{F}_t] = S_t$ for $t = 0, 1, ..., T-1$. The set of equivalent martingale measures is denoted by $\mathcal{M}^e(S)$ and the set of all martingale probability measures by $\mathcal{M}^a(S)$. If the state space Ω is finite and if $P(\omega) > 0$ for all $\omega \in \Omega$, the Radon-Nikodým derivative $\frac{dQ}{dP} \in L^1(\Omega, \mathcal{F}, P)$ can be written as

$$\frac{dQ}{dP}(\omega) = \frac{Q(\omega)}{P(\omega)}. \tag{2.11}$$

Another convenient consequence of the finiteness of Ω is that the set of probability measures is a subset of $\mathbb{R}^N$ where $N = |\Omega|$. One can even say much more about the topology of $\mathcal{M}^a(S)$ in $\mathbb{R}^N$ as the following proposition shows. These topological properties play a crucial role in the dual approach to utility maximization.

Proposition 2.1.1. *Suppose that $|\Omega| = N < \infty$ and $\mathcal{M}^e(S) \neq \emptyset$. Then the following assertions hold:*

(i) $\mathcal{M}^a(S)$ is a bounded, closed, convex polytope in $\mathbb{R}^N$.
(ii) There exist $Q^1, ..., Q^M$ in $\mathcal{M}^a(S)$ such that $\mathcal{M}^a(S) = conv(\{Q^1, ..., Q^M\})$.
(iii) $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$.

Proof. By definition of the convex cone, the second assertion follows directly from the first assertion which in turn is a direct consequence of the definition of $\mathcal{M}^a(S)$. In order to see that $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$, observe that there is at least one $Q^* \in \mathcal{M}^e(S)$ by assumption, and follow the reasoning in the proof of [DS06, Proposition 2.2.9]: For any $Q \in \mathcal{M}^a(S)$ and $0 < \mu \leq 1$, the linear combination $\mu Q^* + (1-\mu)Q$ is clearly an equivalent martingale measure for the price process S which implies that $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$. □

Now, the connection between no-arbitrage and equivalent martingale measures can be formulated:

Theorem 2.1.2 (Fundamental Theorem of Asset Pricing, cf. [DS06, Theorem 2.2.7])**.** *For a financial market S modeled on a finite stochastic base $(\Omega, \mathcal{F}, \mathcal{F}_{t=0}^T, P)$, the folowing are equivalent:*

(i) S satisfies (NA).

(ii) $\mathcal{M}^e(S) \neq \emptyset$.

Utility maximization only makes sense in markets which do not admit arbitrage, otherwise one would end up with infinite wealth. Therefore, the following assumption will stand throughout the remainder of this chapter: $\mathcal{M}^e(S) \neq \emptyset$. According to the Fundamental Theorem of Asset Pricing, this guarantees the absence of arbitrage. The proposition below shows that every contingent claim can be uniquely replicated if and only if $\mathcal{M}^e(S)$ reduces to a singleton $\{Q\}$. Markets with this property are said to be **complete**.

Proposition 2.1.3 (Complete financial markets, cf. [DS06, Corollary 2.2.12])**.** *For a financial market S satisfying the no-arbitrage condition (NA), the following are equivalent:*

(i) $\mathcal{M}^e(S)$ consists of a single element Q.

(ii) Each $f \in L^\infty(\Omega, \mathcal{F}, P)$ may be represented as

$$f = a + (\varphi \cdot S)_T \qquad \text{for some } a \in \mathbb{R} \text{ and } (\varphi^0, \varphi) \in \mathcal{A}(0). \tag{2.12}$$

In this case, $a = \mathbb{E}_Q[f]$, the stochastic integral is unique, and

$$\mathbb{E}_Q[f|\mathcal{F}_t] = \mathbb{E}_Q[f] + (\varphi \cdot S)_t, \quad t = 0, 1, ..., T. \tag{2.13}$$

Economically speaking, this implies that the pay off of any future contingent claim in an arbitrage-free market can be hedged by a self-financing portfolio starting from an initial investment which equals the expected future pay off. Thus, by the economic principle of pricing by arbitrage, today's price of the contingent claim must be equal to the expected future pay off.

This leads to defining the arbitrage-free price of a contingent claim. The intuition is that adding the contingent claim f with price $\pi(f)$ to the financial market does not create an arbitrage opportunity. In the present framework, the contingent claim can be represented as an additional risky asset S_{d+1} where $S_0 = \pi(f)$, $S_T = f$ and $S_t = \mathbb{E}_Q[f|\mathcal{F}_t]$ for $1 \leq t \leq T-1$. Therefore, $\pi(f)$ is called the **arbitrage-free price** of a contingent claim $f \in L^\infty(\Omega, \mathcal{F}, P)$ if the enlarged financial market (S, S_{d+1}) satisfies the no-arbitrage condition, i.e.

$$K^{f,\pi(f)}(0) \cap L^0_+(\Omega, \mathcal{F}, P) = 0 \tag{2.14}$$

where $K^{f,\pi(f)}(x)$ is defined as above as the set of contingent claims attainable at price x in the financial market $(S, S_d + 1)$.

Proposition 2.1.3 shows that $\pi(f) = \mathbb{E}_Q[f]$ in the case of a complete financial market. In general, when turning to financial markets which are not necessarily complete, this nice formula for the arbitrage-free price of a contingent claim does not hold true anymore. However, as it turns out, every contingent claim can be superreplicated, and thus an upper bound for the arbitrage-free price exists. This is the content of the so-called Superreplication Theorem below.

Theorem 2.1.4 (Superreplication Theorem, cf. [DS06, Theorem 2.4.2])**.** *Assume that S satisfies (NA). Then, for $f \in L^\infty(\Omega, \mathcal{F}, P)$, the following formula for the arbitrage-free price of f holds:*

$$\pi(f) = \sup\{\mathbb{E}_Q[f] \,|Q \in \mathcal{M}^e(S)\} \tag{2.15}$$
$$= \max\{\mathbb{E}_Q[f] \,|Q \in \mathcal{M}^a(S)\} \tag{2.16}$$
$$= \min\{a \in \mathbb{R} \mid \textit{there exists } k \in K(0), a + k \geq f\}. \tag{2.17}$$

2.1 Utility Functions and Conjugates

As economic agents do not think in terms of absolute monetary values, it is required to somehow model their preferences. This can be done by employing the notion of utlitiy functions. A **utility function** $U : D \to \mathbb{R} \cup \{-\infty\}$ is a mapping from the set of monetary values to the positively extended real numbers. Here the domain of U is either $D = (-\infty, \infty)$ or $D = (0, \infty)$ depending on whether negative wealth is allowed. If $D = (0, \infty)$, it will be assumed that $U(x) = -\infty$ for $x < 0$. On its domain D, U must satisfy $U(x) > -\infty$.

In addition, U is supposed to be a strictly increasing and strictly concave function. While the strict monotonicity simply means that more is better, the strict concavity is also obvious from an economic point of view: The lower the agent's wealth, the higher the utility gained from an extra unit of wealth. Furthermore, U shall be continuously differentiable and satisfy the so-called **Inada conditions**

$$\lim_{x \downarrow x_0} U'(x) = \infty, \qquad \lim_{x \uparrow \infty} U'(x) = 0, \tag{2.18}$$

where $x_0 \in \{-\infty, 0\}$ denotes the left boundary of D.

As introduced by Arrow [Arr71] and Pratt [Pra64], the **absolute risk aversion** is defined as

$$A(x) = -\frac{U''(x)}{U'(x)}, \tag{2.19}$$

and the **relative risk aversion** as

$$R(x) = xA(x). \tag{2.20}$$

Kramkov and Schachermayer [KS99] developed another measure for utility functions: The **asymptotic elasticity** of a utility function U with domain $D = (0, \infty)$ is defined as

$$\mathrm{AE}(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)}. \tag{2.21}$$

If $\mathrm{AE}(U) < 1$, one says that U has **reasonable asymptotic elasticity**.

The following examples present utility functions that will be referred to in subsequent chapters.

Example 2.1.5 (Logarithmic utility)**.** Let $D = (0, \infty)$ and define $U(x) = \log(x)$. Obviously, the logarithm satisfies the necessary conditions imposed on utility functions. A straightforward calculation yields $A(x) = \frac{1}{x}$ and $R(x) = 1$. It also has reasonable asymptotic elasticity since $\frac{1}{\log(x)} \to 0$ as $x \to \infty$. The logarithmic utility is a special case of the more general family of power utility functions as the following example illustrates. ◁

Example 2.1.6 (Power utility)**.** For $p \in (-\infty, 1)$, consider the utility function

$$U(x) = \begin{cases} \frac{1}{p} x^p, & \text{for } x \neq 0, p \neq 0, \\ \log(x), & \text{for } x \neq 0, p = 0, \end{cases} \quad \text{and} \quad U(0) = \begin{cases} 0, & \text{for } p > 0, \\ -\infty, & \text{for } p \leq 0 \end{cases}. \tag{2.22}$$

It follows that $A(x) = \frac{1-p}{x}$ and $R(x) = 1 - p$. As $R(x)$ is constant, the power utility belongs to the so-called familiy of utility functions with **constant relative risk aversion** (CRRA). The relative risk aversion depends on the parameter p. Hence, p is called the **risk aversion parameter**. Whereas the investor is less risk averse than the log-investor for $0 < p < 1$, $p < 0$ suggests high risk aversion. This utility function has reasonable asymptotic elasticity because $\mathrm{AE}(U) = p < 1$. ◁

Example 2.1.7 (Exponential utility)**.** Define $U(x) = -e^{-ax}$ for a positive constant a on the domain $D = (-\infty, \infty)$. This utility function has the unique property that its absolute risk aversion is constant, as $A(x) = a$. ◁

In the formulation of utility functions above, it was implicitly assumed that the investor's preferences do not depend on time. This is no restriction in the case of maximizing utility from terminal wealth since only the terminal time is relevant. When dealing with problems of maximizing utility from consumption, however, consumption may occur at various times. Therefore, it is necessary to introduce time into the concept of utility functions. As will become evident later, it may be beneficial to additionally allow the utility function to be random. This leads to the following definition (cf. Kallsen and Muhle-Karbe [KMK11, Definition 2.5]): A **utility process** U is a mapping $U : \Omega \times \{0, 1, ..., T\} \times D \to [-\infty, \infty)$, such that $(\omega, t) \mapsto U_t(\omega, x)$ is predictable for every $x \in D$ and U_t is a utility function for every $(\omega, t) \in \Omega \times \{0, 1, ..., T\}$.

For the utility maximization procedures presented in the sequel, it is necessary to adopt some definitions from convex optimization to the notion of utility functions. Given a concave function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, the **conjugate function** V of U is defined by

$$V(y) = \sup_{x \in \mathbb{R}} [U(x) - yx)], \text{ for } y > 0. \tag{2.23}$$

The function $V : \mathbb{R} \to \mathbb{R}$, conjugate to U is said to satisfy the **usual regularity assumptions** if V is finitely valued, differentiable, strictly convex on $(0, \infty)$, and satisfies

$$V'(0) := \lim_{y \downarrow 0} V'(y) = -\infty, \tag{2.24}$$

and the limit-conditions

$$\lim_{y \to \infty} V(y) = \begin{cases} \lim_{x \to 0} U(x), & \text{for } \mathrm{dom}(U) = (0, \infty), \\ \infty, & \text{for } \mathrm{dom}(U) = (-\infty, \infty), \end{cases} \tag{2.25}$$

$$\lim_{y \to \infty} V'(y) = \begin{cases} 0, & \text{for } \mathrm{dom}(U) = (0, \infty), \\ \infty, & \text{for } \mathrm{dom}(U) = (-\infty, \infty). \end{cases} \tag{2.26}$$

As utility functions are concave by definition, every utility function has a conjugate function. The following example illustrates this concept.

Example 2.1.8 (Conjugate functions)**.** Consider the logarithmic utility from Example 2.1.5, i.e. $U(x) = \log(x)$ with $x > 0$. It follows that $U(x) - yx = \log(x) - yx$. Now one seeks to find the supremum of this function over all x. Setting the derivative to zero yields that it attains an extremal point at $x = \frac{1}{y}$. Since the second derivative is negative everywhere, one concludes that $V(y) = \log(\frac{1}{y}) - y\frac{1}{y} = -\log(y) - 1$ for $y > 0$.
Similarly, one obtains the following conjugate functions for the other mentioned utility functions: The power utility, $U(x) = \frac{1}{p}x^p$ for $x > 0$ and $p \in (-\infty, 1) \setminus \{0\}$, has the conjugate function $V(y) = (1 - \frac{1}{p})y^{\frac{p}{p-1}}$ for $y > 0$, and the conjuagate of the exponential utility, $U(x) = -e^{-ax}$ for a positive constant a and $x \in (-\infty, \infty)$, is $V(y) = \frac{y}{a}(1+\log(-\frac{y}{a}))$ for $y > 0$. $\triangleleft$

The following proposition further clarifies the connection between a utility function and its conjugate.

Proposition 2.1.9 (cf. [DS06, Proposition 3.1.2])**.** *The conjugate function V of a utility function U satisfies the inversion formula*

$$U(x) = \inf_{y} [V(y) + yx], \textit{ for } x \in \textit{dom}(U) \tag{2.27}$$

and the usual regularity assumptions. In addition, $-V'(y)$ is the inverse function of $U'(x)$.

Conversely, if V satisfies the usual regularity assumptions, then U defined by (2.27) is a utility function.

2.1 Optimal Portfolios

After having established the definitions of the underlying financial market and utility functions, the portfolio optimiziation problem can now be formulated. First of all, an easy example may serve to introduce this problem.

Example 2.1.10 (Portfolio optimization in the one-period binomial model)**.** Consider a two-element state space $\Omega = \{u, d\}$, where the probability measure P maps $u \in \Omega$ to a fixed $p \in (0, 1)$ and $d \in \Omega$ to $1 - p$. Let $T = 1$ and define the values of the assets by

$$S_0^0 = 1, S_1^0 = 1, \tag{2.28}$$

$$S_0^1 \in \mathbb{R}_+, S_1^1 = \begin{cases} S_0^1 u, & \text{for } \omega = u, \\ S_0^1 d, & \text{for } \omega = d, \end{cases} \tag{2.29}$$

where $u > 1$ and $0 < d < 1$. Recall that the equations in (2.28) are due to the financial market model being in discounted terms. The evolution of the stock price is illustrated in the following binomial tree.

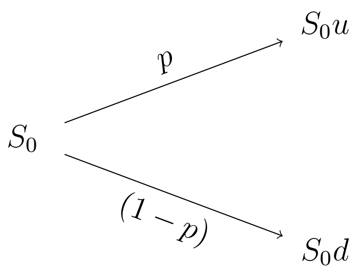

Figure 2.1: One-period binomial model

Now consider an investor with initial endowment $(x, 0)$ who seeks to maximize expected logarithmic utility from terminal wealth which corresponds to the optimization problem

$$(\mathrm{P}) \begin{cases} \text{maximize} & \mathbb{E}[\log(c_1)] = \mathbb{E}[\log(x + (\varphi^1 \cdot S)_1)] \\ \text{subject to} & \varphi^1 \in \mathcal{A}(x), \end{cases} \tag{2.30}$$

where equation (2.5) and $c_0 = 0$ was used. In this one-period model, the only relevant variable in an admissible strategy clearly is the initial trade φ_1^1. In order to satisfy the admissibility criterion $c_1 > 0$, the investor cannot buy more than $\frac{x}{S_0 d - S_0}$ shares. Otherwise, he would end up with a negative terminal wealth with probability $(1 - p)$. He cannot sell more than $\frac{x}{S_0 d - S_0}$ shares for the same reason. All together, the expression above extends to

$$(\mathrm{P}) \begin{cases} \text{maximize} & p\log(x+(S_0u-S_0)\varphi_1^1)+(1-p)\log(x+(S_0d-S_0)\varphi_1^1) \\ \text{subject to} & -\frac{x}{S_0d-S_0} < \varphi_1^1 < \frac{x}{S_0d-S_0}. \end{cases}$$

The constraint may actually be ignored and the maximization performed over all φ_1^1 since every φ_1^1 lying outside the feasible region will lead to an optimal value of $-\infty$. According to the elementary principle of maximizing functions, setting the derivative to zero and observing that the second derivative is strictly negative leads to the optimal value of wealth invested in stock which is

$$\varphi_1^1 = \frac{(pu+(1-p)d-1)x}{(d-1)(u-1)S_0}. \tag{2.31}$$

◁

In the example above, the optimal strategy for a log-utility investor seeking maximal expected utility from terminal wealth in the one-period binomial model was found. Since this special case clearly does not cover all market models and investors' preferences, it is required to generalize the concept of optimal strategies. For the framework with finite time horizon T, there are two types of optimization problems that have been widely studied in the literature:

1. Optimal Consumption

Following Kallsen and Muhle-Karbe [KMK11, Definition 2.5], an admissible portfolio/-consumption pair $((\varphi^0,\varphi),c)$ is said to maximize **expected utility from consumption** if it maximizes

$$\kappa \mapsto \mathbb{E}\left[\sum_{t=0}^{T} U_t(\kappa_t)\right] \tag{2.32}$$

over all admissible portfolio/consumption pairs $((\psi^0,\psi),\kappa)$.

Often problems which involve discounting the consumptions are considered (e.g. see [GMKS11b] or [CSZ12]). Then the objective is to maximize

$$\kappa \mapsto \mathbb{E}\left[\sum_{t=0}^{T} D_t U_t(\kappa_t)\right] \tag{2.33}$$

over all admissible portfolio/consumption pairs $((\psi^0,\psi),\kappa)$ where D_t is a positive predictable discount factor, e.g. $D_t = e^{-\delta t}$. In this case, the parameter δ is called the **impatience rate**. Note that this problem can be traced back to Problem (2.32). To see this, define a new function $\tilde{U}_t := D_t U_t$ for all $t = 0, ..., T$. Since D_t is a constant factor for each t, the function $\tilde{U}_t$ is clearly a utility function, and Problem (2.33) can be expressed as Problem (2.32) where U_t is replaced by $\tilde{U}_t$.

2. Terminal wealth

In some applications, the possibility of consuming wealth at intertemporal dates is ruled out, and only the expected utility of the portfolio's value at the terminal time T is relevant. As regards this case, one says that an admissible portfolio/consumption pair $((\varphi^0, \varphi), c)$ maximizes **expected utility from terminal wealth** if it maximizes

$$\kappa \mapsto \mathbb{E}\left[U_T(\kappa_T)\right] \tag{2.34}$$

over all admissible portfolio/consumption pairs $((\psi^0, \psi), \kappa)$.

Again, this problem can be regarded as a special case of Problem (2.32) by defining the utility process

$$U_t(x) = \begin{cases} -\infty, & \text{for } x < 0, \\ 0, & \text{for } x \geq 0, \end{cases} \qquad \text{for } t \in 0, 1, ..., T-1. \tag{2.35}$$

It is obvious that a candidate for an optimal strategy of this modified problem must satisfy $c_t = 0$ for $t = 0, 1, ..., T-1$. Thus, the set of potential solutions reduces to $(\varphi^0, \varphi) \in \mathcal{A}(x)$.

While the elementary procedure demonstrated in the example above may also be used for these optimization problems in multiperiod models, it gets very complicated as the number of periods increases. For this reason, essentially two other methods can be employed: the **dynamic programming principle** and the **dual approach**.

The first method shall be briefly (and without much mathematical rigor) demonstrated for the problem of maximizing utlity from terminal wealth. For a presentation regarding the problem of maximizing utility from consumption, the interested reader may be referred to [CZ04, Chapter 4.2.4]. Following the methodology of [CZ04, Chapter 4.2.2], one starts by defining the **time-dependent value function**, denoted by $u(t, x)$ which is the expectation at a given time $t \in [0, T]$ of the terminal utility which can be attained by an investor with an initial endowment of $x > 0$ who trades optimally. Explicitly,

$$u(t, x) := \sup_{c \in C^{[t,T]}(x)} \mathbb{E}[U(c_T) \mid \mathcal{F}_t] \tag{2.36}$$

where $C^{[t,T]}$ denotes the set of consumption processes attainable at price x over the period $[t, T]$. The dynamic programming principle then states that under certain conditions

$$u(t, x) = \sup_{c \in \mathcal{C}^{[t,t+1]}(x)} \mathbb{E}[u(t+1, c_{t+1}) \mid \mathcal{F}_t]. \tag{2.37}$$

This equation is also called the **Bellman Equation**. It shows that at time t the investor only needs to find the optimal strategy for the period $[t, t+1]$. Hence, the original multi-period problem reduces to a number of single-period optimization problems. To see this, observe that the value function at the terminal time T obviously satisfies $u(T,x) = U(T,x)$. By using the dynamic programming principle, one can gradually work back in time until the value of $u(0,x)$ is known which enables the computation of the optimal strategy at time $t = 0$.

One of the sufficient conditions for the dynamic programming principle to hold is that the underlying financial market has to satisfy the Markov property, i.e. that the expectation in the right hand side of (2.36) only depends on the values of the financial market at time t. By contrast, the dual approach does not rely on this strong model assumption. Hence, it may be considered the modern way of solving utility maximization problems and shall be presented thoroughly in the next section.

2.1 The Dual Approach

This section shows how the problem of maximizing expected utility from terminal wealth can be solved by following a dual approach from convex optimization. First, it will be assumed that the state space Ω is finite. In doing so, Chapter 3 of Delbaen and Schachermayer [DS06] will be followed by applying the methodology and findings therein to the present framework. If modified slightly, the presented procedure may also be used to find an optimal portfolio/consumption pair which maximizes expected utility from consumption. In this case, however, the calculations become very tedious and the important ideas vanish in a barely manageable plethora of variables and functions.

Following this, the assumption of the state space Ω being finite will be dropped and the general case will be examined. To this end, the main results of Kramkov and Schachermayer [KS99] will be employed.

First of all, assume that $|\Omega| = N < \infty$. As pointed out earlier, the liquidation value at terminal time T of an admissible trading strategy $(\varphi^0, \varphi) \in \mathcal{A}(x)$ is given by $x + (\varphi \cdot S)_T$. This motivates in defining the **value function**

$$u(x) := \sup_{(\varphi^0,\varphi)\in\mathcal{A}(x)} \mathbb{E}_P[U(x + (\varphi \cdot S)_T)], \qquad \text{for } x \in \mathrm{dom}(U). \tag{2.38}$$

This function yields the highest expected utility an investor may achieve provided he trades optimally and is called the **indirect utility function**. It will follow from the subsequent discussion that this is indeed a utility function as defined in Section 2.1.2.

Since the terminal wealth of an admissible strategy starting from a fixed positive inital endowment x is random and depends on $\omega_n \in \Omega$, it can be represented by a random variable $(X_T(\omega_n))_{n=1}^N$. For notational convenience, denote the possibly outcomes $X_T(\omega_n)$ by ξ_n. According to Theorem 2.1.4, the contingent claim X_T can be dominated by a random variable of the form $x + (\varphi \cdot S)_T$ if and only if $\mathbb{E}_Q[X_T] = \sum_{n=1}^N q_n \xi_n \leq x$ for each $Q = (q_1, ..., q_n) \in \mathcal{M}^a(S)$. Therfore, the original problem

$$
(\text{P}) \begin{cases} \text{maximize} & \mathbb{E}_P[U(x + (\varphi \cdot S)_T] \\ \text{subject to} & (\varphi^0, \varphi) \in \mathcal{A}(x), \end{cases} \tag{2.39}
$$

can be reformulated as

$$
(\text{P'}) \begin{cases} \text{maximize} & \sum_{n=1}^N p_n U(\xi_n) \\ \text{subject to} & \sum_{n=1}^N q_n U(\xi_n) \leq x, \qquad \text{for all } Q \in \mathcal{M}^a(S), \\ & (\xi_1, ..., \xi_N) \in \mathbb{R}^N \end{cases} \tag{2.40}
$$

Proposition 2.1.1 guarantees that $\mathcal{M}^a(S) = \text{conv}(\{Q^1, ..., Q^M\})$ where $Q^m = (q_1^m, ..., q_n^m)$ is a martingale measure for $1 \leq m \leq M$. By the properties of convex sets and some minor transformations, it hence suffices to consider

$$
(\text{P''}) \begin{cases} \text{minimize} & -\sum_{n=1}^N p_n U(\xi_n) \\ \text{subject to} & \sum_{n=1}^N q_n^m U(\xi_n) - x \leq 0, \qquad \text{for all } 1 \leq m \leq M, \\ & (\xi_1, ..., \xi_N) \in \mathbb{R}^N. \end{cases} \tag{2.41}
$$

Since the utility function U is concave by definition, (P”) is an inequality constrained convex problem. It can be observed that the zero-vector in $\mathbb{R}^N$ is a feasible solution with $\sum_{n=1}^N q_n^m U(\xi_n) < x$ for all $1 \leq m \leq M$. Thus, the Lagrange duality (see Appendix A.3, in particular Theorem A.3.1) can be employed to find the optimal value of (P”). To this end, consider the Lagrangian of this problem, that is, $L(\xi_1, ..., \xi_N, \eta_1, ..., \eta_M)$ where $(\xi_1, ..., \xi_N) \in \text{dom}(U)^N$ and $(\eta_1, ..., \eta_M) \in \mathbb{R}^M_{>0}$. It follows that

$$
L(\xi_1, ..., \xi_N, \eta_1, ..., \eta_M) = -\sum_{n=1}^N p_n U(\xi_n) + \sum_{m=1}^M \eta_m \left(\sum_{n=1}^N q_n^m U(\xi_n) - x \right) \tag{2.42}
$$

$$
= -\sum_{n=1}^N p_n \left(U(\xi_n) - \sum_{m=1}^M \frac{\eta_m q_n^m}{p_n} \eta_n \right) - \sum_{m=1}^M \eta_m x. \tag{2.43}
$$

This expression can be simplified by defining $y = \eta_1 + ... + \eta_m$, $\mu_m = \frac{\eta_m}{y}$, $\mu = (\mu_1, ..., \mu_M)$ and $Q^\mu = \sum_{m=1}^M \mu_m Q^m$. Since $\sum_{m=1}^M \mu_m = 1$, the latter is a convex combination of elements in $\mathcal{M}^a(S)$. It follows from the topology of $\mathcal{M}^a(S)$ that $Q^\mu \in \mathcal{M}^a(S)$. Thus, the pair (y, Q^μ) is contained within $\mathbb{R}^M_{>0} \times \mathcal{M}^a(S)$ for each Lagrange multiplier $(\eta_1, ..., \eta_M) \in \mathbb{R}^M_{>0}$. Conversely, every vector $(\eta_1, ..., \eta_M) \in \mathbb{R}^M_{>0}$ can be written as $(y\mu_1, ..., y\mu_M)$ which implies that there is a one-to-one correspondence between $\mathbb{R}^M_{>0}$ and $\mathbb{R}^M_{>0} \times \mathcal{M}^a(S)$. Hence, the Lagrangian may be written as

$$L(\xi_1, ..., \xi_N, y, Q) = -\mathbb{E}_P[U(X_T)] + y(\mathbb{E}_Q[U(X_T)] - x) \tag{2.44}$$

$$= -\sum_{n=1}^{N} p_n \left(U(\xi_n) - \frac{y q_n}{p_n} \eta_n \right) - yx \tag{2.45}$$

where $(\xi_1, ..., \xi_N) \in \mathrm{dom}(U)^N$, $y > 0$, $Q = (q_1, ..., q_N) \in \mathcal{M}^a(S)$.

As defined in Appendix A.3, the Lagrange dual function of (P”) is

$$g(y, Q) = \inf_{\xi_1, \dots, \xi_N} L(\xi_1, ..., \xi_N, y, Q), \tag{2.46}$$

and the goal is to maximize it so as to find the optimal value of (P”) which is then given by

$$\sup_{y>0, Q\in\mathcal{M}^a(S)} g(y, Q) = \sup_{y>0, Q\in\mathcal{M}^a(S)} \inf_{\xi_1, \dots, \xi_N} L(\xi_1, ..., \xi_N, y, Q) \tag{2.47}$$

or, equivalently, by

$$\inf_{y>0, Q\in\mathcal{M}^a(S)} -g(y, Q) = \inf_{y>0, Q\in\mathcal{M}^a(S)} \sup_{\xi_1, \dots, \xi_N} -L(\xi_1, ..., \xi_N, y, Q). \tag{2.48}$$

For notational convenience, write

$$\Psi(y, Q) = \sup_{\xi_1, \dots, \xi_N} -L(\xi_1, ..., \xi_N, y, Q) \tag{2.49}$$

$$= \sup_{\xi_1, \dots, \xi_N} \sum_{n=1}^{N} \left(U(\xi_n) - \frac{y q_n}{p_n} \eta_n \right) + yx \tag{2.50}$$

$$= \sum_{n=1}^{N} \sup_{\xi_n} \left(U(\xi_n) - \frac{y q_n}{p_n} \eta_n \right) + yx \tag{2.51}$$

$$= \sum_{n=1}^{N} p_n V\left(y \frac{q_n}{p_n} \right) + yx \tag{2.52}$$

where V denotes the conjugate function of U. In combination with Equation 2.48, this means that the original problem transforms to minimizing the conjugate function of U under the constraints $y > 0, Q \in \mathcal{M}^a(S)$. In doing so, the aim is to first fix $y > 0$ and minimize over $Q \in \mathcal{M}^a(S)$:

For fixed $y > 0$ regard the continuous function which maps Q to $\Psi(y, Q)$. As $\mathcal{M}^a(S)$ is compact, there is a $\hat{Q}(y) = (\hat{q}_1(y), ..., \hat{q}_n(y)) \in \mathcal{M}^a(S)$ where this function attains its minimum. The minimizer $\hat{Q}(y)$ is also unique since the conjugate function V is strictly convex by Proposition 2.1.9. Furthermore, it will be claimed that $\hat{Q}(y)$ is an equivalent martingale measure. To see this, suppose the opposite, that is, $\hat{q}_n(y) = 0$ for some $1 \leq n \leq N$, and fix $Q \in \mathcal{M}^e(S)$. Define $Q^\varepsilon = \varepsilon Q + (1-\varepsilon)\hat{Q}$ for $0 < \varepsilon < 1$. By construction, the probabilities of Q^ε satisfy $q_n^\varepsilon = \varepsilon q_n + (1-\varepsilon)\hat{q}_n > \hat{q}_n$ for $\varepsilon < \min\{q_n \mid 1 \leq n \leq N\}$. Hence, Q^ε is an equivalent martingale measure, and due to the strict convexity of V and $\lim_{y \to 0} V'(y) = -\infty$, it follows for $1 \leq n \leq N$ that $V(q_n^\varepsilon) < V(\hat{q}_n)$. This further implies $\Psi(y, Q^\varepsilon) < \Psi(y, \hat{Q})$ which is a contradiction to the assumption that $\hat{Q}$ is the minimizer of $\Psi(y, Q)$ for fixed y. Therefore $\hat{Q}(y)$ is an equivalent martingale measure.

Define the dual value function $v(y)$ by

$$v(y) := \inf_{Q \in \mathcal{M}^a(S)} \sum_{n=1}^{N} p_n V\left(y \frac{q_n}{p_n}\right) \tag{2.53}$$

$$= \sum_{n=1}^{N} p_n V\left(y \frac{\hat{q}_n(y)}{p_n}\right). \tag{2.54}$$

Since $v(y)$ is a convex combination of V calculated on linearly scaled arguments, $v(y)$ has the same qualitative properties as the conjugate function V listed in its definition above. It can be also observed that $\lim_{y \downarrow 0} v'(y) = -\infty$ and $\lim_{y \uparrow +\infty} v'(y) \geq 0$. Combined with the strict convexity of $v(y)$, these limits imply that for fixed $x \in \mathrm{dom}(u)$ a unique $\hat{y}(x) > 0$ exists such that $v'(\hat{y}(x)) = -x$. A direct calculation yields $\Psi'(y) = v'(y) + x$. Thus, $\Psi'(\hat{y}(x)) = 0$ which due to V being strictly convex implies that $\hat{y}(x)$ is the unique minimizer of $\Psi(y)$.

The following formula for $u(x)$ is an immediate consequence of the two minimizing procedures above:

$$u(x) = \sum_{n=1}^{N} V\left(\hat{y} \frac{\hat{q}_n(y)}{p_n}\right) + \hat{y}x. \tag{2.55}$$

It still remains to calculate $V\left(\hat{y}\frac{\hat{q}_n(y)}{p_n}\right)$. By the definition of the conjugate function, this turns out to be again an optimization problem, namely, maximize

$$\Phi_n(\xi_n) := U(\xi_n) - \hat{y}\frac{\hat{q}_n(y)}{p_n}. \tag{2.56}$$

Differentiating this expression and setting the derivative to zero yields that ξ_n is an extreme point of $\Phi_n(\xi_n)$ if and only if

$$U'(\hat{\xi}_n) = \hat{y}(x)\frac{\hat{q_n}}{p_n} \tag{2.57}$$

or, equivalently,

$$\hat{\xi}_n = I\left(\hat{y}(x)\frac{\hat{q_n}}{p_n}\right) \tag{2.58}$$

where I denotes the inverse function of the utility function U. This is a maximum due to the concavity of $\Phi_n(\xi_n)$ which implies that $(\hat{\xi}_1, ..., \hat{\xi}_N)$ solves the original optimization problem (P"). The following theorem summarizes all the findings from above.

Theorem 2.1.11 (cf. [DS06, Theorem 3.2.1]). *Let the financial market $S = (S_t)_{t=0}^T$ be defined over the finite filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t=0}^T, P)$ and let $\mathcal{M}^e(S) \neq \emptyset$. Let the utility function U satisfy the assumptions in Section 2.1.2 and denote by $u(x)$ and $v(y)$ the value functions*

$$u(x) = \sup_{X_T \in C(x)} \mathbb{E}[U(X_T)], \qquad x \in dom(U), \tag{2.59}$$

$$v(y) = \inf_{Q \in \mathcal{M}^a(S)} \mathbb{E}[V(y\frac{dQ}{dP})], \qquad y > 0. \tag{2.60}$$

The following assertions hold:

(i) The value functions u(x) and v(x) are conjugate and u shares the qualitative properties of U listed in Section 2.1.2.

(ii) The optimizers $\hat{X}_T(x)$ and $\hat{Q}(y)$ in (2.59) and (2.60) exist, are unique, $\hat{Q}(y) \in \mathcal{M}^e(S)$ and satisfy

$$\hat{X}_T(x) = I\left(y\frac{d\hat{Q}(y)}{dP}\right), \qquad y\frac{d\hat{Q}(y)}{dP} = U'(\hat{X}_T(x)), \tag{2.61}$$

where $x \in dom(U)$ and $y > 0$ are related via $u'(x) = y$ or, equivalently, $x = -v'(y)$.

(iii) The following formulae for u' and v' hold true:

$$u'(x) = \mathbb{E}_P[U'(\hat{X}_T(x))], \qquad v'(y) = \mathbb{E}_{\hat{Q}}[V'\left(y\frac{d\hat{Q}(y)}{dP}\right)], \tag{2.62}$$

$$xu'(x) = \mathbb{E}_P[\hat{X}_T(x)U'(\hat{X}_T(x))], \qquad yv'(y) = \mathbb{E}_P[y\frac{d\hat{Q}(y)}{dP}V'\left(y\frac{d\hat{Q}(y)}{dP}\right)]. \tag{2.63}$$

Proof. While (i) and (ii) are evident from the preceeding discussion, it remains to show the assertions in (iii). The formula for $v'(y)$ in 2.62 follows by a direct calculation:

$$\frac{d}{dy}v(y) = \frac{d}{dy}\sum_{n=1}^N p_n V\left(y\frac{\hat{q}_n(y)}{p_n}\right) = \sum_{n=1}^N \hat{q}_n V'\left(y\frac{\hat{q}_n(y)}{p_n}\right) = \mathbb{E}_{\hat{Q}}[V'\left(y\frac{d\hat{Q}(y)}{dP}\right)] \tag{2.64}$$

The formula for the expression $yv'(y)$ then directly follows from (2.64). Finally, the formulae for $u'(x)$ and $xu'(x)$ are direct consequences of the formulae for $u'(x)$ and $xu'(x)$ by emploing the relations displayed in (ii). □

In the special case that the financial market is complete, the set of equivalent martingale measures $\mathcal{M}^e$ reduces to a singleton $\{Q\}$. Due to $\mathcal{M}^e$ being dense in the set of martingale measures $\mathcal{M}^a$ (see Proposition 2.1.1), this implies $\mathcal{M}^a = \{Q\}$. Thus, there is only one potential candidate for the dual optimizer in (2.60), namely Q.

The derivation of the statements in the proof above has demonstrated that the problem of expected utility maximization in the case of a finite probability space can be solved in a quite straightforward way. For general probability spaces, finding the primal and dual optimizers is much more delicate. In fact, the existence of the primal optimizer cannot be guaranteed for all kinds of financial markets and utility functions as Kramkov and Schachermayer [KS99] have shown. The essential notion of their analysis is the asymptotic elasticity of a utility function which was already introduced to this thesis in Section 2.1.2. The remainder of this section is devoted to briefly presenting their main results.

First of all, it is necessary to define suitable domains for the optimization problem. Since the previous definition of the sets of contingent claims superreplicable at price x, denoted by $C(x)$, does not hinge on the finiteness of underlying probability space, $C(x)$ may also serve as the domain for the primal optimizer in this general case. As regards the dual variables, a new domain has to be established: While it was shown above that the dual optimizer is an equivalent martingale measure if the financial market is based on a finite probability space, this may not be the case in general probability spaces as Example 2.1.14 below demonstrates. Therefore it is necessary to work with a larger set of dual variables. As mentioned in [CMKS12, p. 6], define $\mathcal{Y}(y)$ as the set of processes that turn all wealth processes with non-negative terminal pay-off into supermartingales, i.e.,

$$\begin{aligned}\mathcal{Y}(y) = \{Y = (Y)_{t=0}^T \geq 0 \mid Y_0 = y \text{ and } Y(\varphi^0 + \varphi^1 S) = (Y_t(\varphi_t^0 + \varphi_t^1 S_t))_{t=0}^T \\ \text{is a supermartingale for all } (\varphi^0, \varphi^1) \in \mathcal{A}(1)\}.\end{aligned}$$

The set $\mathcal{Y}(1)$ contains the density processes of equivalent martingale measures. Indeed, the density process $Y_t = \mathbb{E}[\frac{dQ}{dP}|\mathcal{F}_t]$ of $Q \in \mathcal{M}^e$ satisfies that $Y_0 = 1$ and that $Y(\varphi^0 + \varphi^1 S)$ is a martingale for all $(\varphi^0, \varphi^1) \in \mathcal{A}(1)$ by the definition of an equivalent martingale measure and the fact that φ^0 and φ^1 are predictable. In particular, it is a supermartingale and hence $Y \in \mathcal{Y}(1)$. Because only the terminal value is relevant for the optimization problem at hand, the set of random variables dominated by a dual element shall be defined by

$$\mathcal{D}(y) = \{h \in L_+^0(\Omega, \mathcal{F}, P) \mid h \leq f \text{ for } f \in \mathcal{Y}(y)\}. \tag{2.65}$$

Having established these suitable domains, the primal and dual value functions can be formulated as

$$u(x) = \sup_{g \in C(x)} \mathbb{E}[U(g)], \tag{2.66}$$

$$v(y) = \inf_{h \in \mathcal{D}(y)} \mathbb{E}[U(h)]. \tag{2.67}$$

In analogy with Theorem 2.1.11, one then obtains the following two theorems. While the first theorem holds true without any further assumptions on the utility function, the statements in the second theorem rely on the property of reasonable asymptotic elasticity. Note that only utility functions on the domain $(0,\infty)$ are considered. However, Kramkov and Schachermayer [KS99, p. 3] indicate that the same results hold also true for utility functions with domain $(-\infty,\infty)$. Whereas the original versions [KS99, Theorem 2.1 and Theorem 2.2] of the two theorems are formulated in a continuous time setting, the abstract versions [KS99, Theorem 3.1 and Theorem 3.2] also hold in the present discrete time framework since the sets $C(x)$ and $\mathcal{D}(x)$ are independent of the time being discrete or continuous due to their abstract definition.

Theorem 2.1.12 (cf. [KS99, Theorem 3.1]). *Assume the utility function U has domain $D=(0,\infty)$ and that $u(x)<\infty$ for some $x>0$. The following assertions hold true:*

(i) The value function satisfies $u(x)<\infty$ for all $x>0$, and there exists $y_0>0$ such that $v(y)$ is finitely valued for $y>y_0$. Furthermore, the value functions u and v are conjugate, u shares the qualitative properties of U listed in Section 2.1.2, the function v is strictly convex on $\{v<\infty\}$, and the functions u' and $-v'$ satisfy:

$$u'(0)=\lim_{x\to 0}u'(x)=\infty \quad \textit{and} \quad v'(\infty)=\lim_{y\to\infty}v'(y)=0. \tag{2.68}$$

(ii) If $v(y)<\infty$, then the optimal solution $\hat{h}(y)\in\mathcal{D}(y)$ in (2.67) exists and is unique.

Theorem 2.1.13 (cf. [KS99, Theorem 3.2]). *In addition to the assumptions of Theorem 2.1.12, also suppose that the asymptotic elasticity of the utility function U is strictly less than one, i.e. $AE(U)=\limsup_{x\to\infty}\frac{xU'(x)}{U(x)}<1$. Then, in addition to the assertions of Theorem 2.1.12, the following assertions hold:*

(i) The dual value function satisfies $v(y)<\infty$ for all $y>0$. Also, u and v are continuously differentiable on $(0,\infty)$, and the functions u' and $-v'$ are strictly decreasing and satisfy:

$$u'(\infty)=\lim_{x\to\infty}u'(x)=0, \quad -v'(\infty)=\lim_{x\to\infty}-v'(x)=\infty. \tag{2.69}$$

The asymptotic elasticity $AE(u)$ of u is less than or equal to the asymptotic elasticity of the utility function U, that is,

$$AE(u)^+\le AE(U)^+<1. \tag{2.70}$$

(ii) The optimizer $\hat{g}(x)\in C(x)$ exists and is unique. The following dual relation holds if $\hat{h}(y)\in\mathcal{D}(y)$ is the dual optimizer, where $y=u'(x)$:

$$\hat{g}(x)=I(\hat{h}(y)). \tag{2.71}$$

Moreover,

$$\mathbb{E}[\hat{g}(x)\hat{h}(y)] = xy. \tag{2.72}$$

(iii) The following relations between u', v' and $\hat{g}$, $\hat{h}$ hold:

$$u'(x) = \mathbb{E}[\frac{\hat{g}(y)U'(\hat{g}(x))}{x}], \quad v'(y) = \mathbb{E}[\frac{\hat{h}(y)V'(\hat{h}(x))}{y}]. \tag{2.73}$$

The authors of these results also show that the requirement of U having reasonable asymptotic elasticity is the minimal condition which implies any of the assertions in the latter theorem. For details, the interested reader may be referred to [KS99, Section 5] where several counterexamples are presented. One of these counterexamples can also be employed to demonstrate that the dual minimizer in $\mathcal{Y}(y)$ may fail to be a martingale. Recall that in the special case of a finite state space Ω the dual minimizer is always an equivalent martingale measure. Later on, this discrepancy will play a crucial role when investigating the existence of shadow prices in discrete time models.

Example 2.1.14 (cf. [KS99, Example 5.1 bis])**.** Consider again a one-period model where the stock price evolves as illustrated in Figure 2.2 below.

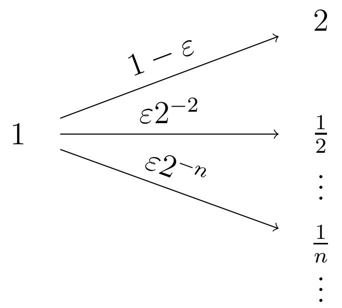

Figure 2.2: Evolution of the stock price

More precisely, define the sequence $(x_n)_{n=0}^{\infty}$ by $x_0 = 2$ and $x_n = \frac{1}{n}$ for $n \geq 1$. Let the corresponding probabilities be $p_0 = 1 - \varepsilon$ and $p_n = \varepsilon 2^{-n}$ for $n \geq 1$ where $0 < \varepsilon < 1$. Furthermore, ε shall be chosen small enough to satisfy

$$\frac{1}{2}\varepsilon + \varepsilon \sum_{n=1}^{\infty} 2^{-n}(1-n) > 0. \tag{2.74}$$

Then the initial price of the stock is given by $S_0 = 1$ and the terminal price S_1 takes on the values $(x_n)_{n=0}^{\infty}$ with probabilities $(p_n)_{n=0}^{\infty}$. The filtration $\mathcal{F}$ shall be the natural filtration generated by S. Since the price process can move both up and down with positive probability, it does not admit arbitrage, and hence $\mathcal{M}^e \neq \emptyset$.

Now consider a log-utility investor who wants to maximize expected utility from terminal wealth in this market. The investor's inital wealth shall amount 1. In this case, the set of admissible strategies consist of the processes $1 + \varphi(S_1 - 1)$ for $-1 \leq \varphi \leq 1$. Indeed, as the stock price may drop arbitrarily close to 0, the investor cannot buy more stock than the inital wealth allows. Otherwise, it could lead to instantaneous bankruptcy at the terminal time. For the same reason, the investor cannot sell more than one stock short. The corresponding optimization problem accordingly formulates as

$$\begin{cases} \text{maximize} & \mathbb{E}[\log(x + \varphi(S_1 - S_0)] \\ \text{subject to} & -1 \leq \varphi \leq 1. \end{cases} \tag{2.75}$$

The first two derivatives of the objective function $f(\varphi) = \mathbb{E}[\log(1 + \varphi(S_1 - 1)]$ are

$$f'(\varphi) = \sum_{n=0}^{\infty} p_n \frac{x_n - 1}{1 + \varphi(x_n - 1)}, \tag{2.76}$$

and

$$f''(\varphi) = -\sum_{n=0}^{\infty} p_n \left(\frac{x_n - 1}{1 + \varphi(x_n - 1)} \right)^2. \tag{2.77}$$

The second derivative is obviously strictly negative on the domain $-1 \leq \varphi \leq 1$. In combination with condition (2.74), the strictly decreasing monotonicity of $f'(\varphi)$ implies for $-1 \leq \varphi \leq 1$ that

$$f'(\varphi) > f'(1) = \frac{1}{2}\varepsilon + \varepsilon \sum_{n=1}^{\infty} 2^{-n}(1 - n) > 0. \tag{2.78}$$

Therefore, the objective function is strictly increasing on its domain and attains its maximum at $\hat{\varphi} = 1$. The primal optimizer $\hat{g}(1) = 1 + \hat{\varphi}(S_1 - 1)$ consequently coincides with S_1. If the initial wealth were x, this would clearly be xS_1. This allows calculating the value function directly:

$$u(x) = \mathbb{E}[\log(xS_1)] = \sum_{n=0}^{\infty} p_n \log(xx_n) = \log(x) + \sum_{n=0}^{\infty} p_n \log(x_n). \tag{2.79}$$

Thus, $u'(1) = 1$. Theorem 2.1.13 states that the dual optimizer $\hat{h}(y)$ with $y = u'(x)$ can be expressed as $\hat{h}(y) = U'(\hat{g}(x))$. In this case, this means $\hat{h}(1) = U'(\hat{g}(1)) = \hat{g}(1)^{-1} = S_1^{-1}$. The crucial observation is that this dual optimizer fails to be a martingale. Indeed, its expectation satisfies the following inequality by condition (2.74):

$$\mathbb{E}[S_1^{-1}] = \sum_{n=0}^{\infty} \frac{p_n}{x_n} = \frac{p_0}{2} + \sum_{n=1}^{\infty} np_n = \frac{1}{2}(1 - \varepsilon) + \varepsilon \sum_{n=1}^{\infty} n2^{-n} \tag{2.80}$$

$$< \frac{1}{2}(1 - \varepsilon) + \frac{1}{2}(1 - \varepsilon) + \varepsilon \sum_{n=1}^{\infty} 2^{-n} = 1. \tag{2.81}$$

This inequality shows that a corresponding dual optimizer $\hat{Y} \in \mathcal{Y}(1)$ with $\hat{Y}_1 = \hat{h}(1)$ is only a supermartingale since $\hat{Y}_0 = 1$ by definition of $\mathcal{Y}(1)$. As a result, it cannot be the density of a martingale measure for the process S. ◁

2.2 The Case with Transaction Costs

In this section, transaction costs will be added to the existing financial market and it will be examined to what extent this modification alters the process of utility maximization.

2.2 The Financial Market Model with Transaction Costs

Consider again a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ with finite time set $\{0, 1, ..., T\}$ and a financial market S defined as in Section 2.1.1. The existence of transaction costs in this market can be expressed as follows: While the **bid price** $\underline{S}_t^i$ of asset i $(i = 1, ..., d)$ is the price an investor receives when selling asset S^i at time t, he has to pay the higher **ask price** $\overline{S}_t^i$ for purchasing it. The interval $[\underline{S}^i, \overline{S}^i]$ is called the **bid-ask spread** of asset i. It will be assumed that trading the riskless asset does not incur transaction costs. In the present setting, this assumption is clearly justified as the riskless asset is perceived as a bank account. Define the **financial market with transaction costs** as the pair $(\underline{S}, \overline{S})$, where $\underline{S} = (\underline{S}^1, ..., \underline{S}^d)$ and $\overline{S} = (\overline{S}^1, ..., \overline{S}^d)$. The following examples give an overview on the various types of transaction costs being widely studied in the literature:

Example 2.2.1 (Proportional transaction costs, e.g. [DN90], [SS94], [CSZ12], [CMKS12])**.** Consider a frictionless price process S and a fixed number $\lambda \in [0, 1)$. The market with transaction costs is then given by the pair $((1 - \lambda)S, S)$. This means that the transaction costs are proportional to the size of the transaction. More generally, fix two constants $\underline{\lambda} \in [0, 1)$ and $\overline{\lambda} \in [0, \infty)$ and define $\underline{S} = (1 - \underline{\lambda})S$ and $\overline{S} = (1 + \overline{\lambda})S$. ◁

Example 2.2.2 (Kabanov's multi-currency market model, e.g. [Kab99], [Sch04], [CS06])**.** Consider again a frictionless price process S. Unlike in Example 2.2.1, it is now assumed that the size of the transaction costs depends on which assets are being exchanged. To this end, define a non-negative $(d+1) \times (d+1)$ matrix $\Lambda = (\lambda^{ij})_{1 \le i,j \le d}$ of transaction cost coefficients. One then has to pay $(1 + \lambda^{ij})$ units of asset i for receiving one unit of asset j. In order to ensure that a chain of exchanges is less profitable than a direct exchange, it is necessary to impose

$$(1 + \lambda^{ij}) \le (1 + \lambda^{ik})(1 + \lambda^{kj}), \text{ for } 1 \le i, j, k \le d + 1. \tag{2.82}$$

The framework described in the beginning of this section fails to cover this very general market model because the bid and ask prices cannot be modeled as a pair $(\underline{S}, \overline{S})$ anymore.

Instead, one defines a so-called **bid-ask matrix** $\Pi = (\pi^{ij})_{1\le i,j,\le d+1}$ with the the properties

(i) $\pi^{ij} > 0$ for every $1 \le i,j \le d+1$,
(ii) $\pi^{ii} = 1$ for every $1 \le i \le d+1$, and
(iii) $\pi^{ij} \le \pi^{ik}\pi^{kj}$ for every $1 \le i,j,k \le d+1$.

The entries π^{ij} represent the ask price of asset j denoted in units of asset i. In framework of this section, the prices of the risky assets are denoted in terms of the bank account which serves as the numéraire. By setting $\pi^{1j} = \overline{S}^j$ for $1 < j \le d+1$, $\pi^{i1} = \frac{1}{\underline{S}^i}$ for $1 < i \le d+1$ and the remaining entries sufficiently high, it follows that the chosen notion of transaction costs in this section can be treated as a special case of this very general market model. ◁

Example 2.2.3 (Fixed transaction costs, e.g. [MP95, Sch95], [Sch95])**.** The size of the transaction costs is a fraction of the portfolio's value, i.e. $\overline{S} = S$ for a frictionless price process S and $\underline{S} = S - \lambda X$ where X denotes the stochastic process representing the value of the portfolio and $\lambda \in [0,1)$ the proportional size of the transaction costs. As in the presence of bid and ask prices the value of a portfolio is not unique, one usually defines the value of the portfolio as the value that can be obtained by liquidating it. While this approach enables rather simple solutions for portfolio optimization problems (e.g. cf. [MP95]), it hardly reflects transaction costs that appear in real financial markets. For a thorough review of this approach, see Korn [Kor04]. ◁

Almost all models and results on shadow prices in the literature are developed under the assumption of proportional transaction costs. Thus, most parts of this thesis will stay in this realm. More specifically, for the remainder of this section, fix a constant $\lambda \in [0,1)$ and define $\underline{S} = (1-\underline{\lambda})S$ and $\overline{S} = S$ for a frictionless price process S. Due to the presence of transaction costs, it is required to reformulate some of the definitions established in the frictionless case. A portfolio/consumption pair $((\varphi^0,\varphi),c)$ is called **self-financing** if

$$\Delta\varphi^0_{t+1} = \underline{S}_t^\top \Delta\varphi^\downarrow_{t+1} - \overline{S}_t^\top \Delta\varphi^\uparrow_{t+1} - c_t \qquad \text{for } t = 0,1,...,T \tag{2.83}$$

where $\Delta\varphi^\downarrow := (\Delta\varphi)^-$ and $\Delta\varphi^\uparrow := (\Delta\varphi)^+$ denote the sale and purchase processes. This equation resembles that the purchases $\Delta\varphi^\uparrow$ which incur at the ask price $\overline{S}$ are subtracted from the bank account, while the sales $\Delta\varphi^\downarrow$ increase the cash holding by the bid price $\underline{S}$. Note that for $\underline{S} = \overline{S}$ the self-financing condition reduces to the already established definition in the case without transaction costs. While **admissibility** is defined as in Section 2.1.1, the set of admissible strategies starting from an initial endowment of $(x,0)$ under proportional transaction costs λ will be denoted by $\mathcal{A}^\lambda(x)$. A formula for the liquidation value of an admissible portfolio/consumption pair $((\varphi^0,\varphi),c) \in \mathcal{A}^\lambda(x)$ at the terminal time T in the presence of transaction costs can be obtained by the same arguments as above:

$$c_T = x + (\varphi^{\downarrow} \cdot \underline{S})_T - (\varphi^{\uparrow} \cdot \overline{S})_T - \sum_{t=1}^{T} c_{t-1} \tag{2.84}$$

where $\varphi^{\downarrow}$ and $\varphi^{\uparrow}$ denote the cumulated sale and purchase processes, i.e.

$$\varphi_t^{\downarrow} = (\varphi_0)^- + \sum_{s=1}^{t} \Delta\varphi_s^{\downarrow}, \quad \varphi_t^{\uparrow} = (\varphi_0)^+ + \sum_{s=1}^{t} \Delta\varphi_s^{\uparrow}, \quad \text{for } t = 1, ..., T. \tag{2.85}$$

As a result, the set of **contingent claims attainable at price x** is defined by

$$K^{\lambda}(x) = \{x + (\varphi^{\downarrow} \cdot \underline{S})_T - (\varphi^{\uparrow} \cdot \overline{S})_T \mid (\varphi^0, \varphi) \in \mathcal{A}^{\lambda}(x)\} \tag{2.86}$$

Similar to the frictionless case, the convex cone $C^{\lambda}(x)$ in $L^{\infty}(\Omega, \mathcal{F}, P)$, called the set of **contingent claims super-replicable at price x**, consists of random terminal pay-offs which can be dominated by an admissible self-financing strategy starting from an initial cash endowment of x, i.e.

$$C^{\lambda}(x) = \{g \in L^{\infty}(\Omega, \mathcal{F}, P) \mid \text{there exists } f \in K^{\lambda}(x) \text{ with } f \geq g\} \tag{2.87}$$

The set of **cosumption processes attainable at price x** is

$$\mathcal{C}^{\lambda}(x) = \{c \mid \text{there is a trading strategy } (\varphi^0, \varphi) \text{ such that } (\varphi^0, \varphi, c) \in \mathcal{A}^{\lambda}(x)\}. \tag{2.88}$$

In the special case $\lambda = 0$, these notations correspond naturally with $K(x)$, $C(x)$ and $\mathcal{C}(x)$ as defined in the frictionless case. Moreover, $K^{\lambda}(x)$ is a subspace of $L^0(\Omega, \mathcal{F}, P)$.

Again, the connection to arbitrage is evident, and one says that the frictious financial market S satisfies the **no-arbitrage condition (NA$^{\lambda}$)** if it does not allow for an arbitrage under transaction costs λ, that is,

$$K^{\lambda}(0) \cap L^0_+(\Omega, \mathcal{F}, P) = 0 \tag{2.89}$$

or, equivalently,

$$C^{\lambda}(0) \cap L^0_+(\Omega, \mathcal{F}, P) = 0. \tag{2.90}$$

The reasonings for this definition are exactly the same as in Section 2.1.1. As it will turn out, in the presence of transaction cost there is also a convenient connection between the no-arbitrage condition and martingale theory.

For reasons of simplicity, from now on only one risky asset will be considered, i.e. $d = 1$. Following [Sch11, Definition 1.6], a **consistent price system** will be defined as a pair $(\tilde{S}, Q)$, such that Q is a probability measure on $\mathcal{F}$ equivalent to P, and $\tilde{S} = (\tilde{S})_{t=0}^{T}$ is a martingale under Q taking its values in the bid-ask spread $[(1-\lambda)S, S]$, i.e.

$$(1-\lambda)S_t \leq \tilde{S} \leq S_t, \qquad \text{for each } \omega \in \Omega. \tag{2.91}$$

Denote the set of consistent price systems by $\mathcal{S}^\lambda$. The set of processes $\tilde{S}$ will be written as $\mathcal{Z}^\lambda$. In the presence of transaction costs, $\mathcal{S}^\lambda$ plays the role of the set of equivalent martingale measures $\mathcal{M}^e$ in the frictionless case: If there exists a consistent price system, then the market is free of arbitrage. This is the main result of the following theorem.

Theorem 2.2.4 (Fundamental Theorem of Asset Pricing, cf. [Sch11, Theorem 1.19])**.** *Fixing the process $S = (S_t)_{t=0}^T$ and transaction costs $0 \leq \lambda < 1$, the following are equivalent:*

(i) The no arbitrage condition (NA^λ) is satisfied.

(ii) There is a consistent price system $(\tilde{S}, Q) \in S^\lambda$.

(iii) There is an $\mathbb{R}^2$-valued P-martingale $(Z_t)_{t=0}^T = (Z_t^0, Z_t^1)_{t=0}^T$ such that $Z_t^0 > 0$, $Z_t^1 > 0$ and

$$\frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t, S_t], \qquad \textit{for } t = 0, ..., T. \tag{2.92}$$

In order to build the bridge from the Fundamental Theorem of Asset Pricing to the arbitrage-free pricing of contingent claims, some essential definitions have to be established first. Following [CMKS12, p. 7], define for $y > 0$

$$\mathcal{B}^\lambda(y) = \{(Y^0, Y^1) \geq 0 \mid Y_0^0 = y,\ \tfrac{Y^1}{Y^0} \in [(1-\lambda)S, S] \text{ and } Y^0\varphi^0 + Y^1\varphi^1 \text{ is a}$$
$$\text{non-negative supermartingale for all } (\varphi^0, \varphi^1) \in \mathcal{A}(1)\}$$

This can be understood as the set of frictionless two-dimensional price processes (Y^0, Y^1) that turn the wealth process corresponding to a self-financing strategy starting at an initial wealth of 1 into a non-negative supermartingale. The set of random variables dominated by a price process in $\mathcal{B}^\lambda(y)$ at the terminal time shall be defined as

$$\mathcal{D}^\lambda(y) = \{h \in L_+^0(\Omega, \mathcal{F}, P) \mid \text{there is } (Y^0, Y^1) \in \mathcal{B}^\lambda(y) \text{ with } h \leq Y_T^0\} \tag{2.93}$$

for $y > 0$. In addition, $\mathcal{M}^\lambda$ denotes the set of probability measures whose densities lie in $\mathcal{D}(1)$, i.e.

$$\mathcal{M}^\lambda = \{Q \mid Y = \tfrac{dQ}{dP} \in \mathcal{D}(1) \text{ and } \mathbb{E}_P[Y] = 1\}. \tag{2.94}$$

Then, the following counterpart of the frictionless Superreplication Theorem can be obtained in order to price assets:

Proposition 2.2.5 (Superreplication Theorem, cf. [Sch11, Corollary 1.15])**.** *Fix the process $S = (S_t)_{t=0}^T$, transaction costs $0 \leq \lambda < 1$, and suppose that (NA^λ) is satisfied. Let $\varphi^0 \in L^\infty(\Omega, \mathcal{F}, P)$ and $x \in \mathbb{R}$ be given. The following are equivalent:*

(i) φ^0 is in $C^\lambda(x)$.

(ii) $\mathbb{E}_Q[\varphi^0] \le x$ *for every* $Q \in \mathcal{M}^\lambda$.

2.2 Optimal Portfolios

The definitions of the optimal portfolios in Section 2.1.3 can evidently be also used for portfolio/consumption pairs in markets with transaction costs. Nevertheless, the process of finding the optimal solution may change dramatically as the following example shows.

Example 2.2.6 (One-period binomial model)**.** Consider again the frictionless stock price S from the one-period binomial model developed in Example 2.1.10, and define the frictious financial market $(\underline{S}, \overline{S})$ by $\underline{S} = (1-\lambda)S$ and $\overline{S} = S$ for a fixed $\lambda \in (0,1)$. Like in the example above, the aim is to derive the optimal strategy of an investor with initial endowment $(x, 0)$ who seeks to maximize expected utility from terminal wealth. In the presence of proportional transaction costs, this objective can be expressed as

$$\text{(P)} \begin{cases} \text{maximize} & \mathbb{E}[\log(c_1)] = \mathbb{E}[\log(x + \varphi_1^{\downarrow}(\underline{S}_0 - \overline{S}_1) + \varphi_1^{\uparrow}(\underline{S}_1 - \overline{S}_0))] \\ \text{subject to} & \varphi_1 \in \mathbb{R} \end{cases} \tag{2.95}$$

where (2.84) and $c_0 = 0$ are used. As explained in Example 2.1.10, the maximization can be performed over all $\varphi_1 \in \mathbb{R}$ without violating the admissibility condition. In order to solve this problem, the idea is to treat sales and purchases seperately. Instead of maximizing over a sales/purchase number $\varphi_1 \in \mathbb{R}$, the maximization shall be done over the amount of sales $\varphi_1^{\downarrow} \in \mathbb{R}_{\ge 0}$ and the amount of purchases $\varphi_1^{\uparrow} \in \mathbb{R}_{\ge 0}$. This leads to the following ordinary convex program over the domain $C = \mathbb{R}^2$:

$$\begin{cases} \text{minimize} & f(\varphi_1^{\downarrow}, \varphi_1^{\uparrow}) := -\mathbb{E}[\log(x + \varphi_1^{\downarrow}(\underline{S}_0 - \overline{S}_1) + \varphi_1^{\uparrow}(\underline{S}_1 - \overline{S}_0))] \\ \text{subject to} & g_1(\varphi_1^{\downarrow}, \varphi_1^{\uparrow}) := -\varphi_1^{\downarrow} \le 0, \\ & g_2(\varphi_1^{\downarrow}, \varphi_1^{\uparrow}) := -\varphi_1^{\uparrow} \le 0. \end{cases} \tag{2.96}$$

While formulating this simple optimization problem as an ordinary convex program might seem like shooting pigeons with canons, it shall serve as a preparation for the proof of the existence of shadow prices in finite probability spaces later on.

According to Proposition A.3.2, an optimal solution to this program exists. Theorem A.3.3 guarantees that the optimal solution is given by $(\overline{\varphi}_1^{\downarrow}, \overline{\varphi}_1^{\uparrow}) \in \mathbb{R}^2$ for which there exist μ_1 and μ_2 such that

(i) $\mu_1 \ge 0$, $\mu_1 \ge 0$, $g_1(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) \le 0$, $g_2(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) \le 0$
and $\mu_1 g_1(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) = 0$, $\mu_2 g_2(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) = 0$,

(ii) $\nabla f(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) + \mu_1 \nabla g_1(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) + \mu_1 \nabla g_2(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) = 0$.

Condition (ii) implies

$$\mu_1 = \frac{d}{d\varphi_1^{\downarrow}} f(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) \qquad \text{and} \qquad \mu_1 = \frac{d}{d\varphi_1^{\downarrow}} f(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}). \tag{2.97}$$

Inserting these two equations into the second line of (i) yields the following candidates for optimal solutions and the corresponding values of μ_1 and μ_2 which can be obtained by direct computations.

– *Case 1*:

$$\hat{\varphi}_1^{\downarrow} = 0, \quad \hat{\varphi}_1^{\downarrow} = -\frac{((1-p)(1-\lambda)d + p(1-\lambda)u - 1)x}{(1-(1-\lambda)d)(1-(1-\lambda)u)S}, \tag{2.98}$$

$$\mu_1 = \frac{(\lambda-2)\lambda S}{(\lambda-1)x}, \quad \mu_2 = 0. \tag{2.99}$$

– *Case 2*:

$$\hat{\varphi}_1^{\downarrow} = 0, \quad \hat{\varphi}_2^{\downarrow} = 0, \tag{2.100}$$

$$\mu_1 = \frac{(pu + (1-p)d - (1-\lambda)S}{x}, \quad \mu_2 = \frac{(1-(1-\lambda)(pu+(1-p)d))S}{x}. \tag{2.101}$$

– *Case 3*:

$$\hat{\varphi}_1^{\downarrow} = -\frac{(1-\lambda+(1-p)d+pu)x}{(1-(1-\lambda)d)(1-(1-\lambda)u)S}, \quad \hat{\varphi}_2^{\downarrow} = 0, \tag{2.102}$$

$$\mu_1 = 0, \quad \mu_2 = \frac{(2-\lambda)\lambda S}{x}. \tag{2.103}$$

In order to obtain the optimal solution, it is further required to check the inequality conditions in the first line of (i). It can be observed that Case 1 is optimal, that is, $\mu_1 \geq 0$, $\mu_1 \geq 0$, $g_1(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) \leq 0$ and $g_2(\hat{\varphi}_1^{\downarrow}, \hat{\varphi}_1^{\uparrow}) \leq 0$, if and only if

$$0 \leq pu + (1-p)d \leq \frac{1}{1-\lambda}. \tag{2.104}$$

For Case 2, the following optimality inequality can be derived:

$$1-\lambda \leq pu + (1-p)d \leq \frac{1}{1-\lambda}. \tag{2.105}$$

Finally, the values in Case 3 are nonnegative and hence optimal if and only if

$$0 \leq pu + (1-p)d \leq 1-\lambda. \tag{2.106}$$

In summary, the optimal amount of wealth invested in the risky asset (i.e. $\hat{\varphi}_1 = \hat{\varphi}_1^{\uparrow} - \hat{\varphi}_1^{\downarrow}$)

in order to attain the highest expected utility depends on the size of the transaction costs and is given by

$$\hat{\varphi}_1 = \begin{cases} \frac{(1-\lambda)(pu+(1-p)d-1)x}{(1-(1-\lambda)d)((1-\lambda)u-1)S}, & \text{for } \frac{1}{1-\lambda} \leq \mathbb{E}[\frac{S_1}{S_0}], \\ 0, & \text{for } 1-\lambda \leq \mathbb{E}[\frac{S_1}{S_0}] \leq \frac{1}{1-\lambda}, \\ -\frac{((1-\lambda)-pu-(1-p)d)x}{((1-\lambda)-d)(u-(1-\lambda))S}, & \text{for } 0 \leq \mathbb{E}[\frac{S_1}{S_0}] \leq 1-\lambda, \end{cases} \tag{2.107}$$

where $\mathbb{E}[\frac{S_1}{S_0}] = pu + (1-p)d$ was used. This allows for a clear economic interpretation and supports what was already hinted in the introduction: Do not trade if the expected returns are canceled out by the transaction costs incurred by a trade. Purchase stock if it yields a higher expected return than the riskless asset plus transaction costs, and sell stock if the expected return plus transaction costs is less than the return of the riskless asset. ◁

2.2 The Dual Approach with Transaction Costs

The goal of this section is to apply the dual appraoch developed in Section 2.1.4 to the the present financial market model with transaction costs. The focus will be placed again on the problem of maximizing expected utility from terminal wealth.

First of all, consider the case of a finite probability space, i.e. $|\Omega| < \infty$. As explained above, the present utility maximization problem can be expressed as

$$(\mathrm{P}) \begin{cases} \text{maximize} & \mathbb{E}_P[U(x + (\varphi^{\uparrow} \cdot \overline{S})_T + (\varphi^{\uparrow} \cdot \underline{S})_T)] \\ \text{subject to} & (\varphi^0, \varphi) \in \mathcal{A}^{\lambda}(x). \end{cases} \tag{2.108}$$

In order for this problem to make sense, it will be assumed throughout this section that the no-arbitrage condition (NA^{λ}) is satisfied.

The primal value function of this problem may be written as

$$u(x) = \sup_{X_T \in C^{\lambda}(x)} \mathbb{E}[U(X_T)] \qquad \text{for } x > 0. \tag{2.109}$$

Now consider a random terminal pay-off $X_T \in L^{\infty}(\Omega, \mathcal{F}, P)$ and denote the possible outcomes $X_T(\omega_n)$ for $\omega_n \in \Omega$ by ξ_n. The Superreplication Theorem for markets with transaction costs guarantees that this contingent claim can be superreplicated if and only if $\mathbb{E}_Q[X_T] \leq x$ for every $Q \in \mathcal{M}^{\lambda}$. By denoting $Q = (q_1, ..., q_N)$, the original problem can therefore be reformulated as

$$\text{(P')} \begin{cases} \text{maximize} & \sum_{n=1}^{N} p_n U(\xi_n) \\ \text{subject to} & \sum_{n=1}^{N} q_n U(\xi_n) \leq x, \qquad \text{for all } Q \in \mathcal{M}^\lambda(S), \\ & (\xi_1, ..., \xi_N) \in \mathbb{R}^N. \end{cases} \tag{2.110}$$

It clearly follows from the definition of the set of probability measures $\mathcal{M}^\lambda(S)$ that it is a compact polyhedron. which means that the infinitely many constraints of (P') can be replaced by finitely many. Denote the extreme points of $\mathcal{M}^\lambda(S)$ by $(Q^1, ..., Q^M)$. It hence suffices to consider

$$\text{(P")} \begin{cases} \text{maximize} & \sum_{n=1}^{N} p_n U(\xi_n) \\ \text{subject to} & \sum_{n=1}^{N} q_n U(\xi_n) \leq x, \qquad \text{for all } Q^m \in \{Q_1, ..., Q^M\}, \\ & (\xi_1, ..., \xi_N) \in \mathbb{R}^N. \end{cases} \tag{2.111}$$

which is exactly the same problem as in the frictionless case. As a result, the same arguments as above can be employed to solve it. In doing so, one obtains the dual problem

$$\text{(D)} \begin{cases} \text{minimize} & \sum_{n=1}^{N} p_n V(y\frac{dQ}{dP}) \\ \text{subject to} & y > 0 \\ & Q = (q_1, ..., q_N) \in \mathcal{M}^\lambda(S). \end{cases} \tag{2.112}$$

The value function corresponding to this dual problem is given by

$$\begin{aligned} v(y) &= \inf\{\mathbb{E}[V(y\tfrac{dQ}{dP})] \mid Q \in \mathcal{M}^\lambda(S)\} & (2.113)\\ &= \inf\{\mathbb{E}[V(yZ_T^0)] \mid Z_T^0 \in \mathcal{D}(1) \text{ and } \mathbb{E}_P[Z_T^0] = 1\} & (2.114)\\ &= \inf\{\mathbb{E}[V(Y_T^0)] \mid \tfrac{Y_T^0}{y} \in \mathcal{D}(1) \text{ and } \mathbb{E}_P[\tfrac{Y_T^0}{y}] = 1\} & (2.115)\\ &= \inf\{\mathbb{E}[V(Y_T^0)] \mid Y_T^0 \in \mathcal{D}(y) \text{ and } \mathbb{E}_P[Y_T^0] = y\} & (2.116)\\ &= \inf\{\mathbb{E}[V(Y_T^0)] \mid Y = (Y^0, Y^1) \in \mathcal{B}(y) \text{ and } \mathbb{E}_P[Y_T^0] = y\}, & (2.117) \end{aligned}$$

and by the same arguments as in the frictionless case one obtains the following results.

Theorem 2.2.7 (cf. [Sch11, Theorem 2.2.]). *Fix $0 \leq \lambda \leq 1$ and suppose that (NA^λ) is satisfied. Let the utility function U satisfy the assumptions in Section 2.1.2 and denote by $u(x)$ and $v(y)$ the primal and dual value functions*

$$\begin{aligned} u(x) &= \sup\{\mathbb{E}[U(X_T)] \mid X_T \in C(x)\}, & x \in \textit{dom(U)}, \quad (2.118)\\ v(y) &= \inf\{\mathbb{E}[V(y\frac{dQ}{dP})] \mid Q \in \mathcal{M}^a(S)\} & (2.119)\\ &= \inf\{\mathbb{E}[V(Y_T^0)] \mid Y = (Y^0, Y^1) \in \mathcal{B}^\lambda(y) \text{ and } \mathbb{E}_P[Y_T^0] = y\}, & y > 0. \quad (2.120) \end{aligned}$$

The following assertions hold:

(i) *The value functions u and v are conjugate and u shares the qualitative properties of U listed in Section 2.1.2.*

(ii) *The optimizers $\hat{X}_T(x)$ and $\hat{Q}(y)$ in (2.118) and (2.119) exist, are unique, $\hat{Q}(y) \in \mathcal{M}^e(S)$ and satisfy*

$$\hat{X}_T(x) = I\left(y\frac{d\hat{Q}(y)}{dP}\right), \qquad y\frac{d\hat{Q}(y)}{dP} = U'(\hat{X}_T(x)), \tag{2.121}$$

where $x \in dom(U)$ and $y > 0$ are related via $u'(x) = y$ or, equivalently, $x = -v'(y)$.

(iii) *The following formulae for u' and v' hold true:*

$$u'(x) = \mathbb{E}_P[U'(\hat{X}_T(x))], \qquad v'(y) = \mathbb{E}_{\hat{Q}}[V'\left(y\frac{d\hat{Q}(y)}{dP}\right)], \tag{2.122}$$

$$xu'(x) = \mathbb{E}_P[\hat{X}_T(x)U'(\hat{X}_T(x))], \quad yv'(y) = \mathbb{E}_P[y\frac{d\hat{Q}(y)}{dP}V'\left(y\frac{d\hat{Q}(y)}{dP}\right)]. \tag{2.123}$$

Proof. Whereas the assertions in (i) and (ii) follow immediately from the preceeding discussion, the formulas in (iii) can be verified by the exact same calculations as in the proof of Theorem 2.1.11. □

The major take away point for the subsequent analysis on the existence on shadow prices in discrete time is that the dual minimizer is a martingale. In passing to financial markets which are based on probability spaces that are not necessarily finite, the results of the theorem above transform to the statements in the theorem below. Note that the utility function is required to satisfy the property of reasonable asymptotic elasticity. The proof for the theorem can be conducted similarly as the proof of its frictionless counterpart, that is, by verifying that the necessary conditions of [KS99, Theorem 3.1 and Theorem 3.1] also hold in the present framework with proportional transaction costs. For details the interested reader may be referred to [CMKS12, Appendix B].

Theorem 2.2.8 (cf. [CMKS12, Theorem 4.2.]). *Fix $0 \leq \lambda \leq 1$ and suppose that S satisfies $(CPS^{\lambda'})$ for some $\lambda' \in [0, \lambda)$. Furthermore, the asymptotic elasticity of U shall be strictly less than one and $u(x) < \infty$ for some $x \in (0, \infty)$. Then:*

(i) *The primal value function u and the dual value function v are conjugate and continuously differentiable on $(0, \infty)$. The functions u and $-v$ are strictly concave, strictly increasing, and satisfy the Inada conditions.*

(ii) *For all $x, y > 0$, the optimizers $\hat{g}(x) \in C^\lambda(x)$ and $\hat{h}(y) \in \mathcal{D}^\lambda(y)$ exist, are unique, and there are $(\hat{\varphi}^0(x), \hat{\varphi}^1(x)) \in \mathcal{A}^\lambda(x)$ and $(\hat{Y}^0(y), \hat{Y}^1(y)) \in \mathcal{B}^\lambda(y)$ such that*

$$\hat{\varphi}^0_{T+1}(x) = \hat{g}(x) \quad \textit{and} \quad \hat{Y}^0_T(y) = \hat{h}(y). \tag{2.124}$$

(iii) For all $x > 0$, let $\hat{y}(x) = u'(x) > 0$ which is the unique solution to

$$v(y) + xy \to \ min!, \qquad for\ y > 0. \tag{2.125}$$

Then, the optimizers $\hat{g}(x)$ and $\hat{h}(\hat{y}(x))$ are related via

$$\hat{g}(x) = I(\hat{h}(\hat{y}(x))), \qquad \hat{h}(\hat{y}(x)) = U'(\hat{g}(x)), \tag{2.126}$$

and $\mathbb{E}[\hat{g}(x)\hat{h}(\hat{y}(x))] = x\hat{y}(x)$. In particular, the process

$$\hat{Y}^0(\hat{y}(x))\hat{\varphi}^0(x) + \hat{Y}^1(\hat{y}(x))\hat{\varphi}^1(x) = \Big(\hat{Y}^0_t(\hat{y}(x))\hat{\varphi}^0_t(x) + \hat{Y}^1_t(\hat{y}(x))\hat{\varphi}^1_t(x)\Big)_{0\leq t\leq T} \tag{2.127}$$

is a martingale for $(\hat{\varphi}^0(x), \hat{\varphi}^1(x)) \in \mathcal{A}^\lambda(x)$ and $(\hat{Y}^0(\hat{y}(x)), \hat{Y}^1(\hat{y}(x))) \in \mathcal{B}^\lambda(\hat{y}(x))$ satisfying (2.124) with $y = \hat{y}(x)$.

(iv) Moreover,

$$\hat{Y}^0(\hat{y}(x))\hat{\varphi}^0(x) + \hat{Y}^1(\hat{y}(x))\hat{\varphi}^1(x) = \hat{Y}^0(\hat{y}(x))\left(x + \hat{\varphi}^1(x)\cdot\tfrac{\hat{Y}^1}{\hat{Y}^0}\right) \tag{2.128}$$

which implies that $\{\Delta\hat{\varphi}^1_{t+1} > 0\} \subset \{\frac{\hat{Y}^1_t}{\hat{Y}^0_t} = S_t\}$ and $\{\Delta\hat{\varphi}^1_{t+1} < 0\} \subset \{\frac{\hat{Y}^1_t}{\hat{Y}^0_t} = (1-\lambda)S_t\}$ for $t = 0, ..., T$.

(v) Finally, $v(y) = \inf\{\mathbb{E}[V(yZ^0_T)] \mid (Z^0, Z^1) \in \mathcal{Z}^\lambda\}$.

Just as in the frictionless case, the major difference between the two theorems above is that while the dual optimizer is always a martingale in a finite financial market, this property might not be the case in general models. The following example illustrates this assertion.

Example 2.2.9. Consider again the same one-period model as in Example 2.1.14: The sequence $(x_n)_{n=0}^\infty$ is defined by $x_0 = 2$ and $x_n = \frac{1}{n}$ for $n \geq 1$. The corresponding probabilites are $p_0 = 1-\varepsilon$ and $p_n = \varepsilon 2^{-n}$ for $n \geq 1$ where $0 < \varepsilon < 1$. Now, proportional transaction costs shall be introduced to this model. For fixed $0 \leq \lambda < 1$, the ask price process $(\overline{S})_{t=0}^1$ is then given by $\overline{S}_0 = 1$ and $\overline{S}_1(\omega_n) = x_n$ for $\omega_n \in \Omega$ while the corresponding bid price is simply $\underline{S} = (1-\lambda)\overline{S}$.

In order to make the arguments from Example 2.1.14 work in the present framework with transaction costs, the weights of the probability measure need to be slightly changed. In this case, ε shall be chosen small enough to satisfy

$$(1-\varepsilon)(\frac{1}{2} - \lambda) + \varepsilon\sum_{n=0}^\infty 2^{-n}(1-\lambda-n) > 0 \tag{2.129}$$

Again, the aim is to find the optimal strategy of a log-utility investor who wants to maximize expected utility from terminal wealth in this market, with the initial wealth being 1. The overwhelming probabilty of the stock going up suggests that the optimal strategy does not involve short selling. This claim can be made precise by following the same procedure as in Example 2.2.6. It therefore suffices to consider the optimization problem

$$\text{(P)} \begin{cases} \text{maximize} & \mathbb{E}[1 + \varphi((1-\lambda)S_1 - S_0)] \\ \text{subject to} & 0 \leq \varphi \leq 1. \end{cases} \tag{2.130}$$

The first two derivatives of the objective function $f(\varphi) = 1 + \varphi((1-\lambda)S_1 - S_0)$ are

$$f'(\varphi) = \sum_{n=0}^{\infty} p_n \frac{(1-\lambda)x_n - 1}{1 + \varphi((1-\lambda)x_n - 1)} \tag{2.131}$$

and

$$f''(\varphi) = -\sum_{n=0}^{\infty} p_n \left(\frac{(1-\lambda)x_n - 1}{1 + \varphi((1-\lambda)x_n - 1)} \right)^2. \tag{2.132}$$

The first derivative is strictly decreasing on its domain because the second derivative is strictly negative. This implies for $0 \leq \varphi \leq 1$ that

$$f'(\varphi) > f'(1) = \frac{1}{1-\lambda} \left((1-\varepsilon)(\frac{1}{2} - \lambda) + \varepsilon \sum_{n=0}^{\infty} 2^{-n}(1 - \lambda - n) \right) > 0 \tag{2.133}$$

where the inequality in (2.129) was used. As a result, the objective function is strictly increasing on its domain which implies that the optimal value is attained at $\hat{\varphi} = 1$ and that the primal optimizer $\hat{g}(1) = 1 + \hat{\varphi}((1-\lambda)S_1 - 1)$ coincides with the bid price $(1-\lambda)S_1$. The value function is then given by

$$u(x) = \mathbb{E}[\log(x(1-\lambda)S_1)] = \sum_{n=0}^{\infty} p_n \log(x(1-\lambda)x_n) \tag{2.134}$$

$$= \log(x) + \sum_{n=0}^{\infty} p_n \log((1-\lambda)x_n), \tag{2.135}$$

and $u'(1) = 1$. According to Theorem 2.2.8, the dual optimizer $\hat{h}(y)$ where $y = u'(x)$ can be expressed as $\hat{h}(y) = U'(\hat{g}(x))$. For the present case with $x = 1$, this means $\hat{h}(1) = U'(\hat{g}(1)) = ((1-\lambda)S_1)^{-1}$. The theorem also guarantees that there is a process $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}^\lambda(1)$ such that $\hat{Y}^0_1 = \hat{h}(1)$. By using condition (2.129), the expected terminal value of this process can be calculated as follows:

$$\mathbb{E}[\hat{Y}_1^0] = \mathbb{E}[\hat{h}(1)] = \mathbb{E}[((1-\lambda)S_1)^{-1}] = \frac{1}{1-\lambda}\sum_{n=0}^{\infty}\frac{p_n}{x_n} \tag{2.136}$$

$$= \frac{1}{1-\lambda}\left(\frac{1-\varepsilon}{2} + \varepsilon\sum_{n=1}^{\infty} n2^{-n}\right) \tag{2.137}$$

$$< \frac{1}{1-\lambda}\left(\frac{1-\varepsilon}{2} + \frac{1-\varepsilon}{2} - \lambda(1-\varepsilon) + (1-\lambda)\varepsilon\right) = 1. \tag{2.138}$$

As every process $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}^\lambda(1)$ satisfies $\hat{Y}_0^0 = 1$ by definition, this calculation shows that $\mathbb{E}[\hat{Y}_1^0] < \hat{Y}_0^0$. Hence, the dual optimizer$(\hat{Y}^0, \hat{Y}^1)$ is a strict supermartingale. $\triangleleft$

Chapter 3

Shadow Price Processes in Discrete Time

In this chapter, the notion of shadow prices will be introduced in discrete time. It will be proved that a shadow price always exists in one-period models. As regards multi-period models, two counterexamples show that a shadow price may fail to exist in general markets. However, if the underlying financial market is based on a finite probability space, existence is guaranteed and a proof for this assertion will be given. Finally, in order to gain additional insights to the existence of shadow prices, connections to the duality theory that was introduced in the previous chapter will be presented.

3.1 The Shadow Price Problem

The previous sections have revealed that portfolio optimization becomes more difficult if transaction costs are taken into account. Thus, it would be beneficial to somehow trace the case with transaction costs back to frictionless markets. The most natural way of doing so is to find a frictionless price process for the risky assets which not only takes values within the bid-ask spread of the original price process but also leads to the same maximal expected utility. The following example demonstrates this approach.

Example 3.1.1 (Shadow price in the one-period binomial model)**.** The financial market model with transaction costs $(\underline{S}, \overline{S})$ shall be the same as the one developed in Example 2.2.6. There, the problem of maximizing expected utility from terminal wealth was solved directly which turned out quite cumbersome, since by the nature of a frictious price process sales and purchases had to be treated seperately. This is not necessary if one follows a shadow price approach. Indeed, assume that it is optimal for the investor to purchase stock at the initial date. As this implies selling stock at the terminal time, the only relevant prices for the optimization problem are $\overline{S}_0$ and $\underline{S}_1$. This motivates in defining

$$\tilde{S}_0 = S_0 \text{ and } \tilde{S}_1 = \begin{cases} (1-\lambda)S_0 u, & \text{for } \omega = u, \\ (1-\lambda)S_0 d, & \text{for } \omega = d, \end{cases} \tag{3.1}$$

which is a frictionless price process lying in the bid-ask spread of the original price process. The evolution of $\tilde{S}$ again pertains to a binomial tree with up-factor $\tilde{u} = (1-\lambda)u$ and down-factor $\tilde{d} = (1-\lambda)d$. Therefore, the optimal value of wealth invested in stock can be determined by the following formula for the frictionless case which was derived in Example 2.1.10:

$$\varphi_1^1 = \frac{(p\tilde{u} + (1-p)\tilde{d} - 1)x}{(\tilde{d}-1)(\tilde{u}-1)S_0}. \tag{3.2}$$

By plugging in $\tilde{u} = (1-\lambda)u$ and $\tilde{d} = (1-\lambda)d$, one obtains

$$\varphi_1^1 = \frac{(p(1-\lambda)u + (1-p)(1-\lambda)d - 1)x}{((1-\lambda)d-1)((1-\lambda)u-1)S_0}. \tag{3.3}$$

This coincides with the optimal solution found in Example 2.2.6 where the derivation was much more difficult. Note that the same arguments can be employed for the case where it is optimal to sell stock short. $\triangleleft$

It is economically obvious that a frictious price process cannot lead to a higher expected utility than a frictionless price process which lies in its bid-ask spread. Before proving this statement in a mathematically rigorous manner, the reader may recall that the latter process coincides with the notion of consistent price processes. As the set of consistent price processes is denoted by $\mathcal{Z}^\lambda$, this leads to the following proposition where $\mathcal{C}(x;\tilde{S})$ stands $\mathcal{C}(x)$ in the frictionless market $\tilde{S}$. The same notation holds for $\mathcal{A}(x;\tilde{S})$.

Proposition 3.1.2. *$\mathcal{C}^\lambda(x) \subseteq \mathcal{C}(x;\tilde{S})$ for each $\tilde{S} \in \mathcal{Z}^\lambda$.*

Proof. Let $c \in \mathcal{C}^\lambda(x)$ which implies that there exists a trading strategy (φ^0, φ) such that $((\varphi^0,\varphi), c) \in \mathcal{A}^\lambda(x)$. By definition of $\mathcal{A}^\lambda(x)$, this portfolio/consumption pair satisfies $(\varphi_0^0, \varphi_0) = (x, 0)$, $(\varphi_{T+1}^0, \varphi_{T+1}) = (0,0)$, $c_T > 0$, and

$$\Delta\varphi_{t+1}^0 \leq \underline{S}_t^\top \Delta\varphi_{t+1}^\downarrow - \overline{S}_t^\top \Delta\varphi_{t+1}^\uparrow - c_t \tag{3.4}$$

for $t = 0, ..., T$. Using the inequality $\underline{S}_t \leq \tilde{S} \leq \overline{S}_t$ yields

$$\Delta\varphi_{t+1}^0 \leq \tilde{S}_t^\top \Delta\varphi_{t+1}^\downarrow - \tilde{S}_t^\top \Delta\varphi_{t+1}^\uparrow - c_t \tag{3.5}$$

$$= \tilde{S}_t^\top \Delta\varphi_{t+1} - c_t \tag{3.6}$$

for all $t = 0, ..., T$. Hence, $((\varphi^0,\varphi), c) \in \mathcal{A}^\lambda(x;\tilde{S})$ and $c \in \mathcal{C}^\lambda(x;\tilde{S})$. $\square$

Now, $u(x;\tilde{S})$ shall denote the value function corresponding to the utility maximization problem in the frictionless market with price $\tilde{S}$, i.e.

$$u(x;\tilde{S}) = \sup\left\{\mathbb{E}\left[\textstyle\sum_{t=0}^T U_t(\kappa_t)\right] \,\middle|\, ((\psi^0,\psi),\kappa) \in \mathcal{A}(x;\tilde{S})\right\}. \tag{3.7}$$

The proposition above then implies

$$u(x) \leq \inf_{\tilde{S}\in\mathcal{Z}} u(x;\tilde{S}) \tag{3.8}$$

where $u(x)$ stands for the value function corresponding to the frictious price process.

The question which now arises is whether there exists a consistent price process which minimizes the left hand side of the inequality in (3.8). The answer is provided in the proposition below. It states necessary conditions for this to be possible, namely that the frictionless price must coincide with the ask price whenever stock is purchased, and with the bid price whenever stock is sold. Again, the economic explanation is straightforward as any price within the bid-ask spread is more favorable for both purchases and sales. Thus, performing the same trading strategy in the frictionless market would lead to a higher expected utility.

Proposition 3.1.3. *Let $c \in \mathcal{C}(x;\tilde{S})$. Then, $c \in \mathcal{C}^\lambda(x)$ if there is a trading strategy (φ^0,φ) such that $((\varphi^0,\varphi),c) \in \mathcal{A}(x;\tilde{S})$ and*

$$\{\varphi^i_{t+1} > 0\} \subseteq \{\tilde{S}^i_t = \overline{S}^i_t\} \text{ and } \{\varphi^i_{t+1} < 0\} \subseteq \{\tilde{S}^i_t = \underline{S}^i_t\}, \tag{3.9}$$

for $t = 0,...,T$ and $i = 1,...,d$.

Proof. Assume $c \in \mathcal{C}(x;\tilde{S})$ satisfies (3.9). Due to the definition of $\mathcal{C}^\lambda(x;\tilde{S})$, this implies for all $t = 0,...,T$ and $i = 1,...,d$

$$\begin{aligned} \Delta\varphi^0_{t+1} &\leq \tilde{S}^\top_t \Delta\varphi^\downarrow_{t+1} - \tilde{S}^\top_t \Delta\varphi^\uparrow_{t+1} - c_t && (3.10)\\ &= \underline{S}^\top_t \Delta\varphi^\downarrow_{t+1} - \overline{S}^\top_t \Delta\varphi^\uparrow_{t+1} - c_t && (3.11) \end{aligned}$$

since $\tilde{S}^i = \overline{S}^i_t$ if $\Delta\varphi^{i,\uparrow}_{t+1} > 0$ and $\tilde{S}^i = \underline{S}^i_t$ if $\Delta\varphi^{i,\downarrow}_{t+1} > 0$. Thus, $c \in \mathcal{C}^\lambda(x)$. ☐

It is now possible to define the central concept of this thesis:

Definition 3.1.4. A price process $\tilde{S} \in \mathcal{Z}$ is called a **shadow price** if $u(x)$=$u(x;\tilde{S})$ for all $x > 0$.

In light of the two propositions above, a candidate for a shadow price $\tilde{S}$ not only must satisfy that $\tilde{S}^i$ takes values in the bid ask spread $[\underline{S}^i,\overline{S}^i]$ for all $1 \leq i \leq d$, but also that there is a solution $((\varphi^0,\varphi^i),c)$ to the corresponding frictionless utility maximization problem

$$\begin{cases} \text{maximize} & \mathbb{E}\left[\sum_{t=0}^T U_t(c_t)\right] \\ \text{subject to} & ((\varphi^0,\varphi^i),c) \in \mathcal{A}(x;\tilde{S}) \end{cases} \tag{3.12}$$

which trades only at bid-ask prices, that is,

$$\{\Delta\varphi^i_{t+1} > 0\} \subseteq \{\tilde{S}^i_t = \overline{S}^i_t\} \text{ and } \{\Delta\varphi^i_{t+1} < 0\} \subseteq \{\tilde{S}^i_t = \underline{S}^i_t\} \tag{3.13}$$

for $t = 0, ..., T$ and $i = 1, ..., d$. The remainder of this chapter is devoted to examining whether such a process exists.

3.2 One-period models

Example 3.2.1 (Continuation of Example 3.1.1)**.** Let the frictionless price process $\tilde{S}$ be defined as in Example 3.1.1 and $((\varphi^0, \varphi), c) \in \mathcal{A}(x; \tilde{S})$ the optimal solution to the problem of maximizing $\mathbb{E}[\log(c_T)]$. It was shown that

$$\varphi_1 = \frac{(p(1-\lambda)u + (1-p)(1-\lambda)d - 1)x}{((1-\lambda)d - 1)((1-\lambda)u - 1)S_0} \tag{3.14}$$

which is positive due to the model assumption $\frac{1}{1-\lambda} \leq pu+(1-p)d$. The optimal strategy is hence given by $\Delta\varphi_1 > 0$ and $\Delta\varphi_2 < 0$ such that it clearly satisfies the necessary condition

$$\{\Delta\varphi_{t+1} > 0\} \subseteq \{\tilde{S}_t = \overline{S}_t\} \text{ and } \{\Delta\varphi_{t+1} < 0\} \subseteq \{\tilde{S}_t = \underline{S}_t\} \tag{3.15}$$

for $t = 0, 1$. Furthermore, the calculations in Example 3.1.1 showed that the optimal solutions of the utility maximization problem in the frictional market $(\overline{S}, \underline{S})$ and in the one corresponding to $\tilde{S}$ coincide, i.e. $u(x) = u(x; \tilde{S})$. Thus, the process $\tilde{S}$ is a shadow price. ◁

Example 3.2.2. Let $(\overline{S}, \underline{S})$ be the financial market with transaction costs over the countable probability space Ω which was developed in Example 2.2.9. As explained there, the environment parameters considered imply that the optimal strategy involves strictly positive purchases of stock. Therefore, an optimal solution to the problem of maximizing expected utility from terminal wealth $((\varphi^0, \varphi), c)$ satisfies $\Delta\varphi_1^1 > 0$ and $\Delta\varphi_2^1 < 0$. This motivates defining a canditate for a shadow price $(\tilde{S})_{t=0}^1$ by $\tilde{S}_0 = \overline{S}_0 = S_0$ and $\tilde{S}_1 = \underline{S}_1 = (1-\lambda)S_0$. The corresponding maximization problem formulates as

$$(\mathrm{P}) \begin{cases} \text{maximize} & \mathbb{E}[1 + \varphi((1-\lambda)S_1 - S_0)] \\ \text{subject to} & 0 \leq \varphi \leq 1. \end{cases} \tag{3.16}$$

Note that this is exaclty the same problem as the one which was considered in Example 2.2.9 after excluding short sales. Therefore, the optimal strategies obviously coincide, and $\tilde{S}$ is a shadow price.

These examples suggest an easy method for finding a shadow price in one-period models: Determine if the optimal strategy in the frictional market involves purchases or sales, and then define the shadow price as the sequence of prices that are relevant for the optimal transaction. This will be made precise in the following proposition. For reasons of notational simplicity, it is stated only for the problem of maximizing utility from terminal wealth and the special case of one stock.

Proposition 3.2.3. *Fix an initial endowment $x > 0$ and suppose that an optimal strategy (φ^0, φ) exists for the frictional market $(\overline{S}, \underline{S})$ with $T = 1$. Then, a shadow price $\tilde{S}$ exists.*

Proof. The proof will be split into the cases $\varphi \geq 0$ and $\varphi < 0$. First, suppose $\varphi \geq 0$, that is, the optimal strategy sells at time $t = 0$. Since there is only one period, the admissibility condition implies that the purchased stock has to be liquidated at the terminal time $t = 1$. Hence,

$$u(x) = U(x + \varphi((1-\lambda)S_1 - S_0)). \tag{3.17}$$

Furthermore, since φ is optimal, it is the solution to the problem

$$\text{(P)} \begin{cases} \text{maximize} & \mathbb{E}[U(x + \varphi^{\uparrow}((1-\lambda)S_1 - S_0) + \varphi^{\downarrow}(S_1 - (1-\lambda)S_0)], \\ \text{subject to} & x + \varphi^{\uparrow}((1-\lambda)S_1 - S_0) + \varphi^{\downarrow}(S_1 - (1-\lambda)S_0) > 0, \\ & \varphi \in \mathbb{R}. \end{cases} \tag{3.18}$$

As the optimal solution satisfies $\varphi \geq 0$ by assumption, the problem (P) is equivalent to

$$\text{(P')} \begin{cases} \text{maximize} & \mathbb{E}[U(x + \varphi((1-\lambda)S_1 - S_0)], \\ \text{subject to} & x + \varphi((1-\lambda)S_1 - S_0) > 0, \\ & \varphi \geq 0. \end{cases} \tag{3.19}$$

Now define a frictionless price process $(Y)_{t=0}^1$ by setting $Y_0 = S_0$ and $Y_1 = (1-\lambda)S_1$. Then the problem (P') is clearly also the optimality problem for the frictionless financial market Y. As a result, the strategy (φ^0, φ) is not only optimal in the financial market with transaction costs but also in the shadow market which implies that Y is a shadow price.

For the other case, namely $\varphi < 0$, analog arguments show that the process $(Z)_{t=0}^1$ defined by $Z_0 = (1-\lambda)S_0$ and $Z_1 = S_1$ is a shadow price. In conclusion, the process $(\tilde{S})_{t=0}^1$ where

$$\tilde{S}_0 = \begin{cases} Y_0, & \text{for } \varphi \geq 0, \\ Z_0, & \text{for } \varphi < 0, \end{cases} \quad \text{and} \quad \tilde{S}_1 = \begin{cases} Y_1, & \text{for } \varphi \geq 0, \\ Z_1, & \text{for } \varphi < 0, \end{cases} \tag{3.20}$$

is a shadow price for the frictional market $(\overline{S}, \underline{S})$. □

3.3 Multi-period models

Whereas the existence of shadow prices in one-period models is guaranteed as shown above, the following counterexample reveals that this does not hold in general multi-period models.

Example 3.3.1 (cf. [BCKMK11, Section 2])**.** Consider a frictional financial market $(\underline{S}, \overline{S})$ with one stock where the bid price $\underline{S}$ decreases deterministically from 3 to 2 to 1, and the ask price $\overline{S}$ evolves as illustrated in the figure below. The model parameter $\varepsilon > 0$ shall be chosen small enough to make the succeeding arguments work.

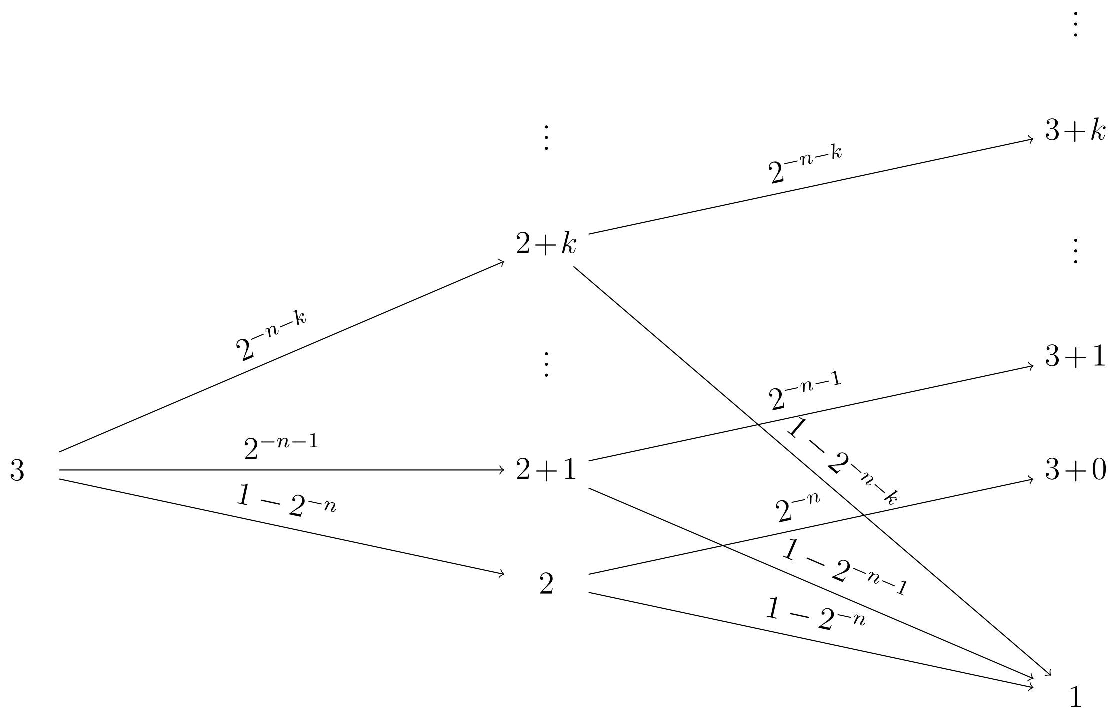

Figure 3.1: Evolution of the ask price in Example 3.3.1.

In this market, an investor with an initial wealth of 1 is seeking to maximize expected utility from terminal wealth. Let $(\hat{\varphi}, \hat{\varphi}^1)$ denote the investor's optimal trading strategy. Since the ask price may move up or down in every period, the present financial market does not admit arbitrage. In particular, the investor's maximal expected utility from terminal wealth is bounded from above.

The admissibility condition $c_2 \geq 0$ forbids short selling the stock at time $t = 0$. Indeed, suppose $\Delta\hat{\varphi}_1 < 0$. As the ask price of the stock may increase arbitrarily high, this transaction leads to negative wealth at time $t = 1$ with positive probability. The resulting negative position in the bank account cannot be eradicated with certainty since the stock may move up or down in the second period leading to $c_2 < 0$ with positive probability. Whereas short selling is ruled out for reasons of solvency, going long is not advisable because the bid price decreases deterministically. As a result, the optimal strategy at time $t = 0$ is to not trade at all.

In the second period, the ask price may either increase by 1 or drop to 1 with overwhelming probability. This means that any losses incurred from short sales are limited. In addition, the overwhelming probability of the future ask price dropping beneath the current bid price makes short selling very attractive. It is thus optimal to sell stock short at time $t = 1$. Summarizing, the optimal strategy satisfies $\Delta\hat{\varphi}_1 = 0$, $\Delta\hat{\varphi}_2 < 0$ and $\Delta\hat{\varphi}_3 > 0$.

Proposition 3.1.3 now guarantees that the only candidate for a shadow price $\tilde{S}$ is given by $\tilde{S}_0 = 3$, $\tilde{S}_1 = \underline{S}_1 = 2$ and $\tilde{S}_2 = \overline{S}_2$. As the shadow price decreases deterministically in the first period, arbitrarily high gains can be obtained with certainty by going short in the stock. In combination with not trading at time $t = 1$, this leads to infinite expected utility from terminal wealth. In the original frictional market, however, this value is bounded. Thus, the candidate process $\tilde{S}$ can hence not be a shadow price. As it was the only candidate, no shadow price exists in this financial market. $\triangleleft$

Because the example above involves unbounded ask prices and deterministically decreasing bid prices, the reader may now ask whether the nonexistence of a shadow price hinges on this abnormality. Since such a price evolution is seldomly found in financial market models, the impact of the counterexample to utility maximization problems is quite insignificant. Thus, there is still hope that a shadow price exists in multi-period models that are widely studied, that is, models with bounded prices, no arbitrage opportunities, and proportional transaction costs. Unfortunately, this reasonable hope is soon shattered by the following counterexample.

Example 3.3.2 (cf. [CMKS12, Section 3])**.** Let $(\underline{S}, \overline{S})$ be a financial market with proportional transaction costs $0 < \lambda < 1$, that is, $\overline{S} = S$ and $\underline{S} = (1 - \lambda)S$ for a price process S. Here, the ask price S evolves as depicted in the figure below where the model parameters ε and $\varepsilon_{n,1}$ $(n \in \mathbb{N}_0)$ are sufficiently small.

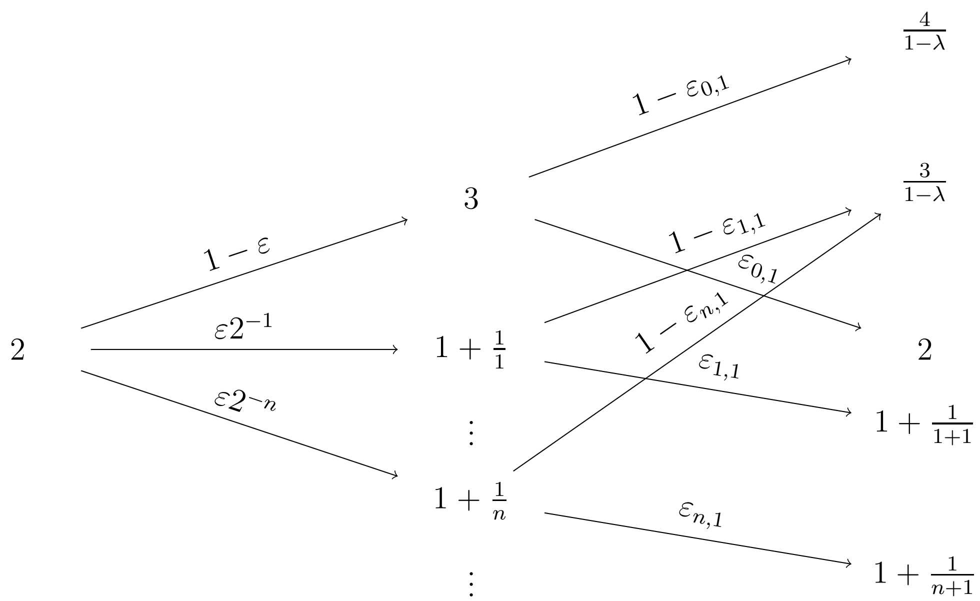

Figure 3.2: Evolution of the ask price in Example 3.3.2

Consider again a log-utility investor facing the problem of maximizing expected utility from terminal wealth. The initial endowment shall be 1. In the first period, decreases in the ask price of the stock are very unlikely. In the case of the stock plummeting, the potential losses are also limited, since it cannot drop below 1. Thus, it is optimal to go long in the stock. As regards the optimal amount of purchases, the investor must consider that negative wealth at time $t = 1$ could lead to bankruptcy since the stock price may move up or down in the second period with positive probability. By denoting the optimal strategy by $(\hat{\varphi}, \hat{\varphi}^1)$, this implies $1 + \Delta\hat{\varphi}_1^1((1-\lambda)S_1 - S_0) \geq 0$. This solvency constraint reduces to $\Delta\hat{\varphi}_1^1 \leq \frac{1}{1-\lambda}$ since S_1 may drop arbitrarily close to 1. Provided ε is small enough, the same formal arguments as in Example 2.2.9 may be employed to show that the optimal amount of purchases is attained at the border of this constraint, i.e. $\Delta\hat{\varphi}_1^1 = \frac{1}{1-\lambda}$.

In the second period, the stock still exhibits an overwhelming tendency to rise such that it is advisable for the investor not to liquidate the long position. In the case of no liquidation at time $t = 1$, the solvency constraint is given by

$$1 + \Delta\hat{\varphi}_2^1((1-\lambda)S_2 - S_0) \geq 0. \tag{3.21}$$

For a given stock price S_1, there is only one potential state of S_2 with $(1-\lambda)S_2 < S_0$, namely the down movement. Hence, it suffices to determine for which values of $\Delta\hat{\varphi}_2^1$ the solvency constraints holds for this state. To this end, suppose $S_1 = 3$. Then the inequality above transforms to

$$1 + \Delta_2^1\hat{\varphi}_2^1((1-\lambda)2 - 2) \geq 0, \tag{3.22}$$

and consequently $\Delta\hat{\varphi} \leq \frac{1}{2\lambda}$. For the remaining stock prices $S_1 = 1 + \frac{1}{n}$ where $n \in \mathbb{N}$, a similar calculation yields that the solvency constraint is

$$\Delta\hat{\varphi}_2^1 \leq \frac{1}{2 - (1-\lambda)(1 + \frac{1}{n+1})}. \tag{3.23}$$

Clearly, both upper boundaries encountered here are strictly higher than the boundary of the solvency constraint in the first period $\Delta\hat{\varphi}_1^1 \leq \frac{1}{1-\lambda}$. As a result, the investor may even increase the holdings in the stock which is in fact optimal if the parameters $\varepsilon_{n,1}$ $(n \in \mathbb{N}_0)$ are sufficiently small. At the terminal time $t = 2$, this long position will be liquidated.

In summary, the optimal strategy $(\hat{\varphi}, \hat{\varphi}^1)$ satisfies $\Delta\hat{\varphi}_1^1 > 0$, $\Delta\hat{\varphi}_2^1 > 0$ and $\Delta\hat{\varphi}_3^1 < 0$. Proposition 3.1.3 can again be employed to find a candidate for a shadow price. It yields that the only candidate is the process $\tilde{S}$ which is defined by $\tilde{S}_0 = 2$, $\tilde{S}_1 = S_1$ and $\tilde{S}_2 = (1-\lambda)S_2$. This process, however, fails to be a shadow price because it does not lead to the same maximal expected utility from terminal wealth. In fact, it allows for a strictly higher maximal expected utility than the original market with transaction costs. The reason for this lies in the solvency constraint of the first period. While the amount of purchases is limited by $\frac{1}{1-\lambda}$ in the original market, the corresponding solvency constraint in the shadow market is

$$1 + \Delta\hat{\varphi}_1^1(\tilde{S}_1 - \tilde{S}_0) \geq 0, \tag{3.24}$$

which reduces to $\hat{\varphi}_1^1 \leq 1$ due to the possibility of $\tilde{S}_1$ going down arbitrarily close to 1. By trading in the shadow market, the investor can therefore purchase more units of stock. If the parameter ε is small enough, this obviously leads to higher expected returns in the first period. As the relevant shadow prices for increasing the long position at time $t = 1$ and liquidating it at the terminal time coincide with the original price process, these additional gains do not vanish in the second period. In conclusion, the only candidate for a shadow price outperforms the frictional price process which implies that no shadow price exists in this example. $\triangleleft$

The two counterexamples above raise the question whether there is any condition on the properties of a multi-period model which guarantees the existence of a shadow price. Both examples have in common that they are based on an infinite probability space. This property turns out to be crucial for the existence of a shadow price as the following theorem shows. It ensures that a shadow price always exists in financial market models which are based on a finite probability space. To this end, the original result of Kallsen and Muhle-Karbe [KMK11] is slightly modified to suit the framework of this thesis. Their version is more general, as it allows discontinuous utility functions, arbitrary initial endowments in both the bank account and stocks, and negative consumption processes.

Theorem 3.3.3 (cf. [KMK11, Theorem 3.2])**.** *Suppose an optimal portfolio/consumption pair $((\varphi^0, \varphi), c)$ exists for the market with bid/ask prices $(\underline{S}, \overline{S})$. Then, if $\mathbb{E}\left[\sum_{t=0}^{T} U_t(c_t)\right] > -\infty$, a shadow price process $\tilde{S}$ exists.*

Proof. The idea of the proof is as follows. First of all, the original problem will be transformed to an equivalent problem in which sales and purchases are treated seperately. By using the finiteness property of the underlying probability space, the latter can be formulated as a finite-dimensional convex minimization problem with convex constraints. For problems of this kind, the Karush-Kuhn-Tucker conditions (see Appendix A.3) can be employed for finding a solution. It turns out that the Kuhn-Tucker vector corresponding to this soultion can be used to construct a candidate for a shadow price. Finally, showing that the optimal solution of the original problem is optimal in the shadow market as well yields that the developed candidate is indeed a shadow price.

Step 1: The $\mathbb{R}^d$-valued predictable process $(\psi)_{t=0}^{T+1}$ in an admissible portfolio/consumption pair $((\psi^0, \psi), \kappa) \in \mathcal{A}^\lambda(x)$ can clearly be decomposed as $\psi = \psi^\uparrow - \psi^\downarrow$ where $(\psi^\uparrow)_{t=0}^{T+1}$ and $(\psi^\downarrow)_{t=0}^{T+1}$ are increasing $\mathbb{R}^d_{\geq 0}$-valued predictable processes with $\psi_0^\uparrow = \psi_0^\uparrow = 0$ and $\psi_0^\downarrow = \psi_0^- = 0$. Since ψ_t denotes the amount of stock held at time t, these processes can be naturally interpreted as cumulated purchases and sales processes. The decomposition allows to express a portfolio/consumption pair $((\psi^0, \psi), \kappa) \in \mathcal{A}^\lambda(x)$ as $((\psi^0, \psi^\uparrow, \psi^\downarrow), \kappa) \in$

$\mathcal{A}^\lambda(x)$. As transaction costs are incurred with every trade, it is not optimal to purchase and sell stock at the same time. Therefore, maximizing over all admissible seperated portfolio/consumption pairs does not lead to a higher maximal expected utility. In fact, the maximum will be attained at the strategy $((\varphi^0, \varphi^\uparrow, \varphi^\downarrow), c)$ where

$$\varphi_t^\uparrow := \sum_{s=1}^{T} \Delta\varphi_t^\uparrow, \quad \varphi_t^\uparrow := \sum_{s=1}^{T} \Delta\varphi_t^\uparrow \tag{3.25}$$

for $t = 1, ..., T+1$. The original problem can hence be transformed to the equivalent problem

$$\text{(P)} \begin{cases} \text{maximize} & \mathbb{E}\left[\sum_{t=0}^{T} U_t(\kappa_t)\right] \\ \text{subject to} & ((\psi^0, \psi^\uparrow, \psi^\downarrow), \kappa) \in \mathcal{A}^\lambda(x). \end{cases} \tag{3.26}$$

Step 2: This modified problem can in turn be reformulated as a finite-dimensional convex minimization problem with convex constraints. To this end, observe that the finiteness of Ω implies that every $\mathcal{F}_t$ can be generated by a corresponding partition of Ω which will be denoted by $F_t^1, ..., F_t^{m_t}$. As $(\psi^\uparrow)_{t=0}^{T+1}$ is an $\mathbb{R}^d_{\geq 0}$-valued, predictable and increasing process with $\psi_0^\uparrow = 0$, the process $\Delta\psi_t^\uparrow$ is $\mathcal{F}_{t-1}$-measurable and takes values in $\mathbb{R}^d_{\geq 0}$ for every $t = 1, ..., T+1$. In particular, it is constant on every set F_{t-1}^j where $j = 1, ..., m_{t-1}$, and hence there is an $\alpha_j \in \mathbb{R}^d_{\geq 0}$ such that $\Delta\psi_t^\uparrow(\omega) = \alpha_j$ for every $\omega \in F_{t-1}^j$. Since this property holds for every $t = 1, ..., T+1$, the whole process $(\Delta\psi^\uparrow)_{t=1}^T$ can be identified with $\mathbb{R}^{m_0 d}_{\geq 0} \times ... \times \mathbb{R}^{m_T d}_{\geq 0}$. The same arguments show that $(\Delta\psi^\downarrow)_{t=1}^T$ can be identified with $\mathbb{R}^{m_0 d}_{\geq 0} \times ... \times \mathbb{R}^{m_T d}_{\geq 0}$ as well. Similarly, the $\mathbb{R}_{\geq 0}$-valued and $\mathcal{F}_t$-measurable consumption process $(\kappa_t)_{t=0}^T$ can be identified with $\mathbb{R}^{m_0}_{\geq 0} \times ... \times \mathbb{R}^{m_T}_{\geq 0}$. By the self-financing equation, the cash position ψ^0 can be expressed as a function of $\psi^\uparrow$, $\psi^\downarrow$ and κ. Thus, in summary, every portfolio/consumption pair $((\psi^0, \psi^\uparrow, \psi^\downarrow), \kappa)$ can be identified with

$$\mathbb{R}^{(2d+1)n}_{\geq 0} := (\mathbb{R}^{m_0 d}_{\geq 0} \times ... \times \mathbb{R}^{m_T d}_{\geq 0}) \times (\mathbb{R}^{m_0 d}_{\geq 0} \times ... \times \mathbb{R}^{m_T d}_{\geq 0}) \times (\mathbb{R}^{m_0}_{\geq 0} \times ... \times \mathbb{R}^{m_T}_{\geq 0}). \tag{3.27}$$

Next, denote $\Delta\psi_t^{\uparrow,j} := \Delta\psi_t^\uparrow(\omega)$ for $t = 0, ..., T$, $j = 1, ..., m_t$ and $\omega \in \mathcal{F}_t^j$. Analog definitions hold for $\Delta\psi^{\uparrow,j}$, κ^j, $\underline{S}^j$ and $\overline{S}^j$. Furthermore, for $j = 1, ..., m_T$, the mappings $h_0^j : \mathbb{R}^{(2d+1)n}_{\geq 0} \to \mathbb{R}$ and $h_j : \mathbb{R}^{(2d+1)n}_{\geq 0} \to \mathbb{R}$ shall be defined by:

$$h_0^j(\Delta\psi^\uparrow, \Delta\psi^\downarrow, \kappa) := x + \sum_{t=1}^{T} \left((\underline{S}_{t-1}^j)^\top \Delta\psi_t^{\downarrow,j} - (\overline{S}_{t-1}^j)^\top \Delta\psi_t^{\uparrow,j} \right) - \sum \kappa_t^j \tag{3.28}$$

and

$$h^j(\Delta\psi^\uparrow, \Delta\psi^\downarrow, \kappa) := \sum_{t=1}^{T+1} (\Delta\psi_t^{\downarrow,j} - \Delta\psi_t^{\uparrow,j}). \tag{3.29}$$

Since the functions h_0^j and h^j evidently yield the terminal position in bonds and stocks respectively, the admissibility condition implies

$$h_0 = 0, \quad h^j = 0 \tag{3.30}$$

for all $j = 1, ..., m_T$. Hence, the portfolio/consumption pairs $((\psi^0, \psi^{\uparrow}, \psi^{\downarrow}), \kappa) \in \mathcal{A}^{\lambda}(x)$ can be identified with the subset of $\mathbb{R}^{(2d+1)n}_{\geq 0}$ that is formed by the constraints in (3.30).

After defining the objective function $f(\Delta\psi^{\uparrow}, \Delta\psi^{\downarrow}, \kappa) := -\mathbb{E}[\sum_{t=1}^{T} U_t(c_t)]$, the findings above imply that problem (P) can be reformulated as

$$\text{(P')} \begin{cases} \text{minimize} & f(\Delta\psi^{\uparrow}, \Delta\psi^{\downarrow}, \kappa) \\ \text{subject to} & h_0^j(\Delta\psi^{\uparrow}, \Delta\psi^{\downarrow}, \kappa) = 0 \text{ for } j = 1, ..., m_T, \\ & h^j(\Delta\psi^{\uparrow}, \Delta\psi^{\downarrow}, \kappa) = 0 \text{ for } j = 1, ..., m_T, \\ & (\Delta\psi^{\uparrow}, \Delta\psi^{\downarrow}, \kappa) \in \mathbb{R}^{(2d+1)n}_{\geq 0}. \end{cases} \tag{3.31}$$

Since all these functions are not only convex on $\mathbb{R}^{(2d+1)n}_{\geq 0}$ but also on $\mathbb{R}^{(2d+1)n}$, the problem (P') is equivalent to minimizing f over $\mathbb{R}^{(2d+1)n}$ subject to $h_0^j = 0$, $h^j = 0$ (for $j = 1, ..., m_T$) and $g_t^{\uparrow,j} \leq 0$, $g_t^{\uparrow,j} \leq 0$, $g_t^{c,j} \leq 0$ (for $t = 0, ..., T$ and $j = 1, ..., m_t$) where the convex mappings $g_t^{\uparrow,j}, g_t^{\downarrow,j}, g_t^{c,j} : \mathbb{R}^{(2d+1)n} \to \mathbb{R}^d$ are given by

$$g_t^{\uparrow,j}(\Delta\psi^{\uparrow}, \Delta\psi^{\downarrow}, \kappa) := -\Delta\psi_{t+1}^{\uparrow,j}, \tag{3.32}$$
$$g_t^{\downarrow,j}(\Delta\psi^{\uparrow}, \Delta\psi^{\downarrow}, \kappa) := -\Delta\psi_{t+1}^{\downarrow,j}, \tag{3.33}$$
$$g_t^{c,j}(\Delta\psi^{\uparrow}, \Delta\psi^{\downarrow}, \kappa) := -\kappa_t. \tag{3.34}$$

According to Theorem A.3.2, a solution to this ordinary convex program exists. As the constraints are differentiable functions, Theorem A.3.3 guarantees that $(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c)$ is optimal if and only if there exist real numbers $\nu^j, \mu^{j,i}$ (for $i = 1, ..., d$ and $j = 1, ..., m_T$) and $\lambda_t^{\downarrow,j,i}$ (for $t = 0, ..., T$, $i = 1, ..., d$ and $j = 1, ..., m_t$) such that the following conditions are satisfied.

(i) For $t = 0, ..., T$, $j = 1, ..., m_t$ and $i = 1, ..., d$, the inequalities $\lambda_t^{\uparrow,j,i} \geq 0$, $\lambda_t^{\downarrow,j,i} \geq 0$, $g_t^{\uparrow,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) \leq 0$, $g_t^{\downarrow,j}(\Delta\varphi^{\downarrow}, \Delta\varphi^{\downarrow}, c) \leq 0$, $g_t^{c,j}(\Delta\varphi^{\downarrow}, \Delta\varphi^{\downarrow}, c) \leq 0$ and the equalities $\lambda_t^{\uparrow,j,i} g_t^{\uparrow,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$, $\lambda_t^{\downarrow,j,i} g_t^{\downarrow,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$, $\lambda_t^{c,j,i} g_t^{c,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ hold.

(ii) $h_0^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ and $h^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ for $j = 1, ..., m_T$.

(iii) The following equation holds:

$$\begin{aligned}
&\nabla f(\Delta\varphi^{\uparrow},\Delta\varphi^{\downarrow},c)+\sum_{j=1}^{m_T}\nu^j\nabla h_0^j(\Delta\varphi^{\uparrow},\Delta\varphi^{\downarrow},c)+\sum_{i=1}^{d}\sum_{j=1}^{m_T}\mu^{j,i}\nabla h^{j,i}(\Delta\varphi^{\uparrow},\Delta\varphi^{\downarrow},c)\\
&+\sum_{t=0}^{T}\sum_{i=1}^{d}\sum_{j=1}^{m_t}\lambda_t^{\uparrow,j,i}\nabla g_t^{\uparrow,j}(\Delta\varphi^{\uparrow},\Delta\varphi^{\downarrow},c)+\sum_{t=0}^{T}\sum_{i=1}^{d}\sum_{j=1}^{m_t}\lambda_t^{\downarrow,j,i}\nabla g_t^{\downarrow,j,i}(\Delta\varphi^{\uparrow},\Delta\varphi^{\downarrow},c)\\
&+\sum_{t=0}^{T}\sum_{i=1}^{d}\sum_{j=1}^{m_t}\lambda_t^{c,j,i}\nabla g_t^{c,j,i}(\Delta\varphi^{\uparrow},\Delta\varphi^{\downarrow},c)=0.
\end{aligned}$$

These optimality conditions can be used to construct a shadow price process as the next step shows.

Step 3: It follows from investigating the components of the gradients in Condition (iii) which correspond to the partial derivatives with respect to c_T^j (for $j=1,...,m_T$) that

$$\frac{\partial}{\partial c_T^j}f(\Delta\varphi^{\uparrow},\Delta\varphi^{\downarrow},c)-\nu^j-\lambda_T^{c,j,i}=0. \tag{3.35}$$

This equality implies $\nu^j<0$ because f is strictly decreasing in c_T^j and $\lambda_t^{c,j,i}\geq 0$. As regards the other partial derivatives, let $t=0,...,T$, $j=1,...,m_t$, $i=1,...,d$, $k=1,...,m_T$, $\omega\in F_t^j$ and $\omega_k\in F_T^k$. Then,

$$\frac{\partial}{\partial\Delta\varphi_{t+1}^{\uparrow,i}(\omega)}h^l(\omega_k)=\begin{cases}1, & \omega_k\in F_t^j \text{ and } i=l,\\ 0, & \text{otherwise,}\end{cases} \tag{3.36}$$

and

$$\frac{\partial}{\partial\Delta\varphi_{t+1}^{\uparrow,i}(\omega)}h_0^l(\omega_k)=\begin{cases}-\overline{S}_t^i(\omega), & \omega_k\in F_t^j \text{ and } i=l,\\ 0, & \text{otherwise,}\end{cases} \tag{3.37}$$

since $F_t^j\subseteq F_T^k$. Analog representations hold for the partial derivatives with respect to $\Delta\varphi_{t+1}^{\downarrow,i}(\omega)$.

As a result, Condition (iii) above implies that

$$\begin{aligned}
0&=\sum_{k:\omega_k\in F_t^j}\mu^{k,i}-\Bigg(\sum_{k:\omega_k\in F_t^j}\nu^k\Bigg)\overline{S}_t^{j,i}-\lambda^{\uparrow,j,i} &(3.38)\\
&=\sum_{k:\omega_k\in F_t^j}\mu^{k,i}-\Bigg(\sum_{k:\omega_k\in F_t^j}\nu^k\Bigg)\left(1+\frac{\lambda_t^{\uparrow,j,i}}{\overline{S}_t^{j,i}\sum_{k:\omega_k\in F_t^j}\nu^k}\right)\overline{S}_t^{j,i}, &(3.39)
\end{aligned}$$

and likewise

$$0 = \sum_{k:\omega_k \in F_t^j} \mu^{k,i} - \Bigg(\sum_{k:\omega_k \in F_t^j} \nu^k \Bigg) \Bigg(1 - \frac{\lambda_t^{\uparrow,j,i}}{\underline{S}_t^{j,i} \sum_{k:\omega_k \in F_t^j} \nu^k} \Bigg) \underline{S}_t^{j,i}. \tag{3.40}$$

By combining these two equations, one obtains

$$\Bigg(1 + \frac{\lambda_t^{\uparrow,j,i}}{\overline{S}_t^{j,i} \sum_{k:\omega_k \in F_t^j} \nu^k} \Bigg) \overline{S}_t^{j,i} = \Bigg(1 - \frac{\lambda_t^{\uparrow,j,i}}{\underline{S}_t^{j,i} \sum_{k:\omega_k \in F_t^j} \nu^k} \Bigg) \underline{S}_t^{j,i} =: \tilde{S}_t^{j,i}. \tag{3.41}$$

Since $\tilde{S} := (\tilde{S}^1, ..., \tilde{S}^d)$ is constant on F_t^j by definition, this defines an adapted process. It also satisfies $\underline{S} \leq \tilde{S} \leq \overline{S}$ because $\lambda_t^{\uparrow,j,i}, \lambda_t^{\downarrow,j,i} \geq 0$ (for $i = 1, ..., d$, $t = 0, ..., T$ and $j = 1, ..., m_t$) and $\nu^k < 0$ (for $k = 1, ..., m_T$). Furthermore, the optimality condition in (i) implies $\lambda_t^{\uparrow,j,i} = 0$ if $\Delta\varphi_{t+1}^{\uparrow,j,i} > 0$ and $\lambda_t^{\downarrow,j,i} = 0$ if $\Delta\varphi_{t+1}^{\downarrow,j,i} > 0$ such that

$$\tilde{S}^i = \overline{S}^i \text{ on } \{\Delta\varphi^{\uparrow,i} > 0\}, \quad \tilde{S}^i = \underline{S}^i \text{ on } \{\Delta\varphi^{\downarrow,i} > 0\}. \tag{3.42}$$

The adapted process $\tilde{S}$ is hence a candidate for a shadow price process. It remains to show that the portfolio/consumption pair (φ, c) is not only optimal in the original market with transaction costs but also in the frictionless market corresponding to $\tilde{S}$.

Step 4: Set $\tilde{\mu}^{j,i} := \mu^{j,i}$, $\tilde{\nu}^j := \nu^j$, $\tilde{\lambda}^{\uparrow,j,i} := 0$, $\tilde{\lambda}^{\downarrow,j,i} := 0$ and $\tilde{\lambda}^{c,j,i} := \lambda^{c,j,i}$ where $j = 1, ..., m_T$, $i = 1, ..., d$ and $j = 1, ..., m_t$. Moreover, define the mappings $\tilde{f}$, $\tilde{h}_0^j$, $\tilde{h}^j$, $\tilde{g}^{\uparrow,i}$, $\tilde{g}^{\downarrow,j}$, $\tilde{g}^{c,j}$ by setting $\underline{S} = \overline{S} = \tilde{S}$ in the original definition of the mappings f, h_0^j, h^j, $g^{\uparrow,i}$, $g^{\downarrow,j}$, $g^{c,j}$. The aim is to show that the optimality conditions in (i), (ii) and (iii) above also hold when the original variables and functions are replaced by the newly defined ones. To begin with, observe that the first condition is clearly satisfied:

(i') For $t = 0, ..., T$, $j = 1, ..., m_t$ and $i = 1, ..., d$, the inequalities $\tilde{\lambda}_t^{\uparrow,j,i} \geq 0$, $\tilde{\lambda}_t^{\downarrow,j,i} \geq 0$, $\tilde{g}_t^{\uparrow,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) \leq 0$, $\tilde{g}_t^{\downarrow,j}(\Delta\varphi^{\downarrow}, \Delta\varphi^{\downarrow}, c) \leq 0$, $\tilde{g}_t^{c,j}(\Delta\varphi^{\downarrow}, \Delta\varphi^{\downarrow}, c) \leq 0$ and the equalities $\tilde{\lambda}_t^{\uparrow,j,i}\tilde{g}_t^{\uparrow,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$, $\tilde{\lambda}_t^{\downarrow,j,i}\tilde{g}_t^{\downarrow,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$, $\tilde{\lambda}_t^{c,j,i}\tilde{g}_t^{c,j}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ hold.

As regards the second condition, the trading constraint (3.42) and the original condition (ii) imply:

(ii') $\tilde{h}_0^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ and $\tilde{h}^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ for $j = 1, ..., m_T$.

It remains to verify the third optimality condition which reduces to a more compact expression because $\tilde{\lambda}^{\uparrow,j,i} = 0$, $\tilde{\lambda}^{\downarrow,j,i} = 0$:

(iii') The following equation holds:

$$\nabla \tilde{f}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{j=1}^{m_T} \tilde{\nu}^j \nabla \tilde{h}_0^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{i=1}^{d}\sum_{j=1}^{m_T} \tilde{\mu}^{j,i} \nabla \tilde{h}^{j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c)$$
$$+ \sum_{t=0}^{T}\sum_{i=1}^{d}\sum_{j=1}^{m_t} \lambda_t^{c,j,i} \nabla \tilde{g}_t^{c,j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0.$$

Because $\tilde{f}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = f(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c)$, one obtains

$$\frac{\partial}{\partial c_t^j} \tilde{f}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) - \nu^j - \lambda_t^{c,j,i} = 0 \tag{3.43}$$

for all $t = 0, ..., T$ due to the equation in (3.35). The partial derivative of the left hand side of the equation in (iii') with respect to $\Delta\varphi_{t+1}^{\uparrow,j,i}$ is

$$\sum_{k:\omega_k \in F_t^j} \mu^{k,i} - \left(\sum_{k:\omega_k \in F_t^j} \nu^k \right) \left(1 + \frac{\lambda_t^{\uparrow,j,i}}{\overline{S}_t^{j,i} \sum_{k:\omega_k \in F_t^j} \nu^k} \right) \overline{S}_t^{j,i} \tag{3.44}$$

which equals 0 by (3.39). The same argument yields that the partial derivative with respect to $\Delta\varphi_{t+1}^{\downarrow,j,i}$ is 0. All together, this proves condition (iii') and shows that (φ, c) is indeed optimal in the shadow market. Hence, the process $\tilde{S}$ is indeed a shadow price. □

While the proof of this theorem finds a shadow price process in a quite straightforward way, it does not give any insights on the reasons why a similar procedure would not work for market models which are based on infinite probability spaces. In order to develop an answer to this question, connections to duality theory have to be established first.

3.4 Connections to Duality

As mentioned in the introductions of [BCKMK11] and [CMKS12], it was Cvitanić and Karatzas [CK96] who first established a connection between the exsitence of a shadow price and duality theory. They prove, in an Itô process setting with proportional transaction costs, that a shadow price exists if the minimizer in a suitable dual problem exists and is a martingale. The following proposition shows that this assertion holds also true in the present framework. It must me noted that both propositions below are stated for shadow prices corresponding to the problem of maximizing utility from terminal wealth. Furthermore, it is assumed that the requirements of Theorem 2.2.8 regarding the reasonable asymptotic elasticity of the utility function and the finiteness property of the primal value function are satisfied.

Proposition 3.4.1. *If a dual minimizer* $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}^\lambda(\hat{y}(x))$ *is a martingale, then* $\hat{Y}^1/\hat{Y}^0$ *is a shadow price.*

Proof. Suppose that $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}^\lambda(\hat{y}(x))$ is a martingale. Then, the process $\hat{Y}^0\varphi^0 + \hat{Y}^1\varphi^1 = \hat{Y}^0(x+\varphi^1\frac{\hat{Y}^1}{\hat{Y}^0})$ is a non-negative local martingale for all $(\varphi^0, \varphi^1) \in \mathcal{A}(x; \frac{\hat{Y}^1}{\hat{Y}^0})$ because the portfolio processes φ^0 and φ^1 are predictable by definition. Since every bounded local martingale is a supermartingale, it follows that $\hat{Y}^0 \in \mathcal{Y}(\hat{y}(x); \frac{\hat{Y}^1}{\hat{Y}^0})$.
As $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}^\lambda(\hat{y}(x))$ is the dual minimizer, Theorem 2.2.8 guarantees that $\hat{Y}^0_T = U'(\hat{\varphi}^0_{T+1})$ and that $\hat{Y}^0\hat{\varphi}^0 + \hat{Y}^1\hat{\varphi}^1 = \hat{Y}^0(x + \varphi^1\frac{\hat{Y}^1}{\hat{Y}^0})$ is a martingale. Therefore, the results about the duality for the frictionless utility maximization problem in Theorem 2.1.12 imply that $(\hat{\varphi}^0, \hat{\varphi}^1) \in \mathcal{A}(x; \frac{\hat{Y}^1}{\hat{Y}^0})$ and $\hat{Y}^0 \in \mathcal{Y}(\hat{y}(x); \frac{\hat{Y}^1}{\hat{Y}^0})$ are the solutions to the frictionless primal and dual problem for the price process $\frac{\hat{Y}^1}{\hat{Y}^0}$ if $\hat{y}(x; \frac{\hat{Y}^1}{\hat{Y}^0}) = \hat{y}(x)$.
It remains to show that this identity indeed holds. By Theorem 2.2.8 and the definition of $\hat{y}(x)$, it follows that $u(x) = v(\hat{y}(x)) + x\hat{y}(x)$. On the other hand, its frictionless counterpart implies

$$u(x; \tfrac{\hat{Y}^1}{\hat{Y}^0}) = v(\hat{y}(x; \tfrac{\hat{Y}^1}{\hat{Y}^0}); \tfrac{\hat{Y}^1}{\hat{Y}^0}) + x\hat{y}(x; \tfrac{\hat{Y}^1}{\hat{Y}^0}) \leq v(\hat{y}(x); \tfrac{\hat{Y}^1}{\hat{Y}^0}) + x\hat{y}(x) \tag{3.45}$$

where the inequality is due to $\hat{y}(x; \frac{\hat{Y}^1}{\hat{Y}^0})$ being the minimizer of the expression $v(y; \frac{\hat{Y}^1}{\hat{Y}^0}) + xy$. All together,

$$v(\hat{y}(x)) + x\hat{y}(x) = u(x) \leq u(x; \tfrac{\hat{Y}^1}{\hat{Y}^0}) \leq v(\hat{y}(x); \tfrac{\hat{Y}^1}{\hat{Y}^0}) + x\hat{y}(x). \tag{3.46}$$

Here the inequality in (3.8) was used. Now, $v(\hat{y}) = \mathbb{E}[V(\hat{Y}^0_T)]$, $x\hat{y}(x) = \mathbb{E}[\hat{\varphi}^0_{T+1}\hat{Y}^0_T] = x\hat{y}(x)$ and $\hat{Y}^0 \in \mathcal{Y}(\hat{y}(x); \frac{\hat{Y}^1}{\hat{Y}^0})$ imply that $v(\hat{y}(x); \frac{\hat{Y}^1}{\hat{Y}^0}) + x\hat{y}(x) \leq v(\hat{y}(x)) + x\hat{y}(x)$ which shows that the inequalities in (3.46) become equalities. Consequently $\hat{y}(x; \frac{\hat{Y}^1}{\hat{Y}^0}) = \hat{y}(x)$. $\square$

This proposition paves the way for a simple proof for the existence of a shadow price in a financial market modeled on a finite stochastic base. Indeed, a direct consequence of Theorem 2.2.7 is that the dual minimizer of the problem of maximizing utility from terminal wealth is a martingale if the underlying probability space is finite. Therefore, a shadow price process always exists in this case due to the proposition above. For financial market models based on general probability spaces, the dual minimizer may fail to be a martingale as Example 2.2.9 shows. Proposition 3.4.1 can hence not be applied in general.
Example 2.2.9 showed that there is a utility maximization problem for which the dual minimizer is not a martingale. As the underlying financial market model only contained one period, a shadow price however exists by Proposition 3.2.3. This shows that the converse of Proposition 3.4.1 is not true. The dual minimizer being a martingale is hence only a sufficient but not a necessary condition for the existence of a shadow price. Nevertheless, a similar assertion holds true, namely, if a shadow price exists, it is necessarily derived from a dual minimizer. This is the content of the proposition below.

Proposition 3.4.2. *If a shadow price $\tilde{S}$ exists, it is given by $\tilde{S} = \hat{Y}^1/\hat{Y}^0$ for a dual minimizer $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}^\lambda(\hat{y}(x))$.*

Proof. By definition of a shadow price, the solution $(\varphi^0, \varphi^1) \in \mathcal{A}(x; \tilde{S})$ to the frictionless utility maximization problem exists. Now suppose the market $\tilde{S} = (\tilde{S})_{t=0}^T$ admits arbitrage. This is obviously equivalent to the existence of an arbitrage opportunity (ξ^0, ξ^1) in one period. Here the arbitrage opportunity (ξ^0, ξ^1) is an admissible trading strategy which only trades in one period and expoits the arbitrage of the financial market $\tilde{S}$. Clearly,

$$\mathbb{E}[U(\varphi^0_{T+1})] < \mathbb{E}[U((\varphi^0 + \xi^0)_{T+1})]. \tag{3.47}$$

As $(\varphi^0+\xi^0, \varphi^1+\xi^1) \in \mathcal{A}(x; \tilde{S})$, this contradicts the assumption that (φ^0, φ^1) is the solution to the frictionless utility maximization problem. Therefore, the shadow market $\tilde{S}$ must be arbitrage-free.

The Fundamental Theorem of Asset Pricing for frictionless markets now implies the existence of an equivalent martingale measure Q for $\tilde{S}$. Since the density processes of equivalent martingale measures are contained in $\mathcal{Y}(y; \tilde{S})$ for all $y > 0$, it follows that $\mathcal{Y}(y; \tilde{S}) \neq \emptyset$ for all $y > 0$. Furthermore, as shown in the proof of Proposition 3.1.2, the inclusion $\mathcal{A}^\lambda(x) \subseteq \mathcal{A}(x; \tilde{S})$ holds for all $x \geq 0$. Hence, for an arbitrary $Y \in \mathcal{Y}(y; \tilde{S})$, the process $Y(\varphi^0 + \varphi^1\tilde{S})$ is a supermartingale for all $(\varphi^0, \varphi^1) \in \mathcal{A}^\lambda(1)$ by definition of $\mathcal{Y}(y; \tilde{S})$. This shows that $\mathcal{Y}(y; \tilde{S})$ can be embedded into $\mathcal{B}^\lambda(y)$ via the mapping

$$\mathcal{Y}(y; \tilde{S}) \to \mathcal{B}^\lambda(y) \tag{3.48}$$

$$Y \mapsto (Y, Y\tilde{S}) \tag{3.49}$$

In particular, one obtains similarly to (3.8):

$$v(y) = \inf_{(Y^0,Y^1)\in\mathcal{B}^\lambda(y)} \mathbb{E}[V(Y^0_T)] \leq \inf_{Y\in\mathcal{Y}(y;\tilde{S})} \mathbb{E}[V(Y_T)] =: v(y; \tilde{S}). \tag{3.50}$$

As in Theorem 2.2.8, let $\hat{y}(x) > 0$ denote the minimizer of

$$v(y) + xy, \tag{3.51}$$

and $\hat{y}(x; \tilde{S}) > 0$ the minimizer of

$$v(y; \tilde{S}) + xy. \tag{3.52}$$

Then Theorem 2.2.8 and its frictionless counterpart Theorem 2.1.12 imply

$$u(x) = v(\hat{y}(x)) + x\hat{y}(x) \tag{3.53}$$

and

$$u(x; \tilde{S}) = v(\hat{y}(x; \tilde{S})) + x\hat{y}(x; \tilde{S}). \tag{3.54}$$

Combining all the findings from above yields

$$\begin{aligned}
u(x) &= v(\hat{y}(x)) + x\hat{y}(x) && (3.55)\\
&\leq v(\hat{y}(x;\tilde{S})) + x\hat{y}(x,\tilde{S}) && (3.56)\\
&\leq v(\hat{y}(x;S))\tilde{S});\tilde{S}) + x\hat{y}(x,\tilde{S}) && (3.57)\\
&= u(x;\tilde{S}) = u(x), && (3.58)
\end{aligned}$$

and hence $\hat{y}(x) = \hat{y}(x;\tilde{S})$. As a result, $(\hat{Y}^0, \hat{Y}^1) := (\hat{Y}, \hat{Y}\tilde{S}) \in \mathcal{B}^\lambda(x)$ is the solution to the frictional dual problem (2.125) where $\hat{Y} \in \mathcal{Y}(\hat{y}(x;\tilde{S}))$ is the solution to its frictionless counterpart (2.67) for $\tilde{S}$. □

Chapter 4

Shadow Price Processes in Continuous Time

This section presents results about the existence of shadow price processes in continuous time. First of all, the underlying financial market model in continuous time will be developed, and the problem of utility maximization of an investor trading in this market will be introduced. By adapting the discrete time version of the shadow price problem established in the previous chapter to the present setting, the definition of shadow price processes in continuous time will be presented. The first result shows that in the case of utility functions with constant relative risk aversion existence is guaranteed whenever the value function is finite. By means of three numerical examples, it will be examined under which model parameters this holds. The second result is formulated in the more general framework of utility functions with reasonable asymptotic elasticity. It shows that shadow prices exist if short selling is ruled out.

The literature for this section is mainly covered by Choi et al. [CSZ12] and Benedetti et al. [BCKMK11]. As the proofs for the main results are very technical and require extensive preparation, they will be omitted, and the interested reader may be referred to the original papers.

4.1 The Financial Market Model

The financial market considered consists of one risk-free asset with price process $S^1 = (S^1_t)_{t\in[0,\infty)}$ and one risky asset whose price is given by

$$dS_t = S_t(\mu dt + \sigma dB_t), \qquad \text{for } t \in [0,T) \text{ with } S_0 > 0 \tag{4.1}$$

where B is the standard Brownian motion, and the parameters $\mu > 0$ and $\sigma > 0$ are assumed to be constants. While μ can be interpreted as the stock's mean rate of return, the parameter σ stands for its volatility. Depending on the context, the time horizon T will either be finite or infinite. Just as in the discrete time setting, the bank account is chosen as the numéraire, that is, $S^0_t = 1$ for all times $t \in [0,\infty)$ and the prices of the risky asset are expressed as multiples of S^0. The transaction costs in this market shall be proportional. To this end, let $\lambda \in (0,1)$ be fixed. Then, the risky asset can be sold at time t for the bid price $\underline{S}_t = (1-\lambda)S_t$ whereas it can be purchased for the ask price $\overline{S}_t = S_t$.

An investor trading in this financial market may allocate his wealth either in the bank account or in the risky asset. Furthermore, he may also consume his wealth. The investor starts with an initial value of $\eta = x$ in the bank account and $\eta^1 = 0$ units of stock. His strategy of investing the initial wealth dynamically in the two securities can be described by the following notion (compare [KMK10, Definition 2.1]).

A **trading strategy** is an $\mathbb{R}^2$-valued predictable process $\varphi = (\varphi^0, \varphi^1)$ of finite variation, where φ_t^0 and φ_t^1 denote the number of shares held in the bank account and in the stock at time t. A **consumption process** is an $\mathbb{R}_{\geq 0}$-valued, adapted stochastic process c satisfying $\int_0^u c_s du < \infty$ a.s. for all $t \geq 0$. The pair (φ, c) of a trading strategy φ and a consumption process c is called **portfolio/consumption pair**. Note that the finiteness requirement of a trading strategy's variation is essential, as positive transaction costs are inccured with every trade. Thus, transactions of infinite variation would lead to instantaneous bankruptcy.

The process φ^1 can be decomposed into one **purchase process** $\varphi^{\uparrow}$ and one **sales process** $\varphi^{\downarrow}$, i.e. $\varphi^1 = \varphi^{\uparrow} - \varphi^{\downarrow}$ where $\varphi^{\uparrow}$ and $\varphi^{\downarrow}$ are increasing and do not grow at the same time. Then, the well-known self-financing property from the discrete time setting transforms to the following in the continuous time case: A portfolio/consumption pair is called **self-financing** if

$$\varphi_t^0 = \varphi_0^0 + \int_0^t \underline{S}_u d\varphi_u^{\downarrow} - \int_0^t \overline{S}_t d\varphi_u^{\uparrow} - \int_0^t c_u du. \tag{4.2}$$

Moreover, the **liquidation value process** of a trading strategy φ is defined as

$$\mathrm{Liq}(\varphi) := \varphi^0 + (\varphi^1)^+ \underline{S} - (\varphi^1)^- \overline{S}. \tag{4.3}$$

Recall that in the discrete time case a trading stragtegy is admissible if the corresponding portfolio can be liquidated into a positive value at the terminal time. This definition clearly cannot be used in a continuous time setting with an infinite time horizon. Instead one employs the following definition: A self-financing portfolio/consumption pair (φ, c) starting with an initial wealth of x is called **admissible** if $(\varphi_0^0, \varphi_0^1) = (x, 0)$ and $\mathrm{Liq}(\varphi_t) \geq 0$ for all $t \in [0,T)$ a.s. The set of all admissible portfolio/consumption pairs is denoted by $\mathcal{A}^{\lambda}(x)$.

The next definition establishes a connection between financial markets with transaction costs and frictionless markets. An Itô-process $\tilde{S}$ is called a **consistent price process** if

$$\tilde{S}_t \in [\underline{S}_t, \overline{S}_t], \qquad \text{for all } t \in [0,T) \text{ a.s.} \tag{4.4}$$

Like in the discrete time case, the set of consistent price processes is denoted by $\mathcal{Z}^\lambda$. For such a frictionless price process $\tilde{S}$, admissibility of a portfolio/consumption pair will be defined as follows. In doing so, a slightly less general definition than the one used in [CSZ12, Section 2.4] is chosen. A portfolio/consumption pair (φ^0, φ, c) starting with an initial wealth of x is called **admissible** for the frictionless price process $\tilde{S}$ if the following four conditions hold:

(i) The components of the trading strategy (φ^0, φ) are predictable, and the consumption process c is adapted and satisfies $c_t \geq 0$ a.s. for all $t \in [0, T)$,

(ii) $\varphi_0^0 = x$,

(iii) The liquidation value V_t of (φ^0, φ, c) satisfies

$$V_t := \varphi_t^0 + \varphi_t \tilde{S} \geq 0 \quad \text{for all } t \in [0, T) \text{ a.s.,} \quad \text{and} \tag{4.5}$$

(iv) (φ^0, φ, c) is self-financing, i.e.

$$V_t = \varphi_0^0 + \int_0^t \varphi_u d\tilde{S}_t - \int_0^t c_u du \quad \text{for all } t \in [0, T) \text{ a.s.} \tag{4.6}$$

The set of of these admissible strategies is denoted by $\mathcal{A}(x; \tilde{S})$.

Furthermore, the set $\mathcal{C}(x; \tilde{S})$ denotes the contingent claims attainable from an initial endowment of x by trading with an admissible strategy in the frictionless market $\tilde{S}$. More precisely,

$$\mathcal{C}(x; \tilde{S}) = \{c \mid \text{there is } (\varphi^0, \varphi) \text{ such that } (\varphi^0, \varphi, c) \in \mathcal{A}(x; \tilde{S})\}. \tag{4.7}$$

The same notion can also be used in the market with transaction costs, that is,

$$\mathcal{C}^\lambda(x) = \{c \mid \text{there is } (\varphi^0, \varphi) \text{ such that } (\varphi^0, \varphi, c) \in \mathcal{A}^\lambda(x)\}. \tag{4.8}$$

4.2 Optimal Portfolios

A description of the two most common versions of the portfolio optimization problem in a continuous time setting can be found in [GMKS11b, Section 1]: On the one hand there is the problem of maximizing the **expected utility from consumption** which is typically formulated for an infinite time horizon, whereas the second version aims to maximize **expected utility from terminal wealth**. Obviously, the latter must be formulated in a setting with a finite time horizon. Explicitly, the two versions are defined as described below. Note that they are essentially the same as the optimal portfolios presented in Section 2.1.3, with the only difference being the continuous time setting.

1. Optimal Consumption

An admissible portfolio/consumption pair (φ, c) is said to maximize **expected utility from consumption** if it maximizes

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t} U(\kappa_t) dt\right] \tag{4.9}$$

over all admissible portfolio/consumption pairs $(\psi, \kappa) \in \mathcal{A}^\lambda(x)$. Here, U denotes a utility function as defined in Section 2.1.2 and the parameter δ stands for the impatience rate. The primal value function of the problem of maximizing expected utility from consumption is hence given by

$$u(x) := \sup_{c \in \mathcal{C}^\lambda(x)} \mathcal{U}(c) \quad \text{where} \quad \mathcal{U}(c) := \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t) dt\right]. \tag{4.10}$$

2. Terminal Wealth

An admissible portfolio/consumption pair (φ, c) is said to maximize **expected utility from terminal wealth** if it maximizes

$$\mathbb{E}\left[\mathrm{Liq}(\varphi_T)\right] \tag{4.11}$$

over all admissible portfolio/consumption pairs $(\psi, \kappa) \in \mathcal{A}^\lambda(x)$.

Another well-studied optimization problem in continuous time is maximizing the growth rate of a portfolio. As this will not be needed in the remainder, the reader who is interested in details shall be referred to [GMKS11b] or [GMK12, Section 3.1] for example.

4.3 The Shadow Price Problem

In this section, the same problem that has already been encountered in the discrete time setting will be presented in continuous time. First of all, the proposition below describes the continuous time counterpart of Proposition 3.1.2. It shows what is intuitively evident, namely that trading in the frictionless market leads to a higher expected utility than trading in the market where transaction costs are present.

Proposition 4.3.1 (cf. [CSZ12, Proposition 2.1.]). *$\mathcal{C}^\lambda \subseteq \mathcal{C}(x; \tilde{S})$ for each $\tilde{S} \in \mathcal{Z}^\lambda$.*

Proof. Fix an $\tilde{S} \in \mathcal{Z}^\lambda$ and let $c \in \mathcal{C}^\lambda$. Then there is a trading strategy (φ^0, φ) such that $(\varphi^0, \varphi, c) \in \mathcal{A}^\lambda$. The self-financing condition, the inequality $\underline{S} \leq \tilde{S} \leq \overline{S}$ and integration by parts imply

$$-\int_0^t c_u du = \varphi_t^0 - \varphi_0^0 + \int_0^t \overline{S}_u d\varphi_u^{\downarrow} - \int_0^t \underline{S}_t d\varphi_u^{\uparrow} \tag{4.12}$$

$$\geq \varphi_t^0 - \varphi_0^0 + \int_0^t \tilde{S}_u d\varphi_u \tag{4.13}$$

$$= \varphi_t^0 - \varphi_0^0 + \tilde{S}_u\varphi_u\Big|_0^t - \int_0^t \tilde{S}_u d\varphi_u \tag{4.14}$$

$$= \varphi_t^0 - \varphi_0^0 + \tilde{S}_t\varphi_t - \tilde{S}_0\varphi_0 - \int_0^t \tilde{S}_u d\varphi_u. \tag{4.15}$$

On the other hand, it follows from the admissibility condition and $\underline{S} \leq \tilde{S} \leq \overline{S}$ that

$$0 \leq \varphi_t^0 + (\varphi_t)^+ \underline{S}_t + (\varphi_t)^- \overline{S}_t \leq \varphi_t^0 + \varphi_t \tilde{S}_t. \tag{4.16}$$

All together, one obtains

$$-\int_0^t c_u du + \int_0^t \varphi_u d\tilde{S}_u + \tilde{S}_0\varphi_0 + \varphi_0^0 \geq \varphi_t^0 + \tilde{S}_t \geq 0. \tag{4.17}$$

Now a new trading strategy $(\tilde{\varphi}^0, \tilde{\varphi})$ shall be defined by setting $\tilde{\varphi} := \varphi$ and

$$\tilde{\varphi}_t^0 := \varphi_0^0 + \int_0^t \tilde{\varphi}_u d\tilde{S}_u - \int_0^t c_u du - \tilde{\varphi}_t \tilde{S}_t + \varphi_0 \tilde{S}_0. \tag{4.18}$$

The inequality in (4.17) then implies

$$\tilde{\varphi}^0 + \tilde{\varphi}\tilde{S}_t = \varphi_0^0 + \int_0^t \tilde{\varphi}_u d\tilde{S}_u - \int_0^t c_u du - \tilde{\varphi}_t \tilde{S}_t + \varphi_0 \tilde{S}_0 + \tilde{\varphi}_t \tilde{S}_t \geq 0 \tag{4.19}$$

which verifies Condition (iii) in the definition of admissibility in the frictionless market $\tilde{S}$. Hence, $(\varphi^0, \varphi, c) \in \mathcal{A}(\tilde{S}; x)$. □

This proposition paves to way to defining shadow prices in continuous time. To begin with, let $\tilde{S}$ be any consistent price process and regard the auxiliary problem

$$u(x; \tilde{S}) := \sup_{c \in \mathcal{C}(\tilde{S}; x)} \mathcal{U}(c) \tag{4.20}$$

where $\mathcal{U}(c)$ is defined as in (4.10). Proposition 4.3.1 then implies

$$u(x) \leq \inf_{\tilde{S} \in \mathcal{Z}} u(\tilde{S}; x). \tag{4.21}$$

Here, $u(x)$ denotes the value function of the original problem as presented in (4.10). Just as in the discrete time setting, the goal is to make the inequality gap in the problem above disappear. In this case, there would exist a frictionless market which leads to the same optimal expected utility as the original market with transaction costs. This motivates the central definition of this chapter:

Definition 4.3.2. A consistent price process $\tilde{S}$ is called a **shadow price** if $u(x) = u(x; \tilde{S})$.

The economic intuition that a shadow price candidate must coincide with the relevant values of the original frictional price process whenever a trade is conducted is also true in the present continuous time setting as the following proposition demonstrates. It is the continuous time version of Proposition 3.1.2.

Proposition 4.3.3 (cf. [CSZ12, Proposition 2.2.]). *Given $\tilde{S} \in \mathcal{Z}$, let $c \in \mathcal{C}(\tilde{S})$ be such that there exists $(\varphi^0, \varphi, c) \in \mathcal{A}(\tilde{S}; x)$ which satisfies the following conditions:*

$$\int_0^t \overline{S}_u d\varphi_u^{\uparrow} = \int_0^t \tilde{S}_u d\varphi^u \quad \text{and} \quad \int_0^t \underline{S}_u d\varphi_u^{\downarrow} = \int_0^t \tilde{S}_u d\varphi^u \tag{4.22}$$

Then, $c \in \mathcal{C}^\lambda(x)$.

The remainder of this chapter is devoted to examining whether a shadow price exists in this framework.

4.4 Existence Results

The first result regarding the existence of shadow price processes in continuous time is formulated for utility functions with constant relative risk aversion. Recall from Section 2.1.2 that these utility functions are given by

$$U(x) = \begin{cases} \frac{1}{p}x^p, & \text{for } x \neq 0, p \neq 0, \\ \log(x), & \text{for } x \neq 0, p = 0, \end{cases} \quad \text{and} \quad U(0) = \begin{cases} 0, & \text{for } p > 0, \\ -\infty, & \text{for } p \leq 0, \end{cases} \tag{4.23}$$

for a fixed $p \in (-\infty, 1)$. The optimization problem that will be considered is maximizing utility from consumption. This problem is said to be **well posed** if for a given initial endowment x the primal value function u satisfies $-\infty < u(x) < \infty$. The following theorem describes the region for the model parameters in which the problem is well posed. It also states the converse, that is, the primal value function is not finite in a model with parameter combinations lying outside of this region.

Theorem 4.4.1 (cf. [CSZ12, Theorem 2.6.]). *Given the initial endowment of $\eta \in \mathbb{R}$, the risk aversion parameter $p \in (-\infty, 1)$, the impatience rate $\delta > 0$, the environment parameters $\mu > 0$, $\sigma > 0$ and transaction costs $\lambda \in (0, 1)$, the following statements are equivalent:*

(i) The problem is well posed.

(ii) The parameters of the model satisfy one of the following three conditions:

- $p \leq 0$,
- $0 < p < 1$ *and* $\mu < \sqrt{\frac{2\delta(1-p)\sigma^2}{p}}$,
- $0 < p < 1$, $\sqrt{\frac{2\delta(1-p)\sigma^2}{p}} \leq \mu < \frac{\delta}{p} + \frac{(1-p)\sigma^2}{2}$ *and* $C(\mu, \sigma, p, \delta) < \log(\frac{1}{1-\lambda})$ *where the function* C *is defined as in Appendix B.1.*

The following example illustrates the region of values of λ and μ for which the well-posedness property holds, given a fixed volatility and investor preferences.

Example 4.4.2. Consider a stock with volatility $\sigma = 0.2$ and an investor with risk aversion $p = 0.5$ and impatience rate $\delta = 0.1$. The proportional transaction costs shall amount $\lambda = 0.05$. According to Theorem 4.4.1 the problem is well posed for all mean rates of return which satisfy

$$\mu < \sqrt{\frac{2\delta(1-p)\sigma^2}{p}} = \sqrt{0.008} =: G. \tag{4.24}$$

Note that this boundary is independent of the size of the transaction costs. In addition, if the mean rate of return μ were to exceed this boundary, the well-posedness property would also hold if

$$G \leq \mu < \frac{\delta}{p} + \frac{(1-p)\sigma^2}{2} = 0.21 \quad \text{and} \quad C(\mu, \sigma, p, \delta) < -\log(1-\lambda) = -\log(0.95). \tag{4.25}$$

The resulting region of feasible values of μ is illustrated in the figure below.

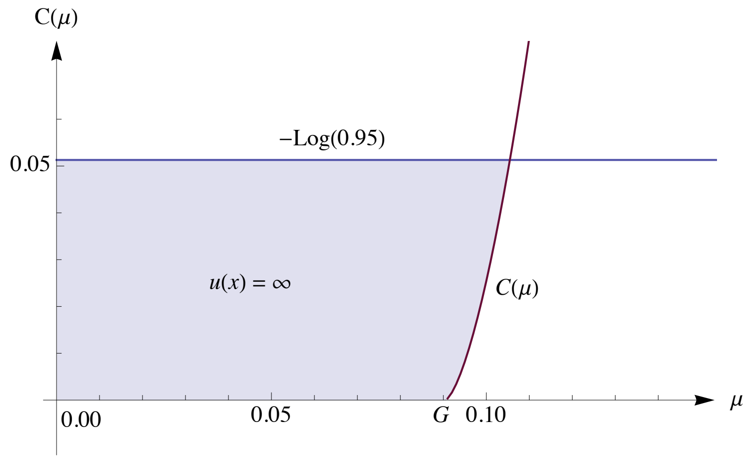

Figure 4.1: The well-posedness region depending on the mean rate of return.

As described in the inequalites above, the well-posedness region is bounded by the functions $-\log(0.95)$ and $C(\mu) := C(\mu, \sigma, p, \delta)$. The highest possible value the mean rate of return may reach without leading to arbitrage is hence attained at the intersection of these functions. A numerical computation of this intersection with 7-digit precision yields that the maximum value of μ is given by 0.1044861. All together, this implies that the problem is well posed for all mean values of return which satisfy

$$\mu < 0.1044861. \tag{4.26}$$

Furthermore, a rise in transaction costs λ forces the boundary $-\log(1-\lambda)$ to go up. The figure suggests that this increases the size of the well-posedness region allowing for higher mean rates of return. Economically, this is perfectly clear: Higher transaction costs have a negative effect on the investor's wealth. Hence, higher mean rates of return of the stock are required to reach the same maximal expected utility. $\triangleleft$

Another interesting task is to fix the size of the transaction costs and examine for which values of μ and σ the problem is well posed. This will be done in the following example.

Example 4.4.3. Consider an investor with risk aversion $p = 0.5$ and impatience rate $\delta = 0.1$ trading in a financial market with proportional transaction costs $\lambda = 0.05$. The second condition in (ii) in the theorem above implies that the well-posedness region includes all pairs (μ, σ) which satisfy

$$\mu < \sqrt{\frac{2\delta(1-p)\sigma^2}{p}} = \sqrt{0.2}\sigma. \tag{4.27}$$

Furthermore, by the third condition in (ii), the problem is well posed if the following inequalities hold:

$$\sqrt{0.2}\sigma \leq \mu < \frac{1}{5} + \frac{\sigma^2}{4} \quad \text{and} \quad C(\mu, \sigma, 0.5, 0.1) < -\log(0.95). \tag{4.28}$$

While the first inequality can be plotted easily, the second is hardly traceable due to the high complexity of the function C. Therefore, a Monte-Carlo simulation will be conducted in order to grasp a feeling for what values of μ and σ this inequality holds. 50000 random pairs (μ, σ) are generated by employing a uniform random distribution on the region $[0, 0.3] \times [0, 0.6]$. The figure below illustrates the results of this experiment and also contains the region which is bounded by the inequality in (4.27). The light blue shaded region derives from the exact solution of the first inequality above and the blue dots from the Monte-Carlo simulation.

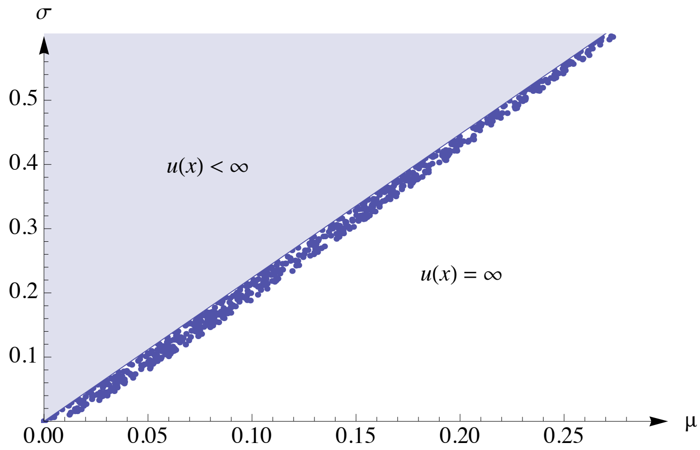

Figure 4.2: The well-posedness region depending on the stock's properties.

The figure indicates that there is an almost linear boundary of the well-posedness region. The feasible pairs (μ, σ) which derive from the the third condition in (ii) only add a small strip to the region. The plot also shows what is clear from an economic point of view: stocks with high mean rates of return and relatively low volatility lead to an infinite expected wealth. For example, given the present investor's preferences and transaction costs, a stock with a mean rate of return of 15% must have a volatility of at least 25% in order to avoid infinite expected wealth. $\triangleleft$

While both examples above deal with parameters regarding the financial market, the next example shall clarify for what kinds of investor preferences the well-posedness property holds.

Example 4.4.4. Let a stock with mean rate of return $\mu = 0.1$ and $\sigma = 0.2$ be given. Trading this stock incurs proportional transaction costs of the size $\lambda = 0.05$. Inserting these numbers into the inequality in the first condition in (ii) above yields that the problem is well posed for all preference pairs (p, δ) with

$$\delta > \frac{5}{4}\sqrt{\frac{p}{1-p}}. \tag{4.29}$$

The third condition transforms to

$$\sqrt{\frac{0.08\delta(1-p)}{p}} \leq 0.1 < \frac{\delta}{p} + 0.02(1-p) \quad \text{and} \quad C(0.1, 0.02, p, \delta) < -\log(0.95). \tag{4.30}$$

Again, as the second inequality can hardly be traced, a Monte-Carlo simulation will be conducted. 50000 random preference pairs (p, δ) are generated by employing a uniform random distribution on the region $[0, 1] \times [0, 0.2]$. The following figure describes the well-posedness region which is implied by the two conditions above. Just as in the example above, the light blue shaded region and the blue dots derive from the exact solution and the Monte-Carlo simulation, respectively.

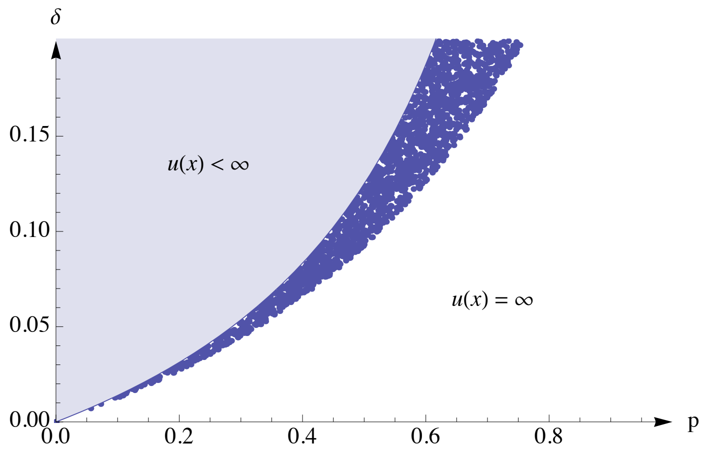

Figure 4.3: The well-posedness region depending on the investor's preferences.

It can be observed that lower levels of risk aversion require higher impatience rates for the problem to be well posed. This connection can be explained economically by considering the extreme case of a very low risk averse investor who also has a low impatience rate. Very high but at the same time unrealistic pay-offs are valued higher for low level risk averse investors compared to higher risk averse ones. Since the value of these unrealistic outliers grows as time proceeds, a high impatience rate is needed to decrease their impact on the expected wealth. Moreover, it can be seen that the boundary of the well-posedness region clearly has a convex shape. ◁

The reader may now wonder why so much effort is put into examining for which model parameters the well-posedness property holds. The answer is given in the following theorem. It shows that there exists a shadow price if the utility maximization problem is well posed. It also describes the form and regularity of the shadow price and shows that it is the solution of a free boundary stochastic differential equation. Finally, it sheds light on the optimal investment/consumption strategy of the utility maximization problem.

Theorem 4.4.5 (cf. [CSZ12, Theorem 2.6.]). *Suppose the initial endowment of $\eta_B \in \mathbb{R}$, the risk aversion parameter $p \in (-\infty, 1)$, the impatience rate $\delta > 0$, the environment parameters $\mu > 0$, $\sigma > 0$ and transaction costs $\lambda \in (0,1)$ satisfy the well-posedness conditions of Theorem 4.4.1. Then the following assertions hold.*

(i) There exist constants $\underline{x}, \overline{x}$ with $0 < \underline{x} < \overline{x}$ and a function $g \in C^2[\underline{x}, \overline{x}]$ such that

(a) $g'(x) > 0$ for $x \in (\underline{x}, \overline{x})$, and g satisfies the equation

$$\inf_{\Sigma,\theta\in\mathbb{R}} \left(\frac{1}{2}\Sigma^2 \frac{x}{g'(x)} - \alpha_q(\Sigma,\theta)x - \beta(\theta)g(x) + \gamma(\theta) \right) = 0 \text{ for } x \in (\underline{x}, \overline{x}) \qquad (4.31)$$

where

$$q = \frac{p}{1-p}, \;\; \alpha_p(\Sigma, \theta) = \theta\sigma - \mu - \Sigma\left(\frac{1}{2}\Sigma + \sigma - \theta(1+q)\right), \tag{4.32}$$

$$\beta(\theta) = (1+q)\left(\delta - \frac{1}{2}q\theta^2\right) \;\; \text{and} \;\; \gamma(\theta) = \begin{cases} \frac{1}{2}\theta^2, & p = 0, \\ \operatorname{sgn}(p), & p \neq 0, \end{cases} \tag{4.33}$$

(b) the following boundary/integral conditions are satisfied:

$$g'(\underline{x}+) = g'(\overline{x}-) = 0 \;\; \text{and} \;\; \int_{\underline{x}}^{\overline{x}} \frac{g'(x)}{x} dx = \log\left(\frac{1}{1-\lambda}\right), \quad \text{and} \tag{4.34}$$

(c) the function $h : [\underline{x}, \overline{x}] \to \mathbb{R}$*, defined by*

$$h(x) = \begin{cases} (1-x)g'(x) + 1, & p = 0, \\ qg(x)(g'(x)+1) - (q+1)xg'(x), & p \neq 0, \end{cases} \tag{4.35}$$

admits no zeros on its domain.

(ii) There exists a shadow price $(\tilde{S}_t)_{t\in[0,\infty)}$*. It is of the form* $\tilde{S}_t = S_t \exp(f(X_t^x))$*, where*

- $f(x) = \underline{y} + \int_{x}^{\overline{x}} \frac{g'(t)}{t} dt$ *for* $x \in [\underline{x}, \overline{x}]$*,*
- *the value of the constant* $\hat{x}$ *is determined as described in Appendix B.2,*
- $X^{\hat{x}}$ *is the unique solution of the SDE presented in Appendix B.3 with* $X_0^{\hat{x}} = \hat{x}$*.*

(iii) The value $u(\eta)$ *and an optimal investment/consumption strategy* $(\hat{\varphi}^0, \hat{\varphi}, \hat{c})$ *for the main problem (4.9) are given by*

$$u(\eta) = \hat{u}(\eta; \hat{x}), \quad (\hat{\varphi}_t^0, \hat{\varphi}_t, \hat{c}_t) = (\hat{\varphi}_t^{0,\hat{x}}, \hat{\varphi}_t^{\hat{x}}, \hat{c}_t^{\hat{x}}), \tag{4.36}$$

where $\hat{x}$*,* $\hat{u}, \hat{\varphi}_t^{0,\hat{x}}, \hat{\varphi}_t^{\hat{x}}$ *and* $\hat{c}_t^{\hat{x}}$ *are defined as described in Appendix B.2.*

While this theorem guarantees the existence of a shadow price, it is unclear whether it can be explicitly calculated. After all, as stated in (ii), an explicit construction would require a solving the SDE presented in Appendix B.3. A mere glance at the functions and variables involved in this SDE indicates that this is hardly manageable. The theorem further restricts general applications as it is only formulated for CRRA utility functions.

For the more general class of utility functions with reasonable asymptotic elasticity, Benedetti et al. [BCKMK11] show that the existence of a shadow price process is guaranteed if the model does not allow short sales. In fact, they prove this assertion in a much more general framework than the one considered here in this chapter, namely in Kabanov's general multi-currency market model (see Example 2.2.2). As their approach requires technical and extensive preparation, details will be omitted. In addition, the scope of this result is quite limited because most well-studied models of financial markets typically allow short selling.

Appendix A

Preliminaries from Convex Analysis

This chapter lays out the technical foundations on which major parts of this thesis are based on. More specifically, important definitions and results from convex analysis will be highlighted. First, the basic topological notions such as affine sets and convex sets will be established. Based on this, convex optimization problems will be introduced and methods for solving them will be presented.

The information is essentially compiled from standard literature in the field of convex analysis such as Rockafellar [Roc97], Borwein and Lewis [BL06], and Boyd and Vandenberghe [BV04]. Proofs for the mentioned assertions will be omitted. They can be found in the references.

A.1 Affine Sets and Functions

Let X denote a vector space throughout this section. A subset $M \in X$ is called an **affine set** if

$$\lambda x + (1 - \lambda) y \in M \tag{A.1}$$

for all $x, y \in M$ and $\lambda \in \mathbb{R}$. This means that the one-dimensional subset generated by the line through two points of an affine set is contained within this set. Thus, the intersection of affine sets is clearly affine again, and one defines the **affine hull** of a set $S \in X$, denoted by $\mathrm{aff}(S)$, as the smallest affine set containing S, i.e.

$$\mathrm{aff}(S) = \bigcap_{\substack{S \subset M \subset X \\ M \text{ affine}}} M. \tag{A.2}$$

Moreover, a linear combination of vectors $x_1, ..., x_n \in X$ with the sum of the coefficients being 1 is called an **affine combination**. One can show that $\mathrm{aff}(S)$ is equal to the set of all affine combinations of elements of S, that is,

$$\mathrm{aff}(S) = \left\{ \sum_{i=1}^{k} \alpha_i x_i | x_i \in S, \alpha_i \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i = 1 \right\}. \tag{A.3}$$

In addition, it is sometimes required to utilize functions between two vector spaces X and Y which map the line through x and y in X to the line through $f(x)$ and $f(y)$ in Y. A function $f : X \to Y$ is said to be an **affine function** if it satisfies this property, i.e.

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y) \qquad \text{for all } x, y \in X \text{ and all } \lambda \in \mathbb{R}. \tag{A.4}$$

A.2 Convex Sets and Functions

The fundamental notion of this chapter starts with the following definition: A subset $C \in X$ is called a **convex set** if

$$\lambda x + (1-\lambda)y \in C \tag{A.5}$$

for all $x, y \in M$ and $\lambda \in [0,1]$. Geometrically speaking, the line connecting two points of a convex set is again in this set. Similar to above, one may define the **convex hull** of a set $S \in X$ by

$$\mathrm{conv}(S) = \bigcap_{\substack{S \subset C \subset X \\ C \text{ convex}}} C. \tag{A.6}$$

A **convex combination** of vectors $x_1, ..., x_n \in X$ is an affine combination where the linear coefficients are additionally nonnegative, that is, a sum of the form

$$\sum_{i=1}^{n} \lambda_i x_i, \qquad \text{where } \lambda_i \geq 0 \text{ and } \sum_{i=1}^{n} \lambda_i = 1. \tag{A.7}$$

Now consider a convex set $C \in \mathbb{R}^n$ and regard it as a subset of its affine hull $\mathrm{aff}(C)$. The interior of C relative to $\mathrm{aff}(C)$ is called the **relative interior** of C and denoted by $\mathrm{ri}(C)$. More precisely,

$$\mathrm{ri}(C) = \{x \in \mathrm{aff}(C) \mid \exists \varepsilon > 0, B_\varepsilon(x) \cap \mathrm{aff}(C) \subset C\} \tag{A.8}$$

where $B_\varepsilon(x)$ is the open ball of radius ε centered on x.

Let f be a function from X into the extended real line $[-\infty, \infty]$. The subset of $X \times \mathbb{R}$ defined by

$$\mathrm{epi}(f) = \{(x, \mu) \mid x \in S, \mu \in \mathbb{R}, \mu \geq f(x)\} \tag{A.9}$$

is called the **epigraph** of f. A function is said to be **convex** if $\mathrm{epi}(f)$ is convex as a subset of $\mathbb{R}^{n+1}$, and **concave** if $-f$ is convex. For a convex function f, the **effective domain**, denoted by $\mathrm{dom}(f)$, is the projection of the epigraph on $\mathbb{R}^n$, i.e.

$$\mathrm{dom}(f) = \{x \in X \mid f(x) < +\infty\}. \tag{A.10}$$

A.3 Constrained Optimization Problems

Having established these basic definitions, the focus will now be placed on optimization problems starting with the differentiable, inequality constrained problem

$$(\mathrm{P}) \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0 \text{ for } i \in I, \\ & x \in C \end{cases} \tag{A.11}$$

where C is a subset of X, I is a finite index set and the **objective function** f and the **inequality constraints** g_i $(i \in I)$ are differentiable. A vector x is called a **feasible solution** to (P) if $x \in C$ and x satisfies the constraints of (P). The infimum of f in C is the **optimal value** in (P) and the points where this infimum is attained are called **optimal solutions** to (P).

The **Lagrangian** of this problem is defined as the function

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x), \tag{A.12}$$

where $\mathrm{dom}(L) = C \times \mathbb{R}^m_{>0}$. A vector $\lambda \in \mathbb{R}^m_{>0}$ is called **dual variable** or **Lagrange multiplier** for $\overline{x}$ if $\overline{x}$ is a critical point of the Lagrangian with respect to λ, i.e.

$$\nabla f(\hat{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\hat{x}) = 0. \tag{A.13}$$

where ∇f denotes the gradient of a function f. In addition, the **Lagrange dual function** is a function $g : \mathbb{R}^m \to \mathbb{R} \cup \{-\infty\}$ with

$$g(\lambda) = \inf_{x \in C} L(x, \lambda) = \inf_{x \in C} \left(f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right). \tag{A.14}$$

Note that this function is concave as it is the pointwise infimum of a family of affine functions of λ. For every $\tilde{x} \in C$ which is a feasible point of (P), it obviously follows that $L(\tilde{x}, \lambda) \leq f(\tilde{x})$ for all $\lambda \geq 0$. Hence,

$$g(\lambda) = \inf_{x \in C} L(x, \lambda) \leq L(\tilde{x}, \lambda) \leq f(\tilde{x}). \tag{A.15}$$

Since this equation holds for all $\lambda \geq 0$ and every feasible point, it yields a lower bound on the optimal value, namely

$$\sup_{\lambda \geq 0} g(\lambda) \leq f(\hat{x}) \qquad \text{for all } \lambda > 0. \tag{A.16}$$

This leads to defining the **Lagrange dual problem** associated with the **primal problem** (P):

$$(\mathrm{D}) \begin{cases} \text{maximize} & g(\lambda) \\ \text{subject to} & \lambda_i \geq 0 \text{ for } 1 \leq i \leq m, \\ & \lambda \in \mathbb{R}^m \end{cases} \tag{A.17}$$

In general, the Lagrange dual problem does not lead to the same optimal value as the primal problem. The difference in the optimal values, i.e. $f(\overline{x}) - g(\overline{\lambda})$, is called the **duality gap**. Consequently, the primal problem can be reduced to the dual problem if the duality gap vanishes. The following theorem states a sufficient condition for this to hold.

Theorem A.3.1 (Dual Attainment, cf. [BL06, Theorem 4.3.7])**.** *If f and g_i $(i \in I)$ are convex and if there exists an $x \in dom(f)$ with $g_i(x) < 0$ for $1 \leq i \leq m$, then the primal and the dual values are equal, and the dual value is attained if finite.*

In general applications, the objective function and the functions pertaining to the constraints may not be differentiable but only convex. This leads to the following definition: An **ordinary convex program** (OCP) is a problem of the form

$$(\mathrm{OCP}) \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g_1(x) \leq 0, ..., g_r(x) \leq 0, \\ & h_1(x) = 0, ..., h_m(x) = 0, \\ & x \in C \end{cases} \tag{A.18}$$

where f and g_i for $1 \leq i \leq r$ are real valued convex functions on C and h_i for $1 \leq i \leq m$ affine functions on C.

For ordinary convex programs, there is an important notion of optimal values: A vector $(\lambda_1, ..., \lambda_{r+m}) \in \mathbb{R}^{r+m}$ is said to be a **Kuhn-Tucker vector** for (OCP) if

$$\lambda_i \geq 0 \qquad \text{for } i = 1, ..., r \tag{A.19}$$

and if the infimum of

$$f + \lambda_1 g_1 + ... + \lambda_r g_r + \lambda_{r+1} h_1 + ... + \lambda_{r+m} h_m \tag{A.20}$$

over C is finite and equal to the optimal value in (OCP). The connection of these definitions becomes clear in the following proposition which guarantees that an optimal solution exists under mild conditions.

Proposition A.3.2 (compare [Roc97, Theorem 28.2])**.** *Let (P) be an ordinary concave program with only affine constraints. If the optimal value in (P) is not $-\infty$ and (P) has* *a feasible solution in ri(C), then a Kuhn-Tucker vector (not necessarily unique) exists for (P).*

In fact, a much more general result exists. However, as it will not be needed in this thesis, the interested reader may be referred to [Roc97, Theorem 28.2.]. The central result of this section is the theorem below. It provides a method of finding the Kuhn-Tucker vector and, consequently, the optimal solution to the optimization problem (P).

Theorem A.3.3 (Karush-Kuhn-Tucker conditions)**.** *Let (P) be an ordinary convex program where the objective function and the constraint functions are differentiable. Let $u \in \mathbb{R}^{r+m}$ and $x^* \in \mathbb{R}^n$ be vectors. Then u is a Kuhn-Tucker vector for (P) and $\hat{x}$ an optimal solution to (P) if and only if $\hat{x}$ and the components λ_i of u satisfy*

(i) $\lambda_i \geq 0$, $g_i(\hat{x}) \leq 0$ *and* $\lambda_i g_i(\hat{x}) = 0$ *for* $i = 1, ..., r$

(ii) $h_i(\hat{x}) = 0$ *for* $i = 1, ..., m$

(iii) $\nabla f(\hat{x}) + \sum_{i=1}^{r} \lambda_i \nabla g_i(\hat{x}) + \sum_{i=r+1}^{r+m} \lambda_j \nabla h_j(\hat{x}) = 0$

Proof. This is a subcase of the more general result which is formulated for subdifferentials instead of gradients. The assertion above follows directly from [Roc97, Theorem 28.3.] by observing that the subdifferential of a differentiable convex function coincides with its gradient. ☐

Appendix B

Explicit Formulae for Chapter 4

B.1 Formulae for Theorem 4.4.1

Let the model parameters $\delta > 0$, $p \in (-\infty, 1)$, $\mu > 0$, $\sigma > 0$ be given. The functions $A(\sigma, p, \delta)$ and $G(\sigma, p, \delta)$ shall denote the arithmetic and geometric mean of the quantities $\frac{2\delta}{p}$ and $(1-p)\sigma^2$. More precisely,

$$A(\sigma, p, \delta) := \frac{\delta}{p} + \frac{(1-p)p^2}{2} \quad \text{and} \quad G(\sigma, p, \delta) := \sqrt{\frac{2\delta(1-p)\sigma^2}{p}}. \tag{B.1}$$

Another auxiliary function will be needed, namely

$$K(\mu, \sigma, p, \delta) := \frac{(1-p)(m - G(\sigma, p, \delta))}{(A(\sigma, p, \delta) - \mu) + p(\mu - G(\sigma, p, \delta))}. \tag{B.2}$$

Furthermore, for $k \in \mathbb{R}$, $X_1(k)$ and $X_2(k)$ stand for the solutions of the quadratic equation

$$a(k)X^2 - b(k)X + c(k) = 0 \tag{B.3}$$

where the functions $a(k)$, $b(k)$ and $c(k)$ are given by

$$a(k) = 2p\delta(1 + k), \tag{B.4}$$

$$b(k) = (2\delta + p(1-p)(2\mu - \sigma^2))k + 2p(1-p)\mu, \tag{B.5}$$

$$c(k) = (1-p)(2\mu + (p^2 - 1)\sigma^2)k + (1-p)^3\sigma^3. \tag{B.6}$$

Then the minimum of $\{X_1(k), X_2(k)\}$ is denoted by $l_d(k)$ whereas $l_u(k)$ stands for the maximum. Having developed all these auxiliary functions, it is finally possible to define the boundary $C(\mu, \sigma, p, \delta)$ of the well-posedness region:

$$C(\mu, \sigma, p, \delta) := \int_0^{K(\mu,\sigma,p,\delta)} k \left(\frac{l_u'(k)}{k - l_u(k)} - \frac{l_d'(k)}{k - l_d(k)} \right) dk. \tag{B.7}$$

B.2 Formulae for Theorem 4.4.5

Let the model parameters $\eta \in \mathbb{R}$, $p \in (-\infty, 1)$, $\delta > 0$, $\mu > 0$, $\sigma > 0$, $\lambda \in (0,1)$ be given such that they satisfy the well-posedness conditions of Theorem 4.4.1. With respect to these parameters, the function g denotes the solution to equation (4.31), and the constants $\underline{x}$, $\overline{x}$ are defined as in Part (i) of the theorem. Moreover, the parameter q and the function $\beta : \mathbb{R} \to \mathbb{R}$ shall be defined as in Part (a), the mapping $h : [\underline{x}, \overline{x}] \to \mathbb{R}$ as in Part (c).

Another pair of variables, namely $[\underline{y}, \overline{y}]$, is given by $\underline{y} = \log(1-\lambda)$ and $\overline{y} = 0$. For $x \in [\underline{x}, \overline{x}]$, the function $f : [\underline{x}, \overline{x}] \to \mathbb{R}$ is then defined by $f(x) = \underline{y} + \int_x^{\overline{x}} \frac{g'(\xi)}{\xi} d\xi$ and the process $(Y_t^x)_{t\in[0,\infty)}$ by $Y_t^x = f(X_t^x)$. Here the process X is the solution of the stochastic differential equation presented in the following section. Building up on this, the price process $\hat{S}$ is defined by $\hat{S}_t^x = S_t e^{Y_t^x}$.

Having established this terminological foundation, define the mapping $r : [\underline{x}, \overline{x}] \to \mathbb{R}$ by

$$r(x) = \begin{cases} -\eta x, & p = 0, \\ -\eta \frac{x}{qg(x)}, & p \neq 0, \end{cases} \tag{B.8}$$

and the constant $\hat{x}$ by

$$\hat{x} := \begin{cases} \overline{x}, & r(x) > 0 \text{ for all } x \in [\underline{x}, \overline{x}], \\ \underline{x}, & r(x) < 0 \text{ for all } x \in [\underline{x}, \overline{x}], \\ \text{a solution to } r(x) = 0, & \text{otherwise.} \end{cases} \tag{B.9}$$

Regarding Part (iv) of the theorem, the function $\hat{u} : \mathbb{R}_{\geq 0} \times [\underline{x}, \overline{x}] \to \mathbb{R}$ is defined by

$$\hat{u}(\eta; x) = \begin{cases} \frac{1}{\delta}(-1 + \log(\delta\eta) + g(x)), & p = 0, \\ \frac{1}{p}\eta^p |g(x)|^{1-p}, & p \neq 0. \end{cases} \tag{B.10}$$

In order to simplify notation the following abbreviations will be used in the sequel:

$$\Pi(x) = \begin{cases} x, & p = 0, \\ \frac{x}{qg(x)}, & p \neq 0, \end{cases} \quad \text{and} \quad K(x) = \begin{cases} \delta, & p = 0, \\ \frac{1}{|g(x)|}, & p \neq 0. \end{cases} \tag{B.11}$$

Now the parameters $\hat{\pi}_t^x$ and $\hat{\kappa}_t^x$ shall be defined by

$$\hat{\pi}_t^x = \Pi(X_t^x) \quad \text{and} \quad \hat{\kappa}_t^x = K(X_t^x). \tag{B.12}$$

Then, one defines

$$\hat{V}_t^x = \eta \mathcal{E}\left(\int_0^\cdot \frac{\hat{\pi}_u^x}{\hat{S}_t^x} d\hat{S}_u^x - \int_0^\cdot \hat{\kappa}_u^x du \right)_t \tag{B.13}$$

where $\mathcal{E}$ denotes the stochastic exponential. Recall that the stochastic exponential of a semimartingale X is definded as a solution to the stochastic differential equation

$$dY_t = Y_t dX_t \tag{B.14}$$

with initial condition $Y_0 = 1$.

Using these notational abbreviations, the optimal strategy $(\hat{\varphi}^{0,x}, \hat{\varphi}^x, c^x)$ can be expressed, for $t \geq 0$, as

$$\hat{c}_t^x = \hat{V}_t^x \hat{\kappa}_t^x, \quad \hat{\varphi}^{0,x} = \hat{V}_t^x(1 - \hat{\pi}_t^x) \quad \text{and} \quad \hat{\varphi}_t^x = \frac{\hat{V}_t^x \hat{\pi}_t^x}{\hat{S}_t^x}. \tag{B.15}$$

B.3 Reflected Skorokhod SDE

Again, the same foundation of variables and functions as in the preceeding section will be employed. One may then define the functions $\hat{\Sigma}, \hat{\Gamma}, \hat{\theta} : [\underline{x}, \overline{x}] \to \mathbb{R}$ by

$$\hat{\Sigma}(x) = \begin{cases} -\frac{\sigma(1-x)g'(x)}{h(x)}, & p = 0, \\ -\frac{\sigma(qg(x)-x)g'(x)}{h(x)}, & p \neq 0, \end{cases} \tag{B.16}$$

$$\hat{\Gamma}(x) = -\frac{x}{g'(x)} \hat{\Sigma}(x), \tag{B.17}$$

and

$$\hat{\theta}(x) = \begin{cases} \frac{\sigma x}{h(x)}, & p = 0, \\ -\frac{\sigma(1-p)x(qg'(x)-1)}{h(x)}, & p \neq 0. \end{cases}. \tag{B.18}$$

The following stochastic differential equations utilizes these functions and is given by

$$\begin{cases} dX_t^x = \left(X_t^x \beta(X_t^x) - q\hat{\theta}(X_t^x)\hat{\gamma}(X_t^x)\right) dt + \hat{\gamma}(X_t^x) dB_t + d\Phi_t^x, \\ X_0^x = x \in [\underline{x}, \overline{x}]. \end{cases} \tag{B.19}$$

This is a so-called Reflected Skorokhod SDE. For more information and results on existence and uniqueness regarding this type of SDE, the reader may be referred to [Sko61]. It turns out, that it has a unique solution because $h(x)$ satisfies the Lipschitz-property.

Appendix C

Curriculum Vitae

Contact

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Education

10/2006 - present:	**University of Vienna**, Vienna Mathematics (Diplom) Major: Probability Theory and Stochastics
10/2008 - 4/2012:	**WU Vienna**, Vienna Business Administration (BSc) Majors: Finance, Transportation Science
2/20011 - 07/2011:	**Fudan University**, Shanghai Joint Study exchange term
09/1997 - 06/2005:	**Bundesgymnasium Blumenstraße**, Bregenz A-Levels with honors

Academic Work Experience

09/2011 - 7/2012:	**WU Vienna**, Vienna Teaching assistant, supporting several advanced finance classes

Presentations

10/2012:	**INFORMS Annual Meeting**, Phoenix, USA Topic: A cost-minimizing heuristic solution algorithm for the Network Sensor Location Problem
10/2012:	**Sci-Net Wissenschaftsforum**, Vienna Topic: Ein kostenminimierender heuristischer Lösungsalgorithmus für das Network Sensor Location Problem

Publication

Österreichische Zeitschrift für Verkehrswissenschaft
Title: Ein kostenminimierender heuristischer Lösungsalgorithmus für das Network Sensor Location Problem, Volume: 3/2012

Languages

German (native speaker), English (fluent), French (conversational), Spanish (basic), Standard Chinese (basic), Japanese (basic)

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