DISSERTATION

Weyl structures for generic rank two distributions in dimension five

Verfasserin

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Abstract

This thesis is concerned with the study of generic rank two distributions on five manifolds, i.e. the study of vector subbundles of the tangent bundle such that iterated Lie brackets of sections of the subbundle of length at most three span all of the tangent bundle. The distributions are treated within the framework of parabolic geometries.

The main results are based on the theory of Weyl structures for parabolic geometries. We consider Weyl structures determined by so-called generalised contact forms and provide an explicit description of the corresponding linear connection on the distribution, the corresponding decomposition of the tangent bundle and the corresponding Rho-tensor. This can be interpreted as a construction of the canonical Cartan connection for a generic rank two distribution in dimension five. The approach leads to formulae for invariants of the distribution, such as the fundamental curvature tensor and certain invariant operators, in terms of generalised contact forms. Furthermore, we discuss Nurowski's conformal structure as a special case of a generalised Fefferman construction. Using our results we obtain a natural description of a metric from Nurowski's conformal class.

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Introduction

Generic rank two distributions in dimension five have a famous history. Elie Cartan studied them in his "five variables paper" [**19**] in 1910. Using his method of equivalence, he naturally associated to a generic rank two distribution $\mathcal{H}$ on a five-dimensional manifold M what is now called a Cartan geometry, i.e. Cartan connection on a principal bundle over the manifold. Every Cartan geometry has a homogeneous model space G/H for a Lie group G and a closed subgroup H. The Cartan connection can be viewed as generalisation of the Maurer Cartan form on the Lie group G. It is defined as an equivariant one-form ω on a P-principal bundle $\mathcal{G}$ with values in the Lie algebra $\mathfrak{g}$ of the group G. The curvature of the Cartan connection, given by $d\omega + \frac{1}{2}[\omega, \omega]$, is a complete obstruction to local equivalence with the homogeneous model.

In this thesis, generic rank two distributions in dimension five are treated within the framework of parabolic geometry. Parabolic geometries are Cartan geometries with homogeneous model spaces of the form G/P, for a semisimple Lie group G and a parabolic subgroup $P \subset G$. These geometries have been intensively studied in recent years. The restriction to semisimple Lie groups and parabolic subgroups allows to use certain algebraic concepts for the study of these geometries. At the same time, the class is broad enough to cover a variety of interesting underlying structures, as explained below. These include conformal structures, projective structures, almost quaternionic structures, CR-structures of hypersurface type and certain generic distributions.

Any regular parabolic geometry gives rise to an underlying geometric structure, which consists of a filtration of the tangent bundle of the underlying manifold with certain properties and, in general, a reduction of structure group of the corresponding graded frame bundle. Conversely, any underlying structure can be prolonged to a unique (up to equivalence) normalised parabolic geometry and this establishes an equivalence of categories. The equivalence to underlying structures goes back to Tanaka, [**38**], the description of underlying structures and the prolongation procedures to parabolic geometries have been further developed, see, [**28**], [**15**] and references therein.

The best studied structures equivalent to parabolic geometries are conformal structures. Here, the filtration of the tangent bundle is trivial and the geometry is equivalent to a reduction of structure group of the tangent bundle to the conformal group $CO(p, q)$. However, in some respect this structure is not typical for a parabolic geometry. In many cases parabolic geometries are essentially determined by the underlying filtrations of the tangent bundle, alone.

Generic rank two distribution in dimension five are one example of such a geometry. A rank two distribution $\mathcal{H}$ in dimension five is said to be generic if for any local frame ξ, η of the distribution, the vector fields ξ, η and $\gamma = [\xi, \eta]$ span a rank three distribution $\mathcal{H}^1$ and the vector fields $\xi, \eta, \gamma, [\gamma, \xi], [\gamma, \eta]$ form a local frame for the entire tangent bundle. That means, $\mathcal{H}$ is a maximally non-integrable rank two distribution. Defining $T^{-1}M = \mathcal{H}$ and $T^{-2}M = \mathcal{H}^1$, the sequence $T^{-1}M \subset T^{-2}M \subset TM$ is a filtration of the tangent bundle by smooth subbundles. The general result on the equivalence of categories applied to these distributions reproduces the famous Cartan result: a distribution $\mathcal{H}$ can be equivalently viewed as a Cartan geometry corresponding to the exceptional Lie group G_2 (more precisely, the split real form) and one of its maximal parabolic subgroups P.

A good starting point for dealing with a parabolic geometry is the study of the homogeneous model. It is well known that the split real form G_2 may be regarded as the automorphism group of the algebra of split octonions $\mathbb{O}_S$. This is an eight dimensional unital algebra with a multiplicative inner product of signature $(4,4)$. The automorphisms of $\mathbb{O}_S$ preserve that inner product. Restricting the action of an automorphism to the space of purely imaginary split octonions leads to an inclusion of G_2 into $SO(3,4)$. The parabolic subgroup P may be defined to be the stabiliser of an isotropic line in $\mathbb{O}_S$. Furthermore, the inclusion of G_2 into $SO(3,4)$, induces a diffeomorphism

$$G_2/P \cong SO(3,4)/\tilde{P},$$

where $\tilde{P}$ is the stabiliser in $SO(3,4)$ of an isotropic line and the latter homogeneous space is the homogeneous model for (pseudo) conformal structures of signature $(2,3)$. It follows, that the homogeneous model G_2/P carries a natural conformal class.

This already suggests a relation to conformal geometry. Indeed, P. Nurowski showed in [**29**], that any generic rank two distribution on a five manifold M determines a canonical conformal structure on M. The result resembles a result by Fefferman [**21**] that says that one can associate to a non-degenerate CR-structure of hypersurface type a natural conformal structure on a circle bundle over the manifold. In [**8**] A. Cap observed that both Fefferman's and Nurowski's results can be rephrased as instances of a generalised Fefferman construction. Such a generalised Fefferman construction can be viewed as a functor between parabolic geometries of different types. Applied to generic rank two distributions, this functor assigns to a parabolic geometry of type (G_2, P) a parabolic geometry of type $(SO(3,4), \tilde{P})$. Another interesting viewpoint is to regard Nurowski's conformal structures associated to generic rank two distributions as holonomy reductions. They are precisely the $(2,3)$-signature conformal structures with conformal holonomy groups contained in G_2.

The purpose of this thesis is to contribute to the investigation of generic rank two distributions in dimension five. The main input is the use of the theory of Weyl structures for parabolic geometries as a tool for the study of these distributions.

There is a well known concept of a Weyl structure in conformal geometry. On a conformal manifold, a Weyl structure is equivalent to a torsion free, linear connection on the tangent bundle, that is compatible with the conformal class. These connections are known as Weyl connections. A nice subclass are the Levi Civita connections for metrics in the conformal class.

Generalising the concept from conformal geometry, a theory of Weyl structures has been introduced for all parabolic geometries by A. Cap and J. Slovak in [**17**]. The choice of a Weyl structure allows to interpret the Cartan connection in terms of geometric data defined on the underlying manifold: A Weyl structure determines a preferred linear connection ∇ on the tangent bundle of the manifold, the Weyl connection, an isomorphism

$\Theta : TM \cong \mathrm{gr}(TM)$ which defines a splitting of the filtration on the tangent bundle, called soldering form, and a distinguished one-form $P \in \Omega^1(M, \mathrm{gr}(T^*M))$, the rho tensor. The theory provides explicit formulae for the transformation of these data under the change of the Weyl structure. Another feature of the theory is the existence of line bundles $\mathcal{L}$ with the property that Weyl structures for a parabolic geometry are in bijective correspondence with linear connections on $\mathcal{L}$. Nowhere vanishing sections of these bundles are called scales. A global scale determines a flat connection on the bundle of scales and thus corresponds to a unique Weyl structure.

We apply the theory of Weyl structures to generic rank two distributions in dimension five. In that case, the bundle $(T^{-2}M/T^{-1}M)^*$ is a natural choice for the line bundle $\mathcal{L}$. The corresponding scales will be used to specify Weyl structures and we introduce a name for these: We define a generalised contact form α to be a smooth section of the bundle $L(T^{-2}M, \mathbb{R})$ such that the kernel of the linear map $\alpha(x) : T^{-2}M \to \mathbb{R}$ equals $\mathcal{H}_x$ for each $x \in M$. Clearly, this is the same as a nowhere vanishing section of $(T^{-2}M/T^{-1}M)^*$. The general theory tells us that the choice of a generalised contact form determines a unique Weyl connection, soldering form and rho tensor representing the normal Cartan connection for a generic rank two distribution in dimension five. An important part of this thesis addresses the problem of describing these data as explicitly as possible in terms of a generalised contact form. This amounts to translating normality of the Cartan connection to conditions on the Weyl connection and the soldering form and the rho tensor and analysing these conditions.

The description of the Weyl structure by means of a generalised contact form has the advantage that it immediately leads to formulae for geometric invariants of a distribution in familiar geometric terms. Some examples are discussed in this thesis. First of all, we can express the fundamental curvature tensor, a certain section of $S^4(\mathcal{H}^*)$, in terms of the curvature of the Weyl connection ∇ and the soldering form. Moreover, we can deduce formulae for certain (known) invariant differential operators in terms of a generalised contact form. The construction of a metric in Nurowski's conformal class presented in this thesis yields yet another application of our results: Using the soldering form, i.e. the decomposition of the tangent bundle associated to a generalised contact form, we derive a formula for a metric in terms of a generalised contact form. Thus, we obtain a description of Nurowski's conformal class which is similar to Lee's description of the Fefferman conformal class, see [**31**]. In the picture of Cartan geometries, it follows immediately from the construction of the metric that the conformal class is indeed the one found by Nurowski and an invariant of the distribution. However, a feature of the description in terms of a generalised contact form is that we can prove directly, without reference to Cartan connections, that the metric rescales conformally when changing the generalised contact form. The latter approach has been pursued in a joint paper [**14**] with A. Cap.

Structure and results. We start with a preliminary chapter on parabolic geometries, focusing on geometries determined by their underlying filtrations. We define Cartan geometries and parabolic geometries. We explain the relationship between parabolic subalgebras and $|k|$-gradings of semisimple Lie algebras. Then we introduce the notions of regularity and normality of parabolic geometries and discuss the equivalence between regular, normal parabolic geometries and underlying structures. Finally, we prove that for

$|k|$-graded Lie algebras that are subject to a certain cohomological condition, underlying structures of regular parabolic geometries of type $(\mathrm{Aut}(\mathfrak{g}), P)$ consist only of the underlying filtrations. The results presented in this chapter imply as a special case the equivalence of categories between generic rank two distributions in dimension five and regular, normal parabolic geometries of type (G_2, P).

The second chapter covers the algebraic material on the Lie group G_2 and the parabolic subgroup P that will be needed in the sequel. The group G_2 is the automorphism group of the algebra of split octonions. Equivalently, it can be described as the stabiliser of an element $\phi \in \Lambda^3(\mathbb{R}^7)^*$ with open $GL(7,\mathbb{R})$-orbit. The parabolic subgroup P is the stabiliser of the real line through a highest weight vector in the seven-dimensional fundamental representation $\mathrm{Im}\mathbb{O}_S$ of G_2. Using the description of the Lie algebra $\mathfrak{g}_2$ as the algebra of derivations of the split octonions $\mathrm{Im}\mathbb{O}_s$, we derive an explicit realisation of $\mathfrak{g}_2$ as a subalgebra in $\mathfrak{so}(3,4)$. We also look at the parabolic subalgebra $\mathfrak{p}$ and the corresponding $|3|$-grading in this picture. Furthermore, we will give several explicit descriptions of the homogeneous model for generic rank two distributions in dimension five, i.e. the homogeneous space G_2/P and its canonical rank two distribution. The homogeneous space can be viewed as the projectivisation $\mathcal{P}(\mathcal{C})$ of the null-cone for the invariant inner product in the seven-dimensional real vector space $\mathrm{Im}(\mathbb{O}_S)$. The distribution has a description in terms of the split octonions, another one in terms of the three-form ϕ.

In the third chapter, we are concerned with Weyl structures for generic rank two distributions in dimension five. We introduce the notion of a generalised contact form and the associated generalised Reeb field. Then we prove that a (local) generalised contact form determines a unique (local) Weyl structure for the geometry. The main part of the third chapter deals with the construction of the Weyl connection ∇, the soldering form Θ and the Rho tensor P associated to a (local) generalised contact form. As an application we derive a formula for the infinitesimal automorphism operator of a distribution in terms of a generalised contact form. We verify that the formula is independent of the choice of the generalised contact form and hence defines an intrinsic operator.

The fourth chapter investigates Nurowski's conformal structures. Our treatment is guided by analogy with the Fefferman construction for CR-manifolds (compare with [**31**], [**11**] and [**12**]). First, we look at Nurowski's construction as a special case of a generalised Fefferman construction. We show that when starting with a regular and normal parabolic geometry of type (G_2, P) the associated conformal geometry is normal as well. Then we discuss Nurowski's conformal structures as holonomy reductions to G_2. As a consequence of normality of the associated conformal geometry, the normal tractor connection for the rank two distribution coincides with the normal conformal tractor connection. This implies that the standard tractor bundle for the Nurowski's conformal structures carries a tractor three-form which is parallel for the normal tractor connection and the conformal holonomy groups for Nurowski's conformal structures are contained in G_2. Conversely, we will see that given a conformal structure of signature $(2,3)$ with conformal holonomy contained in G_2, it is the canonical conformal structure associated to a generic rank two distribution. Finally, we present a construction of a pseudo Riemannian metric g_α contained in Nurowski's conformal class in terms of a generalised contact form. The construction is based on results of Chapter 3, more precisely, the explicit form of the splitting of the filtration $T^{-1}M \subset T^{-2}M \subset TM$.

In the appendix, we determine the dependence of the splitting of the filtration on the chosen generalised contact form. This leads to a direct verification of the fact that changing the generalised contact form causes a conformal rescaling of the metric g_α. Thus, the conformal class of the metric is seen to be independent of the choice of the generalised contact form. We also explain how to recover Nurowski's original formula for a metric from the conformal class, which is related to a certain type of ODE's.

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CHAPTER 1

Parabolic geometries determined by filtrations of the tangent bundle

This chapter is a brief introduction to parabolic geometries and their underlying structures. The main purpose is to establish notation and collect results that will be needed in later parts of this thesis. The survey is based on the work of many people, in our presentation we follow mainly [**8**] and the forthcoming book [**16**]. Since we are ultimately interested in a certain type of distribution, we will focus on parabolic geometries determined by filtrations of the tangent bundle.

1.1. Cartan geometries

Cartan geometries are named after Elie Cartan. In [**19**] he introduced the concept of *espaces généralisés* as an attempt to generalise the concepts of Klein geometry, i.e. geometry of homogeneous spaces, as well as Riemannian geometry. This section will provide only basic information on Cartan geometries. Good and detailed introductions can be found in [**16**] and [**34**].

Consider a Lie group G and a closed subgroup $H \subset G$. Then the homogeneous space G/H comes with the following geometric structure:

- the principal bundle $G \to G/H$,
- the Maurer Cartan form $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$ defined by
$$\omega_{MC}(g)(\xi) = T_g\lambda_{g^{-1}}\xi,$$
where $\lambda_{g^{-1}}$ denotes left multiplication by g^{-1}. (So the Maurer Cartan form is a way to encode the left trivialisation of the tangent bundle of the Lie goup G.)

The automorphisms of this structure, i.e. the principal bundle automorphisms Φ satisfying $\Phi^*\omega_{MC} = \omega_{MC}$, are exactly the left multiplications by elements in the group G.

The notion of a Cartan geometry generalises this situation, introducing a concept of curvature:

Definition 1.1. Let G be a Lie group and $H \subset G$ a closed subgroup. A *Cartan connection* on a H-principal bundle $\mathcal{G}$ is a smooth one-form on $\mathcal{G}$ with values in $\mathfrak{g}$, which satisfies the following properties:

- It is H-equivariant, i.e.
$$(r^h)^*\omega = \mathrm{Ad}(h)^{-1} \circ \omega,$$
where we denote by r the principal H-action on $\mathcal{G}$.
- It reproduces generators of fundamental vector fields, i.e. $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{h}$.
- It trivializes $T\mathcal{G}$, i.e $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

A *Cartan geometry* of type (G, H) on a manifold M is given by a principal H-bundle $\mathcal{G} \to M$ and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$.

DEFINITION 1.2. A morphism between two Cartan geometries of type (G, H) is a principal bundle morphism $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ such that $\Phi^*\omega_2 = \omega_1$.

Since the Maurer Cartan form satisfies the properties required in the definition of a Cartan connection, the principal bundle $G \to G/H$ endowed with the Maurer Cartan form is a Cartan geometry of type (G, H). It is called the *homogeneous model* of Cartan geometries of type (G, H).

DEFINITION 1.3. The two-form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$, defined by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$, is called the *curvature* of the Cartan connection ω.

It is easy to see that the curvature is horizontal and H-invariant. It can be equivalently viewed as the *curvature function* $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$ characterised by

$$\kappa(u)(X + \mathfrak{h}, Y + \mathfrak{h}) := K(\omega(u)^{-1}(X), \omega(u)^{-1}(Y)).$$

Note that the curvature function is well defined since K is horizontal and it is H-equivariant by equivariance of K.

DEFINITION 1.4. A Cartan connection is said to be *torsion free* if and only if the curvature function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$ takes values in the H-submodule $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{h} \subset \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$.

Recall that the Maurer-Cartan form satisfies the Maurer Cartan equation

$$d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0,$$

so the homogeneous model of a Cartan geometry of type (G, H) has always vanishing curvature. Conversely, one can prove

THEOREM 1.5. *A Cartan geometry with vanishing curvature is locally equivalent to the homogeneous model.*

In that sense Cartan geometries can be viewed as curved analogues of homogeneous spaces. For a proof see for instance [**34**].

A lot of interesting geometric structures have an equivalent description as a Cartan geometry of a certain type. A motivating example are of course Riemannian structures. A n-dimensional Riemannian manifold can be interpreted as a torsion-free Cartan geometry of type (G, H), where G is the Euclidean group and $H = O(n)$. The Cartan bundle $\mathcal{G}$ corresponding to the Riemannian structure is the orthonormal frame bundle and the Cartan connection is given by the sum of the soldering form and the Levi-Civita connection viewed as a connection form on $\mathcal{G}$.

Though the concept of a Cartan geometry is very general, the fact that a geometric structure has a description as a Cartan geometry immediately leads to non-trivial information about the geometric structure. To mention an example, the automorphism group of any Cartan geometry of type (G, H) is a Lie group with dimension at most the dimension of G. Hence the same is true for any structure that can be shown to be equivalent to a Cartan geometry.

1.2. Parabolic geometries

Parabolic geometries are Cartan geometries for semisimple Lie goups G and parabolic subgroups P. They provide a conceptual way of studying several interesting geometric structures. Among them are conformal structures, projective structures, almost quaternionic structures, non-degenerate CR-structures of hypersurface type and certain generic distributions.

1.2.1. The definition.

DEFINITION 1.6. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ or $\mathbb{R}$. A $|k|$-grading on $\mathfrak{g}$ is a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we understand $\mathfrak{g}_i = 0$ for $|i| > k$, and the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$.

Any $|k|$-grading gives rise to a filtration

$$\mathfrak{g}^k \subset \mathfrak{g}^{k-1} \subset \cdots \subset \mathfrak{g}^{-k}$$

defined by

$$\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j.$$

We now define parabolic geometries as follows.

DEFINITION 1.7. Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $P \subset G$ be the subgroup defined as the stabiliser of the associated filtration, i.e.

$$P := \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \; \forall i\}.$$

A parabolic geometry is a Cartan geometry of type (G, P) for groups G and P.

Subgroups P as in the theorem are parabolic subgroups, which explains the name parabolic geometry. Homogeneous spaces G/P for semisimple Lie groups and parabolic subgroups are called *generalised flag varieties*. They are always compact.

1.2.2. On $|k|$-gradings of semisimple Lie algebras. Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ be a semisimple Lie algebra endowed with a $|k|$-grading. In the sequel we will use the following notation for subalgebras of $\mathfrak{g}$. We define $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$. It follows immediately from the definition that the grading on the Lie algebra is invariant with respect to the adjoint action of the subalgebra $\mathfrak{g}_0$ on $\mathfrak{g}$ and the associated filtration is even $\mathfrak{p}$-invariant. We state the following observations about $|k|$-gradings, c.f. [**39**]:

- There is a unique element $E \in \mathfrak{g}$ such that $[E, X] = jX$ for all $X \in \mathfrak{g}_j$. It is called *grading element*.
- The Killing form induces an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ which is compatible with the grading. It induces a duality of $\mathfrak{g}_0$-modules between $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ and a duality of $\mathfrak{p}$-modules between $\mathfrak{g}/\mathfrak{g}^{-i+1}$ and $\mathfrak{g}^i$.
- Suppose no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$. Then $\mathrm{ad} : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is injective.

1.2.3. The group level. Let G be a Lie group with Lie algebra $\mathfrak{g}$. In addition to the subgroup $P := \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \ \forall i\}$ we introduce the subgroup of elements preserving the grading on $\mathfrak{g}$, that is

$$G_0 := \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \ \forall i\}.$$

Both P and G_0 are closed subgroups, since they can be written as intersections of normalizers $N_G(\mathfrak{g}_i)$ respectively $N_G(\mathfrak{g}^i)$, which are closed subgroups. A simple argument shows that the Lie algebra of P is $\mathfrak{p}$ and the Lie algebra of G_0 is precisely $\mathfrak{g}_0$. Moreover, we have the following result on the subgroups $G_0 \subset P \subset G$, see e.g. [**15**]:

PROPOSITION 1.8. *The exponential* exp *defines a diffeomorphism from* $\mathfrak{p}_+$ *onto a closed subgroup* $P_+ \subset P$ *and* G *is the semidirect product of* G_0 *and the normal subgroup* P_+*.*

1.2.4. Parabolic subalgebras of and their relation to $|k|$-gradings. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Then a *Borel subalgebra* is a maximal solvable subalgebra of $\mathfrak{g}$. A subalgebra containing a Borel subalgebra is called a *parabolic subalgebra*. Suppose we have fixed a Cartan subalgebra $\mathfrak{h}$ and a set of simple roots Δ^0. Then every root α is a linear combination of elements of Δ^0 with integer coefficients, which are all non-negative or all non-positive. The set of roots with non-negative coefficients is denoted by Δ^+. Recall the well-known root space decomposition

$$\mathfrak{g} = \bigoplus_{-\alpha\in\Delta^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha.$$

The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ is easily seen to be a Borel subalgebra. It is called the *standard Borel subalgebra* (for the given choice of $\mathfrak{h}$ and Δ^0). Likewise, subalgebras containing $\mathfrak{b}$ are called *standard parabolic subalgebras.* Standard parabolic subalgebras are classified by subsets of simple roots: For any set $\Sigma \subset \Delta^0$, let $\mathfrak{p}_\Sigma$ be the sum of the standard Borel subalgebra and all negative root spaces $\mathfrak{g}_{-\alpha}$ for roots α that can be decomposed as a linear combination of simple roots contained in $\Delta^0 \setminus \Sigma$. Then $\mathfrak{p}_\Sigma$ is evidently a standard parabolic subalgebra. Moreover, c.f. [**39**],

THEOREM 1.9. *Assigning to a subset* $\Sigma \subset \Delta^0$ *the parabolic* $\mathfrak{p}_\Sigma$*, induces a bijective correspondence between subsets* $\Sigma \subset \Delta^0$ *and standard parabolic subalgebras* $\mathfrak{p} \subset \mathfrak{g}$*.*

The fact that any two Borel subalgebras are conjugate by an inner automorphism implies that every parabolic subalgebra is conjugate to a standard parabolic one. In view of the above bijection, we can uniquely refer to (the isomorphism class of) a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ by the Dynkin diagram of $\mathfrak{g}$ with nodes corresponding to elements in Σ being replaced by a cross.

The description of standard parabolic subalgebras in terms sets of simple roots allows to establish a correspondence between standard parabolic subalgebras and $|k|$-gradings of $\mathfrak{g}$. For any subset $\Sigma \subset \Delta^0$, we define the Σ-height $\mathrm{ht}_\Sigma(\alpha)$ of a root α to be the sum of all coefficients of elements of Σ in the decomposition of α in a linear combination of simple roots. For a complex semisimple Lie algebra $\mathfrak{g}$, we can use the Σ-height to define a grading as follows:

$$\mathfrak{g}_i = \bigoplus_{\alpha:\mathrm{ht}_\Sigma(\alpha)=i} \mathfrak{g}_\alpha,$$

for $i \neq 0$ and

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha:\mathrm{ht}_\Sigma(\alpha)=0} \mathfrak{g}_\alpha.$$

We thus obtain a $|k|$-grading, such that the sum of all non-negative grading components is precisely $\mathfrak{p}_\Sigma$. Conversely, we have the following result.

THEOREM 1.10. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded complex semisimple Lie algebra. Then $\mathfrak{g}^0 = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ is a parabolic subalgebra. Choosing a Cartan subalgebra and a simple system such that $\mathfrak{g}^0$ is a standard parabolic subalgebra corresponding to a subset Σ, the grading is obtained via the Σ-height.*

A proof of this theorem can be found in [**39**].

There are similar results for real semisimple Lie algebras. Standard parabolic subalgebras of real semisimple Lie algebras are defined via their complexifications. In general, there are less parabolics than for the complexified Lie algebras. They are in bijective correspondence with subsets $\Sigma \subset \Delta_r^0$ of simple restricted roots. Standard parabolic subalgebras can be represented using so-called Satake diagrams with crosses. Also in the real case, we have a correspondence between subsets Σ and $|k|$-gradings of semisimple Lie algebras, given by the Σ-height. We shall not go into any details, because it won't be necessary for this thesis.

The only Lie algebras we will deal with in this thesis are split real forms of complex semisimple Lie algebras and for these, the situation is as simple as in the complex case. By definition, a split real form has a Cartan subalgebra $\mathfrak{h}$ where all the roots are real. Restricting a root to $\mathfrak{h}$ gives a bijection between the (complex) roots Δ and the restricted roots Δ_r. So for split real forms there are as many standard parabolic subalgebras as for their complexifications. The Satake diagram for a parabolic subalgebra of a split real form is exactly the same as the Dynkin diagram for its complexification.

1.2.5. Natural bundles. Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra, let G be a Lie group with Lie algebra $\mathfrak{g}$, and let P be the subgroup determined by the corresponding filtration. Suppose $(\mathcal{G}, \omega)$ is a parabolic geometry of type (G, P). Then any representation $\rho : P \to \mathrm{GL}(\mathbb{V})$ gives rise to an associated vector bundle $\mathcal{G} \times_P \mathbb{V}$, defined to be the quotient of $\mathcal{G} \times \mathbb{V}$ by the relation $(u, v) \sim (u \cdot p, \rho(p^{-1})(v))$. Bundles constructed in this way are called *natural bundles* for the parabolic geometry. Sections of these bundles can be described via P-equivariant functions $f : \mathcal{G} \to \mathbb{V}$. Any P-equivariant map between two representation spaces induces a vector bundle map between the corresponding natural bundles. We discuss some particularly interesting natural bundles.

An important class of natural bundles are the so-called *tractor bundles*. These are natural bundles associated to restrictions of G-representations to the subgroup P. Their importance comes from the fact that, unlike arbitrary natural bundles, tractor bundles always admit natural linear connections.

The tractor bundle corresponding to the restriction of the adjoint action of G to the subgroup P is called the *adjoint tractor bundle*

$$\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}.$$

Since the filtration $\mathfrak{g}^k \subset \cdots \subset \mathfrak{g}^0 \subset \cdots \subset \mathfrak{g}^{-k} = \mathfrak{g}$ is preserved by the action of the group P it gives rise to a natural filtration

$$\mathcal{A}^k M \subset \cdots \subset \mathcal{A}^0 M \subset \cdots \subset \mathcal{A}^{-k} M = \mathcal{A}M$$

of the adjoint tractor bundle.

The quotient $\mathcal{A}M/\mathcal{A}^0 M$ is the associated bundle $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$. The latter bundle can be identified with the tangent bundle TM via the Cartan connection ω. Explicitly, the isomorphism is given by

$$\begin{aligned} \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \;&\cong\; TM \\ [(u, X + \mathfrak{p})] &\mapsto T_u p \cdot \omega(u)^{-1}(X). \end{aligned} \tag{1.1}$$

Since the Killing form defines an isomorphism $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ of P-modules, we obtain an identification $\mathcal{A}^1 M = \mathcal{G} \times_P \mathfrak{p}_+ \cong T^*M$

1.2.6. An example. We work out an example in detail, not only because it illustrates the concepts introduced in this section, but also because it will be important later on. Let us consider $\mathfrak{so}(3,4)$ which is the split real form of the complex semisimple Lie algebra $\mathfrak{so}(7,\mathbb{C})$. Representing $\mathfrak{so}(3,4)$ with respect to the bilinear form corresponding to the matrix

$$\begin{pmatrix} &&&& 1 \\ &&& \mathbb{I}_2 & \\ && -1 && \\ & \mathbb{I}_2 &&& \\ 1 &&&& \end{pmatrix}$$

we obtain:

$$\mathfrak{so}(3,4) = \left\{ \begin{pmatrix} a & P^t & s & Q^t & 0 \\ F & M & V^t & B & -Q \\ r & W & 0 & V & s \\ G & C & W^t & -M^t & -P \\ 0 & -G^t & r & -F^t & -a \end{pmatrix} \right\}.$$

Here, $a, r, s \in \mathbb{R}$, $F, G, P, Q, V, W \in \mathbb{R}^2$ and $M, B, C \in \mathfrak{gl}(2,\mathbb{R})$ with B and C skew symmetric.

We choose as a Cartan subalgebra $\mathfrak{h}$ the subalgebra of diagonal matrices in $\mathfrak{so}(3,4)$. Next we determine the set of restricted roots with respect to this Cartan subalgebra. Let $\phi_i : \mathfrak{h} \to \mathbb{R}$ be the linear functional extracting the ith diagonal entry. Then we obtain the set of restricted roots

$$\Delta_r = \{\pm(\phi_i - \phi_j), \pm(\phi_i + \phi_j), \pm\phi_i, i, j \in \{1,2,3\}, i \neq j\}.$$

The P-block contains the root spaces corresponding to $\phi_1 - \phi_2$ and $\phi_1 - \phi_3$, the entry s is the root space corresponding to ϕ_1, the Q-block contains the root spaces corresponding to $\phi_1 + \phi_2$ and $\phi_1 + \phi_3$, the the block M contains the root spaces corresponding to $\phi_2 - \phi_3$ and $\phi_3 - \phi_2$, the V-block contains the root spaces corresponding to ϕ_2 and ϕ_3, the B-block is the root space corresponding to $\phi_2 + \phi_3$, the F-block contains the root spaces corresponding to $-\phi_1 + \phi_2$ and $-\phi_1 + \phi_3$, the entry r is the root space corresponding to $-\phi_1$, the G-block contains the root spaces corresponding to $-\phi_1 - \phi_2$ and $-\phi_1 - \phi_3$, the

W-block contains the root spaces corresponding to $-\phi_2$ and $-\phi_3$, the C-block is the root space corresponding to $-\phi_2 - \phi_3$.

Let us choose the simple system $\Delta_r^0 = \{\phi_1 - \phi_2, \phi_2 - \phi_3, \phi_3\}$. We want to look at the standard parabolic subalgebra associated to the subset $\Sigma = \{\phi_1 - \phi_2\}$. In Satake diagram notation, this is the parabolic subalgebra $\times\!\!-\!\!\!-\!\!\circ\!\!\Longrightarrow\!\!\circ$.

The roots with Σ-height 1 are $\phi_1 - \phi_2$, $\phi_1 - \phi_3$, ϕ_1, $\phi_1 + \phi_2$ and $\phi_1 + \phi_3$. The negatives of these roots have Σ-height -1 and the remaining roots have Σ-height 0. It follows, that the Σ-height determines a $|1|$-grading (with blocks of sizes 1, 5 and 1).

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ 0 & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}.$$

The parabolic subalgebra corresponding to Σ is the upper block matrix $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

Let us consider the group $G = SO(3,4)$ and the parabolic subgroup $P \subset G$ corresponding to the grading on $\mathfrak{g}$. The invariant bilinear form $\langle , \rangle$ on $\mathbb{V}$ induces an invariant conformal class of bilinear forms of signature $(2,3)$ on $\mathfrak{g}/\mathfrak{p}$, which can be seen as follows: Let $\mathbb{V}$ be the standard representation of $\mathfrak{so}(3,4)$. Then the highest weight space $\mathbb{V}^2$ is generated by the first basis vector in $\mathbb{V}$. The parabolic subalgebra $\mathfrak{p}$ is precisely the stabiliser of that highest weight space. The orthogonal complement to $\mathbb{V}^2$ is the subspace in $\mathbb{V}$ generated by all but the last standard basis vectors. The Lie algebra $\mathfrak{g}$ maps $\mathbb{V}^2$ to $(\mathbb{V}^2)^\perp$ and the action induces an isomorphism

$$\mathfrak{g}/\mathfrak{p} \cong L(\mathbb{V}^2, (\mathbb{V}^2)^\perp/\mathbb{V}^2).$$

Since $\mathbb{V}^2$ is isotropic, the nondegenerate bilinear form of signature $(3,4)$ on $\mathbb{V}$ induces a nondegenerate bilinear form $\langle , \rangle$ of signature $(2,3)$ on the quotient $(\mathbb{V}^2)^\perp/\mathbb{V}^2$. It gives rise to a conformal class of bilinear forms of signature $(2,3)$ on $L(\mathbb{V}^2, (\mathbb{V}^2)^\perp/\mathbb{V}^2)$ and thus on $\mathfrak{g}/\mathfrak{p}$. Note that the isomorphism $\mathfrak{g}/\mathfrak{p} \cong L(\mathbb{V}^2, (\mathbb{V}^2)^\perp/\mathbb{V}^2)$ is equivariant for the respective P-actions and the conformal class on $L(\mathbb{V}^2, (\mathbb{V}^2)^\perp/\mathbb{V}^2)$ is P-invariant. Thus also the conformal class on $\mathfrak{g}/\mathfrak{p}$ is P-invariant.

We have seen that for a parabolic geometry $(\mathcal{G}, \omega)$ of type (G, P), the Cartan connection provides an isomorphism

$$TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}.$$

Hence, parabolic geometries of type (G, P) induce (pseudo) conformal structures of signature $(2,3)$ on the underlying manifolds. Indeed, assuming a normalisation condition on the Cartan connection, this even provides an equivalence of categories, see Example 1.21.

The homogeneous space G/P is the model space for conformal geometry of signature $(2,3)$. It has the following description: Let $\langle , \rangle$ be the invariant inner product of signature $(3,4)$ on the standard representation. Let $\mathcal{C}$ be the nullcone for that inner product and consider the projectivisation of the cone, i.e. the space $\mathcal{P}(\mathcal{C})$ of lines in $\mathcal{C}$. One verifies that the group $SO(3,4)$ acts transitively on $\mathcal{P}(\mathcal{C})$. The group P is the isotropy group of the real line through a highest weight vector in $\mathbb{V}$ (which is a null vector, hence inside $\mathcal{C}$). Thus, the G-action on $\mathcal{P}(\mathcal{C})$ leads to a diffeomorphism $G/P \cong \mathcal{P}(\mathcal{C})$. The conformal structure on $\mathcal{P}(\mathcal{C})$ can also be understood as follows: for any line $\ell \in \mathcal{P}(\mathcal{C})$, the tangent space $T_\ell \mathcal{P}(\mathcal{C})$ can be identified with $\ell^\perp/\ell$. Note that the identification depends on the choice of a vector X spanning ℓ by multiplication with a nonzero real number. The inner product $\langle , \rangle$ induces an inner product on $\ell^\perp/\ell$ and thus gives rise to a well defined conformal class of inner products on $T_\ell\mathcal{P}(\mathcal{C})$. Furthermore, the action of $SO(3,4)$ identifies this group with the group of conformal isometries of $\mathcal{P}(\mathcal{C})$.

1.3. The equivalence to underlying structures

Any parabolic geometry gives rise to an underlying geometric structure, which consists of a filtered manifold and, sometimes, some additional data. Conversely, there are general results on the construction of canonical Cartan connections for these underlying structures.

1.3.1. Filtered manifolds. A *filtered manifold* is a smooth manifold M together with a filtration of the tangent bundle by smooth subbundles

$$T^{-1}M \subset T^{-2}M \cdots \subset T^{-k}M = TM,$$

which is compatible with taking Lie brackets of vector fields, meaning that if $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$ then $[\xi, \eta] \in \Gamma(T^{i+j}M)$. For any filtered manifold, the Lie bracket of vector fields induces a tensorial mapping on the associated graded vector bundle: Let $q_i : T^i \to \mathrm{gr}_i(TM)$ be the natural quotient map and consider the operator $\Gamma(T^iM) \times \Gamma(T^jM) \to \mathrm{gr}_{i+j}(TM)$ defined by $(\xi, \eta) \mapsto q_{i+j}([\xi, \eta])$. We have

$$[f\xi, g\eta] = f(\xi \cdot g)\eta - g(\eta \cdot f)\xi + fg[\xi, \eta],$$

for smooth functions f and g. Since $f(\xi \cdot g)\eta$ and $g(\eta \cdot f)\xi$ are contained in the kernel $T^{i+j+1}M$ of q_{i+j}, we see that the map q_{i+j} is bilinear over smooth functions. Moreover, for $\xi \in \Gamma(T^{i+1}M)$ and $\eta \in \Gamma(T^jM)$, the bracket $[\xi, \eta]$ is contained in the kernel $T^{i+j+1}M$. It follows that the map $\Gamma(T^iM) \times \Gamma(T^jM) \to \mathrm{gr}_{i+j}(TM)$ factors to a bundle map $\mathrm{gr}_i(TM) \times \mathrm{gr}_j(TM) \to \mathrm{gr}_{i+j}(TM)$.

DEFINITION 1.11. The bundle map

$$\mathcal{L} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM)$$

induced by the Lie bracket of vector fields is called *Levi bracket*. For every $x \in M$, the space $\mathrm{gr}_x(TM)$ endowed with $\mathcal{L}_x$ is a nilpotent Lie algebra. It is a called the *symbol algebra* of the filtered manifold at the point x.

Suppose M and $\tilde{M}$ are filtered manifolds such that the filtration components have the same rank for both manifolds. A morphism $f : M \to \tilde{M}$ of filtered manifolds is a local diffeomorphism such that the tangent map Tf preserves the filtrations. It is easy to see that T_xf induces an isomorphism of the corresponding symbol algebras in every point $x \in M$. In particular, symbol algebras are basic invariants for filtered manifolds.

Note that in general, the symbol algebras of a filtered manifold at different points may be different and the bundle $\mathrm{gr}(TM)$ is not locally trivial as a bundle of Lie algebras. Let us suppose $\mathrm{gr}(TM)$ is a locally trivial bundle of Lie algebras modelled on a fixed graded Lie algebra $\mathfrak{a}$. In that case, we can form the natural frame bundle $\mathcal{P}$ of $\mathrm{gr}(TM)$. By definition, the fibre $\mathcal{P}_x$ is the set of all linear isomorphisms $\phi : \mathfrak{a} \to \mathrm{gr}_x(TM)$ compatible with the gradings such that $\phi([X, Y]) = \mathcal{L}_x(\phi(X), \phi(Y))$ for all $X, Y \in \mathfrak{a}$. The structure group of $\mathcal{P}$ is the group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{a})$ of automorphisms of the Lie algebra $\mathfrak{a}$ preserving the grading.

1.3.2. Underlying structures. Suppose $(\mathcal{G} \to M, \omega)$ is a parabolic geometry of type (G, P). Let G_0 and P_+ be the subgroups of G determined by the $|k|$-grading of $\mathfrak{g}$. We explain how to obtain an underlying geometric structure from the parabolic geometry.

To begin with, we have a free P_+-action on $\mathcal{G}$ and we can form the orbit space $\mathcal{G}_0 := \mathcal{G}/P_+$. The projection $p : \mathcal{G} \to M$ factors to a projection $p_0 : \mathcal{G}_0 \to M$ and thus we obtain a smooth principal bundle with structure group $G_0 \cong P/P_+$. We have seen that for any Cartan geometry, the tangent bundle can be identified with the associated bundle $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, where the P-action on $\mathfrak{g}/\mathfrak{p}$ is induced by the adjoint action. The filtration of $\mathfrak{g}$ induces a filtration $\mathfrak{g}^{-1}/\mathfrak{p} \subset \mathfrak{g}^{-2}/\mathfrak{p} \subset \cdots \subset \mathfrak{g}^{-k}/\mathfrak{p} = \mathfrak{g}/\mathfrak{p}$ which is preserved by the P-action. Hence, it gives rise to a filtration

$$T^{-1}M \subset T^{-2}M \cdots \subset T^{-k}M = TM$$

of the tangent bundle by smooth subbundles. Next we can form the associated graded vector bundle, that is

$$\mathrm{gr}(TM) = \mathrm{gr}_{-k}(TM) \oplus \cdots \oplus \mathrm{gr}_{-1}(TM),$$

where $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M$. By construction, $\mathrm{gr}_i(TM)$ is an associated bundle for $\mathcal{G}$ with respect to the P-action on $\mathfrak{g}^i/\mathfrak{g}^{i+1}$. Since the subgroup P_+ acts trivially on $\mathfrak{g}^i/\mathfrak{g}^{i+1}$, the P-action can be viewed as the G_0-action trivially extended to P. Moreover, $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ is isomorphic to $\mathfrak{g}_i$ as a G_0-representation. Hence, $\mathrm{gr}(TM)$ can be identified with the associated bundle

$$\mathrm{gr}(TM) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-.$$

This implies, that the Lie bracket on $\mathfrak{g}_-$ gives rise to a bundle map

$$\{,\} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM),$$

called *algebraic bracket*. The bundle $\mathrm{gr}(TM)$ endowed with $\{,\}$ is a locally trivial bundle of Lie algebras modelled on $\mathfrak{g}_-$.

Definition 1.12. A parabolic geometry $(\mathcal{G} \to M, \omega)$ is called *regular* if the filtration of the tangent bundle obtained via the isomorphism $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ makes the manifold M into a filtered manifold such that the bundle maps $\{,\}$ and $\mathcal{L}$ on the associated graded vector bundle $\mathrm{gr}(TM)$ coincide.

Regularity can be characterised in terms of the curvature of the Cartan connection, see [**16**]:

Proposition 1.13. *A parabolic geometry is regular if and only if the curvature function satisfies* $\kappa(u)(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ *for all* $i, j < 0$ *and* $u \in \mathcal{G}$.

Given a regular parabolic geometry $(\mathcal{G}, \omega)$, we can consider the natural frame bundle $\mathcal{P}$ for $\mathrm{gr}(TM)$. Note that the adjoint action defines a homomorphism $G_0 \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. Suppose no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$. Then this homomorphism is infinitesimally injective, compare with 1.2.2. Next we can assign to any element $u_0 \in \mathcal{G}_0$ with $p_0(u_0) = x$ the map $\mathfrak{g}_- \to \mathrm{gr}_x(TM)$ defined by $X \mapsto [(u_0, X)]$. Thus we obtain a principal bundle morphism $\mathcal{G}_0 \to \mathcal{P}$. This shows that the bundle $\mathcal{G}_0$ is a reduction of structure group of the frame bundle $\mathcal{P}$ with respect to $G_0 \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$.

Summarising, a parabolic geometry determines the following underlying structure:

- a filtered manifold such that the associated graded vector bundle together with the Levi bracket is a bundle of Lie algebras with fibre $\mathfrak{g}_-$ and
- a reduction of structure group of the natural frame bundle of the graded vector bundle $\mathrm{gr}(TM)$ with respect to $G_0 \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$.

These data are called a *regular infinitesimal flag structure.*

Remark 1.14. An infinitesimal flag structure can be equivalently characterised via certain partially defined differential forms on the bundle $\mathcal{G}_0$, see [**15**].

Remark 1.15. We can consider the associated graded vector bundle $\mathrm{gr}(\mathcal{A}M)$ for the natural filtration $\mathcal{A}^k M \subset \cdots \mathcal{A}^0 M \subset \cdots \mathcal{A}^{-k} M = \mathcal{A}M$ of the adjoint tractor bundle, cf. 1.2.5. Then, $\mathrm{gr}(\mathcal{A}M) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}$, we have $\mathrm{gr}_i(\mathcal{A}M) = \mathrm{gr}_i(TM)$ for $i < 0$ and $\mathrm{gr}_i(\mathcal{A}M) = \mathrm{gr}_i(T^*M) = (\mathrm{gr}_{-i}(TM))^*$ for $i > 0$. If no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$, then the component $\mathfrak{g}_0$ can be viewed as a subalgebra in $\mathfrak{gl}(\mathfrak{g}_-)$. Consequently, the bundle $\mathrm{gr}_0(\mathcal{A}M) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_0$ can be regarded as a subbundle in the bundle of endomorphisms of $\mathrm{gr}(TM)$.

1.3.3. The Kostant codifferential, Lie algebra cohomology and a normalisation condition. A parabolic geometry is not uniquely determined by its underlying structure. For a fixed regular infinitesimal flag structure, there are many possible choices of regular Cartan connections inducing it. To obtain uniqueness, a normality condition on the Cartan connection is required. For parabolic geometries there is a conceptual choice of normalisation condition. In order to formulate that condition, we need to introduce the Kostant codifferential, see [**24**]. (For later use we will introduce the Kostant codifferential in more generality than necessary for the purpose of defining normality, i.e. we consider arbitrary representations, instead of restricting to the case of the adjoint representation right from the start.)

Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra and let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\mathbb{V}$ is a finite dimensional G-representation. Consider the standard complex for computing the Lie algebra cohomology $H^*(\mathfrak{g}_-, \mathbb{V})$. By definition, $C^n(\mathfrak{g}_-, \mathbb{V}) = \Lambda^n(\mathfrak{g}_-)^* \otimes \mathbb{V}$ and the differential

$$\partial : C^n(\mathfrak{g}_-, \mathbb{V}) \to C^{n+1}(\mathfrak{g}_-, \mathbb{V})$$

is given by

$$\begin{aligned}\partial\phi(X_0, \cdots, X_n) := &\sum_{i=0}^{n} (-1)^i X_i \cdot \phi(X_0, \cdots, \hat{X}_i, \cdots, X_n)\\ &+ \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], X_0, \cdots \hat{X}_i \cdots \hat{X}_j \cdots, X_n),\end{aligned}$$

where $\cdot$ denotes the infinitesimal action of $\mathfrak{g}$ on $\mathbb{V}$ and the hat over an argument denotes omission. The spaces $\Lambda^n(\mathfrak{g}_-)^* \otimes \mathbb{V}$ are naturally G_0-modules, the differential ∂ is G_0-equivariant and so also the cohomology spaces

$$H^n(\mathfrak{g}_-, \mathbb{V}) = \frac{\ker(\partial : \Lambda^n(\mathfrak{g}_-)^* \otimes \mathbb{V} \to \Lambda^{n+1}(\mathfrak{g}_-)^* \otimes \mathbb{V})}{\mathrm{im}(\partial : \Lambda^{n-1}(\mathfrak{g}_-)^* \otimes \mathbb{V} \to \Lambda^n(\mathfrak{g}_-)^* \otimes \mathbb{V}))}$$

are naturally G_0-modules.

We can also consider the complex computing the cohomology $H^*(\mathfrak{p}_+, \mathbb{V}^*)$. We denote the corresponding differential $\partial_{\mathfrak{p}_+} : C^n(\mathfrak{p}_+, \mathbb{V}^*) \to C^{n+1}(\mathfrak{p}_+, \mathbb{V}^*)$. Recall that the Killing form induces an G_0-module isomorphism $\mathfrak{p}_+ \cong (\mathfrak{g}_-)^*$. Therefore, the spaces $C^n(\mathfrak{g}_-, \mathbb{V})$ and $C^n(\mathfrak{p}_+, \mathbb{V}^*)$ are dual G_0-modules and the differential $\partial_{\mathfrak{p}_+}$ dualises to a homomorphism

$$\partial^* : C^{n+1}(\mathfrak{g}_-, \mathbb{V}) \to C^n(\mathfrak{g}_-, \mathbb{V}),$$

satisfying $\partial^* \circ \partial^* = 0$. This homomorphism is called the *Kostant codifferential*. Dualising the formula for $\partial_{\mathfrak{p}_+}$, we obtain the following formula for the Kostant codifferential on decomposable elements $Z_0 \wedge \cdots \wedge Z_n \otimes v \in \Lambda^{n+1}\mathfrak{p}_+ \otimes \mathbb{V}$:

$$\begin{aligned} \partial^*(Z_0 \wedge \cdots \wedge Z_n \otimes v) = &\sum_{i=0}^{n} (-1)^{i+1} Z_0 \wedge \cdots \wedge \hat{Z}_i \wedge \cdots \wedge Z_n \otimes Z_i \cdot v \\ &+ \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_0 \wedge \cdots \wedge \hat{Z}_i \wedge \cdots \wedge \hat{Z}_j \wedge \cdots \wedge Z_n \otimes v. \end{aligned} \tag{1.2}$$

Due to a result by Kostant, ∂ and ∂^* are adjoint for an inner product on $C^*(\mathfrak{g}_-, \mathbb{V})$ and one obtains an algebraic Hodge-decomposition: Let $\Box := \partial \circ \partial^* = \partial^* \circ \partial$ be the *Kostant Laplacian*. Then

$$C^n(\mathfrak{g}_-, \mathbb{V}) = \mathrm{im}(\partial^*) \oplus \ker(\Box) \oplus \mathrm{im}(\partial)$$

is a decomposition into a direct sum of G_0-modules, with $\mathrm{im}(\partial^*) \oplus \ker(\Box) = \ker(\partial^*)$ and $\ker(\Box) \oplus \mathrm{im}(\partial) = \ker(\partial)$. As an immediate consequence of this result, we see that both $H^n(\mathfrak{g}_-, \mathbb{V}) = \ker(\partial)/\mathrm{im}(\partial)$ and $\ker(\partial^*)/\mathrm{im}(\partial^*)$ can be identified with $\ker(\Box)$ as G_0-modules.

Note that via the identification of $\mathfrak{g}_-$ with $\mathfrak{g}/\mathfrak{p}$, the chain spaces $C^n(\mathfrak{g}_-, \mathbb{V})$ are indeed P-modules. An important point about the Kostant codifferential is that unlike ∂, it is not only a G_0-module homomorphism, but it is a P-module homomorphism. Hence, also the quotient spaces $\ker(\partial^*)/\mathrm{im}(\partial^*)$ are naturally P-modules. Using the explicit formula for the Kostant codifferential, one can verify that the nilpotent subgroup P_+ maps $\ker(\partial^*)$ to $\mathrm{im}(\partial^*)$. Thus, the P-action on the quotient is given by extending the G_0-action trivially on P_+. Another consequence of the P-equivariance of ∂^* is the fact that for any parabolic geometry $(\mathcal{G} \to M, \omega)$ the Kostant codifferential gives rise to vector bundle maps

$$\partial^* : \Lambda^n T^*M \otimes \mathcal{V}M \to \Lambda^{n-1} T^*M \otimes \mathcal{V}M.$$

between natural bundles, where $\mathcal{V}M = \mathcal{G} \times_P \mathbb{V}$.

From now on, instead of an arbitrary representation $\mathbb{V}$ we will consider the adjoint representation $\mathfrak{g}$. Recall that the curvature function is a P-equivariant map $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. Hence, the following definitions make sense:

Definition 1.16. A parabolic geometry is called normal if and only if its curvature function has the property that $\partial^* \circ \kappa = 0$.

Suppose we are given a normal parabolic geometry, i.e. κ takes values in $\ker(\partial^*) \subset \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$.

Definition 1.17. Let $\pi_H : \ker(\partial^*) \to \ker(\partial^*)/\mathrm{im}(\partial^*)$ be the quotient map. Then the composition $\kappa_H = \pi_H \circ \kappa$ is called the harmonic curvature.

Since the curvature function κ is P-equivariant, it can be viewed as a section of the associated bundle $\Lambda^2 T^*M \otimes \mathcal{A}M$, where we use the notation introduced in 1.2.5. Normality means that the section κ is contained in the kernel of the bundle map $\partial^* : \Lambda^n T^*M \otimes \mathcal{A}M \to \Lambda^{n-1} T^*M \otimes \mathcal{A}M$. We have seen above the algebraic spaces $\ker(\partial^*)/\operatorname{im}(\partial^*)$ and $H^n(\mathfrak{g}_-, \mathfrak{g})$ are isomorphic as P-modules, where the P-actions are trivial extensions of the natural G_0-actions. This implies that the bundle $\ker(\partial^*)/\operatorname{im}(\partial^*)$ can be naturally viewed as the associated bundle $\mathcal{G}_0 \times_{G_0} H^n(\mathfrak{g}_-, \mathfrak{g})$. It follows that the harmonic curvature, being a section the quotient bundle, admits a direct interpretation in terms of underlying structures. On the other hand, the harmonic curvature is still a complete obstruction to local flatness. This is an immediate consequence of the following theorem which is an application of the Bianci identity, see e.g. [**16**].

THEOREM 1.18. *Let $(\mathcal{G} \to M, \omega)$ be a regular, normal parabolic geometry. Suppose the curvature satisfies $\kappa(u)(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+l}$ for some $l \geq 1$ and for all $u \in \mathcal{G}$. Let us denote by $\kappa_l(u)$ the homogeneous component of degree l of the bilinear map $\kappa(u)$, i.e. the component mapping $\mathfrak{g}_i \times \mathfrak{g}_j$ to $\mathfrak{g}_{i+j+l}$. Then, $\kappa_l(u)$ is contained in $\ker(\Box)$ and the isomorphism between $\ker(\Box)$ and $\ker(\partial^*)/\operatorname{im}(\partial^*)$ maps $\kappa_l(u)$ to the homogeneous component of degree l of $\kappa_H(u)$.*

It follows, that if $\kappa_H = 0$, then also $\kappa = 0$. Using more sophisticated methods, one can prove an even stronger result. Indeed, κ can be recovered from the harmonic curvature κ_H using the BGG splitting operator, see [**7**].

1.3.4. The equivalence of categories. We have seen that any regular parabolic geometry induces an underlying structure on the manifold. It is easy to see that a morphism of parabolic geometries descends to a morphism of underlying structures.

Conversely, given a filtered manifold such that $\operatorname{gr}(TM)$ with the Levi bracket is a bundle of Lie algebras modelled on $\mathfrak{g}_-$ and a reduction of structure group of the natural frame bundle for $\operatorname{gr}(TM)$ with respect to $G_0 \to \operatorname{Aut}_{\operatorname{gr}}(\mathfrak{g}_-)$, one can construct a regular parabolic geometry inducing the given data. Requiring that all components of homogeneity ≥ 1 of the cohomology space $H^1(\mathfrak{g}_-, \mathfrak{g})$ vanish and that the parabolic geometry be normal, one even obtains uniqueness (up to isomorphism). The construction is functorial. With morphisms being defined in the obvious way, it establishes an equivalence of categories. For a proof, we refer to [**15**] or [**16**].

In the sequel, we will use the following notation: we denote by $H^i(\mathfrak{g}_-, \mathfrak{g})_j$ the component of the cohomology space of homogeneity j and $H^i(\mathfrak{g}_-, \mathfrak{g})^j = \bigoplus_{k \geq j} H^1(\mathfrak{g}_-, \mathfrak{g})_k$. We can state the equivalence result as follows:

THEOREM 1.19. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra such that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ and such that $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = \{0\}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let G_0 and P be the subgroups determined by the grading. Then we have an equivalence of categories between normal, regular parabolic geometries of type (G, P) and regular infinitesimal flag structures of type (G, P).*

REMARK 1.20. The cohomology groups $H^1(\mathfrak{g}_-, \mathfrak{g})$ are to be discussed in 1.4.2. We remark that there are two series of simple graded Lie algebras with $H^1(\mathfrak{g}_-, \mathfrak{g})^1 \neq \{0\}$, cf. [**39**]. These correspond to projective structures and contact projective structures, respectively.

We now discuss cases when the underlying infinitesimal flag structure a parabolic geometry is equivalent to gets simpler. On the one hand, this is the case when the Lie algebra is $|1|$-graded. For $|1|$-graded Lie algebras the filtration of the tangent bundle becomes trivial. Hence, the underlying structure is simply a reduction of structure group of the frame bundle to the group $G_0 \subset GL(\mathfrak{g}_{-1})$.

EXAMPLE 1.21. Let M be a smooth manifold of dimension $n \geq 3$. A (pseudo) conformal structure on M is an equivalence class of pseudo Riemannian metrics, with two metrics g and $\hat{g}$ being equivalent if and only if $\hat{g} = fg$ for a positive smooth function f. That means, it is given by a smooth ray subbundle in S^2T^*M whose fibre at a point x consists of the values of all metrics in the conformal class in the point x. Equivalently, a conformal structure of signature (p, q) can be viewed a a reduction of structure group of the frame bundle for the manifold M to the structure group $CO(p, q)$.

In subsection 1.2.6, we have seen that the grading associated with $\times\!\!-\!\!\!-\!\!\circ\!\Rightarrow\!\circ$ is a $|1|$-grading. Let us consider the group $G = SO(3, 4)$ and let $G_0 \subset G$ be the subgroup determined by the grading. Then any element $g_0 \in G_0$ is of the form

$$\begin{pmatrix} a & & \\ & A & \\ & & a^{-1} \end{pmatrix},$$

where A is contained in $SO(2, 3)$. The adjoint action on $\mathfrak{g}_- \cong \mathbb{R}^5$ is given by $\mathrm{Ad}(g_0)(X) = a^{-1}A(X)$. Hence, it induces an isomorphism between G_0 and $CO(\mathfrak{g}_{-1}) \cong CO(2, 3)$, the conformal group of signature $(2, 3)$.

In general signature we consider $PO(p+1, q+1)$, $p+q \geq 3$, and the grading associated with $\times\!\!-\!\!\!-\!\!\circ\!-\cdots-\!\circ\!\Rightarrow\!\circ$. Thus, we obtain a $|1|$-grading and an isomorphism

$$G_0 \cong CO(p, q)$$

induced by the adjoint action. Hence, Theorem 1.19 implies that there is an equivalence of categories between conformal structures of signature (p, q) and normal parabolic geometries of type $(PO(p + 1, q + 1), P)$, where P is the parabolic subgroup determined by the grading.

The other types of simpler underlying structures - and these are the cases of interest to us - come from geometries, where the underlying $\mathcal{G}_0$ bundle is isomorphic to the frame bundle for $\mathrm{gr}(TM)$. In these cases, the regular, normal parabolic geometry is uniquely determined by the underlying filtration of the tangent bundle. We shall focus on these geometries for the rest of this chapter.

1.4. Parabolic geometries determined by the underlying filtrations

Suppose we have a regular parabolic geometry of type (G, P) with the property that the group G_0 is isomorphic to the group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. In that case, the underlying structure reduces to a filtered manifold, since a reduction of structure group of the frame bundle for $\mathrm{gr}(TM)$ to the group G_0 is just an bundle isomorphism. As an immediate consequence of Theorem 1.19 we obtain :

Corollary 1.22. *Let $\mathfrak{g}$ be a real $|k|$-graded semisimple Lie algebra and suppose that $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = \{0\}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let G_0 and P be the subgroups of G determined by the grading. If $G_0 \cong \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$, there is an equivalence of categories between regular, normal parabolic geometries of type (G, P) and filtered manifolds such that $\mathrm{gr}(TM)$ together with the Levi bracket is a bundle of Lie algebras modelled on $\mathfrak{g}_-$.*

1.4.1. A cohomological condition on the Lie algebra. For any group G with subgroup G_0 isomorphic to $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$, we have an isomorphism $\mathfrak{g}_0 \cong \mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$ on Lie algebra level. This property of a $|k|$-graded semisimple Lie algebra $\mathfrak{g}$ can be rephrased as a cohomological condition:

Proposition 1.23. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra and such that no simple ideal is contained in $\mathfrak{g}_0$. Then the adjoint action induces an isomorphism between $\mathfrak{g}_0$ and the space $\mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$ of Derivations on $\mathfrak{g}_-$ preserving the grading if and only if $H^1(\mathfrak{g}_-, \mathfrak{g})_0 = \{0\}$. .*

Proof. Let us look at the part of the complex one needs to compute the Lie algebra cohomology group $H^1(\mathfrak{g}_-, \mathfrak{g})$:

$$0 \to \mathfrak{g} \xrightarrow{\partial_0} \mathfrak{g}_-^* \otimes \mathfrak{g} \xrightarrow{\partial_1} \Lambda^2 \mathfrak{g}_-^* \otimes \mathfrak{g} \to \dots$$

Here ∂_0 is given by $\partial_0(X)(Y) = [Y, X] = -\mathrm{ad}(X)(Y)$ for $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}_-$ and we have $\partial_1(\phi)(X_0, X_1) = [X_0, \phi(X_1)] + [\phi(X_0), X_1] - \phi([X_0, X_1])$ for $\phi \in \mathfrak{g}_-^* \otimes \mathfrak{g}$ and $X_0, X_1 \in \mathfrak{g}_-$. Elements in $(\mathfrak{g}_-^* \otimes \mathfrak{g})$ of homogeneous degree zero are linear maps $\phi : \mathfrak{g}_- \to \mathfrak{g}_-$ that preserve the grading. If we assume that ϕ is a cocycle of homogeneous degree zero, i.e. $\partial_1(\phi) = 0$, then the formula for ∂_1 shows that ϕ is a derivation on $\mathfrak{g}_-$ that preserves the grading. Vanishing of the component of $H^1(\mathfrak{g}_-, \mathfrak{g})$ of homogeneous degree zero means that every cocycle is actually a coboundary, i.e. of the form $\phi(X) = \mathrm{ad}(A)(X)$ for some $A \in \mathfrak{g}_0$. The assumption that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ implies that the map $\mathrm{ad} : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_-)$ is injective. Thus, the result follows. □

Remark 1.24. The cohomology groups $H^1(\mathfrak{g}_-, \mathfrak{g})$ are easy to deal with. Thanks to Kostant's version of the Bott-Borel-Weil theorem, see Theorem 1.28, the problem whether for a given $|k|$-graded semisimple Lie algebras the component of homogeneity zero of $H^1(\mathfrak{g}_-, \mathfrak{g})$ vanishes or not is not difficult to solve. We will say more about that in 1.4.2.

Suppose that $\mathfrak{g}$ is a Lie algebra such that $\mathfrak{g}_0 \cong \mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$. Next we address the problem of finding a suitable choice of group G such that the subgroup G_0 indeed is isomorphic to the group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. We will see that we can always take $G = \mathrm{Aut}(\mathfrak{g})$.

The Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is $\mathrm{Der}(\mathfrak{g})$ and since we assume that $\mathfrak{g}$ is semisimple the Lie algebra of derivations of $\mathfrak{g}$ is isomorphic to $\mathfrak{g}$ itself. The adjoint action of an element $\Phi \in G$ is given by $\mathrm{Ad}(\Phi)(X) = \Phi(X)$ for any $X \in \mathfrak{g}$. Therefore, the subgroup G_0 equals the group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g})$ of automorphism on $\mathfrak{g}$ that preserve the grading. In the article [**32**] we prove the following result:

Proposition 1.25. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra and such that the adjoint action induces an isomorphism between $\mathfrak{g}_0$ and the space $\mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$ of Derivations on $\mathfrak{g}_-$ preserving the grading. Let G be the group $\mathrm{Aut}(\mathfrak{g})$. Then restriction induces an isomorphism $G_0 \cong \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$.*

Proof. Restricting an automorphism of the Lie algebra $\mathfrak{g}$ to the subalgebra $\mathfrak{g}_-$ gives a homomorphism $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}) \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. We show that it is indeed an isomorphism.

To show injectivity we need to verify that every element $\Phi \in \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g})$ is uniquely determined by its restriction to $\mathfrak{g}_-$. The Killing form is nondegenerate on $\mathfrak{g}_i \times \mathfrak{g}_{-i}$ and thus induces an isomorphism $\mathfrak{g}_i \cong (\mathfrak{g}_{-i})^*$. By invariance of the Killing form under automorphisms, the restriction of Φ to $\mathfrak{g}_i$ corresponds via this isomorphism to the restriction of Φ^{-1} to $\mathfrak{g}_{-i}$. Since Φ is an automorphism we have $\mathrm{ad}(\Phi(A)) = \Phi \circ \mathrm{ad}(A) \circ \Phi^{-1}$ and by assumption $\mathrm{ad} : \mathfrak{g}_0 \to \mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$ is an isomorphism. Hence, the restriction of Φ to $\mathfrak{g}_0$ is uniquely determined by its restriction to $\mathfrak{g}_-$ as well.

It remains to prove surjectivity. Given $\Phi \in \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$ we show that it can be extended to $\tilde{\Phi} \in \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g})$. We define $\tilde{\Phi}(X) := \Phi(X)$ for $X \in \mathfrak{g}_-$. On $\mathfrak{p}_+$ we define $\tilde{\Phi}$ via the duality $\mathfrak{p}_+ \cong (\mathfrak{g}_-)^*$ given by the Killing form, i.e. for $U \in \mathfrak{p}_+$ we have

$$B(\tilde{\Phi}(U), X) = B(U, \Phi^{-1}(X))$$

for all $X \in \mathfrak{g}_-$. Finally, we use that $\mathrm{ad} : \mathfrak{g}_0 \to \mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$ is an isomorphism to define $\tilde{\Phi}$ on $\mathfrak{g}_0$: Note that for any $A \in \mathfrak{g}_0$, the map $\Phi \circ \mathrm{ad}(A) \circ \Phi^{-1}$ is an element of $\mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$ and by definition $\tilde{\Phi}(A)$ is the unique element in $\mathfrak{g}_0$ such that

$$\mathrm{ad}(\tilde{\Phi}(A)) = \Phi \circ \mathrm{ad}(A) \circ \Phi^{-1}.$$

The map $\tilde{\Phi}$ is a linear isomorphism on $\mathfrak{g}$ and preserves the grading. To show that it is contained in $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g})$ we need to verify that it is a Lie algebra homomorphism. This is certainly true for its restriction to $\mathfrak{g}_-$ since it coincides with Φ there. For $A \in \mathfrak{g}_0$ and $X \in \mathfrak{g}_-$ we have

$$[\tilde{\Phi}(A), \tilde{\Phi}(X)] = \mathrm{ad}(\tilde{\Phi}(A))(\tilde{\Phi}(X)) = \Phi \circ \mathrm{ad}(A) \circ \Phi^{-1} \circ \Phi(X) = \tilde{\Phi}([A, X]).$$

For $A, B \in \mathfrak{g}_0$ we make the following computation

$$\begin{aligned} \mathrm{ad}([\tilde{\Phi}(A), \tilde{\Phi}(B)]) &= [\mathrm{ad}(\tilde{\Phi}(A)), \mathrm{ad}(\tilde{\Phi}(B))] = \\ = [\Phi \circ \mathrm{ad}(A) \circ \Phi^{-1}, \Phi \circ \mathrm{ad}(B) \circ \Phi^{-1}] &= \Phi \circ [\mathrm{ad}(A), \mathrm{ad}(B)] \circ \Phi^{-1} = \\ = \Phi \circ \mathrm{ad}([A, B]) \circ \Phi^{-1} &= \mathrm{ad}(\tilde{\Phi}([A, B])). \end{aligned} \tag{1.3}$$

Hence $[\tilde{\Phi}(A), \tilde{\Phi}(B)] = \tilde{\Phi}([A, B])$. Next we take $X \in \mathfrak{g}_{-1}$, $U \in \mathfrak{g}_i$, $i \geq 2$ and $Y \in \mathfrak{g}_{1-i}$. Since Φ is a Lie algebra homomorphism on $\mathfrak{g}_-$ we have $[X, \Phi^{-1}(Y)] = \Phi^{-1}([\tilde{\Phi}(X), Y])$. Using the definition of $\tilde{\Phi}$ on $\mathfrak{p}_+$ and invariance of the Killing form we can compute

$$\begin{aligned} B(\tilde{\Phi}([U, X]), Y) &= B([U, X], \tilde{\Phi}^{-1}(Y)) = B(U, [X, \tilde{\Phi}^{-1}(Y)]) \\ &= B(\tilde{\Phi}(U), [\tilde{\Phi}(X), Y]) = B([\tilde{\Phi}(U), \tilde{\Phi}(X)], Y). \end{aligned} \tag{1.4}$$

It follows that $[\tilde{\Phi}(U), \tilde{\Phi}(X)] = \tilde{\Phi}([U, X])$ as well. For the Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0$ we apply a similar argument. We take $X \in \mathfrak{g}_{-1}$, $U \in \mathfrak{g}_1$ and $A \in \mathfrak{g}_0$. Then we make the same calculation as in the previous case, only that this time we need to verify that $B(\tilde{\Phi}([U, X]), A)$ equals $B([U, X], \tilde{\Phi}^{-1}(A))$:

$$\begin{aligned} B(\tilde{\Phi}([U, X]), A) &= \mathrm{Tr}(\mathrm{ad}(\tilde{\Phi}([U, X])) \circ \mathrm{ad}(A)) \\ &= \mathrm{Tr}(\Phi \circ \mathrm{ad}([U, X]) \circ \Phi^{-1} \circ \mathrm{ad}(A)) \\ &= \mathrm{Tr}(\mathrm{ad}([U, X]) \circ \Phi^{-1} \circ \mathrm{ad}(A) \circ \Phi) \\ &= \mathrm{Tr}(\mathrm{ad}([U, X]) \circ \mathrm{ad}(\tilde{\Phi}^{-1}(A))) \\ &= B([U, X], \tilde{\Phi}^{-1}(A)). \end{aligned}$$

Thus, $\tilde{\Phi}$ is a Lie algebra homomorphism with respect to all brackets $\mathfrak{g}_{-1} \times \mathfrak{g} \to \mathfrak{g}$. Since $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$, the case $\mathfrak{g}_- \times \mathfrak{g} \to \mathfrak{g}$ follows by induction: Suppose we already know $[\tilde{\Phi}(X), \tilde{\Phi}(B)] = \tilde{\Phi}([X, B])$ for all $X \in \mathfrak{g}_i$, $-l < i < 0$, and $B \in \mathfrak{g}$. Consider an element $X = [Y, Z] \in \mathfrak{g}_{-l}$, where $Y \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_{-l+1}$, then

$$\tilde{\Phi}([[Y, Z], B]) = \tilde{\Phi}(-[[B, Y], Z] - [[Z, B], Y]) = [\tilde{\Phi}([Y, Z]), \tilde{\Phi}(B)].$$

Since $\mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$ is dealt with in (1.3), the only remaining case is $\mathfrak{p}_+ \times \mathfrak{p} \to \mathfrak{p}_+$. Take $U \in \mathfrak{g}_i$, $V \in \mathfrak{g}_j$, $i > 0$, $j \geq 0$ and $X \in \mathfrak{g}_{-i-j}$. A computation similar to (1.4) shows that

$$B(\tilde{\Phi}([U, V]), X) = B([\tilde{\Phi}(U), \tilde{\Phi}(V)], X).$$

So $\tilde{\Phi}$ is a Lie algebra homomorphism with respect to these brackets as well.

□

Summarising, we get the following corollary:

Corollary 1.26. *Let $\mathfrak{g}$ be a real $|k|$-graded semisimple Lie algebra such that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ and $H^1(\mathfrak{g}_-, \mathfrak{g})^0 = \{0\}$. Then, for $G = \mathrm{Aut}(\mathfrak{g})$ the subgroup G_0 is isomorphic to $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. Consequently, we have an equivalence of categories between regular, normal parabolic geometries of type (G, P) and filtered manifolds such that the associated graded together with the Levi bracket is a bundle of Lie algebras modelled on $\mathfrak{g}_-$.*

1.4.2. Kostant's theorem. Kostant's version of the Bott-Borel-Weil theorem provides a complete description of the $\mathfrak{g}_0$-module structure of the cohomology spaces $H^i(\mathfrak{g}_-, \mathfrak{g})$ and more generally $H^i(\mathfrak{g}_-, \mathbb{V})$. It even leads to an algorithm for computing these cohomologies. A computer program designed to compute cohomology spaces $H^i(\mathfrak{g}_-, \mathbb{V})$ can be found in [**35**].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{p}$ be a standard parabolic subalgebra for a given choice of Cartan subalgebra $\mathfrak{h}$ and set of simple roots Δ^0. Let $\mathfrak{g}_0$ be the grading component of degree zero. This is the (reductive) Levi subalgebra of $\mathfrak{p}$. It decomposes into a direct sum $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$ of center and semisimple part. Correspondingly, we have a decomposition of the Cartan subalgebra $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}'$ into a direct sum of the center $\mathfrak{z}(\mathfrak{g}_0)$ and a Cartan subalgebra $\mathfrak{h}'$ of the semisimple part $\mathfrak{g}_0^{ss}$. Likewise, the dual space $\mathfrak{h}^*$ decomposes. Note that a weight for $\mathfrak{g}$ is also weight for the reductive subalgebra $\mathfrak{g}_0$.

Let $W_{\mathfrak{g}}$ be the Weyl group of $\mathfrak{g}$. Then, it is not difficult to see that the Weyl group $W_{\mathfrak{p}}$ of the semisimple part $\mathfrak{g}_0^{ss}$ of the Levi subalgebra can be viewed as a subgroup of $W_{\mathfrak{g}}$. For a Weyl group element w, we denote $\Phi_w \subset \Delta^+$ the set of positive roots which are mapped to negative roots by w^{-1}. One can show that $W_{\mathfrak{p}}$ can indeed be characterised as the set of elements $w \in W$ with the property that Φ_w is contained in the set $\Delta^+(\mathfrak{g}_0)$ of positive roots with corresponding root spaces sitting inside $\mathfrak{g}_0$. This motivates the definition:

Definition 1.27. The *Hasse diagram* of $\mathfrak{p}$ is the set of all Weyl group elements w such that Φ_w is contained in the set $\Delta^+(\mathfrak{p}_+)$ of positive roots with corresponding root spaces sitting inside $\mathfrak{p}_+$.

Alternatively, the Hasse diagram $W^{\mathfrak{p}}$ can be defined to be the set of representatives of the right coset space $W_{\mathfrak{g}} \setminus W_{\mathfrak{p}}$ of minimal length. There is a quite simple algorithm to determine its elements. It is based on the following result: Suppose $\mathfrak{p}$ corresponds to the set Σ of simple roots and let $\delta^{\mathfrak{p}}$ be the sum of all fundamental weights corresponding to elements of Σ. Then the map $w \mapsto w^{-1}\delta^{\mathfrak{p}}$ gives a bijection between $W^{\mathfrak{p}}$ and the Weyl group orbit of $\delta^{\mathfrak{p}}$. See [**16**] for more information.

Using the usual notation ρ for the sum of all fundamental weights, we can now state Kostant's version of the Bott-Borel-Weil theorem, see [**24**], as follows:

THEOREM 1.28. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{p}$ a standard parabolic subalgebra. Let $\mathbb{V}$ be a finite-dimensional irreducible $\mathfrak{g}$-representation with highest weight λ and consider the natural $\mathfrak{g}_0$-representation on $H^*(\mathfrak{p}_+, \mathbb{V})$.*

Then the set of irreducible components of that representation is in bijective correspondence with the Hasse diagram $W^{\mathfrak{p}}$. The irreducible component corresponding to a Weyl group element $w \in W^{\mathfrak{p}}$ has highest weight $w \cdot \lambda$, where $\cdot$ denotes the affine action $w \cdot \lambda = w(\lambda + \rho) - \rho$. A highest weight vector is given by the cohomology class of $F_w \otimes v$. Here v is a nonzero weight vector of weight $w(\lambda)$ and F_w is the wedge product of nonzero elements $X_\alpha \in \mathfrak{g}_{-\alpha}$ for each $\alpha \in \Phi_w$. Denoting by $l(w)$ the length of w, the irreducible component corresponding w is contained in $H^{l(w)}(\mathfrak{p}_+, \mathbb{V})$.

For the time being, we are only interested in the case $\mathbb{V} = \mathfrak{g}$. We are concerned with cohomology spaces $H^i(\mathfrak{g}_-, \mathfrak{g})$ rather than $H^i(\mathfrak{p}_+, \mathfrak{g})$, but this is no problem, since we have a $\mathfrak{g}_0$-module homomorphism

$$H^i(\mathfrak{g}_-, \mathfrak{g}) \cong \left(H^i(\mathfrak{p}_+, \mathfrak{g})\right)^* .$$

Using this and Theorem 1.28, one easily obtains the following result:

PROPOSITION 1.29. *Let $\mathfrak{g}$ be a $|k|$-graded complex semisimple Lie algebra, the grading given by the Σ-height, where Σ is a subset of simple roots. Suppose $k \geq 2$ and each $\alpha \in \Sigma$ is orthogonal to the maximal root. Then the cohomology group $H^1(\mathfrak{g}_-, \mathfrak{g})$ is contained in negative homogeneous degrees.*

PROOF. The elements of the Hasse diagram of length one are the reflections s_α for simple roots $\alpha \in \Sigma$. Hence, Kostant's theorem says that irreducible components of $H^1(\mathfrak{p}_+, \mathfrak{g})$ are in bijective correspondence with reflections s_α, $\alpha \in \Sigma$. A highest weight vector for the irreducible component corresponding to a reflection s_α is given by the cohomology class of $F \otimes s_\alpha(A)$, where F is contained in $\mathfrak{g}_{-\alpha}$ and A is contained in the root space corresponding to the maximal root β. Since $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}_{-1}$, the homogeneous degree of such a weight vector is given by $l - 1$, the integer l denoting the Σ-height of $s_\alpha(\beta)$. By assumption, α is orthogonal to β which implies that $s_\alpha(\beta) = \beta$. Hence, the highest weight vector has homogeneous degree $k - 1$. Thus, $H^1(\mathfrak{p}_+, \mathfrak{g})$ is contained in degree $k - 1$ and consequently $H^1(\mathfrak{g}_-, \mathfrak{g}) \cong (H^1(\mathfrak{p}_+, \mathfrak{g}))^*$ is contained in homogeneous degree $1 - k$. In case k is greater or equal to 2, the degree $1 - k$ is negative and the result follows. □

Note that it is also no problem that we will generally deal with real $|k|$-graded Lie algebras, rather than complex ones. We can use the proposition for real Lie algebras as well since

$$H^p_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}_-, \mathfrak{g}^{\mathbb{C}}) \cong H^p_{\mathbb{R}}(\mathfrak{g}_-, \mathfrak{g}) \otimes \mathbb{C}$$

and the isomorphism is compatible with the gradings. For an analogue of Kostant's version of the Bott-Borel-Weil theorem in the real case see [**36**].

EXAMPLE 1.30. Let us consider $\mathfrak{g} = \mathfrak{so}(n{+}1, n)$, $n \geq 3$, and the grading corresponding to $\Sigma = \{\alpha_n\}$. The corresponding Dynkin diagram is $\circ\!-\!\!-\!\circ\!-\cdots-\!\circ\!\Rightarrow\!\times$. It is known that the adjoint representation of $\mathfrak{so}(2n+1, \mathbb{C})$ is isomorphic to $\Lambda^2\mathbb{C}^{2n+1}$. Thus, the maximal root is the second fundamental weight, which is orthogonal to α_n, and we can apply the Proposition 1.29 to see that $H^1(\mathfrak{g}_-, \mathfrak{g})_l = \{0\}$, for all $l \geq 0$.

EXAMPLE 1.31. Next we consider the split real form of the exceptional Lie algebra $\mathfrak{g}_2$. Then there are two simple roots, a longer one α_1 and a shorter one α_2. The set of positive roots is given by

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}.$$

Let us consider the grading corresponding to the subset $\Sigma = \{\alpha_2\} \subset \Delta^0$. In Dynkin diagram notation this is $\circ\!\Rrightarrow\!\times$. In order to see that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is contained in negative homogeneous degree, we only need to verify that α_2 is orthogonal to the maximal root $2\alpha_1 + 3\alpha_2$: Since $2\frac{\langle\alpha_2,\alpha_1\rangle}{\langle\alpha_2,\alpha_2\rangle} = -3$, we see that $\langle\alpha_2, 3\alpha_2 + 2\alpha_1\rangle = 3\langle\alpha_2, \alpha_2\rangle + 2\langle\alpha_2, \alpha_1\rangle = 0$.

REMARK 1.32. Indeed most simple roots are orthogonal to the maximal root. There are two simple roots not orthogonal to the maximal root in case of the root system of $\mathfrak{sl}(n, \mathbb{C})$, $n \geq 3$, and there is only one such root in all of the other types of root systems of complex simple Lie algebras. Gradings corresponding to subsets Σ consisting of all roots not orthogonal to the maximal root are contact gradings, i.e. $|2|$-gradings such that the dimension of $\mathfrak{g}_{-2}$ is equal to one and the bracket $[,] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non-degenerate.

Note that Proposition 1.29 does not apply to $|1|$-gradings, nor does it apply to contact gradings by the above. Indeed, in these cases there actually is cohomology in non-negative degree. In the $|1|$-graded case the filtration of the tangent bundle is trivial, hence it cannot determine a parabolic geometry. Also contact distributions cannot determine parabolic geometries since it is well known that they have infinite-dimensional automorphism groups. Apart from these examples there are two more series of exceptions. Yamaguchi shows in [**39**] that for a complex simple $|k|$-graded Lie algebra, the Lie algebra cohomology space is contained inside negative homogeneous degrees unless

- the Lie algebra is $|1|$-graded,
- the grading is a contact grading
- the graded Lie algebra is isomorphic to $\mathfrak{sl}(n, \mathbb{C})$, $\Sigma = \{\alpha_1, \alpha_i\}$ $1 < i < n$, or $\mathfrak{sp}(n, \mathbb{C})$, $\Sigma = \{\alpha_1, \alpha_l\}$.

1.4.3. On distributions. Let $\mathcal{H} \subset TM$ be a distribution, i.e. a smooth vector subbundle of the tangent bundle. By Frobenius theorem, we know that a distribution is integrable if and only if for vector fields $\xi, \eta \in \Gamma(\mathcal{H})$ the Lie bracket $[\xi, \eta]$ is contained in $\Gamma(\mathcal{H})$ as well. For a non-integrable distribution we can consider the following flag of subspaces of T_xM at each point $x \in M$: We put $\mathcal{H}^1_x := \mathcal{H}_x$. By induction, we define the subspace $\mathcal{H}^j_x$ for $j > 1$ to be the the space generated by values of sections $\eta_i \in \Gamma(\mathcal{H}^{j-1})$ and values of Lie brackets $[\xi_i, \eta_i]$ for $\eta_i \in \Gamma(\mathcal{H}^{j-1})$ and $\xi_i \in \Gamma(\mathcal{H}^1)$ in the point x. Thus, we obtain a flag

$$\mathcal{H}^1_x \subset \mathcal{H}^2_x \subset \mathcal{H}^3_x \cdots .$$

A distribution is called *bracket generating* if for all x there exists an integer j such that $\mathcal{H}^j_x = T_xM$, in other words, if sections of the distribution and iterated Lie brackets of such sections generate all of TM. The sequence of integers $(n_1, n_2, n_3 \cdots)$, where n_i is the dimension of $\mathcal{H}^i_x$, is an invariant of the bracket generating distribution $\mathcal{H}$. It is called the (small) *growth vector* of the distribution at the point x. If all of the $\mathcal{H}^i$ are smooth subbundles of constant rank, then the filtration $\mathcal{H}^1 \subset \mathcal{H}^2 \subset \mathcal{H}^3 \subset \cdots \subset TM$ endows M with the structure of a filtered manifold. We can consider the symbol algebra, see Definition 1.11, of the filtered manifold generated by the distribution $\mathcal{H}$ and call it the symbol algebra of the distribution.

By definition of a $|k|$-grading, the subalgebra $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$ as a Lie algebra. So for a filtration

$$T^{-1}M \subset T^{-2}M \subset \cdots \subset T^{-k}M = TM$$

determined by a regular parabolic geometry, the symbol algebra $\mathrm{gr}_x(TM)$ is generated by the subbundle $T^{-1}_x M$ for all $x \in M$. That means, if we consider sections of $T^{-j}M$ and Lie brackets of the form $[\xi_i, \xi_j]$ for $\xi_i \in \Gamma(T^{-1}M)$ and $\xi_j \in \Gamma(T^{-j}M)$, this set of section spans $T^{-j-1}M$. In particular, $T^{-1}M$ is a bracket generating distribution with growth vector

$$(\dim(T^{-1}M), \dim(T^{-2}M), \cdots, \dim(TM))$$

and symbol algebra $\mathfrak{g}_-$ at each point.

For some $|k|$-graded semisimple Lie algebras, the condition that a distribution generates a filtration with prescribed symbol algebra $\mathfrak{g}_-$ defines an interesting class of distributions. The examples 1.30 and 1.31 of $|k|$-graded Lie algebras with cohomology spaces $H^1(\mathfrak{g}_-, \mathfrak{g})$ contained in negative homogeneous degrees were not chosen randomly. In both cases, distributions generating filtrations with symbol algebras $\mathfrak{g}_-$ are precisely the distributions (of rank 3 in dimension 6 and of rank 2 in dimension 5, respectively) with maximal possible growth vector at each point. In these cases, the condition on a distribution to generate a filtration with symbol algebra $\mathfrak{g}_-$ is a generic condition, i.e. a sufficiently small perturbation of any distribution (of the right rank in the right dimension) will satisfy this condition. Hence, Corollary 1.26 implies that the corresponding parabolic geometries are equivalent to generic distributions.

Example 1.33. We first consider bracket generating distributions with growth vector $(n, n(n+1)/2)$ for $n \geq 3$. This means that smooth sections of such a distribution $\mathcal{H}$ and Lie brackets of two such sections generate the tangent bundle TM. Hence, the Levi bracket

$$\mathcal{L} : \Lambda^2\mathcal{H} \to TM/\mathcal{H}$$

is surjective in every point. Note that the dimension are chosen in such a way that $\Lambda^2\mathcal{H}$ and $TM/\mathcal{H}$ both have rank $n(n-1)/2$. So for dimensional reasons, the Levi bracket is an isomorphism in every point. This condition is satisfied for a generic rank n distribution in dimension $n(n+1)/2$.

The above considerations can be rephrased as the observation that the symbol algebra for a distribution with growth vector $(n, n(n+1)/2)$ is in every point isomorphic to $\mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n$. Now it is a straightforward computation, to show that this graded Lie algebra is isomorphic to the negative part $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ of the Lie algebra corresponding to

$$\circ\!-\!\circ\!-\cdots-\!\circ\!\Rightarrow\!\times.$$

We have seen in Example 1.30 that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is contained in negative homogeneous degrees. Hence we can apply Corollary 1.26. Let G be $\mathrm{Aut}(\mathfrak{so}(n+1, n))$, which is isomorphic to the connected component of the identity in $SO(n+1, n)$ and let P be the subgroup determined by the grading. Then Corollary 1.26 shows that parabolic geometries of type (G, P) are equivalent to distributions $\mathcal{H} \subset TM$ with growth vector $(n, n(n+1)/2)$.

For $n = 3$, these distributions have been looked at in [**6**] and [**1**].

Example 1.34. The next class of generic distributions is the main subject of this thesis. Let $\mathcal{H} \subset TM$ be a rank two distribution in dimension five with maximal possible growth vector $(2, 3, 5)$ at every point. That means, the set of sections ξ_i of the distribution $\mathcal{H}$ and their single Lie brackets $[\xi_i, \xi_j]$ generates a rank three distribution $\mathcal{H}^1 = [\mathcal{H}, \mathcal{H}]$ and adding Lie brackets $[\xi_i, [\xi_j, \xi_k]]$, the span is the whole tangent bundle TM. A detailed discussion of these distributions can be found in the next chapter, see 2.3.1, but we want to mention the equivalence to certain parabolic geometries at this point already.

The symbol algebra of a distribution with growth vector $(2, 3, 5)$ is modelled on the graded Lie algebra $\mathbb{R}^2 \oplus \Lambda^2\mathbb{R}^2 \oplus \mathbb{R}^2$. Let $\mathfrak{g}$ denote the split real form of the complex exceptional Lie algebra. Consider the grading corresponding to $\circ\!\Rrightarrow\!\times$. Then it is easy to see from the root system that the negative part $\mathfrak{g}_-$ is as a graded Lie algebra isomorphic to $\mathbb{R}^2 \oplus \Lambda^2\mathbb{R}^2 \oplus \mathbb{R}^2$. We have already verified that $H^1(\mathfrak{g}_-, \mathfrak{g})$ contained in negative homogeneous degrees. Hence, defining G_2 to be $\mathrm{Aut}(\mathfrak{g})$ and P the parabolic subgroup determined by the grading, there is an equivalence of categories between regular, normal parabolic geometries of type (G_2, P) and distributions in dimension five with growth vector $(2, 3, 5)$.

Finally, we mention yet another example of a parabolic geometry equivalent to a distribution that has been of interest, recently.

Example 1.35. A *quaternionic Heisenberg algebra* of signature (p, q) is the space $\mathbb{H}^n \oplus \mathrm{Im}\mathbb{H}$ with a Lie bracket $[,] : \mathbb{H}^n \times \mathbb{H}^n \to \mathrm{Im}\mathbb{H}$ given by the imaginary part of a quaternionic hermitian form of signature (p, q). A corank three distribution $\mathcal{H} \subset TM$ in dimension $4n + 3$ such that $\mathcal{H} \oplus TM/\mathcal{H}$ endowed with the Levi bracket is a bundle of Lie algebras modelled on a quaternionic Heisenberg algebra is called a *quaternionic contact structure.* For signature $(n, 0)$, these are the structures that have been introduced by O. Biquard, see [**3**].

It turns out that if we consider $\mathfrak{g} = \mathfrak{sp}(p + 1, q + 1)$ with the grading corresponding to the second simple root, then $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ is as a graded Lie algebra isomorphic to the quaternionic Heisenberg algebra of signature (p, q). The adjoint representation of $\mathfrak{sp}(2n, \mathbb{C})$ is isomorphic to $S^2\mathbb{C}^{2n}$. Hence the maximal root is twice the first fundamental weight. Thus, $H^1(\mathfrak{g}_-, \mathfrak{g})_l = \{0\}$, for all $l \geq 0$, also in this case. It follows, that quaternionic contact structures of signature (p, q) are equivalent to regular normal parabolic geometries of type (G, P), where $G = \mathrm{Aut}(\mathfrak{g}) = PSp(p + 1, q + 1)$ and P is the parabolic subgroup determined by the grading.

Quaternionic contact structures appear as conformal infinities of quaternionic Kähler-metrics, see [**3**]. In dimension seven, there are two open orbits for the action of $\mathrm{GL}(4, \mathbb{R} \times \mathrm{GL}(3, \mathbb{R})$ on $L(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$ and correspondingly there are two types of generic rank four distributions, cf. [**13**]. One of these types are quaternionic contact structures.

CHAPTER 2

G_2 and generic rank two distributions in dimension five

The classification of irreducible abstract root systems shows that apart from the four families of classical root systems there are five exceptional ones. The smallest of these is the root system of type G_2. A simple system for this root system consists of two roots, a longer one α_1 and a shorter one α_2, and in terms of these, the set of positive roots is given by

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}.$$

The first realisation of a simple Lie algebra with root system of type G_2 was given in 1893 independently by the two mathematicians Elie Cartan and Friedrich Engel. Their first construction of that Lie algebra was related to a rank two distribution. They considered the distribution in $\mathbb{C}^5$ defined as the kernel of the one-forms

$$\begin{aligned}
\omega_1 &= dx_5 - x_4 dx_2\\
\omega_2 &= dx_3 - x_2 dx_1\\
\omega_3 &= dx_2 - x_4 dx_1
\end{aligned}$$

and claimed that the Lie algebra of vector fields whose flows preserve this rank two distribution is a Lie algebra of type G_2.

The real version of the above distribution is (locally) the flat model for generic rank two distributions in dimension five. Here we mean by a generic rank two distribution on a five manifold a distribution, such that sections ξ_i, ξ_j of the distribution and Lie brackets $[\xi_i, \xi_j]$ of two such sections span a rank three distribution and iterated brackets $[\xi_i, [\xi_j, \xi_k]]$ of at most three sections of the distribution span the entire tangent bundle. Cartan studied these distributions in his "five-variables paper" [**19**] in 1910, where he constructed for the first time what we now call a Cartan geometry of type (G_2, P).

The main part of this chapter is devoted to G_2, the parabolic subgroup P corresponding to the diagram $\circ\!\Rrightarrow\!\times$ and the homogeneous space G_2/P. We will derive an explicit realisation of the Lie algebra $\mathfrak{g}_2$ as a subalgebra in $\mathfrak{so}(3,4)$ to work with in the rest of this thesis. We explain how to recover the equivalence of parabolic geometries of type (G_2, P) to generic rank two distributions in dimension five from the results presented in the first chapter. Furthermore, we will work out some explicit descriptions of the homogeneous model.

Information on G_2 and the homogeneous model of generic rank two distributions in dimension five can also be found in [**4**], [**5**] and my article [**33**]. A lot of it can be traced back to the work of E. Cartan.

2.1. On split octonions and G_2

In this section we will collect some information on G_2 and its relation to the algebra of split octonions. Unless otherwise noted, we will always mean by G_2 the connected Lie group without center with Lie algebra the split real form of the simple exceptional complex Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$.

2.1.1. Split octonions. We start by introducing the notion of a composition algebra. The main reference for this section is [**37**].

DEFINITION 2.1. A *composition algebra* over $\mathbb{R}$ is a real algebra $\mathcal{A}$ with unit E such that there exists a non-degenerate quadratic form $N : \mathcal{A} \to \mathbb{R}$ which is multiplicative, i.e. satisfies $N(XY) = N(X)N(Y)$ for all $X, Y \in \mathcal{A}$.

The Norm N gives rise to a non-degenerate symmetric bilinear form $\langle X, Y\rangle = N(X+Y) - N(X) - N(Y)$. Given a composition algebra, there is a notion of conjugation, defined as the reflection with respect to the to the unit, i.e.

$$\overline{X} = 2\langle X, E\rangle E - X.$$

The bilinear form $\langle\ ,\ \rangle$ can be written in terms of conjugation as

$$\langle X, Y\rangle = \overline{X}Y + \overline{Y}X.$$

Composition algebras are not necessarily associative. However, for all composition algebras a weaker property holds. They are *alternative*, meaning that any subalgebra generated by two elements is associative, or equivalently, that the associator

$$\{X, Y, Z\} := (XY)Z - X(YZ)$$

is skew-symmetric.

Obviously, $\mathbb{R}$ is a composition algebra. It turns out that every real composition algebra can be constructed from $\mathbb{R}$ by repeatedly applying the so-called Cayley-Dickson construction, see [**37**]:

THEOREM 2.2. *Given an associative real composition algebra $\mathcal{A}$ we can define on $\mathcal{A}\times\mathcal{A}$ a product by*

$$(X, Y)(U, V) = (XU + \lambda\overline{V}Y, VX + Y\overline{U}),$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, and a quadratic form N by

$$N(X, Y) = N(X) - \lambda N(Y).$$

Then the new algebra is again a composition algebra.

Starting with $\mathbb{R}$ and choosing $\lambda = -1$ the Cayley-Dickson construction is just the usual way of constructing the complex numbers from the reals. Applying the construction to $\mathbb{C}$, again choosing $\lambda = -1$, one obtains the Hamiltonian quaternions $\mathbb{H}$ and finally applying the construction to $\mathbb{H}$, choosing $\lambda = -1$, one obtains the octonions $\mathbb{O}$. The octonions are not associative. If we keep applying the construction, we get algebras, but these are no longer composition algebras. Up to isomorphism the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only composition algebras with positive definite bilinear form. However in dimensions 2, 4 and 8, there is also a unique composition algebra (up to isomorphism) with indefinite bilinear form. Composition algebras with indefinite bilinear form are called split composition algebras and can be constructed via the Cayley-Dickson process, choosing $\lambda = 1$ instead of -1. The split complex numbers $\mathbb{C}_S$ turn out to be isomorphic to $\mathbb{R}\oplus\mathbb{R}$ with componentwise addition and multiplication and quadratic form $N((X,Y)) = XY$. A nice description of the split quaternions $\mathbb{H}_S$ is given as the algebra of 2×2-matrices with the determinant as quadratic form. A description of the split octonions is given below.

Definition 2.3. The algebra of *split octonions* is the unique 8-dimensional composition algebra over $\mathbb{R}$ with indefinite bilinear form.

An explicit realisation of the split octonions is given by the set of vector matrices $\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}$, where $x, y \in \mathbb{R}^3$ and $\xi, \eta \in \mathbb{R}$, with addition defined componentwise and multiplication defined by

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}\begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix} = \begin{pmatrix} \xi\xi' + \langle x, y'\rangle & \xi x' + \eta' x + y \wedge y' \\ \eta y' + \xi' y - x \wedge x' & \eta\eta' + \langle y, x'\rangle \end{pmatrix},$$

where $\langle\ ,\ \rangle$ denotes the Euclidean inner product on $\mathbb{R}^3$ and $\wedge$ the cross product. A quadratic form satisfying $N(XY) = N(X)N(Y)$ is given by

$$N(\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}) = \xi\eta - \langle x, y\rangle .$$

The corresponding bilinear form reads

$$\left\langle \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}, \begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix} \right\rangle = \xi'\eta + \eta'\xi - \langle x, y'\rangle - \langle x', y\rangle .$$

The unit is the element $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The orthogonal complement to the unit is the set of imaginary split octonions $\mathrm{Im}\mathbb{O}_S$, i.e. the set of all $X \in \mathbb{O}_S$ such that $\bar{X} = -X$. It is given by all vector matrices of the form $\begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix}$. Restricting the bilinear form on the split octonions to $\mathrm{Im}\mathbb{O}_S$ we obtain a bilinear form of signature $(3,4)$. Let us consider the basis $X_1 = \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 & e_3 \\ 0 & 0 \end{pmatrix}$, $X_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X_5 = \begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$, $X_6 = \begin{pmatrix} 0 & 0 \\ -e_2 & 0 \end{pmatrix}$ and $X_7 = \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix}$, e_1, e_2, e_3 denoting the standard basis of $\mathbb{R}^3$. With respect to that basis the bilinear form on $\mathrm{Im}\mathbb{O}_S$ is represented by the matrix

$$J = \begin{pmatrix} & & \mathbb{I}_3 \\ & -1 & \\ \mathbb{I}_3 & & \end{pmatrix},$$

where $\mathbb{I}_3$ denotes the 3×3 unit matrix.

2.1.2. Realising $\mathfrak{g}_2$ in $\mathfrak{so}(3,4)$. Let us denote by $\mathrm{Aut}(\mathbb{O}_S)$ the group of algebra automorphisms of the split octonions. There is a result stating that for any composition algebra, an algebra automorphism necessarily preserves the corresponding bilinear form on the algebra, see [**37**]. It follows that $\mathrm{Aut}(\mathbb{O}_S)$ preserves the bilinear form on $\mathbb{O}_S$. Since any algebra homomorphism maps the identity E onto itself, it leaves invariant the orthogonal complement $\mathrm{Im}\mathbb{O}_S$ to the identity. Restriction to $\mathrm{Im}\mathbb{O}_S$ leads to an injection of $\mathrm{Aut}(\mathbb{O}_S)$

into $SO(\mathrm{Im}\mathbb{O}_S)$) and we can naturally view $\mathrm{Aut}(\mathbb{O}_S)$ as a closed subgroup in $SO(3,4)$. The Lie algebra corresponding to the automorphism group $\mathrm{Aut}(\mathbb{O}_S)$, the algebra $\mathrm{Der}\mathbb{O}_S$ of derivations of the split octonions, can thus be viewed as a subalgebra in $\mathfrak{so}(3,4)$.

Writing $\mathfrak{so}(3,4)$ with respect to the basis $X_1, X_2, X_3, X_4, X_5, X_6, X_7$ from above, it consists of matrices of the following form

$$\mathfrak{so}(3,4) = \left\{ \begin{pmatrix} A & v & B \\ w & 0 & v^t \\ C & w^t & -A^t \end{pmatrix} : C = -C^t, B = -B^t \right\}.$$

Our aim is to determine those matrices in $\mathfrak{so}(3,4)$ that act as derivations on $\mathrm{Im}\mathbb{O}_S$. In order to do so note that an element

$$M = \begin{pmatrix} A & v & B \\ w & 0 & v^t \\ C & w^t & -A^t \end{pmatrix} \in \mathfrak{so}(3,4)$$

acts on $\mathrm{Im}\mathbb{O}_S$ by

$$M \cdot \begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(wx + v^t y) & Ax + \sqrt{2}v\xi + By \\ Cx + \sqrt{2}w\xi - A^t y & -\frac{1}{\sqrt{2}}(wx + v^t y) \end{pmatrix}.$$

Now consider the equation

$$\begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix}.$$

Assuming that M acts as a derivation, i.e.

$$M \cdot \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} M \cdot \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} = M \cdot \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix},$$

implies

$$(Ae_1) \wedge e_2 + e_1 \wedge (Ae_2) = -A^t e_3$$

which is equivalent to A being tracefree. Similarly, the equation

$$\begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -e_i \\ 0 & 0 \end{pmatrix}$$

leads to $Ce_i = \frac{1}{\sqrt{2}}(e_i \wedge v)$, and from

$$\begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}$$

we can deduce that $Be_i = \frac{1}{\sqrt{2}}(e_i \wedge w^t)$. Thus, we get a description of $\mathrm{Der}\mathbb{O}_S$ as matrices in $\mathfrak{so}(3,4)$ satisfying the following additional conditions:

$$A \in \mathfrak{sl}(3,\mathbb{R}), B = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}, C = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

For our purpose it will be convenient to work with a slightly different description. Let us consider a base change, let us use the ordered basis

$$\{X_1, X_6, X_7, X_4, X_2, X_3, X_5\}$$

rather than $\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$. Then the bilinear form on $\mathrm{Im}\mathbb{O}_S$ is represented by the matrix

$$\begin{pmatrix} & & & & 1 \\ & & & \mathbb{I}_2 & \\ & & -1 & & \\ & \mathbb{I}_2 & & & \\ 1 & & & & \end{pmatrix}.$$

The description of $\mathfrak{so}(3,4)$ with respect to this basis is the one used in 1.2.6 and the algebra of derivations of $\mathrm{Im}\mathbb{O}_S$ looks as follows

$$\left\{ \begin{pmatrix} \mathrm{tr}(A) & Z & s & W & 0 \\ X & A & \sqrt{2}\mathbb{J}Z^t & \frac{s}{\sqrt{2}}\mathbb{J} & -W^t \\ r & -\sqrt{2}X^t\mathbb{J} & 0 & -\sqrt{2}Z\mathbb{J} & s \\ Y & -\frac{r}{\sqrt{2}}\mathbb{J} & \sqrt{2}\mathbb{J}X & -A^t & -Z^t \\ 0 & -Y^t & r & -X^t & -\mathrm{tr}(A) \end{pmatrix} \right\} \tag{2.1}$$

with $A \in \mathfrak{gl}(2,\mathbb{R})$, $X, Y, Z, W \in \mathbb{R}^2$, $r, s \in \mathbb{R}$ and $\mathbb{J} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

We can easily see that this is a simple Lie algebra with root system of type G_2. Let us consider the root decomposition of $\mathfrak{so}(3,4)$ first. Choosing as a Cartan subalgebra the set of diagonal matrices, we obtained the set of (restricted) roots

$$\Delta = \{\pm(\phi_i - \phi_j), \pm(\phi_i + \phi_j), \pm\phi_i, i, j \in \{1,2,3\}, i \neq j\},$$

where ϕ_i denotes the linear functional extracting the ith diagonal entry, cf. 1.2.6.

Next we consider the subalgebra of diagonal matrices in $\mathrm{Der}\mathbb{O}_S \subset \mathfrak{so}(3,4)$. Via the adjoint action it acts diagonalisable on $\mathrm{Der}\mathbb{O}_S$ and we get the corresponding eigenspace decomposition. Once we have this eigenspace decomposition, it is not difficult to see that $\mathrm{Der}\mathbb{O}_S$ cannot contain any solvable ideals. Hence it is semisimple, the subalgebra of diagonal matrices is a Cartan subalgebra and we can write down the corresponding root system. As above we denote by ϕ_i the linear functional that extracts the ith diagonal entry. Note, that for diagonal matrices in $\mathrm{Der}\mathbb{O}_S$ the identity $\phi_1 = \phi_2 + \phi_3$ holds. Choosing as simple roots $\alpha_1 = \phi_2 - \phi_3$ and $\alpha_2 = \phi_3 = \phi_1 - \phi_2$ we obtain the following set of positive roots:

$$\begin{aligned} \Delta^+ = \{\, & \alpha_1, \alpha_2, \alpha_1 + \alpha_2 = \phi_2 = \phi_1 - \phi_3, \alpha_1 + 2\alpha_2 = \phi_1 = \phi_2 + \phi_3, \\ & \alpha_1 + 3\alpha_2 = \phi_1 + \phi_3, 2\alpha_1 + 3\alpha_2 = \phi_1 + \phi_2 \,\}. \end{aligned}$$

This is indeed a root system of type G_2.

2.1.3. Fundamental representations of $\mathfrak{g}_2$. The description (2.1) of $\mathfrak{g}_2$ as a subalgebra in $\mathfrak{so}(3,4)$ also enables us to understand the standard representation of $\mathfrak{g}_2 = \mathrm{Der}\mathbb{O}_S$ on $\mathbb{V} = \mathrm{Im}\mathbb{O}_S$ in terms of weights. It turns out that $X_1 = \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}$ is annihilated by all positive root spaces and thus a highest weight vector. Its weight $\phi_1 = \alpha_1 + 2\alpha_2$ is the fundamental weight λ_2 corresponding to the shorter simple root α_2. Other weight vectors and their weights are:

$$\begin{array}{ll} X_6 & \alpha_1+\alpha_2 \\ X_7 & \alpha_2 \\ X_4 & 0 \\ X_3 & -\alpha_2 \\ X_2 & -\alpha_1-\alpha_2 \\ X_5 & -\alpha_1-2\alpha_2. \end{array}$$

Another way to realise this fundamental representation is the following: By invariance of the Killing form, the orthogonal complement of $\mathfrak{g}_2$ in $\mathfrak{so}(3,4)$ with respect to the Killing form is a $\mathfrak{g}_2$-representation. Explicitly, this orthogonal complement is given by all matrices in $\mathfrak{so}(3,4)$ of the form

$$\left\{\begin{pmatrix} \lambda & Z & s & 0 & 0 \\ X & -\lambda\mathbb{I}_2 & -\frac{1}{\sqrt{2}}\mathbb{J}Z^t & -\sqrt{2}s\mathbb{J} & 0 \\ r & \frac{1}{\sqrt{2}}X^t\mathbb{J} & 0 & \frac{1}{\sqrt{2}}Z\mathbb{J} & s \\ 0 & \sqrt{2}r\mathbb{J} & -\frac{1}{\sqrt{2}}\mathbb{J}X & \lambda\mathbb{I}_2 & -Z^t \\ 0 & 0 & r & -X^t & -\lambda \end{pmatrix}\right\}$$

with $X, Z \in \mathbb{R}^2$, $r, s, \lambda \in \mathbb{R}$, $\mathbb{I}_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathbb{J} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This is a 7-dimensional representation, the weights occuring are $\pm\phi_i$, $i = 1, 2, 3$, and 0, so this is again the fundamental representation corresponding to the fundamental weight λ_2.

For the other fundamental representation, note that the fundamental weight λ_1 corresponding to α_1 is the maximal root $2\alpha_1 + 3\alpha_2$. Hence the corresponding fundamental representation is given by the adjoint representation of $\mathfrak{g}_2$.

2.1.4. The invariant 3-form. Note that we can define a 3-form $\phi \in \Lambda^3(\mathbb{V}^*)$ on $\mathbb{V} = \mathrm{Im}\mathbb{O}_S$ via

$$\phi(X, Y, Z) = \langle XY, Z\rangle.$$

Let the coordinate vector of $V \in \mathbb{V}$ with respect to the basis $\{X_1, X_6, X_7, X_4, X_2, X_3, X_5\}$ be

$$\begin{pmatrix} a \\ X \\ b \\ Y \\ c \end{pmatrix}.$$

Then the 3-form ϕ reads

$$\begin{aligned} \phi(V, V', V'') = \; & \tfrac{1}{\sqrt{2}}\left(ba'c'' - b'ac'' - bc'a'' + b'ca'' + ac'b'' - a'cb''\right) \\ & -c''(X^t\mathbb{J}X') - c(X'^t\mathbb{J}X'') + c'(X^t\mathbb{J}X'') \\ & +a''(Y^t\mathbb{J}Y') + a(Y'^t\mathbb{J}Y'') - a'(Y^t\mathbb{J}Y'') \\ & +\tfrac{1}{\sqrt{2}}\left(-bX'^tY'' + b'X^tY'' + bY'^tX'' - b'Y^tX'' + b''Y^tX' - b''Y'^tX\right) \end{aligned}$$

It's easy to see from this description that ϕ is indeed skew symmetric.

REMARK 2.4. Let $\mathbb{V}$ be a 7-dimensional vector space. One can show that there are exactly two open $GL(\mathbb{V})$-orbits in $\Lambda^3(\mathbb{V}^*)$. The stabiliser of an element in one of these orbits is the split real form of G_2 and the stabiliser of an element in the other open orbit is the compact real form of G_2.

2.1.5. The parabolic subalgebra. To avoid confusion, we will from now on denote the Lie algebra of G_2 simply by $\mathfrak{g}$. Then $\mathfrak{g}$ has three standard parabolic subalgebras: the Borel subalgebra and two maximal parabolics. Let us denote by $\mathfrak{p}$ the stabiliser in $\mathfrak{g}$ of the line through the highest weight vector X_1 in the 7-dimensional representation $\mathbb{V}$. Checking what root spaces anihilate X_1, one is immediately lead to a description of $\mathfrak{p}$ as

$$\mathfrak{p} = \underbrace{\mathfrak{g}_{-\alpha_1} \oplus \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1}}_{\mathfrak{g}_0} \oplus \underbrace{\mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}}_{\mathfrak{g}_1} \oplus \underbrace{\mathfrak{g}_{\alpha_1+2\alpha_2}}_{\mathfrak{g}_2} \oplus \underbrace{\mathfrak{g}_{\alpha_1+3\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+3\alpha_2}}_{\mathfrak{g}_3}.$$

So $\mathfrak{p}$ is the standard parabolic subalgebra corresponding to the subset $\Sigma = \{\alpha_2\}$. In Satake diagram notation, this parabolic subalgebra is referred to as $\circ\!\Rrightarrow\!\times$.

In the description (2.1) of $\mathfrak{g}$ with respect to the basis

$$\{X_1, X_6, X_7, X_4, X_2, X_3, X_5\},$$

the parabolic subalgebra $\mathfrak{p}$ is given in a nice block form

$$\mathfrak{p} = \left\{ \begin{pmatrix} \mathrm{tr}A & Z & s & W & 0 \\ & A & \sqrt{2}\mathbb{J}Z^t & \frac{1}{\sqrt{2}}s\mathbb{J} & -W^t \\ & & & -\sqrt{2}Z\mathbb{J} & -s \\ & & & -A^t & -Z^t \\ & & & & -\mathrm{tr}A \end{pmatrix} \right\}.$$

We can also see the corresponding $|3|$-grading in the matrix realisation (2.1): The Y-blocks are contained in $\mathfrak{g}_{-3}$, the r-blocks are contained in $\mathfrak{g}_{-2}$, the X-blocks are contained in $\mathfrak{g}_{-1}$, the A-blocks are contained in $\mathfrak{g}_0$, the Z-blocks are contained in $\mathfrak{g}_1$, the s-blocks are contained in $\mathfrak{g}_2$ and finally, the W-blocks are contained in $\mathfrak{g}_{-1}$. So the grading looks as follows:

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_3 & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_3 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & 0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ 0 & \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}.$$

The grading element E, see 1.2.2, is of the form

$$E = \begin{pmatrix} 2 & & & & \\ & \mathbb{I} & & & \\ & & 0 & & \\ & & & -\mathbb{I} & \\ & & & & -2 \end{pmatrix}. \tag{2.2}$$

2.1.6. The Killing form. The natural choice for an invariant bilinear form on $\mathfrak{g}$ is of course the Killing form. For computations it might be more convenient to use the trace form for matrices as in (2.1), or any suitable multiple of it. Indeed, in Chapter 3, we will work with $\frac{1}{6}$ of the trace form, which we will denote by B. Evaluation on an element, e.g. the grading element, shows that the trace form is $\frac{1}{4}$ of the Killing form and thus B is $\frac{1}{24}$ of the Killing form.

Any invariant bilinear form on the Lie algebra $\mathfrak{g}$ defines dualities $\mathfrak{g}_i \cong (\mathfrak{g}_{-i})^*$. Thinking of matrices as in (2.1), the dualities induced by the form B are given by $\mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathbb{R}$

$$(X, Z) \mapsto ZX$$

and $\mathfrak{g}_{-2} \times \mathfrak{g}_2 \to \mathbb{R}$

$$(r, s) \mapsto \frac{1}{2} rs$$

and $\mathfrak{g}_{-3} \times \mathfrak{g}_3 \to \mathbb{R}$

$$(Y, W) \mapsto \frac{1}{3} WY.$$

2.1.7. The $\mathfrak{p}$-invariant filtration of $\mathbb{V}$. Let us look at the eigenspace decomposition

$$\mathbb{V} = \mathbb{V}_2 \oplus \mathbb{V}_1 \oplus \mathbb{V}_0 \oplus \mathbb{V}_{-1} \oplus \mathbb{V}_{-2}$$

for the grading element E next. Here $\mathbb{V}_2$ is the real one dimensional subspace generated by the the highest weight vector X_1, i.e. the highest weight space $\mathbb{V}_{\alpha_1+2\alpha_2}$. The grading component $\mathbb{V}_1$ is the subspace generated by X_6, X_7, that is the subspace $\mathbb{V}_{\alpha_1+\alpha_2} \oplus \mathbb{V}_{\alpha_2}$, which equals $\mathfrak{g}_{-1} \cdot \mathbb{V}_2$. The component $\mathbb{V}_0$ is the subspace generated by the weight vector X_4 of weight zero, hence $\mathbb{V}_0 = \mathfrak{g}_{-2} \cdot \mathbb{V}_2$, and $\mathbb{V}_{-1}$ is generated by X_2, X_3, hence it is $\mathbb{V}_{-\alpha_1-\alpha_2} \oplus \mathbb{V}_{-\alpha_2} = \mathfrak{g}_{-3} \cdot \mathbb{V}_2$. Finally, $\mathbb{V}_{-2} = \mathbb{V}_{-\alpha_1-2\alpha_2}$ is the real subspace generated by X_5. Obviously, the decomposition is $\mathfrak{g}_0$-invariant, but not $\mathfrak{p}$-invariant.

Next, we put $\mathbb{V}^i = \oplus_{j \geq i} \mathbb{V}_j$. Then we obtain a filtration

$$\mathbb{V}^2 \subset \mathbb{V}^1 \subset \mathbb{V}^0 \subset \mathbb{V}^{-1} \subset \mathbb{V}$$

of $\mathbb{V}$. From the above description of the parabolic $\mathfrak{p}$, we see that the filtration is indeed $\mathfrak{p}$-invariant.

For later use, we make the following observation:

LEMMA 2.5. *The subspace $\mathbb{V}^1$ can be described as the set of all $Y \in \mathrm{Im}\mathbb{O}_S$ such that*

$$X_1 Y = 0,$$

or equivalently, as the set of all $Y \in \mathrm{Im}\mathbb{O}_S$ such that

$$\phi(X_1, Y, Z) = 0$$

for all $Z \in \mathrm{Im}\mathbb{O}_S$.

PROOF. Take an arbitrary element of the imaginary split octonions

$$Y = \begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix} \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi e_1 \\ -x \wedge e_1 & \langle y, e_1 \rangle \end{pmatrix}.$$

The right hand side vanishes if and only if Y has zeros in the diagonal, the entry x is a real multiple of e_1 and y is orthogonal to e_1. But this is the case if and only if Y is contained in the span of X_1, X_6 and X_7, which shows that

$$\mathbb{V}^1 = \{Y \in \mathrm{Im}\mathbb{O}_S : X_1 Y = 0\}.$$

The imaginary split octonions are the orthogonal complement to the unit E. Hence,

$$\phi(X_1, Y, Z) = \langle X_1 Y, Z\rangle = 0$$

for all $Z \in \mathrm{Im}\mathbb{O}_S$ if and only if the product $X_1 Y$ is a real multiple of the identity E. From above we see that this is equivalent to vanishing of $X_1 Y$. □

Note also that $\mathbb{V}^0$ is the orthogonal complement to $\mathbb{V}^1$ and $\mathbb{V}^{-1}$ is the orthogonal complement to $\mathbb{V}^2$.

2.1.8. The subgroups P and G_0. In 1.2.3, we introduced for any semisimple Lie group G and grading on the Lie algebra $\mathfrak{g}$ a parabolic subgroup

$$P := \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \ \forall i\}$$

and a reductive subgroup

$$G_0 := \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \ \forall i\}.$$

We will look at these subgroups more closely in the case of $G = G_2$.

The parabolic P can be viewed as the subgroup of all elements in G_2 that preserve the highest weight space $\mathbb{V}^2$ and G_0 consists of all elements in G_2 that preserve the decomposition $\mathbb{V} = \mathbb{V}_2 \oplus \mathbb{V}_1 \oplus \mathbb{V}_0 \oplus \mathbb{V}_{-1} \oplus \mathbb{V}_{-2}$. This can be seen as follows. Looking at the description of the filtration $\mathbb{V}^2 \subset \mathbb{V}^1 \subset \mathbb{V}^0 \subset \mathbb{V}^{-1} \subset \mathbb{V}$ in terms of ϕ and the inner product $\langle , \rangle$, we see that an element $g \in G_2$ preserving $\mathbb{V}^2$ also preserves the filtration. Note that we can view $\mathfrak{g}$ as a subalgebra in $L(\mathbb{V}, \mathbb{V})$ and the grading of $\mathfrak{g}$ is induced by the grading of $\mathbb{V}$. (Meaning that elements in $\mathfrak{g}_i$ are of homogeneous degree i.) Hence every element that preserves the grading on $\mathbb{V}$ is contained in G_0 and every element that preserves the corresponding filtration, is contained in the parabolic subgroup P. For the converse direction, note that the subspace $\mathbb{V}^2$ can be described as the space of elements $v \in \mathbb{V}$ such that $Z \cdot v = 0$ for all $Z \in \mathfrak{p}_+$. To see that an element $g \in P$ stabilises $\mathbb{V}^2$ we need to show that $Z \cdot g \cdot v = 0$ for all $v \in \mathbb{V}^2$ and $Z \in \mathfrak{p}_+$. This is easy, since $Z \cdot g \cdot v = g \cdot (g^{-1}Zg) \cdot v$ and $g^{-1}Zg \in \mathfrak{p}_+$ for any $Z \in \mathfrak{p}_+$. Let $\mathbb{V}_i = \mathfrak{g}_i \cdot \mathbb{V}^2$ be any other grading component. An element $w \in \mathbb{V}_i$ is of the form $X \cdot v$ for $X \in \mathfrak{g}_i$ and $v \in \mathbb{V}^2$. If g is contained in G_0 it stabilises $\mathfrak{g}_i$ and $\mathbb{V}^2$ and thus we have $g \cdot w = (gXg^{-1})g \cdot v \in \mathbb{V}_i$.

2.2. The homogeneous model

Let G_2 be the automorphism group of the split octonions. Let P be the stabiliser in G_2 of the line through the highest weight vector $X_1 \in \mathbb{V}$. Then P is the parabolic subgroup in G_2 from 2.1.8. In this section we will discuss the homogeneous model for Cartan geometries of type (G_2, P).

Let $p : G_2 \to G_2/P$ be the canonical P-principal bundle. Recall that the grading of $\mathfrak{g}$ leads to a P-invariant filtration of $\mathfrak{g}/\mathfrak{p}$:

$$\mathfrak{g}^{-1}/\mathfrak{p} \subset \mathfrak{g}^{-2}/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}.$$

The associated bundle $G_2 \times_P \mathfrak{g}/\mathfrak{p}$ can be identified with the tangent bundle $T(G_2/P)$ via $[g, A + \mathfrak{p}] \mapsto T_g p\, T_e \lambda_g A$. (Equivalently, the isomorphism can be written in terms of the Maurer Cartan form, since $\omega_{MC}^{-1}(A) = T_e\lambda_g A$.) It follows, that the tangent bundle $T(G_2/P)$ is naturally filtered. The filtration component

$$\mathcal{H} = G \times_P \mathfrak{g}^{-1}/\mathfrak{p}$$

is a rank two distribution on the homogeneous space G_2/P.

To give a different description of G_2/P and the rank two distribution $\mathcal{H}$, we first identify the homogeneous space G_2/P with the pojectivisation of a cone: Let us denote by $\mathbb{V}$ the standard representation of G_2. Recall that we have a G_2-invariant nondegenerate bilinear form $\langle , \rangle$ on $\mathbb{V}$. It follows that the G_2-action preserves the cone of nonzero isotropic vectors

$$\mathcal{C} = \{\, X \in \mathbb{V} : \ X \neq 0, \ \langle X, X \rangle = 0 \,\}.$$

Next we denote by $\mathcal{P}(\mathcal{C})$ the projectivisation of the cone, i.e. the quotient of $\mathcal{C}$ where two elements X, Y are identified if and only if $X = \lambda Y$ for $\lambda \in \mathbb{R}$. Let $\pi : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ be the projection onto the projectivisation. Clearly the action on $\mathcal{C}$ descends to an action on $\mathcal{P}(\mathcal{C})$. Let $X_1 \in \mathbb{V}$ denote the highest weight vector in $\mathbb{V}$. Then X_1 is isotropic, hence the real line through X_1 can be viewed as an element $\pi(X_1) \in \mathcal{P}(\mathcal{C})$. By definition, the parabolic P is the stabiliser of $\pi(X_1)$ in G_2.

Proposition 2.6. *The homogeneous space G_2/P is diffeomorphic to $\mathcal{P}(\mathcal{C})$.*

Proof. Mapping a group element g to $g \cdot X_1$ induces a diffeomorphism between G_2/P and the orbit $G_2 \cdot \pi(X_1) \subset \mathcal{P}(C)$. Since G_2 is 14 dimensional and P is 9 dimensional, the dimension of $G_2 \cdot \pi(X_1)$ is 5. This is the maximal possible dimension, so we conclude that the orbit is open. Moreover, it is well known that quotient of a semisimple Lie group modulo a parabolic subgroup is always compact. Hence the orbit is closed as well. By connectedness of $\mathcal{P}(\mathcal{C})$, the result follows. $\square$

Next note that the tangent space of the cone $\mathcal{C}$ in a point X is the orthogonal complement to X, i.e. $T_X\mathcal{C} = \{Y \in \mathbb{V} : \langle X, Y \rangle = 0\}$. The tangent map $T_X\pi$ of the projection $\pi : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ identifies the tangent space of $T_{\pi(X)}\mathcal{P}(\mathcal{C})$ with the tangent space $T_X\mathcal{C}$ modulo the kernel of $T_X\pi$ which coincides with the real line line $\mathbb{R}X$ through X. It follows that

$$(\mathbb{R}X)^{\perp}/\mathbb{R}X \cong T_{\pi(X)}(\mathcal{P}(\mathcal{C})).$$

Note that the isomorphism depends on the choice of the point X over $\pi(X)$ by multiplication with a nonzero real number.

We want to give an explicit description of $\mathcal{H}_{\pi(X)}$ as a subspace in $T_{\pi(X)}(\mathcal{P}(\mathcal{C}))$. Our first description will be in terms of the split octonions.

Proposition 2.7. *Let $X \in \mathrm{Im}(\mathbb{O}_S)$ be an non-zero isotropic element of the imaginary split octonions and $\pi(X)$ the corresponding point in $\mathcal{P}(\mathcal{C})$. Then the fibre $\mathcal{H}_{\pi(X)}$ can be viewed as the quotient of the subspace of all $Y \in \mathrm{Im}(\mathbb{O}_S)$ such that $XY = 0$ by the real line $\mathbb{R}X$.*

Proof. Let $X_1 \in \mathbb{V}$ be the highest weight vector $\begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}$. Let $\mathbb{V}^2$ denote the real line through X_1 and let $\mathbb{V}^1$ be the subspace

$$\mathbb{V}^1 = \{Y \in \mathrm{Im}\mathbb{O}_S : YX_1 = 0\}.$$

The diffeomorphism $G/P \cong \mathcal{P}(\mathcal{C})$ is induced by the map $\ell^{\pi(X_1)} : G \to \mathcal{P}(\mathcal{C})$, $\ell^{\pi(X_1)}(g) := \pi(g \cdot X_1)$. The tangent map $T_e\ell^{\pi(X_1)} : \mathfrak{g} \to T_{\pi(X_1)}\mathcal{P}(\mathcal{C})$ of $\ell^{\pi(X_1)}$ at the identity reads $T_e\ell^{\pi(X_1)}(A) = A \cdot X_1 + \mathbb{V}^2$. It factors to an isomorphism

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{p} & \cong & (\mathbb{V}^2)^\perp/\mathbb{V}^2 \\ A + \mathfrak{p} & \mapsto & A \cdot X_1 + \mathbb{V}^2. \end{array}$$

This isomorphism maps the filtration component $\mathfrak{g}^{-1}/\mathfrak{p}$ onto the subspace $\mathbb{V}^1/\mathbb{V}^2 \subset T_{p(X_1)}\mathcal{P}(\mathcal{C})$: Lemma 2.5 shows that $\mathbb{V}^1$ is generated by the weight vectors X_1, X_6, X_7, i.e. it is the sum of the following weight spaces

$$\mathbb{V}^1 = \mathbb{V}_{\alpha_1+2\alpha_2} \oplus \mathbb{V}_{\alpha_1+\alpha_2} \oplus \mathbb{V}_{\alpha_2}.$$

The filtration component $\mathfrak{g}^{-1}$ consists of the parabolic subalgebra $\mathfrak{p}$ and the root spaces corresponding to the roots $-\alpha_2$ and $-\alpha_1 - \alpha_2$. The parabolic subalgebra $\mathfrak{p}$ stabilizes the line $\mathbb{V}^2 = \mathbb{V}_{\alpha_1+2\alpha_2}$ and the remaining two root spaces map $\mathbb{V}^2$ onto the weight spaces $\mathbb{V}_{\alpha_1+\alpha_2}$ respectively $\mathbb{V}_{\alpha_2}$. It follows that the isomorphism maps $\mathfrak{g}^{-1}/\mathfrak{p}$ onto $\mathbb{V}^1/\mathbb{V}^2$.

Since $\mathcal{P}(\mathcal{C})$ is homogeneous, any element $\pi(X) \in \mathcal{P}(\mathcal{C})$ is of the form $\pi(X) = g \cdot \pi(X_1)$. The subspace $\mathcal{H}_{\pi(X)}$ equals $(g \cdot \mathbb{V}^1)/\mathbb{R}X$. Moreover G_2 acts as a group of algebra automorphisms and thus we have $X(g \cdot Z) = g \cdot (X_1 Z) = 0$ if and only if $X_1 Z = 0$. This shows that the fibre $\mathcal{H}_{\pi(X)}$ is the quotient of the subspace of all $Y \in \mathrm{Im}(\mathbb{O}_S)$ such that $XY = 0$ by the real line $\mathbb{R}X$.

□

Remark 2.8. Recall that on $\mathbb{V} = \mathrm{Im}(\mathbb{O}_S)$ we have a G_2-invariant inner product $\langle , \rangle$ of signature $(3, 4)$. For any $X \in \mathcal{C}$, it can be restricted to $(\mathbb{R}X)^\perp$ and, since X is isotropic, it induces an inner product of signature $(2, 3)$ on the quotient $(\mathbb{R}X)^\perp/\mathbb{R}X$, which we will also denote by $\langle , \rangle$. We have seen that the tangent map $T_X\pi$ of the projection induces an isomorphism between $(\mathbb{R}X)^\perp/\mathbb{R}X$ and $T_{\pi(X)}\mathcal{P}(\mathcal{C})$. As we vary the element X over a real line, the induced inner product on $T_{\pi(X)}\mathcal{P}(\mathcal{C})$ rescales by a positive real number. Hence, it gives rise to a a well defined conformal class $[g]$ on $\mathcal{P}(\mathcal{C})$. Obviously, the conformal structure on $\mathcal{P}(\mathcal{C})$ is preserved by the G_2-action.

This is not surprising given the inclusion $G_2 \hookrightarrow SO(3, 4)$ and the observation, see 1.2.6, that the projectivisation of the cone $\mathcal{P}(\mathcal{C})$ is also the homogeneous model space for conformal structures of signature $(2, 3)$. We will elaborate on this subject in Chapter 4.

We have mentioned that an alternative way to characterise the group G_2 is to view it as the subgroup in $GL(7, \mathbb{R})$ stabilising a generic 3-form $\phi \in \Lambda^3(\mathbb{V}^*)$. In terms of split octonions this 3-form can be written as $\phi(X, Y, Z) = \langle XY, Z\rangle$. As an immediate corollary of Proposition 2.7 and Lemma 2.5 we obtain the following description of the distribution $\mathcal{H}$:

Corollary 2.9. *For any $X \in \mathbb{V}$ and corresponding point $\pi(X) \in \mathcal{P}(\mathbb{V})$, the fibre of the distribution $\mathcal{H}$ in the point $\pi(X)$ is given by*

$$\mathcal{H}_{\pi(X)} = \{\, Y \in \mathbb{V} : \phi(X, Y, Z) = 0 \;\forall Z \in \mathbb{V} \,\}/\mathbb{R}X,$$

for a 3-form $\phi \in \Lambda^3(\mathbb{V}^*)$ with isotropy group G_2.

We proceed to another corollary of Proposition 2.7. A double cover of the projectivised null cone $\mathcal{P}(\mathcal{C})$ is the space $\mathcal{C}/\mathbb{R}_+$ of rays in the cone $\mathcal{C}$, i.e. the quotient where two vectors are identified if and only if one is a positive multiple of the other. It is diffeomorphic to the quotient G_2/P_o, where P_o is the stabiliser of the ray $\mathbb{R}_+X_1$, i.e. the connected component of the identity of the parabolic subgroup P.

Furthermore, we can identify $\mathcal{C}/\mathbb{R}_+$ with $S^2 \times S^3$. Instead of the light-cone basis $X_1, \cdots, X_7$ we consider the orthogonal basis $Y_1 = \frac{1}{\sqrt{2}}(X_1 + X_5)$, $Y_2 = \frac{1}{\sqrt{2}}(X_2 + X_6)$, $Y_3 = \frac{1}{\sqrt{2}}(X_3 + X_7)$, $Y_4 = X_4$, $Y_5 = \frac{1}{\sqrt{2}}(X_1 - X_5)$, $Y_6 = \frac{1}{\sqrt{2}}(X_2 - X_6)$, $Y_7 = \frac{1}{\sqrt{2}}(X_3 - X_7)$. Then $\langle Y_i, Y_i\rangle = 1$ for $i = 1, 2, 3$ and $\langle Y_j, Y_j\rangle = -1$ for $j = 4, 5, 6, 7$. A vector $V \in \mathrm{Im}(\mathbb{O}_S)$ with coordinate vector (x, α, y) with respect to the basis $\{Y_1, \cdots, Y_7\}$ is contained in the cone $\mathcal{C}$ if and only if $\|y\|^2 + \alpha^2 = \|x\|^2$. Note that the coordinate vector of V with respect to the basis $\{X_1, \cdots, X_7\}$ is $(\frac{1}{\sqrt{2}}(x + y), \alpha, \frac{1}{\sqrt{2}}(x + y))$, hence

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha & x + y \\ y - x & -\alpha \end{pmatrix}.$$

Let $S^2 \subset \mathbb{R}^3$ and $S^3 \subset \mathbb{R}^4$ be the unit spheres. The above observations imply that the map

$$\begin{array}{rcl} S^2 \times S^3 & \to & \mathcal{C} \\ (x, (\alpha, y)) & \mapsto & \begin{pmatrix} \alpha & x + y \\ y - x & -\alpha \end{pmatrix}. \end{array}$$

induces a diffeomorphism $S^2 \times S^3 \cong \mathcal{C}/\mathbb{R}_+$.

As in Proposition 2.7, the subbundle $G \times_{P_o} \mathfrak{g}^{-1}/\mathfrak{p} \subset T(G/P_o)$ can be identified with a distribution $\tilde{\mathcal{H}}$ on $\mathcal{C}/\mathbb{R}_+$ defined in terms of split octonions, i.e. the distribution with fibre

$$\tilde{\mathcal{H}}_{[X]} = \{\, Y \in \mathrm{Im}(\mathbb{O}_S) : XY = 0 \,\}/\mathbb{R}X,$$

for any $[X] \in \mathcal{C}/\mathbb{R}_+$.

Corollary 2.10. *Via the identification $\mathcal{C}/\mathbb{R}_+ \cong S^2 \times S^3$, the above distribution $\tilde{\mathcal{H}}$ on $\mathcal{C}/\mathbb{R}_+$ corresponds to*

$$\tilde{\mathcal{H}}_{(x,(\alpha,y))} = \left\{ \begin{array}{ll} (v, (\beta, w)) \in \mathbb{R}^3 \times \mathbb{R}^4 : & \langle v, x\rangle = 0,\, \beta = \langle v \wedge y, x\rangle\,, \\ & w = \langle v, y\rangle\, x + \alpha(v \wedge x) - \langle y, x\rangle\, v \end{array} \right\},$$

for any $(x, (\alpha, y)) \in S^2 \times S^3$.

Proof. The aim of the proof is to verify that the tangent map of the diffeomorphism $S^2 \times S^3 \cong \mathcal{C}/\mathbb{R}_+$ maps the distribution described in the corollary onto the distribution $\tilde{\mathcal{H}}$ on $\mathcal{C}/\mathbb{R}_+$ defined in terms of split octonions.

We start with an element $(x, (\alpha, y))$ in $S^2 \times S^3$, that is $x \in \mathbb{R}^3$, $(\alpha, y) \in \mathbb{R} \times \mathbb{R}^3$ such that $\|x\|^2 = 1$ and $\alpha^2 + \|y\|^2 = 1$. The tangent space of $S^2 \times S^3$ in $(x, (\alpha, y))$ is given by

$$T_{(x,(\alpha,y))}(S^2 \times S^3) = x^\perp \oplus (\alpha, y)^\perp,$$

i.e. $(v, (\beta, w)) \in T_{(x,(\alpha,y))}(S^2 \times S^3)$ satisfies $\langle x, v\rangle = 0$ and $\alpha\beta + \langle w, y\rangle = 0$. The inclusion $\rho : S^2 \times S^3 \to \mathcal{C}$ maps $(x, (\alpha, y))$ to $X = \begin{pmatrix} \alpha & y + x \\ y - x & -\alpha \end{pmatrix}$ and its tangent map is given by

$$T_{(x,(\alpha,y))}\rho \cdot (v, (\beta, w)) = \begin{pmatrix} \beta & w + v \\ w - v & -\beta \end{pmatrix}.$$

Now we consider the space $\mathcal{H}_{(x,(\alpha,y))}$ of all $(v,(\beta,w))$ where v is any vector orthogonal to x, $\beta = \langle v \wedge y, x\rangle$ and $w = \langle v, y\rangle\, x + \alpha(v\wedge x) - \langle y, x\rangle$. This is a subspace of $T_{(x,(\alpha,y))}(S^2 \times S^3)$ since one easily verifies that $\alpha\beta + \langle w, y\rangle = 0$ and obviously it is two dimensional. We need to verify that it coincides with the subspace of all elements in $T_{(x,(\alpha,y))}(S^2 \times S^3)$ that are mapped via $T_{(x,(\alpha,y))}\rho \cdot (v,(\beta,w))$ into the set of imaginary split octonions Y such that $XY = 0$. Note that by dimensional reasons we are done, if we can show that $\mathcal{H}_{(x,(\alpha,y))}$ contains all elements mapped via $T_{(x,(\alpha,y))}\rho$ into the set of imaginary split octonions Y such that $XY = 0$. This will be done in the following:

Demanding that

$$\begin{pmatrix} \alpha & y+x \\ y-x & -\alpha \end{pmatrix}\begin{pmatrix} \beta & w+v \\ w-v & -\beta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

leads to the following equations

$$\begin{aligned} &\alpha\beta + \langle y-x, v+w\rangle = 0 \\ &\alpha\beta + \langle y+x, w-v\rangle = 0 \\ &\alpha(w-v) - \beta(y-x) + (x+y)\wedge(v+w) = 0 \\ &-\alpha(v+w) + \beta(y+x) - (y-x)\wedge(w-v) = 0. \end{aligned}$$

Expanding these equations an subtracting the second from the first implies

$$\langle v, y\rangle = \langle w, x\rangle$$

and adding the fourth to the third gives

$$-\alpha v + \beta x + y\wedge v + x\wedge w = 0.$$

Now we consider the inner product

$$\langle -\alpha v + \beta x + y\wedge v + x\wedge w, x\rangle = \beta + \langle y\wedge v, x\rangle$$

from which we derive

$$\beta = \langle v\wedge y, x\rangle\,.$$

Next

$$(w\wedge x)\wedge x = \langle w, x\rangle\, x - \langle x, x\rangle\, w = \langle v, y\rangle\, x - w$$

and

$$(w\wedge x)\wedge x = (-\alpha v + \beta x + y\wedge v)\wedge x = -\alpha(v\wedge x) + \langle y, x\rangle\, v$$

which gives

$$w = \langle v, y\rangle\, x + \alpha(v\wedge x) - \langle y, x\rangle\, v.$$

This concludes the proof of the corollary. □

Finally, we relate the result of Proposition 2.7 to a description of G_2 Cartan gave himself. To do so, we look at the condition $XY = 0$ for two imaginary split octonions more carefully: The equation

$$\begin{pmatrix} z & x \\ y & -z \end{pmatrix}\begin{pmatrix} u & v \\ w & -u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

implies

$$zu + \langle x, w\rangle = 0 \tag{2.3}$$
$$zu + \langle y, v\rangle = 0 \tag{2.4}$$
$$zv - ux + y \wedge w = 0 \tag{2.5}$$
$$-zw + uy - x \wedge v = 0. \tag{2.6}$$

Note that if Y is orthogonal to X, i.e. $-2zu - \langle x, w\rangle - \langle y, v\rangle = 0$, the equation

$$\langle x, w\rangle - \langle y, v\rangle = 0 \tag{2.7}$$

is equivalent to (2.3) and (2.4).

In view of Proposition 2.7, equations (2.3), (2.4) and (2.7) imply that the common kernel of the seven one-forms

$$\begin{gathered}
zdx_1 - x_1dz + y_2dy_3 - y_3dy_2\\
zdx_2 - x_2dz + y_3dy_1 - y_1dy_3\\
zdx_3 - x_3dz + y_1dy_2 - y_2dy_1\\
zdy_1 - y_1dz + x_2dx_3 - x_3dx_2\\
zdy_2 - y_2dz + x_3dx_1 - x_1dx_3\\
zdy_3 - y_3dz + x_1dx_2 - x_2dx_1\\
x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2 + x_3dy_3 - y_3dx_3
\end{gathered}$$

defines a distribution on the cone that when pushed down to $T\mathcal{P}(\mathcal{C})$ coincides with $\mathcal{H}$. In his thesis [**18**], Cartan describes G_2 as the subgroup of the general linear group preserving the cone $\mathcal{C}$ and a system of one-forms which is essentially the one above. This is discussed in [**4**].

2.3. Parabolic geometries of type (G_2, P)

We proceed to curved geometries modelled on G_2/P. As usual, let G_2 be the automorphism group of the algebra of split octonions, viewed as a closed subgroup of $SO(3,4)$ and let $P \subset G_2$ be the stabiliser of the real line through the highest weight vector.

2.3.1. The equivalence to generic rank two distributions in dimension five. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G_2. Assume we have fixed a Cartan subalgebra and a set of simple (restricted) roots $\Delta^0 = \{\alpha_1, \alpha_2\}$. Then the positive roots are

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}.$$

Let $\mathfrak{p}$ be the standard parabolic subalgebra corresponding to the subset $\Sigma = \{\alpha_2\}$. The grading on $\mathfrak{g}$ given by the Σ-height is a $|3|$-grading

$$\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3,$$

where

$$\mathfrak{g}_{-3} = \mathfrak{g}_{-3\alpha_2 - 2\alpha_1} \oplus \mathfrak{g}_{-3\alpha_2 - \alpha_1}$$

and

$$\mathfrak{g}_{-2} = \mathfrak{g}_{-2\alpha_2 - \alpha_1}$$

and

$$\mathfrak{g}_{-1} = \mathfrak{g}_{-\alpha_2-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2}.$$

Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $-\alpha_2 - \alpha_1 - \alpha_2 = -2\alpha_2 - \alpha_1$ the Lie bracket

$$[\ ,\ ] : \Lambda^2 \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$$

is surjective and by equality of dimensions it is an isomorphism. Similarly, we see that

$$[\ ,\ ] : \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2} \to \mathfrak{g}_{-3}$$

is an isomorphism.

This already shows what the underlying filtration for any parabolic geometry of type (G_2, P) must look like. The symbol algebra for this filtration is locally trivial with modelling algebra $\mathfrak{g}_-$. Hence, it is of the form

$$T^{-1}M \subset T^{-2}M \subset T^{-3}M = TM,$$

where $T^{-1}M$ has rank two, $T^{-2}M$ has rank three and the the only nontrivial parts of the Levi bracket are

$$\mathcal{L} : \Lambda^2 T^{-1}M \to T^{-2}M/T^{-1}M$$

and

$$\mathcal{L} : T^{-1}M \otimes (T^{-2}M/T^{-1}M) \to TM/T^{-2}M,$$

both of which are isomorphisms. In other words, the underlying structures for parabolic geometries of that type are distributions $\mathcal{H}$ of rank two on five manifolds M, such that sections ξ_i of the distribution $\mathcal{H}$ and Lie brackets $[\xi_i, \xi_j]$ of two such sections generate a rank three distribution $\mathcal{H}^1$ and sections of $\mathcal{H}$ and iterated Lie brackets of at most three such sections generate the entire tangent bundle. These distributions are precisely the distributions with growth vector $(2, 3, 5)$, cf. 1.4.3. A generic rank two distribution in dimension five is of that type.

Indeed, parabolic geometries of type (G_2, P) are already determined by their underlying filtrations. From 2.1.8 we know that the subgroup $G_0 \subset G_2$ determined by the grading of $\mathfrak{g}$ is of the form

$$G_0 = \left\{ \begin{pmatrix} \det M & & & \\ & M & & \\ & & & \\ & & (M^t)^{-1} & \\ & & & (\det M)^{-1} \end{pmatrix} \right\}.$$

Now it is easy to see that the adjoint action of G_0 on $\mathfrak{g}_{-1}$ identifies G_0 with $GL(\mathfrak{g}_{-1})$. On the other hand, since the brackets $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2} \to \mathfrak{g}_{-3}$ are isomorphisms, the group $GL(\mathfrak{g}_{-1})$ is isomorphic to $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. So we have an isomorphism $G_0 \cong \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$ and we can therefore apply Corollary 1.22 to obtain:

THEOREM 2.11. *We have an equivalence of categories between regular, normal parabolic geometries of type (G_2, P) and distributions in dimension five with growth vector $(2, 3, 5)$.*

Given a generic rank two distribution $\mathcal{H}$ we can form the graded frame bundle for the graded vector bundle $\mathrm{gr}(TM) = \mathcal{H} \oplus \mathcal{H}^1/\mathcal{H} \oplus TM/\mathcal{H}^1$. This is obviously isomorphic to the linear frame bundle for the distribution $\mathcal{H}$, i.e.

$$(\mathcal{G}_0)_x = \{ \text{ linear isomorphisms } \phi : \mathfrak{g}_{-1} \to \mathcal{H}_x\}$$

and the G_0 action is given by

$$\phi \cdot g = \phi \circ \mathrm{Ad}(g).$$

The theorem then says that the frame bundle for the distribution $\mathcal{H}$ can be extended to a principal P-bundle, which can be endowed with a canonical Cartan connection ω.

Remark 2.12. As an immediate consequence of the description of generic rank two distributions in dimension five as parabolic geometries, we obtain some information about their automorphism groups. We know that their automorphism groups are always finite dimensional Lie groups with dimension at most 14. For the distribution corresponding to the homogeneous model, the automorphism group is precisely G_2. In particular, G_2 can be realised as the automorphism group of the distributions described in section 2.2. Exploiting the description of generic rank two distributions as parabolic geometries, even more information on the automorphisms of these geometries can be obtained. See [**13**] for that.

Remark 2.13. There is an interesting class of examples of rank two distributions in dimension five, arising from the physical system of two balls rolling on each other. This system is discussed in [**4**], where the reader can find proofs and more information. The reader interested in homogeneous Cartan geometries, such as this one, and their infinitesimal automorphisms is also advised to look at [**26**].

Consider two balls of different sizes, one with radius R and another one with radius r and imagine they are rolling on each other without slipping or spinning. Then the configuration space of this system is the five dimensional manifold $M = SO(3,\mathbb{R}) \times S^2$. The no-slipping and no-spinning condition defines a rank two distribution $\mathcal{D}$ on the space $SO(3,\mathbb{R}) \times S^2$. That means, a smooth curve $c : \mathbb{R} \to SO(3,\mathbb{R}) \times S^2$ represents a rolling motion without slipping and spinning if and only if $c'(t)$ lies in $\mathcal{D}_{c(t)}$ for all t. One can show that these rolling distributions are rank two distributions with growth vector $(2,3,5)$. Furthermore, they are homogeneous: the group $K = SO(3,\mathbb{R}) \times SO(3,\mathbb{R})$ acts transitively on $SO(3,\mathbb{R}) \times S^2$ via

$$(g', g'') \cdot (g, x) = (g' g g''^{-1}, g' x)$$

and the action preserves the distribution $\mathcal{D}$.

Hence, we can identify $SO(3,\mathbb{R}) \times S^2$ with the homogeneous space K/H where H is the isotropy group of the element $(\mathrm{id}, e_3) \in SO(3,\mathbb{R}) \times S^2$. The tangent bundle can be viewed as the associated bundle

$$TM \cong K \times_H \mathfrak{k}/\mathfrak{h}.$$

Here $\mathfrak{k} = \mathfrak{so}(3,\mathbb{R}) \oplus \mathfrak{so}(3,\mathbb{R})$ is the Lie algebra of K and $\mathfrak{h}$ is the Lie algebra of H and H acts via the adjoint action. We can identify $\mathfrak{k}$ with $\mathbb{R}^3 \times \mathbb{R}^3$, $\mathfrak{h}$ with $\mathbb{R}(e_3, e_3) \subset \mathbb{R}^3 \times \mathbb{R}^3$ and the quotient $\mathfrak{k}/\mathfrak{h}$ with the orthogonal complement to $\mathbb{R}(e_3, e_3)$, i.e. the subspace of elements $(\omega', \omega'') \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $\langle \omega', e_3 \rangle + \langle \omega'', e_3 \rangle = 0$. Proposition 2 in [**4**] shows that, in this picture, the rolling distribution $\mathcal{D}$ for radii R and r corresponds to the H-invariant subspace $\mathfrak{k}^{-1}/\mathfrak{h}$ defined by the equations

$$\langle \omega', e_3 \rangle = \langle \omega'', e_3 \rangle = 0$$

and

$$\rho\omega' + \omega'' = 0.$$

For $\rho = 1$ or $\rho = 3$, the above distribution is equivalent to a flat Cartan geometry. In these cases, an embedding of $\mathfrak{k}$ into $\mathfrak{g}_2$ (as the maximal compact subalgebra) leads to an isomorphism $\mathfrak{k}/\mathfrak{h} \cong \mathfrak{g}/\mathfrak{p}$ that maps $\mathfrak{k}^{-1}/\mathfrak{h}$ onto $\mathfrak{g}^{-1}/\mathfrak{p}$.

Remark 2.14. The description of generic rank two distributions in dimension five as parabolic geometries also implies the existence of certain invariant differential operators. The study of invariant operators for parabolic geometries is an active area of research. It is known that many invariant differential operators can be constructed via the so-called BGG-sequence. We introduce these operators only very briefly. The interested reader may consult [**7**] and references therein.

To begin with, there is an important class of natural bundles, corresponding to restrictions of G-representations to the parabolic P. These are called tractor bundles and they always have canonical linear connections, called tractor connections, see 4.3.1. Let $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$ be a tractor bundle corresponding to a G-representation $\mathbb{V}$. Then the tractor connection ∇ on $\mathcal{V}$ induces a covariant exterior derivative

$$d^{\nabla} : \Omega^k(M, \mathcal{V}) \to \Omega^{k+1}(M, \mathcal{V}).$$

Next recall the Kostant codifferential $\partial^* : \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{V} \to \Lambda^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{V}$, introduced in 1.3.3. Since the Kostant codifferential is a P-module homomorphism it defines a bundle map between associated bundles

$$\partial^* : \Omega^k(M, \mathcal{V}) \to \Omega^{k-1}(M, \mathcal{V}).$$

The quotient bundle $\ker(\partial^*)/\mathrm{im}(\partial^*)$ can be naturally viewed as the associated bundle $\mathcal{H}^k := \mathcal{G} \times_P H^k(\mathfrak{g}_-, \mathbb{V})$. Let

$$\pi_H : \Gamma(\ker(\partial^*)) \to \Gamma(\mathcal{H}^k)$$

be the natural projection map on sections.

The key step in the construction of BGG-operators is the construction of the splitting operator. This is a natural differential operator

$$L : \Gamma(\mathcal{H}^k) \to \Gamma(\ker(\partial^*)) \subset \Omega^k(M, \mathcal{V})$$

characterised by the fact that it splits the projection π_H, i.e. $\pi_H(L(\phi)) = \phi$ for all $\phi \in \mathcal{H}^k$, and satisfies $\partial^*(d^{\nabla} L(\phi)) = 0$. The properties of the splitting operator imply that one may define differential operators as

$$D_k := \pi_H \circ d^{\nabla} \circ L : \Gamma(\mathcal{H}^k) \to \Gamma(\mathcal{H}^{k+1})$$

and thus, one obtains a sequence of operators, called BGG-sequence.

The first BGG operator D_0 for the adjoint tractor bundle associated to a generic rank two distribution in dimension five will be discussed in 3.4

2.3.2. The harmonic curvature and torsion freeness. Recall that the harmonic curvature κ_H is a basic invariant of a regular, normal Cartan geometry and a complete obstruction to local flatness. It is defined as the image of κ under the projection onto the quotient $\ker(\partial^*)/\mathrm{im}(\partial^*) \cong \mathcal{G}_0 \times_{G_0} H^2(\mathfrak{g}_-, \mathfrak{g})$. The cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ is the dual representation to $H^2(\mathfrak{p}_+, \mathfrak{g})$ and the latter cohomology can be determined as a $\mathfrak{g}_0$-representation, using Kostant's version of the Bott-Borel-Weil theorem, see Theorem 1.28. The theorem implies that the set of irreducible summands of $H^2(\mathfrak{p}_+, \mathfrak{g})$ is in bijective correspondence with elements of the Hasse diagram of length 2. Moreover, a highest weight vector of the irreducible component corresponding to a Weyl group element w is given by the cohomology class of $F_w \otimes X$, where F_w is the wedge product of one nonzero element from each of the root spaces $\mathfrak{g}_{-\alpha}$ with $\alpha \in \Phi_w$ and X is a nonzero weight vector of weight $w(\beta)$, where β denotes the maximal root.

In case of the Lie algebra $\mathfrak{g}$ of G_2 and the grading corresponds to $\Sigma = \{\alpha_2\}$, the situation is as follows: There is exactly one element of the Hasse diagram of length 2, namely $s_{\alpha_2}s_{\alpha_1}$. The subset $\Phi_{s_{\alpha_2}s_{\alpha_1}} \subset \Delta^+$ is given by $\Phi_{s_{\alpha_2}s_{\alpha_1}} = \{\alpha_2\} \cup \{\alpha_1 + 3\alpha_2\}$. The maximal root is $2\alpha_1 + 3\alpha_2$ and $s_{\alpha_2}s_{\alpha_1}(2\alpha_1 + 3\alpha_2) = s_{\alpha_2}(3\alpha_2 + \alpha_1) = \alpha_1$. It follows that the $\mathfrak{g}_0$-module $H^2(\mathfrak{p}_+, \mathfrak{g})$ is irreducible and a highest weight vector is given by $X \wedge Y \otimes Z$, where $X \in \mathfrak{g}_{-\alpha_2} \subset \mathfrak{g}_{-1}$, $Y \in \mathfrak{g}_{-\alpha_1 - 3\alpha_2} \subset \mathfrak{g}_{-3}$ and $Z \in \mathfrak{g}_{\alpha_1} \subset \mathfrak{g}_0$. Hence, $H^2(\mathfrak{p}_+, \mathfrak{g})$ is contained in $\mathfrak{g}_{-3} \otimes \mathfrak{g}_{-1} \otimes \mathfrak{g}_0$ and is therefore of homogeneous degree -4. Note also, that $Z \in \mathfrak{g}_{-\alpha_1}$ is contained in the semisimple part $\mathfrak{g}_0^{ss}$ of the reductive subalgebra $\mathfrak{g}_0$. Dualising, we obtain:

Proposition 2.15. *Let $\mathfrak{g}$ be the Lie algebra of G_2 and consider the grading corresponding to $\Sigma = \{\alpha_2\}$. Then the cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ is an irreducible $\mathfrak{g}_0$-representation. Viewed as the subspace* $\ker(\Box) \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$, *a generator is given by $X \wedge Y \otimes Z$, where $X \in \mathfrak{g}_{\alpha_2} \subset \mathfrak{g}_1$, $Y \in \mathfrak{g}_{\alpha_1 + 3\alpha_2} \subset \mathfrak{g}_3$ and $Z \in \mathfrak{g}_{-\alpha_1} \subset \mathfrak{g}_0$. Thus,* $\ker(\Box)$ *is contained in*

$$\mathfrak{g}_3 \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_0^{ss}$$

and in particular it is contained in homogeneous degree 4. *Since the harmonic curvature of a regular, normal parabolic geometry of type (G_2, P) takes values in $H^2(\mathfrak{g}_-, \mathfrak{g})$ this implies that κ_H is of homogeneous degree* 4.

Remark 2.16. Since the $\mathfrak{g}_0$-representation $H^2(\mathfrak{p}_+, \mathfrak{g})$ is irreducible with highest weight $-4\alpha_2$ it is isomorphic to $S^4\mathfrak{g}_{-1}$ and the dual representation $H^2(\mathfrak{g}_-, \mathfrak{g})$ is isomorphic to $S^4\mathfrak{g}_1$. Passing to associated bundles, this implies that the harmonic curvature can be viewed as a section of

$$S^4(\mathcal{H}^*) = \mathcal{G}_0 \times_{G_0} S^4(\mathfrak{g}_1).$$

It is the fundamental curvature tensor constructed by Elie Cartan in [**19**].

It turns out that in order to see whether a regular, normal Cartan connection is torsion free, it suffices to look at the harmonic curvature component. This is a consequence of the following result, which is proved using the fact that κ can be recovered from κ_H via the BGG splitting operator:

Proposition 2.17. *Suppose $\mathbb{E} \subset \ker(\partial^*) \subset \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is a P-submodul and define $\mathbb{E}_0 := \mathbb{E} \cap \ker(\Box)$. Let $(\mathcal{G} \to M, \omega)$ be a regular, normal parabolic geometry such that the harmonic curvature has values in $\mathbb{E}_0$. If either ω is torsion free or for any $\phi, \psi \in \mathbb{E}$ we have*

$\partial^*(i_\phi\psi) \in \mathbb{E}$, where $i_\phi\psi$ is the alternation of the map $(X_0, X_1, X_2) \mapsto \phi(\psi(X_0, X_1)+\mathfrak{p}, X_2)$ for $X_i \in \mathfrak{g}/\mathfrak{p}$, then the curvature function κ has values in $\mathbb{E}$.

A proof of this result can be found in [**7**]. (Note that we can view the P-representation $\mathbb{E}$ as a G_0-representation and decompose it into G_0-irreducible components. On each of these irreducible components the Laplacian $\Box$ acts by multiplication with a scalar, which implies that the Laplacian preserves $\mathbb{E}$. This shows that Corollary 3.2 in [**7**] indeed proves the above proposition.) Now we easily obtain

PROPOSITION 2.18. *Let $(\mathcal{G} \to M, \omega)$ be a regular, normal parabolic geometry such that the harmonic curvature takes values in $(\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}) \cap \ker(\Box)$. Then the geometry is torsion free. In particular, regular, normal parabolic geometries of type (G_2, P) are torsion free.*

PROOF. We apply Proposition 2.17 to the P-submodule $\mathbb{E} = (\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}) \cap \ker(\partial^*)$. The only thing left to do is to verify that for any $\phi, \psi \in \mathbb{E}$ we have $\partial^*(i_\phi\psi) \in \mathbb{E}$. But this is easy, since we have $\psi(X_i, X_j) \in \mathfrak{p}$ for all X_i, X_j and thus $i_\phi\psi = 0$.

For regular, normal parabolic geometries of type (G_2, P) Proposition 2.15 shows that the harmonic curvature takes values in $(\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}) \cap \ker(\Box)$. Thus, these geometries are torsion free. $\Box$

CHAPTER 3

Weyl structures associated to generalised contact forms

Let $\mathcal{H}$ be a generic rank two distribution in dimension five and let $\mathcal{H}^1 = [\mathcal{H}, \mathcal{H}]$ be the associated rank three distribution. Suppose α is a smooth section of $L(\mathcal{H}^1, \mathbb{R})$ such that the kernel of $\alpha(x)$ is precisely $\mathcal{H}_x$ for all $x \in M$. Such a section α will be called a generalised contact form. In this chapter we associate to a generalised contact form a splitting Θ of the filtration $\mathcal{H} \subset \mathcal{H}^1 \subset TM$, called soldering form, a linear connection ∇ on the distribution, called Weyl connection, and a one-form $P \in \Omega^1(M, \mathrm{gr}(TM))$ called rho tensor.

How this is done, is based on the theory of Weyl structures for parabolic geometries that has been developed in [**17**]. For any regular parabolic geometry $(\mathcal{G} \to M, \omega)$ associated to a distribution $\mathcal{H}$ and the underlying G_0-principal bundle $\mathcal{G}_0 \to M$, a Weyl structure is defined to be a G_0-equivariant section σ of the projection $\mathcal{G} \to \mathcal{G}_0$. The pullback $\sigma^*\omega$ of the Cartan connection along a Weyl structure splits into three parts: the soldering form, the Weyl connection and the rho-tensor.

The point about generalised contact forms is that they can be used to parametrise Weyl structures for generic rank two distributions in dimension five: The line bundle $\mathrm{gr}_2(T^*M)$ has the property that linear connections on this bundle are in bijective correspondence with Weyl structures. Sections of the bundle $\mathrm{gr}_2(T^*M)$ are precisely generalised contact forms. Hence, any generalised contact form α defines a (flat) connection on the bundle and thus corresponds to a unique Weyl structure for the geometry. In particular, it determines a unique Weyl connection, soldering form and rho tensor. The main aim of this chapter is to characterise these data as explicitly as possible in terms of a generalised contact form.

The results that are achieved are central for this thesis. The characterisation can be used to obtain formulae for geometric invariants of the distribution in terms of generalised contact forms: The harmonic curvature, viewed as a section of $S^4(\mathcal{H}^*)$, can be expressed in terms of ∇ and Θ. A description of an invariant operator whose kernel can be identified with the set of infinitesimal automorphisms of the distribution $\mathcal{H}$ is presented at the end of this chapter. Finally, the formula for a metric in the canonical conformal class that will be determined in the next chapter is based on the explicit form of the splitting of the filtration derived here.

3.1. Generalised contact forms

Before we introduce the notion of a generalised contact form for a generic rank two distribution, we recall some concepts known from contact geometry. This should serve as a motivation for some of the definitions to be given in the sequel.

A *contact structure* on a manifold M of dimension $2n+1$ is given by a corank one distribution $\mathcal{D}$ such that the Levi bracket $\mathcal{L} : \mathcal{D} \times \mathcal{D} \to TM/\mathcal{D}$ is non-degenerate in each point. A *contact form* is a one-form $\theta \in \Omega^1(M)$ defining a contact structure, i.e.

$\mathcal{D} = \{v \in TM : \theta(v) = 0\}$ is a contact distribution. Equivalently, this means that $\theta \wedge (d\theta)^n$ is a volume form. Given a contact form θ, there is a unique vector field ξ on the manifold such that $\theta(\xi) = 1$ and $i_\xi d\theta = 0$. This vector field is known as the *Reeb vector field* associated to a contact form. Note that the Reeb field provides an identification $TM \cong \mathcal{D} \oplus TM/\mathcal{D}$ given by $\zeta \mapsto (\zeta - \theta(\zeta)\xi, \theta(\zeta)\xi)$.

REMARK 3.1. For contact structures there is a version of a Darboux theorem, which implies that locally, all contact structures are isomorphic. However, one can consider contact distributions with additional structure on the distribution $\mathcal{D}$. These structures show up as underlying structures for parabolic geometries. The best known examples are non-degenerate partially integrable almost CR structures of hypersurface type. Here the additional structure is a complex structure $J : \mathcal{D} \to \mathcal{D}$ such that $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in \Gamma(\mathcal{D})$. A contact form for these structures is often called a *pseudo Hermitian structure*. Orientability of the manifold M implies orientability of the quotient $TM/\mathcal{D}$ and thus the existence of global contact forms. See also [**2**], [**11**] and [**31**].

Now we return to our usual setting. Let $\mathcal{H}$ be a generic rank two distribution on a five dimensional manifold M. Defining $T^{-1}M = \mathcal{H}$ and $T^{-2}M = [\mathcal{H}, \mathcal{H}]$, we obtain a filtration

$$T^{-1}M \subset T^{-2}M \subset TM$$

of the tangent bundle by smooth subbundles. For $i = -2, -3$, we denote $q_i : T^iM \to \mathrm{gr}_i(TM)$ the natural quotient map. We define, see also [**14**]:

DEFINITION 3.2. A *generalised contact form* α is a smooth section of the bundle $L(T^{-2}M, \mathbb{R})$ such that the kernel of the linear map $\alpha(x) : T^{-2}_xM \to \mathbb{R}$ equals $\mathcal{H}_x$ for each $x \in M$. A local section of $L(T^{-2}M, \mathbb{R})$ with that property is called a local generalised contact form.

REMARK 3.3. Note that by definition, a generalised contact form is precisely a nowhere vanishing section of the line bundle $\mathrm{gr}_2(T^*M) \cong \Lambda^2\mathcal{H}^*$. In general, this bundle need not admit global nowhere vanishing sections. The existence of a global generalised contact form is equivalent to the fact that the line bundle $\mathrm{gr}_2(T^*M)$ is orientable, or equivalently, that the distribution $\mathcal{H}$ is orientable.

REMARK 3.4. In the following, we will prove directly, i.e. without applying the general theory of Weyl structures, that we can assign to a generalised contact form α a distinguished section $r \in \Gamma(T^{-2}M)$, called generalised Reeb vector field, a distinguished partial connection on the distribution and a canonical extension of the generalised contact form to a one-form on the manifold M. One is actually led to the definition of these data in the course of the characterisation of the soldering form and the Weyl connection associated to a generalised contact form. The ad-hoc introduction of these objects at the beginning is mainly a matter of presentation, as the general theory also ensures their existence, compare with Remark 3.23. The presentation is similar to the joint paper [**14**]. An advantage of this presentation is that it shows how to compute the data for a given generalised contact form, which will be used e.g. in the Appendix B.

Let us consider a section $r \in \Gamma(T^{-2}M)$ which is transversal to $T^{-1}M$, i.e. $r_x \notin T^{-1}_xM$ for all $x \in M$. Then, for any $\gamma, \eta \in \Gamma(T^{-1}M)$ there is a unique section $\nabla_\gamma\eta \in \Gamma(T^{-1}M)$

such that

$$\{\nabla_\gamma \eta, q_{-2}(r)\} = q_{-3}([\gamma,[\eta,r]]). \tag{3.1}$$

The operator

$$\nabla : \Gamma(T^{-1}M) \times \Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$$
$$(\gamma,\xi) \mapsto \nabla_\gamma \xi$$

defines a partial connection on $T^{-1}M$: By construction ∇ is bilinear over $\mathbb{R}$. Moreover, for any $f \in C^\infty(M)$ the simple calculation

$$q_{-3}\left([f\gamma,[\eta,r]]\right) = q_{-3}(f[\gamma,[\eta,r]]) - q_{-3}(([\eta,r]\cdot f)\gamma) = f q_{-3}([\gamma,[\eta,r]])$$

shows that $\nabla_{f\gamma}\eta = f\nabla_\gamma \eta$, and the property $\nabla_\gamma f\eta = f\nabla_\gamma\eta + (\gamma \cdot f)\eta$ follows from

$$q_{-3}([\gamma,[f\eta,r]]) = q_{-3}([\gamma, f[\eta,r] - (r\cdot f)\eta]) = f q_{-3}([\gamma,[\eta,r]]) + (\gamma\cdot f) q_{-3}([\eta,r]).$$

Via the isomorphism $\Lambda^2 T^{-1}M \cong \mathrm{gr}_{-2}(TM)$, the partial connection ∇ on $T^{-1}M$ determines a partial connection

$$\nabla : \Gamma(T^{-1}M) \times \Gamma(\mathrm{gr}_{-2}(TM)) \to \Gamma(\mathrm{gr}_{-2}(TM))$$

characterised by

$$\nabla_\gamma\{\xi,\eta\} = \{\nabla_\gamma\xi,\eta\} + \{\xi,\nabla_\gamma\eta\}. \tag{3.2}$$

for $\gamma,\xi,\eta \in \Gamma(T^{-1}M)$. In the following, we refer to this partial connection on as *the partial connection on* $\mathrm{gr}_{-2}(TM)$ *associated to* r.

We can prove the following proposition:

Proposition 3.5. *Let α be a generalised contact form. Then, there is a unique vector field $r \in \Gamma(T^{-2}M)$ satisfying $\alpha(r) = 1$ such that the partial connection ∇ on $\mathrm{gr}_{-2}(TM)$ associated to r satisfies $\nabla\psi = 0$, where $\psi = q_{-2}(r) \in \Gamma(\mathrm{gr}_{-2}(TM))$.*

Proof. Note that it suffices to prove the result locally to get the global result. Locally, we can always find a vector field $r_0 \in \Gamma(T^{-2}M)$ satisfying $\alpha(r_0) = 1$. Since $\alpha(r_0) = 1$, this vector field is transversal to $T^{-1}M$. Hence, it gives rise to a partial connection ∇ as in (3.1). Any other vector field $r \in \Gamma(T^{-2}M)$ with the property $\alpha(r) = 1$ is of the form $r = r_0 + \delta$ for some $\delta \in \Gamma(T^{-1}M)$. The corresponding partial connection is denoted by ∇^δ. We have

$$\begin{aligned}\{\nabla^\delta_\gamma(\eta), q_{-2}(r)\} &= q_{-3}([\gamma,[\eta,r_0+\delta]]) \\ &= \{\nabla_\gamma(\eta), q_{-2}(r)\} + \{\gamma,\{\eta,\delta\}\}\end{aligned}$$

and thus

$$\nabla^\delta_\gamma\eta = \nabla_\gamma\eta + \alpha([\eta,\delta])\gamma.$$

We claim that there is a unique r such that for the corresponding partial connection $\nabla^\delta q_{-2}(r) = 0$ holds: Choose local smooth sections $\xi,\eta \in \Gamma(T^{-1}M)$ such that $\{\xi,\eta\} = q_{-2}(r)$. Then the two vector fields form a local frame. We compute

$$\begin{aligned}\nabla^\delta_\gamma\{\xi,\eta\} &= \{\nabla^\delta_\gamma\xi,\eta\} + \{\xi,\nabla^\delta_\gamma\eta\} \\ &= \nabla_\gamma\{\xi,\eta\} + \alpha([\xi,\delta])\{\gamma,\eta\} + \alpha([\eta,\delta])\{\xi,\gamma\}.\end{aligned}$$

For fixed ξ,η and δ, the sum of the last two terms on the right hand side defines a bundle map $T^{-1}M \to \mathrm{gr}_{-2}(TM)$. Inserting $\gamma = \xi$, we obtain $\alpha([\xi,\delta])\{\xi,\eta\} = \{\xi,\delta\}$ and inserting

$\gamma = \eta$ we obtain $\{\eta, \delta\}$. It follows that the bundle map is given by $\gamma \mapsto \{\gamma, \delta\}$. Therefore the above equation reads

$$\nabla^{\delta}_{\gamma}\{\xi, \eta\} = \nabla_{\gamma}\{\xi, \eta\} + \{\gamma, \delta\}.$$

Now $\gamma \mapsto -\nabla_{\gamma}\{\xi, \eta\}$ is a bundle map $T^{-1}M \to \mathrm{gr}_{-2}(TM)$. Hence, there is a unique vector field δ such that $\{\gamma, \delta\} = -\nabla_{\gamma}\{\xi, \eta\}$ for all $\gamma \in \Gamma(T^{-1}M)$. This means that for this δ the induced connection satisfies $\nabla^{\delta}\{\xi, \eta\} = 0$. Thus $r = r_0 + \delta$ is the vector field we were looking for.

□

DEFINITION 3.6. Let α be a generalised contact form. The unique section $r \in \Gamma(T^{-2}M)$ such that $\alpha(r) = 1$ and such that $\psi = q_{-2}(r)$ is parallel for the partial connection ∇ associated to r is called the *generalised Reeb vector field* associated to α.

Having proved the existence of a generalised Reeb vector field r, we can next construct a canonical extension of a generalised contact form to a one-form on M.

COROLLARY 3.7. *Let α be a generalised contact form and r the associated generalised Reeb vector field. Then there is a unique one-form $\tilde{\alpha} \in \Omega^1(M)$ such that $\tilde{\alpha}$ coincides with α on $T^{-2}M$ and $i_r d\tilde{\alpha}|_{T^{-1}M} = 0$.*

PROOF. Note that we can write any vector field ζ as

$$\zeta = \zeta_2 + \phi r + [r, \zeta_1],$$

where ζ_1 is the unique vector field in $\Gamma(T^{-1}M)$ such that $q_{-3}(\zeta) = q_{-3}([r, \zeta_1])$, $\phi = \alpha(\zeta - [r, \zeta_1])$ and $\zeta_2 = \zeta - \phi r - [r, \zeta_1]$. Thus, requiring that

$$d\tilde{\alpha}(r, \gamma) = \tilde{\alpha}([\gamma, r]) = 0$$

for all $\gamma \in \Gamma(T^{-1}M)$ uniquely defines an extension of α to all of TM. □

REMARK 3.8. In the following we will use the same notation for both $\alpha \in \Gamma(L(T^{-2}M, \mathbb{R}))$ and its extension to a one-form on the manifold M.

REMARK 3.9. Note that $\zeta \mapsto \zeta - \alpha(\zeta)r$ defines a projection

$$T^{-2}M \to T^{-1}M$$

and $\zeta \mapsto \alpha(\zeta)q_{-2}(r)$ defines a map

$$TM \to \mathrm{gr}_{-2}(TM)$$

extending the natural quotient map q_{-2}. These two maps are part of the soldering form associated to a generalised contact form. Likewise, the partial connection

$$\{\nabla_{\gamma}\eta, q_{-2}(r)\} = q_{-3}([\gamma, [\eta, r]]),$$

where r is the generalised Reeb field, is part of the associated Weyl connection.

3.2. On Weyl structures and scales

In this section, we shall survey the parts of the theory of Weyl structures for parabolic geometries that we will need in the sequel. The reader can find all the proofs of the results stated here in [**17**] and [**16**]. Furthermore, we will show that any generalised contact form determines a unique Weyl structure for generic rank two distributions in dimension five.

3.2.1. Weyl structures. Let $(\mathcal{G} \to M, \omega)$ be a regular parabolic geometry of type (G, P) and let $\mathcal{G}_0 \to M$ be the underlying G_0-principal bundle, cf. 1.3.2.

DEFINITION 3.10. A Weyl structure for the parabolic geometry $(\mathcal{G} \to M, \omega)$ is a global G_0-equivariant section σ of the projection $\pi : \mathcal{G} \to \mathcal{G}_0$. If σ is locally defined, it is called a local Weyl structure.

Note that any Weyl structure defines a reduction of structure group of the P-principal bundle $\mathcal{G} \to M$ to the structure group G_0.

One can show that for any parabolic geometry global Weyl structures exist and the set of all Weyl structures forms an affine space modelled on the space $\Gamma(\mathrm{gr}(T^*M))$ of smooth sections of the associated graded of the cotangent bundle. If σ and $\hat{\sigma}$ are Weyl structures, then there is a unique section $\Upsilon \in \Gamma(\mathrm{gr}(T^*M))$ with corresponding functions $\Upsilon_i : \mathcal{G}_0 \to \mathfrak{g}_i$ such that $\hat{\sigma}(u) = \sigma(u)\exp(\Upsilon_1(u)) \cdots \exp(\Upsilon_k(u))$.

Suppose $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is a fixed Weyl structure. Consider the pullback of the Cartan connection ω along σ. This yields a one-form $\sigma^*\omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$. By equivariance of ω and σ, this one-form is G_0-equivariant. Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ be the $|k|$-grading of the semisimple Lie algebra. We may decompose $\sigma^*\omega = \sigma^*\omega_{-k} + \cdots + \sigma^*\omega_k$ according to its values. For any element $A \in \mathfrak{g}_0$, consider the fundamental vector field $\zeta_A(u) = \frac{d}{dt}|_{t=0} u \cdot \exp(tA)$. Using equivariance of σ, we see that

$$\sigma^*\omega\left(\zeta_A(u)\right) = \omega\left(\zeta_A(\sigma(u))\right) = A.$$

On the one hand, this shows that $\sigma^*\omega_0$ defines a principal connection on the G_0-bundle $\mathcal{G}_0 \to M$, called *Weyl connection.* On the other hand, we see that the components $\sigma^*\omega_i$ for $i \neq 0$ are horizontal. It follows that $\sigma^*\omega_i$ can be viewed as an element in $\Omega^1(M, \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i)$. The part

$$\sigma^*\omega_+ = \sigma^*\omega_1 + \cdots + \sigma^*\omega_k \in \Omega^1(M, \mathrm{gr}(T^*M))$$

is called the *rho tensor* and

$$\sigma^*\omega_- = \sigma^*\omega_{-k} + \cdots + \sigma^*\omega_{-1} \in \Omega^1(M, \mathrm{gr}(TM))$$

is called the *soldering form* associated to σ. The soldering form associated to a Weyl structure can be viewed as an isomorphism

$$TM \to \mathrm{gr}(TM),$$

such that T^iM is mapped to $\oplus_{j \geq i}\mathrm{gr}_j(TM)$ and the component in $\mathrm{gr}_i(TM)$ equals the image of the natural projection $q_i : T^iM \to \mathrm{gr}_i(TM)$ for $i = -1, \cdots, -k$. Hence, it provides a splitting of the filtration on TM.

Since the Weyl connection is a principal bundle connection on $\mathcal{G}_0 \to M$ it induces linear connections on all vector bundles associated to $\mathcal{G}_0$. All these connections are referred to as Weyl connections. In particular, since the soldering form identifies the tangent bundle with the associated bundle $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$, a Weyl structure gives rise to a Weyl connection on the tangent bundle TM.

The general theory also provides explicit formulae for the change of the Weyl connection, the soldering form and the rho tensor as we change the Weyl structure. These transformation rules for general parabolic geometries can be found in [**17**]. The transformation of the Weyl connection on the bundle $\mathrm{gr}_2(T^*M)$ is written down in (3.3) and the other formulae look similar.

REMARK 3.11. The theory of Weyl structures for parabolic geometries generalises well known concepts from conformal geometry. Suppose M is a manifold with a conformal structure. The conformal structure determines a unique (up to equivalence) normal parabolic geometry. Weyl connections for this geometry are torsion free linear connections ∇ on TM compatible with the conformal structure, i.e. $\nabla_\xi g = fg$ for any metric g from the conformal class. A particularly nice subclass of Weyl connections are the Levi Civita connections for metrics from the conformal class. Using terminology that will be explained in the sequel, see 3.15, these are exact Weyl connections, respectively Weyl connections determined by global scales. Since conformal structures are $|1|$-graded, the soldering form is just id_{TM} viewed as a 1-form with values in TM. The rho tensor P determined by a metric g is the classical rho tensor, or Schouten tensor, in conformal geometry, i.e. the trace modification

$$P = \frac{1}{n-2}(\mathrm{Ric} - \frac{1}{2(n-1)}\mathrm{R}g)$$

of the Ricci tensor.

3.2.2. The bundle of scales. Recall that a Weyl connection $\sigma^*\omega_0$ induces linear connections on all bundles associated to $\mathcal{G}_0$. It turns out that for any parabolic geometry the induced Weyl connection on certain associated line bundles uniquely determines the Weyl structure. We will see that for generic rank two distributions in dimension five there is a natural choice for a line bundle with this property, namely the bundle $\mathrm{gr}_2(T^*M) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_2$. Nowhere vanishing sections of this bundle have a nice geometric interpretation. They are generalised contact forms.

PROPOSITION 3.12. *Let $\mathcal{G}_0 \to M$ be the frame bundle for a generic rank two distribution in dimension five and let $(\mathcal{G} \to M, \omega)$ be the corresponding regular, normal parabolic geometry. Suppose the bundle $\mathcal{L} = \mathrm{gr}_2(T^*M)$ is orientable, i.e. global generalised contact forms exist. Then, mapping a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ to the induced linear connection on the line bundle $\mathcal{L}$ defines a bijection between the set of Weyl structures for $(\mathcal{G} \to M, \omega)$ and the set of linear connections on $\mathcal{L}$.*

The proof of the proposition is completely parallel to the proof of Theorem 3.7 in [**17**] for bundles of scales, see also Remark 3.15 below. The key to that proof is that one knows the transformation of the induced Weyl connection under the change of the Weyl structure. Let σ and $\hat{\sigma} = \sigma\exp(\Upsilon_1)\exp(\Upsilon_2)\exp(\Upsilon_3)$ be two Weyl structures, where $(\Upsilon_1, \Upsilon_2, \Upsilon_3) \in \mathrm{gr}(T^*M)$. Let s be a nowhere vanishing section of $\mathcal{L}$ and let ζ be a vector field on M. Denote by ζ_i the component of ζ in $\mathrm{gr}_i(TM)$ under the identification given by the soldering form. Then the Weyl connections corresponding to the two Weyl structures are related by

$$\begin{aligned}\hat{\nabla}_\zeta s = \nabla_\zeta s + \{-\frac{1}{6}\{\Upsilon_1, \{\Upsilon_1, \{\Upsilon_1, \zeta_{-3}\}\}\} + \{\Upsilon_2, \{\Upsilon_1, \zeta_{-3}\}\} - \{\Upsilon_3, \zeta_{-3}\} \\ + \frac{1}{2}\{\Upsilon_1, \{\Upsilon_1, \zeta_{-2}\}\} - \{\Upsilon_2, \zeta_{-2}\} - \{\Upsilon_1, \zeta_{-1}\}, s\}.\end{aligned} \tag{3.3}$$

Using that, Proposition 3.12 can be proved as follows:

PROOF. First note that for any nonzero $s \in \mathfrak{g}_2$ the map $\mathfrak{g}_i \times \mathfrak{g}_{-i} \to \mathfrak{g}_2$ defined by $(X, Y) \mapsto [[X, Y], s]$ is a multiple of $(X, Y) \mapsto B(X, Y)s$, where B denotes the Killing form.

This can be verified directly using the explicit description of the Lie algebra, or it follows from a more general observation: Let $\lambda' : \mathfrak{g}_0 \to \mathfrak{gl}(\mathbb{V})$ be any non-trivial one-dimensional Lie algebra representation of $\mathfrak{g}_0$. Note that the reductive Lie algebra $\mathfrak{g}_0$ decomposes into a direct sum $\mathfrak{g}_0^{ss} \oplus \mathfrak{z}(\mathfrak{g}_0)$ of semisimple part and center. The Killing form B restricts to a non-degenerate bilinear form on the center $\mathfrak{z}(\mathfrak{g}_0)$. Hence, there is an element $E_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$ such that $\lambda'(A)v = B(E_\lambda, A)v$ holds for any $A \in \mathfrak{z}(\mathfrak{g}_0)$. Since the center is one-dimensional, the element $E_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$ is some multiple of the grading element. Since any element A from the semisimple Lie algebra $\mathfrak{g}_0^{ss}$ acts trivially on any one-dimensional representation $\mathbb{V}$, we indeed have $\lambda'(A)v = B(E_\lambda, A)v$ for all $A \in \mathfrak{g}_0$. Now consider elements $Z \in \mathfrak{g}_i$ and $X \in \mathfrak{g}_{-i}$. Invariance of the Killing form implies $B([Z, X], E_\lambda) = -B(X, [Z, E_\lambda])$ and so $\lambda'([Z, X])$ is a multiple of $B(Z, X)$. It follows that the map $\mathfrak{g}_i \times \mathfrak{g}_{-i} \to \mathfrak{g}_2$ given by $(X, Y) \mapsto [[X, Y], s]$ defines a duality pairing and the same is true for the induced bundle map $(\zeta_i, \Upsilon_i) \mapsto \{\{\zeta_i, \Upsilon_i\}, s\}$.

Next we show that mapping a Weyl structure to its induced linear connection on $\mathrm{gr}_2(T^*M)$ is injective: For a section $\zeta \in \Gamma(T^{-1}M)$, formula (3.3) simplifies to $\hat{\nabla}_\zeta s = \nabla_\zeta s - \{\{\Upsilon_1, \zeta_{-1}\}, s\}$. So $\hat{\nabla}_\zeta s = \nabla_\zeta s$ for all $\zeta \in \Gamma(T^{-1}M)$ if and only if $\Upsilon_1 = 0$. Now we assume $\Upsilon_1 = 0$ and consider sections $\zeta \in \Gamma(T^{-2}M)$. Then $\hat{\nabla}_\zeta s = \nabla_\zeta s - \frac{1}{2}\{\{\Upsilon_2, \zeta_{-2}\}, s\}$. Hence, $\hat{\nabla}_\zeta s = \nabla_\zeta s$ for all $\zeta \in \Gamma(T^{-2}M)$ if and only if $\Upsilon_1 = 0$ and $\Upsilon_2 = 0$. Finally, we conclude that $\hat{\nabla}_\zeta s = \nabla_\zeta s$ for all $\zeta \in \mathfrak{X}(M)$ if and only if $\Upsilon = 0$, i.e. $\hat{\sigma} = \sigma$.

Conversely, starting with a fixed Weyl structure and its induced connection ∇, any other linear connection on $\mathrm{gr}_2(TM)$ is of the form $\hat{\nabla}_\zeta s = \nabla_\zeta s + \gamma(\zeta)$ for some $\gamma \in \Omega^1(M)$. We can find a unique section $\Upsilon_1 \in (T^{-1}M)^*$ such that $\gamma(\zeta)s = -\{\{\Upsilon_1, \zeta\}, s\}$ for all $\xi \in \Gamma(T^{-1}M)$. Now we replace σ by $\tilde{\sigma} = \sigma \exp(\Upsilon_1)$. We denote $\tilde{\nabla}$ the connection induced by $\tilde{\sigma}$. Then $\hat{\nabla}_\zeta = \tilde{\nabla}_\zeta + \tilde{\gamma}(\zeta)$ for some $\tilde{\gamma} \in \Omega^1(M)$ and $\tilde{\gamma}(\zeta) = 0$ for all $\zeta \in \Gamma(T^{-1}M)$. Hence, for $\tilde{\gamma}$ we find a section $\Upsilon_2 \in \mathrm{gr}_2(T^*M)$ such that $\tilde{\gamma}(\zeta)s = -\{\{\Upsilon_2, \zeta_{-2}\}, s\}$ for all $\zeta \in \Gamma(T^{-2}M)$. Continuing along these lines, we see that $\hat{\nabla}$ is induced by some Weyl structure $\hat{\sigma} = \sigma(u)\exp(\Upsilon_1(u))\exp(\Upsilon_2(u))\exp(\Upsilon_3(u))$. This proves that we have indeed a bijective correspondence between Weyl structures for $(\mathcal{G}, \omega)$ and linear connections on $\mathrm{gr}_2(TM)$. □

Corollary 3.13. *For any generalised contact form α there exists a unique Weyl structure such that α is parallel for the induced Weyl connection.*

Proof. Suppose we are given a generalised contact form α, that is, a smooth nowhere vanishing sections of the line bundle $\mathcal{L} = \mathrm{gr}_2(T^*M)$. Then α trivialises the line bundle. Hence, it defines a flat linear connection ∇ on $\mathcal{L}$ characterised by $\nabla\alpha = 0$. Now the result follows from the above proposition. □

Remark 3.14. Locally, the result of the above corollary is also true for local generalised contact forms: A smooth local nowhere vanishing section α of the line bundle $\mathrm{gr}_2(T^*M)$ can be equivalently viewed as local generalised contact form. And the proofs of Proposition 3.12 and Corollary 3.13 show that for any local generalised contact form, there is a unique local Weyl structure such that α is parallel for the induced Weyl connection.

Remark 3.15. The proof of Proposition 3.12 can be easily modified such that it indeed shows that for any group homomorphism $\lambda : G_0 \to \mathbb{R}_+$ with non-trivial derivative $\lambda' : \mathfrak{g}_0 \to \mathbb{R}$, the corresponding orientable associated line bundle $\mathcal{L} = \mathcal{G}_0 \times_{G_0} \mathbb{R}$ has the property that mapping a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ to the induced linear connection defines a bijection between the set of Weyl structures and the set of linear connections on $\mathcal{L}$. In formula (3.3) as well as in the proof of Proposition 3.12 we have to replace the algebraic bracket $\{,\} : \mathcal{A}_0 M \times \mathcal{L} \to \mathcal{L}$ by the bundle map induced by the infinitesimal $\mathfrak{g}_0$-representation λ', compare with the proof of Theorem 3.7 in [**17**].

In that paper so-called *bundles of scales* have been introduced for all parabolic geometries. For geometries where the Lie algebra $\mathfrak{g}_0$ has one-dimensional center, these are just orientable line bundles associated to $\mathcal{G}_0$ with respect to non-trivial $\mathfrak{g}_0$-representations. In particular, this is the case for generic rank two distributions in dimension five. So if $\mathrm{gr}_2(T^*M)$ is orientable, i.e. if a global generalised contact form exists, then it is a bundle of scales. Without the assumption of orientability, the line bundles $E[w]$ corresponding to representations $\rho(M) = |\det(M)|^{-w}$ are bundles of scales. Note that the $G_0 = GL(2,\mathbb{R})$-representation on $\mathfrak{g}_2$ is given by $\rho(M)(X) = \det(M)X$. Hence, $E[-2] \cong \mathrm{gr}_2(T^*M) \otimes \mathrm{gr}_2(T^*M)$. The existence of a generalised contact form means that the structure group of $\mathcal{G}_0$ can be reduced to $G_0^{ss} = SL(2,\mathbb{R})$ and thus implies $\mathrm{gr}_2(T^*M) \cong E[-1]$.

Global sections of a bundle of scales are usually called *scales* and Weyl structures determined by scales are referred to as *exact Weyl structures*. So using this terminology, generalised contact forms are scales and the choice of a generalised contact form determines an exact Weyl structure for a generic rank two distribution in dimension five.

REMARK 3.16. To stress the analogy with contact structures once again, let us point out that in the parabolic contact case, assuming orientability, the bundle $(TM/\mathcal{D})^*$ is a bundle of scales. Hence, contact forms are scales and can be used to parametrise Weyl structures.

3.3. Characterisation of the normal Weyl form

In this section we show how to determine the Weyl connection, the soldering form and the rho tensor associated to a given generalised contact form.

3.3.1. The Weyl form and the Weyl curvature. The notion of a Weyl form and the description of the Weyl curvature have also been developed for general parabolic geometries in [**17**]. We restrict our discussion to the case of generic rank two distributions in dimension five, though.

Let $\mathcal{H} = T^{-1}M$ be a generic rank two distribution in dimension five, let $\mathcal{G}_0 \to M$ be the frame bundle for the distribution and let $(\mathcal{G}, \omega)$ be the associated regular, normal parabolic geometry. Suppose $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is a Weyl structure for the geometry. The pullback of the Cartan connection ω splits into parts according to the decomposition of the Lie algebra $\mathfrak{g}$. The pieces $\sigma^*\omega_-$, $\sigma^*\omega_0$ and $\sigma^*\omega_+$ can be viewed as the following data:

- an isomorphism $\Theta : TM \cong \mathrm{gr}(TM)$ of the form

$$\xi \mapsto \pi_{-1}(\xi) + q_{-2}(\pi_{-2}(\xi)) + q_{-3}(\xi),$$

  where $\pi_{-1} : TM \to T^{-1}M$ and $\pi_{-2} : TM \to T^{-2}M$ are projections onto the respective subbundles,

- a principal connection on $\mathcal{G}_0$, or equivalently, a linear connection

$$\nabla : \mathfrak{X}(M) \times \Gamma(T^{-1}M) \to \Gamma(T^{-1}M),$$

- a one-form $P \in \Omega^1(M, \mathrm{gr}(T^*M))$.

The principal connection $\sigma^*\omega_0$ induces Weyl connections on $\mathrm{gr}(TM)$ and $\mathrm{gr}(T^*M)$ which will be also denoted by ∇. Any G_0-equivariant one-form $\tau \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$ which can be viewed as a triple (Θ, ∇, P) as above is called a *Weyl form* for the rank two distribution.

For a Weyl form $\tau \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$, one defines the *Weyl curvature* to be the two form $K_\tau \in \Omega^2(\mathcal{G}_0, \mathfrak{g})$ given by

$$K_\tau(\xi, \eta) := d\tau(\xi, \eta) + [\tau(\xi), \tau(\eta)],$$

where ξ, η are vector fields on $\mathcal{G}_0$. The Weyl curvature is horizontal and G_0-equivariant and thus can be viewed as a two-form on M with values in

$$\mathrm{gr}(\mathcal{A}M) = \mathcal{G}_0 \times_{G_0} \mathfrak{g} = \mathrm{gr}(TM) \oplus \mathrm{End}(\mathcal{H}) \oplus \mathrm{gr}(T^*M),$$

cf. Remark 1.15. The G_0-equivariant function $\mathcal{G}_0 \to \mathfrak{g}$ representing the section $K_\tau(\zeta, \zeta')$ can be written as $d\tau(\zeta^h, \zeta'^h) + [\tau(\zeta^h), \tau(\zeta'^h)]$, for the horizontal lifts ζ^h, ζ'^h with respect to the principal bundle connection τ_0. Note that for any horizontal lift we have $\tau_0(\zeta^h) = 0$ and thus $\tau(\zeta^h) = \tau_-(\zeta^h) + \tau_+(\zeta^h)$.

REMARK 3.17. Via the isomorphism $\Theta : TM \cong \mathrm{gr}(TM)$, we can identify forms $\Omega^2(M, \mathrm{gr}(\mathcal{A}M))$ with smooth sections of the bundle $L(\Lambda^2\mathrm{gr}(TM), \mathrm{gr}(\mathcal{A}M))$. In the following we will sometimes use this identification without explicitly stating it.

Let $\partial^* : \Omega^2(M, \mathrm{gr}(\mathcal{A}M)) \to \Omega^1(M, \mathrm{gr}(\mathcal{A}M))$ be the map induced by the Kostant codifferential. A Weyl form is called normal if its Weyl curvature is ∂^*-closed, i.e. if $\partial^* K_\tau = 0$. Normality of the Cartan connection ω implies that $\partial^* K_{\sigma^*\omega} = 0$ for any Weyl structure σ. Conversely, any normal Weyl form τ is of the form $\sigma^*\omega$ for some Weyl structure σ, see [**17**]. That is:

THEOREM 3.18. *Let ω be the normal Cartan connection. Then a Weyl form τ is of the form $\sigma^*\omega$ for some Weyl structure σ if and only if it is normal.*

Next we split $K_\tau(\xi, \eta)$ into parts and look at the interpretation of the individual parts as tensor fields on the manifold. The first part

$$d\tau_0(\xi, \eta) + [\tau_0(\xi), \tau_0(\eta)]$$

is the curvature of the principal connection τ_0. It can be viewed as

$$R \in \Omega^2(M, \mathrm{End}(\mathcal{H}))$$

given by

$$R(\zeta, \zeta') = \nabla_\zeta \nabla_{\zeta'} - \nabla_{\zeta'} \nabla_\zeta - \nabla_{[\zeta, \zeta']}. \tag{3.4}$$

The next part

$$d\tau_-(\xi, \eta) + [\tau_-(\xi), \tau_-(\eta)] + [\tau_-(\xi), \tau_0(\eta)] + [\tau_0(\xi), \tau_-(\eta)],$$

is the torsion of the Cartan connection $\tau_0+\tau_- \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0\oplus\mathfrak{g}_-)$. Inserting horizontal lifts of vector fields $\zeta,\zeta' \in \mathfrak{X}(M)$ yields

$$d\tau_-(\zeta^h,\zeta'^h) + [\tau_-(\zeta^h),\tau_-(\zeta'^h)] = \\ \zeta^h\cdot\tau_-(\zeta'^h) - \zeta'^h\cdot\tau_-(\zeta^h) - \tau([\zeta^h,\zeta'^h]) + [\tau_-(\zeta^h),\tau_-(\zeta'^h)]$$

This equivariant function corresponds to the element

$$T \in \Omega^2(M, \mathrm{gr}(TM))$$

given by

$$T(\zeta,\zeta') = \nabla_\zeta\Theta(\zeta') - \nabla_{\zeta'}\Theta(\zeta) - \Theta([\zeta,\zeta']) + \{\Theta(\zeta),\Theta(\zeta')\}. \tag{3.5}$$

To interpret the remaining part of K_τ, note that the Weyl connection ∇ on $\mathrm{gr}(T^*M)$ extends to the covariant exterior derivative d^∇ on $\mathrm{gr}(T^*M)$-valued forms. Inserting horizontal lifts, the remaining part reads

$$d\tau_+(\zeta^h,\zeta'^h) + [\tau_+(\zeta^h),\tau_+(\zeta'^h)] + [\tau_-(\zeta),\tau_+(\zeta'^h)] + [\tau_+(\zeta^h),\tau_-(\zeta'^h)],$$

which represents

$$d^\nabla P(\zeta,\zeta') + \{P(\zeta),P(\zeta')\} + \{\Theta(\zeta),P(\zeta')\} + \{P(\zeta),\Theta(\zeta')\}.$$

Let

$$\partial P(\zeta,\zeta') = \{\Theta(\zeta),P(\zeta')\} - \{\Theta(\zeta'),P(\zeta)\} - P(\{\Theta(\zeta),\Theta(\zeta')\})$$

be the bundle map induced by the Lie algebra differential. We introduce the *Cotton-York-tensor* $Y \in \Omega^2(M, \mathrm{gr}(T^*M))$ defined by

$$Y(\zeta,\zeta') := d^\nabla P(\zeta,\zeta') + P(\{\Theta(\zeta),\Theta(\zeta')\}) + \{P(\zeta),P(\zeta')\}. \tag{3.6}$$

Summarising, one has the following expression for the Weyl curvature.

PROPOSITION 3.19. *The Weyl curvature is of the form*

$$K_\tau = (T,R,Y) + \partial(P) \in \Omega^2(M, \mathrm{gr}(TM)\oplus\mathrm{End}(\mathcal{H})\oplus\mathrm{gr}(T^*M))$$

where R, T and Y are defined as in (3.4)*,* (3.5)*,* (3.6)*.*

The following splittings of the Weyl form, the Weyl curvature and its components will be convenient. Recall that the grading on $\mathfrak{g}$ induces a grading on $\mathrm{gr}(\mathcal{A}M)$ such that $\mathcal{A}_sM = \mathrm{gr}_s(TM)$ for $s<0$, $\mathcal{A}_0M = \mathrm{End}(\mathcal{H})$ and $\mathcal{A}_sM = \mathrm{gr}_s(T^*M)$ for $s>0$. We can split any differential form $\Phi \in \Omega^k(M, \mathrm{gr}(\mathcal{A}M))$ according to its values and we denote Φ_s the component of Φ with values in $\mathcal{A}_sM$. The component of T with values in $\mathrm{gr}_s(TM)$ is given by

$$T_s(\zeta,\zeta') = \nabla_\zeta\Theta_s(\zeta') - \nabla_{\zeta'}\Theta_s(\zeta) - \Theta_s([\zeta,\zeta']) + \sum_{l,k<0,l+k=s}\{\Theta_l(\zeta),\Theta_k(\zeta')\}$$

and the component of Y with values in $\mathrm{gr}_s(T^*M)$ is given by

$$Y_s(\zeta,\zeta') = \nabla_\zeta P_s(\zeta') - \nabla_{\zeta'}P_s(\zeta) - P_s([\zeta,\zeta'] - \{\Theta(\zeta),\Theta(\zeta')\}) + \sum_{a,b>0,a+b=s}\{P_a(\zeta),P_b(\zeta')\}.$$

Recall that the soldering form $\Theta : TM \cong \mathrm{gr}(TM)$ identifies forms $\Omega^2(M,\mathrm{gr}(\mathcal{A}M))$ with sections of $L(\Lambda^2\mathrm{gr}(TM),\mathrm{gr}(\mathcal{A}M))$. In particular, we can split a form Φ into homogeneous components. We denote $\Phi^{(r)}$ the homogeneous component of degree r. Note that the

Lie algebra differential ∂ as well as the Kostant codifferential ∂^* are compatible with homogeneities.

3.3.2. The computations. From the previous section, see Theorem 3.18 and Proposition 3.19, we know that a Weyl form given by the data (Θ, ∇, P) is the pullback of the normal Cartan connection along a Weyl structure σ if and only if the normalisation condition

$$\partial^*(T, R, Y) + \partial^* \partial P = 0 \tag{3.7}$$

is satisfied. Furthermore, by Proposition 3.12, a generalised contact form α can be used to specify the Weyl structure σ. That means, there is a unique Weyl form characterised by equation (3.7) and the fact that α is parallel for the associated Weyl connection, i.e.

$$\nabla \alpha = 0. \tag{3.8}$$

This is the Weyl form associated to α. In the following we will analyse the normalisation condition (3.7) and derive a simpler, more explicit description of the Weyl form in terms of α, the natural quotient maps q_i and the Lie brackets of vector fields. The analysis is done homogeneity by homogeneity:

- Analysing homogeneity one of equation 3.7, we characterise the Weyl connection ∇ in direction of the distribution $T^{-1}M$ and the parts
$$\pi_{-1} : T^{-2}M \to T^{-1}M$$
and
$$q_{-2}\pi_{-2} : TM \to \mathrm{gr}_{-2}(TM)$$
of the soldering form.
- Proceeding with homogeneity 2, we characterise the Weyl connection ∇ in direction of $T^{-2}M$, the remaining part
$$\pi_{-1} : TM \to T^{-1}M$$
of the soldering form as well as the component $P^{(2)} : \mathrm{gr}_{-1}(TM) \to \mathrm{gr}_1(T^*M)$ of the rho tensor.
- Taking in account also homogeneity three we obtain a characterisation of the full Weyl connection ∇ and of the components $P^{(3)} : \mathrm{gr}_{-2}(TM) \to \mathrm{gr}_1(T^*M)$ and $P^{(3)} : \mathrm{gr}_{-1}(TM) \to \mathrm{gr}_2(T^*M)$
- Finally, the analysis of remaining homogeneities provides a description of the remaining components of the rho tensor. These are $P^{(4)} : \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_1(T^*M)$, $P^{(4)} : \mathrm{gr}_{-2}(TM) \to \mathrm{gr}_2(T^*M)$, $P^{(4)} : \mathrm{gr}_{-1}(TM) \to \mathrm{gr}_3(T^*M)$, $P^{(5)} : \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_2(T^*M)$, $P^{(5)} : \mathrm{gr}_{-2}(TM) \to \mathrm{gr}_3(T^*M)$ and $P^{(6)} : \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_3(T^*M)$.

Remark 3.20. The following simplification will be very helpful. In Proposition 2.15 we used Kostant's version of the Bott-Borel-Weil theorem to show that lowest homogeneous component of the curvature κ is of homogeneous degree 4. For a normal Weyl form, the corresponding Weyl curvature is the pullback of the normal Cartan connection along a Weyl structure. Hence also the lowest homogeneous component of the Weyl curvature is of degree four. This means, that

$$(T^{(i)}, R^{(i)}, Y^{(i)}) + \partial P^{(i)} = 0, \tag{3.9}$$

for all $i \leq 3$.

Remark 3.21. In the sequel, the duality pairing between $\mathrm{gr}_{-i}(TM) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-i}$ and $\mathrm{gr}_i(T^*M) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$ will be induced by $\frac{1}{24}$ of the Killing form, denoted by B.

Remark 3.22. Finally, note that in general, there may only exist local generalised contact forms. Starting with a local generalised contact form α, the corresponding local Weyl form can also be characterised by equations (3.8) and (3.7) and the forthcoming calculations will give a description of the local Weyl form associated to α.

3.3.2.1. *Homogeneity 1.* By definition, the homogeneous components of degree ≤ 0 of the forms T, R, Y and ∂P vanish. Since R, Y and P have values inside the non-negative grading components of $\mathrm{gr}(\mathcal{A}M)$, we see that the components $R^{(1)}$, $Y^{(1)}$ and $\partial P^{(1)}$ vanish as well. Hence, $K_\tau^{(1)} = T^{(1)}$ and the only non-trivial condition we get from normality in homogeneity one is vanishing of $T^{(1)}$. Vanishing of the component

$$T^{-1}M \times T^{-1}M \to \mathrm{gr}_{-1}(TM)$$

yields

$$T^{(1)}(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - \pi_{-1}([\xi, \eta]) = 0 \tag{3.10}$$

vanishing of the component

$$T^{-1}M \times \mathrm{gr}_{-2}(TM) \to \mathrm{gr}_{-2}(TM)$$

yields

$$T^{(1)}(\xi, q_{-2}(\gamma)) = \nabla_\xi q_{-2}(\gamma) - q_{-2}\pi_{-2}([\xi, \gamma]) + \{\xi, \pi_{-1}(\gamma)\} = 0 \tag{3.11}$$

and vanishing of the component

$$T^{-1}M \times \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_{-3}(TM)$$

yields

$$T^{(1)}(\xi, q_{-3}(\zeta)) = \nabla_\xi q_{-3}(\zeta) - q_{-3}([\xi, \zeta]) + \{\xi, q_{-2}\pi_{-2}(\zeta)\} = 0 \tag{3.12}$$

for any $\xi, \eta \in \Gamma(T^{-1}M)$, $\gamma \in \Gamma(T^{-2}M)$ and $\zeta \in \mathfrak{X}(M)$. These equations uniquely determine the part

$$\nabla : \Gamma(T^{-1}M) \times \Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$$

of the Weyl connection, and the parts

$$\pi_{-1} : T^{-2}M \to T^{-1}M$$

and

$$q_{-2}\pi_{-2} : TM \to T^{-2}M/T^{-1}M$$

of the soldering form.

Since the natural quotient map $q_{-2} : T^{-2}M \to \mathrm{gr}_{-2}(TM)$ defines an isomorphism $\ker(T^{-1}M) \cong \mathrm{gr}_{-2}(TM)$, we find a unique vector field $\hat{r} \in T^{-2}M$ such that $\alpha(\hat{r}) = 1$ and $\pi_{-1}(\hat{r}) = 0$. Then π_{-1} is of the form

$$\pi_{-1}(\eta) = \eta - \alpha(\eta)\hat{r}.$$

The map $q_{-2} \circ \pi_{-2}$ extends the quotient map $q_{-2}(\zeta) = \alpha(\zeta) q_{-2}(\hat{r})$ to TM, hence it can be described as

$$q_{-2}\pi_{-2}(\zeta) = \hat{\alpha}(\zeta) q_{-2}(\hat{r}),$$

for an extension $\hat{\alpha}$ of the generalised contact form to a one-form. Characterising the projection π_{-1} and the map $q_{-2} \circ \pi_{-2}$ amounts to characterising the vector field $\hat{r}$ and the extension $\hat{\alpha}$.

Now let us consider equations (3.11) and (3.12). Since α is parallel for the connection ∇, so is the dual section $q_{-2}(\hat{r})$. That means that choosing $\gamma = \hat{r}$, equation (3.11) implies that

$$q_{-2}\pi_{-2}([\xi, \hat{r}]) = 0$$

for all $\xi \in \Gamma(T^{-1}M)$. For the extension $\hat{\alpha}$, this means that $i_{\hat{r}} d\hat{\alpha}|_{T^{-1}M}$ vanishes. Next we consider (3.12), where we put $\zeta = [\eta, \hat{r}]$ for $\eta \in \Gamma(T^{-1}M)$. Then we have

$$\nabla_\xi \{\eta, q_{-2}(\hat{r})\} - q_{-3}([\xi, [\eta, \hat{r}]]) = 0$$

and since ∇ is compatible with the bracket $\{,\}$, this implies

$$\{\nabla_\xi \eta, q_{-2}(\hat{r})\} = q_{-3}([\xi, [\eta, \hat{r}]])$$

for all $\xi, \eta \in \Gamma(T^{-1}M)$. Hence the part of the Weyl connection ∇ in $T^{-1}M$-directions is precisely the partial connection defined in (3.1) associated to $\hat{r}$. But then $\alpha(\hat{r}) = 1$ and $\nabla q_{-2}(\hat{r}) = 0$ is precisely what characterises the generalised Reeb field associated to α, compare with Proposition 3.5. Finally, we see that $\hat{\alpha}$ is the extension to a one-form as in Corollary 3.7.

Remark 3.23. Note that the following computation shows that $\nabla\alpha = 0$ and equations (3.11) and (3.12) imply that equation (3.10) is holds as well:

$$\begin{aligned}\{\pi_{-1}([\xi, \eta]), q_{-2}(r)\} &= q_{-3}([[\xi, \eta] - \alpha([\xi, \eta])r, r]) = q_{-3}([\xi, \eta], r]) \\ &= q_{-3}([\xi, [\eta, r]]) - q_{-3}([\eta, [\xi, r]]) = \{\nabla_\xi \eta - \nabla_\eta \xi, q_{-2}(r)\}\end{aligned}$$

and thus

$$\nabla_\xi \eta - \nabla_\eta \xi = \pi_{-1}([\xi, \eta]).$$

Hence, the theory implies that there are unique data ∇, π_{-1} and $q_{-2}\pi_{-2}$ characterised by $\nabla\alpha = 0$, (3.11) and (3.12). By the above discussion this is equivalent to the existence of a unique partial connection ∇, a unique vector field $r \in \Gamma(T^{-2}M)$ with $\alpha(r) = 1$, and a unique extension to a one-form $\alpha \in \Omega^1(M)$ such that

- $\nabla q_{-2}(r) = 0$,
- $\{\nabla_\xi \eta, q_{-2}(r)\} = q_{-3}([\xi, [\eta, r]])$ for all $\xi, \eta \in \Gamma(T^{-1}M)$
- and $i_r d\alpha|_{T^{-1}M} = 0$.

In particular, referring to the general theory there is no need to prove Proposition 3.5 and Corollary 3.7.

3.3.2.2. *Homogeneity 2.* Having determined the conditions on the normal Weyl form we get from vanishing of the components of K_τ of homogeneity one, we proceed to homogeneity two.

First, we introduce new notation. Let $R(\xi,\eta) = \nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]}$ be the curvature of the Weyl connection ∇. We introduce a map

$$\Phi : \Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$$

characterised by

$$\begin{aligned} d\alpha(\xi,\eta)\Phi(\gamma) &= -R(\xi,\eta)\gamma - d\alpha(\xi,\eta)\nabla_r\gamma \\ &= -(\nabla_\xi\nabla_\eta\gamma - \nabla_\eta\nabla_\xi\gamma - \nabla_{\pi_{-1}([\xi,\eta])}\gamma) \end{aligned} \tag{3.13}$$

for any $\xi,\eta,\gamma \in \Gamma(T^{-1}M)$. Note that Φ depends only on the part of the Weyl connection ∇ that we have determined already. We define a second map

$$\Psi : \Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$$

by

$$\{\Psi(\gamma), q_{-2}(r)\} = \frac{1}{2} q_{-3}([r,[\gamma,r]]). \tag{3.14}$$

Here r is the generalised Reeb field associated to α.

Remark 3.24. The map Φ can be rewritten as follows. Suppose we have a frame $\xi,\eta \in \Gamma(T^{-1}M)$ such that $\{\xi,\eta\} = q_{-2}(r)$. Then, using the characterisation of ∇ in $T^{-1}M$-directions, we have

$$\{\Phi(\gamma), q_{-2}(r)\} = \nabla_\xi q_{-3}([\eta,[\gamma,r]]) - \nabla_\eta q_{-3}([\xi,[\gamma,r]]) - q_{-3}([\pi_{-1}([\xi,\eta]),[\gamma,r]])$$

We further use (3.12), i.e.

$$\nabla_\xi q_{-3}([\eta,[\gamma,r]]) = q_{-3}([\xi,[\eta,[\gamma,r]]]) - \{\xi, q_{-2}\pi_{-2}([\eta,[\gamma,r]])\}$$

and $\pi_{-1}([\xi,\eta]) = [\xi,\eta] - r$ and the Jakobi identity to see that

$$\{\Phi(\gamma), q_{-2}(r)\} = -\{\xi, q_{-2}\pi_{-2}([\eta,[\gamma,r]])\} + \{\eta, q_{-2}\pi_{-2}([\xi,[\gamma,r]])\} + q_{-3}([r,[\gamma,r]])$$

and thus

$$\Phi(\gamma) = (-\alpha([\eta,[\gamma,r]])\xi + \alpha([\xi,[\gamma,r]])\eta) + 2\Psi(\gamma).$$

Next, we look at the conditions on the Weyl form that follow from vanishing of the homogeneous component of degree two of the curvature K_τ. We get four additional equations. Vanishing of the component

$$T^{-1}M \times T^{-1}M \to \mathcal{A}_0 M$$

yields

$$(R^{(2)} + \partial P^{(2)})(\xi,\eta) = \nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]} + \{\xi, P^{(2)}(\eta)\} - \{\eta, P^{(2)}(\xi)\} = 0. \tag{3.15}$$

The other components of $K_\tau^{(2)}$ are the torsion parts $\mathrm{gr}_{-1}(TM) \times \mathrm{gr}_{-2}(TM) \to \mathrm{gr}_{-1}(TM)$, $\mathrm{gr}_{-1}(TM) \times \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_{-2}(TM)$ and $\mathrm{gr}_{-2}(TM) \times \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_{-3}(TM)$. Vanishing of these implies

$$(T^{(2)} + \partial P^{(2)})(\xi, q_{-2}(r)) = -\nabla_r\xi - \pi_{-1}([\xi,r]) - \{q_{-2}(r), P^{(2)}(\xi)\} = 0 \tag{3.16}$$

and

$$\begin{aligned}(T^{(2)}+\partial P^{(2)})(\xi, q_{-3}(\zeta)) = \\ \nabla_\xi q_{-2}\pi_{-2}(\zeta) - q_{-2}\pi_{-2}([\xi,\zeta]) + \{\xi, \pi_{-1}(\zeta)\} + \{P^{(2)}(\xi), q_{-3}(\zeta)\} = 0\end{aligned} \tag{3.17}$$

and

$$T^{(2)}(r, q_{-3}(\zeta)) = \nabla_r q_{-3}(\zeta) - q_{-3}([r,\zeta]) + \{q_{-2}(r), \pi_{-1}(\zeta)\} = 0 \tag{3.18}$$

where $\xi, \eta \in \Gamma(T^{-1}M)$, $\zeta \in \mathfrak{X}(M)$ and $r \in \Gamma(T^{-2}M)$ is the Reeb field corresponding to the generalised contact form α.

In order to analyse these equations, we first relate the terms containing the rho tensor $P^{(2)}$. Since $\{,\}$ is induced by the Lie bracket on $\mathfrak{g}$, this suggests looking at some Lie algebra identities. Consider $\mathfrak{g}$ realised as in (2.1). Let X_1, X_2 be contained in the grading component $\mathfrak{g}_{-1}$ and let Z be contained in $\mathfrak{g}_1$ and let r be contained in $\mathfrak{g}_{-2}$. Then the following identities hold:

$$[[Z, X_1], r] = -B(Z, X_1)r \tag{3.19}$$

$$[[Z, X_1], X_2] = B(Z, X_1)X_2 - 3B(Z, X_2)X_1 \tag{3.20}$$

$$[Z, [X_1, X_2]] = 4(B(Z, X_1)X_2 - B(Z, X_2)X_1). \tag{3.21}$$

Here B denotes $\frac{1}{24}$ of the Killing form on the Lie algebra $\mathfrak{g}$, in our realisation, this is $\frac{1}{6}$ of the trace form. Passing to associated bundles, the form B induces the duality pairing between $\mathrm{gr}_{-i}(TM)$ and $\mathrm{gr}_i(T^*M)$.

Let us consider equation (3.15) now. Since $q_{-2}(r)$ is parallel for the connection ∇, the equation implies

$$\{\{\xi, P^{(2)}(\eta)\} - \{\eta, P^{(2)}(\xi)\}, q_{-2}(r)\} = 0.$$

The map $(\xi, \eta) \mapsto \{\xi, P^{(2)}(\eta)\} - \{\eta, P^{(2)}(\xi)\}$ is induced by the component $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0$ of the Lie bracket. So by Lie algebra identity (3.19) we have $P^{(2)}(\eta)(\xi) = P^{(2)}(\xi)(\eta)$. Using this symmetry as well as Lie algebra identities (3.20) and (3.21), we can now determine the action of $\{\xi, P^{(2)}(\eta)\} - \{\eta, P^{(2)}(\xi)\}$ on $T^{-1}M$, obtaining

$$\xi' \mapsto -\frac{3}{4}\{P^{(2)}(\xi'), \{\xi, \eta\}\}. \tag{3.22}$$

Hence,

$$\{P^{(2)}(\xi), \{\xi, \eta\}\} = -\frac{4}{3}\{\{\xi, P^{(2)}(\eta)\} - \{\eta, P^{(2)}(\xi)\}, \xi\}. \tag{3.23}$$

Assuming that $\{\xi, \eta\} = q_{-2}(r)$, the left hand side of this equation equals

$$\nabla_r \xi - \pi_{-1}([r, \xi])$$

by (3.16), while (3.15) implies that the right hand side equals

$$\frac{4}{3}(\nabla_\xi \nabla_\eta \xi - \nabla_\eta \nabla_\xi \xi - \nabla_{\pi_{-1}([\xi,\eta])}\xi) = \frac{4}{3}(\Phi(\xi) - \nabla_r \xi).$$

Using this, equation (3.23) reads as

$$\nabla_r \xi - \pi_{-1}([r, \xi]) = \frac{4}{3}(\Phi(\xi) - \nabla_r \xi).$$

Thus,

$$\nabla_r \xi = \frac{3}{7}\pi_{-1}([r,\xi]) + \frac{4}{7}\Phi(\xi). \tag{3.24}$$

Putting $\zeta = [r,\xi]$ and inserting this expression for $\nabla_r \xi$ into equation (3.18) yields

$$\frac{10}{7}\{q_{-2}(r), \pi_{-1}([r,\xi])\} + \frac{4}{7}\{q_{-2}(r), \Phi(\xi)\} + q_{-3}([r,[\xi,r]]) = 0.$$

We can now determine the components of the normal Weyl form in homogeneity two. We have

$$\pi_{-1}([r,\xi]) = \frac{7}{5}\Psi(\xi) - \frac{2}{5}\Phi(\xi). \tag{3.25}$$

Inserting this into equation (3.24), we obtain

$$\nabla_r \xi = \frac{3}{5}\Psi(\xi) + \frac{2}{5}\Phi(\xi). \tag{3.26}$$

Equation (3.16) then allows to determine the component $P^{(2)}$ of the rho tensor as

$$\begin{aligned}\{P^{(2)}(\xi), q_{-2}(r)\} &= \pi_{-1}([\xi,r]) + \nabla_r \xi \\ &= \frac{4}{5}(\Phi(\xi) - \Psi(\xi)).\end{aligned}$$

Computing the Lie bracket, compare with (3.40), yields

$$\{\{P^{(2)}(\xi), q_{-2}(r)\}, \eta\} = -4P^{(2)}(\xi)(\eta)q_{-2}(r)$$

and since

$$\{\Phi(\xi) - \Psi(\xi), \eta\} = -d\alpha(\Phi(\xi) - \Psi(\xi), \eta)q_{-2}(r)$$

this implies

$$P^{(2)}(\xi)(\eta) = \frac{1}{5}d\alpha(\Phi(\xi) - \Psi(\xi), \eta). \tag{3.27}$$

3.3.2.3. *Homogeneity 3.* If we write out the conditions on the Weyl form that follow from vanishing of the homogeneous components of degree three of the curvature K_τ, we obtain five new equations. We will consider two of those: First, vanishing of the component

$$T^{-1}M \times \mathrm{gr}_{-2}(TM) \to \mathcal{A}_0 M$$

reads

$$\begin{aligned}&(R^{(3)} + \partial P^{(3)})(\xi, q_{-2}(r)) = \\ &\qquad \nabla_\xi \nabla_r - \nabla_r \nabla_\xi - \nabla_{[\xi,r]} + \{\xi, P^{(3)}(r)\} - \{q_{-2}(r), P^{(3)}(\xi)\} = 0\end{aligned} \tag{3.28}$$

and from vanishing of the component

$$\mathrm{gr}_{-2}(TM) \times \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_{-2}(TM)$$

we get

$$\begin{aligned}&(T^{(3)} + \partial P^{(3)})(q_{-2}(r), q_{-3}(\zeta)) = \\ &\qquad \nabla_r q_{-2}\pi_{-2}(\zeta) - q_{-2}\pi_{-2}([r,\zeta]) - \{q_{-3}(\zeta), P^{(3)}(r)\} = 0\end{aligned} \tag{3.29}$$

where $\xi, \eta \in \Gamma(T^{-1}M)$, $\zeta \in \mathfrak{X}(M)$ and r is the generalised Reeb vector field.

As in homogeneity two, we first try to find relations between terms containing $P^{(3)}$. Consider Lie algebra elements $X \in \mathfrak{g}_{-1}$, $Z \in \mathfrak{g}_1$, $r \in \mathfrak{g}_{-2}$ and $s \in \mathfrak{g}_2$. We have

$$[[r, s], r] = 2B(r, s)r \tag{3.30}$$

and

$$[[r, X], Z] = 3B(X, Z)r. \tag{3.31}$$

and

$$[[r, s], X] = B(r, s)X \tag{3.32}$$

Since $q_{-2}(r)$ is parallel for the connection ∇, we can use (3.28) to see that $\{q_{-2}(r), P^{(3)}(\xi)\} - \{\xi, P^{(3)}(r)\}$ acts trivially on $\mathrm{gr}_{-2}(TM)$. This means that

$$\{\{q_{-2}(r), P^{(3)}(\xi)\}, q_{-2}(r)\} = \{\{\xi, P^{(3)}(r)\}, q_{-2}(r)\}.$$

Lie algebra identity (3.30) implies

$$\{\{q_{-2}(r), P^{(3)}(\xi)\}, q_{-2}(r)\} = 2P^{(3)}(\xi)(q_{-2}(r))q_{-2}(r)$$

and applying (3.19) we see that

$$\{\{\xi, P^{(3)}(r)\}, q_{-2}(r)\} = P^{(3)}(r)(\xi)q_{-2}(r)$$

holds. In view of these equations, we have

$$P^{(3)}(\xi)(q_{-2}(r)) = \frac{1}{2}P^{(3)}(r)(\xi).$$

Let us consider equation (3.29) now. We have already determined $q_{-2}\pi_{-2}$, so we know that $q_{-2}\pi_{-2}([[r, \xi], r]) = \alpha([[r, \xi], r])q_{-2}(r)$ and $q_{-2}\pi_{-2}([r, \xi]) = 0$. Thus, (3.29) yields

$$\{\{q_{-2}(r), \xi\}, P^{(3)}(r)\} = \alpha([[r, \xi], r])q_{-2}(r).$$

Moreover, by (3.31) we have

$$\{\{q_{-2}(r), \xi\}, P^{(3)}(r)\} = 3P^{(3)}(r)(\xi)q_{-2}(r).$$

Together, this implies

$$P^{(3)}(r)(\xi) = \frac{1}{3}\alpha([[r, \xi], r]). \tag{3.33}$$

Since $P^{(3)}(\xi)(q_{-2}(r)) = \frac{1}{2}P^{(3)}(r)(\xi)$, we further obtain

$$P^{(3)}(\xi)(q_{-2}(r)) = \frac{1}{6}\alpha([[r, \xi], r]). \tag{3.34}$$

By equation (3.28) we have

$$\nabla_{[r,\xi]}\eta = \nabla_r\nabla_\xi\eta - \nabla_\xi\nabla_r\eta + \{\{q_{-2}(r), P^{(3)}(\xi)\} - \{\xi, P^{(3)}(r)\}, \eta\}.$$

Applying Lie algebra identities (3.32) and (3.20) leads to

$$\begin{aligned}\{\{q_{-2}(r), P^{(3)}(\xi)\} - \{\xi, P^{(3)}(r)\}, \eta\} =& \\ P^{(3)}(\xi)(q_{-2}(r))\eta + P^{(3)}(r)(\xi)\eta - 3P^{(3)}(r)(\eta)\xi.&\end{aligned}$$

Now we can use the formulae for $P^{(3)}(\xi)(q_{-2}(r))$ and $P^{(3)}(r)(\xi)$ to obtain

$$\nabla_{[r,\xi]}\eta = \nabla_r\nabla_\xi\eta - \nabla_\xi\nabla_r\eta - \alpha([[r, \eta], r])\xi + \frac{1}{2}\alpha([[r, \xi], r])\eta. \tag{3.35}$$

3.3.2.4. *Homogeneity 4.* At this point we have computed the Weyl connection ∇, the soldering form Θ as well as some components of the rho-tensor. A straightforward way to deal with the remaining components of the rho-tensor is discussed in 3.3.2.5. However, in homogeneity four it is still a reasonable choice to proceed in the same manner as before in order to obtain a simpler expression for $P^{(4)}$.

On the one hand side, we are no longer in the situation that all components of the curvature K_τ vanish by normality. But the situation in homogeneous degree four is still fairly simple since we know that there is only one non-vanishing curvature component in that degree: We have seen that the harmonic curvature component of the normal Cartan connection takes values in

$$H^2(\mathfrak{g}_-, \mathfrak{g}) \subset \mathfrak{g}_3 \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_0^{ss}.$$

Hence the lowest homogeneous component of K_τ is the curvature component

$$\mathrm{gr}_{-3}(TM) \times \mathrm{gr}_{-1}(TM) \to \mathcal{A}_0 M$$

given by

$$\begin{aligned}(R^{(4)} + \partial P^{(4)})(q_{-3}(\zeta), \xi) = \\ \nabla_{\zeta_{-3}} \nabla_\xi - \nabla_\xi \nabla_{\zeta_{-3}} - \nabla_{[\zeta_{-3},\xi]} + \{q_{-3}(\zeta), P^{(4)}(\xi)\} - \{\xi, P^{(4)}(\zeta_{-3})\}.\end{aligned}$$

Here $\xi \in \Gamma(T^{-1}M)$, $\zeta \in \mathfrak{X}(M)$ and we denote by ζ_{-3} the vector field $\zeta - \alpha(\zeta)r - \pi_{-1}(\zeta)$. Note that the adjoint action of the semisimple part $\mathfrak{g}_0^{ss}$ on the grading component $\mathfrak{g}_2$ is trivial. Hence,

$$\{(R^{(4)} + \partial P^{(4)})\,(q_{-3}(\zeta), \xi)\,, q_{-2}(r)\} = 0$$

and since $q_{-2}(r)$ is parallel for ∇, this implies

$$\{\{q_{-3}(\zeta), P^{(4)}(\xi)\}, q_{-2}(r)\} = \{\{\xi, P^{(4)}(\zeta_{-3})\}, q_{-2}(r)\}. \tag{3.36}$$

The other components of K_τ of degree four vanish by normality. We consider two of those components:

$$\mathrm{gr}_{-3}(TM) \times \mathrm{gr}_{-2}(TM) \to \mathrm{gr}_{-1}(TM)$$

given by

$$\begin{aligned}(T^{(4)} + \partial P^{(4)})(q_{-3}(\zeta), q_{-2}(r)) = \\ - \pi_{-1}([\zeta_{-3}, r]) + \{q_{-3}(\zeta), P^{(4)}(r)\} - \{q_{-2}(r), P^{(4)}(\zeta_{-3})\} = 0\end{aligned} \tag{3.37}$$

and

$$\mathrm{gr}_{-2}(TM) \times \mathrm{gr}_{-1}(TM) \to \mathrm{gr}_1(T^*M)$$

given by

$$\begin{aligned}(Y^{(4)} + \partial P^{(4)})(q_{-2}(r), \xi) = \nabla_r P^{(2)}(\xi) - \nabla_\xi P^{(3)}(r) \\ - P^{(2)}(\pi_{-1}([r, \xi])) + \{q_{-2}(r), P^{(4)}(\xi)\} - \{\xi, P^{(4)}(r)\} = 0.\end{aligned} \tag{3.38}$$

To determine the components of the rho-tensor in homogeneity four, we will use the following Lie algebra identities. Consider $X, X_1, X_2 \in \mathfrak{g}_{-1}$, $Z \in \mathfrak{g}_1$, $r \in \mathfrak{g}_{-2}$, $s \in \mathfrak{g}_2$, $Y \in \mathfrak{g}_{-3}$ and $W \in \mathfrak{g}_3$. We have

$$[[r, X], s] = -3B(r, s)X \tag{3.39}$$

and

$$[[r,Z],X]=4B(X,Z)r. \tag{3.40}$$

and

$$[[Y,W],r]=3B(Y,W)r \tag{3.41}$$

and

$$[[X_1,s],[X_2,r]]=-3B(s,r)[X_1,X_2]. \tag{3.42}$$

Let us consider

$$\begin{aligned}&\{(T^{(4)}+\partial P^{(4)})(q_{-3}(\zeta),q_{-2}(r)),\xi\}=\\&\qquad-\{\pi_{-1}([\zeta_{-3},r]),\xi\}+\{\{q_{-3}(\zeta),P^{(4)}(r)\}-\{q_{-2}(r),P^{(4)}(\zeta_{-3})\},\xi\}=0\end{aligned}$$

Suppose ζ is of the form $[r,\eta]$ for some $\eta\in\Gamma(T^{-1}M)$ and apply Lie algebra identities (3.40) and (3.39). Then the equation reads

$$\{\pi_{-1}([[r,\eta]_{-3},r]),\xi\}=3P^{(4)}(r)(q_{-2}(r))\{\xi,\eta\}-4P^{(4)}(\{q_{-2}(r),\eta\})(\xi)q_{-2}(r), \tag{3.43}$$

where $[r,\eta]_{-3}=[r,\eta]-\pi_{-1}([r,\eta])$. Next, looking at $\{(Y^{(4)}+\partial P^{(4)})(q_{-2}(r),\xi),\{q_{-2}(r),\eta\}\}$ and making use of Lie algebra identities (3.41) and (3.42) we obtain

$$\begin{aligned}&\{\nabla_r P^{(2)}(\xi)-\nabla_\xi P^{(3)}(r)-P^{(2)}(\pi_{-1}([r,\xi])),\{q_{-2}(r),\eta\}\}=\\&\qquad-3P^{(4)}(\xi)(\{q_{-2}(r),\eta\})q_{-2}(r)+3P^{(4)}(r)(q_{-2}(r))\{\xi,\eta\}\end{aligned} \tag{3.44}$$

Applying Lie algebra identities (3.19) and (3.41) to equation (3.36) implies

$$P^{(4)}(\{q_{-2}(r),\eta\})(\xi)=3P^{(4)}(\xi)(\{q_{-2}(r),\eta\}). \tag{3.45}$$

Using that, we can combine equations (3.43) and (3.44) to obtain

$$\begin{aligned}P^{(4)}(\xi)(\{q_{-2}(r),\eta\})q_{-2}(r)&=\frac{1}{9}(\{\nabla_r P^{(2)}(\xi)-\nabla_\xi P^{(3)}(r)\\&\quad-P^{(2)}(\pi_{-1}([r,\xi])),\{q_{-2}(r),\eta\}\}-\{\pi_{-1}([[r,\eta]_{-3},r]),\xi\})\end{aligned}$$

and by Lie algebra identity (3.31) this yields

$$\begin{aligned}P^{(4)}(\xi)(\{q_{-2}(r),\eta\})=&-\frac{1}{3}\left(\nabla_r P^{(2)}(\xi)-\nabla_\xi P^{(3)}(r)-P^{(2)}(\pi_{-1}([r,\xi]))\right)(\eta)\\&+\frac{1}{9}d\alpha\,(\pi_{-1}([[r,\eta]_{-3},r]),\xi)\end{aligned} \tag{3.46}$$

But then, it is easy to determine $P^{(4)}:\mathrm{gr}_{-3}(TM)\to\mathrm{gr}_1(T^*M)$ and $P^{(4)}:\mathrm{gr}_{-2}(TM)\to\mathrm{gr}_2(T^*M)$, from equations (3.45) and (3.43) respectively. We have

$$\begin{aligned}P^{(4)}(\{q_{-2}(r),\eta\})(\xi)=&-\left(\nabla_r P^{(2)}(\xi)-\nabla_\xi P^{(3)}(r)-P^{(2)}(\pi_{-1}([r,\xi]))\right)(\eta)\\&+\frac{1}{3}d\alpha\,(\pi_{-1}([[r,\eta]_{-3},r]),\xi)\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}P^{(4)}(r)(\{\xi,\eta\})=&-\frac{4}{3}\left(\nabla_r P^{(2)}(\xi)-\nabla_\xi P^{(3)}(r)-P^{(2)}(\pi_{-1}([r,\xi]))\right)(\eta)\\&+\frac{1}{9}d\alpha\,(\pi_{-1}([[r,\eta]_{-3},r]),\xi)\,.\end{aligned} \tag{3.48}$$

3.3.2.5. *Homogeneity ≥ 5.* It remains to compute the components of the rho-tensor of degree ≥ 5. These are $P^{(5)} : \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_2(T^*M)$, $P^{(5)} : \mathrm{gr}_{-2}(TM) \to \mathrm{gr}_3(T^*M)$ and $P^{(6)} : \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_3(T^*M)$.

We can proceed as follows. From the previous calculations we know the Weyl connection ∇ and the soldering form Θ and thus T and R, cf. (3.4) and (3.5). Now suppose we have constructed all components of the normal Weyl form of homogeneity smaller than n for some $n \geq 4$. Then also $Y^{(n)}$ is known since it can be expressed in terms of Θ, ∇ and homogeneous components of P of degree $< n$, and we have already determined those by assumption. The normalisation condition in homogeneity n has the form

$$\partial^*(T^{(n)} + R^{(n)} + Y^{(n)}) + \partial^*\partial(P^{(n)}) = 0.$$

Recall that the map ∂^* induced by the Kostant codifferential preserves homogeneities. Since $P^{(n)}$ is of homogeneous degree greater than 3, this implies that $\partial^* P^{(n)} = 0$, i.e. $P^{(n)}$ is contained in the kernel of ∂^*. We also know that the positive homogeneous components of the cohomology space $\ker(\partial^*)/\mathrm{im}(\partial^*) = H^1(\mathfrak{g}_-, \mathfrak{g})$ vanish, see Example 1.31. That means $P^{(n)}$ is indeed contained in the image of ∂^*. Since $\partial^* \circ \partial^* = 0$, the Kostant Laplacian $\Box = \partial\partial^* + \partial^*\partial$ coincides on $\mathrm{im}(\partial^*)$ with $\partial^*\partial$. Hence, the normalisation condition can be rewritten as

$$\partial^*(T^{(n)} + R^{(n)} + Y^{(n)}) + \Box(P^{(n)}) = 0.$$

The Kostant Laplacian is invertible on the image of ∂^*. Thus, the component of the rho-tensor of degree n can be computed as

$$P^{(n)} = -\Box^{-1}\left(\partial^*(T^{(n)} + R^{(n)} + Y^{(n)})\right). \tag{3.49}$$

The term $\partial^*(T^{(n)} + R^{(n)} + Y^{(n)})$ can be easily obtained from the data constructed in lower homogeneities using formulae (3.4), (3.5), (3.6) and the formula for the Kostant codifferential. It remains to compute the inverse of the Kostant Laplacian. This is also not difficult, since one knows that the Kostant Laplacian acts by a scalar on each $\mathfrak{g}_0$-isotypical component of $(\mathfrak{g}_-)^* \otimes \mathfrak{g}$ (this is a crucial step in the proof of Kostant's Bott-Borel-Weil theorem, see [**24**]) and there is a formula for the scalar. Note that we compute the Kostant Laplacian on $\mathfrak{g}_-^* \otimes \mathfrak{g}$ rather than on $\mathfrak{p}_+^* \otimes \mathfrak{g}$. This implies that in the formula for $\Box$ we have to take negatives of lowest weights instead of highest weights: If λ denotes the maximal root of $\mathfrak{g}$ and ρ the sum over all fundamental weights and ν the lowest weight of a $\mathfrak{g}_0$-irreducible component in $\mathfrak{g}_-^* \otimes \mathfrak{g}$, then $\Box$ acts by multiplication with $\frac{1}{2}(||-\nu+\rho||^2 - ||\lambda+\rho||^2)$ on that component. It is also possible to compute the right scalars directly from the normalisation condition. We demonstrate this in homogeneity five.

In homogeneity five there are two irreducible $\mathfrak{g}_0$-components in $(\mathfrak{g}_-)^* \otimes \mathfrak{g}$, namely $(\mathfrak{g}_{-2})^* \otimes \mathfrak{g}_3$ and $(\mathfrak{g}_{-3})^* \otimes \mathfrak{g}_2$, and the two components are isomorphic. To compute the Kostant Laplacian on these components we take an arbitrary element, say $A \in (\mathfrak{g}_{-2})^* \otimes \mathfrak{g}_3$. We consider a basis of $\mathfrak{g}_-$ consisting of elements $X_1, X_2 \in \mathfrak{g}_{-1}$, $r = [X_1, X_2] \in \mathfrak{g}_{-2}$ and $Y_1 = [X_1, r], Y_2 = [X_2, r] \in \mathfrak{g}_{-3}$ and dual elements $Z_1, Z_2 \in \mathfrak{g}_1$, $s \in \mathfrak{g}_2$ and $W_1, W_2 \in \mathfrak{g}_3$ with respect to the bilinear form B, i.e. $\frac{1}{24}$ of the Killing form. Then we can write

$$A = s \otimes A(r)$$

and

$$\begin{aligned}\partial A =& - Z_1 \wedge Z_2 \otimes A(r) + Z_1 \otimes s \otimes [X_1, A(r)] + Z_2 \otimes s \otimes [X_2, A(r)]\\ &+ W_1 \otimes s \otimes [Y_1, A(r)] + W_2 \otimes s \otimes [Y_2, A(r)].\end{aligned}$$

Now we use formula (1.2) for the Kostant codifferential on decomposable elements to obtain

$$\begin{aligned}\partial^* \partial A =&[Z_1, Z_2] \otimes A(r) - [Z_1, s] \otimes [X_1, A(r)] - [Z_2, s] \otimes [X_2, A(r)]\\ &- s \otimes [Z_1, [X_1, A(r)]] - s \otimes [Z_2, [X_2, A(r)]]\\ &- s \otimes [W_1, [Y_1, A(r)]] + W_1 \otimes [s, [X_1, A(r)]]\\ &- s \otimes [W_2, [Y_2, A(r)]] + W_2 \otimes [s, [X_2, A(r)]].\end{aligned}$$

Next we have to do some Lie algebra computations. These show that

$$\begin{aligned}-[Z_1, s]\otimes[X_1, A(r)] - [Z_2, s] \otimes [X_2, A(r)]+&\\ W_1 \otimes [s, [X_1, A(r)]] + W_2 \otimes [s, [X_2, A(r)]] &= 0\end{aligned}$$

and

$$[Z_1, Z_2] \otimes A(r) = -4s \otimes A(r)$$

and

$$-s \otimes [Z_1, [X_1, A(r)]] - s \otimes [Z_2, [X_2, A(r)]] = -3s \otimes A(r)$$

and

$$-s \otimes [W_1, [Y_1, A(r)]] - s \otimes [W_2, [Y_2, A(r)]] = -9s \otimes A(r).$$

summarising, we have

$$\Box A = \partial^* \partial A = -16A.$$

Henceforth, $\partial^* : \Lambda^2 T^*M \otimes \mathcal{A}M \to T^*M \otimes \mathcal{A}M$ will denote the bundle map induced by the algebraic codifferential defined by means of B. Then, the component of the rho tensor of degree five is given by

$$P^{(5)} = \frac{1}{16}\partial^*(T^{(5)} + R^{(5)} + Y^{(5)}). \tag{3.50}$$

Finally, let us discuss homogeneity six. The component of the rho-tensor of degree six can be considered as an element of $\mathrm{gr}_3(T^*M) \otimes \mathrm{gr}_3(T^*M)$. The $\mathfrak{g}_0$-representation $\mathfrak{g}_3 \otimes \mathfrak{g}_3$ decomposes into two irreducible components

$$\mathfrak{g}_3 \otimes \mathfrak{g}_3 = S^2\mathfrak{g}_3 \oplus \Lambda^2\mathfrak{g}_3.$$

Using either the formula for the Kostant Laplacian in terms of weights or proceeding as above, we see that $\Box$ acts on $S^2\mathfrak{g}_3$ by multiplication with -12 and on $\Lambda^2\mathfrak{g}_3$ by multiplication with -18. Hence,

$$\begin{aligned}P^{(6)}(\zeta)(\zeta') =&\frac{1}{24}\Big(\partial^*(R^{(6)} + Y^{(6)})(\zeta)(\zeta') + \partial^*(R^{(6)} + Y^{(6)})(\zeta')(\zeta)\Big)\\ &+ \frac{1}{36}\Big(\partial^*(R^{(6)} + Y^{(6)})(\zeta)(\zeta') - \partial^*(R^{(6)} + Y^{(6)})(\zeta')(\zeta)\Big)\end{aligned} \tag{3.51}$$

where the dualities are again defined by means of the bilinear form B.

3.3.3. Summary of the results. Let $\mathcal{H}$ be a generic rank two distribution on a five-dimensional manifold. Suppose α is a generalised contact form for the distribution $\mathcal{H}$. Then there is a unique Weyl form associated to α. This is a triple (Θ, ∇, P) consisting of a splitting of the filtration $T^{-1}M \subset T^{-2}M \subset TM$, the soldering form, a linear connection ∇ on the distribution, the Weyl connection and a one-form $P \in \Omega^1(M, \mathrm{gr}(T^*M))$ called rho tensor. The explicit description of these data has been determined in the preceding subsection. We summarise our findings in the following proposition.

THEOREM 3.25. Let α be a generalised contact form. Let r denote the generalised Reeb field associated to α characterised in Proposition 3.5 and think of α as being extended to a one-form as in Corollary 3.7. Then the Weyl connection

$$\nabla : \mathfrak{X}(M) \times \Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$$

associated to the generalised contact form α has the following description: For any $\gamma, \xi \in \Gamma(T^{-1}M)$ the section $\nabla_\gamma \xi \in \Gamma(T^{-1}M)$ is characterised by

$$\{\nabla_\gamma \xi, q_{-2}(r)\} = q_{-3}([\gamma, [\xi, r]]).$$

We have

$$\nabla_r \xi = \frac{3}{5}\Psi(\xi) + \frac{2}{5}\Phi(\xi),$$

where Φ and Ψ are maps $\Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$ characterised by

$$d\alpha(\xi, \eta)\Phi(\gamma) = -(\nabla_\xi \nabla_\eta \gamma - \nabla_\eta \nabla_\xi \gamma - \nabla_{\pi_{-1}([\xi,\eta])}\gamma)$$

for any $\xi, \eta, \gamma \in \Gamma(T^{-1}M)$, respectively by

$$\{\Psi(\gamma), q_{-2}(r)\} = \frac{1}{2}q_{-3}([r, [\gamma, r]]).$$

We further have

$$\nabla_{[r,\gamma]}\xi = \nabla_r \nabla_\gamma \xi - \nabla_\gamma \nabla_r \xi - \alpha([[r, \xi], r])\gamma + \frac{1}{2}\alpha([[r, \gamma], r])\xi.$$

Since we can write any vector field as

$$\zeta = [r, \zeta_1] + \alpha(\zeta)r + \zeta_2,$$

where ζ_1 is the unique vector field in $\Gamma(T^{-1}M)$ such that $q_{-3}(\zeta) = \{q_{-2}(r), \zeta_1\}$ and $\zeta_2 = \zeta - \alpha(\zeta)r - [r, \zeta_1] \in \Gamma(T^{-1}M)$, this completely describes the Weyl connection. The soldering form Θ associated to α is given by the isomorphism

$$\begin{array}{rcl} TM & \cong & T^{-1}M \oplus \mathrm{gr}_{-2}(TM) \oplus \mathrm{gr}_{-3}(TM) \\ \zeta & \mapsto & \pi_{-1}(\zeta) + q_{-2}(\pi_{-2}(\zeta)) + q_{-3}(\zeta), \end{array}$$

where

$$\pi_{-1}(\zeta) = -\frac{2}{5}\Phi(\zeta_1) + \frac{7}{5}\Psi(\zeta_1) + \zeta_2$$

and

$$q_{-2}\pi_{-2}(\zeta) = \alpha(\zeta)q_{-2}(r).$$

Formulae for the rho tensor are given in (3.27), (3.34), (3.33), (3.46), (3.48), (3.47), (3.50) and (3.51).

Given a local generalised contact form α, the same formulae describe the corresponding local Weyl connection, soldering form and rho tensor.

Remark 3.26. The linear connection ∇ on $T^{-1}M$ from the above proposition can be extended to all of $\mathrm{gr}(TM)$ requiring that the Levi bracket $\{,\}$ is parallel. Via the soldering form $TM \cong \mathrm{gr}(TM)$, the connection on $\mathrm{gr}(TM)$ gives rise to a linear connection on TM. Thus, we obtain a description of the Weyl connection on the tangent bundle of the manifold M associated to a generalised contact form.

Remark 3.27. Changing our perspective, the results summarised in Theorem 3.25 can be interpreted as the construction of a normal Cartan connection for a generic rank two distribution in dimension five.

The data (Θ, ∇, P) that we have characterised in the above proposition define a Weyl form $\tau = \tau_- + \tau_0 + \tau_+ \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$ on the frame bundle $\mathcal{G}_0$ for the distribution and one verifies that the Weyl form is indeed normal. Moreover, one can extend the frame bundle $\mathcal{G}_0$ to the P-principal bundle $\mathcal{G} = \mathcal{G}_0 \times P_+$ and equivariantly extend the Weyl form to a Cartan connection ω on $\mathcal{G}$. Normality of τ implies that ω is normal.

The results also lead to a description of the harmonic curvature for a generic rank two distribution in dimension five:

Theorem 3.28. *Let $\mathcal{H}$ be a generic rank two distribution on a five manifold M. Suppose $\alpha \in \mathrm{gr}_2(T^*M)$ is a generalised contact form, r the corresponding Reeb vector field, let ∇ be the linear connection on $\mathcal{H}$ and $\pi_{-1} : TM \to \mathcal{H}$ the projection associated to α as in Theorem 3.25. Consider the curvature tensor*

$$\mathrm{gr}_{-3}(TM) \times \mathcal{H} \to L(\mathcal{H}, \mathcal{H})$$

given by

$$\begin{aligned} &R^{(4)}(\{q_{-2}(r), \zeta_1\}, \xi) = \\ &\quad \nabla_{[r,\zeta_1]}\nabla_\xi - \nabla_{\pi_{-1}([r,\zeta_1])}\nabla_\xi - \nabla_\xi\nabla_{[r,\zeta_1]} + \nabla_\xi\nabla_{\pi_{-1}([r,\zeta_1])} - \nabla_{[[r,\zeta_1],\xi]} + \nabla_{[\pi_{-1}([r,\zeta_1]),\xi]}. \end{aligned}$$

Then the section of $S^4\mathcal{H}^*$ defined as the complete symmetrisation of

$$(\xi_1, \xi_2, \xi_3, \xi_4) \mapsto d\alpha\left(\xi_1, R^{(4)}(\{q_{-2}(r), \xi_2\}, \xi_3)(\xi_4)\right),$$

for $\xi_1, \xi_2, \xi_3, \xi_4 \in \Gamma(\mathcal{H})$, is an invariant for the generic rank two distribution and a complete obstruction to local equivalence with the homogeneous model.

Proof. We know that the harmonic curvature κ_H is an invariant for the generic rank two distribution and a complete obstruction to local equivalence with the homogeneous model. By 2.15, the harmonic curvature takes values in

$$\ker(\Box) \subset \mathfrak{g}_3 \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_0.$$

Hence, for any Weyl structure, it is represented by the component

$$\mathrm{gr}_{-3}(TM) \times \mathrm{gr}_{-1}(TM) \to \mathcal{A}_0 M$$

of the associated Weyl curvature. If the Weyl structure is determined by a contact form α this curvature component can be written as

$$\begin{aligned} (R^{(4)} + \partial P^{(4)})(q_{-3}(\zeta), \xi) &= \nabla_{[r,\zeta_1]}\nabla_\xi - \nabla_{\pi_{-1}([r,\zeta_1])}\nabla_\xi - \nabla_\xi\nabla_{[r,\zeta_1]} \\ &\quad + \nabla_\xi\nabla_{\pi_{-1}([r,\zeta_1])} - \nabla_{[[r,\zeta_1],\xi]} + \nabla_{[\pi_{-1}([r,\zeta_1]),\xi]} + \{q_{-3}(\zeta), P^{(4)}(\xi)\} - \{\xi, P^{(4)}(\zeta)\}, \end{aligned}$$

where $\xi \in \Gamma(T^{-1}M)$, $\{q_{-2}(r), \zeta_1\} \in \Gamma(\mathrm{gr}_{-3}(TM))$, and ∇, $P^{(4)}$ and π_{-1} are the data characterised in Theorem 3.25.

Next, observe that the Lie bracket induces isomorphisms $\mathfrak{g}_3 \cong \mathfrak{g}_2 \otimes \mathfrak{g}_1$ and $\mathfrak{g}_2 \otimes \mathfrak{g}_{-1} \cong \mathfrak{g}_1$ and the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ leads to an inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$. Via these identifications, the space $\ker(\Box) \subset \mathfrak{g}_3 \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_0$ can be viewed as the the Cartan product of the $\mathfrak{g}_1$'s, i.e. the irreducible component $S^4\mathfrak{g}_1 \subset \bigotimes^4 \mathfrak{g}_1$, see Remark 2.16. Passing to associated bundles, this implies that the harmonic curvature can be viewed a section of $S^4(\mathcal{H}^*)$. Up to a multiple, this section can be written as

$$(\xi_1, \xi_2, \xi_3, \xi_4) \mapsto \{\alpha, \xi_4\} \Big((R^{(4)} + \partial P^{(4)})(\{q_{-2}(r), \xi_1\}, \xi_2)(\xi_3) \Big).$$

Since the tensor is totally symmetric, we have

$$\begin{aligned} &\{\xi_4, \alpha\}((R^{(4)} + \partial P^{(4)})(\{q_{-2}(r), \xi_1\}, \xi_2)(\xi_3)) \\ &\qquad = \frac{1}{4!} \sum_{\sigma \in S_4} \{\xi_4, \alpha\}((R^{(4)} + \partial P^{(4)})(\{q_{-2}(r), \xi_1\}, \xi_2)(\xi_3)). \end{aligned}$$

Note that $\partial P^{(4)}$, being a section of $\mathrm{im}(\partial)$, is contained in the kernel of the projection onto $\ker(\Box) \cong S^4(\mathcal{H}^*)$, i.e. it is contained in the kernel of the symmetrisation map. Hence, on the right hand side of the above equation the terms containing the rho-tensor cancel. Finally, recall that the duality pairing is induced from an invariant bilinear form B, which implies $\{\alpha, \xi\}(\eta) = \alpha(\{\xi, \eta\}) = -d\alpha(\xi, \eta)$ for all $\xi, \eta \in \Gamma(\mathcal{H})$. Together, this proves the proposition. $\Box$

3.4. An easy application

It is an easy application of the description of the Weyl structure summarised in Theorem 3.25 to write down some invariant differential operators in terms of a generalised contact form. To close out this chapter, we chose to derive a formula for a particularly nice differential operator, the so-called infinitesimal automorphism operator. This is an operator whose kernel can be identified with the set of infinitesimal automorphisms of the distribution, i.e. vector fields $\zeta \in \mathfrak{X}(M)$ whose local flows preserve the distribution $\mathcal{H}$. It is known, see [**9**], that the infinitesimal automorphism operator is the first BGG operator for the adjoint tractor bundle, i.e.

$$D : \Gamma(\mathcal{G} \times_P H^0(\mathfrak{g}_-, \mathfrak{g})) \to \Gamma(\mathcal{G} \times_P H^1(\mathfrak{g}_-, \mathfrak{g})).$$

Recall that the P-actions on the cohomology spaces are the trivial extensions of the natural G_0-actions. Kostant's Bott-Borel-Weil theorem can be used to determine the above cohomology spaces as G_0-representations. We have an isomorphism $H^0(\mathfrak{g}_-, \mathfrak{g}) \cong \mathfrak{g}_{-3}$. Since the Lie bracket defines an isomorphism

$$\mathfrak{g}_{-3} \cong \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2},$$

we can decompose

$$(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-3} \cong (\mathfrak{sl}(\mathfrak{g}_{-1}) \otimes \mathfrak{g}_{-2}) \oplus (\mathbb{R} \otimes \mathfrak{g}_{-2}).$$

and $H^1(\mathfrak{g}_-,\mathfrak{g})$ turns out to be isomorphic to the irreducible component $\mathfrak{sl}(\mathfrak{g}_{-1})\otimes\mathfrak{g}_{-2}$. Hence, the infinitesimal automorphism operator is defined between sections of the bundles

$$D:\Gamma(\mathrm{gr}_{-3}(TM))\to\Gamma(\mathfrak{sl}(\mathcal{H})\otimes\mathrm{gr}_{-2}(TM)).$$

Given a Weyl connection we can easily construct such an operator. First we consider the Weyl connection in directions tangent to the distribution $\mathcal{H}$ applied to sections of $\mathrm{gr}_{-3}(TM)$. This defines an operator

$$\nabla:\Gamma(\mathrm{gr}_{-3}(TM))\to\Gamma(\mathcal{H}^*\otimes\mathrm{gr}_{-3}(TM))=\Gamma(\mathcal{H}^*\otimes\mathcal{H}\otimes\mathrm{gr}_{-2}(TM))$$

which we compose with the projection

$$\Gamma(\mathcal{H}^*\otimes\mathcal{H}\otimes\mathrm{gr}_{-2}(TM))\to\Gamma(\mathfrak{sl}(\mathcal{H})\otimes\mathrm{gr}_{-2}(TM))$$

onto the tracefree part. What we want to do here is to write down this operator in terms of a generalised contact form α and then to verify that the formula is independent of the choice of generalised contact form. We will also explain how to see directly that sections in the kernel of the differential operator correspond to infinitesimal automorphisms of the distribution.

Suppose α is a generalised contact form for a generic rank two distribution $\mathcal{H}$. Let r be the generalised Reeb field associated to α, choose $\xi,\eta\in\Gamma(\mathcal{H})$ such that $\{\xi,\eta\}=q_{-2}(r)$, and let ∇ be the Weyl connection associated to α. Then, for any $q_{-3}(\zeta)=\{q_{-2}(r),\zeta_1\}\in\mathrm{gr}_{-3}(TM)$, the trace free part of $\nabla(q_{-3}(\zeta))$ maps γ to

$$\left\{q_{-2}(r),\nabla_\gamma\zeta_1-\frac{1}{2}(d\alpha(\eta,\nabla_\xi\zeta_1)+d\alpha(\nabla_\eta\zeta_1,\xi))\gamma\right\}\tag{3.52}$$

The description (3.1) of the Weyl connection yields

$$\nabla_\gamma(\{q_{-2}(r),\zeta_1\})=-q_{-3}([\gamma,[\zeta_1,r]]).$$

Note that by (3.10) we have

$$\nabla_\xi\zeta_1=\nabla_{\zeta_1}\xi+\pi_{-1}([\xi,\zeta_1])$$

and since $\nabla_{\zeta_1}\{\xi,\eta\}=0$ we have

$$d\alpha(\nabla_{\zeta_1}\xi,\eta)+d\alpha(\xi,\nabla_{\zeta_1}\eta)=0.$$

Hence, (3.52) can be rewritten as

$$-q_{-3}\left([\gamma,[\zeta_1,r]]\right)+\frac{1}{2}\left(d\alpha(\eta,\pi_{-1}([\xi,\zeta_1]))+d\alpha(\pi_{-1}([\eta,\zeta_1]),\xi)\right)\{\gamma,q_{-2}(r)\}.$$

Using that $d\alpha(\gamma_1,\gamma_2)=-\alpha([\gamma_1,\gamma_2])$ and $\pi([\gamma_1,\gamma_2])=[\gamma_1,\gamma_2]-\alpha([\gamma_1,\gamma_2])r$ for all $\gamma_1,\gamma_2\in\Gamma(T^{-1}M)$ and the Jakobi identity, this can be further simplified to

$$-q_{-3}\left([\gamma,[\zeta_1,r]]\right)+\frac{1}{2}(\alpha([[\xi,\eta],\zeta_1])-\eta\cdot d\alpha(\xi,\zeta_1)+\xi\cdot d\alpha(\eta,\zeta_1))\{\gamma,q_{-2}(r)\}.\tag{3.53}$$

It is straightforward to compute the transformation of the above expression as we vary the contact form. Most of the necessary calculations are carried out in Appendix A. First we observe that if we replace α by $\hat{\alpha}=-\alpha$, also the Reeb field changes sign, i.e. $\hat{r}=-r$. The vector field $\hat{\zeta}_1=\zeta_1$ remains the same. It follows, that formula (3.53) remains the same. Next we rescale α by a positive smooth function, say $\hat{\alpha} = e^f\alpha$. Lemma A.2 shows that the generalised Reeb field corresponding to the new $\hat{\alpha}$ is given by

$$\hat{r} = e^{-f}r + \delta,$$

where $\delta \in \Gamma(T^{-1}M)$ is the unique vector field such that

$$\{\gamma, \delta\} = 4e^{-f}df(\gamma)q_{-2}(r)$$

for all $\gamma \in \Gamma(T^{-1}M)$. By Lemma A.3 the canonical extension of α to a one-form transforms as

$$\hat{\alpha}(\zeta) = e^f(\alpha(\zeta) + 3df(\zeta_1)).$$

Since $q_{-2}(\hat{r}) = e^{-f}q_{-2}(r)$, we have $\hat{\zeta}_1 = e^f\zeta_1$. Using this, we compute that

$$-q_{-3}([\gamma, [\hat{\zeta}_1, \hat{r}]]) = -q_{-3}([\gamma, [\zeta_1, r]]) - 3df(\zeta_1)\{\gamma, q_{-2}(r)\}. \tag{3.54}$$

Take $\hat{\xi} = e^{-\frac{f}{2}}\xi$ and $\hat{\eta} = e^{-\frac{f}{2}}\eta$, then a simple computation shows that

$$\begin{aligned}\hat{\alpha}([[\hat{\xi}, \hat{\eta}],\hat{\zeta}_1]) = {}& e^f\alpha([[\xi, \eta], \zeta_1])+\\ & e^f(df(\zeta_1) + \tfrac{1}{2}df(\eta)\alpha([\xi, \zeta_1]) - \tfrac{1}{2}df(\xi)\alpha([\eta, \zeta_1]) + 3df(\zeta_1)).\end{aligned}$$

We compute

$$\hat{\eta} \cdot d\hat{\alpha}(\hat{\xi}, \hat{\zeta}_1) = e^f\eta \cdot d\alpha(\xi, \zeta_1) + \tfrac{3}{2}e^f df(\eta)d\alpha(\xi, \zeta_1).$$

Using that

$$\zeta_1 = d\alpha(\eta, \zeta_1)\xi - d\alpha(\xi, \zeta_1)\eta$$

the two previous equations imply that

$$\begin{aligned}&\frac{1}{2}(\hat{\alpha}([[\hat{\xi}, \hat{\eta}],\hat{\zeta}_1]) - \hat{\eta} \cdot d\hat{\alpha}(\hat{\xi}, \hat{\zeta}_1) + \hat{\xi} \cdot d\hat{\alpha}(\hat{\eta}, \hat{\zeta}_1))\{\gamma, q_{-2}(\hat{r})\} =\\ &\qquad\frac{1}{2}(\alpha([[\xi, \eta], \zeta_1]) - \eta \cdot d\alpha(\xi, \zeta_1) + \xi \cdot d\alpha(\eta, \zeta_1)) + 6df(\zeta_1))\{\gamma, q_{-2}(r)\}.\end{aligned} \tag{3.55}$$

Finally, using (3.54) and (3.55), we see that formula (3.53) does not depend on the choice of contact form. Hence, we have constructed an invariant differential operator associated to the distribution $\mathcal{H}$.

Next we show directly that elements of the kernel of this differential operator are indeed in bijective correspondence with infinitesimal automorphisms $\zeta \in \mathfrak{X}(M)$ of the distribution $\mathcal{H}$.

Suppose $\zeta \in \mathfrak{X}(M)$ is an infinitesimal automorphism. Then the Lie derivative $\mathcal{L}_\zeta$ preserves the distribution, that means $[\zeta, \gamma]$ is contained in $\Gamma(\mathcal{H})$ for all $\gamma \in \Gamma(\mathcal{H})$. In particular, we have $q_{-3}([\zeta, \gamma]) = 0$. Recall, that we can write any vector field ζ as $\zeta = [r, \zeta_1] + \alpha(\zeta)r + \zeta_2$ and using this decomposition we have

$$q_{-3}([\zeta, \gamma]) = q_{-3}([[r, \zeta_1], \gamma]) + \{\alpha(\zeta)r, \gamma\}.$$

The characterisation of the Weyl connection shows that

$$\nabla_\gamma q_{-3}(\zeta) = \{q_{-2}(r), \nabla_\gamma\zeta_1\} = q_{-3}([\gamma, [r, \zeta_1]]).$$

Combining the two equations we see that $q_{-3}([\zeta, \gamma]) = 0$ implies that

$$\nabla_\gamma q_{-3}(\zeta) = \{\alpha(\zeta)q_{-2}(r), \gamma\}.$$

In other words, $\nabla q_{-3}(\zeta)$ is pure trace and thus $q_{-3}(\zeta)$ is contained in the kernel of the differential operator D.

Conversely, suppose $\{q_{-2}(r), \zeta_1\} \in \Gamma(\mathrm{gr}_{-3}(TM))$, for some $\zeta_1 \in \Gamma(\mathcal{H})$, is a section contained in the kernel of the differential operator D. Since $\nabla\{q_{-2}(r), \zeta_1\}$ is pure trace, we can find a unique function $f \in C^\infty(M)$ such that

$$\nabla_\gamma q_{-3}(\zeta) = f\{q_{-2}(r), \gamma\}$$

for all $\gamma \in \Gamma(\mathcal{H})$. Note that by construction $q_{-3}([\gamma, [r, \zeta_1] + fr]) = 0$, which means that $[\gamma, [r, \zeta_1] + fr]$ is a section of $\mathcal{H}^1$. Hence, we can find a unique element $\zeta_2 \in \Gamma(\mathcal{H})$ characterised by

$$\{\gamma, \zeta_2\} = -q_{-2}([\gamma, [r, \zeta_1] + fr])$$

for all $\gamma \in \Gamma(\mathcal{H})$. The vector field

$$\zeta = [r, \zeta_1] + fr + \zeta_2$$

is an infinitesimal automorphism, since we built it such that $q_{-3}([\gamma, \zeta]) = 0$ as well as $q_{-2}([\gamma, \zeta]) = 0$, i.e. $[\gamma, \zeta] \in \Gamma(\mathcal{H})$ for all $\gamma \in \Gamma(\mathcal{H})$. Indeed, we have seen that ζ is the unique infinitesimal automorphism such that the projection of ζ onto the quotient $\mathrm{gr}_{-3}(TM)$ coincides with the section $\{q_{-2}(r), \zeta_1\}$. Hence, we have established a bijective correspondence between infinitesimal automorphisms and sections in the kernel of the operator D.

Remark 3.29. Another variation of the same story: Suppose α is a contact form for a generic rank two distribution in dimension five. If we replace α by $\hat{\alpha} = e^f\alpha$, the corresponding Weyl connection in $T^{-1}M$-directions transforms as

$$\hat{\nabla}_\eta \xi = \nabla_\eta \xi - df(\eta)\xi + 3df(\xi)\eta$$

for any $\xi \in \Gamma(T^{-1}M)$, see (A.8) for a direct verification. Suppose $E[w]$ is a density bundle of weight w, i.e.

$$\hat{\nabla}_\eta \gamma = \nabla_\eta \gamma + w df(\eta)\gamma$$

for any $\gamma \in E[w]$, cf. Remark 3.15. Then,

$$\begin{aligned}\hat{\nabla}_\eta(\xi \otimes \gamma) &= (\hat{\nabla}_\eta \xi \otimes \gamma) + (\xi \otimes \hat{\nabla}_\eta \gamma) \\ &= \nabla_\eta(\xi \otimes \gamma) + (w-1)df(\eta)(\xi \otimes \gamma) + 3df(\xi)(\eta \otimes \gamma).\end{aligned}$$

Using the index notation $\nabla_a \xi^b$ for a section of $\mathcal{H}^* \otimes \mathcal{H} \otimes E[w]$ and defining $\Upsilon = df$ the above transformation reads as

$$\hat{\nabla}_a \xi^b = \nabla_a \xi^b + (w-1)\Upsilon_a \xi^b + 3\Upsilon_c \xi^c \delta_a^b.$$

Taking the obvious trace we obtain

$$\hat{\nabla}_c \xi^c = \nabla_c \xi^c + (6 + w - 1)\Upsilon_c \xi^c$$

and for the tracefree part

$$\hat{\nabla}_a \xi^b - \frac{1}{2}(\hat{\nabla}_c \xi^c)\delta_a^b = \nabla_a \xi^b - \frac{1}{2}(\nabla_c \xi^c)\delta_a^b + (3 - \tfrac{1}{2}(6 + w - 1))\Upsilon_c \xi^c \delta_a^b.$$

This shows: If we choose the weight $w = 1$, the formula for the tracefree part is independent of the choice of contact form. But given a global generalised contact form, we have an isomorphism $E[1] \cong \mathrm{gr}_{-2}(TM)$, cf. Remark 3.15. Hence, the invariant operator is the infinitesimal automorphism operator. Note that choosing a different weight, namely $w = -5$, the trace part defines an operator which is independent of the choice of generalised contact form.

CHAPTER 4

On Nurowski's conformal structure

Nurowki in his paper [**29**] looks at ordinary differential equations that give rise to conformal structures. One of the examples he studies is based on Elie Cartan's famous paper [**19**], namely he considers equations of the form

$$z' = F(x, y, y', y'', z)$$

with $\dfrac{\partial^2 F}{\partial (y'')^2} \neq 0$, where $y = y(x)$ and $z = z(x)$ are functions of x. The differential equation is encoded in the system of one-forms

$$\omega_1 = dz - F(x, y, p, q, z)dx,$$

$$\omega_2 = dy - pdx$$

$$\omega_3 = dp - qdx$$

on a manifold J with coordinates $(x, y, p = y', q = y'', z)$. The common kernel of the one-forms defines a generic rank two distribution $\mathcal{H}$. Solutions of the equation $z' = F(x, y, y', y'', z)$ correspond to curves $c : \mathbb{R} \to J$ tangential to $\mathcal{H}$. (Note that for the equation $z' = q^2$, the one-forms $\omega_1, \omega_2, \omega_3$ coincide with the forms in the beginning of Chapter 2.)

Using the canonical Cartan connection for such a distribution, Nurowski shows that the distribution $\mathcal{H}$ determines a natural conformal structure of signature $(2, 3)$ on J. He gives a long formula for a metric in the conformal class in terms of the function F and its derivatives. He also computes the canonical Cartan connection for the conformal structure and observes that after reduction it can be identified with the Cartan connection for the distribution. In particular, the so-called conformal holonomy group of the conformal structure has to be a subgroup of the exceptional group G_2. The conformal structures determined by generic rank two distributions in dimension five are referred to as Nurowski's conformal structures.

The study of these conformal structures is the subject of this chapter. We interpret Nurowski's result as a special case of a Fefferman construction, that is a construction relating parabolic geometries of different types. We give a proof of the fact that conformal structures obtained via this particular Fefferman construction can be characterised by the existence of a parallel tractor three-form on the standard tractor bundle, or equivalently, by the fact that their conformal holonomy is contained in G_2. The main result of this chapter is the construction of a pseudo-Riemannian metric from Nurowski's conformal class in terms of a generalised contact form α and the associated generalised Reeb field. For the construction, we will use the splitting of the filtration on TM associated to α derived in Chapter 3.

4.1. The canonical conformal structure

Let us recall: We have a realisation of G_2 as a subgroup in $SO(3,4)$. The parabolic subgroup P in G_2 is the stabiliser of the one-dimensional real subspace $\mathbb{V}^2$ through a highest weight vector in the 7 dimensional representation $\mathbb{V}$. We have seen that $\mathbb{V}^2$ is isotropic with respect to the invariant inner product on $\mathbb{V}$. Its stabiliser in $SO(3,4)$ is a maximal parabolic subgroup $\tilde{P}$ of $SO(3,4)$. It follows, that the inclusion $G_2 \to SO(3,4)$ leads to an injective, smooth map

$$G_2/P \to SO(3,4)/\tilde{P}.$$

Both spaces have the same dimension, implying that the map is actually an open embedding. Moreover, it is well known that a quotient of a semisimple Lie group modulo a parabolic subgroup is always compact. By connectedness, this implies that $G_2/P \to SO(3,4)/\tilde{P}$ is a diffeomorphism. The space $SO(3,4)/\tilde{P}$ is the homogeneous model for conformal structures of signatur $(2,3)$, see 1.2.6. Hence, it carries a conformal structure preserved by the action of $SO(3,4)$ (and on an infinitesimal level, $\mathfrak{so}(3,4)/\tilde{\mathfrak{p}}$ carries a $\tilde{P}$-invariant conformal class of bilinear forms). The above diffeomorphism gives rise to a conformal structure on G_2/P. Compare with Remark 2.8.

The situation in the curved case is not much more difficult: Let M be a 5-dimensional manifold with a generic rank two distribution $\mathcal{H}$ and let $(\mathcal{G},\omega)$ be the corresponding regular, normal parabolic geometry. Let us denote by $\mathfrak{g}$ the Lie algebra of G_2 and by $\tilde{\mathfrak{g}}$ the Lie algebra of $SO(3,4)$. The main point is to remember that for any Cartan geometry, the Cartan connection induces an isomorphism

$$TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}.$$

This means, we only need to observe that we have a P-invariant class of inner products of signature $(2,3)$ on $\mathfrak{g}/\mathfrak{p}$, to conclude that we have a conformal structure of signature $(2,3)$ on the manifold M. But this is easy, now: The derivative of the diffeomorphism $G_2/P \cong SO(3,4)/\tilde{P}$ induces an P-equivariant isomorphism

$$\mathfrak{g}/\mathfrak{p} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}.$$

The space $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ carries a $\tilde{P}$-invariant conformal class of inner products of signature $(2,3)$ and via the isomorphism, this conformal class gives rise to one on $\mathfrak{g}/\mathfrak{p}$. A P-invariant class of inner products can be regarded as a P-invariant $\mathbb{R}_+$-subspace in $S^2(\mathfrak{g}/\mathfrak{p})^*$ which induces a ray subbundle in S^2T^*M and thus a conformal structure on M.

Remark 4.1. The conformal class on $\mathfrak{so}(3,4)/\tilde{\mathfrak{p}}$, and then also on $\mathfrak{g}/\mathfrak{p}$, can be understood as being induced from the inner product $\langle , \rangle$ of signature $(3,4)$ on the standard representation $\mathbb{V}$ via the identification $\mathfrak{so}(3,4)/\tilde{\mathfrak{p}} \cong L(\mathbb{V}^2, (\mathbb{V}^2)^\perp/\mathbb{V}^2)$. Compare with 1.2.6. A bilinear form in the conformal class on $L(\mathbb{V}^2, (\mathbb{V}^2)^\perp/\mathbb{V}^2)$ is given by $(\phi,\psi) \mapsto \langle \phi(v), \psi(v) \rangle$ for a choice of a nonzero vector $v \in \mathbb{V}^2$.

To describe the conformal class of bilinear forms of signature $(2,3)$ on $\mathfrak{g}/\mathfrak{p}$, we choose representatives $A, A' \in \mathfrak{g}_-$ for elements $[A], [A'] \in \mathfrak{g}/\mathfrak{p}$, where A is the matrix

$$\begin{pmatrix} X & & & \\ r & -\sqrt{2}X^t\mathbb{J} & & \\ Y & -\frac{r}{\sqrt{2}}\mathbb{J} & \sqrt{2}\mathbb{J}X & \\ & -Y^t & r & -X^t \end{pmatrix}.$$

In this picture, a bilinear form from the conformal class is given by

$$\mathbf{b}([A],[A']) = X^tY' - rr' + Y^tX'. \tag{4.1}$$

Remark 4.2. Suppose $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is a Weyl structure for the parabolic geometry $(\mathcal{G}, \omega)$ corresponding to a rank two distribution $\mathcal{H}$. For any tangent vector $\zeta_x \in T_xM$ we choose a point $u_0 \in \mathcal{G}_0$ with $p(u_0) = x$ and a lift $\hat{\zeta}_{u_0} \in T_{u_0}\mathcal{G}_0$. Let $\sigma^*\omega_-$ be the soldering form, i.e. the component of $\sigma^*\omega$ with values in $\mathfrak{g}_-$. Then, a metric form the conformal class associated to the distribution maps two tangent vectors $\zeta_x, \zeta'_x \in T_xM$ to

$$\mathbf{b}\left(\sigma^*\omega_-(u_0)(\hat{\zeta}_{u_0}), \sigma^*\omega_-(u_0)(\hat{\zeta'}_{u_0})\right)$$

for some inner product $\mathbf{b}$ in the conformal class (4.1) on $\mathfrak{g}_-$. Note, that the conformal class does not depend on the full Cartan connection ω but only on the soldering form $\sigma^*\omega_-$.

Remark 4.3. The conformal structure introduced here coincides with the one found by Pawel Nurowski in [**29**]. In his construction, Nurowski starts with a degenerate bilinear form on $\mathfrak{g}$, that induces a bilinear form in the conformal class on $\mathfrak{g}/\mathfrak{p}$. Via the normal Cartan connection, the bilinear form on $\mathfrak{g}$ gives rise to a degenerate metric on $\mathcal{G}$ and he observes that this metric descends to a well defined conformal class of metrics of signature $(2,3)$ on the underlying manifold M.

4.2. A Fefferman construction

A conceptual way to understand Nurowski's conformal structure is to view it as a result of a so-called (generalised)Fefferman construction.

It was Fefferman's idea to associate to a CR manifold a conformal structure on a S^1-bundle over the manifold and to study the geometry of the CR manifold via this construction. There is a Cartan geometric formulation of the Fefferman construction that can be generalised, giving rise to Fefferman constructions on various parabolic geometries, see [**8**]. In particular, assigning Nurowski's conformal structure to a generic rank two distribution in dimension five can be viewed as a (simple) instance of a Fefferman construction. Hence, we will briefly explain Fefferman constructions in general:

Suppose we have two pairs of semisimple Lie groups and parabolic subgroups $(\tilde{G}, \tilde{P})$ and (G, P) and an inclusion

$$i : G \to \tilde{G}$$

such that the G-orbit of $e\tilde{P}$ in $\tilde{G}/\tilde{P}$ is open and P contains $Q := \tilde{P} \cap G$. In particular, this means that we have an isomorphism $\mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$.

Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P). Then the *Fefferman space* $\tilde{M}$ is defined as

$$\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P P/Q.$$

Note that $\mathcal{G} \to \tilde{M}$ is a principal fibre bundle with structure group Q and we can view ω as a Cartan connection on that bundle. We can extend the structure group of the Q-principal bundle $\mathcal{G} \to \tilde{M}$ to obtain a $\tilde{P}$-principal bundle $\tilde{\mathcal{G}} \to \tilde{M}$, where

$$\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P}.$$

We identify $\mathcal{G}$ with a subspace in $\tilde{\mathcal{G}}$ mapping an element $u \in \mathcal{G}$ to the class $[(u, e)]$. Moreover, we can uniquely extend the Cartan connection ω on $\mathcal{G}$ to a Cartan connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ such that $\tilde{\omega}$ restricts to ω on $T\mathcal{G} \subset T\tilde{\mathcal{G}}$: For any $u \in \mathcal{G}$ and tangent vector $\tilde{\xi}_u \in T_u\tilde{\mathcal{G}}$, we may choose $\xi_u \in T_u\mathcal{G}$ such that $T_u\tilde{p} \cdot \tilde{\xi}_u = T_u p \cdot \xi_u$ and decompose $\tilde{\xi}_u$ as $\xi_u + \zeta_{\tilde{A}}(u)$ for some fundamental vector field $\zeta_{\tilde{A}}$. We define $\tilde{\omega}(u)$ by requiring that it reproduces generators of fundamental vector fields, i.e.

$$\tilde{\omega}(u)(\xi_u + \zeta_{\tilde{A}}(u)) := \omega(u)(\xi_u) + \tilde{A}.$$

Using the fact that ω reproduces generators of fundamental vector fields, we see that this is indeed well defined. Since any $\tilde{u} \in \tilde{\mathcal{G}}$ is of the form $u \cdot \tilde{g}$ for some $u \in \mathcal{G}$ and $\tilde{g} \in \tilde{P}$ and ω is equivariant, we can further define $\tilde{\omega}$ on all of $\tilde{\mathcal{G}}$ by equivariant extension.

For any morphism $F : \mathcal{G}_1 \to \mathcal{G}_2$ between parabolic geometries $(\mathcal{G}_1 \to M_1, \omega_1)$ and $(\mathcal{G}_2 \to M_2, \omega_2)$ of type (G, P), the map $F \times \mathrm{id}_{\tilde{P}}$ induces a morphism between the geometries $(\tilde{\mathcal{G}}_1 \to \tilde{M}_1, \tilde{\omega}_1)$ and $(\tilde{\mathcal{G}}_2 \to \tilde{M}_2, \tilde{\omega}_2)$ of type $(\tilde{G}, \tilde{P})$ obtained via the Fefferman construction. So the construction is functorial.

There are particularly simple examples of generalised Fefferman constructions. These are the cases where the subgroup $Q = G \cap \tilde{P}$ is a parabolic subgroup in G (note that this is not the case in general) and we take P to be that subgroup. Then, assuming that G acts transitively on G/P, we obtain a diffeomorphism $G/P \cong \tilde{G}/\tilde{P}$. The Fefferman construction in these cases simply amounts to an extension of structure group of the P-principal bundle $\mathcal{G} \to M$ to a $\tilde{P}$-principal bundle $\tilde{\mathcal{G}} \to M$ and an equivariant extension of the Cartan connection. Both the Cartan geometry of type (G, P) we started with and the new Cartan geometry of type $(\tilde{G}, \tilde{P})$ are defined over the same manifold M. Nurowski's construction is of this type, see Example 4.5.

Example 4.4. The classical Fefferman result says that given any nondegenerate CR structure of hypersurface type on a manifold M, we obtain a natural conformal structure on the total space of a S^1 bundle over M. The result can be recovered from the general construction: The groups corresponding to CR structures are $G = SU(p + 1, q + 1)$ and P the stabiliser of an isotropic complex line in the standard representation $\mathbb{C}^{p+q+2}$. The group $SU(p + 1, q + 1)$ preserves a Hermitian form on the standard representation. We can consider the real part of that Hermitian form which defines a nondegenerate bilinear form of signature $(2p + 2, 2q + 2)$. Since the bilinear form is invariant under the action of $SU(p + 1, q + 1)$, we obtain an inclusion of $SU(p + 1, q + 1)$ into $\tilde{G} = SO(2p + 2, 2q + 2)$. We define $\tilde{P} \subset SO(2p+2, 2q+2)$ to be the stabiliser of an isotropic real line ℓ and we put $Q = \tilde{P} \cap SU(p + 1, q + 1)$. Then Q is contained in the stabiliser P of the complex span of the line ℓ and G/Q is diffeomorphic to $\tilde{G}/\tilde{P}$. Moreover, note that $P/Q \cong \mathbb{R}P^1 \cong S^1$. Let $(\mathcal{G} \to M, \omega)$ be the parabolic geometry corresponding to the CR structure. Then the

Fefferman space $\mathcal{G}/Q = \mathcal{G} \times_P P/Q$ is a S^1-bundle over M and the construction gives rise to a canonical conformal class on that bundle.

EXAMPLE 4.5. The Nurowski conformal structure can also be understood as a special case of this construction. First we associate to a generic rank two distribution in dimension five the canonical regular, normal parabolic geometry of type (G_2, P). Then we apply the above construction for the natural inclusion of $G = G_2$ into $\tilde{G} = SO(3,4)$. The parabolic subgroups P and $\tilde{P}$ are the respective stabilisers of an isotropic line in the standard representation. Note that this implies that $Q = P$ and thus $M = \tilde{M}$. Therefore, rather than on a bundle over the manifold, the conformal structure is in this case obtained on the manifold itself. Recall that the underlying conformal structure for the conformal parabolic geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ is obtained via the isomorphism $TM \cong \tilde{\mathcal{G}} \times_{\tilde{P}} \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ induced by the Cartan connection $\tilde{\omega}$. By construction of $(\tilde{\mathcal{G}}, \tilde{\omega})$, this is the same conformal structure that we obtain from the isomorphism $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ induced by ω as explained in the previous section.

EXAMPLE 4.6. In [**20**] it is shown that there are two more series of examples of Fefferman constructions such that $\tilde{M} = M$: The first of these starts with a bracket generating distribution with growth vector $(n, n(n+1)/2)$ for $n \geq 3$, see 1.33, and it implies a spinorial structure on the manifold. In dimension six, the almost spinorial structure is equivalent to a conformal structure of signature $(3,3)$. The 6-dimensional case has been worked out by R. Bryant, see [**6**], and it has been looked at as a parabolic Fefferman construction by S. Armstrong in [**1**]. The second series of examples associates projective structures to contact projective structures, cf. [**22**].

EXAMPLE 4.7. Finally, I would like to mention a Fefferman construction very similar to the original one that associates a conformal structure to a quaternionic contact structure. This construction has been studied by J. Alt in his thesis.

4.2.1. Normality of the conformal Cartan connection. The Fefferman construction does not say anything about regularity or normality of the resulting parabolic geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$. In our case, i.e. Example 4.5, the question of regularity is trivial. A $|1|$-graded parabolic geometry is automatically regular and conformal geometries are $|1|$-graded. However, to see that normality of ω implies normality of $\tilde{\omega}$, requires a proof.

We first relate the curvatures of the Cartan geometries $(\mathcal{G}, \omega)$ and $(\tilde{\mathcal{G}}, \tilde{\omega})$. We have seen that the inclusion $i' : \mathfrak{g} \to \tilde{\mathfrak{g}} = \mathfrak{so}(3,4)$ leads to an isomorphism $\mathfrak{g}/\mathfrak{p} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Let $\kappa : \mathcal{G} \to L(\Lambda^2\mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ be the curvature function for the parabolic geometry $(\mathcal{G}, \omega)$ associated with a generic rank two distribution in dimension five and let $\tilde{\kappa} : \tilde{\mathcal{G}} \to L(\Lambda^2\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}, \tilde{\mathfrak{g}})$ be the curvature function for the corresponding conformal geometry. Equivariance of $\tilde{\kappa}$ together with the fact that every element in $\tilde{\mathcal{G}}$ is of the form $u \cdot \tilde{g}$, with $u \in \mathcal{G}$ and $\tilde{g} \in \tilde{P}$, implies that $\tilde{\kappa}$ is determined by its restriction to $\mathcal{G} \subset \tilde{\mathcal{G}}$. Hence, it is determined by the commuting diagram:

$$\begin{tikzcd} \Lambda^2(\mathfrak{g}/\mathfrak{p}) \arrow[r, "\kappa"] \arrow[d] & \mathfrak{g} \arrow[d] \\ \Lambda^2(\mathfrak{so}(3,4)/\tilde{\mathfrak{p}}) \arrow[r, "\tilde{\kappa}"] & \mathfrak{so}(3,4) \end{tikzcd} \tag{4.2}$$

Recall that a parabolic geometry is called normal if and only if its curvature has the property that $\partial^*(\kappa) = 0$. On decomposable elements $U \wedge V \otimes A \in \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ the Kostant codifferential ∂^*, see (1.2), reads

$$\partial^*(U \wedge V \otimes A) = U \otimes [V, A] - V \otimes [U, A] - [U, V] \otimes A.$$

A reformulation using some linear algebra, cf. [**16**], yields:

Lemma 4.8. *For any $\psi \in L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$ we can compute $\partial^*\psi$ viewed as a map $\mathfrak{g}/\mathfrak{p} \to \mathfrak{g}$ as follows. We choose a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}_-$ and a dual basis $\{Z_1, \cdots, Z_n\}$ of $\mathfrak{p}_+$ with respect to the Killing form. Then,*

$$\partial^*\psi(X + \mathfrak{p}) = 2\sum_i [Z_i, \psi(X + \mathfrak{p}, X_i + \mathfrak{p})] - \sum_i \psi([Z_i, X] + \mathfrak{p}, X_i + \mathfrak{p}),$$

for any $X \in \mathfrak{g}$.

Applying the lemma we see that $\partial^*\kappa(u) = 0$ is equivalent to

$$\sum_i [Z_i, \kappa(u)(X, X_i)] - \frac{1}{2}\sum_i \kappa(u)([Z_i, X], X_i) = 0,$$

for all $X \in \mathfrak{g}$ and basis $\{X_i\}$, $\{Z_i\}$ as in the lemma. We can now prove the following result.

Proposition 4.9. *Let $(\mathcal{G} \to M, \omega)$ be a regular, normal parabolic geometry of type (G_2, P). Then the extended Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ of the associated conformal geometry $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ is normal as well.*

Proof. We denote by $\partial^*_{\mathfrak{p}_+} : \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ the Kostant codifferential dualising the differential $\partial_{\mathfrak{p}_+} : (\mathfrak{p}_+)^* \otimes \mathfrak{g} \to \Lambda^2(\mathfrak{p}_+)^* \otimes \mathfrak{g}$. Likewise, $\partial^*_{\tilde{\mathfrak{p}}_+} : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ denotes the Kostant codifferential dualising the differential $\partial_{\tilde{\mathfrak{p}}_+} : (\tilde{\mathfrak{p}_+})^* \otimes \tilde{\mathfrak{g}} \to \Lambda^2(\tilde{\mathfrak{p}_+})^* \otimes \tilde{\mathfrak{g}}$. Let us consider the map

$$\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}} \overset{\partial^*_{\tilde{\mathfrak{p}}_+}}{\to} (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}, \tag{4.3}$$

where the first arrow represents the map induced by the isomorphism $\mathfrak{g}/\mathfrak{p} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ and the inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$. Then this map is a P-homomorphism, hence its kernel is a P-submodule. The aim of this proof is to verify that $\tilde{\omega}$ is normal, i.e. $\partial^*_{\tilde{\mathfrak{p}}_+}(\tilde{\kappa}(u)) = 0$ for all $u \in \tilde{\mathcal{G}}$. Diagram (4.2) shows that this is equivalent to verifying that κ takes values in the kernel of the above homomorphism (4.3). This reformulation allows to apply Proposition 2.17 to simplify the problem: Recall that the lowest homogeneous component of $\kappa(u)$ can be identified with the harmonic curvature component $\kappa_H(u)$ and has values in $\ker(\square) \subset \mathfrak{g}_1 \otimes \mathfrak{g}_3 \otimes \mathfrak{g}_0$. See Theorem 1.18 and Proposition 2.15. Proposition 2.17 implies that it suffices to show that $\kappa_H(u)$ is contained in the kernel of homomorphism (4.3), to conclude that $\kappa(u)$ is contained in the kernel. So this is what we are going to do.

First, we choose a basis of $\mathfrak{g}_-$ consisting of root vectors $X_1, X_2 \in \mathfrak{g}_{-3}$, $X_3 \in \mathfrak{g}_{-2}$ and $X_4, X_5 \in \mathfrak{g}_{-1}$. Then we obtain an induced basis of $\mathfrak{g}/\mathfrak{p}$ as well as an induced basis of $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Next we determine the dual basis of $\mathfrak{p}_+$ respectively of $\tilde{\mathfrak{p}}_+$ with respect to the Killing form. We denote these by $Z_1, \ldots, Z_5 \in \mathfrak{p}_+$ respectively $\tilde{Z}_1, \ldots, \tilde{Z}_5 \in \tilde{\mathfrak{p}}_+$. In 2.1.3 we observed that taking any invariant bilinear form, the Lie algebra $\tilde{\mathfrak{g}}$ decomposes into an orthogonal direct sum $\mathfrak{g} \oplus \mathbb{V}$, where $\mathbb{V}$ is a $\mathfrak{g}$-representation isomorphic to the standard representation. By construction of Z_j and $\tilde{Z}_j$, the difference $Z_j - \tilde{Z}_j$ is contained in the orthogonal complement to $\mathfrak{g}$ that is in the space $\mathbb{V} \subset \tilde{\mathfrak{g}}$. Recall that the $\mathfrak{g}_0$-representation $\mathbb{V}$ decomposes as $\mathbb{V}_2 \oplus \mathbb{V}_1 \oplus \mathbb{V}_0 \oplus \mathbb{V}_{-1} \oplus \mathbb{V}_{-2}$. It is easy to see that the map $Z_j \mapsto Z_j - \tilde{Z}_j$ sends $\mathfrak{g}_3$ to zero and defines $\mathfrak{g}_0$-modul isomorphisms $\mathfrak{g}_2 \cong \mathbb{V}_2$ and $\mathfrak{g}_1 \cong \mathbb{V}_1$. (To see that, one verifies for instance that both maps $\mathfrak{g}_2 \cong \mathbb{V}_2$ and $\mathfrak{g}_1 \cong \mathbb{V}_1$ are given by the adjoint action of a suitable element $F \in \mathbb{V}_0 \subset \tilde{\mathfrak{g}}_0$ and for any such element $[\mathfrak{g}_0, F] = 0$.)

The harmonic curvature component κ_H is contained in the kernel of the Kostant codifferential $\partial^*_{\mathfrak{p}}$ which can be reformulated, by Lemma 4.8, as

$$\sum_j [Z_j, \kappa_H(u)(X_i, X_j)] - \sum_j \frac{1}{2}\kappa_H(u)([Z_j, X_i], X_j) = 0$$

for all X_i and all $u \in \mathcal{G}$. Let us consider the summand $\sum_j \frac{1}{2}\kappa_H(u)([Z_j, X_i], X_j)$. For $j = 1, 2$, the bracket $[Z_j, X_i]$ is contained in $\mathfrak{p}$ which is why $\kappa_H(u)([Z_j, X_i], X_j)$ vanishes. For $j = 3, 4, 5$, note that neither $[Z_j, X_i]$ nor X_j are contained in $\mathfrak{g}_{-3}$ and thus $\kappa_H(u)([Z_j, X_i], X_j) = 0$ in those cases. It follows that

$$\sum_j \frac{1}{2}\kappa_H(u)([Z_j, X_i], X_j) = 0$$

and thus

$$\sum_j [Z_j, \kappa_H(u)(X_i, X_j)] = 0.$$

Since the summands $\sum_{j=1,2}[Z_j, \kappa_H(u)(X_i, X_j)] \in \mathfrak{g}_{-3}$, $[Z_3, \kappa_H(u)(X_i, X_3)] \in \mathfrak{g}_{-2}$ and $\sum_{j=4,5}[Z_j, \kappa_H(u)(X_i, X_j)] \in \mathfrak{g}_{-1}$ are contained in different grading components, we further know that these parts vanish separately.

Now we come to the equation that we want to verify. We want to show that

$$\sum_j [\tilde{Z}_j, \kappa_H(u)(X_i, X_j)] - \sum_j \frac{1}{2}\kappa_H(u)([\tilde{Z}_j, X_i], X_j) = 0$$

holds for all X_i and all $u \in \mathcal{G}$, where $\kappa_H(u)(\tilde{X}, \tilde{Y})$ for $\tilde{X}, \tilde{Y} \in \tilde{\mathfrak{g}}$ is to be understood as $\kappa_H(u)(X, Y)$ for $X, Y \in \mathfrak{g}$ such that $X + \mathfrak{p} = \tilde{X} + \tilde{\mathfrak{p}}$ and $Y + \mathfrak{p} = \tilde{Y} + \tilde{\mathfrak{p}}$. Since $\tilde{\mathfrak{g}}$ is $|1|$-graded and $\tilde{Z}_j \in \tilde{\mathfrak{g}}_1$, we know that $[\tilde{Z}_j, X] \in \tilde{\mathfrak{g}}^0 = \tilde{\mathfrak{p}}$ for all $X \in \tilde{\mathfrak{g}}$. Thus, $\kappa_H(u)([\tilde{Z}_j, X_i], X_j) = 0$ for all i and hence

$$\sum_j \kappa_H(u)([\tilde{Z}_j, X_i], X_j) = 0.$$

It remains to consider $\sum_j [\tilde{Z}_j, \kappa_H(u)(X_i, X_j)]$. For $j = 1, 2$, we have $\tilde{Z}_j = Z_j$ and the sum vanishes by what we have seen above. For $j = 3$, we have $[\tilde{Z}_j, \kappa_H(u)(X_i, X_j)] = 0$, since κ_H is nontrivial only on $\mathfrak{g}_{-1} \times \mathfrak{g}_{-3}$. For $j = 4, 5$, notice that we have

$$\sum_{j=4,5} [\tilde{Z}_j, \kappa_H(u)(X_i, X_j)] = \sum_{j=4,5} [Z_j, \kappa_H(u)(X_i, X_j)] + \sum_{j=4,5} [\tilde{Z}_j - Z_j, \kappa_H(u)(X_i, X_j)].$$

We have seen that $\sum_{j=4,5}[Z_j, \kappa_H(u)(X_i, X_j)]$ vanishes, We have observed that $\Psi(Z_j) = \tilde{Z}_j - Z_j$ defines an $\mathfrak{g}_0$-modul isomorphism $\mathfrak{g}_1 \cong \mathbb{V}_1$. But then, since $\kappa_H(u)(X_i, X_j)$ has values in $\mathfrak{g}_0$, the sum $\sum_{j=4,5}[\tilde{Z}_j - Z_j, \kappa_H(u)(X_i, X_j)] = \Psi(\sum_{j=4,5}[Z_j, \kappa_H(u)(X_i, X_j)])$ vanishes as well. $\square$

4.3. A conformal holonomy reduction

In this section, we will introduce the standard tractor bundle and the standard tractor connection for conformal structures and for generic rank two distributions in dimension five. We will see that among all conformal structures of signature $(2,3)$, the structures arising from Fefferman constructions of generic rank two distributions can be characterised by the fact that their conformal holonomy reduces to G_2.

4.3.1. Tractor bundles and tractor connections. Let us briefly introduce the notions of a tractor bundle and a tractor connection.

Given any Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), we can consider the extended bundle

$$\hat{\mathcal{G}} = \mathcal{G} \times_P G$$

for the action of P on G by left multiplication. Then $\hat{\mathcal{G}} \to M$ is obviously a G-principal bundle, the G-action given by $[(u,g)] \cdot g' := [(u, gg')]$. Moreover, the Cartan connection ω uniquely extends to a principal connection $\hat{\omega}$ on $\hat{\mathcal{G}}$ that restricts to ω on $T\mathcal{G} \subset T\hat{\mathcal{G}}$: For $u \in \mathcal{G}$, any tangent vector $\hat{\xi}_u \in T_u\hat{\mathcal{G}}$ can be written as the sum of a tangent vector $\xi_u \in T_u\mathcal{G}$ and a fundamental vector field $\zeta_X(u)$ for some $X \in \mathfrak{g}$. One defines $\hat{\omega}(\hat{\xi}_u) := \omega(\xi_u) + X$ and then equivariantly extends $\hat{\omega}$ to all of $\hat{\mathcal{G}}$. It follows, that there are induced linear connections on all associated bundles to $\hat{\mathcal{G}}$.

Now let $\rho : G \to GL(\mathbb{W})$ be a representation of the group G. We can consider the restriction of ρ to the subgroup P. Then we can identify the associated bundles

$$\mathcal{G} \times_P \mathbb{W} = \hat{\mathcal{G}} \times_G \mathbb{W}.$$

Consequently, the principal bundle connection $\hat{\omega} \in \Omega^1(\hat{\mathcal{G}}, \mathfrak{g})$ induces a linear connection $\nabla^{\mathcal{W}}_{\xi} s$ on the associated bundle $\mathcal{W} = \mathcal{G} \times_P \mathbb{W}$. Explicitly, that linear connection is given as follows. Let $f : \mathcal{G} \to \mathbb{W}$ be the P-equivariant function corresponding to a section $s \in \Gamma(\mathcal{W})$. Then the function corresponding to $\nabla^{\mathcal{W}}_{\xi} s$ is given by

$$u \mapsto (\tilde{\xi} \cdot f)(u) + \rho(\omega(\tilde{\xi}))(f(u))$$

for any lift $\tilde{\xi} \in \mathfrak{X}(\mathcal{G})$ of the vector field $\xi \in \mathfrak{X}(M)$, see e.g. [**16**].

Definition 4.10. Associated bundles $\mathcal{G} \times_P \mathbb{W}$ corresponding to restrictions of representations of the group G to the subgroup P are called *tractor bundles*, the induced connections are called *tractor connections*. A tractor connection that comes from a normal Cartan connection is called a normal tractor connection.

For any matrix group we have a standard representation $\mathbb{V}$ and correspondingly a standard tractor bundle $\mathcal{T} = \mathcal{G} \times_G \mathbb{V}$. As explained before, the Cartan connection induces a standard tractor connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$.

4.3.2. The standard tractor bundle for generic rank two distributions. Let us return to our usual setting. We consider a generic rank two distribution $\mathcal{H}$ in dimension five. Such a distribution determines a canonical parabolic geometry $(\mathcal{G}, \omega)$ of type (G_2, P), where P is the stabiliser of the highest weight space in the seven dimensional standard representation $\mathbb{V}$. Associated to a parabolic geometry of this type is a normal conformal geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ of type $(SO(3,4), \tilde{P})$. We can form the standard tractor bundles $\mathcal{T} = \mathcal{G} \times_P \mathbb{V}$ and $\tilde{T} = \mathcal{G} \times_{\tilde{P}} \mathbb{V}$. These bundles are can be identified via $[(u,X)] \mapsto [(j(u),X)]$, where j denotes the inclusion $\mathcal{G} \to \tilde{\mathcal{G}}$. Also the Tractor connections induced by respectively ω and $\tilde{\omega}$ coincide. Since both ω and the Cartan connection $\tilde{\omega}$ are normal, the corresponding tractor connection is the normal tractor connection for the generic rank two distribution as well as for the corresponding conformal structure.

Let us think of the standard tractor bundle as a conformal tractor bundle first. The invariant inner product of signature $(3,4)$ on $\mathbb{V}$ gives rise to a bundle metric h of that signature on $\mathcal{T}$, i.e. $h \in \Gamma(S^2(\tilde{\mathcal{T}}^*))$ corresponds to the constant, $\tilde{P}$-equivariant function $f : \tilde{\mathcal{G}} \to S^2(\mathbb{V}^*)$ given by $f(u) = \langle , \rangle$. Note that, $(\tilde{\xi} \cdot f)(u) + \tilde{\omega}(\tilde{\xi})(f(u)) = 0$ for all $u \in \tilde{\mathcal{G}}$ and $\tilde{\xi} \in \mathfrak{X}(\tilde{\mathcal{G}})$, since f is constant and $\mathfrak{so}(3,4)$ stabilizes $\langle , \rangle$. This shows that h is parallel for the tractor connection $\nabla^{\mathcal{T}}$. The parabolic subgroup $\tilde{P}$ is defined as the stabiliser of a one-dimensional isotropic subspace $\mathbb{V}^2$ in $\mathbb{V}$. Hence, this subspace gives rise to an isotropic line subbundle $\mathcal{T}^2$ in $\mathcal{T}$.

Next we can refine the structure on the tractor bundle using the fact that it even is an associated bundle for $\mathcal{G}$. Recall from 2.1.8 that the parabolic P stabilizes a filtration $\mathbb{V} = \mathbb{V}^{-2} \supset \mathbb{V}^{-1} \supset \mathbb{V}^0 \supset \mathbb{V}^1 \supset \mathbb{V}^2$. Correspondingly, we obtain a finer filtration

$$\mathcal{T} = \mathcal{T}^{-2} \supset \mathcal{T}^{-1} \supset \mathcal{T}^0 \supset \mathcal{T}^1 \supset \mathcal{T}^2$$

of the standard tractor bundle. Moreover, the G_2-action leaves invariant a 3-form $\phi \in \Lambda^3(\mathbb{V}^*)$, which gives rise to a tractor 3-form $\Omega \in \Gamma(\Lambda^3(\mathcal{T}^*))$. The tractor 3-form Ω corresponds to the constant P-equivariant function $g : \mathcal{G} \to \Lambda^3(\mathbb{V}^*)$ defined by $g(u) = \phi$. We have $(\tilde{\xi} \cdot g)(u) + \omega(\tilde{\xi})(g(u)) = 0$ for all $u \in \mathcal{G}$ and $\tilde{\xi} \in \mathfrak{X}(\mathcal{G})$, since g is constant and $\mathfrak{g}$ stabilizes ϕ, that is $\nabla^{\mathcal{T}}\Omega = 0$.

We have seen in 2.5 that we can describe the filtration $\mathbb{V} = \mathbb{V}^{-2} \supset \mathbb{V}^{-1} \supset \mathbb{V}^0 \supset \mathbb{V}^1 \supset \mathbb{V}^2$ in terms of 3-form ϕ. Consequently, we obtain a description of the filtration on the tractor bundle in terms of Ω. The subbundle $\mathcal{T}^1$ is the set of all $v \in \mathcal{T}$ such that $\Omega(w, v, \cdot) = 0$ for all $w \in \mathcal{T}^2$. The subbundle $\mathcal{T}^0$ is the orthogonal complement to $\mathcal{T}^1$ with respect to the bundle metric h and $\mathcal{T}^{-1}$ is the orthogonal complement to $\mathcal{T}^2$.

Note also, that since $\mathfrak{g}/\mathfrak{p} \cong L(\mathbb{V}^2, \mathbb{V}^{-1}/\mathbb{V}^2)$ is an isomorphism of P-representation, we obtain an identification

$$TM \cong L(\mathcal{T}^2, \mathcal{T}^{-1}/\mathcal{T}^2)$$

and the distribution $\mathcal{H} = \mathcal{G} \times_P \mathfrak{g}^{-1}/\mathfrak{p}$ can be identified with $L(\mathcal{T}^2, \mathcal{T}^1/\mathcal{T}^2)$. Summarising, we have seen:

PROPOSITION 4.11. *Let M be a five dimensional manifold and let $\mathcal{T}$ be the standard tractor bundle for the conformal structure associated to a generic rank two distribution $\mathcal{H}$. Let ∇ be the normal tractor connection on $\mathcal{T}$ and let $\mathcal{T}^2 \subset \mathcal{T}$ be the distinguished line subbundle in $\mathcal{T}$. Then $\mathcal{T}$ carries a tractor three form $\Omega \in \Gamma(\Lambda^3\mathcal{T}^*)$ such that $\nabla\Omega = 0$ and the stabiliser of Ω_x in $GL(\mathcal{T}_x)$ is a Lie group with Lie algebra the split real form of the exceptional Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$. The distribution $\mathcal{H}$ can be recovered as*

$$L(\mathcal{T}^2, \mathcal{T}^1/\mathcal{T}^2) \subset L(\mathcal{T}^2, \mathcal{T}^{-1}/\mathcal{T}^2) \cong TM,$$

where $\mathcal{T}^1 = \{v \in \mathcal{T} : \Omega(w, v, \cdot) = 0 \; for \; all \; w \in \mathcal{T}^2\}$.

REMARK 4.12. The parallel three form on the standard tractor bundle projects to what has been named a normal conformal Killing form in [**25**]. This is a weighted two-form on the manifold which is special solution of an overdetermined system of partial differential equations, called the conformal Killing equation on forms.

For a conformal manifold M, a conformal Killing operator can be understood as the first operator D_0 in the BGG-sequence associated to a tractor bundle $\mathcal{V} = \Lambda^j \mathcal{T}^*$. See Remark 2.14 for a sketch of the BGG-sequence. Elements in the kernel of the invariant differential operator D_0 are called conformal Killing forms. In the case $j = 2$, solutions correspond to conformal Killing fields, i.e. infinitesimal conformal isometries. We consider the case $j = 3$. Here, the projection from $\Lambda^3\mathbb{V}^*$ onto the cohomology space $H^0(\mathfrak{g}_-, \Lambda^3\mathbb{V}^*)$ induces a bundle projection $\pi_H : \Lambda^3\mathcal{T}^* \to \Lambda^2 T^*M \otimes E[3]$, where $E[3]$ is a density bundle. Now suppose $\phi \in \Gamma(\Lambda^3\mathcal{T}^*)$ a section such that $\partial^*(\nabla\phi) = 0$. Then, by characterisation of the splitting operator, $\phi = L(\pi_H(\phi))$ and thus $D_0(\pi_H(\phi)) = \pi_H\nabla\phi = 0$. In particular, this explains that the projection $\pi_H(\phi) \in \Lambda^2 T^*M \otimes E[3]$ of a parallel section $\phi \in \Gamma(\Lambda^3\mathcal{T}^*)$ is contained in the kernel of the operator D_0, i.e. $\pi_H(\phi)$ is a special conformal 2-Killing form.

4.3.3. Conformal holonomy inside $\mathbf{G_2}$. We have seen, that any Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ can be extended to a principal connection $\hat{\omega}$ on a G-principal bundle $\hat{\mathcal{G}}$. The holonomy group of a Cartan geometry can be defined as the holonomy group of the principal bundle connection $\hat{\omega}$. For a smooth manifold M with a conformal structure of signature (p, q) and corresponding normal parabolic geometry $(\mathcal{G} \to M, \omega)$ this leads to the following notion of conformal holonomy:

DEFINITION 4.13. The *conformal holonomy group* for a conformal manifold M is defined as the holonomy group of the principal bundle connection $\hat{\omega}$.

It is well known that for any faithful representation $\rho : G \to End(\mathbb{W})$, the holonomy group of a principal bundle connection is isomorphic to the holonomy group of the induced linear connection on the associated bundle $\hat{\mathcal{G}} \times_G \mathbb{W}$. Hence, the conformal holonomy group can be identified with the holonomy group of the normal tractor connection ∇ on the conformal standard tractor bundle $\mathcal{T}$. The reader can find an introduction to holonomy theory of Cartan geometries with focus on conformal holonomy in [**2**]. Note, that in that paper a slightly different notion of holonomy groups is introduced (using the development of a Cartan connection). But it is shown there that the connected components of the holonomy groups defined in either way coincide.

One of the well known facts about holonomy groups is that for any G-representation $\mathbb{W}$, holonomy-invariant elements of $\mathbb{W}$ are in bijective correspondence with parallel sections of the associated bundle $\hat{\mathcal{G}} \times_G \mathbb{W}$. In particular, conformal holonomy being contained in G_2 is equivalent to the existence of a parallel section $\Omega \in \Gamma(\Lambda^3\mathcal{T}^*)$ such that the isotropy subgroup of Ω_x in $\mathrm{GL}(\mathcal{T}_x)$, for any $x \in M$, is isomorphic to G_2.

We have already seen that the standard tractor bundle for a five manifold endowed with a generic rank two distribution carries a three-form with isotropy group G_2, which is parallel for the normal conformal tractor connection. That means, we know that the conformal holonomy group for the associated conformal structure is contained in G_2. We are going to show that this characterises Nurowski's conformal structures among all conformal structures of signature $(2, 3)$.

REMARK 4.14. The conformal Cartan bundle $\tilde{\mathcal{G}}$ can be recovered from the conformal tractor bundle as a so-called adapted frame bundle. For any $x \in M$ one defines the fibre $\tilde{\mathcal{G}}_x$ to be the set of all orthogonal isomorphisms $u : \mathbb{V} \to \mathcal{T}_x$ such that $\phi(\mathbb{V}^2) = \mathcal{T}_x^2$. Composition from the right defines a free, transitive $\tilde{P}$-action on $\tilde{\mathcal{G}}_x$ and the union $\tilde{\mathcal{G}} = \cup_{x\in M}\tilde{\mathcal{G}}_x$ can be made into a smooth $\tilde{P}$-principal bundle. By construction, $\mathcal{T} = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{V}$. Also the Cartan connection on $\tilde{\mathcal{G}}$ can be recovered from the tractor connection. There is a unique Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ such that for any $s \in \Gamma(\mathcal{T})$ and corresponding function $f : \tilde{\mathcal{G}} \to \tilde{\mathfrak{g}}$, the function corresponding to $\nabla^{\mathcal{T}}_{\xi}$ is given by $u \mapsto (\tilde{\xi} \cdot f)(u) + \tilde{\omega}(\tilde{\xi})(f(u))$, for any $\xi \in \mathfrak{X}(M)$ and lift $\tilde{\xi} \in \Gamma(\tilde{\mathcal{G}})$. The reader is referred to [**10**] for more information.

LEMMA 4.15. *Let M be a five dimensional smooth manifold with a conformal structure of signature $(2,3)$. Let $\mathcal{T}$ be the corresponding standard tractor bundle endowed with the normal standard tractor connection. Suppose $\mathcal{T}$ carries a parallel three form $\Omega \in \Gamma(\Lambda^3\mathcal{T}^*)$ with isotropy group of Ω_x, for all $x \in M$, a Lie group isomorphic to G_2. Let $\tilde{\mathcal{G}} \to M$ be the conformal Cartan bundle viewed as the adapted frame bundle for $\mathcal{T}$ as in the above Remark 4.14. Define $\mathcal{G}_x$ to be the subset of those maps $u \in \tilde{\mathcal{G}}_x$ satisfying*

$$\Omega_x(u(X), u(Y), u(Z)) = \phi(X, Y, Z)$$

for all $X, Y, Z \in \mathbb{V}$ and set $\mathcal{G} = \cup_{x\in M}\mathcal{G}_x$. Then the natural inclusion $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ defines a reduction of the conformal Cartan bundle to the structure group $P \subset G_2$. The normal conformal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ reduces to a regular Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. In particular, the conformal structure determines a generic rank two distribution $\mathcal{H}$ on M.

PROOF. First note that $\mathcal{G} = \cup_{x\in M}\mathcal{G}_x \subset \tilde{\mathcal{G}}$ is a principal bundle with structure group $P = \tilde{P} \cap G_2$. By construction, the adapted frame bundle $\tilde{\mathcal{G}}$ is an extension of the adapted frame bundle $\mathcal{G}$, i.e. $\tilde{\mathcal{G}} = \mathcal{G} \times_P \tilde{P}$. The inclusion $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ obviously defines a reduction of structure group.

Next we show that the conformal Cartan connection $\tilde{\omega}$ restricts to a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Let $g : \tilde{\mathcal{G}} \to \Lambda^3(\mathbb{V}^*)$ be the equivariant function corresponding to Ω. Then, by definition of $\mathcal{G}$, the restriction of g to $\mathcal{G}$ is the constant function $u \mapsto \phi$. Now take any $x \in M$ and $u \in \mathcal{G}$ such that $p(u) = x$. Let $\xi \in \mathfrak{X}(M)$ be a vector field and choose a (local) lift $\tilde{\xi} \in \mathfrak{X}(\mathcal{G})$ around u tangent to $\mathcal{G}$. Then the integral curves for $\tilde{\xi}$ are contained in $\mathcal{G}$, so along these curves g is constant and thus $(\tilde{\xi} \cdot g)(u) = 0$. Since $\nabla_{\xi}\Omega = 0$, this implies that $\tilde{\omega}(\tilde{\xi})(u)$ stabilizes ϕ and is thus contained in the Lie algebra $\mathfrak{g}$ of the exceptional Lie group G_2. So $\tilde{\omega}$ restricted to $\mathcal{G}$ has values in $\mathfrak{g}$. It remains to show that it is a Cartan connection. First, $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is injective and then even bijective for dimensional reasons. It is P-equivariant by equivariance of $\tilde{\omega}$ and it reproduces generators of fundamental vector fields since $\tilde{\omega}$ has this property.

Since $\tilde{\omega}$ is a normal conformal Cartan connection it is torsion free, i.e. its curvature $\tilde{\kappa}$ takes values in $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$. It follows, that ω is torsion free and in particular, it is regular. Hence the Cartan geometry $(\mathcal{G}, \omega)$ determines an underlying generic rank two distribution $\mathcal{H} = \mathcal{G} \times_P \mathfrak{g}^{-1}/\mathfrak{p}$.

□

Note that the conformal structure induced by the restricted Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is the conformal structure we started with in Lemma 4.15. On the other hand, Nurowski's conformal structure associated to the underlying distribution $\mathcal{H}$ is the conformal structure induced by the (unique up to isomorphism) normal Cartan connection for $\mathcal{H}$ and we have not seen yet that ω is indeed normal. For our purpose the following observation is sufficient:

LEMMA 4.16. *Let $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ be the restriction of the conformal Cartan connection $\tilde{\omega}$ from Lemma* 4.15*. Then ω induces the same conformal structure on the manifold M as a normal Cartan connection for the underlying rank two distribution $\mathcal{H}$.*

PROOF. The idea of the proof is to show that there is a normal Cartan connection $\omega^N \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that the difference $\omega^N - \omega$ is contained in $\Omega^1(M, \mathcal{A}M)^3$, i.e. is of homogeneous degree ≥ 3. Then we know that the two Cartan connections have the same soldering form and thus induce the same conformal structure on M. This proves the lemma.

Let κ and be the curvature of the Cartan connection ω and let $\tilde{\kappa}$ be the curvature of the conformal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$. By construction,

$$\kappa(u)(X, Y) = \tilde{\kappa}(u)(\tilde{X}, \tilde{Y})$$

for any $u \in \mathcal{G}$, $\tilde{X}, \tilde{Y} \in \tilde{\mathfrak{g}}$ and $X, Y \in \mathfrak{g}$ such that $X + \mathfrak{p} = \tilde{X} + \tilde{\mathfrak{p}}$ and $Y + \mathfrak{p} = \tilde{Y} + \tilde{\mathfrak{p}}$. Torsion-freeness of the normal conformal Cartan connection $\tilde{\omega}$ implies torsion-freeness of ω. That means $\kappa(u)$ is contained in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$. The only component of $\kappa(u)$ of homogeneous degree < 3 that does not vanish simply because of torsion-freeness is contained in

$$\mathfrak{g}_1 \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_0.$$

We show that this component vanishes as well.

Let

$$\partial^*_{\tilde{\mathfrak{p}}_+} : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$$

be the Kostant codifferential describing the conformal normality condition, i.e. normality of $\tilde{\omega}$ means that $\partial^*_{\tilde{\mathfrak{p}}_+} \tilde{\kappa} = 0$. Let us choose basis as in the proof of Proposition 4.9: We take $X_1, X_2 \in \mathfrak{g}_{-3}$, $X_3 \in \mathfrak{g}_{-2}$ and $X_4, X_5 \in \mathfrak{g}_{-1}$, these elements give a basis of $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Furthermore, we consider the dual basis $\tilde{Z}_1, \ldots, \tilde{Z}_5 \in \tilde{\mathfrak{p}}_+$ with respect to the Killing form. Then $\partial^*_{\tilde{\mathfrak{p}}_+} \tilde{\kappa} = 0$ reads

$$\sum_{i=1}^{5} [\tilde{Z}_j, \tilde{\kappa}(u)(X, X_j)] = 0$$

for any $u \in \tilde{\mathcal{G}}$. Now let us consider a point $u \in \mathcal{G}$ and a Lie algebra element $X \in \mathfrak{g}$, which means that $\tilde{\kappa}(u)(X, X_j)$ equals $\kappa(u)(X, X_j)$. Furthermore, we decompose

$$\kappa(u) = (\kappa(u))_0 + (\kappa(u))_+$$

according to its values in $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$. Then the above sum can be written as

$$\sum_{i=1}^{5} [\tilde{Z}_j, (\kappa(u))_0(X, X_j)] + \sum_{i=1}^{5} [\tilde{Z}_j, (\kappa(u))_+(X, X_j)] = 0.$$

Recall that we have a $\mathfrak{g}$-modul decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{V}$, see 2.1.3, and the projection $\pi_{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ maps $\tilde{Z}_j$ to Z_j, where Z_j is the basis element in $\mathfrak{p}_+$ dual to $X_j \in \mathfrak{g}_-$. Hence,

$$\sum_{i=1}^{5}[Z_j, (\kappa(u))_0(X, X_j)] + \sum_{i=1}^{5}[Z_j, (\kappa(u))_+(X, X_j)] = 0.$$

The only terms in that sum that are contained in the grading component $\mathfrak{g}_1$ are $[\tilde{Z}_4, (\kappa(u))_0(X, X_4)]$ and $[\tilde{Z}_5, (\kappa(u))_0(X, X_5)]$ and thus

$$[Z_4, (\kappa(u))_0(X, X_4)] + [Z_5, (\kappa(u))_0(X, X_5)] = 0.$$

Inserting $X = X_4$ this implies

$$[Z_5, (\kappa(u))_0(X_4, X_5)] = 0$$

and inserting $X = X_5$ implies

$$[Z_4, (\kappa(u))_0(X_5, X_4)] = 0,$$

which shows that $(\kappa(u))_0(X_4, X_5)$ vanishes. But then $(\kappa(u))_0(X, Y) = 0$ for any $X, Y \in \mathfrak{g}_{-1}$.

This implies that κ is contained in homogeneous degree ≥ 3. But then also $\omega^N - \omega$ is contained in homogeneous degree ≥ 3. This follows from a general result which is stated below as Proposition 4.17. (It can also be deduced from the calculations done in Chapter 2: The curvature of the normal Weyl form is contained in homogeneous degree ≥ 4, cf. Proposition 2.15, and in 3.3.2, homogeneities 1 and 2, we demonstrated that the assumption that the components of the Weyl curvature of homogeneous degree ≤ 2 vanish, uniquely determines the components of the Weyl form of homogeneous degree ≤ 2.)

□

PROPOSITION 4.17. *Let $(\mathcal{G} \to M, \omega)$ be a regular parabolic geometry with curvature κ and suppose $\partial^*(\kappa) \in \Omega^1(M, \mathcal{A}M)^\ell$ for some $\ell \geq 1$. Then there is a normal Cartan connection ω^N such that $\omega^N - \omega \in \Omega^1(M, \mathcal{A})^\ell$.*

A proof of this proposition can be found in [**16**]. Combining Proposition 4.11, Lemma 4.15 and Lemma 4.16, we arrive at the following holonomy characterisation of Nurowski's conformal structures:

THEOREM 4.18. *Let M be a five-dimensional smooth manifold with a conformal structure of signature $(2, 3)$. Then the conformal holonomy* $\mathrm{Hol}(\omega)$ *is a subgroup of the split real form G_2 if and only if the conformal structure is the canonical conformal structure associated to a generic rank two distribution on M.*

4.4. A description in terms of a generalised contact form

Let M be a five-dimensional manifold endowed with a generic rank two distribution. Suppose α is a generalised contact form for the distribution $\mathcal{H}$, i.e. a smooth section of $L(\mathcal{H}^1, \mathbb{R})$ with kernel the distribution $\mathcal{H}$. In this section, we will construct a metric from Nurowski's conformal class associated to α.

We know from Chapter 3 that any generalised contact form corresponds to a unique Weyl structure σ for the associated parabolic geometry. We have determined the soldering form Θ of the normal Cartan connection corresponding to a choice of generalised contact form α. It can be viewed as an isomorphism

$$TM \cong \mathrm{gr}(TM)$$

given by

$$\zeta \mapsto q_{-3}(\zeta) + q_{-2}(\pi_{-2}(\zeta)) + \pi_{-1}(\zeta).$$

Here $q_{-2} \circ \pi_{-2} : TM \to \mathrm{gr}_{-2}(TM)$ is defined as $q_{-2}(\pi_{-2}(\zeta)) = \alpha(\zeta) q_{-2}(r)$, where r is the generalised Reeb field and α is the canonical extension of the generalised contact form to a one-form satisfying $i_r d\alpha = 0$, see Proposition 3.5 and Corollary 3.7. The definition of $\pi_{-1} : TM \to \mathrm{gr}_{-1}(TM)$ was given in Proposition 3.25.

To construct a metric from the conformal class, we first look for a description of the bilinear form (4.1) on $\mathfrak{g}_-$ that can be translated into geometric terms to give a bundle metric on $\mathrm{gr}(TM)$. Then, via the soldering form $\Theta : TM \cong \mathrm{gr}(TM)$ this bundle metric gives rise to a metric on the manifold.

Let B be $\frac{1}{6}$ of the trace form on $\mathfrak{g}$. Suppose s is an element in $\mathfrak{g}_2$. Let A, A' be elements in $\mathfrak{g}_-$ and denote by $X, X' \in \mathfrak{g}_{-1}$, $r, r' \in \mathfrak{g}_{-2}$ and $Y, Y' \in \mathfrak{g}_{-3}$ their grading components. Then the bilinear form (4.1) can be rewritten as

$$\langle A, A' \rangle = B([s,X],[s,Y']) - 4B(s,r)B(s,r') + B([s,Y],[s,X']). \tag{4.4}$$

The verification is a direct calculation: using the explicit matrix realisation (2.1) we compute that

$$B([s,X],[s,Y']) = s^2 X^t Y'$$

and

$$B(s,r)B(s,r') = \frac{s^2}{4} r r'.$$

Passing to the associated bundle $\mathrm{gr}(TM) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$, formula (4.4) can be directly translated into a formula for a bundle metric on $\mathrm{gr}(TM)$. Recall a generalised contact form can be equivalently viewed as a section $\alpha \in \Gamma(\mathrm{gr}_2(T^*M))$. Note that the Lie bracket $[,]$ induces the algebraic bracket $\{,\}$ and the bilinear form B translates to the duality pairing between $\mathrm{gr}_2(T^*M)$ and $\mathrm{gr}_{-2}(TM)$. Let $\xi, \xi' \in \Gamma(\mathcal{H})$ and $\gamma, \gamma' \in \Gamma(\mathrm{gr}_{-2}(TM))$ and $\eta, \eta' \in \Gamma(\mathrm{gr}_{-3}(TM))$. Then, the induced bundle metric on $\mathrm{gr}(TM)$ reads as

$$((\xi,\gamma,\eta),(\xi',\gamma',\eta')) \mapsto \{\alpha,\xi\}\left(\{\alpha,\eta'\}\right) - 4\alpha(\gamma)\alpha(\gamma') + \{\alpha,\xi'\}\left(\{\alpha,\eta\}\right). \tag{4.5}$$

The pullback of the above metric along the isomorphism $\Theta : TM \cong \mathrm{gr}(TM)$ defines a pseudo Riemannian metric on M given by

$$\begin{aligned} g_\alpha(\zeta,\zeta') = \{\alpha, \pi_{-1}(\zeta)\}\left(\{\alpha, q_{-3}(\zeta')\}\right) - 4\alpha(\alpha(\zeta)q_{-2}(r))\alpha(\alpha(\zeta')q_{-2}(r)) \\ + \{\alpha, \pi_{-1}(\zeta)\}\left(\{\alpha, q_{-3}(\zeta')\}\right) \end{aligned} \tag{4.6}$$

for any vector fields $\zeta, \zeta' \in \mathfrak{X}(M)$.

Thinking about the construction, it is clear that g_α is a metric from Nurowski's conformal class. In particular, the conformal class of g_α is independent of the choice of α, i.e. changing α leads to a conformal rescaling of the metric.

THEOREM 4.19. *Let M be a five dimensional manifold with a generic rank two distribution $T^{-1}M \subset TM$. Let α be a generalised contact form canonically extended to a one-form on M and denote by $\pi_{-1} : TM \to T^{-1}M$ the associated projection onto the* *distribution. Let* ζ_1 *be the unique vector field in* $\Gamma(T^{-1}M)$ *such that* $q_{-3}(\zeta) = \{q_{-2}(r), \zeta_1\}$*. Then*

$$g_\alpha(\zeta, \zeta') = d\alpha(\zeta_1, \pi_{-1}(\zeta')) - \frac{4}{3}\alpha(\zeta)\alpha(\zeta') + d\alpha(\zeta_1', \pi_{-1}(\zeta)), \tag{4.7}$$

defines a metric in Nurowski's conformal class. In particular, g_α *is a pseudo-Riemannian metric of signature* $(2,3)$ *and its conformal class is an invariant for the distribution.*

Proof. By construction, the metric g_α as in (4.6) is contained in Nurowski's conformal class. It remains to rewrite the expression for g_α to obtain formula (4.7). Since $\alpha(q_{-2}(r)) = 1$, we have

$$g_\alpha(\zeta, \zeta') = \{\alpha, \pi_{-1}(\zeta)\}\left(\{\alpha, q_{-3}(\zeta')\}\right) - 4\alpha(\zeta)\alpha(\zeta') + \{\alpha, \pi_{-1}(\zeta)\}\left(\{\alpha, q_{-3}(\zeta')\}\right).$$

Let us write $\zeta \in \mathfrak{X}(M)$ as

$$\zeta = \zeta_2 + \alpha(\zeta)r + [r, \zeta_1],$$

where ζ_1 is the unique vector field in $\Gamma(T^{-1}M)$ such that $q_{-3}(\zeta) = q_{-3}([r, \zeta_1])$. Now the metric reads

$$g_\alpha(\zeta, \zeta') = \{\alpha, \pi_{-1}(\zeta)\}(\{\alpha, \{q_{-2}(r), \zeta_1'\}\}) - 4\alpha(\zeta)\alpha(\zeta') + \{\alpha, \pi_{-1}(\zeta')\}(\{\alpha, \{q_{-2}(r), \zeta_1\}\}).$$

A simple computation in the Lie algebra $\mathfrak{g}$ shows that

$$\{\alpha, \{q_{-2}(r), \zeta_1\}\} = 3\alpha(r)\zeta_1$$

compare with (3.39). It follows from invariance of the Killing form that

$$\{\alpha, \xi\}(\eta) = \alpha([\xi, \eta])$$

for any $\xi, \eta \in \Gamma(T^{-1}M)$. Thus,

$$g_\alpha(\zeta, \zeta') = 3\alpha([\pi_{-1}(\zeta), \zeta_1']) - 4\alpha(\zeta)\alpha(\zeta') - 3\alpha([\pi_{-1}(\zeta'), \zeta_1]).$$

Rewriting $\alpha([\pi_{-1}(\zeta), \zeta_1'])$ as $d\alpha(\zeta_1', \pi_{-1}(\zeta))$, and dividing everything by 3, we obtain

$$g_\alpha(\zeta, \zeta') = d\alpha(\zeta_1, \pi_{-1}(\zeta')) - \frac{4}{3}\alpha(\zeta)\alpha(\zeta') + d\alpha(\zeta_1', \pi_{-1}(\zeta))$$

and this proves the theorem. □

Remark 4.20. Note that the metric g_α defines a nondegenerate pairing between the two isotropic subbundles $\mathcal{H}$ and $\ker(\pi_{-1}) \cap \ker(\alpha)$ and it is negative definite on the line bundle spanned by the generalised Reeb field r associated to α.

Remark 4.21. The description (4.7) of Nurowski's conformal structure can be compared to Lee's description of the Fefferman conformal structure on a S^1-bundle over a CR-manifold. In [**31**] Lee presents a characterisation of the Fefferman metric by means of a pseudo-Hermitian structure, i.e. a contact form. He verifies directly that the metric rescales conformally when changing the contact form on the CR-manifold. It is certainly also possible to show that the metric (4.7) rescales when we change the generalised contact form. This is done in Appendix A.

Remark 4.22. We can recover Nurowski's original formula, given in [**29**], from the description (4.7) of a metric in the conformal class. This is explained in Appendix B.

APPENDIX A

Dependence on the generalised contact form

Let $T^{-1}M$ be a generic rank two distribution on a smooth five-dimensional manifold. We have seen in Chapter 4 that such a distribution determines a natural conformal structure on the manifold. For the construction of this conformal structure we treated generic rank two distributions as parabolic geometries and used the general theory for these geometries: The construction of the metric g_α as in (4.7) is based on the theory of Weyl structures and the characterisation of Weyl structures determined by generalised contact forms given in Chapter 3. However, once we have obtained a description of a natural geometric object associated to the distribution in terms of a generalised contact form, to prove that it does not depend on the choice of the generalised contact form is a matter of direct verification. For Nurowski's conformal structure that means that starting from formula (4.7) we can prove that a generic rank two distribution carries a canonical conformal structure without reference to Cartan connections or Weyl structures. This approach has been taken up in the joint article [**14**] with A. Cap. We will follow this article very closely in this appendix. Note, that the calculations presented here are also used for the verification that the operator defined in 3.4 does not depend on the choice of generalised contact form.

Suppose $\alpha \in \Gamma(L(T^{-1}M, \mathbb{R}))$ is a generalised contact form for the rank two distribution $T^{-1}M$. We have seen in Chapter 3 that α determines a generalised Reeb field r, an extension of α to a one-form on M, which we also denote by α, and a partial connection ∇ on the distribution given by

$$\nabla_\gamma \xi = q_{-3}([\gamma, [\xi, r]])$$

for $\xi, \gamma \in \Gamma(T^{-1}M)$, cf. Proposition 3.5 and Proposition 3.7. Let ζ_1 be the unique vector field in $\Gamma(T^{-1}M)$ such that $q_{-3}(\zeta) = \{q_{-2}(r), \zeta_1\}$ and $\zeta_2 = \zeta - \alpha(\zeta)r - [r, \zeta_1] \in \Gamma(T^{-1}M)$. Then, we can associate to every generalised contact form a bundle projection $\pi_{-1} : TM \to T^{-1}M$ characterised as in Proposition 3.25, i.e.

$$\pi_{-1}(\zeta) = -\frac{2}{5}\Phi(\zeta_1) + \frac{7}{5}\Psi(\zeta_1) + \zeta_2,$$

where Φ and Ψ are maps $\Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$ characterised by

$$d\alpha(\xi, \eta)\Phi(\gamma) = -(\nabla_\xi \nabla_\eta \gamma - \nabla_\eta \nabla_\xi \gamma - \nabla_{\pi_{-1}([\xi,\eta])}\gamma)$$

respectively by

$$\{\Psi(\gamma), q_{-2}(r)\} = \frac{1}{2} q_{-3}([r, [\gamma, r]])$$

for any $\xi, \eta, \gamma \in \Gamma(T^{-1}M)$. We can now define a map

$$g_\alpha(\zeta, \zeta') = d\alpha(\zeta_1, \pi_{-1}(\zeta')) - \frac{4}{3}\alpha(\zeta)\alpha(\zeta') + d\alpha(\zeta'_1, \pi_{-1}(\zeta)), \tag{A.1}$$

on vector fields $\zeta, \zeta' \in \mathfrak{X}(M)$. In the sequel, we shall verify that g_α indeed defines a pseudo Riemannian metric and we shall investigate its dependence on α.

PROPOSITION A.1. *The map g_α defined in (A.1) is a pseudo-Riemannian metric of signature* $(2, 3)$.

PROOF. Obviously, g_α is a smooth, symmetric bilinear bundle map. The rank two distribution $T^{-1}M$ is isotropic for the metric g_α, since for any $\zeta \in \Gamma(T^{-1}M)$ we know that ζ_1 and $\alpha(\zeta)$ vanish. Likewise, $\ker(\pi_{-1}) \cap \ker(\alpha)$ is an isotropic rank two distribution, which is transversal to $T^{-1}M$. The metric g_α induces a nondegenerate pairing between the two isotropic subbundles. This can be seen as follows: For $\zeta' \in \Gamma(\ker(\pi_{-1}) \cap \ker(\alpha))$ and $\zeta \in \Gamma(T^{-1}M)$, we have $g_\alpha(\zeta, \zeta') = d\alpha(\zeta'_1, \zeta)$. This vanishes for all ζ if and only if $\zeta'_1 = 0$ and hence $\zeta' \in \Gamma(T^{-2}M)$. But then the assumption $\alpha(\zeta') = \pi_{-1}(\zeta') = 0$ implies that $\zeta' = 0$. Note that r spans a rank one subbundle transversal to the two isotropic subbundles on which g_α is negative definite. Hence, the metric has signature $(2, 3)$. □

In order to understand the dependence of g_α on the generalised contact form α, we first study the dependence of the data used in the construction. The generalised contact α we start with will be replaced by another generalised contact form denoted $\hat{\alpha}$. All data associated with $\hat{\alpha}$ will be referred to using a hat.

Let us first see what happens if we replace α by $\hat{\alpha} = -\alpha$. If we replace in the defining equation

$$\{\nabla_\gamma \xi, q_{-2}(r)\} = q_{-3}([\gamma, [\xi, r]])$$

the vector field r by $-r$ we obtain the same partial connection ∇. Hence, $-r$ is a vector field parallel for the induced connection and satisfies $\hat{\alpha}(-r) = 1$. This shows that the generalised Reeb field associated to $\hat{\alpha}$ is $\hat{r} = -r$ and $\hat{\nabla} = \nabla$. The extension of a contact form to a one-form $\alpha \in \Omega^1(M)$ was characterised by $i_r d\alpha(\xi) = 0$ for all $\xi \in \Gamma(T^{-1}M)$. It follows, that $\hat{\alpha} = -\alpha$ holds for the extensions to one-forms as well. Decomposing $\zeta \in \mathfrak{X}(M)$ as $\zeta = [r, \zeta_1] + \alpha(\zeta)r + \zeta_2$, where ζ_1 is the unique vector field in $\Gamma(T^{-1}M)$ such that $q_{-3}(\zeta) = q_{-3}([r, \zeta_1])$, we obtain $\hat{\zeta}_1 = \zeta_1$ and $\hat{\zeta}_2 = \zeta_2$. Furthermore, we have $\hat{\Phi} = -\Phi$ and $\hat{\Psi} = -\Psi$ and thus $\hat{\pi}_{-1} = \pi_{-1}$. Putting the results together, we conclude that $g_{-\alpha} = g_\alpha$.

Any generalised contact form is obtained from a fixed one by multiplication with a nowhere vanishing smooth function. In view of the above paragraph, it thus remains to determine the transformation of the metric resulting from a rescaling of α by a positive smooth function. Again, we deal with the transformation of r, the extension α and the projection π_{-1} first.

LEMMA A.2. *Let α be a generalised contact form and let $\hat{\alpha}$ be another generalised contact form given by $\hat{\alpha} = e^f\alpha$ for some smooth function $f \in C^\infty(M, \mathbb{R})$. Then the associated generalised Reeb vector fields are related by*

$$\hat{r} = e^{-f} r + \delta,$$

where $\delta \in \Gamma(T^{-1}M)$ is the unique vector field such that

$$\{\gamma, \delta\} = 4e^{-f} df(\gamma) q_{-2}(r)$$

for all $\gamma \in \Gamma(T^{-1}M)$.

PROOF. Since $\hat{\alpha}(\hat{r}) = e^f\alpha(\hat{r}) = 1$, the generalised Reeb vector field $\hat{r}$ is of the form $\hat{r} = e^{-f}r + \delta$ for some $\delta \in \Gamma(T^{-1}M)$. Let us put $r_0 = e^{-f}r$ and denote by ∇^0 the partial connection associated to r_0. Then we know from the proof of Proposition 3.5 that δ is characterised by $-\nabla^0_\gamma q_{-2}(r_0) = \{\gamma, \delta\}$. Hence, once we show that $-\nabla^0_\gamma q_{-2}(r_0) = 4e^{-f}df(\gamma)q_{-2}(r)$, we prove the lemma.

Using that $\gamma \cdot e^{-f} = -e^{-f}df(\gamma)$ for every $\gamma \in \Gamma(T^{-1}M)$, we compute

$$q_{-3}([\gamma, [\xi, r_0]]) = e^{-f}\left(q_{-3}([\gamma, [\xi, r]]) - df(\gamma)\{\xi, q_{-2}(r)\} - df(\xi)\{\gamma, q_{-2}(r)\}\right).$$

Thus, the defining equation of ∇^0 implies

$$\nabla^0_\gamma\xi = \nabla_\gamma\xi - df(\gamma)\xi - df(\xi)\gamma.$$

Consequently, for the induced connection on $\mathrm{gr}_{-2}(TM)$ we obtain

$$\nabla^0_\gamma\{\xi, \eta\} = \nabla^0_\gamma\{\xi, \eta\} - 2df(\gamma)\{\xi, \eta\} - df(\xi)\{\gamma, \eta\} - df(\eta)\{\xi, \gamma\}.$$

Let ξ, η be smooth sections of $T^{-1}M$ such that $\{\xi, \eta\} = q_{-2}(r)$. Then ξ, η form a frame for $T^{-1}M$. Such a frame being fixed, the sum of last two terms on the right hand side defines a bundle map $T^{-1}M \to \mathrm{gr}_{-2}(TM)$. Inserting $\gamma = \xi$ and $\gamma = \eta$ in this bundle map, we obtain $-df(\gamma)\{\xi, \eta\}$. Hence, the induced connection on $\mathrm{gr}_{-2}(TM)$ is characterised by

$$\nabla^0_\gamma\{\xi, \eta\} = \nabla_\gamma\{\xi, \eta\} - 3df(\gamma)\{\xi, \eta\}.$$

Since $\nabla_\gamma\{\xi, \eta\} = 0$, this implies

$$\nabla^0_\gamma q_{-2}(r_0) = \nabla^0_\gamma e^{-f}\{\xi, \eta\} = -4df(\gamma)e^{-f}\{\xi, \eta\}$$

and this proves the lemma. □

LEMMA A.3. *The canonical extensions of α respectively $\hat{\alpha} = e^f\alpha$ to one-forms on M are related by*

$$\hat{\alpha}(\zeta) = e^f(\alpha(\zeta) + 3df(\zeta_1)),$$

where $\zeta_1 \in \Gamma(T^{-1}M)$ is characterised by $q_{-3}(\zeta) = \{q_{-2}(r), \zeta_1\}$.

PROOF. It is easy to see that the right hand side of equation defines an extension of $\hat{\alpha}$. To show that it defines the extension introduced in Corollary 3.7, we need to verify that it vanishes on vector fields of the form $[\hat{r}, \gamma]$ for $\gamma \in \Gamma(T^{-1}M)$.

As in the previous lemma we put $r_0 = e^{-f}r$, then $\hat{r} = r_0 + \delta$ and

$$[\hat{r}, \gamma] = e^{-f}[r, \gamma] + e^{-f}df(\gamma)r + [\delta, \gamma]. \tag{A.2}$$

Since $\alpha([r, \gamma]) = 0$, we have

$$e^f\alpha([\hat{r}, \gamma]) = df(\gamma) + e^f\alpha([\delta, \gamma])$$

Recall that δ is characterised by $\{\gamma, \delta\} = -\nabla^0_\gamma q_{-2}(r_0)$ and in the proof of Lemma A.2 we have seen that $\nabla^0_\gamma q_{-2}(r_0) = -4df(\gamma)q_{-2}(r_0)$. Hence,

$$\alpha([\delta, \gamma]) = -4e^{-f}df(\gamma) \tag{A.3}$$

and consequently $e^f\alpha([\hat{r}, \gamma]) = -3df(\gamma)$, which implies $\hat{\alpha}([\hat{r}, \gamma]) = 0$. □

Lemma A.4. *Let α be a generalised contact form and consider $\hat{\alpha} = e^f \alpha$ for $f \in C^\infty(M, \mathbb{R})$. Then the projections $TM \to T^{-1}M$ associated to α respectively $\hat{\alpha}$ are related by*

$$\hat{\pi}_{-1}(\zeta) = \pi_{-1}(\zeta) - \tfrac{3}{2} df(r)\zeta_1 - \tfrac{3}{2} e^f df(\zeta_1)\delta - e^f \alpha(\zeta)\delta,$$

where $\zeta_1 \in \Gamma(T^{-1}M)$ is the unique vector field such that $q_{-3}(\zeta) = \{q_{-2}(r), \zeta_1\}$.

Proof. Let ζ be a vector field on M. Any generalised contact form determines a decomposition

$$\zeta = [r, \zeta_1] + \alpha(\zeta)r + \zeta_2,$$

where ζ_1 is characterised by $q_{-3}(\zeta) = \{q_{-2}(r), \zeta_1\}$. We will first compare the decompositions with respect to the generalised contact forms α and $\hat{\alpha}$. Since $q_{-2}(\hat{r}) = e^{-f} q_{-2}(r)$, we have $\hat{\zeta}_1 = e^f \zeta_1$. We have seen in Lemma A.2 that $\hat{r} = e^{-f} r + \delta$, where δ is characterised by $\{\gamma, \delta\} = 4e^{-f} df(\gamma) q_{-2}(r)$. Choosing $\gamma = \delta$, we see that this implies $df(\delta) = 0$. Now we use formula (A.2) to obtain

$$[\hat{r}, \hat{\zeta}_1] = [r, \zeta_1] + df(r)\zeta_1 + df(\zeta_1)r + e^f [\delta, \zeta_1]. \tag{A.4}$$

By Lemma A.3, $\hat{\alpha}(\zeta) = e^f(\alpha(\zeta) + 3df(\zeta_1))$. Thus,

$$\hat{\alpha}(\zeta)\hat{r} = (\alpha(\zeta) + 3df(\zeta_1))(r + e^f \delta).$$

The above results imply that $\hat{\zeta}_2 - \zeta_2$ is given by

$$-e^f \alpha(\zeta)\delta - df(r)\zeta_1 - 3e^f df(\zeta_1)\delta - 4df(\zeta_1)r - e^f [\delta, \zeta_1]. \tag{A.5}$$

We shall determine $\hat{\Psi}(\hat{\zeta}_1) - \Psi(\zeta_1)$ next. By definition $\{\hat{\Psi}(\gamma), q_{-2}(\hat{r})\} = \frac{1}{2} q_{-2}([\hat{r}, [\gamma, \hat{r}]])$. Using (A.4) we compute

$$\begin{aligned} q_{-3}([\hat{r}, [\hat{\zeta}_1, \hat{r}]]) &= e^{-f} q_{-3}([r, [\zeta_1, r]]) + 2q_{-3}([\delta, [\zeta_1, r]]) - q_{-3}([\zeta_1, [\delta, r]]) \\ &\quad - e^f \{\delta, \{\delta, \zeta_1\}\} - e^{-f} df(r)\{q_{-2}(r), \zeta_1\} - df(\zeta_1)\{\delta, q_{-2}(r)\}. \end{aligned}$$

The second and the third summand on the right hand side can be rewritten in terms of the partial connection ∇. Then we see that $\hat{\Psi}(\hat{\zeta}_1) - \Psi(\zeta_1)$ equals

$$e^f \nabla_\delta \zeta_1 - \frac{1}{2} e^f \nabla_{\zeta_1} \delta + \frac{3}{2} e^f df(\zeta_1)\delta + \frac{1}{2} df(q_{-2}(r))\zeta_1. \tag{A.6}$$

Recall that $\hat{\Phi}$ is characterised by

$$\hat{\nabla}_\xi \hat{\nabla}_\eta \gamma - \hat{\nabla}_\eta \hat{\nabla}_\xi \gamma - \hat{\nabla}_{[\xi,\eta] - \hat{\alpha}([\xi,\eta])\hat{r}} \gamma = \hat{\alpha}([\xi, \eta])\hat{\Phi}(\gamma), \tag{A.7}$$

for $\xi, \eta, \gamma \in \Gamma(T^{-1}M)$. In order to compute $\hat{\Phi}(\hat{\zeta}_1) - \Phi(\zeta_1)$ we first determine the relation between the partial connection ∇ associated with α and the partial connection $\hat{\nabla}$ associated with $\hat{\alpha}$. We use equation (A.2) to compute $q_{-3}([\xi, [\eta, \hat{r}]])$ and as a result we obtain

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta - df(\xi)\eta + 3df(\eta)\xi. \tag{A.8}$$

Next we verify directly that for $\xi, \eta, \gamma \in \Gamma(T^{-1}M)$ the difference $\hat{\nabla}_\xi \hat{\nabla}_\eta \gamma - \nabla_\xi \nabla_\eta \gamma$ can, up to terms symmetric in ξ and η, be expressed as

$$3df(\nabla_\eta \gamma)\xi - (\xi \cdot df(\eta))\gamma + 3(\xi \cdot df(\gamma))\eta + 3df(\gamma)\nabla_\xi \eta + 9df(\gamma)df(\eta)\xi. \tag{A.9}$$

Since $[\xi,\eta]\in\Gamma(T^{-2}M)$, we have

$$\hat\alpha([\xi,\eta])\hat r=\alpha([\xi,\eta])(r+e^f\delta).$$

Using this and $df(\delta)=0$ we compute that

$$\hat\nabla_{([\xi,\eta]-\hat\alpha([\xi,\eta])\hat r)}\gamma-\nabla_{([\xi,\eta]-\alpha([\xi,\eta])r)}\gamma$$

is given by

$$-df([\xi,\eta]-\alpha([\xi,\eta])r)\gamma+3df(\gamma)([\xi,\eta]-\alpha([\xi,\eta])r)-\hat\alpha([\xi,\eta])\left(\nabla_\delta\gamma+3df(\gamma)\delta\right). \quad \text{(A.10)}$$

Next we derive a few helpful identities. First, expanding $ddf(\xi,\eta)=0$, we obtain

$$\xi\cdot df(\eta)-\eta\cdot df(\xi)-df([\xi,\eta]-\alpha([\xi,\eta])r)=\alpha([\xi,\eta])df(r). \quad \text{(A.11)}$$

Jakobi identity yields

$$q_{-3}([\gamma_1,[\gamma_2,r]])-q_{-3}([\gamma_2,[\gamma_1,r]])=q_{-3}([[\gamma_1,\gamma_2],r])$$

for any vector fields $\gamma_1,\gamma_2\in\Gamma(T^{-1}M)$. Moreover, $q_{-3}([[\gamma_1,\gamma_2],r])=\{\zeta,q_{-2}(r)\}$ for $\zeta=[\gamma_1,\gamma_2]-\alpha([\gamma_1,\gamma_2])r\in\Gamma(T^{-1}M)$. This gives the identity

$$\nabla_{\gamma_1}\gamma_2-\nabla_{\gamma_2}\gamma_1=[\gamma_1,\gamma_2]-\alpha([\gamma_1,\gamma_2])r. \quad \text{(A.12)}$$

To obtain a formula for $\hat\Phi(\gamma)-\Phi(\gamma)$, we choose $\xi,\eta\in T^{-1}M$ such that $\{\xi,\eta\}=q_{-2}(r)$. We take (A.9), then subtract the analogous terms with ξ and η exchanged and further subtract (A.10). Using (A.11) and (A.12), we obtain

$$\begin{aligned}3df(\nabla_\eta\gamma)\xi-&3df(\nabla_\xi\gamma)\eta+3(\xi\cdot df(\gamma))\eta-3(\eta\cdot d_1f(\gamma))\xi\\&-\alpha([\xi,\eta])df(r)\gamma+9df(\gamma)(df(\eta)\xi-df(\xi)\eta)\\&+\hat\alpha([\xi,\eta])(\nabla_\delta\gamma+3df(\gamma)\delta).\end{aligned} \quad \text{(A.13)}$$

Inserting (A.12) into (A.11), we get

$$df(\nabla_\eta\gamma)-\eta\cdot df(\gamma)=df(\nabla_\gamma\eta)-\gamma\cdot df(\eta)-\alpha([\eta,\gamma])df(r).$$

Using this and the analogous formula for ξ instead of η, we see that the first line of (A.13) can be rewritten as

$$\begin{aligned}3(\gamma\cdot df(\xi))\eta-&3d_1f(\nabla_\gamma\xi)\eta-3(\gamma\cdot df(\eta))\xi+3df(\nabla_\gamma\eta)\xi\\&+3df(r)(\alpha([\xi,\gamma])\eta-\alpha([\eta,\gamma])\xi).\end{aligned} \quad \text{(A.14)}$$

The bundle map $\gamma\mapsto\alpha([\xi,\gamma])\eta-\alpha([\eta,\gamma])\xi$ coincides with $\alpha([\xi,\eta])\gamma$ for $\gamma=\xi$ and $\gamma=\eta$, and since $\{\xi,\eta\}$ is nowhere vanishing by assumption, this holds for all γ. Similarly, we see that the bundle map $T^{-1}M\to\operatorname{gr}_{-2}(TM)$ given by $\gamma\mapsto df(\xi)\{\gamma,\eta\}-df(\eta)\{\gamma,\xi\}$ coincides with $df(\gamma)\{\xi,\eta\}$. By definition of δ, this implies

$$4(df(\xi)\eta-df(\eta)\xi)=\alpha([\xi,\eta])e^f\delta. \quad \text{(A.15)}$$

It follows that

$$\begin{aligned}4(df(\nabla_\gamma\xi)\eta-df(\eta)\nabla_\gamma\xi+df(\xi)\nabla_\gamma\eta-df(\nabla_\gamma\eta)\xi)=\\ \alpha([\nabla_\gamma\xi,\eta])e^f\delta-\alpha([\nabla_\gamma\eta,\xi])e^f\delta.\end{aligned}$$

Recall that the induced connection on $\operatorname{gr}_{-2}(TM)$ is characterised by

$$\nabla_\gamma\{\xi,\eta\}=\{\nabla_\gamma\xi,\eta\}+\{\xi,\nabla_\gamma\eta\}.$$

Since $\nabla_\gamma q_{-2}(r)$ vanishes and we have $\{\gamma_1, \gamma_2\} = \alpha([\gamma_1, \gamma_2])q_{-2}(r)$ for all $\gamma_1, \gamma_2 \in \Gamma(T^{-1}M)$, we see that

$$\alpha([\nabla_\gamma \xi, \eta])e^f\delta - \alpha([\nabla_\gamma \eta, \xi])e^f\delta$$

equals

$$(\gamma \cdot \alpha([\xi, \eta]))e^f\delta.$$

Hence,

$$4(df(\nabla_\gamma \xi)\eta - df(\eta)\nabla_\gamma \xi + df(\xi)\nabla_\gamma \eta - df(\nabla_\gamma \eta)\xi) = (\gamma \cdot \alpha([\xi, \eta]))e^f\delta. \tag{A.16}$$

We apply ∇_γ to (A.15). This yields

$$\begin{aligned} 4(df(\xi)\nabla_\gamma \eta + &(\gamma \cdot df(\xi))\eta - df(\eta)\nabla_\gamma \xi - (\gamma \cdot df(\eta))\xi) = \\ &\alpha([\xi, \eta])e^f\nabla_\gamma \delta + \alpha([\xi, \eta])df(\gamma)e^f\delta + (\gamma \cdot \alpha([\xi, \eta]))e^f\delta. \end{aligned} \tag{A.17}$$

Using equation (A.16) this can be simplified to

$$\begin{aligned} df(\nabla_\gamma \eta)\xi &- (\gamma \cdot df(\eta))\xi - df(\nabla_\gamma \xi)\eta + (\gamma \cdot df(\xi))\eta \\ &= \frac{1}{4}\hat{\alpha}([\xi, \eta])(df(\gamma)\delta + \nabla_\gamma \delta). \end{aligned} \tag{A.18}$$

Now we put all results together. Then we obtain

$$\hat{\Phi}(\gamma) = \Phi(\gamma) + 2e^{-f}df(r)\gamma + \frac{3}{2}df(\gamma)\delta + \nabla_\delta \gamma + \frac{3}{4}\nabla_\gamma \delta.$$

Inserting $\gamma = \hat{\zeta}_1 = e^f\zeta_1$, and using $df(\delta) = 0$ we see that $\hat{\Phi}(\hat{\zeta}_1) - \Phi(\zeta_1)$ is given by

$$3df(r)\zeta_1 + \frac{3}{2}e^f df(\zeta_1)\delta + e^f\nabla_\delta \zeta_1 + \frac{3}{4}e^f\nabla_{\zeta_1}\delta \tag{A.19}$$

To compute $\hat{\pi}_1(\zeta) - \pi_1(\zeta)$ we multiply this by $-\frac{2}{5}$ and then add $\frac{7}{5}$ times (A.6) and add (A.5). This gives

$$\begin{aligned} e^f(\nabla_\delta \zeta_1 - \nabla_{\zeta_1}\delta - [\delta, \zeta_1]) &- 4df(\zeta_1)r \\ &- \frac{3}{2}df(r)\zeta_1 - \frac{3}{2}e^f df(\zeta_1)\delta - e^f\alpha(\zeta)\delta. \end{aligned} \tag{A.20}$$

Using (A.12), the first line simplifies to

$$-(e^f\alpha([\delta, \zeta_1]) + 4df(\zeta_1))r,$$

and we have seen in the proof of Lemma A.3 that this vanishes.

□

Having all these lemmas at hand, it is now quite easy to prove that changing α by a positive smooth function leads to a conformal rescaling of the metric:

THEOREM A.5. *Let α be a generalised contact form and consider a rescaling $\hat{\alpha} = e^f\alpha$ for a positive smooth function f. Then the corresponding metrics are related by*

$$g_{\hat{\alpha}} = e^{2f}g_\alpha.$$

In particular, the conformal class of the metric depends on the distribution, only.

PROOF. By definition,

$$g_{\hat{\alpha}}(\zeta,\zeta') = d\hat{\alpha}(\hat{\zeta}_1, \hat{\pi}_{-1}(\zeta')) - \frac{4}{3}\hat{\alpha}(\zeta)\hat{\alpha}(\zeta') + d\hat{\alpha}(\hat{\zeta'}_1, \hat{\pi}_{-1}(\zeta)).$$

Using Lemma A.3, we see that $\alpha, \hat{\alpha} \in \Omega^1(M)$ are related by

$$\hat{\alpha}(\zeta) = e^f(\alpha(\zeta) + 3df(\zeta_1)).$$

Since in the definition of g_α the exterior derivative $d\alpha$ is only applied to elements of $\Gamma(T^{-1}M)$, it rescales by e^f when passing to $\hat{\alpha}$. By Lemma A.4 we have

$$\hat{\pi}_{-1}(\zeta) = \pi_{-1}(\zeta) - \frac{3}{2}df(r)\zeta_1 - \frac{3}{2}e^f df(\zeta_1)\delta - e^f\alpha(\zeta)\delta.$$

Hence,

$$d\hat{\alpha}(\hat{\zeta}_1, \hat{\pi}_1(\zeta')) - e^{2f}d\alpha(\zeta_1, \pi_1(\zeta'))$$

is given by

$$-e^{2f}\left(\frac{3}{2}d_2f(r)d\alpha(\zeta_1,\zeta_1') + e^f(\alpha(\zeta') + \frac{3}{2}df(\zeta_1'))d\alpha(\zeta_1,\delta)\right).$$

The first summand in the bracket is skew symmetric in ζ and ζ' and hence it does not contribute to the final result. In the proof of Lemma A.3 we have seen that

$$e^f d\alpha(\zeta_1,\delta) = -e^f\alpha([\zeta_1,\delta]) = -4df(\zeta_1).$$

Inserting this, the result follows by a simple direct computation.

□

REMARK A.6. The results on the transformation of the Reeb vector field, the extension of α and the projection π_{-1} can be reformulated using the algebraic bracket $\{,\}$ on $\mathcal{A}M$, cf. 1.3.2. Using the explicit description (2.1) of the Lie algebra $\mathfrak{g}_2$ from Chapter 2, we obtain:

$$\{\ ,\ \} : \mathrm{gr}_1(T^*M) \times \mathrm{gr}_{-2}(TM) \to T^{-1}M$$

is characterised by

$$\{\phi, \{\xi,\eta\}\} = 4(\phi(\xi)\eta - \phi(\eta)\xi) \tag{A.21}$$

for $\phi \in \mathrm{gr}_1(T^*M)$ and $\xi, \eta \in T^{-1}M$,

$$\{\ ,\ \} : \mathrm{gr}_1(T^*M) \times \mathrm{gr}_{-3}(TM) \to \mathrm{gr}_{-2}(TM)$$

is characterised by

$$\{\phi, \{\xi,\gamma\}\} = 3\phi(\xi)\gamma \tag{A.22}$$

for $\phi \in \mathrm{gr}_1(T^*M)$, $\xi \in T^{-1}M$, and $\gamma \in \mathrm{gr}_{-2}(TM)$ and

$$\{\ ,\ \} : \mathrm{gr}_2(T^*M) \times \mathrm{gr}_{-3}(TM) \to T^{-1}M$$

is characterised by

$$\{\psi, \{\xi,\gamma\}\} = -3\psi(\gamma)\xi, \tag{A.23}$$

for $\psi \in \mathrm{gr}_2(T^*M)$, $\gamma \in \mathrm{gr}_{-2}(TM)$, and $\xi \in T^{-1}M$.

Using these brackets, we can rewrite Lemmata $A.2$, $A.3$ and $A.4$. As usual we use a hat for data associated to $\hat{\alpha} = e^f\alpha$. Then we obtain

$$\hat{r} = e^{-f}r + e^{-f}\{d_1f, q_{-2}(r)\}$$

and

$$\hat{\alpha}(\zeta)q_{-2}(\hat{r}) = \alpha(\zeta)q_{-2}(r) - \{d_1f, q_{-3}(\zeta)\},$$

where $d_1 f \in \Gamma((T^{-1}M)^*)$ is the restriction of df to $T^{-1}M$. Next we introduce the section $d_2 f \in \Gamma(\mathrm{gr}_2(T^*M))$ defined by $d_2 f(q_{-2}(\zeta)) = \alpha(\zeta) r \cdot f$ for all $\zeta \in \Gamma(T^{-2}M)$. Note, that in contrast to the form $d_1 f$, this one does depend on the generalised contact form α. Now the transformation of the projection reads

$$\hat{\pi}_{-1}(\zeta) = \pi_{-1}(\zeta) - \{d_1 f, \alpha(\zeta) q_{-2}(r)\} - \frac{1}{2}\{d_2 f, q_{-3}(\zeta)\} + \frac{1}{2}\{d_1 f, \{d_1 f, q_{-3}(\zeta)\}\}.$$

This is in accordance with the general formula for the transformation of a soldering form resulting from the change of a Weyl structure given in Proposition 3.9 of [**17**]

APPENDIX B

Comparison with Nurowski's formula

The aim of the second part of the appendix is to relate our description (4.7) of a metric in the conformal class to Nurowski's original formula. As mentioned at the beginning of Chapter 4, Nurowski's investigation starts with differential equations of the form

$$z' = F(x, y, y', y'', z)$$

where $F_{y''y''} \neq 0$ and $y = y(x)$ and $z = z(x)$ are functions in x. Introducing new variables $p = y'$ and $q = y''$, the kernel of the system of one-forms

$$\begin{aligned}\omega^1 &= dy - pdx\\ \omega^2 &= dz - F(x, y, p, q, z)dx\\ \omega^3 &= dp - qdx\end{aligned}$$

defines a generic rank two distribution on a five dimensional manifold. Solutions of the differential equation correspond to curves tangent to the distribution.

Computing the exterior derivatives of the one-forms ω^1, ω^2 and ω^3, we obtain

$$d\omega^1 = -dp \wedge dx = dx \wedge \omega^3,$$

$$d\omega^3 = -dq \wedge dx$$

and

$$\begin{aligned}d\omega^2 &= -F_y dy \wedge dx - F_p dp \wedge dx - F_q dq \wedge dx - F_z dz \wedge dx\\ &= F_y dx \wedge \omega^1 + F_p dx \wedge \omega^3 + F_q d\omega^3 + F_z dx \wedge \omega^2.\end{aligned} \tag{B.1}$$

Hence, replacing the form ω^2 by

$$\tilde{\omega}^2 = \omega^2 - F_q \omega^3,$$

yields

$$d\tilde{\omega}^2 = d\omega^2 - dF_q \wedge \omega^3 - F_q d\omega^3. \tag{B.2}$$

The common kernel of the new system $\tilde{\omega}^1 = \omega^1$, $\tilde{\omega}^2$ and $\tilde{\omega}^3 = \omega^3$ is still the rank two distribution $\mathcal{H}$. But the new system has the property that $d\tilde{\omega}^1$ and $d\tilde{\omega}^2$ are contained in the ideal spanned by the one forms $\tilde{\omega}^1$, $\tilde{\omega}^2$ and $\tilde{\omega}^3$. Since $d\tilde{\omega}^i(\xi, \eta) = -\tilde{\omega}^i([\xi, \eta])$ for $\xi, \eta \in \Gamma(\mathcal{H})$ this implies that the kernel of $\tilde{\omega}^1$ and $\tilde{\omega}^2$ is precisely the derived rank three distribution $\mathcal{H}^1$.

To construct a representative in the conformal class, Nurowski extends new system of one-forms to the coframe

$$\begin{aligned}
\tilde{\omega}^1 &= dy - pdx \\
\tilde{\omega}^2 &= dz - Fdx - F_q(dp - qdx) \\
\tilde{\omega}^3 &= dp - qdx \\
\tilde{\omega}^4 &= dq \\
\tilde{\omega}^5 &= dx.
\end{aligned} \tag{B.3}$$

He writes down the following metric, see [**30**],

$$\begin{aligned}
&g_F = (DF_{qq}^2 F_{qq}^2 + 6DF_q DF_{qqq} F_{qq}^2 - 6DF_{qqq} F_q F_{qq}^2 - \\
&3DDF_{qq} F_{qq}^3 + 9DF_{qp} F_{qq}^3 - 9F_{pp} F_{qq}^3 + \\
&9DF_{qz} F_q F_{qq}^3 - 18F_{pz} F_q F_{qq}^3 + 3DF_z F_{qq}^4 - \\
&6DF_q F_{qq}^2 F_{qqp} + 6F_q F_{qq}^2 F_{qqp} - 8DF_q DF_{qq} F_{qq} F_{qqq} + \\
&8DF_{qq} F_p F_{qq} F_{qqq} + 3DDF_q F_{qq}^2 F_{qqq} - 3DF_p F_{qq}^2 F_{qqq} - \\
&3DF_z F_q F_{qq}^2 F_{qqq} + 4(DF_q)^2 F_{qqq}^2 - 8DF_q F_p F_{qqq}^2 - \\
&3(DF_q)^2 F_{qq} F_{qqqq} + 4F_p^2 F_{qqq}^2 + 6DF_q F_p F_{qq} F_{qqqq} - \\
&3F_p^2 F_{qq} F_{qqqq} - 6DF_q F_q F_{qq}^2 F_{qqz} + 6F_p F_q F_{qq}^2 F_{qqz} - \\
&3DF_q F_{qq}^3 F_{qz} + 12F_p F_{qq}^3 F_{qz} + 3F_{qq}^2 F_{qqq} F_y - \\
&6DF_{qqq} F_q F_{qq}^2 F_z + 4DF_{qq} F_z^3 + 6F_q F_{qq}^2 F_{qqp} F_z + \\
&8DF_{qq} F_q F_{qq} F_{qqq} F_z - 4DF_q F_{qq}^2 F_{qqq} F_z - \\
&9F_{qp} F_{qq}^3 F_z + F_p F_{qq}^2 F_{qqq} F_z - 8DF_q F_q F_{qqq}^2 F_z + \\
&8F_p F_q F_{qqq}^2 F_z + 6DF_q F_q F_{qq} F_{qqqq} F_z + 3F_q F_{qq}^3 F_{qz} F_z^2 - \\
&3F_q^2 F_{qq} F_{qqqq} F_z^2 - 9F_q^2 F_{qq}^3 F_{zz})(\tilde{\omega}^1)^2 + \\
&(6DF_{qqq} F_{qq}^2 - 6F_{qq}^2 F_{qqp} - 8DF_{qq} F_{qq} F_{qqq} + \\
&8DF_q F_{qqq}^2 - 8F_p F_{qqq}^2 - 6DF_q F_{qq} F_{qqqq} + \\
&6F_p F_{qq} F_{qqqq} - 6F_p F_{qq}^2 F_{qqz} + 6F_{qq}^3 F_{qz} + \\
&2F_{qq}^2 F_{qqq} F_z - 8F_q F_{qqq}^2 F_z + 6F_q F_{qq} F_{qqqq} F_z)\tilde{\omega}^1 \tilde{\omega}^2 + \\
&(10DF_{qq} F_{qq}^3 - 10DF_q F_{qq}^2 F_{qqq} + 10F_p F_{qq}^2 F_{qqq} - \\
&10F_{qq}^4 F_z + 10F_q F_{qq}^2 F_{qqq} F_z)\tilde{\omega}^1 \tilde{\omega}^3 + \\
&30F_{qq}^4 \tilde{\omega}^1 \tilde{\omega}^4 + (30DF_q F_{qq}^3 - 30F_p F_{qq}^3 - 30F_q F_{qq}^3 F_z)\tilde{\omega}^1 \tilde{\omega}^5 + \\
&(4F_{qqq}^2 - 3F_{qq} F_{qqqq})(\tilde{\omega}^2)^2 - 10F_{qq}^2 F_{qqq} \tilde{\omega}^2 \tilde{\omega}^3 + \\
&30F_{qq}^3 \tilde{\omega}^2 \tilde{\omega}^5 - 20F_{qq}^4 (\tilde{\omega}^3)^2
\end{aligned}$$

in terms of this coframe.

In the sequel, we explain how to derive this formula from (4.7). First, note that the restriction of the one-form $\tilde{\omega}_3$ to the rank three distribution $\mathcal{H}^1$ is an element α of $\Gamma(L(\mathcal{H}^1, \mathbb{R}))$ with kernel the rank two distribution. That is, it is a generalised contact form for the distribution $\mathcal{H}$. We need to construct the data associated to a generalised contact α in Chapter 3: the generalised Reeb field r, the extension of the generalised contact form to a one-form on the manifold and the projection $\pi_{-1} : TM \to \mathcal{H}$.

There are a few things we can do first: We differentiate the coframe (B.3) which yields, compare with (B.1) and (B.2),

$$\begin{aligned} d\tilde{\omega}^1 &= \tilde{\omega}^5 \wedge \tilde{\omega}^3 \\ d\tilde{\omega}^2 &= F_y \tilde{\omega}^5 \wedge \tilde{\omega}^1 + F_z \tilde{\omega}^1 \wedge \tilde{\omega}^2 + ((F_z F_q + F_p)\tilde{\omega}^5 - dF_q) \wedge \tilde{\omega}^3 \\ d\tilde{\omega}^3 &= \tilde{\omega}^5 \wedge \tilde{\omega}^4 \\ d\tilde{\omega}^4 &= 0 \\ d\tilde{\omega}^5 &= 0, \end{aligned} \tag{B.4}$$

and we compute the dual frame for the coframe (B.3) which is given by

$$\begin{aligned} \xi_1 &= \partial_y, \\ \xi_2 &= \partial_z, \\ \xi_3 &= \partial_p + F_q \partial_z, \\ \xi_4 &= \partial_q, \\ \xi_5 &= \partial_x + p\partial_y + q\partial_p + F\partial_z. \end{aligned}$$

The vector fields ξ_5 and ξ_4 span the rank two distribution $\mathcal{H}$ and ξ_5, ξ_4 and ξ_3 span $\mathcal{H}^1$. Here, ξ_5 is what Nurowski calls D and in the sequel we will use the two notations simultaneously. Now we would like to determine for an arbitrary vector field $\zeta \in \mathfrak{X}(M)$ the unique vector field $\zeta_1 = \tilde{\omega}^4(\zeta_1)\xi_4 + \tilde{\omega}^5(\zeta_1)\xi_5 \in \Gamma(\mathcal{H})$ such that $\{q_{-2}(r), \zeta_1\} = q_{-3}(\zeta)$. Note that $q_{-2}(\xi_3) = \tilde{\omega}^3(\xi_3) q_{-2}(r) = q_{-2}(r)$. Hence, $\zeta_1 \in \Gamma(\mathcal{H})$ is characterised by the equations

$$\begin{aligned} \tilde{\omega}^1([\xi_3, \zeta_1]) &= \tilde{\omega}^1(\zeta) \\ \tilde{\omega}^2([\xi_3, \zeta_1]) &= \tilde{\omega}^2(\zeta). \end{aligned}$$

Using (B.4), we compute that

$$\tilde{\omega}^1([\xi_3, \zeta_1]) = -d\tilde{\omega}^1(\xi_3, \zeta_1) = \tilde{\omega}^5(\zeta_1)$$

and

$$\tilde{\omega}^2([\xi_3, \zeta_1]) = -d\tilde{\omega}^2(\xi_3, \zeta_1) = \tilde{\omega}^5(\zeta_1)(F_z F_q + F_p - \mathrm{D}F_q) - \tilde{\omega}^4(\zeta_1) F_{qq}$$

and thus

$$\begin{aligned} \tilde{\omega}^5(\zeta_1) &= \tilde{\omega}^1(\zeta) \\ \tilde{\omega}^4(\zeta_1) &= -\frac{1}{F_{qq}}(\tilde{\omega}^2(\zeta) + \tilde{\omega}^1(\zeta)(DF_q - F_p - F_q F_z)). \end{aligned} \tag{B.5}$$

Next, we would like to determine the generalised Reeb field corresponding to α. This is done in several steps following the proof of Proposition 3.5. Since ξ_3 is a vector field tangent to $\mathcal{H}^1$ with the property that $\alpha(\xi_3) = 1$, the Reeb vector field is of the form $r = \xi_3 + \delta$ for some $\delta \in \Gamma(\mathcal{H})$. To compute δ, we first associate to ξ_3 a partial connection ∇ on $\mathcal{H}$ via $\{\nabla_\gamma \eta, q_{-2}(\xi_3)\} = q_{-3}([\gamma, [\eta, \xi_3]])$ for any $\gamma, \eta \in \Gamma(\mathcal{H})$. In other words, $\nabla_\gamma \eta$ equals $-[\gamma, [\eta, \xi_3]]_1$, i.e. it is the vector field $-\zeta_1$ for $\zeta = [\gamma, [\eta, \xi_3]]$. Using (B.5), it thus remains to compute $\tilde{\omega}^i([\gamma,[\eta,\xi_3]])$ for $i=1,2$ in order to determine ∇ and doing so we obtain

$$\nabla_{\xi_4}\xi_4 = (\frac{1}{F_{qq}}F_{qqq})\xi_4$$

$$\nabla_{\xi_5}\xi_4 = -\frac{1}{F_{qq}}(F_zF_{qq} - D\cdot F_{qq})\xi_4,$$

$$\nabla_{\xi_4}\xi_5 = -\frac{1}{F_{qq}}(F_{zq}F_q + F_zF_{qq} + F_{pq} - \partial_q DF_q)\xi_4,$$

and

$$\nabla_{\xi_5}\xi_5 = \frac{1}{F_{qq}}(F_y + (F_z)^2F_q + F_zF_p - 2F_zDF_q + DF_zF_q - DF_p + DDF_q)\xi_4.$$

In the sequel ∇ will also denote the induced partial connection $\mathrm{gr}_{-2}(TM)$ associated to ξ_3. Then the proof of Proposition 3.5 shows that the generalised Reeb field is given by

$$r = \xi_3 + \delta,$$

where $\delta = \tilde{\omega}^4(\delta)\xi_4 + \tilde{\omega}^5(\delta)\xi_5 \in \Gamma(\mathcal{H})$ is characterised by $\{\delta,\gamma\} = \nabla_\gamma q_{-2}(\xi_3)$. Note that $d\tilde{\omega}^3 = \tilde{\omega}^5\wedge\tilde{\omega}^4$ implies $\{\xi_4,\xi_5\} = q_{-2}(\xi_3)$ and thus $\nabla_\gamma q_{-2}(\xi_3) = \{\nabla_\gamma\xi_4,\xi_5\} + \{\xi_4,\nabla_\gamma\xi_5\}$. Hence to compute δ we use can the equations

$$d\tilde{\omega}^3(\delta,\xi_4) = d\tilde{\omega}^3(\nabla_{\xi_4}\xi_4,\xi_5) + d\tilde{\omega}^3(\xi_4,\nabla_{\xi_4}\xi_5) = d\tilde{\omega}^3((\frac{1}{F_{qq}}F_{qqq})\xi_4,\xi_5)$$

$$d\tilde{\omega}^3(\delta,\xi_5) = d\tilde{\omega}^3(\nabla_{\xi_5}\xi_4,\xi_5) + d\tilde{\omega}^3(\xi_4,\nabla_{\xi_5}\xi_5) = d\tilde{\omega}^3(\frac{1}{F_{qq}}(DF_{qq} - F_zF_{qq})\xi_4,\xi_5)$$

and thus we obtain

$$\tilde{\omega}^5(\delta) = -\frac{1}{F_{qq}}F_{qqq}$$

and

$$\tilde{\omega}^4(\delta) = \frac{1}{F_{qq}}(DF_{qq} - F_{qq}F_z).$$

Remark B.1. Note that it is now also easy to determine the Weyl connection associated to the contact form $\alpha = \tilde{\omega}^3$ in directions $\gamma \in \Gamma(\mathcal{H})$. Following the proof of Proposition 3.5 it is given by $\nabla^\delta_\gamma\eta = \nabla_\gamma\eta + \alpha([\eta,\delta])\gamma$.

To get the right extension $\tilde{\alpha}$ of the generalised contact form $\alpha = \tilde{\omega}^3$ recall that $\tilde{\alpha}$ is characterised by $i_r d\tilde{\alpha}(\xi) = 0$ for all $\xi \in \Gamma(\mathcal{H})$, hence

$$d\tilde{\alpha}(\xi_4, r) = d\tilde{\alpha}(\xi_5, r) = 0.$$

Writing the extension as

$$\tilde{\alpha} = \tilde{\omega}^3 + s_1\tilde{\omega}^1 + s_2\tilde{\omega}^2$$

and using

$$d\tilde{\alpha} = \tilde{\omega}^5\wedge\tilde{\omega}^4 + ds_1\wedge\tilde{\omega}^1 + s_2F_y\tilde{\omega}^5\wedge\tilde{\omega}^1 + s_2F_z\tilde{\omega}^5\wedge\tilde{\omega}^2$$
$$+ s_2(F_zF_q + F_p + s_1)\tilde{\omega}^5\wedge\tilde{\omega}^3 - s_2dF_q\wedge\tilde{\omega}^3 + ds_2\wedge\tilde{\omega}^2.$$

we compute that this condition on $\tilde{\alpha}$ implies

$$s_2 = -\frac{1}{F_{qq}}\tilde{\omega}^5(\delta)$$

and

$$s_1 = -\frac{1}{F_{qq}}(DF_q - F_p - F_qF_z)\tilde{\omega}^5(\delta) - \tilde{\omega}^4(\delta).$$

The projection $\pi_{-1} : TM \to \mathcal{H}$ is the identity on ξ_5, ξ_4 and $\pi_{-1}(\xi_3) = -\delta$. To compute π_{-1} on ξ_1 and ξ_2 one can use Remark 3.24 which provides the following description of the projection

$$\pi_{-1}(\zeta) = \frac{2}{5}(d\tilde{\alpha}(\xi_5, [r, \zeta_1])\xi_4 - d\tilde{\alpha}(\xi_4, [r, \zeta_1])\xi_5) + \frac{3}{10}([r, [r, \zeta_1]])_1 + \zeta_2,$$

where $\zeta_2 = \zeta - \tilde{\alpha}(\zeta)r - [r, \zeta_1]$ and $([r, [r, \zeta_1]])_1$ is the unique section of $\mathcal{H}$ such that $\{q_{-2}(r), ([r, [r, \zeta_1]])_1\} = q_{-3}([r, [r, \zeta_1]])$. Looking at the formula shows that the computations will be straightforward using the previous results, but the expressions will get more involved.

To compare formula (4.7) to Nurowski's formula, we insert the vector fields ξ_i into the metric g_α. There are a few immediate observations: Looking at formula (4.7), we see that $g_\alpha(\xi, \eta)$ as well as $g_\alpha(\xi, r)$ vanishes for all vector fields $\xi, \eta \in \Gamma(\mathcal{H})$. Thus, when writing the metric g_α with respect to the coframe $\tilde{\omega}^i$, it is clear that $(\tilde{\omega}^5)^2$, $(\tilde{\omega}^4)^2$, $\tilde{\omega}^4\tilde{\omega}^5$, $\tilde{\omega}^3\tilde{\omega}^4$ and $\tilde{\omega}^3\tilde{\omega}^5$ have vanishing coefficients. Furthermore, formula (4.7) shows that $g_\alpha(r, r) = -\frac{4}{3}$. Since in Nurowki's formula $(\tilde{\omega}^3)^2$ has coefficient $-20F_{qq}^4$, we expect that we have to multiply our formula by $15F_{qq}^4$ to obtain Nurowski's formula. The verification is direct computation. We carry out the computations for those coefficients that can be obtained easily from what we have done so far.

For $\xi_2 = \partial_z$ the unique vector field $(\partial_z)_1 \in \Gamma(\mathcal{H})$ such that $\{q_{-2}(r), (\partial_z)_1\} = q_{-3}(\partial_z)$ is given by

$$(\partial_z)_1 = -\frac{1}{F_{qq}}\xi_4.$$

Hence, $g_\alpha(\xi_2, \xi_5) = d\tilde{\omega}^3(-\frac{1}{F_{qq}}\xi_4, \xi_5) = \frac{1}{F_{qq}}$ and $g_\alpha(\xi_2, \xi_4) = d\tilde{\omega}^3(-\frac{1}{F_{qq}}\xi_4, \xi_4) = 0$. Next,

$$\pi_{-1}(\xi_3) = \xi_3 - r = -\delta$$

and

$$g_\alpha(\xi_2, \xi_3) = d\tilde{\omega}^3(-\frac{1}{F_{qq}}\xi_4, -\delta) - \frac{4}{3}\tilde{\alpha}(\xi_2)\tilde{\alpha}(\xi_3) = -\frac{1}{F_{qq}}\tilde{\omega}^5(\delta) - \frac{4}{3}s_2 = -\frac{1}{3F_{qq}^2}F_{qqq}.$$

Moreover,

$$(\partial_y)_1 = -\frac{1}{F_{qq}}(\mathrm{D}F_q - F_p - F_qF_z)\xi_4 + \xi_5$$

and thus $g_\alpha(\xi_1, \xi_5) = \frac{1}{F_{qq}}(\mathrm{D}F_q - F_p - F_qF_z)$ and $g_\alpha(\xi_1, \xi_4) = 1$. Now we also see that

$$d\tilde{\omega}^3((\partial_y)_1, \pi_{-1}(\xi_3)) = -\frac{1}{F_{qq}}(\mathrm{D}F_q - F_p - F_qF_z)\tilde{\omega}^5(\delta) - \tilde{\omega}^4(\delta)$$

and

$$\tilde{\alpha}(\xi_1) = -\frac{1}{F_{qq}}(\mathrm{D}F_q - F_p - F_qF_z)\tilde{\omega}^5(\delta) - \tilde{\omega}^4(\delta)$$

and thus

$$\begin{aligned} g_\alpha(\xi_1, \xi_2) &= d\tilde{\omega}^3((\partial_y)_1, \pi_{-1}(\xi_3)) - \tfrac{4}{3}\tilde{\alpha}(\xi_1)\tilde{\alpha}(\xi_3) \\ &= -\frac{1}{3}\left((\mathrm{D}F_q - F_p - F_qF_z)\frac{1}{F_{qq}^2}F_{qqq} + \frac{1}{F_{qq}}(DF_{qq} - F_{qq}F_z)\right). \end{aligned}$$

This implies that the coefficients of $\tilde{\omega}^2\tilde{\omega}^5$, $\tilde{\omega}^2\tilde{\omega}^4$, $\tilde{\omega}^2\tilde{\omega}^3$, $\tilde{\omega}^1\tilde{\omega}^5$, $\tilde{\omega}^1\tilde{\omega}^4$ and $\tilde{\omega}^1\tilde{\omega}^3$ are what they are supposed to be.

Finally, we compute the shortest of the coefficients involving π_{-1} on an element not contained in $\mathcal{H}^1$, namely we compute

$$g_\alpha(\xi_2,\xi_2) = 2d\tilde{\omega}^3((\partial_z)_1,\pi_{-1}(\xi_2)) - \frac{4}{3}\tilde{\alpha}(\xi_2)\tilde{\alpha}(\xi_2):$$

We have

$$[r,(\partial_z)_1] = \left(-r\cdot\tfrac{1}{F_{qq}} + \tfrac{1}{F_{qq}}\partial_q(\tilde{\omega}^4(\delta))\right)\xi_4 + \xi_2 + \tfrac{1}{F_{qq}}(\partial_q(\tilde{\omega}^5(\delta)))\xi_5 + \tfrac{1}{F_{qq}}\tilde{\omega}^5(\delta)\xi_3$$

and

$$d\tilde{\alpha}(\xi_4,[r,(\partial_z)_1]) = -\tfrac{1}{F_{qq}}\partial_q(\tilde{\omega}^5(\delta)) - s_2\tilde{\omega}^5(\delta) - \partial_q(\tfrac{1}{F_{qq}}\tilde{\omega}^5(\delta)).$$

The coefficient of ξ_5 in the decomposition of $([r,[r,(\partial_z)_1]])_1$ turns out to be

$$\tilde{\omega}^5(([r,[r,(\partial_z)_1]])_1) = \tfrac{1}{F_{qq}}(\partial_q(\tilde{\omega}^5(\delta))) - \tfrac{1}{F_{qq}}(\tilde{\omega}^5(\delta))^2$$

and we compute that

$$\begin{aligned}\xi_2 - \tilde{\alpha}(\xi_2)r - [r,(\partial_z)_1] = &\left((r\cdot\tfrac{1}{F_{qq}}) - \tfrac{1}{F_{qq}}\partial_q(\tilde{\omega}^4(\delta)) + \tfrac{1}{F_{qq}}\tilde{\omega}^4(\delta)\tilde{\omega}^5(\delta)\right)\xi_4\\ &+\left(-\tfrac{1}{F_{qq}}\partial_q(\tilde{\omega}^5(\delta)) + \tfrac{1}{F_{qq}}(\tilde{\omega}^5(\delta))^2\right)\xi_5.\end{aligned}$$

Note that

$$\partial_q(\tilde{\omega}^5(\delta)) = \frac{(F_{qqq})^2 - F_{qqqq}F_{qq}}{(F_{qq})^2}$$

and

$$(\tilde{\omega}^5(\delta))^2 = \left(\frac{F_{qqq}}{F_{qq}}\right)^2.$$

Using that we obtain

$$2d\tilde{\omega}^3((\partial_z)_1,\pi_{-1}(\xi_2)) = -\frac{1}{5}\frac{F_{qqqq}F_{qq}}{(F_{qq})^4} + \frac{8}{5}\frac{(F_{qqq})^2}{(F_{qq})^4}$$

and

$$-\frac{4}{3}\tilde{\alpha}(\xi_2)\tilde{\alpha}(\xi_2) = -\frac{4}{3}\frac{(F_{qqq})^2}{(F_{qq})^4}$$

and thus

$$g_\alpha(\xi_2,\xi_2) = -\frac{1}{5}\frac{F_{qqqq}F_{qq}}{(F_{qq})^4} + \frac{4}{15}\frac{(F_{qqq})^2}{(F_{qq})^4}.$$

Computing $g_\alpha(\xi_1,\xi_2)$ and $g_\alpha(\xi_1,\xi_1)$ along the same lines, one should recover the remaining two (long) coefficients from Nurowski's formula.

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Abstract (deutsch)

Diese Dissertation beschäftigt sich mit dem Studium generischer Distributionen vom Rang zwei auf fünf-dimensionalen Mannigfaltigkeiten, wobei mit Distributionen hier Teilvektorbündel des Tangentialbündels gemeint sind. Die Untersuchung der Distributionen erfolgt im Rahmen der Theorie parabolischer Geometrien.

Unter einer parabolischen Geometrie versteht man eine Cartan-Geometrie mit einer verallgemeinerten Flaggenmannigfaltigkeit als homogenem Modell. In einer klassischen Arbeit aus dem Jahr 1910 bewies E. Cartan, dass man einer generischen Rang zwei Distribution auf einer fünf-dimensionalen Mannigfaltigkeit eine kanonische Cartan-Geometrie zuordnen kann. Das homogene Modell dieser Geometrie ist eine verallgemeinerte Flaggenmannigfaltigkeit, genauer, der homogene Raum G_2/P, wobei G_2 die zerfallende reelle Form der komplexen exzeptionellen Lie-Gruppe und P eine maximal parabolische Untergruppe von G_2 bezeichnet. Die Betrachtung der Distributionen als parabolische Geometrien zeigt unter anderem, dass sie lokale Invarianten und endlich-dimensionale Automorphismengruppen besitzen.

Der zentrale Beitrag dieser Dissertation basiert auf der von A. Cap und J. Slovak entwickelten Theorie der Weylstrukturen für parabolische Geometrien. Die Wahl einer Weylstruktur ermöglicht die Interpretation der kanonischen Cartan-Konnexion einer generischen Rang zwei Distribution durch Daten auf der fünf-dimensionalen Mannigfaltigkeit, genauer, durch eine lineare Konnexion auf der Distribution, eine Zerlegung des Tangentialbündels der Mannigfaltigkeit und einen Tensor, genannt Rho-Tensor. In dieser Arbeit betrachten wir Weyl-Strukturen, die durch sogenannte verallgemeinerte Kontaktformen festgelegt sind. Wir erarbeiten eine möglichst explizite Beschreibung der zugehörigen linearen Konnexion, der zugehörigen Zerlegung des Tangentialbündels und des zugehörigen Rho-Tensors. Die Konstruktion dieser Daten ermöglicht die Darstellung von geometrischen Invarianten der Distribution mit Hilfe von verallgemeinerten Kontaktformen. So können die fundamentale Krümmungs-Invariante sowie gewisse invariante Differentialoperatoren durch die konstruierten Daten beschrieben werden. Schließlich liefern die Resultate auch eine neue natürliche Beschreibung einer Metrik aus Nurowskis konformer Klasse, die einer generischen Rang zwei Distribution in Dimension fünf zugeordnet werden kann.

Lebenslauf

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Publikationen

- *Split octonions and generic rank 2 distributions in dimension 5*, Arch. Math. (Brno) 42 Suppl. 329-339.
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Preprint

- *On Nurowski's Conformal Structure Associated to a Generic Rank Two Distribution in Dimension Five* (gemeinsam mit A. Čap), Preprint ESI 1963.