

Comparison of two different formalisms for relativistic elasticity theory

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Abstract

The relativistic theory of elasticity is developed and its formulations as given by Carter and Quintana [4] and by Beig and Schmidt [3] are compared. Then, the nonrelativistic limit is performed, and the resulting non-relativistic elasticity theory is investigated. Linear perturbations are also considered in this case, especially the so-called *trivial displacements*, that are special perturbations which satisfy the equations of motion trivially. Finally, the eigenmodes of a self-gravitating sphere are calculated in the non-relativistic case as an example.

Zusammenfassung

Die relativistische Elastizitätstheorie wird entwickelt und ihre Formulierungen von Carter und Quintana [4] und von Beig und Schmidt [3] werden verglichen. Dann wird der nichtrelativistische Limes durchgeführt, und die resultierende nicht-relativistische Elastizitätstheorie untersucht. Lineare Störungen werden auch in diesem Fall betrachtet, insbesondere die sogenannten *trivialen Störungen*, das sind spezielle Störungen, die die Bewegungsgleichungen trivial erfüllen. Als abschließendes Beispiel werden die Eigenmoden einer selbst-gravitierenden Kugel im nichtrelativistischen Fall ausgerechnet.

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Chapter 1

Introduction

The theory of elasticity is one of the oldest fields of theoretical physics; it was developed back in the 17th century by Bernoulli, Cauchy and Euler, to name just a few. Here also roots the concept of *tensors* that is so central to relativity. A mathematically clean presentation of the nonrelativistic theory can be found in the textbook by Marsden and Hughes [13].

When special and general relativity emerged in 1905 and 1915, this meant of course a program to formulate every physical theory in a relativistic version, so also a relativistic elasticity theory (sometimes called *relasticity* for short) was needed, and indeed such a theory was presented pretty fast by Herglotz in 1911 [8] (covering the special relativistic case) and Nordström in 1916 [15] (already for the general-relativistic background). The following years, the field was quiet, since perfect fluids seemed sufficient as matter model in general relativity. It was as late as 1973 that Carter and Quintana renewed interest in relasticity [4]. One of the main goals of this thesis is to compare their approach with more recent work by Kijowski and Magli [11, 12] and Beig and Schmidt [3].

Apart from the obvious theoretical interest in relativistic elasticity as a building block to complete relativistic physics, there is also practical use in astrophysics: we know nowadays that neutron stars have a solid crust [16].

The structure of this thesis is as following: in the chapters 2 and 3, the relativistic elasticity theory is developed. We first view the kinematical framework, introducing the body manifold $\mathcal{B}$ and the basic fields f^A as mappings from the spacetime to the points of the body occupying it. Then, the dynamics of the theory is developed: we view relativistic elasticity as a Lagrangian field theory, with the energy density of the material as the Lagrange density. It turns out that the covariant divergence freeness condition for the energy momentum tensor is equivalent to the Lagrange equations of motion for this theory.

In chapter 4, we discuss how the relativistic equations of motion can be linearized using perturbation formalism. Eulerian and Lagrangian variations are introduced and their connection is discussed.

In chapter 5, the non-relativistic limit is performed (via $\frac{1}{c} \rightarrow 0$) on the Lagrange density. This yields the Lagrange density for the nonrelativistic version of elastodynamics.

This theory is examined further in chapter 6, where we first discuss the kinematical framework in the nonrelativistic case and then derive the equations of motion from the action principle. Finally, we also discuss the linearization of

these equations via perturbation theory. We then discuss special perturbations that satisfy the linearized equations of motions trivially: firstly, ones that are generated by Killing vector fields on the body manifold $\mathcal{B}$ for homogeneous and isotropic materials. These are related to the so-called *trivial displacements* in the literature (see e.g. [7]). Secondly, there is a similar result for perturbations generated by Killing vector fields on Newtonian space $\mathcal{S}$. This is because the Lagrange density (and hence also the equations of motion) are invariant under Euclidean motions (which are generated by the Killing vector fields). Finally, we use the theory developed so far to calculate the eigenmodes of a self-gravitating elastic sphere, a problem that goes back to Cauchy, who solved this in the case without gravity back in 1829.

There is also an appendix, in which some mathematical results are threatened shortly which may not be familiar to everyone (at least they were not familiar to me while writing this thesis).

Finally, some words on the conventions used: lowercase greek letters ($\mu, \nu, \lambda, \dots$) denote spacetime indices, capital latin letters ($A, B, C, \dots$) denote indices for the body manifold. Lowercase latin letters ($i, j, k, \dots$) denote general indices, as they are for example used in general mathematical statements that are independent of the dimension. The spacetime metric $g_{\mu\nu}$ is assumed to be of signature $(-, +, +, +)$.

Chapter 2

Kinematics

2.1 Configuration

All descriptions of elastic bodies make - at least implicitly - use of a 3-dimensional body manifold $\mathcal{B}$, which can be thought of as the abstract collection of all points (“molecules”) in the considered continuous medium. Some authors assume $\mathcal{B}$ to be equipped with a positive-definite metric γ_{AB} , which describes the distances in the body in relaxed state. Since it is possible, however, to formulate large parts of the theory without a body metric, we will try to do so; we will only assume the body manifold $\mathcal{B}$ to be equipped with a volume form Ω_{ABC} , that describes the particle density in the body.

Let $(\mathcal{M}, g_{\mu\nu})$ be a general-relativistic spacetime with a metric $g_{\mu\nu}$. Let $\mathcal{M}' \subseteq \mathcal{M}$ be the open subset of the space-time through which the elastic body passes. Then the configuration of the elastic body is completely specified through a smooth mapping

$$f : \mathcal{M}' \rightarrow \mathcal{B} \quad (2.1)$$

$$x^\mu \mapsto f^A(x^\mu) \quad (2.2)$$

that assigns points in the body manifold (“molecules” of the body) to given spacetime-points. We call a given $f^A(x^\mu)$ *configuration* for short as well. Since the coordinate system on the body manifold doesn’t necessarily change when changing spacetime coordinates, the f^A transform as three scalar fields, i.e.:

$$\mathcal{L}_\xi f^A = \xi^\mu \partial_\mu f^A \quad (2.3)$$

and of course $\nabla_\mu f^A = \partial_\mu f^A$ as well. The same will be true for all other objects that only carry body manifold indices $A, B, C, \dots$ as well.

This is the so-called *space-time description*, which is somehow more natural in the relativistic setting. In non-relativistic elastodynamics the *material description* is used more frequently; here mappings $\phi : \mathcal{B} \rightarrow \mathcal{M}$ are used to describe the elastic body, which are inverse to the configuration f^A in an appropriate sense.

The inverse image $f^{-1}(X^A)$ of a given point $X^A \in \mathcal{B}$ is supposed to be a timelike curve in $\mathcal{M}'$; these are the world-lines of the “molecules”. This means (by the implicit function theorem) that the *deformation gradient* $f^A_{, \mu} \equiv \partial_\mu f^A$

must have full rank everywhere:

$$\text{rg}(f^A{}_{,\mu}) = 3 \quad (2.4)$$

and that the null space of $f^A{}_{,\mu}$ is timelike with respect to the spacetime metric.

Of course the world lines of the “molecules” must not intersect and fill the whole $\mathcal{M}'$ (formally speaking the inverse images of all points in $\mathcal{B}$ form a timelike congruence). This enables us to define a 4-velocity field u^μ , which is the uniquely given vector field that is everywhere:

- orthogonal to the deformation gradient: $f^A{}_{,\mu} u^\mu = 0$
- normalised: $g_{\mu\nu} u^\mu u^\nu = -1$ and
- future-pointing.

The rank condition on the deformation gradient is of course equivalent to the condition that $f^A{}_{,\mu}(x)$ is an isomorphism between $[u_x^\mu]^\perp = \{v^\mu \in T_x \mathcal{M} | v^\mu g_{\mu\nu} u^\nu = 0\}$ (the subspace orthogonal to u^μ) and $T_{f(x)} \mathcal{B}$.

2.2 Strain

The strain is defined the following way:

$$h^{AB} := f^A{}_{,\mu} g^{\mu\nu} f^B{}_{,\nu} \quad (2.5)$$

Note that this is no push-forward of the spacetime metric (although it looks similar), since f^A is not a diffeomorphism. That's why h^{AB} in general is not a tensor field on $\mathcal{B}$, since it still depends on the spacetime coordinates x^μ .

The strain is obviously a symmetric matrix in each spacetime point. Furthermore, it is even positive definite, since the deformation gradients $f^A{}_{,\mu}$ project out the timelike (negative) parts of the spacetime metric, and leave behind only the positive definite rest. Because of the full-rank condition of the deformation gradient $f^A{}_{,\mu}$, also h^{AB} has rank three.

Since h^{AB} is symmetric and positive definite, it is “almost” (apart from the fact that it depends on spacetime points) a metric on $\mathcal{B}$. As such, it describes the distances of neighbouring points in $\mathcal{B}$ in their local rest frame in spacetime.

Also, the positive definiteness of h^{AB} is the reason that there exists an inverse, which is denoted by h_{AB} , that is of course symmetric and positive definite too.

Next, we would like to define a corresponding tensor $h_{\mu\nu}$ (note that this is *not* the same $h_{\mu\nu}$ as the one used by Kijowski and Magli [11, 12]!). This can be done in two equivalent ways:

$$h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu \equiv f^A{}_{,\mu} f^B{}_{,\nu} h_{AB} \quad (2.6)$$

To show the equivalence of both definitions, we first note that both are symmetric. So it is sufficient to contract one index on both sides with a basis of $T\mathcal{M}$. We use our standard basis u^μ and $f^{i\mu}$ for that purpose. The former is obviously annihilated by both, the latter gives $f^i{}_{,\nu}$ on both sides:

$$(g_{\mu\nu} + u_\mu u_\nu) f^i{}_{,\lambda} g^{\lambda\mu} = f^i{}_{,\nu} + f^i{}_{,\mu} u^\mu u_\nu = f^i{}_{,\nu} \quad (2.7)$$

$$f^A{}_{,\mu} f^B{}_{,\nu} h_{AB} f^i{}_{,\lambda} g^{\lambda\mu} = h^{Ai} h_{AB} f^B{}_{,\nu} = f^i{}_{,\nu} \quad (2.8)$$

In other words, $h^\mu{}_\nu = g^{\mu\lambda}h_{\lambda\nu}$ annihilates u^μ and leaves its orthogonal subspace (spanned by the $f^{i\mu}$) untouched, so it is the orthogonal projection operator to the tangent subspace orthogonal to u^μ , i.e. $h^\mu{}_\lambda h^\lambda{}_\nu = h^\mu{}_\nu$. This can also be shown directly by calculation using both definitions:

$$h^\mu{}_\lambda h^\lambda{}_\nu = (\delta^\mu{}_\lambda + u^\mu u_\lambda)(\delta^\lambda{}_\nu + u^\lambda u_\nu) = \delta^\mu{}_\nu + u^\mu u_\nu = h^\mu{}_\nu \quad (2.9)$$

$$h^\mu{}_\lambda h^\lambda{}_\nu = g^{\mu\rho}h_{\rho\lambda}g^{\lambda\sigma}h_{\sigma\nu} = g^{\mu\rho}f^A{}_{,\rho}f^B{}_{,\lambda}h_{AB}g^{\lambda\sigma}f^C{}_{,\sigma}f^D{}_{,\nu} = \quad (2.10)$$

$$= g^{\mu\rho}f^A{}_{,\rho}h^{BC}h_{AB}f^D{}_{,\nu}h_{CD} = g^{\mu\rho}f^A{}_{,\rho}h_{AB}f^B{}_{,\nu} = h^\mu{}_\nu \quad (2.11)$$

The orthogonality of the projection operator $h^\mu{}_\nu$ is of course equivalent with the symmetry of $h_{\mu\nu}$.

Since h_{AB} is positive definite, and $f^A{}_{,\mu}$ is an isomorphism between $[u_x^\mu]^\perp$ and $T_{f(x)}\mathcal{B}$, $h_{\mu\nu}$ acts as a positive definite metric on $[u_x^\mu]^\perp$. Also an inverse $h^{\mu\nu} = g^{\mu\lambda}h_{\lambda\rho}g^{\rho\nu}$ can be defined that satisfies $h^{\mu\lambda}h_{\lambda\nu} = h^\mu{}_\nu$ because of the projection operator property ($h^\mu{}_\nu$ is of course the identity on $[u_x^\mu]^\perp$). For orthogonal spacetime tensors (i.e. such for which all contractions with u^μ vanish) $h^{\mu\nu}$ and $h_{\mu\nu}$ can be used instead of $g^{\mu\nu}$ and $g_{\mu\nu}$ for raising and lowering of indices.

Because $f^A{}_{,\mu}$ is an isomorphism between $[u_x^\mu]^\perp$ and $T_{f(x)}\mathcal{B}$, we can also define an inverse $f_A{}^\mu$ that satisfies

$$f^A{}_{,\mu}f_B{}^\mu = \delta_B^A \quad (= h_B^A) \quad (2.12)$$

$$f_A{}^\mu f^A{}_{,\nu} = h_\nu^\mu \quad (2.13)$$

using h^{AB} this can be explicitly written down as

$$f_A{}^\mu = h_{AB}f^B{}_{,\nu}g^{\nu\mu} \quad (2.14)$$

because

$$f^A{}_{,\mu}f_B{}^\mu = f^A{}_{,\mu}h_{BC}f^C{}_{,\nu}g^{\nu\mu} = h_{BC}h^{AC} = \delta_B^A \quad (2.15)$$

$$f_A{}^\mu f^A{}_{,\nu} = h_{AB}f^B{}_{,\lambda}g^{\lambda\mu}f^A{}_{,\nu} = h_{\nu\lambda}g^{\lambda\mu} = h_\nu^\mu \quad (2.16)$$

Finally, we note that

$$h^{AB} = f^A{}_{,\mu}h^{\mu\nu}f^B{}_{,\nu} \quad (2.17)$$

since $f^A{}_{,\mu}u^\mu = 0$. This means that there is a 1-1 mapping between the h^{AB} s and $h^{\mu\nu}$ s.

Finally, we calculate the partial derivatives of h^{AB} for further use:

$$\begin{aligned} \frac{\partial h^{AB}}{\partial g^{\mu\nu}} &= f^A{}_{,\mu}f^B{}_{,\nu} \\ \frac{\partial h^{AB}}{\partial f^C{}_{,\lambda}} &= \delta_C^A \delta_\mu^\lambda f^B{}_{,\nu}g^{\mu\nu} + \delta_C^B \delta_\lambda^\nu f^A{}_{,\nu}g^{\mu\nu} = \delta_C^A f^{B\mu} + \delta_C^B f^{A\mu} = 2\delta_C^{(A} f^{B)\mu} \end{aligned} \quad (2.18)$$

2.3 Matter current and particle density

With the help of the volume form Ω_{ABC} on $\mathcal{B}$, we can give an explicit formula for u^μ : we define $\omega_{\mu\nu\lambda}$ as the pull-back of Ω_{ABC} to $\mathcal{M}$:

$$\omega = f^*\Omega \quad (2.19)$$

$$\omega_{\mu\nu\rho} = f^A{}_{,\mu}f^B{}_{,\nu}f^C{}_{,\rho}\Omega_{ABC} \quad (2.20)$$

and use the Hodge operator to define J^μ (we use $-*$ per convention, to make the coordinate formulae more beautiful):

$$J = -*\omega \quad (2.21)$$

$$J^\mu = \frac{1}{3!}\epsilon^{\mu\nu\lambda\rho}\omega_{\nu\lambda\rho} = \frac{1}{3!}\epsilon^{\mu\nu\lambda\rho}f^A{}_{,\nu}f^B{}_{,\lambda}f^C{}_{,\rho}\Omega_{ABC} \quad (2.22)$$

Per definition, $\omega_{\mu\nu\lambda}$ is orthogonal to u^μ (i.e. $\omega_{\mu\nu\lambda}u^\mu = 0$), since $f^A{}_{,\mu}u^\mu = 0$ identically. In consequence, J^μ is proportional to u^μ , since $f^D{}_{,\mu}J^\mu = 0$, because $\epsilon^{\mu\nu\lambda\rho}f^A{}_{,\nu}f^B{}_{,\lambda}f^C{}_{,\rho}f^D{}_{,\mu}$ is the determinant of a 4×4 matrix with two identical columns. Without loss of generality, we can assume that J^μ is future-pointing, otherwise we could simply change the orientation of Ω_{ABC} on $\mathcal{B}$, i.e. replace Ω_{ABC} by $-\Omega_{ABC}$. In other words, we want f^A to be orientation-preserving.

Furthermore we note that J^μ is conserved:

$$\nabla_\mu J^\mu = \delta J = -*d*J = *d\omega = *df^*\Omega = *f^*d\Omega = 0 \quad (2.23)$$

since the volume form Ω on $\mathcal{B}$ is closed. For people who don't like differential forms, we also give a coordinate-based proof:

$$\begin{aligned} \nabla_\mu J^\mu &= \nabla_\mu \left(\frac{1}{3!}\epsilon^{\mu\nu\lambda\rho}\omega_{\nu\lambda\rho} \right) = \frac{1}{3!}\epsilon^{\mu\nu\lambda\rho}\nabla_\mu (f^A{}_{,\nu}f^B{}_{,\lambda}f^C{}_{,\rho}\Omega_{ABC}) = \\ &= \frac{1}{3!}\epsilon^{\mu\nu\lambda\rho} (f^A{}_{,\nu;\mu}f^B{}_{,\lambda}f^C{}_{,\rho}\Omega_{ABC} + f^A{}_{,\nu}f^B{}_{,\lambda;\mu}f^C{}_{,\rho}\Omega_{ABC} + \\ &\quad f^A{}_{,\nu}f^B{}_{,\lambda}f^C{}_{,\rho;\mu}\Omega_{ABC} + f^A{}_{,\nu}f^B{}_{,\lambda}f^C{}_{,\rho}\Omega_{ABC,D}f^D{}_{,\mu}) = 0 \end{aligned} \quad (2.24)$$

The first equality holds because $\nabla_\mu \epsilon^{\mu\nu\lambda\rho} = 0$; the second one, because $\epsilon^{\mu\nu\lambda\rho}f^A{}_{,\nu;\mu} = 0$, since the covariant derivative is torsion free and $f^A{}_{,\nu}f^B{}_{,\lambda}f^C{}_{,\rho}\Omega_{ABC,D}f^D{}_{,\mu} = 0$ because $\epsilon^{\mu\nu\lambda\rho}f^A{}_{,\nu}f^B{}_{,\lambda}f^C{}_{,\rho}f^D{}_{,\mu}$ is a vanishing determinant again (see explanation after equation (2.22)).

The conservation of J^μ justifies the definition of a particle density $n > 0$ (since both u^μ and J^μ are future-pointing) by

$$J^\mu = nu^\mu. \quad (2.25)$$

Next we would like to derive an explicit formula for $n = \sqrt{-J^\mu J_\mu}$. For this, the strain will already be very useful:

$$\begin{aligned} n^2 &= -J^\mu J_\mu = -\frac{1}{3!}\epsilon^{\mu\nu\lambda\rho}\omega_{\nu\lambda\rho}\frac{1}{3!}\epsilon_{\mu\nu'\lambda'\rho'}\omega^{\nu'\lambda'\rho'} = \\ &= \frac{1}{3!}\frac{1}{3!}\delta^{\nu\lambda\rho}_{\nu'\lambda'\rho'}\omega_{\nu\lambda\rho}\omega^{\nu'\lambda'\rho'} = \frac{1}{3!}\omega_{\nu\lambda\rho}\omega^{\nu\lambda\rho} = \\ &= \frac{1}{3!}g^{\nu\nu'}g^{\lambda\lambda'}g^{\rho\rho'}f^A{}_{,\nu}f^B{}_{,\mu}f^C{}_{,\lambda}\Omega_{ABC}f^{A'}{}_{,\nu'}f^{B'}{}_{,\mu'}f^{C'}{}_{,\lambda'}\Omega_{A'B'C'} = \\ &= \frac{1}{3!}h^{AA'}h^{BB'}h^{CC'}\Omega_{ABC}\Omega_{A'B'C'} = \det_\Omega(h^{AB}) \end{aligned} \quad (2.26)$$

The indices in the calculation above can be read as “abstract indices”; the determinant relative to a volume form is an invariant geometric concept. If we introduce coordinates, we can rewrite the volume form the following way:

$$\frac{1}{3!}\Omega_{ABC}(X)dX^AdX^BdX^C = \Omega(X)dX^1 \wedge dX^2 \wedge dX^3 \quad (2.27)$$

Then we get

$$n = \Omega(f(x))\sqrt{\det h^{AB}} = \frac{\Omega(f(x))}{\sqrt{\det h_{AB}}} \quad (2.28)$$

Where $\det$ now denotes the standard determinant of the matrix h^{AB} in a given basis. This shows that n is the ratio of the materials volume form Ω_{ABC} to the metric volume form generated by h_{AB} .

So the particle number density n as a function only depends on the strain h^{AB} (or equivalently h_{AB} of course) and the configuration (via the volume form of the body) that is given as an explicit formula. By plugging the equations (2.22), (2.25) and (2.26) together, we even get an explicit formula for u^μ .

By differentiating $n^2 = \det_\Omega(h^{AB})$ with respect to h^{CD} using (7.1) from the appendix, we get $2n \frac{\partial n}{\partial h^{CD}} = \det_\Omega(h^{AB}) h_{CD}$ (since h_{CD} is symmetric) or (after reinserting $n^2 = \det_\Omega(h^{AB})$ and renaming indices):

$$\frac{\partial n}{\partial h^{AB}} = \frac{1}{2} n h_{AB} \quad (2.29)$$

Using $\frac{\partial h^{AB}}{\partial h_{CD}} = -h^{AC} h^{BD}$ (see (7.1.3) in the appendix) and the chain rule $\frac{\partial n}{\partial h_{CD}} = \frac{\partial n}{\partial h^{AB}} \frac{\partial h^{AB}}{\partial h_{CD}}$ we further get (after renaming indices):

$$\frac{\partial n}{\partial h_{AB}} = -\frac{1}{2} n h^{AB} \quad (2.30)$$

Chapter 3

Dynamics

3.1 The Lagrangian

The dynamics is given by an action principle. As Lagrangian density we use the rest-frame energy density ρ (which is of course non-negative), just as it is in relativistic hydrodynamics, too. This is viewed as function of our field variables, the configuration $f^A(x^\lambda)$, its derivatives, the configuration gradient $f^A_{,\mu}(x^\lambda)$ and the spacetime location x^λ - but we assume that the dependency on x^λ is only implicitly via the spacetime metric $g_{\mu\nu}(x^\lambda)$. So our action looks the following:

$$S_M[f^A] = \int_{\mathcal{M}} \rho(f, \partial f; g) \sqrt{-\det(g)} d^4x \quad (3.1)$$

3.1.1 Basic assumption and consequences

$\rho(f, \partial f; g)$ is assumed to be covariant under spacetime diffeomorphisms, i.e. that it transforms as scalar for a given configuration. Mathematically, this means

$$\mathcal{L}_\xi \rho(x^\nu) = \xi^\mu \partial_\mu \rho(f^A(x^\nu), f^A_{,\lambda}(x^\nu); g_{\lambda\sigma}(x^\nu)) \quad (3.2)$$

for arbitrary vector fields $\xi^\mu(x^\nu)$ on $\mathcal{M}$. But on the other hand, $\mathcal{L}_\xi \rho$ can be written as

$$\mathcal{L}_\xi \rho = \frac{\partial \rho}{\partial f^A} \mathcal{L}_\xi(f^A) + \frac{\partial \rho}{\partial f^A_{,\mu}} \mathcal{L}_\xi(f^A_{,\mu}) + \frac{\partial \rho}{\partial g^{\mu\nu}} \mathcal{L}_\xi(g^{\mu\nu}) \quad (3.3)$$

By writing out both expressions, we get the following equation:

$$\begin{aligned} & \frac{\partial \rho}{\partial f^A} \xi^\mu \partial_\mu f^A + \frac{\partial \rho}{\partial f^A_{,\nu}} \xi^\mu \partial_\mu f^A_{,\nu} + \frac{\partial \rho}{\partial g^{\nu\lambda}} \xi^\mu \partial_\mu g^{\nu\lambda} = \\ &= \frac{\partial \rho}{\partial f^A} \xi^\mu \partial_\mu f^A + \frac{\partial \rho}{\partial f^A_{,\nu}} (\xi^\mu \partial_\mu f^A_{,\nu} + (\partial_\nu \xi^\mu) f^A_{,\mu}) + \\ & \quad + \frac{\partial \rho}{\partial g^{\nu\lambda}} (\xi^\mu \partial_\mu g^{\nu\lambda} - (\partial_\mu \xi^\nu) g^{\mu\lambda} - (\partial_\mu \xi^\lambda) g^{\nu\mu}) \end{aligned} \quad (3.4)$$

This yields:

$$\frac{\partial \rho}{\partial f^A_{,\nu}} (\partial_\nu \xi^\mu) f^A_{,\mu} = 2 \frac{\partial \rho}{\partial g^{\mu\lambda}} (\partial_\nu \xi^\mu) g^{\nu\lambda} \quad (3.5)$$

Since ξ^μ is arbitrary, we get:

$$\frac{\partial \rho}{\partial f^A{}_{,\nu}} f^A{}_{,\mu} = 2 \frac{\partial \rho}{\partial g^{\mu\lambda}} g^{\lambda\nu} \quad (3.6)$$

or (after pulling down one index):

$$2 \frac{\partial \rho}{\partial g^{\mu\nu}} = f^A{}_{,\mu} g_{\nu\lambda} \frac{\partial \rho}{\partial f^A{}_{,\lambda}} \quad (3.7)$$

Since the left hand side of this equation is obviously symmetric, we conclude that the right hand side must be symmetric in μ and ν , too. By contracting the last equation with u^μ , we get as another corollary:

$$\frac{\partial \rho}{\partial g^{\mu\nu}} u^\mu = 0 \quad (3.8)$$

From this we conclude that

$$\rho(X^A, f^A{}_{,\mu}; g^{\mu\nu}) = \rho(X^A, f^A{}_{,\mu}; g^{\mu\nu} + u^\mu u^\nu) = \rho(X^A, f^A{}_{,\mu}; h^{\mu\nu}) \quad (3.9)$$

since $\rho(X^A, f^A{}_{,\mu}; g^{\mu\nu} + s u^\mu u^\nu)$ does not depend on $s \in \mathbb{R}$:

$$\frac{d}{ds} \rho(X^A, f^A{}_{,\mu}; g^{\mu\nu} + s u^\mu u^\nu) = \frac{\partial \rho}{\partial g^{\mu\nu}} u^\mu u^\nu = 0 \quad (3.10)$$

because of equation (3.8). Inserting $s = 0$ and $s = 1$ yields the claim (3.9).

Equation (2.17) tells us that h^{AB} contains the same information as $h^{\mu\nu}$, so we can write $\rho = \sigma(X^A, f^A{}_{,\mu}, h^{AB}(f^A{}_{,\mu}, g^{\mu\nu}))$ as function. Inserting σ in equation (3.6) yields (using (2.18))

$$\begin{aligned} \frac{\partial \rho}{\partial f^A{}_{,\nu}} f^A{}_{,\mu} &= \left(\frac{\partial \sigma}{\partial f^A{}_{,\nu}} + \frac{\partial \sigma}{\partial h^{BC}} \frac{\partial h^{BC}}{\partial f^A{}_{,\nu}} \right) f^A{}_{,\mu} = \\ &= \frac{\partial \sigma}{\partial f^A{}_{,\nu}} f^A{}_{,\mu} + 2 \frac{\partial \sigma}{\partial h^{AB}} f^A{}_{,\mu} f^B{}_{,\lambda} g^{\lambda\nu} \\ 2 \frac{\partial \rho}{\partial g^{\mu\lambda}} g^{\lambda\nu} &= 2 \frac{\partial \sigma}{\partial h^{AB}} \frac{\partial h^{AB}}{\partial g^{\mu\lambda}} g^{\lambda\nu} = 2 \frac{\partial \sigma}{\partial h^{AB}} f^A{}_{,\mu} f^B{}_{,\lambda} g^{\lambda\nu} \end{aligned} \quad (3.11)$$

so we get

$$\frac{\partial \sigma}{\partial f^A{}_{,\nu}} f^A{}_{,\mu} = 0 \quad (3.12)$$

Using the full-rank condition of $f^A{}_{,\mu}$ (or, more explicitly, by multiplying with $f_B{}^\mu$), we see that σ doesn't depend on the deformation gradient explicitly, so

$$\rho = \rho(X^A, h^{AB}) \quad (3.13)$$

Doing the calculation (3.11) again with $\rho(X^A, h^{AB})$ (i.e. using that $\frac{\partial \rho}{\partial f^A{}_{,\mu}} = 0$), we regain equation (3.6). This means that (3.2), (3.6) and (3.13) are equivalent.

There is also a “physicist’s proof” for (3.13): we view the deformation gradient as three vectors $\partial_\mu f^1, \partial_\mu f^2, \partial_\mu f^3$. The only way of forming scalars out of this vectors are their six different scalar products. But these are just the components of h^{AB} .

Because ρ is just a function of X^A and h^{AB} , and we already know that the same is true for the particle density n , we are now able to decompose ρ in the following way:

$$\rho = n \epsilon \quad (3.14)$$

ϵ is called *stored energy function* and of course just depends on X^A and h^{AB} as well. It is the total rest frame energy per particle.

3.1.2 Euler-Lagrange equations

In Lagrangian field theory we have generally an action $S[\phi^i]$ depending on some fields ϕ^i in the following form:

$$S[\phi^i] = \int_{\mathcal{M}} \mathcal{L}(\phi^i, \nabla \phi^i, x^\mu) d^4x = \int_{\mathcal{M}} \tilde{\mathcal{L}}(\phi^i, \nabla \phi^i, x^\mu) \sqrt{-\det(g)} d^4x \quad (3.15)$$

Here we use the following notation for the Lagrangian density: $\mathcal{L} = \tilde{\mathcal{L}} \sqrt{-\det(g)}$, where $\tilde{\mathcal{L}}$ is a scalar and $\mathcal{L}$ is a scalar density.

Then the Euler-Lagrange Equations have the following form:

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi^i)} \right) = 0 \quad (3.16)$$

which is generally equivalent to the following equation (see e.g. [17], chapter 11.3, page 189):

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \phi^i} - \nabla_\mu \left(\frac{\partial \tilde{\mathcal{L}}}{\partial (\nabla_\mu \phi^i)} \right) = 0 \quad (3.17)$$

Now we go back to relativistic elasticity theory ($\tilde{\mathcal{L}} = \rho$, $\phi^i = f^A$, $\nabla_\mu \phi^i = \partial_\mu f^A$, since the f^A transform as scalars). Here this equivalence is especially easy to see, since our fields f^A transform as scalars under spacetime diffeomorphisms.

First, we finally write down our equations of motion, the Euler - Lagrange equations of the action principle (3.1):

$$\sqrt{-\det(g)} \frac{\rho}{\partial f^A} - \partial_\mu \left(\sqrt{-\det(g)} \frac{\partial \rho}{\partial (\partial_\mu f^A)} \right) = 0 \quad (3.18)$$

By simply dividing by $\sqrt{-\det(g)}$ we get:

$$\frac{\rho}{\partial f^A} - \frac{1}{\sqrt{-\det(g)}} \partial_\mu \left(\sqrt{-\det(g)} \frac{\partial \rho}{\partial (\partial_\mu f^A)} \right) = 0 \quad (3.19)$$

which obviously transforms as a covector on $\mathcal{B}$. But this is the same as

$$\frac{\rho}{\partial f^A} - \nabla_\mu \left(\frac{\partial \rho}{\partial (\partial_\mu f^A)} \right) = 0 \quad (3.20)$$

by the standart identity $\nabla_\mu v^\mu = \frac{1}{\sqrt{-\det(g)}} \partial_\mu \left(\sqrt{-\det(g)} v^\mu \right)$ for the divergence.

3.2 Energy-Momentum tensor

3.2.1 Definition

In General Relativity, there are two possible ways to define an Energy-Momentum tensor: on the one hand, there is the Hilbert Energy-Momentum tensor, that is the source term in the Einstein field equations (as result from the Hilbert action principle) and therefore symmetric by birth:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-\det(g)}} \frac{\delta S}{\delta g^{\mu\nu}(x)} \quad (3.21)$$

Which yields in our case (the Lagrangian just depends on the fields, their first derivatives and the metric):

$$T_{\mu\nu} = 2 \frac{\partial \rho}{\partial g^{\mu\nu}} - \rho g_{\mu\nu} \quad (3.22)$$

On the other hand there is the canonical Energy-Momentum tensor that is familiar from special relativity:

$$T_{\nu}^{\mu} = \frac{\partial \rho}{\partial f^A_{,\mu}} f^A_{,\nu} - \rho \delta_{\nu}^{\mu} \quad (3.23)$$

after pulling down one index we get:

$$T_{\mu\nu} = f^A_{,\nu} g_{\mu\lambda} \frac{\partial \rho}{\partial f^A_{,\lambda}} - \rho g_{\mu\nu} \quad (3.24)$$

which is surprisingly identical with the symmetric Energy-Momentum tensor (3.22) because of equation (3.7)!

3.2.2 Divergence

Next, we calculate the divergence of T_{ν}^{μ} using (3.23):

$$\begin{aligned} \nabla_{\mu} T^{\mu}_{\nu} &= \nabla_{\mu} \left(\frac{\partial \rho}{\partial (\partial_{\mu} f^A)} \partial_{\nu} f^A \right) - \partial_{\nu} \rho = \left(\nabla_{\mu} \frac{\partial \rho}{\partial (\partial_{\mu} f^A)} \right) (\partial_{\nu} f^A) + \\ &+ \frac{\partial \rho}{\partial (\partial_{\mu} f^A)} (\nabla_{\mu} (\partial_{\nu} f^A)) - \frac{\partial \rho}{\partial f^A} (\partial_{\nu} f^A) - \frac{\partial \rho}{\partial (\partial_{\sigma} f^A)} \partial_{\nu} \partial_{\sigma} f^A - \frac{\partial \rho}{\partial g^{\lambda\rho}} \partial_{\nu} g^{\lambda\rho} \end{aligned} \quad (3.25)$$

The first and the third term together are the Euler - Lagrange equation in the form (3.20) multiplied with $f^A_{,\nu}$, the other terms cancel out: in the last term we use $\partial_{\nu} g^{\lambda\rho} = -(\Gamma^{\lambda}_{\nu\sigma} g^{\sigma\rho} + \Gamma^{\rho}_{\nu\sigma} g^{\lambda\sigma})$, which is true because $g^{\lambda\rho}_{;\nu} = 0$, and equation (3.6):

$$\begin{aligned} -\frac{\partial \rho}{\partial g^{\lambda\rho}} \partial_{\nu} g^{\lambda\rho} &= \frac{\partial \rho}{\partial g^{\lambda\rho}} (\Gamma^{\lambda}_{\nu\sigma} g^{\sigma\rho} + \Gamma^{\rho}_{\nu\sigma} g^{\lambda\sigma}) = 2 \frac{\partial \rho}{\partial g^{\lambda\rho}} g^{\rho\sigma} \Gamma^{\lambda}_{\nu\sigma} = \\ &= \frac{\partial \rho}{\partial f^A_{,\sigma}} f^A_{,\lambda} \Gamma^{\lambda}_{\nu\sigma} \end{aligned} \quad (3.26)$$

Next, we write out the covariant derivative in the second term of (3.25) using Christoffel symbols and get the following:

$$\frac{\partial \rho}{\partial(\partial_\mu f^A)} (\nabla_\mu (\partial_\nu f^A)) = \frac{\partial \rho}{\partial(\partial_\mu f^A)} (\partial_\sigma \partial_\nu f^A - \Gamma^\lambda_{\sigma\nu} f^A_{,\lambda}) \quad (3.27)$$

Indeed, the two Christoffel symbol terms cancel out because $\Gamma^\lambda_{\nu\sigma} = \Gamma^\lambda_{(\nu\sigma)}$, and the two second-order partial derivatives cancel out because $\partial_\nu \partial_\sigma = \partial_{(\nu} \partial_{\sigma)}$. So we get as final result:

$$\nabla_\mu T^\mu_\nu = \left(\nabla_\mu \left(\frac{\partial \rho}{\partial(\partial_\mu f^A)} \right) - \frac{\rho}{\partial f^A} \right) f^A_{,\nu} \quad (3.28)$$

So we see the Energy-Momentum Tensor is conserved on-shell. Another important observation is that $\nabla_\mu T^\mu_\nu u^\nu = 0$ identically.

3.2.3 Stress tensors

We know that

$$T_{\mu\nu} u^\nu = -\rho u_\mu \quad (3.29)$$

according to (3.8) and (3.22). This suggests that it is possible to rewrite $T_{\mu\nu}$ in the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + P_{\mu\nu} \quad (3.30)$$

where the *Cauchy stress tensor* $P_{\mu\nu}$ is symmetric and orthogonal to u^μ . Indeed, we get it in that form if we insert $g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu$ in (3.22):

$$T_{\mu\nu} = \rho u_\mu u_\nu + 2 \frac{\partial \rho}{\partial g^{\mu\nu}} - \rho h_{\mu\nu} \quad (3.31)$$

so

$$P_{\mu\nu} = 2 \frac{\partial \rho}{\partial g^{\mu\nu}} - \rho h_{\mu\nu} \quad (3.32)$$

The form (3.30) of the Energy-Momentum Tensor also justifies the interpretation of our Lagrangian ρ as the total energy density in the materials rest frame.

To get another nice form of $P_{\mu\nu}$ we use (3.14)

$$\frac{\partial \rho}{\partial g^{\mu\nu}} = \frac{\partial(n\epsilon)}{\partial h^{\mu\nu}} = n \frac{\partial \epsilon}{\partial h^{\mu\nu}} + \epsilon \frac{\partial n}{\partial h^{\mu\nu}} = n \frac{\partial \epsilon}{\partial h^{\mu\nu}} + \frac{1}{2} \rho h_{\mu\nu} \quad (3.33)$$

to get

$$P_{\mu\nu} = 2n \frac{\partial \epsilon}{\partial h^{\mu\nu}} \quad (3.34)$$

This can be further rewritten as

$$P_{\mu\nu} = 2n \frac{\partial \epsilon}{\partial h^{\mu\nu}} = 2n \frac{\partial \epsilon}{\partial h^{AB}} \frac{\partial h^{AB}}{\partial h^{\mu\nu}} = n \tau_{AB} f^A_{,\mu} f^B_{,\nu} \quad (3.35)$$

where

$$\tau_{AB} := 2 \frac{\partial \epsilon}{\partial h^{AB}} \quad (3.36)$$

is the *second Piola-Kirchhoff stress tensor*, which is symmetric by definition. This is again (similar to the strain h_{AB}) not a tensor on the body manifold, because it is a function on the spacetime $\mathcal{M}$, not $\mathcal{B}$.

For completeness we also mention the *first Piola-Kirchhoff stress tensor* $\sigma^\mu{}_A$:

$$\sigma^\mu{}_A := \frac{\partial \epsilon}{\partial f^A{}_{,\mu}} \quad (3.37)$$

This is connected to τ_{AB} via the chain rule in the following way:

$$\sigma^\mu{}_A := \frac{\partial \epsilon}{\partial f^A{}_{,\mu}} = \frac{\partial \epsilon}{\partial h^{BC}} \frac{\partial h^{BC}}{\partial f^A{}_{,\mu}} = \tau_{AB} f^{B\mu} \quad (3.38)$$

This also shows $\sigma^\mu{}_A u_\mu = 0$.

3.2.4 Stress tensor decomposition, Hydrodynamics

We define a new quantity η^{AB} conformal to h^{AB} according to

$$\eta^{AB} = n^{-2/3} h^{AB} \quad (3.39)$$

It obviously satisfies the constraint

$$\det_\Omega(\eta^{AB}) = 1 \quad (3.40)$$

because of (2.26). That's why $\{n, \eta^{AB}\}$ contains the same information about the strain as h^{AB} : n denotes how much the material is compressed and η^{AB} contains the information about the deformation aside from the compression.

We now express the Cauchy stress tensor using n and η^{AB} :

$$\begin{aligned} P_{\mu\nu} &= 2n \frac{\partial \epsilon}{\partial h^{\mu\nu}} = 2n \left(\frac{\partial \epsilon}{\partial n} \frac{\partial n}{\partial h^{\mu\nu}} + \frac{\partial \epsilon}{\partial \eta^{\lambda\rho}} \frac{\partial \eta^{\lambda\rho}}{\partial h^{\mu\nu}} \right) = \\ &= n^2 \frac{\partial \epsilon}{\partial n} h_{\mu\nu} + 2n \left(\frac{\partial \epsilon}{\partial h^{\lambda\rho}} - \frac{1}{3} \frac{\partial \epsilon}{\partial h^{\lambda\rho}} h^{\lambda\rho} h_{\mu\nu} \right) \end{aligned} \quad (3.41)$$

Here we have used

$$\frac{\partial n}{\partial h^{\mu\nu}} = \frac{1}{2} n h_{\mu\nu} \quad (3.42)$$

$$\frac{\partial \epsilon}{\partial \eta^{\mu\nu}} = \frac{\partial \epsilon}{\partial h^{\lambda\rho}} \frac{\partial h^{\lambda\rho}}{\partial \eta^{\mu\nu}} = \frac{\partial \epsilon}{\partial h^{\mu\nu}} n^{2/3} \quad (3.43)$$

$$\frac{\partial \eta^{\lambda\rho}}{\partial h^{\mu\nu}} = -\frac{2}{3} n^{-5/3} \frac{\partial n}{\partial h^{\mu\nu}} h^{\lambda\rho} + n^{-2/3} \delta_\mu^\lambda \delta_\nu^\rho = \quad (3.44)$$

$$= n^{-2/3} \left(\delta_\mu^\lambda \delta_\nu^\rho - \frac{1}{3} h_{\mu\nu} h^{\lambda\rho} \right) \quad (3.45)$$

This shows we can decompose the stress tensor into the *isotropic pressure* p :

$$p = \frac{1}{3} P_{\mu\nu} h^{\mu\nu} = n^2 \frac{\partial \epsilon}{\partial n} = n \frac{\partial \rho}{\partial n} - \rho \quad (3.46)$$

which is directly related to the particle density n , and a trace-free anisotropic tensor $\pi_{\mu\nu} = P_{\mu\nu} - p h_{\mu\nu}$. The pressure is also equivalent to the familiar expression from Thermodynamics:

$$p = -\frac{\partial U}{\partial V} = -N \frac{\partial \epsilon}{\partial V} = -N \frac{\partial \epsilon}{\partial n} \frac{\partial n}{\partial V} = n^2 \frac{\partial \epsilon}{\partial n} \quad (3.47)$$

Now we can also see that relativistic Hydrodynamics is just the special case of our theory in which the Lagrangian only depends on n , but not on $\eta_{\mu\nu}$. Then the Energy-Momentum tensor takes the familiar form

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \quad (3.48)$$

and as the equations of motion we can use $T^{\mu\nu}{}_{;\nu} = 0$ because of (3.28).

3.2.5 Carter-Quintana equation of motion

Carter and Quintana [4] use $T^{\mu\nu}{}_{;\nu} = 0$ as equation of motion, which is equivalent to our Euler-Lagrange equations, as we have already seen in subsection 3.2.2 (more generally, they use $T^{\mu\nu}{}_{;\nu} = f^\mu$, where the force density f^μ is due to other fields like the electromagnetic field, but we aren't interested in external forces). More precisely, they use the spatial projection, because $u_\mu T^{\mu\nu}{}_{;\nu} = 0$ identically, as we have already seen too. Nevertheless, we want to prove it again in the Carter-Quintana way. In this calculation, we will use the following trick twice:

$$P^{\mu\nu} u_\nu = 0 \quad (3.49)$$

$$(P^{\mu\nu} u_\nu)_{;\lambda} = 0 \quad (3.50)$$

$$P^{\mu\nu}{}_{;\lambda} u_\nu = -P^{\mu\nu} u_{\nu;\lambda} \quad (3.51)$$

Now we can decompose the divergence of the pressure tensor in a part proportional to u^μ and a part orthogonal to it:

$$\begin{aligned} P^{\mu\nu}{}_{;\nu} &= P^{\mu\nu}{}_{;\lambda} h^\lambda{}_\nu - P^{\mu\nu}{}_{;\lambda} u^\lambda u_\nu = \\ &= P^{\sigma\nu}{}_{;\lambda} h^\lambda{}_\nu h^\mu{}_\sigma - P^{\sigma\nu}{}_{;\lambda} h^\lambda{}_\nu u^\mu u_\sigma + P^{\mu\nu} u_{\nu;\lambda} u^\lambda = \\ &= P^{\sigma\nu}{}_{;\lambda} h^\lambda{}_\nu h^\mu{}_\sigma + P^{\sigma\lambda} u_{\sigma;\lambda} u^\mu + P^{\mu\nu} u_{\nu;\lambda} u^\lambda \end{aligned} \quad (3.52)$$

Now we write down the divergence of the energy-momentum tensor, already in decomposed form:

$$T^{\mu\nu}{}_{;\nu} = (\dot{\rho} + \rho\theta + P^{\sigma\lambda}\theta_{\sigma\lambda}) u^\mu + (\rho\dot{u}^\nu + P^{\mu\nu}\dot{u}_\nu + P^{\sigma\nu}{}_{;\lambda} h^\lambda{}_\nu h^\mu{}_\sigma) \quad (3.53)$$

where $\theta_{\nu\lambda} := u_{(\nu;\lambda)}$ and $\theta = g^{\mu\nu}\theta_{\mu\nu} = h^{\mu\nu}\theta_{\mu\nu}$.

The part proportional to u^μ in (3.53) automatically vanishes because of $P_{\mu\nu} = 2\frac{\partial\rho}{\partial g^{\mu\nu}} - \rho h_{\mu\nu}$ (equation (3.32)). The orthogonal part of (3.53) can be rewritten as

$$(\rho h^{\mu\nu} + P^{\mu\nu}) \dot{u}_\nu = -D_\nu P^{\mu\nu} \quad (3.54)$$

Here D_μ denotes the Levi-Civita connection specified by the metric $h_{\mu\nu}$. It can be expressed by

$$D_\mu T^{\nu\dots}{}_{\lambda\dots} = h_\mu{}^\rho h^\nu{}_\sigma h_\lambda{}^\tau \dots \nabla_\rho T^{\sigma\dots}{}_{\tau\dots} \quad (3.55)$$

It is obvious that D_μ is linear, satisfies the Leibnitz rule and is torsion free. Compatibility with the metric $h_{\mu\nu}$ can be seen as following:

$$D_\mu h_{\nu\lambda} = h_\mu{}^\rho h_\nu{}^\sigma h_\lambda{}^\tau \nabla_\rho h_{\sigma\tau} = h_\mu{}^\rho h_\nu{}^\sigma h_\lambda{}^\tau (u_{\sigma;\rho} u_\tau + u_\sigma u_{\tau;\rho}) = 0 \quad (3.56)$$

For the special case of the ideal fluid ($P^{\mu\nu} = P h^{\mu\nu}$) this yields the well-known relativistic Euler equation:

$$(\rho + P) \dot{u}^\mu = -h^{\mu\nu} \partial_\nu P \quad (3.57)$$

3.2.6 Recovering $\mathcal{B}$ and f^A

When one uses u^μ as field variable evolving with the Carter-Quintana equation of motion (3.54), the question is how the full setup including the body manifold $\mathcal{B}$ and the configuration mappings f^A can be restored. One possibility is to define new fields $e_\mu^{(i)}$, which form a basis of the orthogonal space to u^μ (i.e. $e_\mu^{(i)} u^\mu = 0$ and will be identified with the configuration gradient $f^A{}_\mu$ later on, and also use the strain tensor h^{AB} as an independent field. So the fields of the theory become

$$\{u^\mu(x), e_\mu^{(i)}(x), h^{ij}(x)\} \quad (3.58)$$

which have to satisfy the constraints

$$u^\mu u^\nu g_{\mu\nu} = -1 \quad (3.59)$$

$$e_\mu^{(i)} u^\mu = 0 \quad (3.60)$$

$$\iota_\Sigma^* \left(\nabla_{[\mu} e_{\nu]}^{(i)} \right) = 0 \quad (3.61)$$

The last of these equations means that the exterior derivative of the $e_\mu^{(i)}$, calculated on one time-like slice vanishes.

Next, we need to find equations of motions for our fields (3.58). To get one for $e_\mu^{(i)}$, we remark that we have the equation

$$\mathcal{L}_u f^A{}_\mu = \nabla_\nu f^A{}_\mu u^\nu + f^A{}_\nu \nabla_\mu u^\nu = 0 \quad (3.62)$$

from our original formulation, which follows immediately from $\nabla_\mu (f^A{}_\nu u^\nu) = 0$ and $f^A{}_{;\mu\nu} = f^A{}_{;\nu\mu}$; rewritten in our new terms, this becomes

$$u^\nu \nabla_\nu e_\mu^{(i)} = -e_\nu^{(i)} \nabla_\mu u^\nu \quad (3.63)$$

This is the wanted equation of motion for the $e_\mu^{(i)}$. Using it, we can show that the constraint (3.60) propagates:

$$u^\nu \nabla_\nu (e_\mu^{(i)} u^\mu) = -e_\nu^{(i)} u^\mu \nabla_\mu u^\nu + e_\mu^{(i)} u^\nu \nabla_\nu u^\mu = 0 \quad (3.64)$$

Using (3.63) in the other direction on $\nabla_\nu (u^\mu e_\mu^{(i)})$, we get

$$\nabla_\nu (u^\mu e_\mu^{(i)}) = (\nabla_\nu u^\mu) e_\mu^{(i)} + u^\mu \nabla_\nu e_\mu^{(i)} = -u^\mu \nabla_\mu e_\nu^{(i)} + u^\mu \nabla_\nu e_\mu^{(i)} = 0 \quad (3.65)$$

or, in short

$$u^\mu \nabla_{[\mu} e_{\nu]}^{(i)} = 0 \quad (3.66)$$

Using Cartan's magic formula we see

$$\mathcal{L}_u \nabla_{[\mu} e_{\nu]}^{(i)} = 0 \quad (3.67)$$

which means that the constraint (3.61) propagates and the $e_\mu^{(i)}$ become closed one-forms on the whole space-time $\mathcal{M}$, which is necessary for them to be gradients of functions f^A .

The definition of the strain tensor becomes in current terms

$$h^{ij} g_a = e_\mu^{(i)} e_\nu^{(j)} g^{\mu\nu} \quad (3.68)$$

From this, we can get an equation of motion for h^{ij} by differentiating it with respect to $u^\lambda \nabla_\lambda$ and inserting (3.63), which yields

$$u^\mu \nabla_\mu h^{ij} = -2e_\mu^{(i)} e_\nu^{(j)} \nabla^{(\mu} u^{\nu)} \quad (3.69)$$

The equation of motion for u^μ is given by $\nabla_\mu T^{\mu\nu} = 0$, as usual; the three independent equations can be projected out by contraction with $e_\mu^{(i)}$, and the stress tensor is given by

$$P_{\mu\nu}(h^{kl}) = e_\mu^{(i)} e_\nu^{(j)} \tau_{ij}(h^{kl}) \quad (3.70)$$

with the second Piola-Kirchhoff stress tensor given by

$$\tau_{ij}(h^{kl}) = 2n(h^{kl}) \frac{\partial \epsilon(h^{kl})}{\partial h^{ij}} \quad (3.71)$$

as usual. So in sum the equation of motion takes the following form in the current terms

$$e_\lambda^{(i)} g^{\mu\lambda} \nabla^\nu \left(\rho(h^{kl}) u_\mu u_\nu + e_\mu^{(i)} e_\nu^{(j)} \tau_{ij}(h^{kl}) \right) = 0 \quad (3.72)$$

In sum, the equations (3.72), (3.63) and (3.69) taken together form a complete set of equations of motion for the fields (3.58) satisfying the constraints (3.59)-(3.61).

To restore the body manifold from a solution of this equation system, one has to introduce a coordinate system of the form (t, x^i) on $\mathcal{M}$ such that $u^\mu \partial_\mu = \partial_t$. The body manifold $\mathcal{B}$ can then be defined as the quotient of $\mathcal{M}$ by the equivalence relation defined by the flowlines. Because the $e_\mu^{(i)}$ are closed one-forms, the f^A can be defined via $\partial_\mu f^i = e_\mu^{(i)}$.

3.3 Relativistic Elasticity as gauge theory

Kijowski and Magli have pointed out in [12] that the two different approaches for an equations of motion for relativistic elasticity theory can be seen gauge-type theory. In their approach, u^μ is seen as the basic field, and the f^A s are the gauge potentials. To see this more nicely, they introduce a material metric $\gamma_{\mu\nu}$ on the spacetime that describes distances between body points in their rest frame in the relaxed state (i.e. if their corresponding small portion of the body had been extracted and relaxed). Therefore $\gamma_{\mu\nu}$ has to be symmetric, orthogonal to u^μ ($\gamma_{\mu\nu} u^\mu = 0$) and positive semidefinite, i.e. $\gamma_{\mu\nu}$ has the signature $(0, +, +, +)$. This means that $\gamma_{\mu\nu}$ has 9 independent parameters.

If a $\gamma_{\mu\nu}$ is given, we can easily restore u^μ from it; we simply define it as the unique timelike, future oriented, normalized vector that is annihilated by $\gamma_{\mu\nu}$.

There are further kinematical conditions that are imposed on $\gamma_{\mu\nu}$: the material metric is “frozen” in the material; mathematically this means that its Lie derivative with respect to u^μ has to vanish:

$$\mathcal{L}_u \gamma_{\mu\nu} = 0 \quad (3.73)$$

Because u^μ is a function of $\gamma_{\mu\nu}$, this condition can be considered as imposed on $\gamma_{\mu\nu}$ alone.

The 10 equations (3.73) are not independent; they satisfy the following 4 equations identically

$$u^\mu \mathcal{L}_u \gamma_{\mu\nu} = \mathcal{L}_u (u^\mu \gamma_{\mu\nu}) - \gamma_{\mu\nu} \mathcal{L}_u u^\mu \equiv 0 \quad (3.74)$$

This means there are only 6 independent conditions in (3.73), which reduces the number of independent functions in $\gamma_{\mu\nu}$ to 3. Now both $\gamma_{\mu\nu}$ and u^μ have the same number of degrees of freedom, which means that they are equivalent. They can be both viewed as fundamental fields of our theory.

The condition (3.73) is purely kinematical; its analog in Electrodynamics would be the homogeneous Maxwell equations. And similar to Electrodynamics, we can introduce a potential, and define the basic field of the theory in terms of the potential in a way that satisfies the kinematical conditions identically. Here the potentials become the mappings to the material space (which is assumed to be equipped with a material metric G_{AB}), and the field $\gamma_{\mu\nu}$ is defined as the pull-back of G_{AB} under f^A :

$$\gamma_{\mu\nu}(x) = f^A{}_{,\mu}(x) f^B{}_{,\nu}(x) G_{AB}(f(x)) \quad (3.75)$$

This way (3.73) is satisfied identically

$$\begin{aligned} \mathcal{L}_u \gamma_{\mu\nu} &= u^\lambda \gamma_{\mu\nu,\lambda} + \gamma_{\lambda\nu} u^\lambda{}_{,\mu} + \gamma_{\mu\lambda} u^\lambda{}_{,\nu} = u^\lambda (\gamma_{\mu\nu,\lambda} - \gamma_{\lambda\nu,\mu} - \gamma_{\mu\lambda,\nu}) = \\ &= u^\lambda (G_{AB,C} f^C{}_\lambda f^A{}_\mu f^B{}_\nu + G_{AB} f^A{}_\mu f^B{}_\nu + G_{AB} f^A{}_\mu f^B{}_{\nu\lambda} \\ &\quad - G_{AB,C} f^C{}_\mu f^A{}_\lambda f^B{}_\nu - G_{AB} f^A{}_\lambda f^B{}_\nu - G_{AB} f^A{}_\lambda f^B{}_{\nu\mu} \\ &\quad - G_{AB,C} f^C{}_\nu f^A{}_\mu f^B{}_\lambda - G_{AB} f^A{}_\mu f^B{}_\lambda - G_{AB} f^A{}_\mu f^B{}_{\lambda\nu}) = 0 \end{aligned} \quad (3.76)$$

because the second term cancels with the fifth and the third one with the last; all other vanish due to $f^A{}_\lambda u^\lambda = 0$. Thus, (3.73) is analogue to the homogeneous Maxwell equations $F_{[\mu\nu;\lambda]} = 0$, which also become identities when using the potential A_μ .

The whole theory is obviously invariant under coordinate changes on the material space, which therefore take the role of gauge transformations, so the fields f^A can be regarded as gauge potentials for the elasticity field $\gamma_{\mu\nu}$ (or, equivalently, the velocity field u^μ).

3.4 Isotropic materials - Eigenvalue and eigenvector formalism

In the case where the body manifold $\mathcal{B}$ has certain symmetries, the elastic problem simplifies. Firstly, if the stored energy function ϵ only depends on the strain h^{AB} , but not directly on the configuration f^A , the material is said to be *homogeneous*.

If one has a metric G_{AB} on the body, one can further define what an *isotropic* material is: here we demand that the stored energy function only depends on h^{AB} via its invariants (which can be expressed as $h^{AB} G_{AB}$, $h^{AB} h^{CD} G_{BC} G_{AD}$, $\det_\Omega h^{AB}$, for example). Another way of choosing invariants of the strain tensor taken from [9] is to view the quantity

$$k^\mu{}_\nu = g^{\mu\lambda} \gamma_{\lambda\nu} = h^{\mu\lambda} \gamma_{\lambda\nu} \quad (3.77)$$

where $\gamma_{\mu\lambda}$ is the pull-back of the material metric G_{AB} , like in the last section (cf. equation (3.75)). $k^\mu{}_\nu$ is obviously flowline orthogonal and can therefore be diagonalized with respect to the positive-definite metric $h_{\mu\nu}$ within the three-dimensional space orthogonal to u^μ . As usual, we get a orthonormal basis e_i^μ of eigenvectors (eigenvectors to different eigenvalues are orthogonal automatically, and eigenvectors to the same eigenvalue can be chosen orthogonal). Because both $h^{\mu\lambda}$ and $\gamma_{\lambda\nu}$ are positive definite, all eigenvalues are positive, and we can write them as n_i^2 ; these can be seen as linear particle densities along the principal directions, which are given by the eigenvectors e_i^μ . This interpretation is also supported by the fact that we also have

$$\det k^\mu{}_\nu = \det (h^{AB} G_{BC}) = \det h^{AB} = n^2 \quad (3.78)$$

because Ω is the metric volume element belonging to G_{AB} , so the particle density n can be written as the product of the linear particle densities n_i :

$$n = n_1 n_2 n_3 \quad (3.79)$$

Also the stress tensor becomes diagonal when using the basis e_i^μ :

$$P_{\mu\nu} = 2n \frac{\partial \epsilon}{\partial h^{\mu\nu}} = 2n \frac{\partial \epsilon}{\partial k^\lambda{}_\rho} \frac{\partial k^\lambda{}_\rho}{\partial h^{\mu\nu}} = 2n \frac{\partial \epsilon}{\partial k^\lambda{}_\rho} \delta^\lambda_{(\mu} \delta^\sigma_{\nu)} \gamma_{\sigma\rho} = \quad (3.80)$$

$$= 2n \gamma_{\rho(\nu} \frac{\partial \epsilon}{\partial k^\mu{}_\rho)} = 2n \sum_{i=1}^3 n_i^2 e_{i\rho} e_{i(\nu} \sum_{j=1}^3 e_{j\mu)} e_j^\rho \frac{\partial \epsilon}{\partial (n_j^2)} = \quad (3.81)$$

$$= 2n \sum_{i=1}^3 n_i^2 e_{i\mu} e_{i\nu} \frac{\partial \epsilon}{\partial n_i} \frac{\partial n_i}{\partial (n_i^2)} = \sum_{i=1}^3 n n_i \frac{\partial \epsilon}{\partial n_i} e_{i\mu} e_{i\nu} \quad (3.82)$$

So the stress tensor can be written in diagonal form

$$P_{\mu\nu} = \sum_{i=1}^3 p_i e_{i\mu} e_{i\nu} \quad (3.83)$$

with

$$p_i = \sum_{i=1}^3 n n_i \frac{\partial \epsilon}{\partial n_i} \quad (3.84)$$

The p_i are called the *principal pressures*.

Chapter 4

Perturbation Formalism

4.1 Material (Lagrangian) description

Until now we have worked with the Space-Time (Eulerian) description of Relativity, which means that we have used the mappings f^A from $\mathcal{M}$ to $\mathcal{B}$. This is the more natural way in the relativistic setting. In the Material description, we use time-dependant mappings F^i from $\mathcal{B}$ to $\mathcal{M}$, which involves choosing a foliation on $\mathcal{M}$.

We're only introducing some basic facts of the Lagrangian picture that are useful when working with perturbations; a more throughout description can be found in [18].

So let's assume there exists a foliation of our space-time $\mathcal{M}$ by spacelike hypersurfaces, i.e. $\mathcal{M} = I \times \Sigma$, where $I \subset \mathcal{R}$ and Σ is a three dimensional spacelike manifold. By assumption, $f^A_{,i}$ is a full-rank 3×3 -matrix. This means that the configuration is a diffeomorphism between Σ_t and $\mathcal{B}$ for each instant of time t . So we can define an inverse $F^i(t, X^B)$ (called *deformation*) for each t that satisfies

$$f^A(t, F^i(t, X^B)) = X^A \quad (4.1)$$

$$F^i(t, f^A(t, x^j)) = x^i \quad (4.2)$$

One must not forget that this definition is dependant on the choice of foliation $\mathcal{M} = I \times \Sigma$. We call the $F^i(t, X^B)$ the deformation and its derivative $F^i_{,A}(t, X^B)$ the deformation gradient. Obviously, we have

$$f^A_{,i} F^i_{,B} = \delta^A_B \quad (4.3)$$

$$F^i_{,A} f^A_{,j} = \delta^i_j \quad (4.4)$$

Furthermore, if we differentiate equation (4.1) with respect to t , we get

$$\dot{f}^A + f^A_{,i} \dot{F}^i = 0 \quad (4.5)$$

Since the 4-velocity u^μ is per definiton annihilated by $f^A_{,\mu}$, this means that the vector field

$$v^\mu(t, x^i) \partial_\mu = \partial_t + \dot{F}^j(t, f^A(t, x^i)) \partial_j \quad (4.6)$$

is proportional to u^μ , but not necessarily normalized ($v^\mu v_\mu \neq -1$ in general).

4.2 Perturbation theory

This treatment is based on [6].

When doing perturbation theory, one assumes that the configuration of the system one is interested in is always nearby an equilibrium configuration which is known. To make that more formal, one introduces a family of solutions indexed by a parameter λ in a smooth way:

$$Q(\lambda) = \{g_{\mu\nu}(\lambda), u^\mu(\lambda), \rho(\lambda), P_{\mu\nu}(\lambda), \dots\} \quad (4.7)$$

where $Q(0)$ is the equilibrium solution. Now we compare $Q(0)$ with the perturbed variables $Q(\lambda)$ in first order in λ :

$$\delta Q = \frac{d}{d\lambda} Q(\lambda)|_{\lambda=0} \quad (4.8)$$

These so-called Eulerian perturbations (denoted with δ) are constructed by comparing the Q s in the corresponding points of spacetime (so $\delta x^\mu = 0$). In regions filled with matter, one can additionally define Lagrangian perturbations ΔQ . Here the quantities Q belonging to the same particle are compared, i.e as measured by an observer swimming with the matter.

For that purpose, one uses that inside the matter a family of diffeomorphisms Φ_λ is uniquely given that maps the particle trajectories of $Q(0)$ to the ones of $Q(\lambda)$, which is given by $\Phi_\lambda(x^\mu) = (t, F_\lambda^i(t, f_0^A(x^\mu)))$. The generating vector field $\xi^\mu(x^\nu) = \delta\Phi^\mu(x^\nu) = \Delta x^\mu$ (the "Lagrangian displacement") can be seen as the vector connecting the unperturbed particle coordinates with the perturbed ones. More explicitly, ξ^μ is given by:

$$\eta^i(t, X^A) = \frac{d}{d\lambda} F_\lambda^i(t, X^A)|_{\lambda=0} \quad (4.9)$$

$$\xi^\mu \partial_\mu = \eta^i(t, f_0^A(t, x^j)) \partial_i \quad (4.10)$$

Now we can define the Lagrangian perturbations ΔQ :

$$\Delta Q = \frac{d}{d\lambda} (\Phi_\lambda^* Q(\lambda))|_{\lambda=0} = (\delta + \mathcal{L}_\xi) Q \quad (4.11)$$

The problem with the Eulerian perturbations is that they contain an gauge degree of freedom because one can match different points of the perturbed and unperturbed spacetimes, whereas the Lagrangian perturbations are uniquely determined (up to a replacement $\xi^\mu \rightarrow \xi^\mu + f u^\mu$, where f is an arbitrary scalar field) and therefore often preferred. To fix that last degree of freedom, one can restrict the timelike part of ξ^μ , for example by the requirement $\xi^\mu u_\mu = 0$.

4.2.1 Perturbing v^μ

We calculate the perturbation of the quantity v^μ defined in section 4.1:

$$(\delta v^\mu) \partial_\mu = (\delta v^i)(t, x) \partial_i \quad (4.12)$$

$$(\delta v^i)(t, x^j) = \delta \dot{F}^i(t, f^A(t, x^j)) + \dot{F}^i{}_{,A}(t, f(t, x^j)) \delta f^A(t, x^j) \quad (4.13)$$

Perturbing equation (4.1), we get:

$$\delta f^A = -f^A{}_i \delta F^i \quad (4.14)$$

Inserting that in (4.13) and using the definition of η^i yields:

$$\delta v^i(t, x^j) = \dot{\eta}^i(t, f^A(t, x^j)) - \dot{F}^i{}_{,A}(t, f(t, x)) f^A{}_j(t, x) \eta^j(t, f(t, x)) \quad (4.15)$$

$$= \dot{\eta}^i(t, f^A(x)) - \partial_j (v^i(t, f(t, x))) \eta^j(t, f(t, x)) \quad (4.16)$$

This is equivalent to $-\mathcal{L}_\xi v$, as can be seen by the following calculation:

$$\begin{aligned} -(\mathcal{L}_\xi v) &= -[\xi, v] = [v, \xi] = [\partial_t + v^i \partial_i, \xi^j \partial_j] = \\ &= \left(\dot{\eta}^i + \eta^i{}_{,A} \dot{f}^A + v^j \partial_j \xi^i - \xi^j \partial_j v^i \right) \partial_i = (\dot{\eta}^i - \xi^j \partial_j v^i) \partial_i \end{aligned} \quad (4.17)$$

because $v^j \partial_j \xi^i = v^j \eta^i{}_{,A} \dot{f}^A{}_j = -\eta^i{}_{,A} \dot{f}^A$, so we get

$$\delta v^\mu = -[\xi, v]^\mu = -(\mathcal{L}_\xi v)^\mu \quad (4.18)$$

$$\Delta v^\mu = \delta v^\mu + (\mathcal{L}_\xi v)^\mu = 0 \quad (4.19)$$

Using that, we can now calculate Δu^μ using $u^\mu = (-g_{\nu\lambda} v^\nu v^\lambda)^{-1/2} v^\mu$:

$$\Delta u^\mu = \frac{1}{2} (-g_{\nu\lambda} v^\nu v^\lambda)^{-3/2} \Delta g_{\nu\lambda} v^\nu v^\lambda v^\mu = \frac{1}{2} \Delta g_{\nu\lambda} u^\nu u^\lambda u^\mu \quad (4.20)$$

Chapter 5

Non-relativistic Limit

We conduct the non-relativistic limit directly in the action principle, like in [3] and [18] (a similar treatment for the special relativistic case can also be found in [17]). To get convergence, we have to multiply the action principle (3.1) with $\frac{1}{c}$; to get the nonrelativistic Lagrangian in the standard form, we also have to take the negative of it:

$$S_M[f^A] = -\frac{1}{c} \int_{\mathcal{M}} n(f^A, h^{AB}) \epsilon(f^A, h^{AB}) \sqrt{-\det(g_{\mu\nu})} d^4x \quad (5.1)$$

Now we develop all items of the Lagrangian density in power series in $\frac{1}{c}$. We start with n . To get an appropriate expression for the strain, we use the ADM decomposition of the metric (see e.g. [14]):

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + g_{ij}^{(3)} (dx^i + Y^i dt) (dx^j + Y^j dt) \quad (5.2)$$

$$g^{\mu\nu} \partial_\mu \partial_\nu = -\frac{1}{N^2} (\partial_t - Y^i \partial_i)^2 + g_{(3)}^{ij} \partial_i \partial_j \quad (5.3)$$

Here $g_{(3)}^{ij}$ is the inverse to $g_{ij}^{(3)}$; note that this is not the spatial part of the inverse metric, but $g_{ij}^{(3)} - \frac{Y^i Y^j}{N^2}$ is. Inserting the ADM representation of the inverse metric in the definition of the relativistic strain tensor yields

$$h^{AB} = f^A_{,i} f^B_{,j} \left(g_{(3)}^{ij} - \frac{1}{N^2} Y^i Y^j \right) - \frac{1}{N^2} \dot{f}^A \dot{f}^B + \frac{2}{N^2} f^{(A}_{,i} \dot{f}^{B)} Y^i \quad (5.4)$$

We now use $\dot{f}^A = -f^A_{,i} \dot{F}^i = -f^A_{,i} V^i$ (equation (4.5)) and get

$$h^{AB} = f^A_{,i} f^B_{,j} \left(g_{(3)}^{ij} - \frac{Y^i Y^j}{N^2} - \frac{V^i V^j}{N^2} - \frac{2Y^{(i} V^{j)}}{N^2} \right) \quad (5.5)$$

After introducing the abbreviations

$$W^i = \frac{1}{N} (V^i + Y^i) \quad (5.6)$$

$$W^A = f^A_{,i} W^i \quad (5.7)$$

$$k^{AB} = f^A_{,i} f^B_{,j} \left(g_{(3)}^{ij} \right) \quad (5.8)$$

the strain can be written as

$$h^{AB} = k^{AB} - W^A W^B \quad (5.9)$$

The Determinant of h^{AB} in this decomposition can be calculated using Sylvester's determinant theorem

$$n^2 = \det_{\Omega} (k^{AB} - W^A W^B) = \det_{\Omega} (k^{AC} (\delta_C^B - k_{CD} W^D W^B)) = \quad (5.10)$$

$$= \det_{\Omega} (k^{AB}) (1 - W^A W^B k_{AB}) \quad (5.11)$$

Obviously, the inverse of k^{AB} is given by

$$k_{AB} = F^i{}_A F^j{}_B g_{ij}^{(3)} \quad (5.12)$$

If we insert that, we finally get for n

$$n = \sqrt{\det_{\Omega} k^{AB}} \sqrt{1 - W^i W^j g_{ij}} \quad (5.13)$$

The volume element takes the following form in the ADM formalism

$$\sqrt{-\det g_{\mu\nu}} d^4 x = N \sqrt{\det g_{ij}^{(3)}} dt d^3 x \quad (5.14)$$

This is because $g^{00} = \frac{1}{\det g_{\mu\nu}} \text{cofactor}_{00}$ (see equation (7.4)), and we have $g^{00} = -\frac{1}{N^2}$ and $\text{cofactor}_{00} = \det g_{ij}^{(3)}$. For N we have the following formula:

$$N = c \exp U/c^2 \quad (5.15)$$

where U is the Newtonian gravitational potential, and the factor c comes from $dx^0 = c dt$. This can be expanded in a Taylor series in $\frac{1}{c}$ immediately:

$$N = c \left(1 + \frac{U}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right) \right) \quad (5.16)$$

Now, as a first step to the non-relativistic limit, we choose a foliation with $Y^i = 0$. Then $g_{ij}^{(3)}$ becomes the spatial part of the inverse metric, and k^{AB} becomes the non-relativistic version of the strain tensor. For W^i we get

$$W^i = \frac{V^i}{N} = \frac{V^i}{c} + \mathcal{O}\left(\frac{1}{c^2}\right) \quad (5.17)$$

so equation (5.9) implies that

$$h^{AB} = k^{AB} + \mathcal{O}\left(\frac{1}{c^2}\right) \quad (5.18)$$

Therefore, the first term $\kappa := \sqrt{\det_{\Omega} k^{AB}}$ in equation (5.13) is the nonrelativistic expression for the particle number density, and the factor $\sqrt{1 - W^i W^j g_{ij}}$ can be interpreted as relativistic correction. κ can be seen as the ratio between the pulled back volume form from the body and the induced metric volume form on the spacelike hypersurfaces

$$f^A{}_i f^B{}_j f^C{}_k \Omega_{ABC}(f(x)) = \kappa \epsilon_{ijk}(x) \quad (5.19)$$

This can be seen by multiplying this equation with itself and summing over the space indices, which gives back the original term for κ .

Using (5.17), the relativistic correction term can be developed in a power series in $1/c$:

$$\sqrt{1 - W^i W^j g_{ij}} = 1 - \frac{V^2}{2c^2} + \mathcal{O}\left(\frac{1}{c^4}\right) \quad (5.20)$$

Finally, we need to split the stored energy function in the rest energy and the potential energy change caused by deformations to get a finite limit.

$$\epsilon(h^{AB}) = mc^2 + e(h^{AB}) \quad (5.21)$$

m denotes the mass per particle; we will assume it to be constant. Since h^{AB} is equal to k^{AB} up to Terms $\mathcal{O}\left(\frac{1}{c^2}\right)$, the same is true for $e(h^{AB})$:

$$\epsilon(h^{AB}) = mc^2 + e(k^{AB}) + \mathcal{O}\left(\frac{1}{c^2}\right) \quad (5.22)$$

Putting all this together, we get:

$$\begin{aligned} S_M[f^A] &= - \int_{\mathcal{M}} \kappa \left(1 - \frac{V^2}{2c^2} + \mathcal{O}\left(\frac{1}{c^4}\right) \right) \left(mc^2 + e(k^{AB}) + \mathcal{O}\left(\frac{1}{c^4}\right) \right) \\ &\quad \left(1 + \frac{U}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right) \right) \sqrt{\det g_{ij}} dt d^3x = \\ &= \int_{\mathcal{M}} \kappa \left(-mc^2 + \frac{mV^2}{2} - e(k^{AB}) - mU + \mathcal{O}\left(\frac{1}{c^2}\right) \right) \sqrt{\det g_{ij}} dt d^3x \end{aligned} \quad (5.23)$$

The term proportional to mc^2 does not converge; nevertheless it doesn't contribute to the equations of motion and can therefore be discarded. So we get as our action principle for nonrelativistic elasticity theory:

$$S_M[f^A] = \int_{\mathcal{M}} \kappa \left(\frac{mV^2}{2} - e(k^{AB}) - mU \right) dt d^3x \quad (5.24)$$

The total action of the system can be obtained by adding the gravitational action $S_G[g_{\mu\nu}]$ to the matter action $S_M[f^A, g_{\mu\nu}]$; it is well-known that the former is given basically by the scalar curvature R of the metric $g_{\mu\nu}$:

$$S_G[g_{\mu\nu}] = \frac{c^4}{16\pi G} \int_{\mathcal{M}} R \sqrt{-\det(g_{\mu\nu})} d^4x \quad (5.25)$$

We multiply this with $-\frac{1}{c}$ as well and get in the Newtonian limit (see [2]):

$$S_G[U] = -\frac{1}{8\pi G} \int_{\mathcal{M}} \partial_i U \partial_j U \delta^{ij} dt d^3x \quad (5.26)$$

where the gravitation is described by the Newtonian potential U . Similar to the relativistic case, one can add up the gravitational action and the matter action

to get a variational principle for the whole system

$$S[f^A, U] = \int_{\mathcal{M}} \Lambda \, dt \, d^3x \quad (5.27)$$

$$\Lambda = \Lambda_{kin} + \Lambda_{elast} + \Lambda_{pot} + \Lambda_{grav} \quad (5.28)$$

$$\Lambda_{kin} = \frac{1}{2} \rho v^i v^j \delta_{ij} \quad (5.29)$$

$$\Lambda_{elast} = -ne \quad (5.30)$$

$$\Lambda_{pot} = -\rho U \quad (5.31)$$

$$\Lambda_{grav} = -\frac{1}{8\pi G} \partial_i U \partial_j U \delta^{ij} \quad (5.32)$$

Note that we have already renamed the non-relativistic particle density κ to n here, and that we have introduced the abbreviation $\rho = mn$, as is usual in non-relativistic elasticity.

Chapter 6

Non-relativistic theory

6.1 Kinematics

Newtonian physics lives on Euclidean space $\mathcal{S}$, which is essentially $\mathbb{R}^3$ endowed with the standard metric. So the configuration is a time dependant function $f : \mathcal{S} \rightarrow \mathcal{B}$, the deformation $\phi : \mathcal{B} \rightarrow \mathcal{S}$ is its inverse (of course time dependant as well), i.e. $f \circ \phi = \text{id}_{\mathcal{B}}$ and $\phi \circ f = \text{id}_{f^{-1}(\mathcal{B})}$.

For the derivatives of the configuration and the deformation we use the short notation $f^A_{\mu} = \partial_{\mu} f^A$ and $\phi^i_A = \partial_A \phi^i$, where $\mu = 0, 1, 2, 3$ and $f^A_0 = \dot{f}^A$ means the time derivative. Obviously, f^A_i and ϕ^i_A are inverse matrices:

$$f^A_i(t, x) \phi^j_A(t, f(t, x)) = \delta^j_i \quad (6.1)$$

$$f^A_i(t, \phi(t, X)) \phi^i_B(t, X) = \delta^A_B \quad (6.2)$$

From differentiating (6.1) we get the useful identities

$$\partial_k (\phi^i_A) = \phi^i_{AC} f^C_k = -\phi^i_B f^B_{jk} \phi^j_A \quad (6.3)$$

$$\partial_B (f^A_i) = f^A_{ik} \phi^k_B = -f^A_j \phi^j_{CB} f^C_i \quad (6.4)$$

The nonrelativistic strain tensor is defined via (this is the limit of the K^{AB} from the last section)

$$H^{AB} = f^A_i \delta^{ij} f^B_j \quad (6.5)$$

and its inverse is given by

$$H_{AB} = \phi^i_A \delta_{ij} \phi^j_B \quad (6.6)$$

H_{AB} is often called the *right Cauchy-Green tensor* in the classic elasticity literature.

The velocity field $v^{\mu} \partial_{\mu} = \partial_t + v^i \partial_i$ is defined via

$$f^A_{\mu} v^{\mu} = \dot{f}^A + f^A_i v^i = 0 \quad (6.7)$$

From (6.7) we get an explicit formula for v^i :

$$v^i = -\phi^i_A \dot{f}^A \quad (6.8)$$

On the other hand, differentiating $f^A(t, \phi(t, X)) = X^A$ with respect to t yields

$$\dot{f}^A(t, \phi(t, X)) + f^A_i \dot{\phi}^i(t, X) = 0 \quad (6.9)$$

so we get another formula for v^i by comparing this with (6.7):

$$v^i(t, x) = \dot{\phi}^i(t, f(t, x)) \quad (6.10)$$

As in the relativistic case (5.19), the particle density is defined by the relation

$$f^A_i f^B_j f^C_k \Omega_{ABC}(f(x)) = n(x) \epsilon_{ijk}(x) \quad (6.11)$$

which results in

$$n = \sqrt{\det_{\Omega} H^{AB}} \quad (6.12)$$

If we are using Cartesian coordinates in flat space, this reduces to

$$n = \det f^A_i \quad (6.13)$$

From that immediately follows for the derivation of n (compare equation (7.1))

$$\frac{\partial n}{\partial f^A_i} = n \phi^i_A \quad (6.14)$$

As already mentioned at the end of the last section, it is common in non-relativistic elastodynamics to denote the mass density ρ . It is given by

$$\rho = nm \quad (6.15)$$

where m is the mass of a single particle, which we will assume to be constant. Do not confuse this with the energy density ρ in the relativistic case!

As in the relativistic theory, we can form a conserved current out of the particle density. We have

$$\partial_\mu (nv^\mu) = 0 \quad (6.16)$$

because

$$\partial_\mu (nv^\mu) = v^\mu \partial_\mu n + n \partial_i v^i = v^\mu n \phi^i_A f^A_{i\mu} + n \partial_i v^i = -n \phi^i_A f^A_{\mu} \partial_i v^\mu + n \partial_i v^i = 0 \quad (6.17)$$

Further on, we have the similar equation

$$\partial_i (n \phi^i_A) = \frac{\partial n}{\partial f^B_j} f^B_{ji} \phi^i_A + n \partial_i \phi^i_A = n \phi^j_B f^B_{ij} \phi^i_A + n \partial_i \phi^i_A = 0 \quad (6.18)$$

because $\phi^j_B f^B_{ij} \phi^i_A = -\partial_i \phi^i_A$, as can be seen by differentiating equation (6.1). Since ρ and n just differ by the constant factor m , these equations are also true if one replaces n with ρ .

The stress tensor is defined via a stored energy function $\epsilon = \epsilon(f^A, H^{AB})$. First we have the second Piola-Kirchhoff tensor

$$\tau_{AB} = 2 \frac{\partial \epsilon}{\partial H^{AB}} \quad (6.19)$$

from which we get the Cauchy stress tensor by

$$\tau_{ij} = n f^A_i f^B_j \tau_{AB} \quad (6.20)$$

Again, those definitions are completely analogous to the relativistic case. Further on, the quantity

$$\tau_A{}^i = n\tau_{AB}f^B{}_j\delta^{ij} \quad (6.21)$$

will be useful later on in the variational process, because

$$\frac{\partial\epsilon}{\partial f^A{}_i} = \frac{\partial\epsilon}{\partial H^{BC}} \frac{\partial H^{BC}}{\partial f^A{}_i} = \frac{1}{2}\tau_{BC}2\delta_A^{(B}f^{C)i} = \frac{1}{n}\tau_A{}^i \quad (6.22)$$

The connection of $\tau_A{}^i$ to the Cauchy stress is obviously given by $\tau_i{}^j = f^A{}_i\tau_A{}^j$.

Note that our definition of the stress tensor differs by a minus sign to the one more commonly used in non-relativistic elasticity literature. This is because the Lagrangian description is more frequently used in classical elasticity, so H_{AB} (see equation (6.6)) is the more common form of the strain tensor, and then the natural definition for the stress tensor is

$$2\frac{\partial\epsilon}{\partial H_{AB}} \quad (6.23)$$

which differs by a factor of $-H_{AC}H_{BD}$ (see equation (7.14)) from our definition (6.19).

6.2 Dynamics

The equations of motion are obtained by varying the action (5.27). Variation with respect to U yields the poisson equation (only Λ_{grav} contains $\partial_i U$, and U is only in Λ_{pot}):

$$\partial_i \frac{\partial\Lambda}{\partial(\partial_i U)} - \frac{\partial\Lambda}{\partial U} = -\frac{1}{4\pi G}\partial_i\delta^{ij}\partial_j U + \rho = 0 \quad (6.24)$$

$$\partial_i\partial^i U = 4\pi G\rho \quad (6.25)$$

Variation of $\Lambda_{kin} = \frac{1}{2}\rho v_i v^i$ yields

$$\partial_\mu \frac{\partial\Lambda_{kin}}{\partial f^A{}_\mu} - \frac{\partial\Lambda_{kin}}{\partial f^A} = \partial_t \frac{\partial\Lambda_{kin}}{\partial \dot{f}^A} + \partial_k \frac{\partial\Lambda_{kin}}{\partial \dot{f}^A{}_k} = -\partial_t(\rho v_i \phi^i{}_A) + \quad (6.26)$$

$$+\partial_k \left(\frac{1}{2}\rho \phi^k{}_A v^2 - \rho v_i \phi^i{}_A \phi^k{}_B \dot{f}^B \right) \quad (6.27)$$

where we have used $v^i = -\phi^i{}_A \dot{f}^A$ and $\frac{\partial\phi^i{}_B}{\partial \dot{f}^A{}_k} = -\phi^i{}_A \phi^k{}_B$ (see equation (7.14)). If we further plug $\partial_\mu(\rho v^\mu) = 0$ and $\partial_k(\rho \phi^k{}_A) = 0$ into that expression, we get

$$-\rho\partial_t(v_i\phi^i{}_A) + \frac{1}{2}\rho\phi^k{}_A\partial_k v^2 - \rho v^k\partial_k(v_i\phi^i{}_A) = \quad (6.28)$$

$$-\rho\dot{v}_i\phi^i{}_A - \rho v_i\dot{\phi}^i{}_A - \rho v_i\phi^i{}_{AB}\dot{f}^B + \rho\phi^k{}_A v_i\dot{\phi}^i{}_B \dot{f}^B - \quad (6.29)$$

$$-\rho v^k v_{i,k}\phi^i{}_A - \rho v^k v_i\phi^i{}_{AB}\dot{f}^B{}_k \quad (6.30)$$

The second and the fourth terms cancel, and the third and the sixth term vanish because of $f^A{}_\mu v^\mu = 0$; the remains can be combined into

$$-\rho v^\mu \partial_\mu v_i \phi^i{}_A \quad (6.31)$$

When varying Λ_{elast} , we need to take some more care: n is only defined on $f^{-1}(\mathcal{B})$ (strictly speaking, we should multiply it with the appropriate characteristic function $\chi_{f^{-1}(\mathcal{B})}$). This will give us some boundary condition:

$$\delta \int_{f^{-1}(\mathcal{B})} \Lambda_{elast} dt d^3x = \int_{f^{-1}(\mathcal{B})} \frac{\partial \Lambda_{elast}}{\partial f^A} \delta f^A + \frac{\partial \Lambda_{elast}}{\partial f^A_i} \delta f^A_i dt d^3x = \quad (6.32)$$

$$= \int_{f^{-1}(\mathcal{B})} \frac{\partial \Lambda_{elast}}{\partial f^A} \delta f^A + \partial_i \left(\frac{\partial \Lambda_{elast}}{\partial f^A_i} \delta f^A \right) - \partial_i \frac{\partial \Lambda_{elast}}{\partial f^A_i} \delta f^A dt d^3x = \quad (6.33)$$

$$= \int_{f^{-1}(\mathcal{B})} \left(\frac{\partial \Lambda_{elast}}{\partial f^A} - \partial_i \frac{\partial \Lambda_{elast}}{\partial f^A_i} \right) \delta f^A dt d^3x + \quad (6.34)$$

$$\int_{\partial f^{-1}(\mathcal{B})} \frac{\partial \Lambda_{elast}}{\partial f^A_i} n_i \delta f^A dt d^2A \quad (6.35)$$

so we get (using (6.18)) the following contribution to the Euler equations

$$\partial_\mu \frac{\partial \Lambda_{elast}}{\partial f^A_\mu} - \frac{\partial \Lambda_{elast}}{\partial f^A} = \partial_j \left(-n \phi^j_A \epsilon - n \frac{1}{n} \tau_A^j \right) + n \frac{\partial \epsilon}{\partial f^A} = \quad (6.36)$$

$$= -n \phi^j_A \left(\frac{\partial \epsilon}{\partial f^B} f^B_j + \frac{\partial \epsilon}{\partial H^{BC}} 2f^B_{kj} f^C_l \delta^{kl} \right) - \partial_j \tau_A^j + n \frac{\partial \epsilon}{\partial f^A} = \quad (6.37)$$

$$= -\phi^j_A \tau_B^k f^B_{kj} - \partial_j \tau_A^j = -\phi^i_A \partial_j (f^B_i \tau_B^j) = -\phi^i_A \partial_j \tau_i^j \quad (6.38)$$

plus the boundary condition

$$\frac{\partial \Lambda_{elast}}{\partial f^A_i} n_i|_{\partial f^{-1}(\mathcal{B})} = \left(-n \phi^i_A \epsilon - n \frac{1}{n} \tau_A^i \right) n_i|_{\partial f^{-1}(\mathcal{B})} = 0 \quad (6.39)$$

n vanishes on the boundary anyways, that's why we only need to keep the second term; after multiplication with $-f^A_i$ we get the final form of the boundary condition

$$\tau_i^j n_j|_{\partial f^{-1}(\mathcal{B})} = 0 \quad (6.40)$$

The normal covector n_i on $\partial f^{-1}(\mathcal{B})$ can be expressed as the pull-back of a normal covector n_A on $\partial \mathcal{B}$:

$$n_i(t, x^k) = f^A_i(t, x^k) n_A(f^B(t, x^k)) \quad (6.41)$$

Finally, the variation of Λ_{pot} contributes

$$\partial_i \frac{\partial \Lambda_{pot}}{\partial f^A_i} = -\partial_i (\rho \phi^i_A U) = -\rho \phi^i_A \partial_i U \quad (6.42)$$

where we have used (6.18) for the second equality. putting all this together, we get

$$-\rho v^\mu \partial_\mu v_i \phi^i_A - \partial_j \tau_i^j \phi^i_A - \rho \phi^i_A \partial_i U = 0 \quad (6.43)$$

or, after multiplication with $-f^A_k \delta^{kl}$ (and renaming the index afterwards):

$$\rho v^\mu \partial_\mu v^i + \partial_j \tau^{ij} + \rho \partial^i U = 0 \quad (6.44)$$

This equation, together with the boundary condition (6.40) and the Poisson equation (6.25) fully determines nonrelativistic elastodynamics coupled to gravity.

6.3 Variations

If we apply the perturbation formalism to the equations of motion derived in the last section, we get

$$\delta\rho\frac{\partial v^i}{\partial t} + \rho\frac{\partial\delta v^i}{\partial t} + \delta\rho v^j\partial_j v^i + \rho\delta v^j\partial_j v^i + \rho v^j\partial_j\delta v^i + \partial_j\delta\tau^{ij} + \delta\rho\partial^i U + \rho\partial^i\delta U = 0 \quad (6.45)$$

$$\nabla^2\delta U = 4\pi G\delta\rho \quad (6.46)$$

To perturb the boundary condition (6.40), it is more convenient to transform it to the body manifold:

$$\tau^{ij}(\phi(t, X))f^A_i(\phi(t, X))n_A(X)|_{\partial\mathcal{B}} = 0 \quad (6.47)$$

applying the perturbation formalism to this equation gives

$$(\delta\tau^{ij}f^A_in_A + \tau^{ij}{}_{,k}\delta\phi^k f^A_in_A + \tau^{ij}\delta f^A_in_A + \tau^{ij}f^A_{ik}\delta\phi^k n_A)|_{\partial\mathcal{B}} = 0 \quad (6.48)$$

where everything is understood to be evaluated at the same points as in (6.47). For $\delta\phi^k$ we use the relationship

$$\delta\phi^k = -\phi^k_B\delta f^B \quad (6.49)$$

which is a direct consequence of applying the perturbation operator δ to the identity $\phi^k(f(x)) = x^k$. Now inserting (6.49) to (6.47) yields

$$(\delta\tau^{ij}f^A_in_A - \tau^{ij}{}_{,k}\phi^k_B\delta f^B f^A_in_A + \tau^{ij}\delta f^A_in_A - \tau^{ij}f^A_{ik}\phi^k_B\delta f^B n_A)|_{\partial\mathcal{B}} = 0 \quad (6.50)$$

From perturbing the definitions (6.15), (6.5), (6.19) and (6.20) we get

$$\delta\rho = m\delta n = m\frac{\partial n}{\partial f^A_i}\delta f^A_i = mn\phi^i_A\delta f^A_i \quad (6.51)$$

$$\delta H^{AB} = \delta f^A_i\delta^{ij}f^B_j + f^A_i\delta^{ij}\delta f^B_j = 2\delta f^{(A}_i f^{B)}_j\delta^{ij} \quad (6.52)$$

$$\delta\tau_{AB} = 2\frac{\partial^2\epsilon}{\partial H^{AB}\partial H^{CD}}\delta H^{CD} + \tau_{AB,C}\delta f^C \quad (6.53)$$

$$\delta\tau_{ij} = \delta n f^A_i f^B_j \tau_{AB} + n\delta f^A_i f^B_j \tau_{AB} + n f^A_i \delta f^B_j \tau_{AB} + n f^A_i f^B_j \delta\tau_{AB} \quad (6.54)$$

To get an expression for δv^μ , we use the defining equation for v^μ (6.7), and get

$$\delta f^A_\mu v^\mu + f^A_\mu \delta v^\mu = 0 \quad (6.55)$$

which implies

$$\delta v^i = -\phi^i_A \delta f^A_\mu v^\mu = -\phi^i_A \delta \dot{f}^A - \phi^i_A \delta f^A_j v^j \quad (6.56)$$

The generating vector field of a perturbation is given by

$$\xi^i(t, x) = \delta\phi^i(t, f(t, x)) = -\phi^i_A(t, f(t, x))\delta f^A(t, x) \quad (6.57)$$

where the second equality comes from perturbing $f^A(t, \phi(t, X)) = X^A$. Using that, we can verify the formula

$$\Delta v^i = \delta v^i + \mathcal{L}_\xi v^i = \dot{\xi}^i \quad (6.58)$$

from Friedman and Schutz [7]: we simply add (6.56) to the Lie derivative

$$\begin{aligned}\mathcal{L}_\xi v^i &= \xi^j \partial_j v^i - v^j \partial_j \xi^i = \\ &= -\phi^j_A \delta f^A \dot{\phi}^i_B f^B_j + v^j \phi^i_{AB} f^B_j \delta f^A + v^j \phi^i_A \delta f^A_j = \\ &= -\dot{\phi}^i_A \delta f^A - \phi^i_{AB} \dot{f}^B \delta f^A + \phi^i_A \delta f^A_j v^j\end{aligned}\quad (6.59)$$

and get the same expression as from differentiating (6.57)

$$\dot{\xi}^i = -\dot{\phi}^i_A \delta f^A - \phi^i_{AB} \dot{f}^B \delta f^A - \phi^i_A \delta \dot{f}^A \quad (6.60)$$

The reason that the result in the non-relativistic case is different to the relativistic result ($\Delta v^\mu = 0$, (4.19)) is that the four-dimensional Lie derivative (4.17) differs from the three-dimensional Lie derivative (6.59).

Finally, we note that another way to write (6.58) is

$$\delta v^i = (\partial_t + \mathcal{L}_v) \xi^i \quad (6.61)$$

Similarly to the above, we also see $\delta \rho = -\partial_i (\rho \xi^i)$:

$$\begin{aligned}-\partial_i (\rho \xi^i) &= \phi^i_A \delta f^A \frac{\partial \rho}{\partial f^B_j} f^B_{ji} - \rho (-\partial_i (\phi^i_A) \delta f^A - \phi^i_A \delta f^A_i) = \\ &= \rho \phi^j_B f^B_{ji} \phi^i_A \delta f^A - \rho \phi^i_B f^B_{ji} \phi^j_A \delta f^A + \rho \phi^i_A \delta f^A_i = \delta \rho\end{aligned}\quad (6.62)$$

When working with variations around a static configuration, one can use the portion of $\mathcal{S}$ occupied by matter as body manifold. This means the unperturbed f^A become the identity and we therefore also have

$$\begin{aligned}f^A_i &= \delta^A_i \\ \phi^i_A &= \delta^i_A\end{aligned}\quad (6.63)$$

Then, the relation of δf^A and the displacement vector ξ^i (6.57) simplifies to

$$\begin{aligned}\delta f^A &= -\delta^A_j \xi^j \\ \delta f^A_i &= -\delta^A_j \xi^j_{,i}\end{aligned}\quad (6.64)$$

Thus, we can express the variation of the strain tensor (6.52) via the displacement vector field ξ^i

$$\begin{aligned}\delta H^{AB} &= \delta f^A_i f^B_j \delta^{ij} + f^A_i \delta f^B_j \delta^{ij} = -\delta^A_k \xi^k_{,i} \delta^B_j \delta^{ij} - \delta^A_i \delta^B_k \xi^k_{,j} \delta^{ij} = \\ &= -\delta^A_i \delta^B_j (\xi^{i,j} + \xi^{j,i})\end{aligned}\quad (6.65)$$

This expression is sometimes called the *infinitesimal strain tensor*. In a similar way, the variations of the Cauchy stress tensor (6.54) becomes

$$\begin{aligned}\delta \tau_{ij} &= -\partial_k (n \xi^k) \delta^A_i \delta^B_j \tau_{AB} - n \delta^A_k \xi^k_{,i} \delta^B_j \tau_{AB} - n \delta^A_i \delta^B_k \xi^k_{,j} \tau_{AB} + \\ &\quad + n \delta^A_i \delta^B_j \left(-\frac{1}{2} \tilde{C}_{ABCD} \delta^C_k \delta^D_l (\xi^{k,l} + \xi^{l,k}) - \tau_{AB,C} \delta^C_k \xi^k \right) = \\ &= -\partial_k \xi^k \tau_{ij} - \tau_{kj} \xi^k_{,i} - \tau_{ik} \xi^k_{,j} - \tilde{C}_{ijkl} \xi^{k,l} - \tau_{ij,k} \xi^k = \\ &= -\left(\tau_{ij} \delta_{kl} + \tau_{kj} \delta_{il} + \tau_{ik} \delta_{lj} + \tilde{C}_{ijkl} \right) \xi^{k,l} - \tau_{ij,k} \xi^k = \\ &= -C_{ijkl} \xi^{k,l} - \tau_{ij,k} \xi^k\end{aligned}\quad (6.66)$$

where we have used the abbreviations

$$\tilde{C}_{ABCD} = 4 \frac{\partial^2 \epsilon}{\partial H^{AB} \partial H^{CD}} \quad (6.67)$$

$$\tilde{C}_{ijkl} = n \delta_i^A \delta_j^B \delta_k^C \delta_l^D \tilde{C}_{ABCD} \quad (6.68)$$

$$C_{ijkl} = \tilde{C}_{ijkl} + \tau_{ij} \delta_{kl} + \tau_{kj} \delta_{il} + \tau_{ik} \delta_{lj} \quad (6.69)$$

$\tilde{C}_{ABCD}$ is called the *elasticity tensor*.

6.3.1 Perturbations generated by a Killing field on $\mathcal{B}$

We want to show that the following perturbed quantities satisfy the perturbed equations of motion (6.45), (6.46) identically:

$$\delta f^A(t, x) = \eta^A(f(t, x)) \quad (6.70)$$

$$\delta U(t, x) = 0 \quad (6.71)$$

Here η^A is a Killing vector field on the body manifold $\mathcal{B}$ that is tangential to $\partial\mathcal{B}$, i.e.

$$\eta^A n_A|_{\partial\mathcal{B}} = 0 \quad (6.72)$$

Further we assume the body to be homogeneous and isotropic. For a fluid it is already sufficient to assume that $\eta^A{}_{,A} = 0$.

The first step in proving the claim is to show that all perturbed quantities that enter the perturbed equations of motion (6.45), (6.46) vanish. First, we get through simple calculations:

$$\delta f^A{}_\mu = \eta^A{}_{,B} f^B{}_\mu \quad (6.73)$$

$$\delta \rho = mn \phi^i{}_A \eta^A{}_{,B} f^B{}_i = mn \eta^A{}_{,B} \delta_A^B = 0 \quad (6.74)$$

$$\delta v^i = -\phi^i{}_A \delta f^A{}_\mu v^\mu = -\phi^i{}_A \eta^A{}_{,B} f^B{}_\mu v^\mu = 0 \quad (6.75)$$

$$\delta H^{AB} = \eta^A{}_{,C} f^C{}_i \delta^{ij} f^B{}_j + f^A{}_i \delta^{ij} \eta^B{}_{,C} f^C{}_j = \quad (6.76)$$

$$= \eta^A{}_{,C} H^{CB} + \eta^B{}_{,C} H^{AC} \quad (6.77)$$

We have not yet used that η^A is a Killing vector, only that it is divergence free (for $\delta \rho = 0$). This already yields the result for the perfect fluid, since in this case we have $\delta p = 0$ with $\delta \rho = 0$, and thus also $\delta \tau^{ij} = 0$.

To extend the result to an isotropic solid, we look at the perturbations of the invariants of the strain. They vanish as well, where we now have to explicitly use $\eta_{A,B} = \eta_{[A,B]}$:

$$\begin{aligned} \delta H^A{}_A &= \delta H^{AB} \delta_{AB} = (\eta^A{}_{,C} H^{CB} + \eta^B{}_{,C} H^{AC}) \delta_{AB} = \\ &= \eta_{B,C} H^{CB} + \eta_{A,C} H^{AC} = 0 \\ \delta (H^A{}_B H^B{}_A) &= 2 \delta H^{AB} H^{CD} \delta_{BC} \delta_{AD} = \\ &= 2 (\eta^A{}_{,E} H^{EB} + \eta^B{}_{,E} H^{AE}) H^{CD} \delta_{BC} \delta_{AD} = \\ &= 2 (\eta_{D,E} H^{EB} \delta_{BC} H^{CD} + \eta_{C,E} H^{AE} \delta_{AD} H^{CD}) = 0 \\ \delta (H^A{}_B H^B{}_C H^C{}_A) &= 3 \delta H^{AB} H^{CD} H^{EF} \delta_{BC} \delta_{DE} \delta_{AE} = \\ &= 3 (\eta^A{}_{,G} H^{GB} + \eta^B{}_{,G} H^{AG}) H^{CD} H^{EF} \delta_{BC} \delta_{DE} \delta_{AF} = \\ &= 6 \eta_{F,G} H^{GB} \delta_{BC} H^{CD} \delta_{DE} H^{EF} = 0 \end{aligned} \quad (6.78)$$

The assumption that the body is homogeneous and isotropic means that the stored energy function ϵ only depends on the invariants of the strain. Thus, we can rewrite the second Piola-Kirchhoff stress tensor as

$$\tau_{AB} = 2 \sum_{a=1}^3 \frac{\partial \epsilon}{\partial q_a} \frac{\partial q_a}{H^{AB}} \quad (6.79)$$

where the q_a ($a = 1, 2, 3$) are the invariants of H^{AB} . Consequently, we get

$$\tau_{ij} = 2n \sum_{a=1}^3 \frac{\partial \epsilon}{\partial q_a} f^A{}_i f^B{}_j \frac{\partial q_a}{H^{AB}} \quad (6.80)$$

For the variations of the first part of this expression we get:

$$\delta \left(\frac{\partial \epsilon}{\partial q_a} \right) = \frac{\partial^2 \epsilon}{\partial q_a \partial q_b} \delta q_b = 0 \quad (6.81)$$

because $\delta q_b = 0$ (equation (6.78)).

If we use $q_a = \text{tr}((H^{AB} \delta_{BC})^a)$ as the simplest choice for the invariants (like we have already done above in (6.78)), we get

$$\frac{\partial q_1}{H^{AB}} = \delta_{AB} \quad (6.82)$$

$$\frac{\partial q_2}{H^{AB}} = 2\delta_{AC} H^{CD} \delta_{DB} \quad (6.83)$$

$$\frac{\partial q_3}{H^{AB}} = 3\delta_{AC} H^{CD} \delta_{DE} H^{EF} \delta_{FB} \quad (6.84)$$

so the second part of (6.80) can be expressed via the pull-back of the body metric $\gamma_{ij} = f^A{}_i f^B{}_j \delta_{AB}$:

$$f^A{}_i f^B{}_j \frac{\partial q_a}{H^{AB}} = a \delta_{ik} (\delta^{kl} \gamma_{lj})^a \quad (6.85)$$

The variation of γ_{ij} vanishes:

$$\begin{aligned} \delta \gamma_{ij} &= \delta f^A{}_i f^B{}_j \delta_{AB} + f^A{}_i \delta f^B{}_j \delta_{AB} = \\ &= \eta^A{}_{,C} f^C{}_i f^B{}_j \delta_{AB} + f^A{}_i \eta^B{}_{,C} f^C{}_j \delta_{AB} = \\ &= \eta_{B,C} f^C{}_i f^B{}_j + \eta_{A,C} f^A{}_i f^C{}_j = 0 \end{aligned} \quad (6.86)$$

and with it $(\delta^{ij} \gamma_{jk})^a$ for all a as well. Thus, we get the desired result

$$\delta \tau_{ij} = 0 \quad (6.87)$$

To finish the proof of the claim, we also have to show that the perturbed boundary condition (6.50) is identically satisfied by the perturbations (6.70). To do so, we insert the perturbed quantities (6.70) and (6.73) into (6.50) and get

$$(\delta \tau^{ij} \eta^A{}_{,B} f^B{}_i - \tau^{ij}{}_{,k} \phi^k{}_B \eta^B f^A{}_i n_A + \tau^{ij} \eta^A{}_{,B} f^B{}_i n_A - \tau^{ij} f^A{}_{,ik} \phi^k{}_B \eta^B n_A) |_{\partial \mathcal{B}} = 0 \quad (6.88)$$

The first term vanishes, since $\delta\tau^{ij} = 0$. The third term can be rewritten using the Gauss equation, because both η^A and $\tau^{ij}f^B{}_i$ are tangential to $\partial\mathcal{B}$ by the assumption (6.72) and the unperturbed boundary condition (6.47); we can work with an arbitrary extension of n_A to an open neighbourhood of $\partial\mathcal{B}$.

$$\eta^A{}_{,B}n_A\tau^{ij}f^B{}_i|_{\partial\mathcal{B}} = -\tau^{ij}f^B{}_i\eta^A n_{A,B}|_{\partial\mathcal{B}} = -\tau^{ij}f^B{}_i\eta^A n_{B,A}|_{\partial\mathcal{B}} \quad (6.89)$$

The first equality comes from applying $\tau^{ij}f^B{}_i\partial_B$ to the orthogonality condition (6.72), the second one is the Gauss equation. Finally, we exchange the role of the summation indices A and B in the said term. Thus, (6.88) becomes

$$(-\tau^{ij}{}_{,k}\phi^k{}_B\eta^B f^A{}_i n_A - \tau^{ij}f^A{}_{ik}\phi^k{}_B\eta^B n_A - \tau^{ij}f^A{}_i n_{A,B}\eta^B)|_{\partial\mathcal{B}} = 0 \quad (6.90)$$

This is automatically satisfied, since this is the same as we get when applying $-\eta^B\partial_B$ to the unperturbed boundary condition (6.47).

The perturbations (6.70) are equal to the "trivial displacements" used in [7]:

$$\xi^i(t, x) = \delta\phi^i(t, f(t, x)) = -\phi^i{}_A(t, f(t, x))\eta^A(f(t, x)) \quad (6.91)$$

Thus we see ξ^i is just minus the pull-back of η^A :

$$\xi^i\partial_i = -f^*(\eta^A\partial_A) \quad (6.92)$$

The operator $(\partial_t + \mathcal{L}_v)$ applied to a pull-back vanishes identically (this is another way to show $\delta v^i = 0$, according to equation (6.61)). Here is the explicit calculation for vectors:

$$\begin{aligned} (\partial_t + \mathcal{L}_v)\phi^i{}_A(t, f(t, x))\eta^A(f(t, x)) &= \dot{\phi}^i{}_A\eta^A + \phi^i{}_{AB}\dot{f}^B\eta^A + \phi^i{}_A\eta^A{}_{,B}\dot{f}^B + \\ &+ v^j\phi^i{}_{AB}f^B{}_j\eta^A + v^j\phi^i{}_A\eta^A{}_{,B}f^B{}_j - \phi^i{}_A\eta^A v^i{}_{,j} = 0 \end{aligned} \quad (6.93)$$

The second and forth and third and fifth term cancel because of (6.7) and $v^i{}_{,j} = \dot{\phi}^i{}_B f^B{}_j$ because of (6.8), so the first and the sixth term cancel as well. For one-forms, the calculation looks like this:

$$\begin{aligned} (\partial_t + \mathcal{L}_v)f^A{}_i(t, x)\eta_A(f(t, x)) &= \dot{f}^A{}_i\eta_A + f^A{}_{iA,B}\dot{f}^B + v^j f^A{}_{ij}\eta_A + \\ &+ v^j f^A{}_{iA,B}f^B{}_j + f^A{}_{ij}\eta_A v^j{}_{,i} = 0 \end{aligned} \quad (6.94)$$

Again, because of (6.7) the second and fourth term cancel; the remaining terms vanish, as one can see by differentiating (6.7).

Divergence free vector fields are corresponding to closed two-forms by Hodge duality, which again correspond to exterior derivatives of one-forms by the Poincaré lemma. In index notation, the exterior derivative of a one-form ζ_A is given by $\partial_{[A}\zeta_{B]}$, and its corresponding vector field is given by

$$\eta^C = \Omega^{CAB}\partial_{[A}\zeta_{B]} = \Omega^{CAB}\partial_A\zeta_B \quad (6.95)$$

If that is pulled back to $\mathcal{S}$, this yields

$$\eta^i = \frac{1}{n}\Omega^{ijk}\partial_j\zeta_k \quad (6.96)$$

which is (up to a constant factor of $\frac{1}{m}$) just the formula given in [7]. Also, as a pull-back $\zeta^k\partial_k = f^*(\zeta^A\partial_A)$ satisfies equation (6.94).

6.3.2 Perturbations generated by a Killing field on $\mathcal{S}$

We want to show that the following perturbed quantities satisfy the perturbed equations identically:

$$\delta f^A = \mathcal{L}_\xi f^A = f^A_{,i} \xi^i \quad (6.97)$$

$$\delta U = \mathcal{L}_\xi U = U_{,i} \xi^i \quad (6.98)$$

where ξ^i is a Killing vector field on the Galileian (Euclidean) space $\mathcal{S}$.

In this case we get for the other perturbed quantities:

$$\delta f^A_{,i} = f^A_{,ij} \xi^j + f^A_{,j} \xi^j_{,i} = \mathcal{L}_\xi f^A_{,i} \quad (6.99)$$

$$\delta \dot{f}^A = \dot{f}^A_{,i} \xi^i = \mathcal{L}_\xi \dot{f}^A \quad (6.100)$$

$$\delta \rho = mn \phi^i_A \delta f^A_{,i} = mn \phi^i_A (f^A_{,ij} \xi^j + f^A_{,j} \xi^j_{,i}) = \quad (6.101)$$

$$= \frac{\partial \rho}{\partial f^A_{,i}} f^A_{,ij} \xi^j = \mathcal{L}_\xi \rho \quad (6.102)$$

$$\delta v^i = -\phi^i_A \delta \dot{f}^A - \phi^i_A \delta f^A_{,j} v^j = \quad (6.103)$$

$$= -\phi^i_A \dot{f}^A_{,k} \xi^k - \phi^i_A f^A_{,jk} \xi^k v^j - \phi^i_A f^A_{,k} \xi^k_{,j} v^j = \quad (6.104)$$

$$= \phi^i_A f^A_{,j} v^j_{,k} \xi^k - \xi^i_{,j} v^j = v^i_{,j} \xi^j - \xi^i_{,j} v^j = \mathcal{L}_\xi v^i \quad (6.105)$$

$$\delta H^{AB} = (f^A_{,ik} \xi^k + f^A_{,k} \xi^k_{,i}) \delta^{ij} f^B_{,j} + f^A_{,i} \delta^{ij} (f^B_{,jk} \xi^k + f^B_{,k} \xi^k_{,j}) = \quad (6.106)$$

$$= H^{AB}_{,k} \xi^k + f^A_{,k} f^B_{,j} (\xi^{k,j} + \xi^{j,k}) = H^{AB}_{,k} \xi^k = \mathcal{L}_\xi H^{AB} \quad (6.107)$$

$$\delta \tau^{AB} = \frac{\partial \tau^{AB}}{\partial f^C} \delta f^C + \frac{\partial \tau^{AB}}{\partial H^{CD}} \delta H^{CD} = \frac{\partial \tau^{AB}}{\partial x^i} \xi^i = \mathcal{L}_\xi \tau^{AB} \quad (6.108)$$

$$\delta \tau_{ij} = \delta (n f^A_{,i} f^B_{,j} \tau_{AB}) = \mathcal{L}_\xi \tau_{ij} \quad (6.109)$$

In equation (6.103) we have used $-\dot{f}^A_{,k} - f^A_{,jk} v^j = f^A_{,j} v^j_{,k}$, which follows from differentiating the defining equation of v^μ (6.7). Also note that with (6.109), a similar equality holds for the contravariant stress tensor, i.e. $\delta \tau^{ij} = \mathcal{L}_\xi \tau^{ij}$, because $\mathcal{L}_\xi \delta^{ij} = 0$ by the assumption that ξ^i is a Killing vector field.

We first show that the perturbed Poisson equation (6.46) follows from the unperturbed one (6.25) by applying $\mathcal{L}_\xi$ to it. Since we already know that $\delta \rho = \mathcal{L}_\xi \rho$, we just have to show a similar equation for δU :

$$\nabla^2 \delta U = \nabla^2 (U_{,i} \xi^i) = (\nabla^2 U)_{,i} \xi^i + 2U_{,ij} \xi^{i,j} + U_{,i} \nabla^2 \xi^i = \mathcal{L}_\xi (\nabla^2 U) \quad (6.110)$$

Here, the last two terms in the decomposition of the Laplace operator vanish since ξ^i is a Killing vector, i.e. $\xi^{i,j} = \xi^{[i,j]}$, and second derivations of a Killing vector fields are proportional to the Riemann curvature and therefore vanish in flat space. So indeed the Poisson equation (6.46) is satisfied with the unperturbed one (6.25).

Similarly, applying $\mathcal{L}_\xi$ to the unperturbed equation of motion (6.44) yields the equation of motion for the perturbations (6.45):

$$\begin{aligned} \mathcal{L}_\xi (v^\mu \partial_\mu v^i) &= \mathcal{L}_\xi \left(\frac{\partial v^i}{\partial t} + v^j \partial_j v^i \right) = \\ &= \frac{\partial \mathcal{L}_\xi v^i}{\partial t} + \mathcal{L}_\xi (v^j) \partial_j v^i + v^j \mathcal{L}_\xi (\partial_j v^i) = \\ &= \frac{\partial \delta v^i}{\partial t} + \delta v^j \partial_j v^i + v^j \partial_j \delta v^i \end{aligned} \quad (6.111)$$

because Lie derivatives and time derivatives commute and

$$\begin{aligned}\mathcal{L}_\xi (\partial_j v^i) &= v^i{}_{,jk} \xi^k + v^i{}_{,k} \xi^k{}_{,j} - v^k{}_{,j} \xi^i{}_{,k} = \\ &= v^i{}_{,kj} \xi^k + v^i{}_{,k} \xi^k{}_{,j} - v^k{}_{,j} \xi^i{}_{,k} - v^k \xi^i{}_{,kj} = \\ &= \partial_j (v^i{}_{,k} \xi^k - v^k \xi^i{}_{,k}) = \partial_j (\mathcal{L}_\xi v^i) = \partial_j \delta v^i\end{aligned}\quad (6.112)$$

Also

$$\mathcal{L}_\xi (\partial_j \tau^{ij}) = \tau^{ij}{}_{,jk} \xi^k - \tau^{kj}{}_{,j} \xi^i{}_{,k} = \partial_j (\mathcal{L}_\xi \tau^{ij}) = \partial_j (\delta \tau^{ij}) \quad (6.113)$$

since

$$\begin{aligned}\partial_j (\mathcal{L}_\xi \tau^{ij}) &= \partial_j (\tau^{ij}{}_{,k} \xi^k - \tau^{kj} \xi^i{}_{,k} - \tau^{ik} \xi^j{}_{,k}) = \\ &= \tau^{ij}{}_{,kj} \xi^k + \tau^{ij}{}_{,k} \xi^k{}_{,j} - \tau^{kj}{}_{,j} \xi^i{}_{,k} - \tau^{kj} \xi^i{}_{,kj} - \\ &\quad - \tau^{ik}{}_{,j} \xi^j{}_{,k} - \tau^{ik} \xi^j{}_{,kj} = \tau^{ij}{}_{,jk} \xi^k - \tau^{kj}{}_{,j} \xi^i{}_{,k}\end{aligned}\quad (6.114)$$

The fourth and the sixth term vanish, since they contain second derivations of the Killing vector field, and the second and the fifth term cancel out.

Also for the gradient of U , the perturbation and the differentiation commute:

$$\begin{aligned}\mathcal{L}_\xi (\partial^i U) &= U^{,i}{}_{,j} \xi^j - U^{,j} \xi^i{}_{,j} = U^{,j}{}_{,i} \xi^j + U^{,j} \xi_{j,i} = \\ &= \partial^i (U_{,j} \xi^j) = \partial^i \delta U\end{aligned}\quad (6.115)$$

To finish the proof of the claim, we still need to show that also the perturbed boundary condition (6.50) are identically satisfied. To do so, we insert the perturbations (6.97), $\delta \tau^{ij} = \mathcal{L}_\xi \tau^{ij} = \tau^{ij}{}_{,k} \xi^k - \xi^i{}_{,k} \tau^{kj} - \xi^j{}_{,k} \tau^{ik}$ and $\delta f^A{}_i = \mathcal{L}_\xi f^A{}_i = f^A{}_{ik} \xi^k + f^A{}_k \xi^k{}_{,i}$ and get

$$\begin{aligned}((\tau^{ij}{}_{,k} \xi^k - \xi^i{}_{,k} \tau^{kj} - \xi^j{}_{,k} \tau^{ik}) f^A{}_i n_A - \tau^{ij}{}_{,k} \phi^k{}_B f^B{}_l \xi^l f^A{}_i n_A + \\ + \tau^{ij} (f^A{}_{ik} \xi^k + f^A{}_k \xi^k{}_{,i}) n_A - \tau^{ij} f^A{}_{ik} \phi^k{}_B f^B{}_l \xi^l n_A)|_{\partial \mathcal{B}} = 0\end{aligned}\quad (6.116)$$

In this equation, the third term vanishes with the unperturbed boundary condition (6.40), i.e. $\tau^{ik} f^A{}_i n_A = 0$; the remaining terms cancel in pairs (the first with the fourth, the second with the sixth and the fifth with the seventh), so indeed with the perturbations (6.97) the perturbed boundary condition (6.50) is identically satisfied.

From a more general point of view the fact that the perturbations (6.97) identically satisfy the perturbed equations (6.45), (6.46) follows from the invariance of the Lagrange function (5.27) under Euclidean motions $\sigma : \mathcal{S} \rightarrow \mathcal{S}$.

$$\mathcal{L}[\sigma^* f^A, \sigma^* U](x) = \mathcal{L}[f^A, U](\sigma(x)) \quad (6.117)$$

This is true since under Euclidean motions the quantities of the theory behave as following:

$$\begin{aligned}f^A(x^i) &\longrightarrow f^A(\sigma(x^i)) \\ f^A{}_i(x^k) &\longrightarrow \sigma^j{}_{,i} f^A{}_j(\sigma(x^k)) \\ \phi^i(X^A) &\longrightarrow \sigma^i(\phi^k(X^A)) \\ \phi^i{}_A(X^B) &\longrightarrow \sigma^i{}_j \phi^j{}_A(X^B) \\ v^i(x^k) &\longrightarrow \sigma^i{}_j v^j(\sigma(x^k)) \\ v^i(x^k) v^j(x^k) \delta_{ij} &\longrightarrow v^i(\sigma(x^k)) v^j(\sigma(x^k)) \delta_{ij} \\ U_{,i}(x^k) &\longrightarrow \sigma^i{}_{,j} U_{,j}(\sigma(x^k)) \\ U_{,i}(x^k) U_{,j}(x^k) \delta^{ij} &\longrightarrow U_{,i}(\sigma(x^k)) U_{,j}(\sigma(x^k)) \delta^{ij}\end{aligned}\quad (6.118)$$

because the derivatives $\sigma^i_{,j}$ of Euclidean motions are orthogonal matrices, that leave the Euclidean metric δ_{ij} invariant. Thus, all parts of the Lagrangian behave as scalars under Euclidean motions as well.

With the Lagrangian, also the equations of motion $\mathcal{F}[f^A, U]$ is invariant under σ :

$$\mathcal{F}[f^A, U] = 0 \implies \mathcal{F}[\sigma^* f^A, \sigma^* U] = 0 \quad (6.119)$$

Let's view an one-parametric family of Euclidean motions σ_λ now. If we differentiate (6.119) with respect to the parameter λ and set it to zero afterwards, we get

$$\delta\mathcal{F}[\mathcal{L}_\xi f^A, \mathcal{L}_\xi U] = 0 \quad (6.120)$$

where ξ^i is the Killing vector field generating σ_λ and the perturbed equations $\delta\mathcal{F}$ are understood to be evaluated at the fixed solution of the unperturbed equation.

6.4 Self-gravitating isotropic sphere

As an example, we view the radial oscillations of a self-gravitating isotropic sphere with radius a and constant density ρ . This problem (in the case without gravity) was first solved by Cauchy in 1829; we follow the treatment of [5].

We first look at the static background solution: since we assume it to be a sphere with constant density ρ , the gravitational field strength $g(r) = U'(r)$ can be immediately written down:

$$g(r) = \begin{cases} \frac{4\pi}{3} G \rho r & \text{for } r < a \\ \frac{4\pi}{3} G \rho \frac{a^3}{r^2} & \text{for } r \geq a \end{cases} \quad (6.121)$$

Thus, the Newtonian potential U is given by

$$U(r) = \begin{cases} \frac{4\pi}{3} G \rho \left(\frac{r^2}{2} - \frac{3a^2}{2} \right) & \text{for } r < a \\ -\frac{4\pi}{3} G \rho \frac{a^3}{r} & \text{for } r \geq a \end{cases} \quad (6.122)$$

where the integration constant was chosen so that U is continuous at $r = a$.

Further we assume that the background configuration we are perturbing around is *hydrostatic*, i.e. that the unperturbed stress is simply determined by the pressure p :

$$\tau_i^j = p \delta_i^j \quad (6.123)$$

Then, the equation of motion (6.44) for the background configuration reduces to

$$\partial_i p + \rho \partial_i U = 0 \quad (6.124)$$

This integrates to

$$p(r) = \frac{4\pi}{3} G \rho^2 (a^2 - r^2) / 2 \quad (6.125)$$

where we have chosen the integration constant so that $p(a) = 0$.

A constitutive relation that is compatible with this assumptions is

$$\epsilon(H^{AB}, f^A) = \frac{1}{2} p (H^{AB} - \delta^{AB}) \delta_{AB} + \frac{1}{2} \tilde{C}_{ABCD} (H^{AB} - \delta^{AB}) (H^{CD} - \delta^{CD}) \quad (6.126)$$

with

$$\tilde{C}_{ABCD} = \frac{1}{4} \left(\left(\kappa - \frac{2}{3} \mu - p \right) \delta_{AB} \delta_{CD} + (\mu - p) (\delta_{AC} \delta_{BD} + \delta_{AD} \delta_{BC}) \right) \quad (6.127)$$

where $p = p(\sqrt{f^A f^B \delta_{AB}})$ is given by relation (6.125) with $\sqrt{f^A f^B \delta_{AB}}$ inserted for r (i.e. (6.126) is not homogeneous), and κ and λ are constants. Note that here we have made use of the considerable freedom one has in choosing a constitutive relation. The first order term in H^{AB} was chosen to give the hydrostatic pressure (6.125), while the second order term was chosen in such a way that we get the particularly simple stress-strain relation (6.136) with a constant tensor C_{ijkl} (equation (6.130)). This choice was made in accordance with the literature (i.e. [5]), but cannot be fully justified from the theory. Also, we have neglected terms of higher orders than two.

The Ansatz (6.126) gives

$$\tau_{AB} = 2 \frac{\partial \epsilon}{\partial H^{AB}} \big|_{H^{AB}=\delta^{AB}} = p \delta_{AB} \quad (6.128)$$

which of course reproduces the hydrostatic form of the Cauchy stress tensor (6.123), and

$$4 \frac{\partial^2 \epsilon}{\partial H^{AB} \partial H^{CD}} \big|_{H^{AB}=\delta^{AB}} = \tilde{C}_{ABCD} \quad (6.129)$$

which leads to

$$C_{ijkl} = \left(\kappa - \frac{2}{3} \mu \right) \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (6.130)$$

according to equation (6.69). Again, we have identified the body manifold $\mathcal{B}$ with $f^{-1}(\mathcal{B})$, as described in section 6.3. In particular, with $f^A_i = \delta^A_i$, we also have $n = 1$, so the constant density ρ is simply given by the mass per particle m .

Now that we have set up the background solution, we can view perturbations around it. First we look at equation (6.45), the main equation of motion; since we are perturbing around a static solution, i.e. $v^i = 0$, it simplifies considerably:

$$\rho \frac{\partial \delta v_i}{\partial t} + \nabla_j \delta \tau_i^j + \delta \rho \partial_i U + \rho \partial_i \delta U = 0 \quad (6.131)$$

Since we are going to use spherical coordinates, we now use ∇ or semicolons to denote the affine connection of $\mathcal{S}$. Another simplification due to $v^i = 0$ is

$$\delta v^i = \Delta v^i = \dot{\xi}^i \quad (6.132)$$

For the stress tensor we use (6.66)

$$\delta \tau_i^j = -C_i^j{}_{kl} \xi^{k;l} - \tau_i^j{}_{;k} \xi^k \quad (6.133)$$

Because of the assumption that the background solution is hydrostatic, the second term in (6.133) reduces to

$$\tau_i^j{}_{;k} \xi^k = p_{,k} \xi^k \delta_i^j \quad (6.134)$$

To express $p_{,k}$, we use equation (6.124); inserting this in (6.131) yields

$$\rho \frac{\partial^2 \xi_i}{\partial t^2} - \nabla_j (C_i^j{}_{kl} \xi^{k;l}) + \partial_i (\rho \xi^j U_{,j}) + \delta \rho \partial_i U + \rho \partial_i \delta U = 0 \quad (6.135)$$

For $C_i^j{}_{kl} \xi^{k;l}$, we use the isotropic constitutive relation (6.130)

$$C_i^j{}_{kl} \xi^{k;l} = \left(\kappa - \frac{2}{3} \mu \right) \nabla_k \xi^k \delta_i^j + \mu (\nabla_i \xi^j + \delta_{il} \nabla_m \xi^l \delta^{mj}) \quad (6.136)$$

which gives

$$\begin{aligned} \nabla_j (C_i^j{}_{kl} \xi^{k;l}) &= \left(\kappa - \frac{2}{3} \mu \right) \xi^k{}_{;ki} + \mu (\xi^j{}_{;ij} + \delta_{il} \xi^l{}_{;mj} \delta^{mj}) = \\ &= \left(\kappa + \frac{1}{3} \mu \right) \xi^k{}_{;ki} + \mu \nabla^2 \xi_i \end{aligned} \quad (6.137)$$

where of course entered that Newtonian (=Euclidean) space is flat and derivatives therefore can be exchanged.

We apply a separational ansatz to (6.135)

$$\xi^i(x^k, t) = \xi^i(x^k) \exp i\omega t \quad (6.138)$$

This just results in the replacement of the first term in (6.135) by $-\omega^2 \rho \xi^i$

$$-\omega^2 \rho \xi^i - \nabla_j (C_i^j{}_{kl} \xi^{k;l}) + \partial_i (\rho \xi^j U_{,j}) + \delta \rho \partial_i U + \rho \partial_i \delta U = 0 \quad (6.139)$$

This way, we have turned the problem into an eigenvalue problem; we are searching for the *eigenfunctions* $\xi(x^k)$ to the corresponding *eigenfrequencies* ω .

In the perturbed Poisson equation (6.46), we can express $\delta \rho$ using (6.62) to get

$$\nabla_i \nabla^i \delta U = -4\pi G \nabla_i (\rho \xi^i) \quad (6.140)$$

The equation of motion (6.139) and the Poisson equation (6.140) still need to be supplemented by the boundary condition (6.48):

$$(\delta \tau_i^j f^A{}_j n_A + \tau_i^j{}_{,k} \xi^k f^A{}_j n_A + \tau_i^j \delta f^A{}_j n_A - \tau_i^j f^A{}_{jk} \phi^k{}_B \delta f^B n_A) |_{\partial \mathcal{B}} = 0 \quad (6.141)$$

Also this simplifys considerably because perturbing around a hydrostatic state. Firstly, the unperturbed hydrostatic stress tensor τ_i^j vanishes on $\partial \mathcal{B}$, so only the first two terms in (6.141) remain. Using (6.133) this becomes

$$C_i^j{}_{kl} \xi^{k;l} n_j |_{r=a} = 0 \quad (6.142)$$

with the normal vector

$$n_j dx^j = dr \quad (6.143)$$

Here again the identification of $\mathcal{B}$ with the portion of $\mathcal{S}$ occupied by the unperturbed configuration comes into play; $\partial \mathcal{B}$ is then obviously given by $r = a$.

Next, we apply the simplifying assumption that the perturbations should be spherical symmetric, i.e. the displacement vector and δU are given by

$$\begin{aligned} \xi^i(x^k) \partial_i &= \xi(r) \partial_r \\ \delta U &= \delta U(r) \end{aligned} \quad (6.144)$$

It is noteworthy that by the restriction to spherical symmetric perturbations, we eliminate the possibility of *trivial displacements* as discussed in section 6.3.1. This is because the only Killing vector fields on a spherical symmetric and bounded manifold $\mathcal{B}$ are the ones generating rotations (i.e. given by $\eta^A(X) = \epsilon^{ABC} n_B X_C$ for a fixed n_B determining the rotational axis), which are not spherically symmetric themselves, since the choice of an axis breaks the symmetry.

Using (6.144), $\xi^j U_{,j}$ simplifies to ξg and therefore we insert $\partial_i (\rho \xi^j U_{,j}) dx^i = -\delta \rho g dr + \rho \xi g_{,r} dr$ into (6.139), where we have used the expression (6.62) for $\delta \rho$. The first term in this expression just cancels with the fourth term in (6.139). For the second one we get $\rho \xi g_{,r} = \frac{4\pi}{3} G \rho^2 \xi$ after using 6.121.

Using the spherical symmetry assumption, the Poisson equation (6.140) becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \delta U_{,r}) = -4\pi G \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho \xi) \quad (6.145)$$

This can be readily integrated once, which gives

$$\delta U_{,r} dr = -4\pi G \rho \xi dr \quad (6.146)$$

Here, the integration constant was set to zero since both δU and ξ have to vanish at the origin because of the symmetry assumption. Combining all this into (6.139) yields

$$-\rho \omega^2 \xi dr - \nabla_j (C_i{}^j{}_{kl} \xi^{k;l}) dx^i - \frac{8\pi}{3} G \rho^2 \xi dr = 0 \quad (6.147)$$

The expression (6.137) for $\nabla_j (C_i{}^j{}_{kl} \xi^{k;l})$ can be evaluated in spherical coordinates by standard formulas of vector analysis (dashes are used to denote derivatives with respect to r):

$$\xi^i{}_{;j} \frac{\partial}{\partial x^i} \otimes dx^j = \xi' \frac{\partial}{\partial r} \otimes dr + \frac{\xi}{r} \left(\frac{\partial}{\partial \theta} \otimes d\theta + \frac{\partial}{\partial \phi} \otimes d\phi \right) \quad (6.148)$$

$$\xi^k{}_{;k} = \xi' + \frac{2}{r} \xi \quad (6.149)$$

$$\xi^k{}_{;ki} dx^i = \left(\xi'' + \frac{2}{r} \xi' - \frac{2}{r^2} \xi \right) dr = \nabla^2 \xi_i dx^i \quad (6.150)$$

so we get

$$\nabla_j (C_i{}^j{}_{kl} \xi^{k;l}) dx^i = \left(\kappa + \frac{4}{3} \mu \right) \left(\xi'' + \frac{2}{r} \xi' - \frac{2}{r^2} \xi \right) dr \quad (6.151)$$

Therefore, only the r component of our eigenvalue equation (6.147) is nontrivial and gives the wanted equation for ξ

$$-\rho \omega^2 \xi - \left(\kappa + \frac{4}{3} \mu \right) \left(\xi'' + \frac{2}{r} \xi' - \frac{2}{r^2} \xi \right) - \frac{8}{3} \pi G \rho^2 \xi = 0 \quad (6.152)$$

which can be rewritten as

$$\xi'' + \frac{2}{r} \xi' + \left(\gamma^2 - \frac{2}{r} \right) \xi = 0 \quad (6.153)$$

using the abbreviation

$$\gamma^2 = \rho \left(\omega^2 + \frac{8}{3} \pi G \rho \right) \left(\kappa + \frac{4}{3} \mu \right)^{-1} \quad (6.154)$$

Equation (6.153) is (up to a rescaling $x = \gamma r$) just the spherical Bessel equation of degree 1, so we get as solution the rescaled spherical Bessel function j_1 :

$$\xi(r) = j_1(\gamma r) = \frac{\sin(\gamma r)}{\gamma^2 r^2} - \frac{\cos(\gamma r)}{\gamma r} \quad (6.155)$$

The boundary condition (6.142) becomes

$$0 = \left(\kappa - \frac{2}{3} \mu \right) \left(\xi'(a) + \frac{2}{a} \xi(a) \right) + 2\mu \xi'(a) = \quad (6.156)$$

$$= \left(\kappa + \frac{4}{3} \mu \right) \left(\xi'(a) + \frac{2}{a} \xi(a) \right) - \frac{4\mu}{a} \xi(a) \quad (6.157)$$

after inserting equation (6.136) for $C_i^j{}_{kl} \xi^{k;l}$ and using the vector calculus relations (6.149) and (6.148). If we further insert the solution (6.155), we get:

$$\cot(\gamma a) = \frac{1}{\gamma a} - \frac{a}{4\mu} \left(\kappa + \frac{4}{3} \mu \right) \gamma \quad (6.158)$$

This equation is satisfied for certain discrete γ 's, which in turn gives the eigenfrequencies ω as they are inserted into equation (6.154).

Chapter 7

Appendix

7.1 Some mathematical formulae

7.1.1 Derivative of the determinant

For the determinant $\det(A)$ of a matrix A , the following formula holds:

$$\frac{\partial \det(A)}{\partial A_{ij}} = \det(A)(A^{-1})_{ji} \quad (7.1)$$

This can be seen by writing the determinant using the Laplace expansion formula:

$$\det(A) = \sum_i A_{ij} C_{ij} = \sum_j A_{ij} C_{ij} \quad (7.2)$$

Here the C_{ij} are the so-called cofactors given by $C_{ij} = (-1)^{i+j} M_{ij}$ (M_{ij} ; M_{ij} are the minors of A , i.e. the determinants of the submatrices one gets after removing the i th row and j th column from A). Differentiating (7.2) one gets:

$$\frac{\partial \det(A)}{\partial A_{ij}} = C_{ij} \quad (7.3)$$

Then, by expressing the C_{ij} from the inverse matrix formula

$$(A^{-1})_{ij} = \frac{1}{\det(A)} C_{ji} \quad (7.4)$$

one gets exactly (7.1).

For a matrix that depends on some variable x (for example a second-order tensor field), we get by the chain rule

$$\frac{\partial \det(A)}{\partial x} = \frac{\partial \det(A)}{\partial A_{ij}} \frac{\partial A_{ij}}{\partial x} = \det(A)(A^{-1})_{ji} \frac{\partial A_{ij}}{\partial x} \quad (7.5)$$

In the special case of the metric tensor this means:

$$\frac{\partial g}{\partial x^\lambda} = g g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \quad (7.6)$$

7.1.2 Divergence

The divergence of a vector field is given as:

$$\operatorname{div}(v^i) = \nabla_i v^i \equiv v^i{}_{;i} \quad (7.7)$$

Written out explicitly in components, this is $v^i{}_{;i} = v^i{}_{,i} + \Gamma^i{}_{ij} v^j$. The expression $\Gamma^i{}_{ij}$ can be simplified using (7.6):

$$\begin{aligned} \Gamma^i{}_{ij} &= \frac{g^{ik}}{2} (g_{ki,j} + g_{kj,i} - g_{ij,k}) = \frac{g^{ik}}{2} (g_{ki,j} + 2g_{j[k,i]}) = \\ &= \frac{g^{ik}}{2} \frac{\partial g_{ki}}{\partial x^j} = \frac{1}{2g} \frac{\partial g}{\partial x^j} = \frac{\partial \ln \sqrt{g}}{\partial x^j} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j} \end{aligned} \quad (7.8)$$

Now we insert this in (7.7) and get:

$$v^i{}_{;i} = v^i{}_{,i} + \Gamma^i{}_{ij} v^j = \frac{\partial v^i}{\partial x^i} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^i} v^i = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} v^i) \quad (7.9)$$

so we get as final result:

$$\operatorname{div}(v^i) = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} v^i) \quad (7.10)$$

Without proof we note that this is also the same as the codifferential of the one-form associated with v^i :

$$\operatorname{div}(v^i) = \delta(g_{ij} v^j) \quad (7.11)$$

7.1.3 Derivation of the inverse matrix

We consider a matrix h_{AB} with inverse h^{AB} , i.e.

$$h^{AE} h_{EF} = \delta_F^A \quad (7.12)$$

and want to calculate $\frac{\partial h^{AB}}{\partial h_{CD}}$. For that purpose we differentiate (7.12) with respect to h_{CD} and get:

$$\frac{\partial h^{AE}}{\partial h_{CD}} h_{EF} + h^{AE} \delta_E^{(C} \delta_F^{D)} = 0 \quad (7.13)$$

Multiplication with h^{FB} yields:

$$\frac{\partial h^{AB}}{\partial h_{CD}} = -h^{A(C} h^{D)B} \quad (7.14)$$

by exchanging the role of upper and lower indices this can be rewritten as:

$$\frac{\partial h_{AB}}{\partial h^{CD}} = -h_{A(C} h_{D)B} \quad (7.15)$$

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