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Abstract

The mathematical theory of wave propagation for various physical phenomena has produced a certain class of nonlinear evolution equations, which can be solved using the Inverse Scattering Transform [IST]. The IST is a very sophisticated mathematical theory which provides an analytical solution, namely the soliton solution, for the initial-value problem for the particular nonlinear wave equation. The most important ones are the Korteweg-de Vries [KdV] equation, the sine-Gordon equation and the nonlinear Schrödinger equation.

In this thesis an overview of the IST is presented by solving the initial-value problem for the KdV equation. For this purpose the mathematical discussion starts with the modified KdV [mKdV] equation, which was the decisive link in the historical development of the IST. Solutions of the mKdV equation are mapped into solutions of the KdV equation using the Miura transformation. This nonlinear transformation reveals an ingenious connection between the soliton solutions of the KdV equation and the classical scattering problem of quantum mechanics.

The next step contains the introduction of the Gelfand-Levitan-Marchenko [GLM] theory, which constitutes the theoretical backbone of the IST. Using the famous GLM integral equation one can solve the initial-value problem for reflectionless potentials and generate the N-soliton solution.

Zusammenfassung

Partielle Differentialgleichungen spielen sowohl in der mathematischen als auch in der theoretischen Physik bei der Modellierung bestimmter Phänomene eine zentrale Rolle. Anders als im linearen Fall, der sich durch die Fourier-Transformation lösen lässt, existiert für den nichtlinearen Fall keine allgemeine Lösungsmethode.

Die inverse Streutheorie [Inverse Scattering Transform] ermöglicht es, bestimmte nichtlineare partielle Differentialgleichungen analytisch zu lösen. Die Anwendung dieser Methode auf eine bestimmte nichtlineare Evolutionsgleichung setzt die Existenz eines linearen Spektralproblems voraus, das sich als nichtlineare Transformation für die entsprechende nichtlineare Gleichung eignet.

1967 konnten Greene, Gardner, Kruskal und Miura [27] zeigen, dass im Falle der Korteweg-de Vries-Gleichung das lineare Spektralproblem aus dem eindimensionalen Streuproblem für den Sturm-Liouville-Operator besteht. Die funktionalanalytischen Arbeiten der berühmten Charkower Schule um Gelfand, Levitan und Marchenko bilden den mathematischen Kern der inversen Streutheorie.

List of Acronyms

AKNS	Ablowitz-Kaupp-Newell-Segur
FPU	Fermi-Pasta-Ulam
GGKM	Greene-Gardner-Kruskal-Miura
GLM	Gelfand-Levitan-Marchenko
IST	Inverse Scattering Transform
KdV	Korteweg-de Vries
mKdV	modified Korteweg-de Vries
NLS	Nonlinear Schrödinger

CONTENTS

It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of great beauty and power. As time goes on, it becomes increasingly evident that the rules that the mathematician finds interesting are the same as those that nature has chosen.

Paul Adrien Maurice Dirac

1 A Historical Survey

> To understand a science it is necessary to know its history.
>
> Auguste Comte

1.1 The Discovery of Solitary Waves

The first historically significant observation of a so-called solitary wave was made in the first half of the nineteenth century [1834] by the Scottish engineer John Scott Russell[1]. Russell carried out a project on behalf of the Union Canal Society of Edinburgh to determine the most suitable design parameters for canal barges. One day while experimenting on the different types of boats, Russell made one of the most fundamental discoveries in fluid dynamics. It took another decade of thorough research before he gave a report on his work to the British Association for the Advancement of Science [1]. The following quotation is taken from his report [1]:

> *I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary wave elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed on the horseback, and overtook it still rolling on at a rate of some eight to nine miles an hour, preserving its*

[1] John Scott Russell [1808–1882]. Scottish engineer; Fellow of the Royal Society of Edinburgh.

original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834 was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation…

Using a self-designed and self-constructed water tank for his experiments[2] Russell made further experimental and theoretical investigations of his important discovery and obtained the following properties [1]:

1. The experimental proof of the existence of a solitary wave.

2. Given a certain initial mass of water one can generate different types of solitary waves.

3. The solitary wave interaction suggests a special type of nonlinearity[3].

4. The velocity v of the solitary wave is empirically given by

$$v=\sqrt{g(h+\xi)} \tag{1.1}$$

   where g is the acceleration of gravity, h depends on the geometry of the water tank [canal][4] and ξ is the amplitude of the wave.

5. From (1.1) one concludes that a large-amplitude solitary wave propagates faster than a low-amplitude solitary wave.

Despite of his experimental research, Russell was aware of the lack of a consistent mathematical theory describing his discovery. Although he was convinced that his work was of fundamental importance for the theory of fluid dynamics[5], the contemporary [British] scientific community thought otherwise, because the phenomenon could not be explained within the theoretical framework of hydrodynamics[6].

[2] Russell dropped different weights at one end of his water tank; see Figure 1.
[3] We refer to the Zabusky-Kruskal experiment in section 1.2.
[4] The term geometry is synonymous for the depth of the water tank [canal].
[5] Despite Russell's effort and obsession his observation remained unexplained during his lifetime.
[6] The famous British physicist and royal astronomer Sir George Airy expressed the opinion that solitary waves could not exist.

Apart from Russell several leading British mathematicians and physicists including Lord Rayleigh [2] and Stokes [3] failed to solve the “solitary puzzle”.

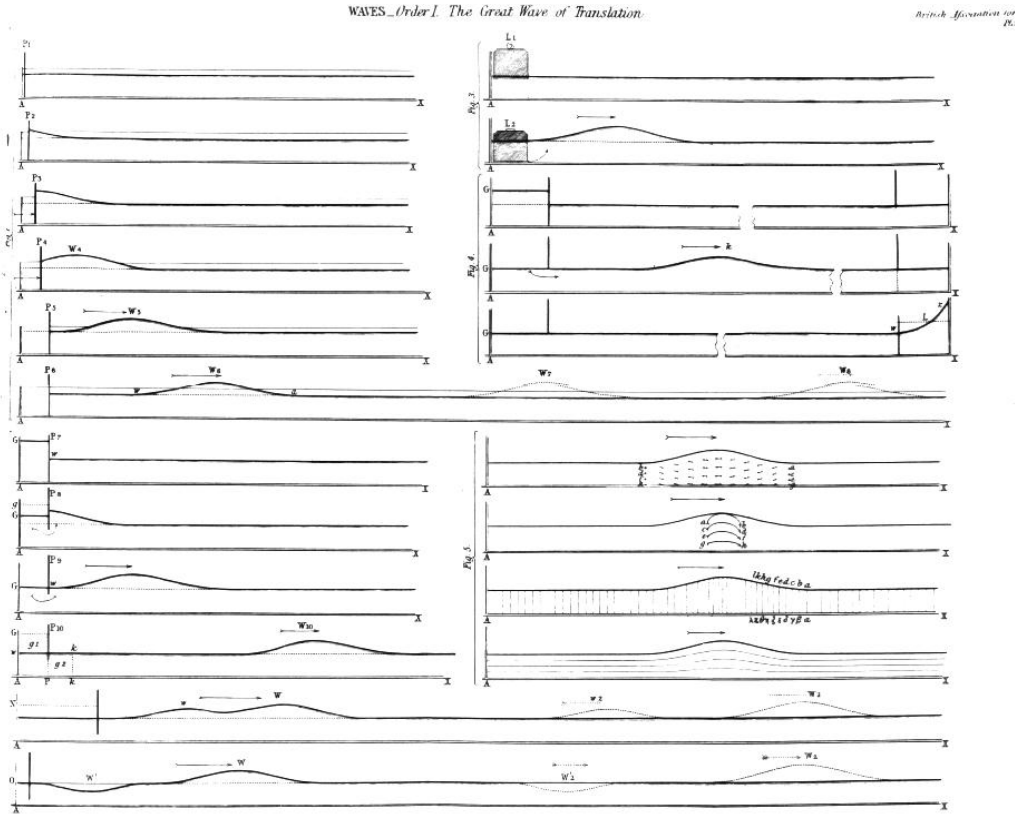

Figure 1 – Russell’s sketch from his experimental studies on solitary waves [1].

It took more than six decades between Russell’s discovery and a further development of mathematical fluid dynamics to obtain an appropriate explanation of the phenomenon. The decisive breakthrough was achieved by the two Dutch mathematicians Diederik Johannes Korteweg and Gustav de Vries[7] in 1895 [4], although the French mathematician Joseph Boussinesq [5-7] obtained a similar equation[8].

Remark. For a detailed historical discussion on the Korteweg-de Vries equation we refer to [8-13].

[7] D. J. Korteweg was a well-known Dutch mathematician at that time, and G. de Vries wrote his doctoral thesis under his supervision.

[8] In contrast to Boussinesq, Korteweg and de Vries were able to find the solitary wave solutions.

Korteweg and de Vries proposed a mathematical theory which explained the solitary wave observed by Russell. Their main result was the nonlinear partial differential equation [14]

$$\frac{\partial\eta}{\partial t}=\frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial\zeta}\left(\frac{2}{3}\alpha\eta+\frac{1}{2}\eta^2+\frac{1}{3}\sigma\frac{\partial^2\eta}{\partial\zeta^2}\right) \quad , \tag{1.2}$$

where

- l depends on the geometry of the canal [water tank][9],
- η is the elevation of the water surface,
- g is the acceleration of gravity,
- α is a constant related to the [uniform] motion of the fluid,
- $\sigma=\frac{l^3}{3}-\frac{Tl}{\rho g}$ is a constant,
- T is the surface tension of the fluid,
- ρ is the density of the fluid,
- ζ is a coordinate chosen to be moving with the wave along the canal [water tank].

The [historical] Korteweg-de Vries equation (1.2), hereafter abbreviated the KdV equation, describes long shallow water waves and verifies mathematically the existence of solitary waves in accordance with Russell's empirical results [14, 15]. Appropriate scaling transforms (1.2) into the canonical form[10, 11]

$$u_t \pm 6uu_x + u_{xxx} = 0 \tag{1.3}$$

used in the modern literature. Both the experimental fact of the existence of a solitary wave, and the theoretical fact of a consistent mathematical description were later recognized as extremely important. Nevertheless, they disappeared from the scientific scene for another six decades until they were rediscovered.

[9] We refer to Equation (1.1).
[10] We use the standard notation for partial derivatives.
[11] The factor ± 6 is a standard choice, but it can always be rescaled.

1.2 The Fermi-Pasta-Ulam Problem

It was not until the middle of the 20th century and the emergence of the first computers that the scientific community started to understand and appreciate the significance of Russell's fundamental work, leading to the modern theory of solitons.

In 1954, Enrico Fermi, John Pasta and Stanisław Ulam [FPU] carried out one of the first and most fundamental numerical computations in the history of physics. Their pioneering work contributed to the birth of computational and nonlinear physics [16]. They used the Los Alamos MANIAC computer[12] to investigate a problem from statistical mechanics – namely the so-called thermalization process of a one-dimensional solid. Fermi, Pasta and Ulam decided to study the dynamics of energy equipartition in a [slightly] nonlinear crystal lattice. According to a hypothesis by the Dutch physicist Debye [1914], it was expected that the nonlinear interactions among the normal modes of the linear system would lead to an equal distribution of the energy among all modes, with in agreement to Boltzmann´s ergodic hypothesis. To their surprise, the numerical computations were inconsistent with the theoretical prediction. The following quotation is taken from Ulam´s autobiography [17]:

> *Computers were brand new; in fact the Los Alamos Maniac was barely finished…. As soon as the machines were finished, Fermi, with his great common sense and intuition, recognized immediately their importance for the study of problems in theoretical physics, astrophysics, and classical physics. We discussed this at length and decided to formulate a problem simple to state, but such that a solution would require a lengthy computation which could not be done with pencil and paper or with existing mechanical computers…. We found a typical one…the consideration of an elastic string with two fixed ends, subject not only to the usual elastic force of stress proportional to strain, but having, in addition, a physically correct nonlinear term…. The question was to find out how…the entire motion would eventually thermalize….John Pasta, a recently arrived physicist, assisted us in the task of flow diagramming, programming, and running the problem on the Maniac….*

[12] The MANIAC [Mathematical Analyzer, Numerical Integrator And Computer] computer was originally built for the Manhattan project.

The problem turned out to be felicitously chosen. The results were entirely different qualitatively from what even Fermi, with his great knowledge of wave motion, had expected.

We briefly want to sketch the FPU problem following the presentation given in [18, 19]; for a detailed discussion we refer to [14, 20].
FPU used a linear chain consisting of 64 identical particles, i.e. oscillators, of unit mass on the line with fixed end points and nearest-neighbour interactions. The FPU chain is given by the FPU Hamiltonian [18]

$$H=\sum_{i=0}^{N}\frac{1}{2}p_i^2+\sum_{i=0}^{N}\frac{1}{2}\left(u_{i+1}-u_i\right)^2+\frac{\alpha}{3}\sum_{i=0}^{N}\left(u_{i+1}-u_i\right)^3 \quad , \tag{1.4}$$

where u_i is the displacement of particle i with respect to its equilibrium position, and p_i is its momentum. Without any problems we can set the stiffness of the FPU chain and the FPU lattice constant to one.
The FPU parameter $\alpha = 1$ is a measure for the nonlinear contribution to the FPU interaction potential, and the two ends of the FPU chain are chosen to be fixed, $u_0 = u_{N+1} = 0$.
Using the standard Fourier approach, the FPU Hamiltonian (1.4) can be rewritten as [18]

$$H=\frac{1}{2}\sum_{k=1}^{N}\left(\dot{A}_k^2+\omega_k^2A_k^2\right)+\frac{\alpha}{3}\sum_{k,l,m=1}^{N}c_{klm}A_kA_lA_m\omega_k\omega_l\omega_m \quad , \tag{1.5}$$

where

$$A_k=\sqrt{\frac{2}{N+1}}\sum_{i=1}^{N}u_i\sin\left(\frac{ik\pi}{N+1}\right) \tag{1.6}$$

are the normal modes of the displacements with the frequencies

$$\omega_k^2=4\sin^2\left(\frac{k\pi}{2(N+1)}\right) . \tag{1.7}$$

FPU expected that, because of the last term in the FPU Hamiltonian (1.5) which emerges from the nonlinearity in the FPU interaction potential, the energy fed into the ground mode $[k=1]$ should gradually distribute among the other higher modes, until the steady state, i.e. equipartition of energy, is reached.

At the beginning of the simulation the FPU system seemed to be ergodic in accordance with the theory of statistical mechanics, because it converged to a steady state. But one day, by accident, the program execution was continued after the system had reached the steady state. To their surprise the authors noticed that after a while the system departed from "equilibrium". They discovered that the system has an almost periodic behaviour in time, which means that the initial state is almost perfectly recovered after a certain recurrence period[13]. This phenomenon is known as FPU recurrence.

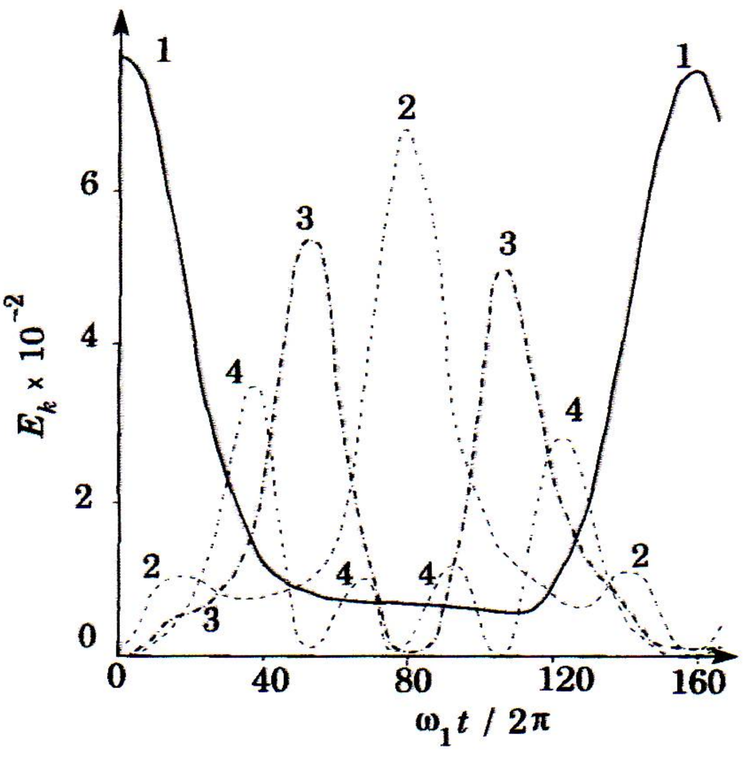

Figure 2 – The FPU recurrence: time evolution of the energy [kinetic and potential] of the lowest modes [16, 18, 19].

Here is a quotation from the original FPU report [16]:

> *Let us say here that the results of our computations show features which were, from the beginning, surprising to us. Instead of a gradual, continuous flow of energy from the first mode to the higher modes, all of the problems showed an entirely different behaviour.*

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[13] After 157 periods of the ground mode approximately 97% of the energy was back to the initial state [16, 18, 19]; see Figure 2.

> *Starting in one problem with a quadratic force and a pure sine wave as the initial position of the string, we did indeed observe initially a gradual increase of energy in the higher modes as predicted. Mode 2 starts increasing first, followed by mode 3, and so on. Later on, however, this gradual sharing of energy among the successive modes ceases. Instead, it is one or the other mode that predominates. For example, mode 2 decides, as it were, to increase rather rapidly at the cost of the others. At one time it has more energy than all the others put together! Then mode 3 undertakes this role. It is only the first few modes which exchange energy among themselves, and they do this in a rather regular fashion. Finally, at a later time, mode 1 comes back within one percent of its initial value, so that the system seems to be almost periodic.*

Ulam described the situation as followed [17]:

> *I know that Fermi considered this to be, as he said, "a minor discovery." And when he was invited a year later to give the Gibbs lectures (a great honorary event at the annual American mathematical society meeting), he intended to talk about it. He became ill before the meeting, and his lecture never took place….*
> *The results were truly amazing. There were many attempts to find the reasons for this periodic and regular behaviour, which was the starting point of what is now a large literature on nonlinear vibrations. Martin Kruskal, a physicist in Princeton, and Norman Zabusky, a mathematician at Bell Labs, wrote papers about it.*

The solution of the FPU problem - The Zabusky-Kruskal Experiment

The attempt to find a precise physical and mathematical explanation for the FPU recurrence led to an intensive research on lattice equations [18]

$$\ddot{u}_i = \left(u_{i+1} + u_{i-1} - 2u_i\right) + \alpha\left[\left(u_{i+1} - u_i\right)^2 - \left(u_i - u_{i-1}\right)^2\right] \qquad (1.8)$$

derived from the FPU Hamiltonian (1.4).

Remark. Fermi, Pasta and Ulam considered two different cases referred to as the α- and β- models, which arise from cubic and quadratic terms in the interaction potential [14].

Two different scientific schools of thought were involved in the solution of this problem. On the one hand, applied mathematicians and mathematical physicists used the famous theorem by Kolmogorov, Arnold and Moser [21-23][14]. Despite an enormous effort, this path led only to a qualitative explanation of the FPU recurrence. Furthermore a result obtained by the two Soviet mathematical physicists Izrailev and Chirikov [24] showed that for strong-enough perturbation, the FPU recurrence is destroyed and a fast convergence to thermal equilibrium takes place.

The other school of thought used a method called reductive perturbation theory[15]. This approach led to the numerical solution of the FPU recurrence by the two American scientists Zabusky and Kruskal [25]. Their fundamental contribution was to recognize the deep relation between non-destructive soliton interactions[16] and the FPU recurrence. The behaviour of the KdV equation showed the exchange of energy among the few modes observed in the FPU experiment. The initial condition generates a series of solitons which preserve their shapes and differing velocities and, from time to time, they come back to the initial state [14].

Remark. For a detailed physical and mathematical discussion using the reductive perturbation theory we refer to [14, 15, 19, 20, 26].

1.3 The Golden Age of Soliton Theory

The remarkable numerical and theoretical results obtained by Zabusky and Kruskal [1965] were followed by the ingenious and pioneering work of Gardner, Greene, Kruskal and Miura [1967]. They proposed a nonlinear version of the linear Fourier transform method for solving the initial-value problem for the KdV equation (1.3) and published their result, known as

[14] The KAM theorem states that most orbits of a [slightly] perturbed integrable Hamiltonian system remain quasi-periodic [18].

[15] Under certain physical and mathematical approximations the KdV equation (1.3) can be derived from the FPU lattice equation (1.8).

[16] This special nonlinear interaction of the solitary wave solutions of the KdV equation led Zabusky and Kruskal to introduce the term soliton, because of their particle-like behaviour [14].

the Inverse Scattering Transform [IST], in a very brief and concise Physical Review Letter [27][17]. Shortly after that [1972], by introducing an extended mathematical version of the inverse scattering transform, the Soviet mathematicians Zakharov and Shabat [28][18] were able to find a solution for the initial-value problem for the [cubic] Nonlinear Schrödinger equation [14]

$$i\,u_t + u_{xx} + 2|u|^2\,u = 0 \quad . \tag{1.9}$$

In the following years they further developed and generalized their results and established a method called the Zakharov-Shabat Inverse Method, which they also applied to a variety of other nonlinear evolution equations [29-31]. At about the same time [1973][19], an American group of applied mathematicians proposed a similar method for solving the initial-value problem for the [geometrical] sine-Gordon equation [14]

$$u_{xt} = \sin u \quad . \tag{1.10}$$

Their ingenious and pioneering work was published in a series of papers by Ablowitz, Kaup, Newell and Segur [32, 33] and is nowadays known as the AKNS Method[20, 21].

In 1968 the Hungarian mathematician Peter Lax introduced an ingenious method for solving nonlinear evolution equations with linear operators so that the eigenvalues of the linear operator are constants of the motion for the nonlinear evolution equation [34].

We conclude our historical tour d´horizon with the discussion of the contribution to the theory of solitons by the so-called Japanese school. In the mid-1960´s the Japanese theoretical physicist Toda studied a class of certain potentials for the FPU Hamiltonian (1.4) such that the regarding FPU lattice equation has soliton solutions [35-37]. Another very important contribution to the theory of solitons, namely the algebraic soliton theory, was developed by the Japanese mathematician Hirota [38-43].

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[17] The so-called GGKM paper is one of the most fundamental papers in soliton theory.

[18] Zakharov and Shabat were members of the prestigious Kharkov Mathematics Institute.

[19] In 1972 Newell and his colleagues organized the first conference on solitons. Because of the cold war Zakharov and Shabat were not allowed to leave the Soviet Union. So they sent their results by post [14, 15].

[20] In 1974 Ablowitz, Kaup, Newell and Segur provided a generalisation of their work [14].

[21] Both, the Zakharov-Shabat and the AKNS Method, are intensively used to study the physical and mathematical properties of a variety of nonlinear evolution equations.

Since the mid-1970´s, the soliton theory has become an established part in several branches of applied mathematics and theoretical physics. Current soliton research is focused on the further mathematical and theoretical development and the application to a variety of problems from different research areas [19, 44].

2 Integrable Nonlinear Evolution Equations

True laws of nature cannot be linear.

Albert Einstein

2.1 Preliminaries

During the second half of the 20th century it has become an established fund of knowledge in theoretical physics that qualitatively new physical phenomena, which cannot be derived by using standard perturbation techniques, can emerge from the concept of nonlinearity. One of the most fundamental results in the field of classical mathematical physics[22, 23] is the discovery that a certain class of nonlinear evolution equations, mainly in $(1+1)$ dimensions[24], are completely integrable. They are characterized by the following remarkable properties [45]:

1. They have particle-like solutions.

2. These solutions interact with each other in a non-destructive way[25].

3. The corresponding initial-value problem can be solved analytically using the inverse scattering transform[26].

4. If the system of equations is formulated as a Lagrangian field theory, it has infinitely many conservation laws.

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[22] In the Classical Theory of Hamiltonian Systems.

[23] In this connection we should mention the fundamental contribution made by the Soviet school of mathematical physics [Arnold, Bogolyubov, Faddeev, Kolmogorov, Sinai].

[24] One variable represents a spatial dimension whereas the other variable is associated with time.

[25] We refer to the Kruskal-Zabusky experiment in section 1.2.

[26] We refer to Chapter 3.

The best-known examples are

1. the Korteweg-de Vries equation,
2. the sine-Gordon equation,
3. the [cubic] Nonlinear Schrödinger equation,

which are of fundamental importance for many physical applications.

Remark. For a discussion of various applications we refer to [19, 44-46]. The basic principles of wave theory may be found in [47, 48].

We now turn our attention to a rather encyclopaedic overview[27] on the elementary physical and mathematical properties of the three previously mentioned nonlinear evolution equations.

2.2 The Korteweg-de Vries Equation

The KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \quad , \tag{2.1}$$

1. is a nonlinear dispersive partial differential equation, which is used as a mathematical model to describe such physical phenomena as the propagation of waves in a weakly nonlinear and dispersive medium, anharmonic nonlinear lattices and hydromagnetic waves in a cold plasma [19, 45-47].

2. A soliton solution requires the interplay between the nonlinear term uu_x and the dispersive term u_{xxx}[28].

3. Conservation laws are one of the most fundamental concepts of physics expressing the conservation of physical quantities. During the intensive period of attempting to solve the KdV equation

[27] The presentation reflects the author's opinion.

[28] The nonlinear term tends to localise the wave, whereas the dispersive term spreads it out. For the emergence of soliton solutions this equilibrium has to be stable [14, 15, 19].

exactly, it was discovered that there exists an infinite sequence of nontrivial conservation laws, which we briefly want to discuss by directly following the presentation given in [15]:

Definition 2.1. [15] *A conservation law associated with a differential equation in (1+1) dimensions is given by the expression*

$$\Theta_t + \Xi_x = 0 \quad , \tag{2.2}$$

where Θ and Ξ are functions of u, x, t and derivatives of u. Θ is called the conserved density, and Ξ is called the flux.

If Θ and Ξ_x are integrable on $(-\infty, \infty)$, such that

$$\Xi \;\to\; \text{constant} \qquad \text{for } \; |x| \to \infty \quad , \tag{2.3}$$

then we can integrate equation (2.2) to obtain

$$\Pi = \int_{-\infty}^{\infty} \Theta \, dx = \text{constant} \qquad \Rightarrow \qquad \Pi_t = 0 \;\; . \tag{2.4}$$

Π is called a constant of the motion.

We examine the first three conservation laws for the KdV equation which correspond to the conservation of mass, momentum and energy, respectively.

Conservation of mass

The first conservation law can be directly interferred from the KdV equation (2.1)

$$u_t - \left(3u^2 + u_{xx}\right)_x = 0 \tag{2.5}$$

with

$$\Theta = u \qquad \text{and} \qquad \Xi = u_{xx} - 3u^2 \;\; . \tag{2.6}$$

Thus, if Θ and Ξ_x are integrable on $(-\infty,\infty)$ and u satisfies the KdV equation, then

$$\Pi = \int_{-\infty}^{\infty} u\,dx = \text{constant} \quad . \tag{2.7}$$

Conservation of momentum

The second conservation law is obtained if we multiply the KdV equation by u,

$$\left(\frac{1}{2}u^2\right)_t + \left(uu_{xx} - \frac{1}{2}u_x^2 - 2u^3\right)_x = 0 \tag{2.8}$$

and, therefore,

$$\Pi = \int_{-\infty}^{\infty} u^2\,dx = \text{constant} \quad . \tag{2.9}$$

Conservation of energy

For the third conservation law we make the following ansatz

$$3u^2\left(u_t - 6uu_x + u_{xxx}\right) + u_x\left(u_t - 6uu_x + u_{xxx}\right)_x = 0 \quad , \tag{2.10}$$

which can be rewritten as

$$\left(u^3 + \frac{1}{2}u_x^2\right) + \left(-\frac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 + u_xu_{xxx} - \frac{1}{2}u_{xx}^2\right) = 0 \quad . \tag{2.11}$$

Hence

$$\Pi = \int_{-\infty}^{\infty}\left(u^3 + \frac{1}{2}u_x^2\right)dx = \text{constant} \quad . \tag{2.12}$$

(2.7), (2.9) and (2.12) are just the first three conservation laws of an infinite set, where each successive conservation law contains an increasing power of u. For a futher discussion we refer to [15].

4. The KdV equation (2.1) has different classes of solutions. We briefly want to examine the most important ones.

2.2.1 Permanent Profile Solutions[29]

The following mathematical derivation is taken from [19]:

We make the standard ansatz

$$u(x,t) = f(x-ct) \equiv \phi(\xi) \quad . \tag{2.13}$$

Substitution of (2.13) into the KdV equation (2.1) leads to the ordinary differential equation

$$-c\frac{d\phi}{d\xi} + 6\phi\frac{d\phi}{d\xi} + \frac{d^3\phi}{d\xi^3} = 0 \quad . \tag{2.14}$$

Integration can be done directly since (2.14) can be written as

$$\frac{d}{d\xi}\left(-c\phi + 3\phi^2 + \frac{d^2\phi}{d\xi^2}\right) = 0 \quad , \tag{2.15}$$

which implies

$$-c\phi + 3\phi^2 + \frac{d^2\phi}{d\xi^2} = C_1 \quad , \tag{2.16}$$

where C_1 is an integration constant.

Multiplying (2.16) by $\frac{d\phi}{d\xi}$ and integrating again with respect to ξ, we obtain

$$-\frac{c}{2}\phi^2 + \phi^3 + \frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 = C_1\phi + C_2 \quad , \tag{2.17}$$

where C_2 is a second integration constant.

[29] These special solutions are also known as travelling wave solutions [14, 15, 19, 47].

The KdV soliton solution is a spatially localized solution, which requires the following boundary conditions: $\phi \to 0,\ \frac{d\phi}{d\xi} \to 0,\ \frac{d^2\phi}{d\xi^2} \to 0,$ for $|\xi| \to \infty$; this implies that $C_1 = C_2 \equiv 0$ and, therefore, Equation (2.17) yields

$$\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 = c\phi^2 - 2\,\phi^3 \quad . \tag{2.18}$$

The resulting equation (2.18) is a Ricatti equation which is solved by using standard techniques from the theory of ordinary differential equations [49]

$$\phi(\xi) = \frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}\xi\right) \quad . \tag{2.19}$$

Using (2.13) we finally get

$$u(x,t) = f\,(x-ct) = \frac{c}{2}\ \mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct-\delta)\right) \quad , \tag{2.20}$$

where δ is an arbitrary constant.

We can conclude two important consequences from the solution (2.20):

1. The solitary wave solution exists only for $c > 0$ [30].

2. The propagation speed c is strictly proportional to the wave amplitude[31].

Remark. More general solutions can be found if the integration constants C_1 and C_2 do not have to vanish. These solutions are represented by Jacobian elliptic functions and are known as cnoidal waves. For a detailed discussion we refer to [15].

2.2.2 Multisoliton Solutions

For the detailed discussion we refer the reader to Chapter 3 section 3.7.

[30] Any soliton solution (2.20) propagates to the right.
[31] We refer to Equation (1.1).

2.3 The sine-Gordon Equation

The following section is taken from [14]:

The [physical] sine-Gordon equation

$$\phi_{xx} - \phi_{tt} = \mu^2 \phi \quad , \tag{2.21}$$

can be derived as the Euler-Lagrange equation emerging from a nonlinear extension of the Klein-Gordon Lagrangian density [50]

$$L_{KG}(\phi) = \frac{1}{2}\left(\phi_x^2 - \phi_t^2\right) + \frac{1}{2}\mu^2 (1 - \cos\phi) \quad . \tag{2.22}$$

The sine-Gordon equation plays an important role in many branches of theoretical physics. It provides one of the simplest models of unified field theory, it can be found in solid state physics, and, furthermore it is used in theoretical biology to investigate the DNA dynamics.
In soliton theory the following form of Equation (2.21) is used[32]

$$\phi_{xx} - \phi_{tt} = \sin\phi \quad . \tag{2.23}$$

The [geometrical] sine-Gordon equation (1.10) has an even older history than the KdV equation. The Swedish mathematician Bäcklund [1875] introduced it while considering a problem in differential geometry [51].
In the 1940´s, the Soviet physicists Frenkel and Kontorova [52] derived the equation (2.23) by studying a problem from crystal dynamics [19].

Soliton Solutions

The sine-Gordon equation (2.21) has the following 1-soliton solution [14]

$$\phi(x,t) = 4 \arctan \exp\left[\gamma(x - vt) + \delta\right] \quad , \tag{2.24}$$

where

[32] The [geometrical] sine-Gordon equation (1.10) can be derived from (2.21) using light cone coordinates instead of space-time coordinates [14].

$$\gamma^2 = \frac{1}{1-v^2} \quad , \tag{2.25}$$

and δ is an arbitrary constant.
The 1-soliton solution referring to the positive root for γ is called a kink whereas the 1-soliton solution referring to the negative root for γ is called an antikink. Since the solution approaches 0 as $x \to -\infty$ and 2π as $x \to +\infty$ it describes a twist in the field ϕ.

2.4 The Nonlinear Schrödinger Equation[33, 34]

The [cubic] Nonlinear Schrödinger Equation[35] [NLS Equation] [14],

$$i\psi_t + \psi_{xx} + \kappa|\psi|^2 \psi = 0 \quad , \tag{2.26}$$

is a nonlinear version of the Schrödinger equation in two dimensions for the complex field ψ. It contains a balance between linear dispersion, which tends to spread out the wave packet, and the effect of the cubic nonlinearity, which emerges from the self interaction of the wave with itself. The NLS equation is used in fields such as hydrodynamics, nonlinear optics, nonlinear acoustics, plasma waves and biomolecular dynamics.

Soliton Solution

The following mathematical derivation is taken from [53]:

We consider the case for $\kappa = 1$; a permanent profile solution is assumed to have the form

$$\psi(x,t) = \phi(x-ct)\exp\left[i\theta(x-ct) + i\sigma t\right] \equiv \phi(\xi)\exp\left[i\theta\xi + i\sigma t\right] \tag{2.27}$$

[33] This section is for the sake of completeness and concludes our encyclopaedic overview.
[34] Once again we refer to [14] for our following discussion.
[35] There is a class of nonlinear Schrödinger equations used in the literature.

where $\phi(x-ct)\equiv\phi(\xi)$ and $\theta(x-ct)\equiv\theta(\xi)$ are real-valued functions, c is the propagation speed and σ is some number.
Substituting (2.27) into (2.26) and differentiating with respect to ξ yields

$$\begin{aligned}\psi_t &= -c\frac{d\phi}{d\xi}\exp[i\theta+i\sigma t]+\left(-ic\frac{d\theta}{d\xi}+i\sigma\right)\phi\exp[i\theta+i\sigma t]\\ \psi_{xx} &= \frac{d^2\phi}{d\xi^2}\exp[i\theta+i\sigma t]+2i\frac{d\theta}{d\xi}\frac{d\phi}{d\xi}\exp[i\theta+i\sigma t]\\ &\quad +i\frac{d^2\theta}{d\xi^2}\phi\exp[i\theta+i\sigma t]-\left(\frac{d\theta}{d\xi}\right)^2\phi\exp[i\theta+i\sigma t]\ .\end{aligned} \tag{2.28}$$

Applying (2.28) to (2.26) and calculating the real and imaginary parts, we obtain

$$c\frac{d\theta}{d\xi}\phi+\frac{d^2\phi}{d\xi^2}-\left(\frac{d\theta}{d\xi}\right)^2\phi+\phi^3+\sigma\phi=0 \qquad \text{real part} \tag{2.29}$$

$$-c\frac{d\phi}{d\xi}+2\frac{d\theta}{d\xi}\frac{d\phi}{d\xi}+\frac{d^2\theta}{d\xi^2}\phi=0 \qquad \text{imaginary part}\ . \tag{2.30}$$

From Equation (2.30) we derive

$$\frac{d\theta}{d\xi}=\frac{1}{2}\left(c+\frac{\alpha}{\phi^2}\right)\quad , \tag{2.31}$$

where α is an arbitrary constant. Using (2.31), we can show that (2.29) satisfies

$$\left(\frac{d\Phi}{d\xi}\right)^2=-2\left(\Phi^3-2\left(\sigma-\frac{c^2}{4}\right)\Phi^2+\beta\Phi+\frac{\alpha^2}{2}\right)\quad , \tag{2.32}$$

where we have defined $\Phi\equiv\phi^2$, and β is another arbitrary constant.
In analogy to the KdV equation we determine the 1-soliton solution by taking $\alpha=\beta\equiv 0$ and obtain

$$\psi(x,t)=\zeta\,\mathrm{sech}\left(\frac{\zeta(x-ct)}{\sqrt{2}}\right)\exp\left(\frac{ic(x-ct)}{2}+i\sigma t\right) \tag{2.33}$$

for all $\zeta^2=2\left(\sigma-\frac{c^2}{4}\right)>0$.

3 The Inverse Scattering Transform for the KdV Equation

Physical Laws should have mathematical beauty.

Paul Adrien Maurice Dirac

3.1 The Miura Transformation and the modified KdV Equation

The main topic of our work is the solution of the initial-value problem for the KdV equation [54]

$$\begin{aligned} &u_t - 6uu_x + u_{xxx} = 0 \ , \quad x \in (-\infty, \infty) \ , \quad t > 0 \\ &u(x,0) = u_0(x) \ . \end{aligned} \tag{3.1}$$

We will see that the solution reveals an ingenious connection with the classical scattering problem of quantum mechanics. This connection is the main idea of the Inverse Scattering Transform. The decisive inspiration for the IST came from a remarkable result due to Miura [55].

Theorem 3.1. [54] *Let $v(x,t)$ be a solution of the modified KdV equation,*

$$v_t - 6v^2 v_x + v_{xxx} = 0 \quad , \tag{3.2}$$

then the function $u(x,t)$ defined by the Miura transformation

$$u = v^2 + v_x \tag{3.3}$$

satisfies the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \ .$$

Proof. [15] Direct substitution of (3.3) into the KdV equation gives

$$2vv_t + v_{xt} - 6\left(v^2 + v_x\right)\left(2vv_x + v_{xx}\right) + 6v_x v_{xx} + 2vv_{xxx} + v_{xxxx} = 0 \quad . \tag{3.4}$$

Rewriting the derivates, we find

$$\left(2v + \frac{\partial}{\partial x}\right)\left(v_t - 6v^2 v_x + v_{xxx}\right) = 0 \quad . \tag{3.5}$$

Therefore, if v is a solution of the modified KdV equation (3.2) then the Miura transformation defines a solution for the KdV equation. ■

The Miura transformation is a Riccati equation for v, which can be solved by using the Cole-Hopf transformation [15]

$$v = \frac{\psi_x}{\psi} \tag{3.6}$$

for some differentiable function $\psi(x;t) \neq 0$ [36]. Thus we get for (3.3)

$$\psi_{xx} - u\psi = 0 \quad . \tag{3.7}$$

Using

Theorem 3.2. [54] *The KdV equation is Galilean invariant; the transformation*

$$u(x,t) \to \lambda + u(x + 6\lambda t, t) \quad , \qquad -\infty < \lambda < \infty \tag{3.8}$$

leaves Equation (2.1) unchanged for arbitrary real λ*.*

we can replace u by $u - \lambda$ and finally obtain

$$\psi_{xx} + (\lambda - u)\psi = 0 \quad . \tag{3.9}$$

This is the Sturm-Liouville problem of the Schrödinger Scattering Problem for ψ with potential u and eigenvalue λ. Thus, the Miura transformation reveals a connection between solutions of the KdV equation and the potential in the Schrödinger equation!

[36] The semicolon indicates that t is a parameter.

3.2 The Schrödinger Scattering Problem

This section summarizes the main results from the quantum mechanical analysis of the scattering problem for the Schrödinger equation [56-60][37]:

1. The one dimensional Schrödinger equation emerged in theoretical physics with the birth of wave mechanics [61, 62]

$$\psi_{xx} + (\lambda - u(x))\psi = 0 \quad , \qquad x \in R \quad . \tag{3.10}$$

The real function $u(x)$ is the potential and λ is the spectral parameter, which is interpreted as the energy of the state ψ.

2. For the existence of solutions it is required that the potential $u(x)$ is integrable[38]

$$\int_{-\infty}^{\infty} |u(x)|\, dx < \infty \quad . \tag{3.11}$$

We search for values of λ for which there exist solutions $\psi(x)$ of Equation (3.10) which are bounded for $|x| \to \infty$. The set of all eigenvalues is the spectrum corresponding to a given potential $u(x)$.

3. The spectral analysis consists of two parts, namely

3.1. Bound states - Discrete spectrum: $\lambda < 0$.

- For each potential satisfying (3.10) there exists a finite number of discrete eigenvalues[39, 40]

$$\lambda = \lambda_n = -\kappa_n^2 \quad , \quad \kappa_n \in R_+ \ ; \quad n = 1, \ldots, N \tag{3.12}$$

with the ordering

$$0 < \kappa_1 < \ldots < \kappa_N \quad . \tag{3.13}$$

[37] We only list results which are important for our discussion.
[38] For a detailed discussion on the rigorous mathematical requirements we refer to [54].
[39] This follows from the Sturm-Liouville theory [20].
[40] We do not consider degenerate eigenvalues.

- The corresponding eigenfunctions $\psi_n(x) \in L^2(R)$ are

$$\int_{-\infty}^{\infty} \psi_n^2(x)\,dx = 1 \quad . \tag{3.14}$$

- The asymptotic behaviour of these eigenfunctions is given by[41]

$$\psi_n(x) \sim \begin{cases} c_n \exp(-\kappa_n x) & \text{for } x \to +\infty \\ \hat{c}_n \exp(\kappa_n x) & \text{for } x \to -\infty \end{cases} \quad . \tag{3.15}$$

- The normalization coefficients $c_1, \ldots, c_n$ are defined by[42]

$$c_n = \lim_{x \to +\infty} \exp(\kappa_n x)\psi_n(x) \quad . \tag{3.16}$$

3.2. Scattering solutions - Continuous spectrum: $\lambda > 0$.

- For

$$\lambda = k^2 \quad , \quad k \in R_+ \tag{3.17}$$

the corresponding eigenfunctions[43] $\hat{\psi}(x;k)$ behave asymptotically as

$$\hat{\psi}(x;k) \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{for } x \to +\infty \\ a(k)e^{-ikx} & \text{for } x \to -\infty \end{cases} \quad . \tag{3.18}$$

- The complex constants $a(k)$, $b(k)$ can be determined uniquely from a given potential $u(x)$.
  $a(k)$ is the transmission coefficient and $b(k)$ is the reflection coefficient.

4. Definition 3.1. [54] *The spectrum of the Schrödinger equation, together with the set $S = \{-\kappa_n^2, c_n, b(k)\}$ is called the scattering data of a given potential $u(x)$.*

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[41] The subscript *n* denotes the *n*th eigenfunction.
[42] The normalization coefficients are fixed by the normalization condition (3.11).
[43] The continuous spectrum has generalized eigenfunctions.

3.3 The Inverse Scattering Problem

Eigenvalue problems like the Sturm-Liouville problem (3.10) have been intensively studied by mathematicians and mathematical physicists of the 19th century [63]. But it was not until the middle of the 20th century that the inverse problem was solved by the Soviet mathematicians Gelfand and Levitan [64]. The inverse scattering problem consists of determining the potential $u(x)$ from its scattering data.

The following mathematical treatise is directly taken from [15][44]:

Our starting point is the classical wave equation

$$\phi_{xx} - \phi_{zz} = 0 \quad . \tag{3.19}$$

Using the Fourier transform method we obtain the following eigenvalue problem

$$\psi_{xx} + k^2\psi = 0 \quad . \tag{3.20}$$

We are only interested in the scattering solutions of (3.20), i.e.

$$\psi \sim e^{ikx} \qquad \text{for } x \to +\infty \quad . \tag{3.21}$$

For this case it is convenient to write $\phi(x,z)$ in the following form

$$\phi(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_k(x) e^{-ikz} dk = \delta(x-z) + K(x,z) \tag{3.22}$$

where $K(x,z) = 0$ if $z < x$, and $K(x,z)$ is a solution of (3.19).
Using (3.22) the scattering solution (3.21) yields

$$\psi(x;k) = \int_{-\infty}^{\infty} \phi(x,z) e^{ikz} dz = e^{ikx} + \int_{x}^{\infty} K(x,z) e^{ikz} dz \quad . \tag{3.23}$$

By analogy with Equation (3.20), we discuss the eigenvalue problem for the continuous spectrum

[44] We try to work out the main idea of the Gelfand-Levitan Theory without going into mathematical details.

$$\psi_{xx} + \left(k^2 - u\right)\psi = 0 \tag{3.24}$$

with a given potential $u(x)$. Therefore, we assume the following ansatz for the solution

$$\psi^+(x;k) = e^{ikx} + \int_x^\infty K(x,z)\, e^{ikz} dz \ . \tag{3.25}$$

Remark. The subscript $+$ denotes the asymptotic case $x \to +\infty$.

It is convenient to rewrite (3.25) as

$$\psi^+(x;k) = e^{ikx}\left(1 + \frac{i\hat{K}}{k} - \frac{\hat{K}_z}{k^2}\right) - \frac{1}{k^2}\int_x^\infty K_{zz}\, e^{ikz} dz \tag{3.26}$$

with $\hat{K} = K\big|_{z=x}$; (3.26) is obtained after integration by parts, provided that $K(x,z) \to 0$ and $K_z(x,z) \to 0$ as $z \to +\infty$ for fixed x.

The Sturm-Liouville problem (3.24) becomes

$$\begin{aligned} 0 &= \psi^+_{xx} + \left(k^2 - u\right)\psi^+ \\ &= -e^{ikx}\left(u + 2\left[\hat{K}_x + \hat{K}_z\right]\right) + \int_x^\infty \left(K_{xx} - K_{zz} - u(x)K\right) e^{ikz} dz \ . \end{aligned} \tag{3.27}$$

We notice that (3.27) is satisfied if

$$K_{xx} - K_{zz} - u(x)K = 0 \tag{3.28}$$

for $z > x$, and

$$u(x) = -2\left[K_x(x,x) + K_z(x,x)\right] = -2\frac{d\hat{K}}{dx} \tag{3.29}$$

The formula (3.29) together with the asymptotic conditions $K(x,z) \to 0$ and $K_z(x,z) \to 0$ as $z \to +\infty$, defines the problem for $K(x,z)$.

The idea of the Gelfand-Levitan Theory is to invert equation (3.26), if possible, to obtain $K(x,z)$ from a known solution $\psi^+(x;k)$, and, hence, use (3.29) to reconstruct the potential $u(x)$.

The continuous eigenfunction $\hat{\psi}(x;k)$ can be reconstructed from the function $\psi^+(x;k)$ by using the ansatz

$$\hat{\psi} = \left[\psi^{+}\right]^{*} + b(k)\psi^{+} \qquad (3.30)$$

which gives the correct asymptotic behaviour as $x \to +\infty$. From (3.26) and (3.30) we obtain

$$\hat{\psi}(x;k) = e^{-ikx} + b(k)e^{ikx} + \int_{-\infty}^{\infty} K(x,z)e^{-ikz}dz + b(k)\int_{-\infty}^{\infty} K(x,z)e^{ikz}dz \qquad (3.31)$$

where $K(x,z) = 0$ if $z < x$. Rewriting and inverting this equation yields

$$K(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\hat{\psi}(x;k) - e^{-ikx} - b(k)e^{ikx} - b(k)\int_{-\infty}^{\infty} K(x,\xi)e^{ik\xi}d\xi\right\}e^{ikz}dk \qquad (3.32)$$

where ξ is the variable for the inner integration.
Following the idea of Gelfand and Levitan we introduce a function $B(x)$ defined by

$$B(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikx}dk \qquad (3.33)$$

which is an integral along the real k-axis.
Using (3.32) and (3.33) and interchanging the order of integration we obtain

$$K(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\hat{\psi} - e^{-ikx}\right\}e^{ikz}dk - B(x+z) - \int_{-\infty}^{\infty} K(x,\xi)B(\xi+z)d\xi \quad . \qquad (3.34)$$

A straightforward calculation applying Cauchy's theorem to the first integral on the right-hand side of (3.34) gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\hat{\psi} - e^{-ikx}\right\}e^{ikz}dk = 0 \quad . \qquad (3.35)$$

Given the function $B(x)$ we finally obtain the famous Gelfand-Levitan-Marchenko [GLM] integral equation

$$K(x,z) + B(x+z) + \int_{x}^{\infty} K(x,\xi)B(\xi+z)d\xi = 0 \qquad (3.36)$$

for the function $K(x,z)$.

So far we only considered the case for the continuous spectrum. We complete our analysis by discussing the general case, where we have a discrete as well as a continuous spectrum. The difference emerges from the evaluation of the first integral on the right-hand side of (3.34).

Using the Cauchy residue theorem we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\left\{\hat{\psi}-e^{-ikx}\right\}e^{ikz}dk &= 2\pi i\sum_{i=1}^{N}R_n \\ &= -2\pi\sum_{i=1}^{N}c_n^2 e^{-\kappa_n z}\left\{e^{-\kappa_n x}+\int_{x}^{\infty}K(x,\xi)\,e^{-\kappa_n \xi}d\xi\right\} \quad ,\end{aligned} \tag{3.37}$$

where R_n is the residue of $\hat{\psi}\,e^{ikz}$ at $k=i\kappa_n$; $n=1,\dots,N$.

If we insert (3.37) into (3.34) we obtain the GLM integral equation (3.36) provided that $B(x)$ is redefined by

$$B(\zeta)=\sum_{i=1}^{N}c_n^2 e^{-\kappa_n \zeta}+\frac{1}{2\pi}\int_{-\infty}^{\infty}b(k)\,e^{ik\zeta}dk \quad . \tag{3.38}$$

This completes our analysis of the inverse scattering problem. Thus, we see that, given the scattering data $S=\left\{-\kappa_n^2, c_n, b(k)\right\}$, the potential $u(x)$ can be completely determined from $K(x,z)$ by solving the GLM integral equation (3.36).

3.4 The Initial Value Problem for the KdV Equation

In our previous analysis we discussed the classical Sturm-Liouville problem and the corresponding inverse scattering problem. For the purpose of our forthcoming discussion we systematically summarize the results here.

1. Given the initial value problem for the KdV model,

$$\begin{aligned}&u_t-6uu_x+u_{xxx}=0 \ , \quad x\in(-\infty,\infty) \ , \quad t>0 \\ &u(x,0)=u_0(x) \quad ,\end{aligned} \tag{3.39}$$

the Miura transformation (3.3) and the Galilean Invariance (3.8) lead to the Sturm-Liouville problem (3.10).

2. If $u(x)$ is the potential for the Sturm-Liouville problem (3.10), then we can reconstruct the potential $u(x)$ from the scattering data $S=\left\{-\kappa_n^2, c_n, b(k)\right\}$. The set $S=\left\{-\kappa_n^2, c_n, b(k)\right\}$ is given by the asymptotic behaviour of the eigenfunctions ψ:

2.1 Discrete spectrum: $\lambda<0$: $\lambda=\lambda_n=-\kappa_n^2$; $\kappa_n \in R_+$; $n=1,\dots,N$

$$\psi_n(x) \sim \begin{cases} c_n \exp(-\kappa_n x) & \text{for } x \to +\infty \\ \hat{c}_n \exp(\kappa_n x) & \text{for } x \to -\infty \end{cases} .$$

The normalization coefficients $c_1,\dots,c_n$ are defined by

$$c_n = \lim_{x\to+\infty} \exp(\kappa_n x)\psi_n(x) .$$

2.2 Continuous spectrum: $\lambda>0$: $\lambda=k^2$; $k\in R_+$

$$\hat{\psi}(x;k) \sim \begin{cases} e^{-ikx}+b(k)e^{ikx} & \text{for } x \to +\infty \\ a(k)e^{-ikx} & \text{for } x \to -\infty \end{cases} .$$

The reflection coefficient $b(k)$ can be determined uniquely from a given potential $u(x)$.

3. We know that

$$u(x)=-2\left[K_x(x,x)+K_z(x,x)\right]=-2\frac{d\hat{K}}{dx} ,$$

where $K(x,z)$ is a solution of the GLM integral equation

$$K(x,z)+B(x+z)+\int_x^\infty K(x,\xi)B(\xi+z)\,d\xi=0$$

with $B(x)$ defined by

$$B(\zeta)=\sum_{i=1}^{N} c_n^2 e^{-\kappa_n\zeta}+\frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ik\zeta}\,dk .$$

The solution of the initial value problem (3.39) can be understood as follows. At $t=0$ the potential $u(x,0)=u_0(x)$ is assumed to be known and so we can solve the Sturm-Liouville problem (3.10) for this potential and obtain the scattering data

$$S(0)=\left\{-\kappa_n^2, c_n, b(k)\right\} \quad . \tag{3.40}$$

If the time evolution of the set $S(0)$ can be determined we get the set

$$S(t)=\left\{-\kappa_n(t), c_n(t), b(k;t)\right\} \,, \qquad t>0 \tag{3.41}$$

and, therefore, we can reconstruct $u(x,t)$ for $t>0$.

3.5 Time Evolution of the Scattering Data

The following section is also directly taken from [15]:

Once again we consider the Sturm-Liouville problem (3.10) and assume that $u(x,t)$ is a solution of the KdV equation. Differentiation of (3.10) with respect to x yields

$$\psi_{xxx} - u_x\psi + (\lambda - u)\psi_x = 0 \tag{3.42}$$

and with respect to t gives

$$\psi_{xxt} + (\lambda_t - u_t)\psi + (\lambda - u)\psi_t = 0 \quad . \tag{3.43}$$

For the following discussion it is convenient to define

$$\Psi(x,t) = \psi_t + u_x\psi - 2(u+2\lambda)\psi_x \tag{3.44}$$

and to use the identity

$$\left(\psi_x\Psi - \psi\Psi_x\right)_x = \psi_{xx}\Psi - \psi\Psi_{xx} \quad . \tag{3.45}$$

Evaluation of (3.45) gives

$$\left(\psi_x\Psi - \psi\Psi_x\right)_x = \psi^2\left\{\lambda_t - u_t + 6uu_x - u_{xxx}\right\} \quad , \tag{3.46}$$

where we use (3.10), (3.42) and (3.43). Since $u(x,t)$ is a solution of the KdV equation, we finally obtain

$$\left(\psi_x \Psi - \psi \Psi_x\right)_x = \lambda_t \psi^2 \quad . \tag{3.47}$$

Equation (3.47) is the Gardner-Greene-Kruskal-Miura [GGKM] equation. We will study this equation in the following sections for the discrete as well as for the continuous spectrum in order to evaluate the time evolution of the scattering data.

Remark. The following subsections 3.5.1 and 3.5.2 are also directly taken from [15].

3.5.1 Discrete Spectrum

We have $\lambda = \lambda_n = -\kappa_n^2$; $\kappa_n \in R_+$; $\psi = \psi_n$; $n = 1,\ldots,N$; Integration of (3.47) over all x gives

$$\left[\psi_{nx}\Psi_n - \psi_n \Psi_{nx}\right]_{-\infty}^{\infty} = -\left(\kappa_n^2\right)_t \int_{-\infty}^{\infty} \psi_n^2 \, dx = -\left(\kappa_n^2\right)_t \tag{3.48}$$

where Ψ_n is the evaluation of Ψ for κ_n and ψ_n. Since the eigenfunctions ψ_n and Ψ_n decay exponentially as $|x| \to \infty$, we obtain

$$\left(\kappa_n^2\right)_t = 0 \qquad \Rightarrow \qquad \kappa_n = \text{constant} \quad . \tag{3.49}$$

Thus we have the remarkable result that the discrete spectrum is invariant in time.

Inserting (3.49) into (3.47) and integration with respect to x yields

$$\psi_{nx}\Psi_n - \psi_n \Psi_{nx} = \theta_n(t) \quad , \tag{3.50}$$

where $\theta_n(t)$; $(n = 1,\ldots,N)$ are arbitrary functions of t. Again, the application of the exponential decay of ψ_n and Ψ_n for $|x| \to \infty$ leads to $\theta_n = 0$ for each n and for all t.

Hence, from (3.50) and after a further integration, we obtain

$$\frac{\Psi_n}{\psi_n} = \zeta_n(t) \quad , \tag{3.51}$$

where $\zeta_n(t)$; $(n = 1,\dots,N)$ are also arbitrary functions of t.

Multiplication of (3.51) by ψ_n^2 yields

$$\psi_n\left[\psi_{nt} + u_x\psi_n - 2u\psi_{nx} + 4\kappa_n^2\psi_{nx}\right] = \zeta_n\psi_n^2 \quad . \tag{3.52}$$

On the other hand, using (3.10) gives

$$\frac{1}{2}\left(\psi_n^2\right)_t + \left\{u\psi_n^2 - 2\psi_{nx}^2 + 4\kappa_n^2\psi_n^2\right\}_x = \zeta_n\psi_n^2 \quad . \tag{3.53}$$

Again, the integration over all x leads to

$$\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\psi_n^2\,dx\right) = \zeta_n\int_{-\infty}^{\infty}\psi_n^2\,dx \quad \Rightarrow \quad \zeta_n = 0 \tag{3.54}$$

for each n and for all t.

Therefore we obtain

$$\Psi_n = \psi_{nt} + u_x\psi_n - 2\left(u - 2\kappa_n^2\right)\psi_{nx} = 0 \quad . \tag{3.55}$$

Equation (3.55) is the time evolution equation for the eigenfunctions $\psi_n(x;t)$ of the discrete spectrum. Using the asymptotic behaviour

$$u(x,t) \to 0 \quad \text{and} \quad \psi_n(x;t) \sim c_n(t)e^{-\kappa_n x} \quad \text{for } x \to +\infty \quad , \tag{3.56}$$

the time evolution equation (3.55) gives

$$\psi_{nt} + u_x\psi_n - 2\left(u - 2\kappa_n^2\right)\psi_{nx} = \frac{d}{dt}c_n(t) - 4\kappa_n^3 c_n(t) = 0 \tag{3.57}$$

or

$$c_n(t) = c_n(0)\exp\left(4\kappa_n^3 t\right) \quad , \tag{3.58}$$

where $c_n(0)$; $(n = 1,\dots,N)$ are the normalization coefficients determined at $t = 0$.

3.5.2 Continuous Spectrum

We have $\lambda = k^2$; $k \in R_+$; since k may take any real value, we discuss the time evolution for a fixed k. For the continuous eigenfunction $\hat{\psi}(x;k)$ integration of the GGKM Equation (3.47) gives

$$\hat{\psi}_x \hat{\Psi} - \hat{\psi}\hat{\Psi}_x = \sigma(t;k) \quad , \tag{3.59}$$

where $\sigma = \sigma(t;k)$ is an arbitrary function, and $\hat{\Psi}$ is the evaluation of Ψ for the continuous eigenfunction. With the asymptotic behaviour (3.18) and with (3.44) we obtain

$$\hat{\Psi}(x,t;k) \sim \left(\frac{da}{dt} + 4aik^3\right) e^{-ikx} \tag{3.60}$$

as $x \to -\infty$ and, thus,

$$\hat{\psi}_x \hat{\Psi} - \hat{\psi}\hat{\Psi}_x \ \to\ 0 \qquad \text{for } x \to -\infty \ . \tag{3.61}$$

So we see that $\sigma(t;k) = 0$ for all t. By analogy with the discrete case a further integration yields

$$\frac{\hat{\Psi}}{\hat{\psi}} = \varsigma(t;k) \qquad \Rightarrow \qquad \hat{\Psi} = \varsigma\hat{\psi} \ . \tag{3.62}$$

With the asymptotic behaviour (3.18) and (3.60) Equation (3.62) requires that

$$\frac{da}{dt} + 4\, aik^3 t = \varsigma a \ . \tag{3.63}$$

The corresponding asymptotic case $x \to +\infty$ gives

$$\hat{\psi}(x;t,k) \sim e^{-ikx} + b(k;t)\, e^{ikx} \tag{3.64}$$

and

$$\hat{\Psi}(x,t;k) \sim \frac{db}{dt} e^{ikx} + 4\, ik^3 \left(e^{-ikx} - be^{ikx}\right) \ . \tag{3.65}$$

So equation (3.62) requires that

$$\frac{db}{dt}e^{ikx} + 4ik^3\left(e^{-ikx} - be^{ikx}\right) = \varsigma\left(e^{-ikx} + be^{ikx}\right) \quad . \tag{3.66}$$

Because of the linear independence of the exponential functions $e^{\pm ikx}$, Equation (3.66) implies

$$\frac{db}{dt} - 4bik^3 = \varsigma b \quad \text{and} \quad \varsigma(t;k) = 4ik^3 \quad . \tag{3.67}$$

We finally obtain from (3.63) and (3.66)

$$\frac{da}{dt} = 0 \quad \Rightarrow \quad a(k;t) = a(k;0) \quad \text{for } t \geq 0 \tag{3.68}$$

and

$$\frac{db}{dt} = 8bik^3 \quad \Rightarrow \quad b(k;t) = b(k;0)\exp\left(8ik^3t\right) \quad \text{for } t \geq 0 \quad . \tag{3.69}$$

Equations (3.68) and (3.69) describe the time evolution of the scattering coefficients, although only the reflection coefficient varies with time.

3.6 Summary of the Method of Solution

At this stage we briefly want to summarize our results concerning the inverse scattering transform for the solution of the initial-value problem of the KdV equation.

1. We start with the initial-value problem (3.39) for the KdV equation with $u(x,0) = u_0(x)$. It is assumed that $u_0(x)$ is a sufficiently well-behaved function.

2. We solve the Sturm-Liouville problem (3.10) to determine the the scattering data $S = \left\{-\kappa_n^2, c_n, b(k)\right\}$.

3. Using the GGKM Equation (3.47) the time evolution of the scattering data is given by (3.49), (3.58) and (3.69).

4. The function $B(\zeta)$, defined by (3.38), is given by

$$\begin{aligned} B(\zeta;t) &= \sum_{n=1}^{N} c_n^2(0)\exp\left(8\kappa_n^3 t - \kappa_n\zeta\right) \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k;0)\exp\left(8ik^3t + ik\zeta\right)dk \quad . \end{aligned} \tag{3.70}$$

Note that B depends on the parameter t.

5. Therefore, the GLM integral equation (3.36) becomes

$$K(x,z;t) + B(x+z;t) + \int_x^{\infty} K(x,\xi;t)B(\xi+z;t)d\xi = 0 \quad . \tag{3.71}$$

6. The solution of the KdV equation is given by

$$u(x,t) = -2\frac{d}{dx}K(x,x;t) \quad . \tag{3.72}$$

This completes our discussion of the inverse scattering transform for solving the KdV equation. With this method we reduce the problem of solving a nonlinear partial differential equation to the problem of solving two linear problems, namely a second order ordinary differential equation and a linear integral equation. One has to be aware that the solution of the two linear problems could be technically very difficult.

3.7 Soliton Solutions[45]

3.7.1 Reflectionless Potentials

In this section we provide the general mathematical technique to generate the N-soliton solution for the KdV equation for a reflectionless potential, i.e. $b(k) = 0$ for all k.

[45] For this section we again directly rely on the presentation given in [15].

Our starting point is the GLM integral equation

$$K(x,z)+B(x+z)+\int_x^{\infty} K(x,\xi)B(\xi+z)\,d\xi=0 \quad .$$

Suppose that $B(x+z)$ is a separable function, i.e.

$$B(x+z)=\sum_{n=1}^{N}\Lambda_n(x)\Psi_n(z) \tag{3.73}$$

with N being finite.

Therefore, Equation (3.36) can be written as

$$K(x,z)+\sum_{n=1}^{N}\Lambda_n(x)\Psi_n(z)+\sum_{n=1}^{N}\Psi_n(z)\int_x^{\infty}K(x,\xi)\Lambda_n(\xi)\,d\xi=0 \quad , \tag{3.74}$$

and the solution has the form

$$K(x,z)=\sum_{n=1}^{N}\Omega_n(x)\Psi_n(z) \quad . \tag{3.75}$$

Inserting (3.75) into equation (3.74) yields

$$\Omega_n(x)+\Lambda_n(x)+\sum_{m=1}^{N}\Omega_m(x)\int_x^{\infty}\Psi_m(\xi)\Lambda_n(\xi)\,d\xi=0 \quad . \tag{3.76}$$

(3.76) is a system of N algebraic equations for the unknown functions $\Omega_n(x)$.

Now we apply these results to discuss the N-soliton solution.

We take the initial profile as

$$u(x,0)=-N(N+1)\operatorname{sech}^2 x \quad . \tag{3.77}$$

The corresponding Sturm-Liouville problem (3.10) can be transformed into the associated Legendre equation

$$\frac{d}{d\vartheta}\left\{(1-\vartheta^2)\frac{d}{d\vartheta}\psi\right\}+\left\{N(N+1)+\frac{\lambda}{(1-\vartheta^2)}\right\}\psi=0 \quad , \tag{3.78}$$

where N is a positive integer.

For the discrete spectrum the only bounded solutions for $\vartheta \in [-1;1]$ occur if $\kappa_n = n$; $n = 1, 2, \ldots, N$. Furthermore, these solutions are proportional to the associated Legendre functions $P_N^n(\vartheta)$ given by

$$P_N^n(\vartheta) = (-1)^n \sqrt{\left(1-\vartheta^2\right)^n} \frac{d^n}{d\vartheta^n} P_N(\vartheta) \tag{3.79}$$

with

$$P_N(\vartheta) = \frac{1}{N!2^N} \frac{d^N}{d\vartheta^N} \left(\vartheta^2 - 1\right)^N \tag{3.80}$$

being the Legendre polynomial of degree N.

We have N discrete eigenvalues $\lambda = \lambda_n = -\kappa_n^2$; $\kappa_n \in R_+$; $n = 1, \ldots, N$, and the discrete eigenfunctions have the asymptotic form

$$\psi_n(x) \sim c_n e^{-\kappa_n x} \quad \text{for } x \to +\infty \ . \tag{3.81}$$

Using the associated Legendre functions (3.79) we obtain

$$\psi_n(x) \propto P_N^n(\tanh x) \quad \text{and} \quad c_n(t) = c_n(0) e^{4n^3 t} \ . \tag{3.82}$$

The function $B(\zeta)$, defined by (3.38), is given by

$$B(\zeta; t) = \sum_{n=1}^{N} c_n^2(0) e^{\left(8n^3 t - n\zeta\right)} \ , \tag{3.83}$$

and the GLM integral equation (3.71) becomes

$$\begin{aligned} &K(x,z;t) + \sum_{n=1}^{N} c_n^2(0) e^{\left\{8n^3 t - n(x+z)\right\}} \\ &+ \int_x^\infty K(x,\xi;t) \sum_{n=1}^{N} c_n^2(0) e^{\left\{8n^3 t - n(\xi+z)\right\}} d\xi = 0 \ . \end{aligned} \tag{3.84}$$

The solution is given by the following ansatz:

$$K(x,z;t) = \sum_{n=1}^{N} \Omega_n(x,t) e^{-nz} \ . \tag{3.85}$$

It is convenient to rewrite the integral equation as an algebraic system

$$AL + G = 0 \quad \text{or} \quad A_{nm} L_m + G_n = 0 \ , \tag{3.86}$$

where L, G are column vectors with elements L_m and $G_n = c_n^2(0)e^{\{8n^3t\text{-}nx\}}$, respectively. The $N \times N$ matrix A has the elements

$$A_{nm} = \delta_{nm} + \frac{c_n^2(0)}{m+n} e^{\{8n^3t-(m+n)x\}} \quad . \tag{3.87}$$

Then the solution is given by

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\log(\det A) \quad . \tag{3.88}$$

3.7.2 General Case

For the general case, i.e. if $b(k) \neq 0$, the GLM integral equation has no solution for K in closed form. The discussion is then based on numerical and asymptotic analysis.

3.8 A Mathematical Survey

This section provides a systematic, but rather encyclopaedic mathematical overview of the results we obtained so far.

The following mathematical discussion is taken from [54]:

Let $u(x,t)$ be defined as a solution of the initial value problem

$$\begin{aligned} &u_t - 6uu_x + u_{xxx} = 0 \ , \quad x \in (-\infty,\infty) \ , \quad t > 0 \\ &u(x,0) = u_0(x) \ . \end{aligned} \tag{3.89}$$

1. Bona and Smith [65] proved the existence of a classical solution if $u_0(x)$, and its derivatives up to fourth one are square-integrable.

2. Further results on the existence and regularity of solutions were obtained by Tanaka [66] and Cohen [67].

3. From the physical point of view it is very important to notice that if the decay of $u_0(x)$ and its derivatives is sufficiently rapid, then the solution $u(x,t)$, $t>0$ will be smooth.

The following theorems are the mathematical framework for the IST:

Theorem 3.3. [54] *Solutions of the KdV equation that decay sufficiently rapidly are uniquely determined by the initial data.*

Proof. [54] Let u and $\overline{u}$ be two solutions of the initial value problem (3.89), and consider

$$w = u - \overline{u} \quad . \tag{3.90}$$

Substitution yields the following linear equation for w,

$$w_t = 6uw_x + 6\overline{u}_x w - w_{xxx} \quad . \tag{3.91}$$

Multiplication by w and integration over all x gives

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2 dx = 6\int_{-\infty}^{\infty} uww_x dx + 6\int_{-\infty}^{\infty} \overline{u}_x w^2 dx - \int_{-\infty}^{\infty} ww_{xxx} dx \quad . \tag{3.92}$$

If w, w_x and w_{xx} tend to zero for $|x| \to \infty$, then the last term on the right-hand side of (3.92) vanishes. Integration by parts of the first term leads to

$$\frac{d}{dt}\int_{-\infty}^{\infty} w^2 dx = 12\int_{-\infty}^{\infty}\left(\overline{u}_x - \frac{1}{2}u_x\right)w^2 dx \le 12M\int_{-\infty}^{\infty} w^2 dx \quad , \tag{3.93}$$

where we used

$$\left|\overline{u}_x - \frac{1}{2}u_x\right| \le M \ , \quad x \in (-\infty, \infty) \quad . \tag{3.94}$$

From the inequality (3.94) it follows that

$$\int_{-\infty}^{\infty} w^2 dx \le \left[\int_{-\infty}^{\infty} w^2 dx\right]\Bigg|_{t=0} e^{12Mt} \quad . \tag{3.95}$$

However, for $t=0$, w is equal to zero, because u and $\overline{u}$ satisfy the same initial value problem. Hence, $w=0$ for $t \ge 0$ which proves the uniqueness. ■

Theorem 3.4. [54] *Let* $u(x,t)$ *be a solution of the KdV equation which satisfies the condition*

$$\int_{-\infty}^{\infty} |u(x)||x|^k \, dx < \infty \ , \quad k = 0,1,2 \tag{3.96}$$

and which has the property that for $\xi = 1,2,3$

$$\frac{\partial^{\xi} u(x,t)}{\partial x^{\xi}} \tag{3.97}$$

is bounded for $|x| \to \infty$*. Then the corresponding spectrum of the Schrödinger equation is invariant in time.*

Proof. We refer the reader to [54].

Theorem 3.5. [54] *Let the potential* $u(x,t)$ *satisfy the KdV equation and let* $\lambda = \lambda(t)$ *be a family of isolated eigenvalues with corresponding eigenfunctions* $\psi(x,t)$*. Then one has the following relation*

$$\left(\frac{\partial^2}{\partial x^2} - (u-\lambda)\right)\Phi = -\lambda_t \psi \tag{3.98}$$

with

$$\Phi = \psi_t - 2(u+2\lambda)\psi_x + u_x\psi \ . \tag{3.99}$$

Proof. We refer the reader to [54].

Theorem 3.6. [54] *Let the potential* $u(x,t)$ *be as specified in Theorem 3.4. Let* λ *be any point of the spectrum and* $\psi(x,t)$ *the corresponding eigenfunction. Then the function* Φ *defined by*

$$\Phi = \psi_t - 2(u+2\lambda)\psi_x + u_x\psi \tag{3.100}$$

satisfies the Schrödinger equation

$$\left[\frac{\partial^2}{\partial x^2} - (u-\lambda)\right]\Phi = 0 \ . \tag{3.101}$$

Proof. We refer the reader to [54].

Theorem 3.7. [54] *Let the potential* $u(x,t)$ *satisfy the conditions of Theorem 3.4, and, furthermore*

$$\lim_{|x|\to\infty} u(x,t) = \lim_{|x|\to\infty} u_x(x,t) = 0 \qquad (3.102)$$

uniformly with respect to t *on any compact time interval. Let* $\lambda = -\kappa_n^2$ *be any discrete eigenvalue, with a corresponding eigenfunction* $\psi(x,t)$ *normalized by*

$$\int_{-\infty}^{\infty} \psi_n^2(x,t)\,dx = 1 \ , \qquad \psi_n(x,t) > 0 \quad \text{for } x \to \infty \ . \qquad (3.103)$$

Then the normalization coefficient c_n *defined by*

$$c_n = \lim_{x\to+\infty} \exp(\kappa_n x)\psi_n(x) \qquad (3.104)$$

is given by

$$c_n(t) = c_n(0)\exp\left(4\kappa_n^3 t\right) \ . \qquad (3.105)$$

Proof. We refer the reader to [54].

Theorem 3.8. [54] *Let the potential* $u(x,t)$ *satisfy the conditions of Theorem 3.7, and, and let* $\lambda = k^2$ *be any point of the continuous spectrum. The reflection coefficient* $b(k;t)$ *is given by*

$$b(k;t) = b(k;0)\exp\left(8ik^3 t\right) \ . \qquad (3.106)$$

Proof. We refer the reader to [54].

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