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Simplicial Replacement of Certain Model Categories

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Introduction

In the two influential books [Qui67] and [Qui69] Quillen introduced the concept of model categories which provides a very general framwork for homotopy theory. Since then the theory of model categories has developed rapidly and became extremely successful. Nowadays, if one wants to study homotopical properties of an object of interest, the standard approach is to set up a model structure on it. Quillen defined a model category as a category with three distinguished classes of maps, called *weak equivalences, fibrations* and *cofibrations*, satisfying five simple axioms. Moreover, a model category is called *closed*, if any two of the three distinguished classes of maps determine the third. Quillen also showed that a model category is closed if and only if all three of its distinguished classes of maps are closed under retracts. It turns out that the class of weak equivalences is always closed under retracts. This implies that the requirement to be closed is not a serious one, because one can always turn a model category into a closed model category by closing the other two classes of maps under retracts.

In the early works mentioned above, two of the five defining axioms of a model category are quite weak compared with the modern definitions. The first axiom assumes only the existence of finite limits and colimits, but not general small ones. The last axiom assumes the existence of factorizations of maps, but not their functoriality. This approach has the advantage to include more categories, such as the category of finitely generated chain complexes, but at the same time these generalizations make the theory very complicated. Nowadays, if one is talking about a model category, then one usually means a closed model category which satisfies the more restrictive axioms, i.e. which is also complete, cocomplete and permits a functorial factorization.

In order to understand the motivation for introducing the model category, one goes back to the starting point of abstract homotopy theory, which is the study of the localization of a category. This means that for a choosen class of morphisms (the weak equivalences), one constructs a new category, called the homotopy category, by formally inverting all the weak equivalences. In general, after the localization with respect to a class of morphisms the class of maps between two objects does not need to be a proper set anymore, but if the class of the weak equivalences is part of a model structure, this is the case. The nice behaviour of the passage from a model category to its homotopy category is due to the existence of the class of cofibrations and the class of fibrations, which have the property to induce lifts and extend maps. Despite the importance of cofibrations and fibrations, the authors of [DHKS04] realized that the class of weak equivalences already determines the homotopy theory in the sense that the presence of weak equivalences allows one to define the homotopy notations.

In the book [DHKS04], homotopical categories are introduced as a generalization of the model categories. A homotopical category is defined as a category with only one distinguished class of map, called the weak equivalences, which includes all identities and have the two-of-six property. On the first sight, the weak equivalences here seem to be different from the weak equivalences in a model category, but in [DHKS04] the authors showed that the homotopical categories really include all model categories. The theory of the homotopical categories provides a better understanding of the nature of the weak equivalences in a model category by isolating the ideas which only involve this class of maps. Furthermore, many terms in the theory of model categories, such as approximations, homotopical uniqueness or homotopical completeness and cocompleteness, are now presented in a new light. They become more abstract and more general, but simpler at the same time.

Beside generalizations of the theory of model categories, there is also an intense study of model categories with additional properties. These properties are necessary to deal with problems arising from special situations. In particular the classes of enriched model categories are studied for many purposes. Its most prominent member are simplicial model categories, i.e. model categories which are enriched over the category of simplicial sets. Since for each pair of objects in a simplicial model category the maps between them is not only a set, but a simplicial set, called the function complex between the objects, it is possible to capture "homotopy structure of higher order". Among other good properties, many complicated formulas such as the one for homotopy colimit and limits or the concept of homotopical maps become quite simple in a simplicial model structure.

Therefore, it is often advantageous to know whether there is a "simplicial replacement" for a model category one is working with. First we have to clarify what we mean by a simplicial replacement. In the theory of categories, there is the notion of an equivalence of categories. Since the passage of a model category to its homotopy category is a (2-pseudo-)functor (usually denoted by Ho) from the (2-)category of model categories to the (2-)category of categories, one can ask which conditions imply that two homotopy categories become equivalent after the localization. This leads us to the notation of a Quillen pair which is a pair of adjoint functors with homotopical meaning. More precisely, a left Quillen functor is a left adjoint which also preserves cofibrations and trivial cofibrations. A right Quillen functor is defined dually. If $\tilde{C}$ and $\hat{D}$ denotes a cofibrant and a fibrant approximation respectively and if (F, G) is a Quillen pair such that the pair of total derived functors $(LF, RG) = (Ho\tilde{C} \circ HoF, Ho\hat{D} \circ HoG)$ induces an equivalences between categories, then it is called a Quillen equivalence. This terminology allows us to define a simplicial replacement as a simplicial model category which is Quillen equivalent to the original one.

Moreover, the definition of model categories is too general such that it is impossible to find a simplicial replacement for every model category. It seems that even the question which properties a model category must have in order to admit a simplicial replacement is too difficult to be answered. Nevertheless the question can be answered affirmatively for particular classes of model categories.

For example, Dugger showed in his paper [Dug00] that every combinatorial model category is Quillen equivalent to a simplical model category. A combinatorial model category was introduced by J. Smith in [Smi] as a cofibrantly generated model category whose underlying category is locally presentable. The existence of the simplicial replacement is proved by giving a small presentation. If $\mathcal{S}$ denotes the model category of simplicial sets, then a small presentation for a model category $\mathcal{M}$ is a tuple $(\mathcal{C}, S, \mathrm{Re}, \mathrm{Sing})$, where $\mathcal{C}$ is a small category, S a set of maps in the category $\mathcal{S}^{\mathcal{C}^{op}}$ equipped with the Bousfield-Kan model structure and (Re, Sing) is a Quillen pair

$$\mathrm{Re}\colon\ \mathcal{S}^{\mathcal{C}^{op}} \longleftrightarrow \mathcal{M} : \mathrm{Sing}$$

such that

1. The images of maps in S under the total left derived functor of Re are isomorphisms;
2. The induced Quillen pair $L_S\mathcal{S}^{\mathcal{C}^{op}} \longleftrightarrow \mathcal{M}$ is a Quillen equivalence, where $L_S\mathcal{S}^{\mathcal{C}^{op}}$ denotes the left Bousfield localization with respect to S.

The Bousfield-Kan model structure on $\mathcal{S}^{\mathcal{C}^{op}}$ is cofibrantly generated, simplicial and proper. In [Dug01b] and [Dug00] this category is called the *universal model category*, because it satisfies the following property:

Proposition. *[Dug01b, Proposition 2.3.] Let $\mathcal{C}$ be a small category and $\mathcal{M}$ be a model category. For every functor $H\colon \mathcal{C} \to \mathcal{M}$, there exists a Quillen pair* (Re, Sing) *and a natural weak equivalence $\tau\colon \mathrm{Re} \circ \iota \circ Y \to H$ in the diagram:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Y"] \arrow[drr, "H"'] & Set^{\mathcal{C}^{op}} \arrow[r, "\iota"] & \mathcal{S}^{\mathcal{C}^{op}} \arrow[d, shift right, "\mathrm{Re}"'] \\ & & \mathcal{M} \arrow[u, shift right, "\mathrm{Sing}"'] \end{tikzcd}$$

where the functor Y denotes the Yoneda embedding while ι is induced by the natural embedding $Set \to \mathcal{S}$.

In [Dug00], Dugger proved that for every combinatorial model category it is possible to choose a set S in $\mathcal{S}^{\mathcal{C}^{op}}$ such that the model category has a small presentation. The left Bousfield localization of $\mathcal{S}^{\mathcal{C}^{op}}$ with respect to a set S preserves properties such as being left proper and simplicial. Therefore, as a consequence of the existence of a small presentation, every combinatorial model category has a left proper simplical replacement.

In summary we could say that a simplicial replacement is found in two steps. First one constructs the universal model category which can be viewed as a free object which enjoys many nice properties. The difficult part is then to find an appropriate set of maps S in it and to impose relations on the universal model category by localizing with respect to S. Since the left Bousfield localization may not always exist, one has to work with special classes of model categories. Moreover, it is often not obvious how to define a set S and a Quillen pair (Re, Sing) such that the localized model category becomes Quillen equivalent to the original one.

However, due to Hirschhorn and Smith, it is known that at least for two major classes of model categories a left Bousfield localization exists.

Theorem (J. Smith)**.** *If $\mathcal{M}$ is a left proper combinatorial model category and if S is a set of morphisms in $\mathcal{M}$, then the left Bousfield localization $L_S\mathcal{M}$ of $\mathcal{M}$ exists and it is again left proper and combinatorial.*

Proof. Cf. [Bar10, Theorem 2.11.]. □

Theorem (P. S. Hirschhorn)**.** *If $\mathcal{M}$ is a left proper cellular (see. Definition 1.9.1) model category and if S is a set of morphisms in $\mathcal{M}$, then the left Bousfield localization $L_S\mathcal{M}$ of $\mathcal{M}$ exists and it is again left proper and cellular.*

Proof. Cf. [Hir03, Theorem 4.1.1.]. □

In this diploma thesis we study the methods Dugger used in [Dug01a] in order to prove the existence of a simplicial replacement for model categories which are either left proper combinatorial or left proper cellular.

Chapter 1 begins with the definition of a model category as a homotopical category satisfying additional axioms. Then we recall some standard terminology such as resolutions, homotopy function complexes and properness. We also mention some properties of cofibrantly generated, Reedy or Bousfield-Kan model structure.

In Chapter 2 we define the three simplicial axioms and the categorical action on the category of Δ^{op}-diagrams $T^{\Delta^{op}}$ for a cocomplete category T. The categorical action on a Reedy model category is almost simplicial, it already satisfies two simplicial axioms. The key observation of this chapter is the Proposition 2.2.7 which states that the last simplicial axiom is also satisfied, if the fibrant objects are the simplicial resolutions. The general theory of the left Bousfield localization says that for a set S of maps an object Z is fibrant in the localized model structure if and only if the induced morphism of homotopy function complexes $\operatorname{map}(B, Z) \to \operatorname{map}(A, Z)$ is a weak equivalence for every $f\colon A \to B$ in S. This fact gives a hint that the Reedy model structure may become simplicial after localization. At the end of the chapter we verify that the localized model category $L_S\mathcal{M}$ is simplicial and Quillen equivalent to the original model category $\mathcal{M}$. Most of the proofs we need here are due to Dugger, but compared to [Dug01a] we will not use the Bousfield-Kan model structure as a stepping stone.

The left Bousfield localization guarantees the existance of a simplicical replacement and it provides an explicit description of the class of cofibrations and the class of weak equivalences. However, the fibrations there are only defined by the lifting property and the general theory does not give us any further information about these maps. In [RSS01] the authors prove that under certain conditions it is possible to give fibrations in a model category a precise description and this is the topic of Chapter 3.

The idea and the main theorems in Chapter 3 are due to the authors of [RSS01], our attempt is to explain the motivation behind the definitions. As in [RSS01], the plan is to introduce the canonical model category as a model category with reasonable properties such that it may become a simplicial replacement. Once more, it seems to be crucial to require the fibrant objects to be the simplicial resolutions. If $\mathcal{M}$ is a model category, then a canonical model category is a model category on $\mathcal{M}^{\Delta^{op}}$ such that:

1. The Reedy weak equivalences are weak equivalences.
2. The Reedy cofibrations are the cofibrations.
3. The simplicial resolutions are the fibrant objects.

The notation is general enough to contain the simplicial replacements we studied in Chapter 2. The authors of [RSS01] showed that if there exist a canonical model category, then it is uniquely defined by these three conditions. It is then easily checked that the simplicial replacement $L_S\mathcal{M}$

of $\mathcal{M}$ we obtained in the previous chapter is a canonical model category. Hence, describing the fibrations of a canonical model category is the same as describing the fibrations in $L_S\mathcal{M}$. The existence of a canonical model category and the description of the fibrations are given at the same time by the construction of a realization model category which is in fact a cofibrantly generated canonical model category where the weak equivalences, the cofibrations and the fibrations have the following form.

1. The weak equivalences are the realization weak equivalences (see Definition 2.3.10).

2. The cofibrations are the Reedy cofibrations.

3. The fibrations are the equifibered Reedy fibrations (see Definition 3.2.1).

The main effort is to prove the existence of a realization model category and we do this in two steps.

First we find the sets of generating cofibration and trivial cofibrations. As a motivation we recall a theorem in [Hir03]:

Theorem. *Let $\mathcal{M}$ be a cofibrantly generated model category with I and J as the sets of generating cofibrations and trivial cofibrations respectively. If the codomains of maps in I and J are small relative to all I-cell complexes and all relative J-cell complexes respectively, then the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$ is cofibrantly generated.*

This hypothesis concerning the codomains ensures that one can apply the small object argument on the induced sets of generating cofibrations and trivial cofibrations of the Reedy model category.

As in the previous chapter the idea is to modify the Reedy model structure by changing the weak equivalences and the fibrations in such a way that it becomes simplicial. Since the cofibrations stay unchanged and the fibrations in the realization model category are particularly Reedy fibrations, it is reasonable to try to construct a realization model structure just by adding maps to the set of generating trivial cofibrations as defined in the theorem above. In [RSS01] the authors suggest to add maps which also involve the set of generating trivial cofibrations in $\mathcal{M}$. This process of adding maps violates the conditions of the theorem above. In order to render the process harmless, we follow [RSS01] and strengthen the definition of a cofibrantly generated model category $\mathcal{M}$:

Definition. *If I and J denotes the sets of generating cofibrations and trivial cofibrations respectively, then $\mathcal{M}$ is also cofibrantly generated in the sense of [RSS01], if two additional conditions are satisfied:*

1. *The codomains of maps in I and J are small relative to I (see Definition 1.3.9).*

2. *The domains and codomains of maps in I are cofibrant.*

What we did Theorem 3.2.10 is to re-prove the theorem about the cofibrant generation of the Reedy model structure under the assumption that the model category $\mathcal{M}$ is cofibrantly generated in the strenghtened sense.

In the second part of the chapter, we verify that the hypothesis in the Recognition Lemma (see Theorem 1.3.15) which is a theorem due to D. M. Kan and gives sufficient conditions for a category to admit a cofibrantly generated model structure. Once the conditions are checked, a realization model category exists. At the end of the chapter we show that the realization model category is a simplicial one.

Last but not at least, I would like to thank Ass.-Prof. Dr. Stefan Haller for his patience and innumerous discussions during the course of this work.

Chapter 1

Model Categories and Homotopical Categories

This chapter is devoted to many important concepts of abstract homotopy theory, which are indispensable for the proofs in the other chapters. Some of the theorems presented here are of great theoretical importance, just as many ideas in abstract homotopy theory have influenced many other fields of mathematics. They certainly deserve a more precise introduction than we could give here.

We start this chapter with the definition of *homotopical categories*. In the following section, *model categories* are introduced as categories which satisfy certain axioms. Our definition is the one given in [DHKS04]. Althought it differs from the usual ones, as presented in [Hir03] or [Hov99], Proposition 1.2.3 reveals that every model category in the sense of [DHKS04] is also a model category in the usual sense.

In Section 1.3. some terminologies such as *transfinite compositions* (see Definition 1.3.1) and *smallness* (see Definition 1.3.3 and Definition 1.3.9) are discussed. They are indispensable for the discussion of the *small object argument* (see Proposition 1.3.13) which provides a functorial factorization on a cocomplete category.

After defining *cofibrantly generated model categories* (see Definition 1.3.14), we present the *Recognition Lemma* (see Theorem 1.3.15). It gives the sufficient conditions for the existance of a cofibrantly generated model structure on a category. We list some model categories, whose cofibrantly generated model structure are verified by using the Recognition Lemma.

In Section 1.4., we mention that if a model category is cofibrantly generated, then there exists a model structure on the category of diagrams in it (see Theorem 1.4.6). Moreover, this model structure is also cofibrantly generated and it is called the *Bousfield-Kan model structure* (see Definition 1.4.7). The reason for the existence of this model structure is the so-called *Lifting Lemma* (see Theorem 1.4.1).

In Section 1.5. we give the definition of a *direct category* (see 1.5.1) and an *inverse category* (see 1.5.6). For any model category $\mathcal{M}$, these categories give rise to establish a model structure on the category of diagrams in $\mathcal{M}$. In order to describe cofibrations and fibrations, the *latching functor* and the *matching functor* are introduced in Definition 1.5.3 and Definition 1.5.8. As a generalization of the concept of the direct and inverse categories, we define a *Reedy category* in Definition 1.5.11, which also provides a model structure on the category of functors from a Reedy category into a model category. In Proposition 1.5.16 we give a characterisation of the Reedy model structure.

A Reedy model structure allows to define *cosimplicial resolutions* and *simplicial resolutions* (see Definition 1.6.1) in Section 1.6. These objects enable the construction of *homotopy function complexes*. As an application, Proposition 1.7.6 shows that a map is a weak equivalence if and only if it induces weak equivalences between homotopy function complexes.

In Section 1.8. we define *left proper* and *right proper* model categories (see Definition 1.8.1). In theorems 1.8.3 and 1.8.4 we point out that if a model category is left or right proper, then the Bousfield-Kan model structure as well as the Reedy model structure inherit this property.

Two important classes of model categories are introduced in Section 1.9. These are the *cellular model categories*, of P. S. Hirschhorn, (see Definition 1.9.1) and the *combinatorial model categories*, of J. Smith, (see Definition 1.9.7). Due to them we know that if a model category is cellular or combinatorial, then the Reedy model structure is again cellular or combinatorial respectively (see Theorem 1.9.2 and Theorem 1.9.10).

1.1 Homotopical Categories

A *model category* is a category with three distinguished classes of morphisms, the cofibrations, fibrations and weak equvialences. Moreover, a model category satisfies certain axioms which allow one to do homotopy theory in a very abstract way. This concept was first introduced by Quillen in his fundamental works [Qui67] and [Qui69]. The class of cofibrations and fibrations on the one side and the class of weak equvialences on the other side have very different nature. One realized that while the homotopy notations can be defined just in the presence of weak equivalences, the cofibrations and fibrations are needed if one wants to work with the homotopy theory and and prove statements. In order to isolate the properties, which are really induced by weak equivalences, the authors of [DHKS04] define a new class of categories, the *homotopical categories*. These categories have only one distinguished class of maps, called weak equivalences, and it is shown that they generalize the notation of model categories.

We believe that thinking of a model category as a homotopical category with more structure may be helpful. Therefore, although we will not need any properties of homotopical categories explicitly, we will give the definition and discuss some basic results. For a discussion of the theory of model categories from a homotopical point of view we warmly recommend the book [DHKS04].

Most of the definitions and propositions in the following two sections can be found in [DHKS04].

Definition 1.1.1 (Homotopical category)**.** [DHKS04] A *homotopical category* is a category $\mathcal{M}$ with a distinguished class W of maps, called *weak equivalences*, such that:

1. W contains all *identity maps* of $\mathcal{M}$;
2. W has the *two-of-six property.* This means, for very three maps f, g and h in $\mathcal{M}$ for which the two compositons $g \circ f$ and $h \circ g$ exist and are in W, the four maps f, g, h and $h \circ g \circ f$ are also in W.

These two properties are equivalent to:

1.′ W contains all identity maps of $\mathcal{M}$.

2.′ For every map g in $\mathcal{M}$, if there exist maps f and h in $\mathcal{M}$ such that the compositions $g \circ f$ and $h \circ g$ exist and are in W, then g is also in W. A class of maps with this property is said to have the *weak invertibility property.*

3.′ W has the *two-of-three property.* This means, for every two maps f and g in $\mathcal{M}$ such that their composition $g \circ f$ exists, if two of the three maps f, g and $g \circ f$ are in W, then the third is also in W.

One readily verifies that the part 1.′ and the part 2.′ together imply that every isomorphism in $\mathcal{M}$ is also in W. Moreover, the part 3.′ guarantees that W is a subcategory of $\mathcal{M}$.

1.2 Model Categories

Definition 1.2.1 (Model category)**.** A *model category* $\mathcal{M}$ is a category with a model structure, i.e. three distinguished classes of maps, *weak equivalences, fibrations* and *cofibrations* satisfying the following five axioms:

1. *Limit axiom:* The category $\mathcal{M}$ is *complete* and *cocomplete.*

2. *Two-of-six axiom:* The weak equivalences have the *two-of-six property.*

3. *Retract axiom:* The cofibrations and fibrations are *closed under retracts*, i.e. if there is a commutative diagram in $\mathcal{M}$ of the form

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] & X \arrow[r, "q"] \arrow[d, "g"] & A \arrow[d, "f"] \\ A' \arrow[r, "p'"'] & X' \arrow[r, "q'"'] & A' \end{tikzcd}$$

such that the compositions $q \circ p$ and $q' \circ p'$ are the identities id_A and $id_{A'}$ respectively, and the map g: $X \to X'$ is fibration or a cofibration, then so is the map f: $A \to A'$.

4. *Lifting axiom:* Given the commutative solid arrow diagram in $\mathcal{M}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "k"] & Y \end{tikzcd}$$

the dotted arrow exists if either

(a) i is a cofibration and p is a trivial fibration, i.e. p is both a fibration and a weak equivalence, or

   (b) i is a trivial cofibration, i.e. it is both a cofibration and a weak equivalence, and p is a fibration.

5. *Factorization axiom:* A map f in $\mathcal{M}$ admits two *functorial factorizations*
   (a) $f = q \circ i$, where i is a cofibration and q is a trivial fibration, and
   (b) $f = p \circ j$, where p is a fibration and j is a trivial cofibration.

Here, a *functorial factorization* is a pair of functors (Cof, Fib): $\mathrm{Mor}(\mathcal{M}) \to \mathrm{Mor}^2(\mathcal{M})$ such that $dom \circ Cof = dom$, $cod \circ Cof = dom \circ Fib$, $cod \circ Fib = cod$, where dom assigns each map to its domain and cod assigns each map to its codomain. Furthermore, we require that for every map f in $\mathcal{M}$, we can write f as a composite: $f = Cof(f) \circ Fib(f)$.

The definition above is different from the usual definition of a model category. Here we strengthened the second axiom by requiring the two-of-six property instead of the two-of-three property. At the same time we weakened the retract axiom by requiring that only the class of fibrations and the class of cofibrations need to closed by retracts, but not the weak equivalences. The following two propositions tell us that the new definition of a model category provide the same categories as the old one and that it allows us to view a model category as a homotopical category.

Proposition 1.2.2. *The above notation of a model category agrees with that of a closed model category in the sense of Quillen [Qui69].*

Proof. A proof can be found in [DHKS04, 9.3]. □

Proposition 1.2.3. *[DHKS04] Every model category $\mathcal{M}$ is a homotopical category.*

Proof. The existence of weak equivalences and of the two-of-six axiom are part of the definition of a model category. Therefore, it is only to show that for every object X in $\mathcal{M}$, the identity map id_X is a weak equivalence in $\mathcal{M}$. By the factorization axiom we can write $id_X = q \circ i$, such that q is a trivial fibration. Now the result follows from applying the two-of-three property (which is weaker than the two-of-six property) to the map $q = id_X \circ q$. □

1.3 Cofibrantly Generated Model Categories

All the definitions in this section are standard, they can be found in most of the books about model categories (e.g. [Hir03], [Hov99] or [MP71]).

1.3.1 The small object argument

Definition 1.3.1. [Hir03] Given a cocomplete category $\mathcal{M}$, let $\mathcal{D}$ be a class of maps in $\mathcal{M}$ and let λ be an ordinal.

1. A *λ-sequence of maps in $\mathcal{D}$* is a functor $X\colon \lambda \to \mathcal{M}$, such that every morphism in this diagram $X_\alpha \to X_{\alpha+1}$ is a map in $\mathcal{D}$, for all $\alpha + 1 < \lambda$, and for every limit ordinal $\kappa < \lambda$ the induced map $\mathrm{colim}_{\alpha<\kappa} X_\alpha \to X_\kappa$ is an isomorphism.

2. A *transfinite composition* of a λ-sequence X of maps in $\mathcal{D}$ is the map

$$X_0 \to \operatorname{colim}_{\alpha<\lambda} X_\alpha.$$

3. If $\mathcal{C}$ is a subcategory of $\mathcal{M}$, then a *tranfinite composition* of maps in $\mathcal{C}$ is then a transfinite composition of a λ-sequence X of maps in $\operatorname{Mor}(\mathcal{C})$.

Definition 1.3.2 (Regular cardinal)**.** Let λ be a cardinal and let S be a set such that:

1. The cardinality of S is less than the cardinality of λ.

2. For every $s \in S$ there is a set A_s with lower cardinality than λ.

Then λ is a *regular cardinal* if the cardinality of the sum $\coprod_{s\in S} A_s$ is less than the cardinality of λ for all such S and A_s.

Definition 1.3.3. [Hir03] Let $\mathcal{M}$ be a cocomplete category and $\mathcal{C}$ be a subcategory in $\mathcal{M}$. Furthermore, let λ be a cardinal.

1. An object Z in $\mathcal{M}$ is *λ-small relative to* $\mathcal{C}$, if for every regular cardinal $\mu \geq \lambda$ and every μ-sequence X_α in $\mathcal{C}$, the canonical map: $\operatorname{colim}_{\alpha<\mu} \mathcal{M}(Z, X_\alpha) \to \mathcal{M}(Z, \operatorname{colim}_{\alpha<\mu} X_\alpha)$ is an isomorphism.

2. An object Z in $\mathcal{M}$ is *small relative to* $\mathcal{C}$, if Z is λ-small relative to $\mathcal{C}$ for some cardinal λ.

3. An object Z in $\mathcal{M}$ is *small*, if it is small relative to $\mathcal{M}$.

Notation 1.3.4. Because we will use the lifting property very often, we now introduce a useful short cut. Let I and J be two classes of maps in a category. We will use the phrase "(I, J) has the lifting property" as short hand for "every map in the class I has the left lifting property with respect to every map in the class J." The notation (f, J) is used, if f is the only element in I and the same goes for (I, g).

Definition 1.3.5. [Hov99] Let $\mathcal{M}$ be category and $\mathcal{D}$ be a class of maps in $\mathcal{M}$.

1. A map f in $\mathcal{M}$ is *I-injective*, if (I, f) has the lifting property. For brevity, we will denote the class of I-injective maps by I-inj.

2. A map f in $\mathcal{M}$ is *I-projective*, if (f, I) has the lifting property. For brevity, we will denote the class of I-projective maps by I-proj.

3. A map f in $\mathcal{M}$ is an *I-cofibration*, if (f,I-inj) has the lifting property. In other words the class of I-cofibrations is the class (I-inj)-proj and we will denote this class by I-cof.

4. A map f in $\mathcal{M}$ is an *I-fibration*, if (I-proj,f) has the lifting property. In other words the class of I-fibrations is the class (I-proj)-inj and we will denote this class by I-fib.

5. An object A in $\mathcal{M}$ is *I-projective* or *I-cofibrant* if the map $\emptyset \to A$ exists and it is an element of I-proj or I-cof respectively.

6. An object X in $\mathcal{M}$ is *I-injective* or *I-fibrant* if the map $X \to *$ exists and it is an element of I-inj or I-fib respectively.

Example 1.3.6. If $\mathcal{M}$ is a model category and I is the class of cofibrations, then by definition the class of trivial fibrations is I-inj and the class of cofibrations is I-cof $= I$. If I is the class of fibrations in $\mathcal{M}$, then the class of trivial cofibrations is I-proj and the class of fibrations is I-fib $= I$.

Lemma 1.3.7. *Let $I \subseteq J$ be two classes of maps in a category. Then we have:*

1. *J-inj $\subseteq$ I-inj and J-proj $\subseteq$ I-proj.*
2. *$I \subseteq I$-cof and $I \subseteq I$-fib.*
3. *I-inj = (I-cof)-inj and I-proj = (I-fib)-proj.*
4. *I-cof = (I-cof)-cof and I-fib=(I-fib)-fib.*

Proof. All the inclusions follow easily from the definition. □

Definition 1.3.8. Let $\mathcal{M}$ be a cocomplete category and let I be a set of maps in $\mathcal{M}$.

1. A *relative I-cell complex* is a transfinite composition of pushouts of maps in I.
2. An object A is an *I-cell complex* if the map $\emptyset \to A$ is a relative I-cell complex.

Definition 1.3.9. Let $\mathcal{M}$ be a cocomplete category and let I be a set of maps in $\mathcal{M}$.

1. For a cardinal λ, an object Z in $\mathcal{M}$ is *λ-small relative to I*, if Z is λ-small relative to all I-cell complexes.
2. An object Z is *small relative to I* if there exists a cardinal λ such that Z is λ-small relative to I.

Remark 1.3.10. In this paper we usually use the same notation as in [Hir03], but many authors use another terminology. Especially, the relative I-cell complexes are also known as regular I-cofibrations.

Lemma 1.3.11. *Let $\mathcal{M}$ be a cocomplete category and I a set of maps in $\mathcal{M}$. If an object is small relative to I, then it is also small relative to any set of relative I-cell complexes.*

Proof. If J is an arbitrary set of relative I-cell complexes, then the result follows from the fact that every relative J-cell complex is a relative I-cell complex. □

Definition 1.3.12 (D. M. Kan)**.** Let $\mathcal{M}$ be a complete category and let I be a set of maps in $\mathcal{M}$. We say that I *permits the small object argument* if the domains of the maps in I are small relative to I.

Proposition 1.3.13 (The small object argument)**.** *Let $\mathcal{M}$ is a cocomplete category and I a set of maps it. If I permits the small object argument, then there exists a functorial factorization (α, β) of maps in $\mathcal{M}$ such that for every f in $\mathcal{M}$, the map $\alpha(f)$ is relative I-cell complex, while $\beta(f)$ is I-inj and $\alpha(f) \circ \beta(f) = f$.*

Proof. This is a standard result and we refer the reader to the sources: [Hir03], [Hov99] or [DHK97]. □

1.3.2 Cofibrantly generated model structure

Although the axioms for a model category are quite simple, it is often very difficult to define a model structure on a given category. In most cases the main difficulty is to find a functorial factorization. One possiblity to overcome this obstacle is to use the small object argument due to D. M. Kan, see Proposition 1.3.13. Moreover, Kan proved in the so called *Recognition Lemma* (see Theorem 1.3.15) that a few conditions are already sufficient for establishing a model structure on a category. This result is not only of theoretical importance, it is also powerful tool which can be used to prove that many well known categories provide a model structure by verifying the conditions in the Recognition Lemma. Moreover, a model structure obtained in this way enjoys the nice property that cofibrations and fibrations are controlled by two sets of maps. These model categories are called the *cofibrantly generated model categories* which are defined below.

Definition 1.3.14 (Cofibrantly generated model category)**.** A model category $\mathcal{M}$ is *cofibrantly generated*, if there are two sets of maps I and J in $\mathcal{M}$ such that:

1. The sets I and J permit the small object argument;
2. The class of trivial fibrations is I-inj;
3. The class of fibrations is J-inj.

If the sets I and J satisfy these three conditions, then they are called the *set of generating cofibrations* and the *set of generating trivial cofibrations* respectively.

If the hypotheses above are satisfied, then it is clear that the cofibrations are the I-cof and the trivial cofibrations are the J-cof.

Since the cofibrations and fibrations in a cofibrantly generated model category are well understood, one always tries to establish a cofibrantly generated model structure on the model category one is working with. The usual method to do this is to use the following theorem due to D. M. Kan, which is called the *Recognition Lemma*, because it gives the necessary and sufficient conditions for a category $\mathcal{C}$ to admit a cofibrantly generated model structure.

Theorem 1.3.15 (Recognition Lemma)**.** *[D. M. Kan] Let $\mathcal{C}$ be a complete and cocomplete category and let W be a class of maps which is closed under retracts and satisfies the two-of-three axiom. If I and J are sets of maps in $\mathcal{C}$ such that*

1. *the sets I and J permit the small object argument,*
2. *J-cof $\subseteq$ I-cof $\cap\, W$ and I-inj $\subseteq$ J-inj $\cap\, W$, and*
3. *I-cof $\cap\, W \subseteq$ J-cof or J-inj $\cap\, W \subseteq$ I-inj;*

Then there is a cofibrantly generated model structure on $\mathcal{C}$ in which I and J are the generating cofibrations and generating trivial cofibrations respectively and W is the class of weak equivalences.

Proof. [DHK97, 8.1.]. □

Lemma 1.3.16. *Suppose $\mathcal{M}$ is a cofibrantly generated model category and suppose I and J are the sets of generating cofibrations and trivial cofibrations respectively, then there exists a set $\bar{J}$ such that $\bar{J}$-cof $= J$-cof. Moreover, if the domains and the codomains of the maps in J are small relative to I, then the domains and the codomains of the maps in $\bar{J}$ are also small relative to I.*

Proof. The existence of a functorial factorization follows from the small object argument, see Proposition 1.3.13. By using the factorization, we can assume that a map j in J is a composition $j = p \circ \bar{j}$, such that $\bar{j}$ is a relative I-cell complex and p be an I-injective. Let $\bar{J}$ denotes the set of all such maps $\bar{j}$. By construction the set $\bar{J}$ consists of I-cell complexes. We show that we can replace J by $\bar{J}$ by proving that the set $\bar{J}$ generates the trivial cofibrations.

Since every I-injective map is a trivial fibration, the two-of-three axiom implies that the maps in $\bar{J}$ are weak equivalences, and so they are trivial cofibrations. Every J-injective is a fibration, therefore every J-injective is also a $\bar{J}$-injective. For the other inclusion, use the retract argument to show that every map j is a retract of $\bar{j}$. Hence, every $\bar{J}$-injective is also a J-injective.

For the last statement, note that the maps in J and $\bar{J}$ have the same domains and that the codomains of maps of $\bar{J}$ are small relative to I by [Hir03, Proposition 10.4.8.], if the codomains of maps in J are. □

1.3.3 Examples of cofibrantly generated model categories

In this section we will see that the most prominent examples of model categories are cofibrantly generated. The proofs of the existence of their model structures use the Recognition Lemma explicitly. Because of their length we will not give the proofs here. The original proofs are due to D. G. Quillen and can be found in [Qui67, II. 3]. A more readable version is written in [Hov99, Sections 2.3, 2.4, 3.2-3.6].

Example 1.3.17 (The category of simplicial sets). The category of simplicial sets $\mathcal{S}$ is a cofibrantly generated model category .

1. The set of generating cofibrations $I_{\mathcal{S}}$ are the maps:

$$\partial\Delta[n] \to \Delta[n], \quad n \geq 0.$$

2. The set of generating trivial cofibrations $J_{\mathcal{S}}$ are the maps:

$$\Lambda\Delta[n,m] \to \Delta[n], \quad n > 0 \quad \textit{and} \quad 0 \leq m \leq n.$$

Here $\partial\Delta[n]$ denotes the boundary of the n-simplex and $\Lambda\Delta[n,m]$ denotes the m-horn of $\Delta[n]$, i.e. it is the largest subobject not containing the $(n-1)$-simplex $d_m id_{[n]}$.

For a proof, see [Hov99, Sections 3.2-3.6].

Remark 1.3.18. Of course there is a pointed version for the category of pointed simplicial sets $\mathcal{S}^*$. In this case set of generating cofibrations $I_{\mathcal{S}^*}$ are the maps: $\partial\Delta[n]^+ \to \Delta[n]^+$, $n \geq 0$ and the set of generating trivial cofibrations $J_{\mathcal{S}}$ are the maps: $\Lambda\Delta[n,m]^+ \to \Delta[n]^+$, $n > 0$ and $0 \leq m \leq n$. Here $\Delta[n]^+ = \Delta[n] \sqcup *$, where $*$ denotes the terminal object of the category.

Example 1.3.19 (The category of topological spaces). The category of topological spaces Top is a cofibrantly generated model category.

1. The set of generating cofibrations I_{Top} are the maps:

$$\mid \partial\Delta[n] \mid \to \mid \Delta[n] \mid, \quad n \geq 0.$$

2. The set of generating trivial cofibrations J_{Top} are the maps:

$$\mid \Lambda\Delta[n,m] \mid \to \mid \Delta[n] \mid, \quad n > 0 \quad and \quad 0 \leq m \leq n.$$

Here $\mid - \mid$ denotes the realization functor.

For a proof, see [Hov99, Section 2.4].

Remark 1.3.20. For the model category category of pointed topological spaces, the set of generating cofibrations I_{Top^*} are the maps: $\mid \partial\Delta[n]^+ \mid \to \mid \Delta[n]^+ \mid$, $n \geq 0$ and the set of generating trivial cofibrations $J_{\mathcal{S}}$ are the maps: $\mid \Lambda\Delta[n,m]^+ \mid \to \mid \Delta[n]^+ \mid$, $n > 0$ and $0 \leq m \leq n$.

Example 1.3.21 (The category chain complexes of modules over a ring). Let $Ch(R)$ denote the category of chain complexes of modules over a ring R. The objects there are the chain complexes of left R-modules and the morphisms are the chain maps.

If S^n denotes the chain complex which is the module R in the degree n and 0 otherwise, and if D^n denotes the chain complex which is the module R in the degrees $n-1$ or n and 0 otherwise, while the differential ∂_n is the identity id_R, then the category $Ch(R)$ is a cofibrantly generated model category.

1. The set of generating cofibrations $I_{Ch(R)}$ are the maps:

$$S^{n-1} \to D^n.$$

2. The set of generating trivial cofibrations $J_{Ch(R)}$ are the maps:

$$0 \to D^n.$$

For a proof, see [Hov99, Section 2.3].

1.4 Bousfield-Kan model structure

If $\mathcal{M}$ is a cofibrantly generated model category and $\mathcal{C}$ is a small category, then the *Lifting Lemma* (see Theorem 1.4.1) asserts that the model structure on $\mathcal{M}$ induces a cofibrantly generated model structure on the category of $\mathcal{C}$-diagrams $\mathcal{M}^{\mathcal{C}}$, which is called the *Bousfield-Kan model structure* (see Definition 1.4.7). This lift is natural in the sense that the weak equivalences are the objectwise weak equivalences and that the set of generating cofibrations and trivial cofibrations of $\mathcal{M}^{\mathcal{C}}$ are defined by the set of generating cofibrations and fibrations of $\mathcal{M}$. Moreover, there exists a Quillen pair connecting the model categories $\mathcal{M}$ and $\mathcal{M}^{\mathcal{C}}$.

The following fundamental theorem is due to D. M. Kan and it is the reason for the existence of the Bousfield-Kan model structure.

Theorem 1.4.1 (Lifting Lemma)**.** *Let $\mathcal{M}$ be a cofibrantly generated model category with I and J as its sets of generating cofibrations and trivial cofibrations. Let $\mathcal{N}$ be a complete and cocomplete* *category and let $F\colon \mathcal{M} \longleftrightarrow \mathcal{N} : G$ be a pair of adjoint functors. If $FI = \{Fi\colon i \in I\}$ and $FJ = \{Fj\colon j \in J\}$ and if the following hold:*

1. *The sets FI and FJ permit the small object argument.*

2. *The right adjoint functor G takes relative FJ-cell complexes to weak equivalences.*

Then the category $\mathcal{N}$ admits a cofibrantly generated model structure with FI and FJ as its sets of generating cofibrations and trivial cofibrations, and the weak equivalences are the maps that G takes to weak equivalences in $\mathcal{M}$. Moreover, the adjoint functors F and G form a Quillen pair respect to this model structure.

Proof. The proof is essentially a verification of the conditions in the Recognition Lemma 1.3.15, by using the adjointness. A detailed version can be found in [Hir03, Theorem 11.3.2.]. □

According to the Lifting Lemma, one way to define a model structure on $\mathcal{M}^{\mathcal{C}}$ is to find a pair of adjoint functors with nice properties between a cofibrantly generated model category $\mathcal{N}$ and the category $\mathcal{M}^{\mathcal{C}}$. This idea leads to the definition of *free $\mathcal{C}$-diagram* (see Definition 1.4.4).

Definition 1.4.2. Let $\mathcal{M}$ be a cocomplete category and A an object of $\mathcal{M}$. Let $\mathcal{C}$ be small category and c an object of $\mathcal{C}$. Then a *free diagram generated by A at c* is the object F_c^A in $\mathcal{M}^{\mathcal{C}}$ defined by

$$F_c^A(-) = \coprod_{\mathcal{C}(c,-)} A.$$

Remark 1.4.3. Every c in $\mathcal{C}$ induces a functor $F_c^-\colon\ X \mapsto F_c^X$ which is the left adjoint to the *evaluation functor ev_c*: $X \mapsto X(c)$. This pair of adjoint functors:

$$F_c^-\colon\ \mathcal{M} \longleftrightarrow \mathcal{M}^{\mathcal{C}} : ev_c$$

gives rise to a natural isomorphism:

$$\mathcal{M}^{\mathcal{C}}(F_c^X, Y) \cong \mathcal{M}(X, Y(c)).$$

If $\mathcal{C}$ is a category, then $\mathcal{C}^*$ denotes the discrete category which has the same objects as $\mathcal{C}$ but no other morphisms than the identities.

Definition 1.4.4 (Free diagrams)**.** Let $\mathcal{C}$ be small category and $\mathcal{C}^*$ the discrete category. If $\mathcal{M}$ is a cocomplete category and A an object of $\mathcal{M}^{\mathcal{C}^*}$, then the *free diagram in $\mathcal{M}$ generated by A* is the object $F(A)$ in $\mathcal{M}^{\mathcal{C}}$, which is defined by

$$F(A) = \coprod_{c \in \mathcal{C}^*} F_c^{A_c}.$$

The free diagrams $F(A)$ give rise to define a pair of adjoint functors:

$$F\colon\ \mathcal{M}^{\mathcal{C}^*} \longleftrightarrow \mathcal{M}^{\mathcal{C}} : G,$$

where F denotes the functor which assigns to every object A in $\mathcal{M}^{\mathcal{C}^*}$ the object $F(A)$ in $\mathcal{M}^{\mathcal{C}}$ and G denotes the forgetful functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}^*}$.

If $\mathcal{C}$ is a small category and $\mathcal{M}$ is a cofibrantly generated model category, then the category $\mathcal{M}^{\mathcal{C}^*}$ which is a product of $\mathcal{M}$ is again a cofibrantly genereated model category. Even more is true if the sets I and J are the sets of generating cofibrations and trivial cofibrations respectively, then it happens that the sets $\coprod_{c\in\mathcal{C}} I \times \Pi_{d\neq c} id_{\emptyset}$ and $\coprod_{c\in\mathcal{C}} J \times \Pi_{d\neq c} id_{\emptyset}$ are the sets of the generating cofibrations and trivial cofibrations of $\mathcal{M}^{\mathcal{C}^*}$ respectively. A close observation reveals that the images of these sets are exactly the sets $F^I_{\mathcal{C}}$ and $F^J_{\mathcal{C}}$ as defined in Definition 1.4.5.

As a consequence of this observation, the Lifting Lemma tells us that the pair of adjoint functors (F, G) induces a cofibrantly generated model structure on $\mathcal{M}^{\mathcal{C}}$. This is the statement of Theorem 1.4.6.

Definition 1.4.5. Let $\mathcal{C}$ be a small category, let $\mathcal{M}$ be a cofibrantly generated model category. If I is its set of generating cofibrations and if J is its set of generating trivial cofibrations, then one defines $F^I_{\mathcal{C}}$ to be the set of maps:

$$F^A_c \to F^B_c$$

where $c \in \mathcal{C}$ and $A \to B$ is a map in I. The set $F^J_{\mathcal{C}}$ is defined similarly.

Theorem 1.4.6. *If $\mathcal{C}$ is a small category and $\mathcal{M}$ is a cofibrantly generated model category with I and J as the sets of generating cofibrations and trivial cofibrations respectively, then there is a cofibrantly generated model structure on $\mathcal{M}^{\mathcal{C}}$ such that:*

1. *The set of generating cofibrations is $F^I_{\mathcal{C}}$.*
2. *The set of generating trivial cofibrations is $F^J_{\mathcal{C}}$.*
3. *A map is a weak equivalence if and only if it is an objectwise weak equivalence.*
4. *A map is a fibration if and only if it is an objectwise fibration.*
5. *A map is a cofibration if and only if it is a retract of a transfinite composition of pushouts of elements of $F^I_{\mathcal{C}}$.*

Proof. [Hir03, Theorem 11.6.1]. □

Definition 1.4.7 (Bousfield-Kan model structure)**.** The model structure on $\mathcal{M}^{\mathcal{C}}$ in the theorem above is called the *Bousfield-Kan model structure.*

Proposition 1.4.8. *In the Bousfield-Kan model structure every cofibration is in particular an objectwise cofibration.*

Proof. We refer the reader to [Hir03, Proposition 11.6.2.]. □

1.5 Reedy Model Categories

In the last section we saw that if a model category $\mathcal{M}$ is cofibrantly generated, then for every small category $\mathcal{C}$ there exists a model structure called the Bousfield-Kan model structure on $\mathcal{M}^{\mathcal{C}}$. For an arbitrary model category $\mathcal{M}$ this is, in general, not true any more.

In this section we introduce small categories with a degree function on them. These categories are called *Reedy categories* (see Definition 1.5.11). The major advantage of Reedy categories is that if a diagram is indexed by a Reedy category, then it can by analyzed inductively. This property allows one to extend a diagram step by step. Therefore, as we will see in Proposition 1.5.13, for every model category $\mathcal{M}$ and Reedy category $\mathcal{C}$ the category $\mathcal{M}^{\mathcal{C}}$ admits a model structure.

Before we give the definition of a Reedy category, we discuss two special cases of them.

Definition 1.5.1 (Direct categories)**.** A small $\mathcal{C}$ is called a *direct category* if the following properties are satisfied:

1. For some ordinal λ, there exists a function $d\colon Ob(\mathcal{C}) \to \lambda$, called the *degree function*;
2. Every nonidentity map in $\mathcal{C}$ raises the degree.

Recall that for any category $\mathcal{N}$ and any object n in $\mathcal{N}$, the comma category $(\mathcal{N} \downarrow n)$ is defined as the category in which the objects are the maps $n' \to n$ for every object n' in $\mathcal{N}$ and the morphisms are the obvious commuting triangles. Moreover, every comma category comes with a forgetful functor $\iota\colon (\mathcal{N} \downarrow n) \to \mathcal{N}$.

Definition 1.5.2 (Latching category)**.** Suppose $\mathcal{C}$ is a direct category, then its *latching category* $\partial(\mathcal{C} \downarrow c)$ is the maximal subcategory of $(\mathcal{C} \downarrow c)$ which does not contain the identity maps $id_c \in \mathcal{C}$.

A latching category gives rise to the definition of latching objects and latching maps which are needed in the definition of the direct model structure.

Definition 1.5.3 (Latching functor)**.** Let $\mathcal{M}$ be a cocomplete category. If $\mathcal{C}$ is a direct category and $\iota\colon (\mathcal{C} \downarrow c) \to \mathcal{C}$ denotes the forgetful functor, then the *latching functor* L_c is defined as a composite of two functors:

$$\mathcal{M}^{\mathcal{C}} \xrightarrow{\iota^*} \mathcal{M}^{\partial(\mathcal{C}\downarrow c)} \xrightarrow{\text{colim}} \mathcal{M}\ ,$$

where ι^* is induced by the forgetful functor ι. For a $\mathcal{C}$-diagram in the cocomplete category $\mathcal{M}$, its image

$$L_cX = \text{colim}_{\partial(\mathcal{C}\downarrow c)}\, \iota^* X$$

is called the *latching object* of X at c while the natural map

$$L_cX \to X(c)$$

is called the *latching map* of X at c.

Proposition 1.5.4. *If $\mathcal{C}$ is a direct category and $\mathcal{M}$ is a model category, then there exists a model structure on $\mathcal{M}^{\mathcal{C}}$, called the direct model structure, such that a map $f\colon X \to Y \in \mathcal{M}^{\mathcal{C}}$*

1. *is a weak equivalence if and only if it is an objectwise weak equivalence,*
2. *is a (trivial) cofibration if and only if, for every object $c \in \mathcal{C}$, the induced map*
$$X(c) \sqcup_{L_cX} L_cY \to Y(c) \in \mathcal{M}$$
is a (trivial) cofibration, and
3. *is a (trivial) fibration if and only if it is an objectwise (trivial) fibration.*

Proof. The main idea of the proof is to use the degree function to construct a factorization of maps inductively. A detailed proof can be found in [Hov99, Theorem 5.1.3.]. □

Corollary 1.5.5. *In the direct model structure every cofibration is in particular an objectwise cofibration.*

Proof. [Hov99, Remark 5.1.7.]. □

The dual concept of direct categories is the notation of inverse categories.

Definition 1.5.6 (Inverse categories)**.** A category $\mathcal{C}$ is called an *inverse category* if and only if $\mathcal{C}^{op}$ is a direct category.

As in the case of direct categories, a matching category gives rise the definition of matching objects and matching maps which are needed in the definition of an inverse model structure.

Definition 1.5.7 (Matching category)**.** The *matching category* $\partial(\mathcal{C} \downarrow c)$ of an inverse category $\mathcal{C}$ is the latching category of $\mathcal{C}^{op}$.

Definition 1.5.8 (Matching functor)**.** Let $\mathcal{M}$ be a complete category. If $\mathcal{C}$ is an inverse category and $\iota\colon \mathcal{C} \to (c \downarrow \mathcal{C})$ denotes the forgetful functor, then the *matching functor* M_c is defined as a composite of two functors:

$$\mathcal{M}^{\mathcal{C}} \xrightarrow{\iota^*} \mathcal{M}^{\partial(c\downarrow\mathcal{C})} \xrightarrow{\lim} \mathcal{M} \ ,$$

where ι^* is induced by the forgetful functor ι. For a $\mathcal{C}$-diagram in the complete category $\mathcal{M}$, its image

$$M_c X = \lim_{\partial(c\downarrow\mathcal{C})} \iota^* X$$

is called the *matching object* of X at c while the natural map

$$X(c) \to M_c X$$

is called the *matching map* of X at c.

By dualzing Proposition 1.5.4 every model category admits an inverse model structure.

Proposition 1.5.9. *If $\mathcal{C}$ is an inverse category and $\mathcal{M}$ is a model category, then there exists a model structure on $\mathcal{M}^{\mathcal{C}}$, called the inverse model structure, such that a map $f\colon X \to Y \in \mathcal{M}^{\mathcal{C}}$*

1. *is a weak equivalence if and only if it is an objectwise weak equivalence,*
2. *is a (trivial) cofibration if and only if it is an objectwise (trivial) cofibration, and*
3. *is a (trivial) fibration if and only if for every object $c \in \mathcal{C}$ the induced map*

$$X(c) \to Y(c) \times_{M_c Y} M_c X \in \mathcal{M}$$

is a (trivial) fibration.

Proof. [Hov99, Theorem 5.1.3.]. □

Corollary 1.5.10. *In the inverse model structure every fibration is in particular an objectwise fibration.*

Proof. [Hov99, Remark 5.1.7.]. ☐

The concept of Reedy categories, which was first studied by C. L. Reedy in [Ree74], is a generalization of the notation of direct and inverse categories.

Definition 1.5.11. [Hir03] A *Reedy category* is a triple $(\mathcal{C}, \overrightarrow{\mathcal{C}}, \overleftarrow{\mathcal{C}})$ consisting of two small subcategories $\overrightarrow{\mathcal{C}}$ and $\overleftarrow{\mathcal{C}}$, such that

1. there exists a *degree function* d,
2. every nonidentity map in $\overrightarrow{\mathcal{C}}$ raises the degree,
3. every nonidentity map in $\overleftarrow{\mathcal{C}}$ lowers the degree,
4. every map $f \in \mathcal{C}$ admits a *functorial factorization* $f = \overrightarrow{f}\overleftarrow{f}$ with $\overrightarrow{f} \in \overrightarrow{\mathcal{C}}$ and $\overleftarrow{f} \in \overleftarrow{\mathcal{C}}$.

Remark 1.5.12. From the definition it is clear that the subcategories $\overrightarrow{\mathcal{C}}$ and $\overleftarrow{\mathcal{C}}$ of a Reedy category $\mathcal{C}$ are direct and inverse categories respectively and every direct or inverse category is in particular a Reedy category. In the sequel, if we are talking about Reedy categories, then by abuse of notation a latching L_c will be the composite of the functors

$$\mathcal{M}^{\mathcal{C}} \xrightarrow{incl^*} \mathcal{M}^{\overrightarrow{\mathcal{C}}} \xrightarrow{L_c} \mathcal{M},$$

where $incl^*$ is induced by the inclusion functor $incl\colon \overrightarrow{\mathcal{C}} \to \mathcal{C}$ and L_c denotes the latching functor of the direct category $\overrightarrow{\mathcal{C}}$. The matching functor M_c of a Reedy category is defined similarly.

Proposition 1.5.13. *For every model category $\mathcal{M}$ and Reedy category $\mathcal{C}$, the category $\mathcal{M}^{\mathcal{C}}$ admits a model structure, called the* Reedy model structure, *such that a map is a weak equivalence, a cofibration or a fibration if and only if it is so viewed as a map in $\mathcal{M}^{\overrightarrow{\mathcal{C}}}$ and $\mathcal{M}^{\overleftarrow{\mathcal{C}}}$.*

As a consequence of the definition of a Reedy model structure, we have the following corollary.

Corollary 1.5.14. *Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category.*

1. *If a map is a Reedy cofibration, then it is an objectwise cofibration.*
2. *If a map is a Reedy fibration, then it is an objectwise fibration.*

Notation 1.5.15. For a Reedy model category $\mathcal{N}$, we sometimes use the notation $(\mathcal{N})_R$ to emphesize the Reedy model structure on it.

If one really wants to work with Reedy model categories, one needs more than just the existence of the Reedy model structure.

Proposition 1.5.16. *Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category. In the Reedy model structure a map $f\colon X \to Y \in \mathcal{M}^{\mathcal{C}}$*

1. *is a weak equivalence if and only if it is an objectwise weak equivalence,*
2. *is a (trivial) cofibration if and only if, for every object $c \in \mathcal{C}$, the induced map*

$$X(c) \sqcup_{L_cX} L_cY \to Y(c) \in \mathcal{M}$$

is a (trivial) cofibration, and

3. is a (trivial) fibration if and only if, for every object $c \in \mathcal{C}$, the induced map

$$X(c) \to Y(c) \times_{M_c Y} M_c X \in \mathcal{M}$$

is a (trivial) fibration.

Proof. See [Hov99, Chapter 5]. □

The proposition above not only gives us formulas for checking the Reedy cofibrations and the Reedy fibrations, but also reveals that the Reedy model structure of $\mathcal{M}^{\mathcal{C}}$ is controlled by the model structure of $\mathcal{M}$. It is folklore that the Reedy model category $\mathcal{M}^{\mathcal{C}}$ inherits many properties such as being left or right proper (see Definition 1.8.1) or being simplicial (see Definition 2.1.6) from the model category $\mathcal{M}$.

Remark 1.5.17. Since the Reedy weak equivalences are the objectwise weak equivalences and since the Reedy fibrations are objectwise fibrations, the evaluation functor ev_c: $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}$ preserves fibrations and trivial fibrations. Thus the adjoint pair of functors (F_c^-, ev_c) in remark 1.4.3 is even a Quillen pair with respect to a Reedy model structure on $\mathcal{M}^{\mathcal{C}}$. If the small category $\mathcal{C}$ is the cosimplicial category Δ^{op}, then the free functor F_0^- is the constant functor c, therefore in particular there exists a Quillen pair:

$$c\colon \mathcal{M} \longleftrightarrow \mathcal{M}^{\Delta^{op}} : ev_0.$$

It will play an essential role in the proofs of the main theorems in the next two chapters.

1.6 Cosimplicial and simplicial resolutions

Definition 1.6.1. Let $\mathcal{M}$ be a model category.

1. For an object Z in $\mathcal{M}$, a *cosimplical resolution* of Z is a cofibrant approximation of the constant diagram cZ in the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$, i.e. it is a weak equivalence $\tilde{Z} \to cZ$ such that $\tilde{Z}$ is Reedy cofibrant.
2. For an object Z in $\mathcal{M}$, a *simplical resolution* of Z is a fibrant approximation of the constant diagram cZ in the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$, i.e. it is a weak equivalence $cZ \to \hat{Z}$ such that $\hat{Z}$ is Reedy fibrant.

In a Reedy model structure the weak equivalences are the objectwise weak equivalences, therefore the maps $\tilde{Z}(n) \to Z$ are weak equivalences for all n in Δ^{op} and by the two-of-three axiom every face map d_n and every degeneracy map s_n is a weak equivlance. As in the paper [RSS01], we will call objects with this property the *homotopical constants.* Using this terminology the cosimplicial resolutions in $\mathcal{M}^{\Delta^{op}}$ are the homotopical constant and Reedy cofibrant objects, while the simplicial resolutions are the homotopical constant and Reedy fibrant objects.

Proposition 1.6.2. *[Hir03, Proposition 16.2.1.] Given two model categories $\mathcal{M}$, $\mathcal{N}$ and a Quillen pair $F\colon \mathcal{M} \longleftrightarrow \mathcal{N} : G$, the following assertions hold.*

1. *If $\tilde{A} \to cA$ is a cosimplicial resolution of a cofibrant object A, then $F\tilde{A} \to cFA$ is a cosimplicial resolution of the object FA.*

2. *If $cX \to \hat{X}$ is a simplicial resolution of a fibrant object X, then $cGX \to G\hat{X}$ is a simplicial resolution of the object GX.*

Proof. The assertions are dual to each other, therefore we only need to prove the first one.

Since $\tilde{A}$ is a cosimplicial resolution, it is Reedy cofibrant and by Proposition 1.5.16 the latching maps $L_n(\tilde{A}) \to \tilde{A}(n)$ are cofibrations in $\mathcal{M}$ for every n. The functor F is a left Quillen functor, therefore it commutes with colimits and preserves cofibrations. This means that the maps $L_n(F\tilde{A}) \to F\tilde{A}(n)$ are cofibrations in $\mathcal{N}$ and therefore $F\tilde{A}$ is Reedy cofibrant.

The object A is cofibrant by assumption and for every n the object $\tilde{A}(n)$ is cofibrant, since $\tilde{A}$ is Reedy cofibrant. A left Quillen functor preserves trivial cofibrations, hence it preserves weak equivalences between cofibrant objects. It follows that $F\tilde{A} \to cFA$ is an objectwise weak equivalence, because $\tilde{A} \to cA$ is an objectwise weak equivalence between cofibrant objects. $\square$

1.7 Homotopy function complexes

The cosimplicial and simplicial resolutions allow us to introduce a homotopy function complex which is a very useful tool in the homotopy theory. After its definition we will see some applications. Among others, the concept of the homotopy function complexes simplify many proofs of important theorems.

Definition 1.7.1 (Left homotopy function complex)**.** Let $\mathcal{M}$ be a model category and let X and Y be two objects of $\mathcal{M}$. A *left homotopy function complex* is a simplicial set $\mathcal{M}(\underline{\tilde{X}}, \hat{Y})$ defined by

$$\mathcal{M}(\underline{\tilde{X}}, \hat{Y})(n) = \mathcal{M}(\underline{\tilde{X}}(n), \hat{Y}),$$

where $\underline{\tilde{X}}$ is a cosimplicial resolution of X and $\hat{Y}$ is a fibrant approximation to Y.

Definition 1.7.2 (Right homotopy function complex)**.** Let $\mathcal{M}$ be a model category and let X and Y be two objects of $\mathcal{M}$. A *right homotopy function complex* is a simplicial set $\mathcal{M}(\tilde{X}, \underline{\hat{Y}})$ defined by

$$\mathcal{M}(\tilde{X}, \underline{\hat{Y}})(n) = \mathcal{M}(\tilde{X}, \underline{\hat{Y}}(n)),$$

where $\tilde{X}$ is a cofibrant approximation to X and $\underline{\hat{Y}}$ is a simplicial resolution of Y.

Definition 1.7.3 (Two-sided homotopy function complex)**.** Let $\mathcal{M}$ be a model category and let X and Y be two objects of $\mathcal{M}$. A *two-sided homotopy function complex* is a simplicial set $diag\mathcal{M}(\underline{\tilde{X}}, \underline{\hat{Y}})$ defined by

$$diag\mathcal{M}(\underline{\tilde{X}}, \underline{\hat{Y}})(n) = \mathcal{M}(\underline{\tilde{X}}(n), \underline{\hat{Y}}(n)),$$

where $\underline{\tilde{X}}$ is a cosimplicial resolution of X and $\underline{\hat{Y}}$ is a simplicial resolution of Y.

Remark 1.7.4. From now on, when we say *homotopy function complex from X to Y* we mean either a left, a right or a two-sided homotopy function complex from X to Y. We will use $\mathrm{map}(X, Y)$ to denote a homotopy function complex from X to Y.

The following theorems can be found in [Hir03]. They tell us that a map is a weak equivalence if and only if it induces weak equivalences between some homotopy function complexes.

Theorem 1.7.5. *Let $\mathcal{M}$ be a model category.*

1. *Let A be an object and let $f\colon X \to Y$ be a map in $\mathcal{M}$. If f induces a weak equivalence of certain homotopy function complexes* $\mathrm{map}(A, f)\colon \mathrm{map}(A, X) \to \mathrm{map}(A, Y)$*, then f induces a weak equivalence for any other homotopy function complexes.*

2. *Let X be an object and let $g\colon A \to B$ be a map in $\mathcal{M}$. If g induces a weak equivalence of certain homotopy function complexes* $\mathrm{map}(g, X)\colon \mathrm{map}(B, X) \to \mathrm{map}(A, X)$*, then g induces a weak equivalence for any other homotopy function complexes.*

Proof. [Hir03, Theorem 17.5.31.]. □

Theorem 1.7.6. *Suppose $\mathcal{M}$ is a model category and $f\colon I \to J$ is a map in $\mathcal{M}$, then the following assertions are equivalent:*

1. *f is a weak equivalence.*

2. *For every object A, the induced morphism* $\mathrm{map}(A, f)\colon \mathrm{map}(A, I) \to \mathrm{map}(A, J)$ *is a weak equivalence in $\mathcal{S}$.*

3. *For every cofibrant object A, the induced morphism* $\mathrm{map}(A, f)\colon \mathrm{map}(A, I) \to \mathrm{map}(A, J)$ *is a weak equivalence in $\mathcal{S}$.*

4. *For every object X, the induced morphism* $\mathrm{map}(f, X)\colon \mathrm{map}(J, X) \to \mathrm{map}(I, X)$ *is a weak equivalence in $\mathcal{S}$.*

5. *For every fibrant object X, the induced morphism* $\mathrm{map}(f, X)\colon \mathrm{map}(J, X) \to \mathrm{map}(I, X)$ *is a weak equivalence in $\mathcal{S}$.*

Proof. [Hir03, Theorem 17.7.7.]. □

In [Dug01a] Dugger proves a generalized form of the Theorem 1.7.6.

Proposition 1.7.7 (D. Dugger)**.** *Let $\mathcal{M}$ be model category and $f\colon A \to B$ a map in $\mathcal{M}$. If the induced map $\pi_0\mathcal{M}(f, \underline{Z})\colon \pi_0\mathcal{M}(B, \underline{Z}) \to \pi_0\mathcal{M}(A, \underline{Z})$ is an isomorphism for every simplicial resolutions $\underline{Z}$ in $\mathcal{M}^{\Delta^{op}}$, then f is a weak equivalence.*

1.8 Proper Model Categories

In this diploma thesis, we are dealing with left proper model categories. Therefore we want to recall its definition and some basic results. The statements and their proofs are adapted to our demands, they are often not in the most general form which can be found in [Hir03].

Definition 1.8.1. Let $\mathcal{M}$ be a model category.

1. $\mathcal{M}$ is called *left proper*, if every pushout of a weak equivalence along a cofibration is a weak equivalence.

2. $\mathcal{M}$ is called *right proper*, if every pullback of a weak equivalence along a fibration is a weak equivalence.

3. $\mathcal{M}$ is called *proper*, if it is both left proper and right proper.

Example 1.8.2. The category of simplicial sets $\mathcal{S}$ and the category of pointed simplicial sets $\mathcal{S}_*$ are proper model categories. Moreover, the category of topological spaces Top and the category of pointed topological spaces Top_* are proper model categories.

If a small category $\mathcal{C}$ and a model category $\mathcal{M}$ are given, then under certain assumptions there exists a Bousfield-Kan or a Reedy model structure on the function category $\mathcal{M}^{\mathcal{C}}$. Now, one can ask, if the original model category $\mathcal{M}$ was left, right or proper, does the $\mathcal{M}^{\mathcal{C}}$ inherits these nice properties and the next two theorems provide an affirmative answer to this question.

Theorem 1.8.3. *Let $\mathcal{C}$ be a small category and let $\mathcal{M}$ be a cofibrantly generated model category. If $\mathcal{M}$ is left proper, right proper or proper then the model category $\mathcal{M}^{\mathcal{C}}$ equipped with the Bousfield-Kan model structure is also left proper, right proper or proper, respectively.*

Proof. Pullbacks and pushouts in $\mathcal{M}^{\mathcal{C}}$ are constructed objectwise. According to 1.4.8 a cofibration in the Bousfield-Kan model structure is particularly an objectwise cofibration. If $\mathcal{M}$ is left proper, then the pushout of an objectwise weak equivalence along a cofibration is an objectwise equivalence. Hence $\mathcal{M}^{\mathcal{C}}$ is also left proper.

Since a fibration in the Bousfield-Kan structure is by definition an objectwise fibration, a similar argumentation shows that $\mathcal{M}^{\mathcal{C}}$ is right proper, provided that $\mathcal{M}$ is. □

Theorem 1.8.4. *Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category. If $\mathcal{M}$ is left proper, right proper or proper then the Reedy model category $\mathcal{M}^{\mathcal{C}}$ is also left proper, right proper or proper, respectively.*

Proof. The proof is exactly the same as for the Bousfield-Kan structure in the theorem above, because according to Proposition 1.5.14 the cofibrations and the fibrations in the Reedy model structure are in particular objectwise cofibrations and fibrations. □

Proposition 1.8.5 (D.M.Kan)**.** *Let $\mathcal{M}$ be a left proper model category and let $f\colon A \to B$ be a cofibration. Suppose p $X \to Y$ is a fibration and $\tilde{f}\colon \tilde{A} \to \tilde{B}$ is a cofibrant approximation to f such that $\tilde{f}$ is a cofibration. If p has the right lifting property with respect to $\tilde{f}$, then p has the right lifting property with respect to f.*

Proof. The assumption yields a diagram

$$\begin{tikzcd} \tilde{A} \arrow[r, "i_A", "\cong"'] \arrow[d, "\tilde{f}"] & A \arrow[r, "a"] \arrow[d, "f"] & X \arrow[d, "p"] \\ \tilde{B} \arrow[r, "i_B", "\cong"'] & B \arrow[r, "b"] & Y \end{tikzcd}$$

For any such pair of maps a and b, we have to find a map c: $B \to X$ such that $c \circ f = a$ and

$p \circ c = b$. Let P denote the pushout $\tilde{B} \coprod_{\tilde{A}} A$, then we get a diagram

$$\begin{tikzcd}
\tilde{A} \arrow[rr, "i_A", "\cong"'] \arrow[dd, "\tilde{f}"'] & & A \arrow[r, "a"] \arrow[dl, "j"'] \arrow[dd, "f"] & X \arrow[dd, "p"] \\
& P \arrow[urr, dotted, "x"] \arrow[dr, "h"] & & \\
\tilde{B} \arrow[rr, "i_B", "\cong"'] \arrow[ur, "k"] & & B \arrow[r, "b"] \arrow[uur, dotted, "c"] & Y
\end{tikzcd}$$

Since $\tilde{f}$ has the left lifting property with respect to p and the left lifting property is closed under pushouts, we have a map $x\colon P \to X$. The map j is a cofibration since cofibrations are closed under pushouts as well.

Now we consider the category $(A \downarrow \mathcal{M} \downarrow Y)$ of objects of $\mathcal{M}$ under A and over Y. One easily verifies that this category is a model category if one defines maps in it to be a cofibration, fibration or weak equivalence if and only if they have the respective property in $\mathcal{M}$. In this model structure, P and B are cofibrant because j and f are cofibrations. Moreover, X is fibrant because p is a fibration. Since the map k is the pushout of a weak equivalence along the cofibration $\tilde{f}$, it is also a weak equivalence in the left proper model category $\mathcal{M}$. The two-of-three axiom guarantees that h is also a weak equivalence. Since P and B are cofibrant and h is a weak equivalence, there exists a map $P \to X$ if and only if there is a map $y\colon B \to X$. It is a map in $(A \downarrow \mathcal{M} \downarrow Y)$ and as a map in $\mathcal{M}$, it is exactly the map c we are searching for. $\square$

1.9 Model Categories of Type LpCe and LpCo

In this section we give the exact definition of cellular model categories and combinatorial model categories. Many of the following definitions are taken from [Hir03, Sections 10.8, 11.4 and 12.1] and [Dug00, Definition 2.1].

Definition 1.9.1 (Cellular model categories)**.** A model category $\mathcal{M}$ is *cellular*, if it is cofibrantly generated and if there is a set I of generating cofibrations and a set J of generating trivial cofibrations which satisfy the following properties:

1. The domains and codomains of elements of I are compact.

2. The domains of elements of J are small relative to I.

3. The cofibrations are effective monomorphisms.

Terms like “compact” or “effective monomorphism” are of technical nature. Since we do not need their properties explicitely, we will not introduce them. Their precise definitions can be found in [Hir03, Sections 10.8. and 10.9.].

Theorem 1.9.2. *[Hir03] Let $\mathcal{C}$ be a Reedy category. If $\mathcal{M}$ is a cellular model category, then the Reedy model structure on $\mathcal{M}^{\mathcal{C}}$ is a cellular model category.*

Proof. [Hir03, Theorem 15.7.6.]. $\square$

Notation 1.9.3. For brevity, we will use the acronym *LpCe* for model categories which are both left proper and cellular.

Corollary 1.9.4. *Let $\mathcal{C}$ be a Reedy category. If $\mathcal{M}$ is of type LpCe, then the Reedy model structure on $\mathcal{M}^{\mathcal{C}}$ is also of the same type.*

Proof. If $\mathcal{M}$ is left proper, then according to Theorem 1.8.4 the Reedy model category $\mathcal{M}^{\mathcal{C}}$ is also left proper. If $\mathcal{M}$ is cellular, then by Theorem 1.9.2 $\mathcal{M}^{\mathcal{C}}$ is also cellular. □

Definition 1.9.5. [AR94, Definition 1.13.] A poset is called λ-*directed*, if every subset of cardinality smaller than λ has an upper bound. If the idexing category of a diagram is a λ-directed poset, then it is called a λ-*directed diagram* and its colimit is called a λ-*direced colimit*.

Definition 1.9.6. [AR94, Definition 1.17. and 1.13.] Let λ be a regular cardinal and let $\mathcal{C}$ be a cocomplete category.

1. An object X of a category is called λ-*presentable* if for every λ-directed colimit $\operatorname{colim}_{i\in I} Y_i$ the natural morphism
$$\operatorname{colim}_{i\in I}\mathcal{C}(X,Y_i)\to\mathcal{C}(X,\operatorname{colim}_{i\in I}Y_i)$$
is an ismorphism.

2. $\mathcal{C}$ is called *locally* λ-*presentable*, if there exists a set S of λ-presentable objects such that every object in $\mathcal{C}$ is a λ-directed colimit of objects in S.

3. $\mathcal{C}$ is called *locally presentable* if it is locally λ-presentable for some regular cardinal λ.

Definition 1.9.7 (Combinatorial model categories)**.** Let $\mathcal{M}$ be a model category. It is called *combinatorial*, if its model structure is cofibrantly generated and its underlying category is locally presentable.

Combinatorial model categories were introduced by J. Smith in [Smi]. Loosely speaking the homotopy theory of a combinatorial model category is local and beyond sufficiently large cardinals it becomes formal.

To be more precise, Dugger proved in his paper [Dug00] the following proposition.

Proposition 1.9.8 ([Dug00])**.** *If $\mathcal{M}$ is a combinatorial model category and λ a sufficiently large regular cardinal, then the following assertions hold.*

1. *There exist functorial cofibrant and functorial fibrant approximations which preserve λ-filtered colimits.*

2. *λ-filtered colimits preserve weak equivalences.*

We just want to mention these nice properties of combinatorial model categories. Since we will not need them explicitely any more, we do not give any proof and refer the reader to [Dug00, Proposition 2.3.].

Notation 1.9.9. We will use the acronym *LpCo* for model categories which are both left proper and combinatorial.

The next proposition is due to C. Barwick.

Theorem 1.9.10 ([Bar07])**.** *If $\mathcal{C}$ is a Reedy category and $\mathcal{M}$ is a combinatorial model category, then the Reedy model structure on $\mathcal{M}^{\mathcal{C}}$ is a combinatorial model category.*

Proof. [Bar07, Lemma, 3.10.]. □

By Corollary 1.9.4, if $\mathcal{M}$ is of type $LpCe$, then the Reedy model structure on $\mathcal{M}^{\mathcal{C}}$ inherits this property. The corollary below tells us that there is a similar result for model categories of type $LpCo$.

Corollary 1.9.11. *Let $\mathcal{C}$ be a Reedy category. If $\mathcal{M}$ is of type $LpCo$, then the Reedy model structure on $\mathcal{M}^{\mathcal{C}}$ is also of the same type.*

Proof. This follows from Theorem 1.8.4 and Proposition 1.9.10. □

Chapter 2

Simplical Model Categories

The aim of this chapter is to prove the main theorem which ensures the existence of a simplicial model category which is Quillen equivalent to $\mathcal{M}$, provided $\mathcal{M}$ is left proper and cellular or left proper and combinatorial. Among others, the method we are going to use is the left Bousfield localization.

In Section 2.1 we define the *adjunction of two variables* and the *Quillen adjunction of two variables* (see Definition 2.1.1 and 2.1.2). Then a *simplicial model category* (see Definition 2.1.6) is introduced as a model category with a Quillen adjunction of two variables which is compatible with the product in the category of simplicial sets. After a short discussion about equivalent formulations of a simplicial structure, we follow Dugger and show that under some conditions the verification of a simplicial structure can be simplified.

In Section 2.2 we give the definition of categorical simplicial action on $\mathcal{M}^{\Delta}$ and discuss its properties. In particular, we will see that although in general a categorical simplicial action does not induce a simplicial model structure, it does so, if the fibrant objects in $\mathcal{M}^{\Delta}$ are the simplicial resolutions. (see Proposition 2.2.5 and Proposition 2.2.7) This important observation is due to Dugger [Dug01a] and leads us to the idea to construct a new model structure where the fibrant objects are the simplicial resolutions. The method we will use is the left Bousfield localization.

In Section 2.3 we recall the basic properties of a *left Bousfield localization.* We cite a theorem due to Hischhorn [Hir03], which states that model categories of type *LpCe* are closed under the left Bousfield localization (see Theorem 2.3.7). Another theorem due to Smith [Smi] guarantees that the same is true for model categories of type *LpCo.* (see Theorem 2.3.8). Two classes of maps are introduced, the class of *hocolim-equivalences* (see Definition 2.3.9) of Dugger and the class of *realization weak equivalences* (see Definition 2.3.10) as in [RSS01]. At the end of the section, we give the definition of a certain set S at which we will localize in the next section.

In Section 2.4 we verify in Theorem 2.4.2 that the class of hocolim-equivalences equals the class of realization weak equivalences and that they become exactly the weak equivalences in the localized model structure. Moreover, the theorem also reveals that the localized model structure has the desired property that the fibrant objects are the simplicial resolutions. Equipped with these result we prove Theorem 2.4.4 which is due to Dugger, which states that the localized model structure on $\mathcal{M}^{\Delta^{op}}$ is simplicial and Quillen equivalent to $\mathcal{M}$.

2.1 Simplical Model Categories

In this section we introduce the class of simplicial model categories, but before we can do this we have to extend the concept of Quillen functors to bifunctors.

Definition 2.1.1 (Adjunction of two variables)**.** Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{N}$ be categories. An *adjunction of two variables* from $\mathcal{M}_1 \times \mathcal{M}_2$ to $\mathcal{N}$ is a tuple $(\otimes, \mathrm{Hom}_l, \mathrm{Hom}_r, \phi_l, \phi_r)$, such that:

1. $\otimes$, Hom_l and Hom_r are functors:

$$\otimes : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{N},$$

$$\mathrm{Hom}_l : \mathcal{M}_1^{op} \times \mathcal{N} \to \mathcal{M}_2,$$

$$\mathrm{Hom}_r : \mathcal{M}_2^{op} \times \mathcal{N} \to \mathcal{M}_1$$

   and

2. ϕ_l and ϕ_r are natural isomorphisms, such that for all $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$, $n \in \mathcal{N}$ we have:

$$\mathcal{M}_2(m_2, \mathrm{Hom}_l(m_1, n)) \xleftarrow[\cong]{\varphi_l} \mathcal{N}(m_1 \otimes m_2, n) \xrightarrow[\cong]{\varphi_r} \mathcal{M}_1(m_1, \mathrm{Hom}_r(m_2, n))$$

In other words, for every $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$ we have two pairs of adjoint functors:

$$m_1 \otimes - : \mathcal{M}_2 \longleftrightarrow \mathcal{N} : \mathrm{Hom}_l(m_1, -)$$

$$- \otimes m_2 : \mathcal{M}_1 \longleftrightarrow \mathcal{N} : \mathrm{Hom}_r(m_2, -)$$

If $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{N}$ are model categories, we can define

Definition 2.1.2. (Quillen adjunction of two variables) Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{N}$ be model categories. A *Quillen adjunction of two variables* is an adjunction of two variables $(\otimes, \mathrm{Hom}_l, \mathrm{Hom}_r, \phi_l, \phi_r)$ such that for cofibrations $f\colon A \to B$ and $j\colon K \to L$ in $\mathcal{M}_2$ and $\mathcal{M}_1$ respectively, the induced map:

$$f \Box j\colon (A \otimes L) \coprod_{A \otimes K} (B \otimes K) \to B \otimes L$$

is a cofibration in $\mathcal{N}$ which is a weak equivalence if either f or j is.

Notation 2.1.3. For brevity we will often abuse notation and use the term *"bifunctor $\otimes$"* when we mean "the adjunction of two variables $(\otimes, \mathrm{Hom}_l, \mathrm{Hom}_r, \phi_l, \phi_r)$" and use the term *"Quillen bifunctor $\otimes$"* when we mean "Quillen adjunction of two variables $(\otimes, \mathrm{Hom}_l, \mathrm{Hom}_r, \phi_l, \phi_r)$".

Lemma 2.1.4. *Given model categories $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{N}$, let $\otimes$ be a bifunctor from $\mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{N}$. Then the following are equilvant:*

1. *The bifunctor $\otimes$ is a Quillen bifunctor,*

2. *For any cofibration $f\colon A \to B$ in $\mathcal{M}_1$ and any fibration $p\colon X \to Y$ in $\mathcal{N}$, the induced map*

$$\mathrm{Hom}_{l,\Box}(f, p) : \mathrm{Hom}_l(B, X) \to \mathrm{Hom}_l(B, Y) \times_{\mathrm{Hom}_l(A,Y)} \mathrm{Hom}_l(A, X)$$

is a fibration in $\mathcal{M}_2$ which is a weak equivalence if either f or p is.

3. *For any cofibration $j\colon K \to L$ in $\mathcal{M}_2$ and any fibration $p\colon X \to Y$ in $\mathcal{N}$, the induced map*

$$\mathrm{Hom}_{r,\square}(j,p) : \mathrm{Hom}_r(L,X) \to \mathrm{Hom}_r(L,Y) \times_{\mathrm{Hom}_r(K,Y)} \mathrm{Hom}_r(K,X)$$

is a fibration in $\mathcal{M}_1$ which is a weak equivalence if either j or p is.

This lemma can also be found in Hovey's book [Hov99, Lemma 4.2.2.]. Here we present a method which Hirschhorn used in his book [Hir03] in order to prove other results. Since we find the idea not really obvious, we hope that the reader find our proof valuable.

Proof. In the first part of the proof we show that the claims as in the lemma are equivalent to requiring the existence of an arrow in certain diagrams. In the second part we verify that if one of these diagrams exists, then all of them exist.

Part 1

By definition $\otimes$ is a Quillen functor if and only if for any cofibration f in $\mathcal{M}_1$ and any cofibration j in $\mathcal{M}_2$, the map $f\square j$ is a cofibration in $\mathcal{N}$, which is even a trivial cofibration, if f or j is. This means that for any two cofibrations f and j and every fibration p in $\mathcal{N}$, there exists a dotted arrow in the solid arrow diagram in $\mathcal{N}$ of the form

$$\begin{tikzcd} A \otimes L \coprod_{A\otimes K} B \otimes K \arrow[r] \arrow[d, "f\square j"'] & X \arrow[d, "p"] \\ B \otimes L \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

if in addition one of the maps f and j or p is a weak equivalence.

If f is a cofibration in $\mathcal{M}_1$, j is a cofibration in $\mathcal{M}_2$ and p is a fibration in $\mathcal{N}$, then one proves in a similar manner that the second claim in the lemma is equivalent to requiring the existence of the dotted arrow in every solid arrow diagram in $\mathcal{M}_2$ of the form

$$\begin{tikzcd} K \arrow[r] \arrow[d, "j"'] & \mathrm{Hom}_l(B,X) \arrow[d, "\mathrm{Hom}_{l,\square}(f,p)"] \\ L \arrow[r] \arrow[ur, dotted] & \mathrm{Hom}_l(B,Y) \times_{\mathrm{Hom}_l(A,Y)} \mathrm{Hom}_l(A,X) \end{tikzcd} ,$$

where either one of the maps f and p is weak equivalence, or j is a trivial cofibration.

Furthermore, under the same condition on the maps f, j and p the third claim in the lemma is equivalent to requiring the existence of the dotted arrow in every solid arrow diagram in $\mathcal{M}_1$ of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & \mathrm{Hom}_r(L,X) \arrow[d, "\mathrm{Hom}_{r,\square}(j,p)"] \\ B \arrow[r] \arrow[ur, dotted] & \mathrm{Hom}_r(L,Y) \times_{\mathrm{Hom}_r(K,Y)} \mathrm{Hom}_r(K,X) \end{tikzcd}$$

where either one of the maps j and p is weak equivalence, or f is a trivial cofibration.

Part 2

Now, we show that the dotted arrow exists in the first diagram if and only if it exists in the second diagram. Indeed, the first diagram is equilvant to the surjectivity of the following map in the category of sets:

$$\mathcal{N}(B \otimes L, X) \to \mathcal{N}(A \otimes L \coprod_{A\otimes K} B \otimes K, X) \times_{\mathcal{N}(A\otimes L \coprod_{A\otimes K} B\otimes K, Y)} \mathcal{N}(B \otimes L, Y).$$

Using the natural isomorphism φ_l and the property that the functor $\mathcal{N}(-,-) : \mathcal{N}^{op} \times \mathcal{N} \to Set$ commutes with the product, the codomain of the surjective map can be written as the pullback of the following diagram:

$$\begin{tikzcd}
\mathcal{M}_2(L, \mathrm{Hom}_l(A,X)) \arrow[r] \arrow[d] & \mathcal{M}_2(K, \mathrm{Hom}_l(A,X)) \arrow[d] & \mathcal{M}_2(K, \mathrm{Hom}_l(B,X)) \arrow[l] \arrow[d] \\
\mathcal{M}_2(L, \mathrm{Hom}_l(A,Y)) \arrow[r] & \mathcal{M}_2(K, \mathrm{Hom}_l(A,Y)) & \mathcal{M}_2(K, \mathrm{Hom}_l(B,Y)) \arrow[l] \\
& \mathcal{M}_2(L, \mathrm{Hom}_l(B,Y)) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

This is the same as $E \times_G F$, where

$$\begin{aligned}
E &:= \mathcal{M}_2(L, \mathrm{Hom}_l(A,X) \times_{\mathrm{Hom}_l(A,Y)} \mathrm{Hom}_l(B,Y)) \\
F &:= \mathcal{M}_2(K, \mathrm{Hom}_l(B,X)) \\
G &:= \mathcal{M}_2(K, \mathrm{Hom}_l(A,X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B,Y)).
\end{aligned}$$

Thus, to give a surjective map as above is the same as giving the surjective map

$$\mathcal{M}_2(L, \mathrm{Hom}_l(B,X)) \to E \times_G F$$

which is equilvant to the second diagram.

Now the equivalence of first and the second claim in the lemma is obvious. The equivalence between second and the third claim can be proven in a similar way. □

The following inconspicuous lemma which can also be found in [Hov99] will become very useful in the last chapter.

Lemma 2.1.5. *As in the lemma above, let $\otimes$ be a bifunctor from $\mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{N}$ and let J, J' and I be sets of maps in $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{N}$. If $J \Box J' \subseteq I$, then J-cof $\Box$ J'-cof $\subseteq I$.*

Proof. Since $J \Box J' \subseteq I \subseteq I$-cof, $(J \Box J', I\text{-inj})$ has the lifting property and by adjointness, the set $(J, \mathrm{Hom}_r(J', I\text{-inj}))$ has the lifting property as well. Then by definition of J-cof, $(J\text{-cof}, \mathrm{Hom}_r(J', I\text{-inj}))$ has the lifting property as well. Using the adjointness once more, we have that $(J\text{-cof} \Box J', I\text{-inj})$ has the lifting property and this implies that J-cof $\Box J' \subseteq I$-cof. Now, by using the functor Hom_l one readily verifies the inclusion J-cof $\Box$ J'-cof $\subseteq I$. □

Definition 2.1.6 (Simplicial model category)**.** Let $\mathcal{S}$ denote the category of simplicial sets. A *simplicial model category* is a model category $\mathcal{M}$ with a bifunctor $\otimes$: $\mathcal{M} \times \mathcal{S} \to \mathcal{M}$, which is

1. $SM1)$ a cartesian action, i.e. it is compatible with the product in $\mathcal{S}$, and
2. $SM2)$ a Quillen bifunctor.

More explicit, $SM2)$ says that for any cofibration f in $\mathcal{M}$ and any cofibration j in $\mathcal{S}$ the following holds:

1. $SM2)1)$ $f\Box j$ is a cofibration,
2. $SM2)2)$ $f\Box j$ is a trivial cofibration, if f is, and
3. $SM2)3)$ $f\Box j$ is a trivial cofibration, if j is a trivial cofibration.

We will call the axioms $SM2)1)$ to $SM2)3)$ the *simplical model axioms*.

In the case of a simplicial model category the common notations for the functors Hom_l and Hom_r are:

$$\mathrm{Map}(X,Y) := \mathrm{Hom}_l(X,Y) \text{ and } X^K := \mathrm{Hom}_r(K,X),\ \forall X,Y \in \mathcal{M}, K \in \mathcal{S}.$$

For the maps one often uses $p^{\Box j}$ instead of $\mathrm{Hom}_{r,\Box}(j,p)$ and $\mathrm{Map}(f,p)$ instead of $\mathrm{Hom}_{l,\Box}(f,p)$. In the sequel, we will abuse notation and use the same terminology even if the model category is not simplicial.

Remark 2.1.7.

1. Lemma 2.1.4 says that in case of simplicial model categories the condition SM2) has an equivalent dual version.

   $SM2)'$ If $p\colon X \to Y$ is a fibration in $\mathcal{M}$ and $j\colon K \to L$ a cofibration in $\mathcal{S}$, then the induced map: $p^{\Box j} : X^L \to X^K \times_{Y^K} Y^L$ is

   (a') a trivial fibration, if p is.

   (b') It is a fibration, and

   (c') It is a trivial fibration, if j is.

2. In a simplicial model category $\mathcal{M}$ the functor $\otimes$ is by definition a Quillen functor, therefore we have

   (a) for any cofibrant object $A \in \mathcal{M}$ the Quillen pair:

   $$A \otimes - : \mathcal{S} \leftrightarrow \mathcal{M} : \mathrm{Map}(A,-),$$

   (b) for any object $K \in \mathcal{S}$ (every object in $\mathcal{S}$ is cofibrant) the Quillen pair:

   $$- \otimes K : \mathcal{M} \leftrightarrow \mathcal{M} : (-)^K,$$

   (c) for any fibrant object $X \in \mathcal{M}$ the Quillen pair:

   $$X^- : \mathcal{S} \leftrightarrow \mathcal{M}^{op} : \mathrm{Map}(-,X).$$

3. A close examination of the proof of Lemma 2.1.4 reveals that for all (trivial) cofibrations j the map $\mathrm{Hom}_{r,\Box}(j,p)$ has the right lifting property with respect to the map f if and only if

the map $\mathrm{Hom}_{l,\Box}(f,p)$ has the right lifting property with respect to all (trivial) cofibrations. Often $\mathcal{M}_2$ will be a cofibrantly generated model category, in which case it is only necessary to check the lifting property with respect to the set of generating (trivial) cofibrations.

As one can imagine the properties of being a left Quillen functor in two variables are related to each other. The following propositions are due to Dugger and they show that under certain assumptions checking SM2)2) or SM2)3) can be reduced to checking that certain classes of maps are weak equivalences.

Proposition 2.1.8. *Suppose $\mathcal{M}$ is a model category and $\otimes$ is a bifunctor. If $SM2)1)$ holds, then $SM2)2)$ is equivalent to requiring that for any trivial cofibration $f\colon A \to B$ and any $K \in \mathcal{S}$, the induced map $f \otimes id_K : A \otimes K \to B \otimes K$ is a trivial cofibration.*

Proof. If $f\colon A \to B$ is a trivial cofibration in $\mathcal{M}$ and $K \to L$ a cofibration in $\mathcal{S}$, then there is a commuting diagram in $\mathcal{M}$:

$$\begin{tikzcd}
A \otimes K \arrow[r, "f\otimes \mathrm{id}_K", "\cong"'] \arrow[d] & B \otimes K \arrow[d] \arrow[ddr, bend left] & \\
A \otimes L \arrow[r, "h"] \arrow[drr, bend right, "\cong", "f\otimes \mathrm{id}_L"'] & A \otimes L \coprod_{A\otimes K} B \otimes K \arrow[dr, dotted, "f\Box j"] & \\
 & & B \otimes L
\end{tikzcd}$$

The maps $f \otimes id_K$ and $f \otimes id_L$ are weak equivalences by assumption. Since every object in $\mathcal{S}$ is cofibrant, $SM2)1)$ implies that $f \otimes id_K$ is even a trivial cofibration. As a pushout of a trivial cofibration, the map h is also a trivial cofibration and therefore in particular a weak equivalence. By $SM2)1)$ $f\Box j$ is a cofibration and the claim follows form the two-of-three axiom.

Conversely, if we denote the map $\emptyset \to K$ in $\mathcal{S}$ by i_K, the map $f \otimes id_K$ is exactly the map $f\Box i_K$ which is a trivial cofibration by SM2)2). $\Box$

Corollary 2.1.9. *With the assumptions as in the proposition above, $SM2)2)$ holds if and only if the functors $(-)^K$ preserve fibrations between fibrant objects.*

Proof. The proposition above shows that requiring that $SM2)2)$ holds is equivalent to requiring that the functor $- \otimes K$ preserves the trivial cofibrations. By the following adjointness

$$- \otimes K\colon \mathcal{M} \longleftrightarrow \mathcal{M} :(-)^K,$$

the functor $- \otimes K$ preserves the trivial cofibrations if and only if the functor $(-)^K$ preserves the fibrations. It is a standard result that if a right adjoint functor preserves fibrations between fibrant objects, then it preserves all fibrations. $\Box$

Proposition 2.1.10. *[Dug01a, Proposition 3.2.b)] Let $\mathcal{M}$ be a left proper model category and let $\otimes$ be a bifunctor such that $SM2)1)$ and $SM2)2)$ hold. The verification of $SM2)3)$ is equivalent to the requirement that if $j\colon K \to L$ is a trivial cofibration in $\mathcal{S}$ and Z fibrant in $\mathcal{M}$, then the map $g\colon Z^L \to Z^K$ is a weak equivalence.*

Proof. Proving $SM2)3)$ is the same as proving its dual version which states that if p is a fibration and j is a trivial cofibration, then $p^{\Box j}$ is a trivial fibration. This is equivalent to requiring that for every cofibration f in $\mathcal{M}$ $(f, p^{\Box j})$ has the lifting property, or by adjointness $(f\Box j, p)$ has the lifting property. By left properness, according to Proposition 1.8.5, it is enough to verify the lifting property with respect to every cofibrant approximation $\tilde{f}\colon \tilde{A} \to \tilde{B}$ of $f\colon A \to B$ such that $\tilde{f}$ is also a cofibration. Since $SM2)1)$ is satisfied, the map $\tilde{f}\Box j$ is a cofibration. If it is also a weak equivalence, then it is a trivial cofibration, hence $(\tilde{f}\Box j, p)$ has the lifting property and we are done.

By Theorem 1.7.6, checking the weak equivalence of the map $\tilde{f}\Box j$ can be reduced to show the induced map of simplicial sets

$$\operatorname{map}(\tilde{f}\Box j, Z)\colon\ \operatorname{map}(\tilde{B}\otimes L, Z) \to \operatorname{map}(\tilde{A}\otimes L \coprod_{\tilde{A}\otimes K} \tilde{B}\otimes K, Z)$$

is a weak equivalence between homotopy function complexes for every fibrant object Z in $\mathcal{M}$. Theorem 1.7.5 allows us to verify the weak equivalece just for a special right homotopy function complex. Therefore, we have to check that the map $\mathcal{M}(\tilde{B}\otimes L, \underline{Z}) \to \mathcal{M}(\tilde{A}\otimes L \coprod_{\tilde{A}\otimes K} \tilde{B}\otimes K, \underline{Z})$ is a weak equvialence, where $\underline{Z}$ is a simplicial resolution of Z.

Note if $\underline{Z}^L$ is defined by $\underline{Z}^L(n) = \underline{Z}(n)^L$, then for all n the map $\underline{Z}^L(n) \to M_n\underline{Z}^L \cong (M_n\underline{Z})^L$ is a fibration by the dual version of $SM2)2)$ and for all m and n the map $\underline{Z}^L(m) \to \underline{Z}^L(n)$ is a weak equivalence between fibrant objects by assumption. Therefore, if $\underline{Z}$ is a simplicial resolution of Z, $\underline{Z}^L$ is a simplicial resolution of Z^L.

Using the adjointness again, the map in question is naturally isomorphic to

$$\mathcal{M}(\tilde{B}, \underline{Z}^L) \to \mathcal{M}(\tilde{A}, \underline{Z}^L) \times_{\mathcal{M}(\tilde{A},\underline{Z}^K)} \mathcal{M}(\tilde{B}, \underline{Z}^K)$$

which is the map φ in the diagram below

$$\begin{tikzcd}
\mathcal{M}(\tilde{B}, \underline{Z}^L) \arrow[dr, dotted, "\varphi"] \arrow[drr, bend left, "\mathcal{M}(\tilde{B},g)"] \arrow[ddr, bend right] & & \\
& \mathcal{M}(\tilde{A}, \underline{Z}^L) \times_{\mathcal{M}(\tilde{A},\underline{Z}^K)} \mathcal{M}(\tilde{B}, \underline{Z}^K) \arrow[r] \arrow[d] & \mathcal{M}(\tilde{B}, \underline{Z}^K) \arrow[d] \\
& \mathcal{M}(\tilde{A}, \underline{Z}^L) \arrow[r, "\mathcal{M}(\tilde{A},g)"'] & \mathcal{M}(\tilde{A}, \underline{Z}^K)
\end{tikzcd}$$

The map g is a fibration by $SM2)2)'$ and a weak equvialence by assumption, the maps $\mathcal{M}(\tilde{A}, g)$ and $\mathcal{M}(\tilde{B}, g)$ are trivial fibrations. As a pullback of a trivial fibration the map $\mathcal{M}(\tilde{A}, \underline{Z}^L) \times_{\mathcal{M}(\tilde{A},\underline{Z}^K)} \mathcal{M}(\tilde{B}, \underline{Z}^K) \to \mathcal{M}(\tilde{B}, \underline{Z}^K)$ is a trivial fibration too. The two-of-three axiom guarantees that the map φ is a weak equivalence, this is what we want to show. □

Remark 2.1.11. Inspired by the proposition above, we want to construct a model structure, such that for all fibrant objects Z the induced map $Z^L \to Z^K$ is a weak equivalence. Later, Proposition 2.2.7 shows that the class of simplicial resolutions in $\mathcal{M}^{\Delta^{op}}$ are good candidate of being fibrant objects in the new model structure. At the end of this chapter, we will use the localization in order to define this new model structure.

2.2 The Categorical Simplicial Action

For any complete and cocomplete category $\mathcal{M}$ we can define a categorical simplicial action on the category of Δ^{op}-diagrams in $\mathcal{M}$. If $\mathcal{M}$ is a model category, the constructed category $\mathcal{M}^{\Delta^{op}}$ comes with a Reedy model structure induced by the Reedy category structure on Δ^{op} (see Proposition 1.5.13 and Proposition 1.5.16). In this section we will discuss the properties of the categorical simplicial action and the relation between the Reedy model structure and the simplicial model axioms.

First we introduce some useful notations:

Let $\mathcal{T}$ be a complete and cocomplete category.

1. For $S \in Set$ and $X \in \mathcal{T}$, we denote $X \cdot S$ the coproduct $\coprod_{s\in S} X$ in $\mathcal{T}$, i.e. for each element of S, there is a copy of X in the coproduct.
2. Dually, let $X^{\cdot S}$ denote the product $\prod_{s\in S} X$ in $\mathcal{T}$.
3. Given a small category $\mathcal{C}$, a $\mathcal{C}$-diagram X in $\mathcal{T}^{\mathcal{C}}$ and a simplicial set K in $Set^{\mathcal{C}^{op}}$, let $X \otimes K$ denote the resulting coend:
$$X \otimes_{\mathcal{C}} K := \mathrm{coeq}(\coprod_{c\to d\in\mathcal{C}} X(c)\cdot K(d) \rightrightarrows \coprod_{c\in\mathcal{C}} X(c)\cdot K(c)),$$
where coeq denotes the coequalizer. We will use the coend-notation as in [ML71],
$$X \otimes K = \int^{c} X(c)\cdot K(c)$$
4. Given a $\mathcal{C}$-diagram X in $\mathcal{T}^{\mathcal{C}}$ and a simplicial set K in $Set^{\mathcal{C}^{op}}$, let $\hom(K, X)$ denote the resulting end:
$$\hom(K, X) := \mathrm{eq}(\prod_{c\in\mathcal{C}} X(c)^{\cdot K(c)} \rightrightarrows \prod_{c\to d\in\mathcal{C}} X(d)^{\cdot K(c)}),$$
where eq denotes the equalizer. We will use the end-notation as in [ML71],
$$\hom(K, X) = \int_{c} X(c)^{\cdot K(c)}$$

This general construction allows us to define a *categorical simplicial action* on the category $\mathcal{T}^{\Delta^{op}}$.

Definition 2.2.1. Let $\mathcal{T}$ be a complete and cocomplete category. If X and Y are objects in $\mathcal{T}^{\Delta^{op}}$ and K is an object in $\mathcal{S}$, then

1. $X \otimes K$ is the object in $\mathcal{T}^{\Delta^{op}}$ which is given by $X(n)\cdot K(n)$ in degree n and has the obvious face and degeneracy maps.

2. X^K is the object in $\mathcal{T}^{\Delta^{op}}$ which is given by $\hom(K \times \Delta[n], X)$ in degree n and has the face and degeneracy maps induced by the coface and codegeneracy maps in Δ^{op}.

3. $\mathrm{Map}(X, Y)$ is the object in $\mathcal{S}$ whose n-simplices are $\mathcal{T}^{\Delta^{op}}(X \otimes \Delta[n], Y)$.

We will use the term *tensor* if we refer to the functor $\otimes$. Similarly, we will use the term *cotensor* if we refer to the functor $(-)^{(-)}$.

Lemma 2.2.2. *The tensor $\otimes$ as in the definition above induces a categorial simplicial action on $\mathcal{T}^{\Delta^{op}}$, i.e. the tensor $\otimes$: $\mathcal{T}^{\Delta^{op}} \times \mathcal{S} \to \mathcal{T}^{\Delta^{op}}$ is a cartesian action and an adjunction of two variables.*

Proof. The first requirement follows almost directly from the definition: In every degree n we have

$$\begin{aligned}
((X \otimes K) \otimes L)(n) &= (X \otimes K)(n) \cdot L(n) \\
&= (X(n) \cdot K(n)) \cdot L(n) \\
&= (\coprod_{K(n)} X(n)) \cdot L(n) \\
&\cong \coprod_{K(n) \times L(n)} X(n) \\
&= X(n) \cdot (K \times L)(n) \\
&= (X \otimes (K \times L))(n)
\end{aligned}$$

Therefore, the tensor $\otimes$ is compatible with taking products in $\mathcal{S}$. The natural isomorphism φ_l: $\mathcal{T}^{\Delta^{op}}(X \otimes K, Y) \to \mathcal{S}(K, \mathrm{Map}(X, Y))$ is given by a chain of isomorphisms

$$\begin{aligned}
\mathcal{T}^{\Delta^{op}}(X \otimes K, Y) &\cong \mathcal{T}^{\Delta^{op}}(X \otimes \mathrm{colim}_{(\Delta \downarrow K)} \Delta[m], Y) \\
&\cong \mathcal{T}^{\Delta^{op}}(X \otimes \int^m \coprod_{K(m)} \Delta[m], Y) \\
&\cong \int_m \prod_{K(m)} \mathcal{T}^{\Delta^{op}}(X \otimes \Delta[m], Y) \\
&\cong \int_m \mathit{Set}(K(m), \mathcal{T}^{\Delta^{op}}(X \otimes \Delta[m], Y)) \\
&\cong \mathcal{S}(K, \mathcal{T}^{\Delta^{op}}(X \otimes \Delta[-], Y)) \\
&= \mathcal{S}(K, \mathrm{Map}(X, Y))
\end{aligned}$$

where we used, among other things, the isomorphism $K \cong \mathrm{colim}_{(\Delta \downarrow K)} \Delta[m] \cong \int^m \coprod_{K(m)} \Delta[m]$ and the presentation of the set of natural transformations between two functos as an end. Another chain of isomorphism provides the natural isomorphism φ_r: $\mathcal{T}^{\Delta^{op}}(X \otimes K, Y) \cong \mathcal{T}^{\Delta^{op}}(X, Y^K)$:

$$
\begin{aligned}
\mathcal{T}^{\Delta^{op}}(X, Y^K) &\cong \int_m \mathcal{T}(X(m), \hom(K \times \Delta[m], Y)) \\
&\cong \int_m \mathcal{T}(X(m), \int_m \prod_{K(n)} \prod_{\Delta[m](n)} Y(n)) \\
&\cong \int_m \int_n \mathcal{T}(X(m) \cdot \Delta[m](n), \prod_{K(n)} Y(n)) \\
&\cong \int_n \mathcal{T}(X(n), \prod_{K(n)} Y(n)) \\
&\cong \int_n \mathcal{T}(X(n) \cdot K(n), Y(n)) \\
&\cong \mathcal{T}^{\Delta^{op}}(X \otimes K, Y).
\end{aligned}
$$

□

For a model category $\mathcal{M}$, our hope now is that $\otimes$ is even a Quillen bifunctor on $\mathcal{M}^{\Delta^{op}}$. The next proposition shows that $SM2)1)$ and $SM2)2)$ are satisfied, but the last property of a simplicial model category structure seems not to be satisfied in general. It looks as if one has to localize in order to obtain this last property. Before we can give the proof of the proposition, we need a useful lemma.

Lemma 2.2.3. *1. $\forall X \in \mathcal{T}^{\Delta^{op}}, \hom(\Delta[n], X) \cong X_n$*

2. $\forall X \in \mathcal{T}^{\Delta^{op}}$, K, $L \in \mathcal{S}$, $\hom(L, X^K) \cong \hom(K \times L, X)$

Proof. 1. $\hom(\Delta[n], X) = \int_m X(m)^{\Delta[n](m)} = \int_m \prod_{\Delta(m,n)} X(m) \cong \lim_{\Delta^{op}\Delta[n]} X(m) \cong X(n)$.

2. The two functors from $\mathcal{S}^{op}$ to $\mathcal{T}^{\Delta^{op}}$, $X^{(-)}$ and $\hom(-, X)$, commute with colimits in $\mathcal{S}$. Moreover, any object in $\mathcal{S}$ is a colimit of special simplicial sets $\Delta[n]$. Therefore proving the natural isomorphism is reduced to proving

$$
\begin{aligned}
\hom(\Delta[m], X^{\Delta[n]}) &=_{a)} X^{\Delta[n](m)} \\
&\cong_{\text{def.}} \hom(\Delta[n] \times \Delta[m], X),
\end{aligned}
$$

which is exactly part a) and the definition.

□

Remark 2.2.4. Proposition 1.5.16 says that a map $f\colon\ X \to Y$ in $\mathcal{M}^{\Delta^{op}}$ is a Reedy (trivial) fibration if and only if the n-th matching map $X_n \to M_nX \times_{M_nY} Y_n$ is a (trivial) fibration for all n. Furthermore, the matching object M_nX is defined to be $\lim_{\partial(n\downarrow\overleftarrow{\Delta})} \iota^*X$. One easily verifies that it is equal to the coend $\int^m X^{\partial\Delta[n]} = \hom(\partial\Delta[n], X) = X^{\partial\Delta[n]}(0)$.

If we denote the inclusion of the boundary of the standard n-th simplex $\partial\Delta[n] \to \Delta[n]$ by i_n, then, because pushouts in $\mathcal{M}^{\Delta^{op}}$ are constructed componentwise, the n-th matching map of

$f\colon X \to Y$ is the map $f^{\square i_n}(0)\colon X^{\Delta[n]}(0) \to X^{\partial\Delta[n]}(0) \times_{Y^{\partial\Delta[n]}(0)} Y^{\Delta[n]}(0)$ and by definition this map is exactly the map: $\hom(\Delta[n], X) \to \hom(\partial\Delta[n], X) \times_{\hom(\partial\Delta[n],Y)} \hom(\Delta[n], Y)$.

As mentioned earlier, the Reedy model structure almost admits a simplicial model structure. This is exactly the statement of the following proposition. Our proof given here is adapted to our notation and is slightly different to the proof of [Dug01a].

Proposition 2.2.5. *Given a model category $\mathcal{M}$ and a Reedy model structure on $\mathcal{M}^{\Delta^{op}}$, the tensor $\otimes$: $\mathcal{M}^{\Delta^{op}} \times \mathcal{S} \to \mathcal{M}^{\Delta^{op}}$ as in the Definition 2.2.1 satisfies the properties $SM2)1)$ and $SM2)2)$.*

Proof. We will prove the dual version of the claim, which is easier than to prove it directly. Therefore, we need to show that for a fibration $p\colon X \to Y$ in $\mathcal{M}^{\Delta^{op}}$ and a cofibration $j\colon K \to L$ in $\mathcal{S}$, the map

$$p^{\square j}\colon X^L \to X^K \times_{Y^K} Y^L$$

is a fibration, which is even a trivial fibration if p is.

A map is a Reedy fibration in $\mathcal{M}^{\Delta^{op}}$ if and only if its matching maps are fibrations in $\mathcal{M}$ (see Proposition 1.5.16). By the remark above, the n-th matching map of $p^{\square j}$ is the map $(p^{\square j})^{\square i_n}(0)$, where i_n denotes the inclusion of the boundary $\partial\Delta[n] \to \Delta[n]$.

Therefore the map $p^{\square j}$ is a fibration if and only if for every trivial cofibration f in $\mathcal{M}$, $(f, (p^{\square j})^{\square i_n}(0))$ has the lifting property. By the adjointness of the pair (c, ev_0) this is equivalent to requiring that $(cf, (p^{\square j})^{\square i_n})$ has the lifting property. Using the adjointness of the tensor and the cotensor, it is equivalent to requiring $(cf\square i_n\square j, g)$ has the lifting property. Because the maps i_n are the generating cofibraions, it is only necessary to check that $(cf\square i_n, g)$ has the lifting property. Going back by using the same adjointness, we have proved that the n-th matching map $(p^{\square j})^{\square i_n}(0)$ is a fibration if and only if $(p^{\square i_n})(0)$ is a fibration, but this is clear, since it is the n-th matching map of the Reedy fibration g.

By assuming that the map f in the proof was a cofibration, the discussion above shows that $p^{\square j}$ is a trivial fibration if p is. □

Corollary 2.2.6. *Let $\mathcal{M}$ be a model category. For every simplicial set K the functors $(-)^K$ preserves the Reedy fibrations between Reedy fibrant objects in the Reedy model category $\mathcal{M}^{\Delta^{op}}$.*

Proof. According to the proposition above the Reedy model category $\mathcal{M}^{\Delta^{op}}$ satisfies property $SM2)2)$ and Corollary 2.1.9 says that is is equivlant to the fact that the functors $(-)^K$ preserves the Reedy fibrations between Reedy fibrant objects. □

Proposition 2.2.7. *[Dug01a, Proposition 2.2.6.] If $P \in \mathcal{M}^{\Delta^{op}}$ is a simplicial resolution $K \to L$ is a trivial cofibration in $\mathcal{S}$, then $P^L \to P^K$ is a Reedy trivial fibration between simplicial resolution.*

Proof. Proposition 2.2.5 shows that $SM2)2)$ is satisfied for the model category $\mathcal{M}^{\Delta^{op}}$. Therefore, for every trivial cofibration $K \to L$, the map $P^L \to P^K$ is a Reedy fibration in $\mathcal{M}^{\Delta^{op}}$. It only remains to show that the map in question is also a Reedy weak equivalence.

By definition, the n-th degree of this map is

$$\hom(L \times \Delta[n], P) \to \hom(K \times \Delta[n], P).$$

Since $K \times \Delta[n] \to L \times \Delta[n]$ is a trivial cofibration of simplicial sets in $\mathcal{S}$, using a standard result, the problem reduces to showing that $\forall n$, $\hom(\Delta[n], P) \to \hom(\Delta[0], P)$ is a weak equivalence. But this is obvious since it is naturally isomorphic to $P(n) \to P(0)$, which is a weak equivalence by Lemma 2.2.3 and the definition of a simplicial resolution.

Now one has to verify that both P^K and P^L are simplicial resolutions. We will do this for P^K, the proof for P^L works in the same way.

The map $P \to *$ is a Reedy fibration in $M^{\Delta^{op}}$, while $\emptyset \to K$ is a cofibration in $\mathcal{S}$, hence by SM2)2) P^K is a Reedy fibrant object. If the face and the degeneracy maps are weak equivalences then P^K is a simplicial resolution. That means that for all n, the maps

$$P^K(n) = \hom(K \times \Delta[n], P) \to \hom(K \times \Delta[m], P) = P^K(m)$$

have to be weak equivalences. In the first part of our proof, we saw that $P^{(-)}$ takes trivial cofibration in $\mathcal{S}$ to trivial fibration in $\mathcal{M}^{\Delta^{op}}$. In particular, the functor $\hom(-, P)$ which is the 0-th level of the functor $P^{(-)}$ takes trivial cofibration to trivial fibration in $\mathcal{M}$. Then, since $K \times \Delta[m] \to K \times \Delta[n]$ is a weak equivalence between cofibrant objects, using the Ken-Brown-Lemma [DHK97, 3.6] we see that the map in question is a weak equivalence indeed. □

Corollary 2.2.8. *If X is a simplicial resolution in $\mathcal{M}^{\Delta^{op}}$, then $X^{\Delta[-]}$ is a simplicial resolution of X.*

Proof. We have to check the following two properties:

1. X is Reedy fibrant. This is because the n-matching map of $X \to *$ has the form $X^{\Delta[-]}(n) \to M_n(X^{\Delta[-]})$. Since the functor $X^{\Delta[-]}$ is a right adjoint, it commutes with limits. Therefore the matching map is the map $X^{\Delta[n]} \to X^{\partial\Delta[n]}$, which is a fibration by Proposition 2.2.5. This observation reveals that the map $X \to *$ is a Reedy fibration and X is fibrant.

2. For proving the weak equivalence of the map $X^{\Delta[-]} \to X$, we note that $X^{\Delta[0]} \cong X$ and that the maps $X^{\Delta[n]} \to X^{\Delta[0]}$ are weak equivalences by Proposition 2.2.7.

□

The above corollary is needed for the following observation.

Corollary 2.2.9. *Let A be cofibrant in $\mathcal{M}^{\Delta^{op}}$ and let X be a simplicial resolution, then the simplicial set* $\mathrm{Map}(A, X)$ *as defined in Definition 2.2.1 is a homotopy function complex.*

Proof. We could assume that the homotopy function complex $\mathrm{map}(A, X)$ is the simplicial set $\mathcal{M}^{\Delta^{op}}(\tilde{A}, \underline{\hat{X}})$, where $\tilde{A}$ is cofibrant approximation of A and $\underline{\hat{X}}$ is simplicial resolution of X. Since A is already cofibrant, we assume further that A is its cofibrant approximation. Now the claim follows from the isomorphism:

$$\mathrm{Map}(A, X) = \mathcal{M}^{\Delta^{op}}(A \otimes \Delta[-], X) \cong \mathcal{M}^{\Delta^{op}}(A, X^{\Delta[-]}) = \mathcal{M}^{\Delta^{op}}(\tilde{A}, \underline{\hat{X}}) = \mathrm{map}(A, X).$$

□

2.3 A Hocolim Model Structure on Diagrams

We have seen that because of the lack of axiom $SM2)3)$ the Reedy model structur on $M^{\Delta^{op}}$ is not a simplicial one. By using the theory of localizations a more sophisticated model structure can be constructed. A. K. Bousfield studied a special kind of localization in his paper [Bou75] which is now called the Bousfield localization. Since some properties of the left Bousfield localization are essential for the existence of the simplicial replacement, this chapter is dedicated to a better understanding of this powerful tool.

In this section we will often use the same notation as in [Hir03].

Definition 2.3.1. Let $\mathcal{M}$ be a model category, let $Ho\mathcal{M}$ be its homotopy category and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$. A *left localization of* $\mathcal{M}$ *with respect to* $\mathcal{C}$ is a model category $L_{\mathcal{C}}\mathcal{M}$ together with a left Quillen functor $j\colon \mathcal{M} \to L_{\mathcal{C}}\mathcal{M}$ such that

1. For j, its total left derived functor $Lj\colon Ho\mathcal{M} \to HoL_{\mathcal{C}}\mathcal{M}$ (i.e. the right Kan extension of $\gamma_{L_{\mathcal{C}}\mathcal{M}} \circ j\colon \mathcal{M} \to HoL_{\mathcal{C}}\mathcal{M}$ along the localization functor $\gamma_{\mathcal{M}}\colon \mathcal{M} \to Ho\mathcal{M}$, for further information see [ML71] and [KS06]) takes the images in $Ho\mathcal{M}$ of elements of $\mathcal{C}$ into isomorphisms in $HoL_{\mathcal{C}}\mathcal{M}$ and

2. If $\mathcal{N}$ is a model category and $\varphi\colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor such that $L\varphi\colon Ho\mathcal{M} \to \mathcal{N}$ takes the images in $Ho\mathcal{M}$ of elements of $\mathcal{C}$ into isomorphisms in $Ho\mathcal{N}$, then there is a unique left Quillen functor $\delta\colon L_{\mathcal{C}}\mathcal{M} \to \mathcal{N}$ such that $\delta \circ j = \varphi$.

Remark 2.3.2. The definition makes clear that a localiziation of a model category is a construction that adds inverses for the images of elements of $\mathcal{C}$ in the homotopy category $Ho\mathcal{M}$, rather than a "usual" localization which adds inverses for maps in the underlying category. A standard result [Hir03] [Theorem 8.3.10] states that the homotopy category $Ho\mathcal{M}$ of any model category $\mathcal{M}$ is saturated, i.e. a map is weak equivalent in $\mathcal{M}$ if and only if it is isomorphic in $Ho\mathcal{M}$.

Loosely speaking, a localiziation of a model category is adding a class of weak equivalences called the $\mathcal{C}$-local equivalences, which are defined by the given class of maps $\mathcal{C}$. This leads us to the following definition.

Definition 2.3.3 ($\mathcal{C}$-local object and $\mathcal{C}$-local equivalence)**.** Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$.

1. An object W of $\mathcal{M}$ is $\mathcal{C}$*-local* if W is fibrant and for every element $f\colon A \to B$ of $\mathcal{C}$ the induced map of homotopy function complexes $f^*\colon \operatorname{map}(B, W) \to \operatorname{map}(A, W)$ is a weak equivalence.

2. A map $g\colon X \to Y$ in $\mathcal{M}$ is a $\mathcal{C}$*-local equivalence* if for every $\mathcal{C}$-local object W the induced map of homotopy function complexes $g^*\colon \operatorname{map}(Y, W) \to \operatorname{map}(X, W)$ is a weak equivalence.

Remark 2.3.4. Theorem 1.7.5 implies that if this is true for any homotopy function complex then it is true for every homotopy function complex.

A paraphrase of Definition 2.3.3 is that a fibrant object is $\mathcal{C}$-local if it makes every element f of $\mathcal{C}$ looks like a weak equivalence and a map is a $\mathcal{C}$-local equivalence if all $\mathcal{C}$-local objects make it look like a weak equivalence. Sometimes it is possible to construct a localization of a model category $\mathcal{M}$ which keeps the class of cofibrations.

Definition 2.3.5 (The left Bousfield localization)**.** Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a class of maps in $\mathcal{M}$. The left *left Bousfield localization* of $\mathcal{M}$ with respect to $\mathcal{C}$ is a model structure $L_{\mathcal{C}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that

1. the class of weak equivalences of $L_{\mathcal{C}}\mathcal{M}$ equals the class of $\mathcal{C}$-local equivalences of $\mathcal{M}$,
2. the class of cofibrations of $L_{\mathcal{C}}\mathcal{M}$ equals the class of cofibrations of $\mathcal{M}$, and
3. the class of fibrations of $L_{\mathcal{C}}\mathcal{M}$ is the class of maps with the right lifting property with respect to those maps that are both cofibrations and $\mathcal{C}$-local equivalences.

Remark 2.3.6.

1. A left Bousfield localization as in the above definition is really a localization of a model category in the sense of Definition 2.3.1.
2. A left Bousfield localization does not need to exist, but it is well-known that for a model category $\mathcal{M}$ which is either of type *LpCe* or *LpCo* and for any set S of maps in $\mathcal{M}$, the left Bousfield localization of $\mathcal{M}$ with respect to S always exists, which we will denote by $L_S\mathcal{M}$

The following existence theorem due to Hirschhorn is essential for us, since it shows that the class of model categories of type *LpCe* are closed under left Bousfield localization.

Theorem 2.3.7 (P. S. Hirschhorn)**.** *If $\mathcal{M}$ is a model category of type LpCe and if S is a set of morphisms in $\mathcal{M}$, then the left Bousfield localization $L_S\mathcal{M}$ of $\mathcal{M}$ exists and it has the following properties:*

1. *The model category $L_S\mathcal{M}$ is again of type LpCe.*
2. *The weak equivalences in $L_S\mathcal{M}$ are the S-local equivalences.*
3. *The cofibrations in $L_S\mathcal{M}$ are the cofibrations in $\mathcal{M}$.*
4. *The fibrant objects in $L_S\mathcal{M}$ are the S-local objects.*

Proof. See [Hir03, Theorem 4.1.1.]. □

Smith proved a similar theorem for model categories of type *LpCo* in his paper [Smi]. Since he has not published his paper yet, the version presented here is taken from [Bar10]

Theorem 2.3.8 (J. Smith)**.** *If $\mathcal{M}$ is a model category of type LpCo and if S is a set of morphisms in $\mathcal{M}$, then the left Bousfield localization $L_S\mathcal{M}$ of $\mathcal{M}$ exists and has the following properties:*

1. *The model category $L_S\mathcal{M}$ is again of type LpCo.*
2. *The weak equivalences in $L_S\mathcal{M}$ are the S-local equivalences.*
3. *The cofibrations in $L_S\mathcal{M}$ are the cofibrations in $\mathcal{M}$.*
4. *The fibrant objects in $L_S\mathcal{M}$ are the S-local objects.*

Proof. For a proof we refer the reader to [Bar10, Theorem 2.11.]. □

The idea now is to localize our category $\mathcal{M}^{\Delta^{op}}$ with respect to a set of maps. These maps then become weak equivalences after applying the homotopy colimit functor hocolim to them. Before we go any further, we want to recall some basic properties of the homotopy colimit.

Let $\mathcal{M}$ be a model category and $\mathcal{C}$ be a small category. We list some basic properties which can found in [DHK97]. For explicit formulas of homotopy colimits and homotopy limits of diagrams of simplicial sets the book [BK72] is a good source.

1. A homotopy colimit $\mathrm{hocolim}_{\mathcal{C}}$ is a functor which comes with a natural transformation
$$\mathrm{hocolim}_{\mathcal{C}} \to \mathrm{colim}_{\mathcal{C}}$$
such that $\mathrm{hocolim}_{\mathcal{C}}$ sends objectwise weak equivalences between objectwise cofibrant objects to weak equivalences.

2. Its total derived functor $Lhocolim_{\mathcal{C}}$ exists and there is a natural isomorphism $Lhocolim_{\mathcal{C}} \to Lcolim_{\mathcal{C}}$.

One can immediately see a problem arising in this context. For a small indexing category, a weak equivalence $X \to Y$ in $\mathcal{M}^{\mathcal{C}}$ induces a map $\mathrm{hocolim}\, X \to \mathrm{hocolim}\, Y$ in $\mathcal{M}$ which does not need to be a weak equivalence anymore. To overcome this difficulty, Dugger defines the *corrected homotopy colimit* "corhocolim" to be the composite $\mathrm{hocolim} \circ \tilde{C}$, where $\tilde{C}$ denotes a cofibrant approximation functor. With this definition corhocolim really preserves all weak equivalences.

Definition 2.3.9. [Dug01a, Hocolim-equivalence] Let $\mathcal{C}$ be a small category. A map $f\colon X \to Y$ in $\mathcal{M}^{\mathcal{C}}$ is a *hocolim-equivalence* if the induced map $\mathrm{corhocolim}\, X \to \mathrm{corhocolim}\, Y$ is a weak equivalence in $\mathcal{M}$.

In his paper [Dug01a] Dugger introduces the hocolim-equivalences and shows that that these maps become the weak equivalences in the localized model structure which will be our desired simplicial model structure. The authors of [RSS01] use other methods which we are going to describe in the last chapter. They do not use the hocolim-equivalences in their paper, but instead the realization weak equivalences which we define now.

Definition 2.3.10. [RSS01, Realization weak equvialence] Let c denotes the constant functor. A map f in $\mathcal{M}^{\Delta^{op}}$ is a realization weak equivalence, if the induced map $[f, cZ]^{HoReedy}$ is an isomorphism in the homotopy category $Ho(\mathcal{M}^{\Delta^{op}})$ for all objects Z in $\mathcal{M}$.

Remark 2.3.11. As mentioned before, the reason for introducing the hocolim-equivalences or the realization weak equivalences is the idea that these maps should be the weak equivalences in a new model structure of $\mathcal{M}^{\Delta^{op}}$. It is clear from the definitions that an objectwise equivalence is a hocolim-equivalence as well as a realization weak equivalence. The two conditions for being weak equivalences still have to be checked: they are closed under retracts and they satisfy the two-of-three axiom. Fortunately the verification is very easy.

If a map f is a retract of a hocolim-equivalence g, then corhocolim f is a retract of the weak equivalence corhocolim g and by the retract axiom corhocolim f is also a weak equivalence. Since corhocolim is a functor and since the weak equivalences satisfy the two-of-three axiom, the hocolim-equivalences also satisfy this axiom.

In the case of realization weak equivalences the argument works in a similar manner. If the map f is a retract of a realization weak equivalence g, then as a retract of the isomorphism

$[g, cZ]^{HoReedy}$, the map $[f, cZ]^{HoReedy}$ is also an isomorphism, hence f is also a realization weak equivalence. The two-of-three axiom is satisfied since $[-, cZ]^{HoReedy}$ is a functor and isomorphisms satisfy the two-of-three axiom.

Although the definitions of hocolim-equivalences and realization weak equivalences quite differ from each other at first sight, Theorem 2.4.2 reveals that they are actually equal. Before we can show the equality, we need some preparation. The next proposition is the first important observation which provides an equivalent description of realization weak equivalences by using the categorical simplicial action.

Proposition 2.3.12. *[RSS01] A map $f\colon A \to B$ is a realization weak equivalence in $\mathcal{M}^{\Delta^{op}}$ if and only if for every simplicial resolution Z in $\mathcal{M}^{\Delta^{op}}$ and every cofibrant approximation $\tilde{f}$ of f, the morphism* $\mathrm{Map}(\tilde{f}, Z)\colon \mathrm{Map}(\tilde{B}, Z) \to \mathrm{Map}(\tilde{A}, Z)$ *is a weak equivalence.*

Remark 2.3.13. In [RSS01, Proposition 8.9.] the hypothesis is slightly different to ours. There, the cofibrant approximation $\tilde{f}$ is required to be a cofibration. Note that for every map g, there exists a factorization $g = \hat{g} \circ \tilde{g}$, where $\tilde{g}$ is a cofibration and $\hat{g}$ a trivial fibration. Since $\mathrm{Map}(\hat{g}, Z)$ is always a weak equivalence, by the two-of-three axiom $\mathrm{Map}(g, Z)$ is a weak equivalence if and only if $\mathrm{Map}(\tilde{g}, Z)$ is. Therefore, Proposition 2.3.12 is equivalent to Proposition 8.9. in [RSS01].

Proof. Since f is a realization weak equivalence if and only if its cofibrant approximation $\tilde{f}$ is, we only need to prove that $\tilde{f}\colon \tilde{A} \to \tilde{B}$ is a weak equivalence if and only if $\mathrm{Map}(\tilde{f}, Z)\colon \mathrm{Map}(\tilde{B}, Z) \to \mathrm{Map}(\tilde{A}, Z)$ is a weak equivalence.

Observe that for a simplicial resolution Z of an object in $\mathcal{M}$, $\mathrm{Map}(\tilde{f}, Z)$ is a weak equivalence if and only if $\pi_0 \mathrm{Map}(K, \mathrm{Map}(\tilde{f}, Z)) \cong \pi_0 \mathrm{Map}(\tilde{f}, Z^K)$ is a bijection for all simplicial sets K. Then we have the following diagram:

$$\begin{tikzcd}
\pi_0 \mathrm{Map}(\tilde{B}, Z^K) \arrow[r, "\pi_0 \mathrm{Map}(\tilde{f}, Z^K)"] \arrow[d, "i_{\tilde{B}}"] & \pi_0 \mathrm{Map}(\tilde{A}, Z^K) \arrow[d, "i_{\tilde{A}}"] \\
{[\tilde{B}, Z^K]^{Ho(\mathcal{M}_R^{\Delta^{op}})}} \arrow[r] \arrow[d] & {[\tilde{A}, Z^K]^{Ho(\mathcal{M}_R^{\Delta^{op}})}} \arrow[d] \\
{[\tilde{B}, c({Z^K}_0)]^{Ho(\mathcal{M}_R^{\Delta^{op}})}} \arrow[r] & {[\tilde{A}, c({Z^K}_0)]^{Ho(\mathcal{M}_R^{\Delta^{op}})}}
\end{tikzcd}$$

The maps $i_{\tilde{A}}$ and $i_{\tilde{B}}$ are isomorphisms by Corollary 2.2.9 and [Hir03, Theorem 17.7.2], because $\tilde{A}$ and $\tilde{B}$ are cofibrant and Z^K is fibrant. If K is the terminal object $*$, then for every object X in $\mathcal{M}$ there exists a simplicial resolution Z of X and the weak equivalence $cX \to Z = Z*$ implies that if $\mathrm{Map}(\tilde{f}, Z)$ is a weak equivalence then $\tilde{f}$ is a realization weak equivalence.

According to Proposition 2.2.7, Z^K is a simplicial resolution and therefore the map $c({Z^K}_0) \to Z^K$ is a weak equivalence. This map induces the isomorphisms

$$[\tilde{A}, Z^K]^{Ho(\mathcal{M}_R^{\Delta^{op}})} \to [\tilde{A}, c({Z^K}_0)]^{Ho(\mathcal{M}_R^{\Delta^{op}})} \text{ and } [\tilde{B}, Z^K]^{Ho(\mathcal{M}_R^{\Delta^{op}})} \to [\tilde{B}, c({Z^K}_0)]^{Ho(\mathcal{M}_R^{\Delta^{op}})}.$$

Since ${Z^K}_0$ is an object in $\mathcal{M}$, the isomorphisms above and the commuting diagram show that if $\tilde{f}$ is realization weak equivalence, then $\mathrm{Map}(\tilde{f}, Z)\colon \mathrm{Map}(\tilde{B}, Z) \to \mathrm{Map}(\tilde{A}, Z)$ is a weak equivalence. $\square$

In order to use the general machinery of localization, we have to find the right set S we are going to localize at. The auxillary proposition below is needed for its definition.

Proposition 2.3.14. *[Dug01a] If $\mathcal{M}$ is left proper and cofibrantly generated, then there exists a set U such that a map $X \to Y$ is already a weak equivalence provided the induced map of homotopy function complexes* $\operatorname{map}(A, X) \to \operatorname{map}(A, Y)$ *is a weak equivalence for every object A in U.*

Proof. The reader is refered to [Dug01a, Proposition A.5.]. □

Definition 2.3.15. Let U be a set as in the proposition above. In the sequel, let S denote the set of maps of the form $F_j^A \to F_i^A$ in $\mathcal{M}^{\Delta^{op}}$ (see Definition 1.4.2), for every $i \to j$ and every A in U.

Equipped with all these auxiliary results we are going to prove one of the main theorems in the chapter.

2.4 A Hocolim Structure for Reedy Model Categories

In this section, we localize the Reedy model category $\mathcal{M}^{\Delta^{op}}$ with respect to the set of maps S and we derive important properties.

The following proposition will be used in the proof of the second part of Theorem 2.4.2.

Proposition 2.4.1. *[DHK97, Proposition 58.1.] Suppose $\mathcal{M}$ is a model category and suppose $\mathcal{C}$ is a small category, then the pair of adjoint functor* (colim, c) *induces an adjoint pair of total derived functors:*

$$L\operatorname{colim} : \operatorname{Ho}\mathcal{M}^{\mathcal{C}} \longleftrightarrow \operatorname{Ho}\mathcal{M} : Rc,$$

where $L\operatorname{colim}$ *and Rc denote the total left and the total right derived functor of* colim *and c respectively.*

Theorem 2.4.2. *[Dug01a] Let S be a set of maps as defined in Definition 2.3.15. Suppose $\mathcal{M}$ is a model category of type LpCe or LpCo, then*

1. *The fibrant objects in the localized model category $L_S(\mathcal{M}^{\Delta^{op}})_R$ are the simplicial resolutions.*
2. *The weak equivalences in $L_S(\mathcal{M}^{\Delta^{op}})_R$ are the* hocolim*-equivalences.*

Remark 2.4.3. The first statement is proved [Dug01a, Theorem 5.2.] in case $\mathcal{M}^{\Delta^{op}}$ is equipped with the Bousfield-Kan model structure. Compared with Dugger's proof of the second statement, we will not use the Bousfiel-Kan model structure as a stepping stone.

Proof. 1. By Corollary 1.9.4 and Corollary 1.9.11, if $\mathcal{M}$ is of type *LpCe* or *LpCo*, then the Reedy model category $\mathcal{M}^{\Delta^{op}}$ is of the same type. Therefore Theorem 2.3.7 and Theorem 2.3.8 tell us that the left Bousfield localization of $\mathcal{M}^{\Delta^{op}}$ with respect to a set S always exists and that an object X is fibrant in $L_S(\mathcal{M}^{\Delta^{op}})_R$ if and only if it is an S-local object in $\mathcal{M}^{\Delta^{op}}$. By the definition of S-local objects, the object X has to be Reedy fibrant and for every map $F_j^A \to F_i^A$ in S, the induced map of homotopy function complexes

$$\operatorname{map}(F_i^A, X) \to \operatorname{map}(F_j^A, X)$$

has to be a weak equivalence.

We can assume that the homotopy function complexe $\mathrm{map}(F_i^A, X)$ is the right homotopy function complexes $\mathcal{M}(F_i^A, \hat{X})$, for a simplicial resolution $\hat{X}$ of X. Note that A is cofibrant, therefore according to remark 1.5.17 the adjoint pair of functors (F_i^-, ev_i) is a Quillen pair. Hence, the object F_i^A is cofibrant in $\mathcal{M}^{\Delta^{op}}$ and there is no need to use a cofibrant replacement of F_i^A. Furthermore the adjointness induces an isomorphism on each degree

$$\mathcal{M}^{\Delta^{op}}(F_i^A, \hat{X})(n) = \mathcal{M}^{\Delta^{op}}(F_i^A, \hat{X}(n)) \cong \mathcal{M}(A, \hat{X}(i)(n)) = \mathcal{M}(A, \hat{X}(i))(n).$$

This shows that the homotopy function complex $\mathcal{M}^{\Delta^{op}}(F_i^A, \hat{X})$ is weak equivalent to the homotopy function complex $\mathcal{M}(A, \hat{X}(i))$, where $\hat{X}$ is a simplicial resolution of X. This means that the maps we are interested in are weak equivalences if and only if the maps $\mathrm{map}(A, X(i)) \to \mathrm{map}(A, X(j))$ are. By the definition of the set W, this is equivalent to requiring the map $X(i) \to X(j)$ to be a weak equivalence for all $i \to j$.

2. By definiton a map $f\colon A \to B$ is a hocolim-equivalence in $\mathcal{M}^{\Delta^{op}}$ if and only if the induced map $\mathrm{corhocolim}\, A \to \mathrm{corhocolim}\, B$ is a weak equivalence in $\mathcal{M}$. If $\tilde{f}\colon \tilde{A} \to \tilde{B}$ denotes a cofibrant approximation of f, then the map $\mathrm{corhocolim}\, f\colon \mathrm{corhocolim}\, A \to \mathrm{corhocolim}\, B$ equals $\mathrm{hocolim}\, \tilde{f}\colon \mathrm{hocolim}\, \tilde{A} \to \mathrm{hocolim}\, \tilde{B}$.

   Thus, f is a hocolim-equivalence if and only if hocolim $\tilde{f}$ is a weak equivalence in $\mathcal{M}$, which is the same as requiring $[\mathrm{hocolim}\ \tilde{f}, Z]_{Ho(\mathcal{M})}$ to be an isomorphism for every fibrant object Z in $\mathcal{M}$.

   Since the total left derived functor of hocolim is naturally isomorphic to the total left derived functor of colim which is left adjoint to the constant functor c by the proposition above, $[\mathrm{hocolim}\ \tilde{f}, Z]_{Ho(\mathcal{M})}$ is isomorphic to

$$[\tilde{f}, cZ]_{HoReedy}: \ [\tilde{B}, cZ]_{HoReedy} \to [\tilde{A}, cZ]_{HoReedy}.$$

   This observation reveals that f is a hocolim-equivalences if and only if $\tilde{f}$ is a realization weak equivalence. Furthermore, by Proposition 2.3.12 this is the same as requiring that the morphism

$$\mathrm{Map}(\tilde{f}, X): \ \mathrm{Map}(\tilde{B}, X) \to \mathrm{Map}(\tilde{A}, X)$$

   is a weak equivalence for every simplicial resolution X.

   Since $\tilde{A}$ and $\tilde{B}$ are cofibrant and X is fibrant, by Corollary 2.2.9 the simplicial sets $\mathrm{Map}(\tilde{A}, X)$ and $\mathrm{Map}(\tilde{B}, X)$ are homotopy function complexes. Recall that the simplicial resolutions are the S-local objects, therefore f is a hocolim-equivalence if and only if f is an S-local equivalence.

□

We summarize: If $\mathcal{M}$ is a model category of type $(LpCe)$ or $(LpCo)$, then the theory of the left Bousfield localization and Theorem 2.4.2 tell us that it is always possible to define a model structure on $\mathcal{M}^{\Delta^{op}}$ such that

1. the weak equivalences are the hocolim-equivalences,
2. the cofibrations are Reedy cofibrations, and

3. the fibrations are the maps which have the right-lifting-property with respect to all trivial cofibrations.

As above, we will denote this category by $L_S(\mathcal{M}^{\Delta^{op}})_R$ and by using its model structure, we are going to prove the main theorem.

Theorem 2.4.4. *[Dug01a] The categorial simplicial structure makes $L_S(\mathcal{M}^{\Delta^{op}})_R$ into a simplicial model category, for which the usual adjoint pair*

$$c\colon \mathcal{M} \longleftrightarrow L_S(\mathcal{M}^{\Delta^{op}})_R :ev_0$$

is a Quillen equivalence.

Proof. We separate the proof in two parts. In the first part we prove that this Quillen pair is actually a Quillen equivalence. In the second we verify that the three simplicial structure axioms are satisfied.

Part 1

We know from remark 1.5.17 that the pair of adjoint functors (c, ev_0) is a Quillen pair respect to the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$:

$$c\colon \mathcal{M} \longleftrightarrow (\mathcal{M}^{\Delta^{op}})_R :ev_0$$

Moreover, by the definition of a left localization (see Definition 2.3.1) the identity maps also induce a Quillen pair:

$$id\colon (\mathcal{M}^{\Delta^{op}})_R \longleftrightarrow L_S(\mathcal{M}^{\Delta^{op}})_R :id.$$

These two Quillen pairs provide the Quillen pair we are interested in $c\colon \mathcal{M} \longleftrightarrow L_S(\mathcal{M}^{\Delta^{op}})_R :ev_0$. Now we show that this is even a Quillen equivalence. Given a cofibrant object A in $\mathcal{M}$ and a fibrant object X in $L_S(\mathcal{M}^{\Delta^{op}})_R$, i.e. it is a simplicial resolution, we have to verify that the map $A \to ev_0(X) = X(0)$ is a weak equivalence in $\mathcal{M}$ if and only if the map $cA \to X$ is a weak equivalence in $L_S(\mathcal{M}^{\Delta^{op}})_R$, i.e. it is a hocolim-equivalence.

Let $\tilde{A}$ and $\tilde{X}$ denote the result of applying a cofibrant replacement functor in $\mathcal{M}$ to A and objectwise to X respectively. This gives us a diagram

$$\begin{tikzcd} A \arrow[d] & \tilde{A} \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \operatorname{hocolim}(c\tilde{A}) \arrow[d] \\ X(0) & \tilde{X(0)} \arrow[l, "\cong"'] \arrow[r, "\cong"] & \operatorname{hocolim}(\tilde{X}) \end{tikzcd}$$

The diagrams $c\tilde{A}$ and $\tilde{X}$ are objectwise cofibrant and every map in them is a weak equivalence. Thus, Proposition 2.4.6 below tells us that the horizontal maps in the right square are weak equivalences. It follows from the diagram above that the map $A \to X(0)$ is a weak equivalence in $\mathcal{M}$ if and only if $\operatorname{hocolim}(c\tilde{A}) \to \operatorname{hocolim}(\tilde{X})$ is a weak equivalence, i.e. if and only if $cA \to X$ is a hocolim-equivalence. This is what we want.

Part 2

For the simplicial model structure we check the three axioms,

1. Since the left Bousfield localization does not change the cofibrations, part $SM2)1)$ is already proven in Proposition 2.2.5.

2. Corollary 2.1.9 shows that the verification of $SM2)2)$ reduces to verifying that the functors $(-)^K$ preserve fibrations between fibrant objects. Fibrant objects in $L_S(\mathcal{M}^{\Delta^{op}})_R$ are S-local object in $(\mathcal{M}^{\Delta^{op}})_R$. Recall that a map between fibrant objects is a fibration in $L_S(\mathcal{M}^{\Delta^{op}})_R$ if and only if it is a fibration in $(\mathcal{M}^{\Delta^{op}})_R$. Now, the dual version of Proposition 2.2.5 together with Proposition 2.2.7 imply that the functors $(-)^K$ preserve fibrations between fibrant objects.

3. The model category $\mathcal{M}$ was assumed to be left proper, hence the model category $L_S(\mathcal{M}^{\Delta^{op}})_R$ is left proper again. Therefore, by Proposition 2.1.10 one only needs to check that if Z is fibrant in $L_S(\mathcal{M}^{\Delta^{op}})_R$ and $K \to L$ is a trivial cofibration in $\mathcal{S}$, the map $Z^L \to Z^K$ is a weak equivalence. The fibrant object in $L_S(\mathcal{M}^{\Delta^{op}})_R$ are the simplical resolutions and the claim follows from Proposition 2.2.7.

□

Definition 2.4.5 (Homotopically contractible)**.** If $\mathcal{C}$ is catgory then $\mathcal{C}$ is said to be *homotopically contractible*, if the geometric realization of its nerve $N\mathcal{C}$ is.

Due to Chachólski and Scherer [CS98] we have the following proposition. Its proof can also be found in [DHK97]

Proposition 2.4.6. *Suppose $\mathcal{M}$ is a model category and $\mathcal{C}$ is a small homotopically contractible category. Then for every objectwise cofibrant and homotopical constant diagram A in $\mathcal{M}^{\mathcal{C}}$ and every object c in $\mathcal{C}$, the canonical maps $A(c) \to \operatorname{hocolim} A$ are weak equivalences.*

Remark 2.4.7. In contrast to the localized Reedy model category $L_S(\mathcal{M}^{\Delta^{op}})_R$, the localized Bousfield-Kan structure on $L_S(\mathcal{M}^{\mathcal{C}})_{BK}$ is not simplicial with respect to the categorical simplicial action. The trivial fibrations are the same in the localized and unlocalized model structures. If $L_S(\mathcal{M}^{\mathcal{C}})_{BK}$ is a simplicial model category, then for any trivial fibration $p\colon X \to Y$ in $L_S(\mathcal{M}^{\mathcal{C}})_{BK}$ or $(\mathcal{M}^{\mathcal{C}})_{BK}$ and for cofibration $\partial\Delta[n] \to \Delta[n]$ the map $X^{\Delta[n]} \to X^{\partial\Delta[n]} \times_{Y^{\partial\Delta[n]}} Y^{\Delta[n]}$ is a trivial fibration in $L_S(\mathcal{M}^{\mathcal{C}})_{BK}$. The definition of the simplicial structure yields

$$X^{\Delta[n]}(0) \cong \hom(\Delta[n], X) \to (X^{\partial\Delta[n]} \times_{Y^{\partial\Delta[n]}} Y^{\Delta[n]})(0) \cong \hom(\partial\Delta[n], X) \times_{\hom(\partial\Delta[n], Y)} \hom(\Delta[n], Y),$$

where the isomorphisms are induced by $K \times \Delta[0] \cong K$, for K in $\mathcal{S}$. Fibrations and weak equivalences are objectwise, hence it is also a trivial fibration in $\mathcal{M}$. But a close examination reveals that it is also the n-th matching map of the map $p\colon X \to Y$.

The conclusion of this observation is that every trivial fibration in $L_S(\mathcal{M}^{\mathcal{C}})_{BK}$ is also a Reedy trivial fibration. To see that this is not true in general, let $\mathcal{M}$ by the category of the simplicial sets $\mathcal{S}$ and let $\mathcal{C}$ be Δ^{op}. For any nontrivial fibrant simplicial set Z which is also weak equivalent to $*$, the constant simplicial object at Z is fibrant and weak equilvant in $\mathcal{S}^{\Delta^{op}}_{BK}$. But the matching map in degree 1 is $(c_*Z)(1) \to M_1(cZ)$. This is exactly the diagonal map $Z \to Z \times Z$ which is not, in general, a fibration, thus a contradiction to assumption.

Chapter 3

Canonical Model Categories and Realization Model Categories

So far we have shown that for a given model category of type *LpCe* or *LpCo*, one can find a model structure on $\mathcal{M}^{\Delta^{op}}$ such that it becomes simplicial and Quillen equivalent to the original model category. For this result we used the general localization machinery. This method has the advantage that the result could be generalized to other classes of model categories without changing the proof by just developing the theory of the left Bousfield localizations. A disadvantage is that we do not know anything about model categories which do not necessary admit a Bousfield localization. But even if the Bousfield localization exist, as in our case, it is normally extremly difficult to give an explicit description of the class of fibrations. While the cofibrations are fully understood, it seems that the fibrations behave very mysterious in the localized model structure. This was one of the motivation for the authors of [RSS01] to use another method of replacing model categories with simplicial ones. Without mentioning it explicitly, their approach is to use the Recognition Lemma in order to prove the existence of a cofibrantly generated model structure on $\mathcal{M}^{\Delta^{op}}$ in which fibrations have a special form. Moreover, the model category obtained in this way is the model category we are searching for, i.e. it is simplicial and Quillen equivalent to $\mathcal{M}$. In some sense the localization is still the leading idea, but for the sake of a better understanding of the fibrations we never use the general localization machines.

In this chapter we follow the path of the authors of [RSS01] and prove that under certain assumptions (which are different from before) a model categeory is Quillen equivalent to a simplicial one. The proofs here are all due to the authors of [RSS01], we just try to explain them.

In Section 3.1. the notation of a *canonical model category* (see Definition 3.1.1) is introduced. The Theorem 3.1.3 shows that its definition is very restrictive, in the sense that if a canonical model exists at all on $\mathcal{M}^{\Delta^{op}}$, then its model structure is unique.

In Section 3.2. we first introduce a new class of maps, called *equifibered Reedy fibrations* (see Definition 3.2.1), then *realization model categories* (see Definition 3.2.2) are defined as cofibrantly generated model categories such that

1. The cofibrations are the Reedy cofibrations.

2. The fibrations are the equifibered Reedy fibrations

3. The weak equivalences are the realization weak equvialences.

The main part of this section is a preparation for the construction of the realization model category in Section 3.3. The inspiration of all this is a theorem (see Theorem 3.2.10) which states that if a model category $\mathcal{M}$ is cofibrantly generated in the strengthened sense of Definition 3.2.8, so is the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$. The additional requirements in the unusual definition of a cofibrantly generated model category are necessary for the small object argument we use for proving the existence of a realization model category in Theorem 3.3.8.

In Section 3.3. we define of the *Realization Axiom* (see Definition 3.3.7). Then we prove in Theorem 3.3.8 that if a model category is left proper, cofibrantly generated and satisfies the Realization Axiom, then the realization model category exists. The verification of its existence requires the Recognition Lemma. Most of this section is devoted to the verification of the conditions in the Recognition Lemma. The proofs are due to the authors of [RSS01].

In Section 3.4. we obtain that the realization model category is even a simplicial ones by verifing the three simplicial axioms.

In Section 3.5. we show that in a realization model category a weak equivalence between fibrant objects becomes an objectwise one. This result is used in the proof of the Quillen equivalence between $\mathcal{M}$ and the realization model category (see Theorem 3.5.3).

3.1 Canonical Model Categories

The method we used in the previous chapter is inappropriate if we want to control the fibrations. In order to give an explicit description of the fibrations, we go back to the origin of the problem.

What we want is a simplicial model category which is Quillen equivalent to the original model category. For this purpose, we defined the categorical simplicial action and we saw in Proposition 2.2.5 that a Reedy model structure on $\mathcal{M}^{\Delta^{op}}$ equipped with this action is almost a simplicial one. By Proposition 2.1.10 the last simplicial axiom is equivalent to requiring the maps $Z^L \to Z^K$ to be weak equivalences for fibrant objects Z. According to Proposition 2.2.7 this is true if the objects Z are simplicial resolutions.

One question arises quite naturally: Is it possible to change the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$ in such a way, that fibrant objects become the simplicial resolutions? Since we still want use some results of Proposition 2.2.5 it may be wise to keep the cofibrations unchanged.

The reasonable properties which the new model category should have are listed in the following definition.

Definition 3.1.1 (Canonical model category)**.** Let $\mathcal{M}$ be a model category. A *canonical model category on* $\mathcal{M}^{\Delta^{op}}$ is the category $\mathcal{M}^{\Delta^{op}}$ endowed with a model structure, such that

1. The Reedy weak equivalences are weak equivalences.
2. The Reedy cofibrations are the cofibrations.
3. The simplicial resolutions are the fibrant objects.

Example 3.1.2. In the hocolim model structure of $L_S(\mathcal{M}^{\Delta^{op}})_R$ the Reedy weak equivalences are weak equivalences and the Reedy cofibrations are the cofibrations. Moreover, in the Theorem 2.4.2 we proved that the fibrant objects are the simplicial resolutions. Therefore, the model category $L_S(\mathcal{M}^{\Delta^{op}})_R$ is a canonical model category.

Theorem 3.1.3 ([RSS01]). *If the canonical model category exists, it is unique and the weak equivalences are the realization weak equivalences.*

Proof. We assume that the canonical model category exists.

In a model category the fibrant objects determine the weak equivalences, because a map f is a weak equivalence if and only if the map $[f, Z]$ is an isomorphism for all fibrant objects. Since the fibrant objects and the cofibrations in a canonical model category are given by the definition, the cofibrations and the weak equivalences are determined and since the fibrations are also determined by the lifting axiom, the canonical model structure is unique.

The Reedy cofibrations and the Reedy weak equivalences are also cofibrations and weak equivalences in the canonical model category, therefore the identity functors give rise to a Quillen pair

$$id_{Reedy}\colon \mathcal{M}^{\Delta^{op}}_{Reedy} \to \mathcal{M}^{\Delta^{op}}_{Can.} :id_{Can.}$$

which induces an isomorphism of homotopy classes

$$[A, Z]_{HoReedy} \cong [A, Z]_{HoCan.}$$

where A is Reedy cofibrant and Z is fibrant in the canonical model structure, i.e. Z is a simplicial resolution. Since the objectwise weak equivalences are in particular weak equivalences in the canonical model category, this isomorphism can be extended to all objects A in $\mathcal{M}^{\Delta^{op}}$ and Z homotopical constant in $\mathcal{M}^{\Delta^{op}}$. This means that requiring that the map $[f, Z]^{HoCan.}$ is an isomorphism for all homotopical constant objects Z is equivalent to requiring the map $[f, X]^{HoReedy}$ to be an isomorphism. Because the homotopical constant object Z is objectwise weak equivalent to the object in the 0-th level cZ_0 and as Z runs through all homotopical objects, cZ_0 runs through all objects in $\mathcal{M}$, it is also sufficient to require that $[f, cX]^{HoReedy}$ is bijective for all objects X in $\mathcal{M}$. Therefore we have that f is a weak equivalence if and only if it is a realization weak equivalence. □

Remark 3.1.4. As we have seen, the model category $L_S(\mathcal{M}^{\Delta^{op}})_R$ with the hocolim model structure is an example of a canonical model category. The theorem above says that every model structure on $\mathcal{M}^{\Delta^{op}}$ satisfying the properties of Definition 3.1.1 must coincide with the hocolim model structure. Furthermore, it provides another proof that the hocolim-equivalences are the realization weak equivalences (cf. the proof of Theorem 2.4.2).

3.2 Realization Model Categories

In this section we proof the existence of a canonical model structure by changing the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$. By definition the cofibrations are the Reedy cofibrations and by Theorem 3.1.3 the weak equivalences have to be the realization weak equivalences.

Now, we want to find out more about the fibrations. Since the simplicial resolutions should become the fibrant objects in the canonical model category, the maps $p_Z\colon Z \to *$ have to be fibrations there. A simplicial resolution is a homotopical constant, Reedy fibrant object, therefore an object Z is a simplicial resolution if and only if the map $p_Z\colon Z \to *$ is a Reedy fibration and for all n the maps $X_{n+1} \to X_n$ are weak equivalences in $\mathcal{M}$.

If $s_i\colon \Delta[n] \to \Delta[n+1]$ denotes the degeneracy map and f is a map in $\mathcal{M}^{\Delta^{op}}$, then by Definition 2.2.1 and Lemma 2.2.3 the map $f^{\square s_i}(0)\colon X^{\Delta[n+1]}(0) \to Y^{\Delta[n+1]}(0) \times_{Y^{\Delta[n]}(0)} X^{\Delta[n]}(0)$ is exactly the map $(d_i, f_{n+1})\colon X_{n+1} \to X_n \times_{Y_n} Y_{n+1}$. Using this notation, an object Z is a simplicial resolution if and only if the map p_Z is a Reedy fibration and for every n the maps $p_Z{}^{\square s_i}(0)$ are weak equivalences. This observation leads us to the following definition and encourages us to define a class of *realization model categories*.

Definition 3.2.1 (Equifibered Reedy fibration)**.** A map $f\colon X \to Y$ in $\mathcal{M}^{\Delta^{op}}$ is an *equifibered Reedy fibration*, if

1. The map f is a Reedy fibration;
2. The map $f^{\square s_i}(0) = (d_i, f_{n+1})\colon X_{n+1} \to X_n \times_{Y_n} Y_{n+1}$ is a weak equivalence for all n and $0 \leq i \leq n+1$, where d_i denotes the i-th face operator.

Definition 3.2.2 (Realization model category)**.** If $\mathcal{M}$ is a model category, then $\mathcal{M}^{\Delta^{op}}$ is a *realization model category* if the following properties are satisfied:

1. It is cofibrantly generated.
2. The weak equivalences are the realization weak equivalences.
3. The cofibrations are the Reedy cofibrations.
4. The fibrations are the equifibered Reedy fibrations.

Remark 3.2.3. As we will see later, a realization model category should be cofibrantly genereated in the sense of Definition 3.2.8.

One might ask, why we introduce the realization model categories at all, if we are interested in proving the existence of the canonical model categories. Note that the Reedy weak equivalences are in particular realization weak equivalences which are, by definition, the weak equivalences in a realization model category. In the discussion above we saw that the fibrant objects in there are the simplicial resolutions. Thus, a realization model category is a canonical model category, since the cofibrations are the Reedy cofibration and by Theorem 3.1.3 it is even unique. Therefore, proving the existence of the canonical model category is reduced to proving the existence of the realization model category.

We will construct a realization model category explicitly. The method used in the construction of a Reedy model category by the sets of generating cofibrations and trivial cofibrations can be viewed as a inspiration for the proof given here. Hence, we recall some results which can also be found in [Hir03, Section 15.6.].

Definition 3.2.4. [Hir03, Definition 15.6.10.] Let $\mathcal{C}$ be a Reedy category, let c and d be two objects in $\mathcal{C}$. A *boundary* $\partial\mathcal{C}(c,d)$ is a set of maps $f\colon c \to d$ such that there is a factorization $f = \overrightarrow{f}\overleftarrow{f}$ with $\overrightarrow{f} \in \overrightarrow{\mathcal{C}}$, $\overleftarrow{f} \in \overleftarrow{\mathcal{C}}$ and $\overleftarrow{f} \neq id_c$.

Definition 3.2.5. Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a cocomplete category. If c is an object in $\mathcal{C}$ and A is an object in $\mathcal{M}$, then the *boundary* ∂F_c^A *of the free diagram by A generated at* c is defined by

$$\partial F_c^A(d) = \coprod_{\partial\mathcal{C}(c,d)} A,$$

for every object d in $\mathcal{C}$.

Remark 3.2.6. There is a canonical isomorphism $\partial F_c^A \cong \operatorname{colim}_{d\in\partial(c\downarrow\overleftarrow{\mathcal{C}})^{op}} F_d^A$.

While the functor $F_c^-\colon X \mapsto F_c^X$ is the left adjoint to the *evaluation functor* $ev_c\colon X \mapsto X(c)$, the functor $\partial F_c^-\colon X \mapsto \partial F_c^X$ is the left adjoint to the matching object functor $M_c\colon X \mapsto M_c X$.

If $\mathcal{M}$ is a model category, then the induced isomorphism

$$\mathcal{M}^{\mathcal{C}}(\partial F_c^A, X) \cong \mathcal{M}(A, M_c X)$$

implies that for every object c in $\mathcal{C}$ there exists a dotted arrow in the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X_c \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & Y_c \times_{M_c Y} M_c X \end{tikzcd}$$

iff there exists a dotted arrow in the following diagram

$$\begin{tikzcd} F_c^A \coprod_{\partial F_c^A} \partial F_c^B \arrow[r] \arrow[d] & X \arrow[d] \\ F_c^B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Hence, a map $X \to Y$ is a Reedy fibration if and only if it has the right lifting property with respect to all maps of the form $F_c^A \coprod_{\partial F_c^A} \partial F_c^B \to F_c^B$, where $A \to B$ is a cofibration in $\mathcal{M}$. In addition, if $\mathcal{M}$ is assumed to be cofibrantly generated, it is enough to verify the property if $A \to B$ is a generated cofibration.

Definition 3.2.7. Suppose $\mathcal{C}$ is a Reedy category, $\mathcal{M}$ is a model category and I is a set of maps. Then $(F_I^{\mathcal{C}})_{Reedy}$ denotes the set of maps of the form

$$F_c^A \coprod_{\partial F_c^A} \partial F_c^B \to F_c^B,$$

where c is an object in $\mathcal{C}$ and $A \to B$ is a map in I.

Using this definition, we summarize the result of the short discussion above.

Let $\mathcal{C}$ be a Reedy category and $\mathcal{M}$ a cofibrantly generated model category. If I denotes the set of generating cofibrations and J denotes the set of generating trivial cofibrations, then a map is a Reedy fibration if and only if it is in $(F_J^{\mathcal{C}})_{Reedy}$-inj and a map is a Reedy trivial fibration if and only if it is in $(F_I^{\mathcal{C}})_{Reedy}$-inj.

If the sets $(F_I^{\mathcal{C}})_{Reedy}$ and $(F_J^{\mathcal{C}})_{Reedy}$ permit the small objects arguments, then they are the sets of generating cofibrations and generating trivial cofibrations in the Reedy model structure respectively.

In [Hir03, Theorem 15.6.27.] Hirschhorn proved that this is indeed true, if a model category $\mathcal{M}$ is cofibrantly generated and satisfies two additional properties:

1. The domains and the codomains of the sets of generating cofibrations I are small relative to I.

2. The domains and the codomains of the sets of generating trivial cofibrations J are small relative to J.

Since we can not use this result directly, we have to change the hypothesis and the statement in the theorem. First we give a new definition of a cofibrantly generated model category, which is more suitable to our demands.

Definition 3.2.8 (Cofibrantly generated model category in the sense of [RSS01])**.** Let $\mathcal{M}$ be a cofibrantly generated model category (see Definition 1.3.14). If I and J denote the sets of generating cofibrations and trivial cofibrations respectively, then $\mathcal{M}$ is also cofibrantly generated in the sense of [RSS01], if two additional conditions are satisfied:

1. The codomains of maps in I and J are small relative to I (see Definition 1.3.9).
2. The domains and codomains of maps in I are cofibrant.

The first condition is needed for smallness of the domains of the set of generating cofibrations and trivial cofibrations in $\mathcal{M}^{\mathcal{C}}$ and therefore essential for the small object argument. Amongst others, the second condition is necessary in the verification of the existence of a simplicial structure on $\mathcal{M}^{\Delta^{op}}$.

Remark 3.2.9. In the sequel, if we talk about cofibrantly generated model category, we mean cofibrantly generated in the sense of Definition 3.2.8.

Theorem 3.2.10. *[Hir03] Suppose $\mathcal{C}$ is a Reedy category and $\mathcal{M}$ is a cofibrantly generated model category, then the Reedy model structure on $\mathcal{M}^{\mathcal{C}}$ is cofibrantly generated with $(F_I^{\mathcal{C}})_{Reedy}$ and $(F_J^{\mathcal{C}})_{Reedy}$ as its sets of generating cofibrations and trivial cofibrations respectively.*

Proof. At first we take a close look at the map $F_c^A \coprod_{\partial F_c^A} \partial F_c^B \to F_c^B$, where $A \to B$ is a map in I.

By Definition 3.2.5, for every d in $\mathcal{C}$ the object $\partial F_c^A(d) = \coprod_{\partial\mathcal{C}(c,d)} A$ and the object

$$F_c^A \coprod_{\partial F_c^A} \partial F_c^B(d) = F_c^A(d) \coprod_{\partial F_c^A(d)} \partial F_c^B(d) \cong (\coprod_{\mathcal{C}(c,d)\setminus\partial\mathcal{C}(c,d)} A) \sqcup (\coprod_{\partial\mathcal{C}(c,d)} B).$$

It follows that the map $F_c^A \coprod_{\partial F_c^A} \partial F_c^B \to F_c^B$ is isomorphic to of the identity map of the object $\coprod_{\partial\mathcal{C}(c,d)} B$ with the inclusion map $\coprod_{\mathcal{C}(c,d)\setminus\partial\mathcal{C}(c,d)} A \to \coprod_{\mathcal{C}(c,d)\setminus\partial\mathcal{C}(c,d)} B$. Since the last map can be written as a transfinite composition of $A \to B$ which is a map of I, $F_c^A \coprod_{\partial F_c^A} \partial F_c^B \to F_c^B$ is a relative I-cell complexe. In the same way the evaluation of maps in $(F_J^{\mathcal{C}})_{Reedy}$ at an object in $\mathcal{C}$ are relative J-cell complexes.

The next step is to verify that if the domains and codomains of maps in I are small relative to I, then the domains and codomains of maps in $(F_I^{\mathcal{C}})_{Reedy}$ are small relative to $(F_I^{\mathcal{C}})_{Reedy}$. Let X be a λ-sequence of relative $(F_I^{\mathcal{C}})_{Reedy}$-cell complexes. Since for every object c in $\mathcal{C}$, the evaluation of the maps in $(F_I^{\mathcal{C}})_{Reedy}$ are relative I-cell complexes, the evaluation of X at c is a λ-sequence of relative I-cell complexes.

For every object F_c^A, where A is a domain or codomain of a map in I, this observation and the adjointness induce a chain of isomorphisms

$$\mathcal{M}^{\mathcal{C}}(F_c^A, \mathrm{colim}_{i\in I} X_i) \cong \mathcal{M}(A, \mathrm{colim}_{i\in I} X_i(c)) \cong \mathrm{colim}_{i\in I} \mathcal{M}(A, X_i(c)) \cong \mathrm{colim}_{i\in I} \mathcal{M}^{\mathcal{C}}(F_c^A, X_i),$$

where the second isomorphism comes from the hypothesis that A is small relative to I. Therefore, all the objects F_c^A are small relative to $(F_I^{\mathcal{C}})_{Reedy}$. A theorem in [Hir03] guarantees that a colimit of objects which are all small relative to I is also small relative to I. Hence, since the domains of maps in $(F_I^{\mathcal{C}})_{Reedy}$ can be written as colimits of F_c^A (see remark 3.2.6), the domains and codomains of maps in $(F_I^{\mathcal{C}})_{Reedy}$ are small relative to $(F_I^{\mathcal{C}})_{Reedy}$. Similarly, the domains and codomains of maps in $(F_J^{\mathcal{C}})_{Reedy}$ are small relative to $(F_J^{\mathcal{C}})_{Reedy}$.

Now the proof is complete and there exists a cofibrantly generated model structure on $\mathcal{M}^{\mathcal{C}}$ such that $(F_I^{\mathcal{C}})_{Reedy}$ is the generating cofibrations and $(F_J^{\mathcal{C}})_{Reedy}$ is the set of generating trivial cofibrations. $\square$

Remark 3.2.11. Usually we are working with the Reedy category Δ^{op}. In this case, if c denotes the constant functor, then for every n there are following identities:

1. $F_n^A = cA \otimes \Delta[n]$.
2. $\partial F_n^A = cA \otimes \partial\Delta[n]$.
3. $F_n^A \coprod_{\partial F_n^A} \partial F_n^B \to F_n^B = cA \otimes \Delta[n] \coprod_{cA\otimes\partial\Delta[n]} cB \otimes \partial\Delta[n] \to cB \otimes \Delta[n] = c(f)\square i_n$,

where f is the map $A \to B$ and i_n is the inclusion $\partial\Delta[n] \to \Delta[n]$.

Recall that the set of generating cofibrations in $\mathcal{S}$ are the maps $\partial\Delta[n] \to \Delta[n]$ which we denote by $I_{\mathcal{S}}$. Let $I_{\mathcal{M}}$ and $J_{\mathcal{M}}$ denote the sets of generating cofibrations and trivial cofibrations of $\mathcal{M}$ respectively. Using the notation above, Theorem 3.2.10 states that

1. The set $cI_{\mathcal{M}}\square I_{\mathcal{S}}$ is the set of generating cofibrations;
2. The set $cJ_{\mathcal{M}}\square I_{\mathcal{S}}$ is the set of generating trivial cofibrations

of the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$.

It is obvious that the Reedy model category on $\mathcal{M}^{\Delta^{op}}$ is not the realization model category, so why did we study the last theorem?

We did so it because it provides an explicit description of the sets which control the cofibrations and the trivial cofibrations. Compared with the Reedy model structure, the realization model structure has the same cofibrations, but less fibrations which are the Reedy equifibered fibrations. It is reasonable to ask whether it is possible to add special maps to the set of generating trivial cofibrations such that the fibrations become the equifibered Reedy fibrations, while leaving the set of generating cofibrations untouched.

We answer this question affirmatively in the next section.

3.3 Existence of Realization Model Categories

From now on, let $\mathcal{M}$ be a left proper, cofibrantly generated model category with $I_{\mathcal{M}}$ and $J_{\mathcal{M}}$ as sets of generating cofibrations and trivial cofibrations respectively.

We are going to use the Recognition Lemma in oder to endow the category $\mathcal{M}^{\Delta^{op}}$ with a realization model structure. For convenience, we recall the Recognition Lemma.

Theorem 3.3.1 (Recognition Lemma)**.** *[D. M. Kan] Let $\mathcal{C}$ be a complete and cocomplete category and let W be a class of maps which is closed under retracts and satisfies the two-of-three axiom. If I and J are sets of maps in $\mathcal{C}$ such that*

1. *the sets I and J permit the small object argument,*
2. *J-cof $\subseteq$ I-cof $\cap$ W and I-inj $\subseteq$ J-inj $\cap$ W, and*
3. *I-cof $\cap$ W $\subseteq$ J-cof or J-inj $\cap$ W $\subseteq$ I-inj;*

Then there is a cofibrantly generated model structure on $\mathcal{C}$ in which I and J are the generating cofibrations and generating trivial cofibrations respectively and W is the class of weak equivalences.

By the definition of realization model categories, the weak equivalences are the realization weak equivalences and the cofibrations are the Reedy cofibrations. These observations motivate the following definition.

Definition 3.3.2. Let W in the Recognition Lemma denote the class of realization weak equivalences and let the set I denote the set of generating cofibrations $cI_{\mathcal{M}}\Box I_{\mathcal{S}}$.

Now, we only have to find a set J which generates the trivial cofibrations. Since the fibrations are the equifibered Reedy fibrations, we define J to contain all maps such that J-inj is the class of the equifibered Reedy fibration. More explicitly we have:

Lemma 3.3.3. *Let I_d denote the set of degenerency maps $s_i\colon \Delta[n] \to \Delta[n+1]$, for each n and $0 \le i \le n+1$. A map is an equifibered Reedy fibration in $\mathcal{M}^{\Delta^{op}}$ if and only if it has the right lifting property with respect to the sets $cJ_{\mathcal{M}}\Box I_{\mathcal{S}}$ and $cI_{\mathcal{M}}\Box I_d$.*

Proof. An equifibered Reedy fibration f is by definition a Reedy fibration such that for all m the maps $f^{\Box s_i}(0)\colon X_{m+1} \to X_m \times_{Y_m} Y_{m+1}$ are weak equivalence. By Theorem 3.2.11, being a Reedy fibration is equivalent to having the right lifting property with respect to the sets $cJ_{\mathcal{M}}\Box I_{\mathcal{S}}$ Since the map s_i is a cofibration and f a Reedy fibration, the map $f^{\Box s_i}$ is a Reedy fibration by Proposition 2.2.5. In particular, the 0-th level $f^{\Box s_i}(0)$ is a fibration. Therefore a Reedy fibration f is also an equifibered Reedy fibration if and only if $f^{\Box s_i}(0)$ is a trivial fibration and by adjunction this is equivalent to $(cI_{\mathcal{M}}\Box I_d, f)$ having the lifting property. Therefore a map f is an equifibered Reedy fibration if and only if it has the right lifting property with respect to both sets $cJ_{\mathcal{M}}\Box I_{\mathcal{S}}$ and $cI_{\mathcal{M}}\Box I_d$. $\Box$

Now we have found the appropriate sets of generating cofibrations and trivial cofibrations.

Definition 3.3.4. Let $I_{\mathcal{S}}$ be the set of generating cofibrations in $\mathcal{S}$ and let I_d be the set of degenerency maps $s_i\colon \Delta[n] \to \Delta[n+1]$, for each n and $0 \le i \le n+1$.

1. The set I is defined to be the set of generating cofibrations $cI_{\mathcal{M}}\Box I_{\mathcal{S}}$ in the Reedy model category $\mathcal{M}^{\Delta^{op}}$, i.e. it contains all maps of the form:
$$c(f)\Box i_n\colon cA \otimes \Delta[n] \coprod_{cA\otimes\partial\Delta[n]} cB \otimes \partial\Delta[n] \to cB \otimes \Delta[n],$$
for every $f \in I_{\mathcal{M}}$ and $i_n \in I_{\mathcal{S}}$.
2. The set J' is defined to be the set of generating trivial cofibrations $cJ_{\mathcal{M}}\Box I_{\mathcal{S}}$ in the Reedy model category $\mathcal{M}^{\Delta^{op}}$, i.e. it contains all maps of the form:
$$c(f)\Box i_n\colon cA \otimes \Delta[n] \coprod_{cA\otimes\partial\Delta[n]} cB \otimes \partial\Delta[n] \to cB \otimes \Delta[n],$$

for every $f \in J_{\mathcal{M}}$ and $i_n \in I_{\mathcal{S}}$.

3. The set J'' is defined to be the set $cI_{\mathcal{M}} \Box I_d$ which contains all maps of the form:

$$c(f)\Box s_n \colon cA \otimes \Delta[n+1] \coprod_{cA\otimes\Delta[n]} cB \otimes \Delta[n] \to cB \otimes \Delta[n+1],$$

for every $f \in I_{\mathcal{M}}$ and $s_n \in I_d$.

4. Let J be $J' \cup J''$.

A first sign that the definition given here is the correct one is the Proposition 3.3.6. It shows that the last simplicial axiom $SM2)3)$ is satisfied, if the trivial cofibration of simplicial sets is a generating trivial cofibration. As we will see in Theorem 3.4.1, once we know it is true for the set of generating trivial cofibration, it is true for all trivial cofibrations in $\mathcal{S}$.

The lemma below will be used in the proof of the proposition.

Lemma 3.3.5. *If Z is a simplicial resolution, then the elements of* $\mathrm{Map}((cI_{\mathcal{M}}\Box I_{\mathcal{S}})\text{-}cof\,, Z)$ *are fibrations and the elements of* $\mathrm{Map}(cI_{\mathcal{M}}\Box J_{\mathcal{S}}, Z))$ *are trivial fibrations in $\mathcal{S}$.*

Proof. The category of simplicial sets $\mathcal{S}$ is a simplicial model category and the tensor $\otimes$ is a Quillen bifunctor in $\mathcal{S}$, therefore all maps in $I_{\mathcal{S}}\Box J_{\mathcal{S}}$ are trivial cofibrations in $\mathcal{S}$, according to Proposition 2.2.7, for all simplicial resolutions Z the maps $Z^{I_{\mathcal{S}}\Box J_{\mathcal{S}}}$ are trivial fibrations in the Reedy model structure of $\mathcal{M}^{\Delta^{op}}$. Hence, by adjointness the set $(cI_{\mathcal{M}}\Box I_{\mathcal{S}}\Box J_{\mathcal{S}}, Z)$ has the lifting property. By adjointness again, $((cI_{\mathcal{M}}\Box I_{\mathcal{S}})\text{-cof}, Z^{J_{\mathcal{S}}})$ has the lifting property and by Theorem 3.2.11 $(cI_{\mathcal{M}}\Box I_{\mathcal{S}})\text{-cof} = I\text{-cof}$ are the Reedy cofibrations. Therefore, by using the adjointness once more, the set $(J_{\mathcal{S}}, \mathrm{Map}((cI_{\mathcal{M}}\Box I_{\mathcal{S}})\text{-cof}\,, Z)$ has the lifting property. This implies that $\mathrm{Map}((cI_{\mathcal{M}}\Box I_{\mathcal{S}})\text{-cof}\,, Z)$ is a fibration, since $J_{\mathcal{S}}$ is the set of generating trivial cofibrations in $\mathcal{S}$.

If we change the place of $I_{\mathcal{S}}$ and $J_{\mathcal{S}}$, then for the same reason, the maps $J_{\mathcal{S}}\Box I_{\mathcal{S}}$ are still trivial cofibrations. Similar to the paragraph above, by adjointness, first $(cI_{\mathcal{M}}\Box J_{\mathcal{S}}\Box I_{\mathcal{S}}, Z)$ has the lifting property and then $(I_{\mathcal{S}}, \mathrm{Map}(cI_{\mathcal{M}}\Box J_{\mathcal{S}}, Z))$ has the lifting property as well. Hence, the elements of $\mathrm{Map}(cI_{\mathcal{M}}\Box J_{\mathcal{S}}, Z)$ are the trivial fibrations. □

Proposition 3.3.6. *The $(cI_{\mathcal{M}}\Box J_{\mathcal{S}})$-cofibrations are realization weak equivalences.*

Proof. By the lemma above, the maps in $\mathrm{Map}(cI_{\mathcal{M}}\Box J_{\mathcal{S}}, Z))$ are trivial fibrations. Since the assumption was that the domains and codomains of maps in $cI_{\mathcal{M}}$ are cofibrant and since the constant functor c, which is a left Quillen functor, sends cofibrations in $\mathcal{M}$ to cofibrations in $\mathcal{M}^{\Delta^{op}}$ the maps in $cI_{\mathcal{M}}\Box J_{\mathcal{S}}$ are Reedy cofibrations between Reedy cofibrant objects. By Proposition 2.3.12, these maps are even realization weak equivalences.

The next step is to prove that the set of maps which are both cofibrations and realization weak equivalences is closed under pushouts, directed colimits and retracts. In this case not only the maps in $cI_{\mathcal{M}}\Box J_{\mathcal{S}}$ but also the maps in $(cI_{\mathcal{M}}\Box J_{\mathcal{S}})$-cof are realization weak equivalences.

So, let g be a pushout of a map f. Using the left properness of the model category $\mathcal{M}$, for any cofibrant approximation of a map f in $\mathcal{M}^{\Delta^{op}}$ such that f' is a Reedy cofibration there is a map g' such that g' is a cofibrant approximation of g as well as a Reedy cofibration and a pushout of the map f'. Let Z be a simplicial resolution. By Lemma 3.3.5 and Proposition 2.3.12, if f

is a Reedy cofibration and a realization weak equivalence, then $\mathrm{Map}(f', Z)$ is a trivial fibration, for all simplicial resolution Z. As a pullback, $\mathrm{Map}(g', Z)$ is also a trivial fibration and by using Proposition 2.3.12 again, g is a realization weak equivalence.

Let g be a directed colimit of f_i, where every f_i is both a realization weak equivalence. Then $\mathrm{Map}(g, Z)$ is a directed limit of $\mathrm{Map}(f_i, Z)$ for a simplicial resolution Z. Since $\mathrm{Map}(f_i, Z)$ are trivial fibrations and directed limits preserve trivial fibrations, $\mathrm{Map}(g, Z)$ is a trivial fibration and hence g is a Reedy cofibration and a realization weak equivalence.

In the case that g is a retract of a Reedy cofibration and a realization weak equivalence, the proof works in the same way, since retracts also preserve trivial fibrations. □

Up to this point the definitions of I, J and W arise in a natural way, but it seems that the existence of the realization model category cannot be proven unless we postulate that the model category $\mathcal{M}$ also satisfies the third condition in the Recognition Lemma (J-inj $\cap W \subseteq I$-inj). An equivalent formulation is the *Realization Axiom* below.

Definition 3.3.7 (Realization Axiom)**.** If a map in $\mathcal{M}^{\Delta^{op}}$ is both an equifibered Reedy fibration and a realization weak equivalence then it is an objectwise weak equivalence.

Now we are able to state the main result of this chapter.

Theorem 3.3.8 ([RSS01])**.** *If the model category $\mathcal{M}$ is left proper, cofibrantly generated and satisfies the Realization Axiom, then the category $\mathcal{M}^{\Delta^{op}}$ admits a realization model structure.*

In order to help the reader navigate through the proof, we first give a short explaination of the idea. We will mainly follow the path of [RSS01], the only noteworthy difference between our proof of the existence of the realization model category and the proof given in [RSS01] is the usage of the Recognition Lemma (see Theorem 1.3.15). There, the authors of [RSS01] verify the model category axioms explicitly.

Recall that the class of realization weak equivalences denoted by W is closed under retracts and satisfies the two-of-three axiom (see Remark 2.3.11 or Theorem 3.1.3). Moreoever, since $\mathcal{M}$ is a model category and hence complete and cocomplete, the category $\mathcal{M}^{\Delta^{op}}$ is also complete and cocomplete.

Therefore we only have to check the assumptions 1.-3. in the Recognition Lemma and we do this in the following order:

1. The first condition requires that the sets I and J permit the small object argument. The smallness condition for I has already been shown in Theorem 3.2.10 and Corollary 3.3.10 states that J also permits the small object argument.

2. The proof of the second assumption, which states that J-cof $\subseteq I$-cof $\cap\, W$ and I-inj $\subseteq J$-inj $\cap\, W$, is given in Proposition 3.3.12 and Proposition 3.3.11.

3. Finally, for the last part, the inclusion J-inj $\cap\, W \subseteq I$-inj is just an immediate consequence of the Realization Axiom, which is also proven in Proposition 3.3.11.

By definition of the realization model structure, the cofibrations have to be the Reedy cofibrations and the fibrations have to be the equifibered Reedy fibrations, but this is clear by the construction and Lemma 3.3.3. Once the conditions for the Recognition Lemma are verified, the cofibrantly generated model structure on $\mathcal{M}^{\Delta^{op}}$ in the sense of Definition 1.3.14 exists and the weak equivalences are the maps in W, i.e. the realization weak equivalences. The proof of Theorem 3.3.8 shows that this model structure is also cofibrantly generated in the sense of Definition 3.2.8.

Lemma 3.3.9. *The domains and codomains of maps in I and J are small relative to I.*

Proof. In the proof of Theorem 3.2.10 we saw that the domains and codomains of I are small relative to I.

By definition, the domains and codomains of the maps in $J_{\mathcal{M}}$ are small relative to $I_{\mathcal{M}}$. The arguments in Theorem 3.2.10 guarantee that domains and codomains of $I = cI_{\mathcal{M}} \square I_{\mathcal{S}}$ and $J' = cJ_{\mathcal{M}} \square I_{\mathcal{S}}$ are small relative to I.

Therefore, we only need to verify that the domains and codomains of maps in J'' are also small relative to I. By replacing $\partial\mathcal{C}(c,d)$ by $\Delta[m,n]$ in the proof of Theorem 3.2.10 we see that for every m the evaluation of maps in J'' at m is isomorphic to a coproduct of identities and with a coproduct of maps in $I_{\mathcal{M}}$. Then a similar argument as in the theorem reveals that because every domain or codomain A is small relative to $I_{\mathcal{M}}$, the objects $cA \otimes \Delta[n]$ are small relative to J'' for every n. This implies that the domains and codomains of maps in J'' which are colimits of $I_{\mathcal{M}}$-small objects are also small relative to $I_{\mathcal{M}}$. $\square$

For the Recognition we need that J permits the small object argument, but is just a consequence of the lemma above.

Corollary 3.3.10. *The domains of maps in J are small relative to J.*

Proof. Since the domains of maps in J are small relative to I and I permits the small object argument, they are also small relative to all I-cof (see [Hir03, Theorem 10.5.27.]). The maps in I-cof are the Reedy cofibrations, and according to Proposition 2.2.5 $cJ_{\mathcal{M}} \square I_{\mathcal{S}}$ and $cI_{\mathcal{M}} \square I_d$ consist of Reedy cofibrations. Hence, the domains of maps in J are small relative to J. $\square$

Proposition 3.3.11. *If W denotes the class of realization weak equivalences, then there is an equality I-inj $= J$-inj $\cap W$.*

Proof. Let us assume that the map f is both an element of W and J-injective. By Lemma 3.3.3 f is a realization weak equivalence and an equifibered Reedy fibration. Then the Realization Axiom ensures that it is also an objectwise weak equivalence. Thus the map f is a Reedy trivial fibration. By construction, I is identical to the set of generating cofibrations of the Reedy model structure, therefore f is in I-inj.

For the other inclusion, let f be a map in I-inj, i.e. f is a Reedy trivial fibration. We have to show that f is in J-inj, i.e. f is an equifibered Reedy fibration and a map in W. In order for f to be an equifibered Reedy fibration, for each n and each face operator d_i the induced map (d_i, f_{n+1}) has to be a weak equivalence in the following diagram:

$$\begin{tikzcd}
X_{n+1} \arrow[dr, dotted, "{(d_i,f_{n+1})}"] \arrow[drr, bend left, "f_{n+1}"] \arrow[ddr, bend right, "d_i"'] & & \\
& X_n \times_{Y_n} Y_{n+1} \arrow[r, "g"] \arrow[d] & Y_{n+1} \arrow[d] \\
& X_n \arrow[r, "f_n"] & Y_n
\end{tikzcd}$$

By our assumption f is a Reedy trivial fibration, therefore for each n the map $f_n\colon X_n \to Y_n$ is a trivial fibration. The map g in this diagram is the pullback of a trivial fibration and therefore it is in particular a weak equivalence. By the two-of-three axiom the map (d_i, f_{n+1}): $X_{n+1} \to X_n \times_{Y_n} Y_{n+1}$ is also a weak equivalence. Since every weak equivalence is also a realization equivalence, we have shown that a Reedy trivial fibration is I-injective and a realization weak equivalence. □

Proposition 3.3.12. *J-cof $\subseteq$ I-cof $\cap\, W$.*

Proof. According to Proposition 3.3.11, J-inj $\supseteq$ I-inj, hence J-cof $\subseteq$ I-cof.

In order to show that a J-cofibration is a realization weak equivalence, one observe that a J-cofibration is a retract of a relative J-cell complex. Since J' is the set of generating trivial cofibrations, the maps built from J' are Reedy cofibrations and in particular realization weak equivalences.

In case of J'', observe that the maps in I_d are trivial cofibrations in $\mathcal{S}$, hence they are $J_{\mathcal{S}}$-cof. Then Lemma 2.1.5 provides following inclusions:

$$cI_{\mathcal{M}} \square I_d \subseteq cI_{\mathcal{M}} \square J_{\mathcal{S}}\text{-cof} \subseteq cI_{\mathcal{M}}\text{-cof} \square J_{\mathcal{S}}\text{-cof} \subseteq (cI_{\mathcal{M}} \square J_{\mathcal{S}})\text{-cof} .$$

By Proposition 3.3.6 $(cI_{\mathcal{M}} \square J_{\mathcal{S}})$-cof are realization weak equivalences and the proof is complete. □

The following corollary is just a conclusion of everything what we have proved.

Proof. We have already verified every assumption in the Recognition Lemma, hence there exists a cofibrantly generated model structure in the sense of Definition 1.3.14. As mentioned in the proof of Theorem 3.2.10, the domains and codomains of I are cofibrant. By Lemma 3.3.9 the domains and codomains of maps in I and J are both small relative to I. In conclusion, the cofibrantly generated model structure on $\mathcal{M}^{\Delta^{op}}$ is even cofibrantly generated in the sense of Definition 3.2.8. □

The next proposition shows that the Realization Axiom is tightly linked to equifibered Reedy fibrations.

Proposition 3.3.13. *If $\mathcal{M}$ is left proper, cofibrantly generated and the canonical model category exists, then the Realization Axiom 3.3.7 is satisfied if and only if the equifibered Reedy fibrations are the fibrations in $\mathcal{M}^{\Delta^{op}}$.*

Proof. If the Realization Axiom is satisfied, then Theorem 3.3.8 asserts that the realization model category exists, in which, by definition, the fibrations are the equifibered Reedy fibrations.

For the other implication, if the equifibered Reedy fibrations are the fibrations, then a map which is both a Reedy fibration and a realization weak equivalence is also a trivial fibration in the canonical model category. The Realization Axiom follows, since every trivial fibration is a Reedy trivial fibration and particularly an objectwise weak equivalence. □

Since the simplicial replacement we considered in the previous chapter is a canonical model category, the following corollary of the above proposition is the result we are looking for.

Corollary 3.3.14. *If $\mathcal{M}$ is a model category such that*

1. *$\mathcal{M}$ is cofibrantly generated in the sense of Definition 3.2.8;*

2. *$\mathcal{M}$ is of type LpCe or LpCo;*

3. *$\mathcal{M}$ satisfies the Realization Axiom;*

Then the fibrations in the localized model category $L_S\mathcal{M}^{\Delta^{op}}$ are the equifibered Reedy fibration.

3.4 Realization Model Categories are Simplicial

Theorem 3.4.1 ([RSS01])**.** *If the model category is left proper, cofibrantly generated and satisfies the Realization Axiom, then the realization model structure in Theorem 3.3.8 is a simplicial one.*

Proof. Shortly after the definition of the categorical simplicial action, we proved that the bifunctor $\otimes$ is compatible with product in $\mathcal{S}$. The only thing left to verify are the axioms SM2)1)-SM2)3) are also satisfied.

1. Since the cofibrations in the realization model structure are the Reedy cofibrations, by Proposition 2.2.5 the property SM2)1) is satisfied.

2. In order to prove SM2)2), let f be a trivial cofibration in the realization model structure and j be a cofibration in $\mathcal{S}$. Pushouts, retracts and transfinite composition preserve trivial cofibrations, therefore we only need to prove SM2)2) in the cases where f is a map in J.

   If f is in J' then f is a Reedy trivial cofibration and applying Proposition 2.2.5 the map $f\Box i$ is a Reedy trivial cofibration, hence in particular a trivial cofibration in the realization model structure.

   If f is in $J'' = cI_{\mathcal{M}}\Box I_d$, then we have

$$cI_{\mathcal{M}}\Box I_d\Box I_{\mathcal{S}} \subseteq cI_{\mathcal{M}}\text{-cof } \Box J_{\mathcal{S}}\text{-cof } \subseteq (cI_{\mathcal{M}}\Box J_{\mathcal{S}})\text{-cof },$$

   where we used Lemma 2.1.5 and the fact that the maps in I_d are trivial cofibrations in the simplicial model category $\mathcal{S}$. Thus, we have the following inclusion

$$(J''\Box I_{\mathcal{S}})\text{-cof } = (cI_{\mathcal{M}}\Box I_d\Box I_{\mathcal{S}})\text{-cof } \subseteq (cI_{\mathcal{M}}\Box J_{\mathcal{S}})\text{-cof}$$

   and $(cI_{\mathcal{M}}\Box J_{\mathcal{S}})$-cof consists of realization weak equivalences by Proposition 3.3.6.

3. For the last property to check, let f be a cofibration and let j be a trivial fibration. The cofibrations in the realization model category are the $(cI_{\mathcal{M}}\Box I_{\mathcal{S}})$-cof and the trivial fibrations in $\mathcal{S}$ are the $J_{\mathcal{S}}$-cof. We show the following inclusion:

$$(cI_{\mathcal{M}}\Box I_{\mathcal{S}})\text{-cof } \Box J_{\mathcal{S}}\text{-cof } \subseteq (cI_{\mathcal{M}}\Box J_{\mathcal{S}})\text{-cof }.$$

   Since $J_{\mathcal{S}}$ consists of trivial cofibrations and $\mathcal{S}$ is a simplicial model category, $I_{\mathcal{S}}\Box J_{\mathcal{S}}$ is contained in $J_{\mathcal{S}}$-cof. Now, by applying Lemma 2.1.5 we have the inclusions:

$$cI_{\mathcal{M}}\Box I_{\mathcal{S}}\Box J_{\mathcal{S}} \subseteq cI_{\mathcal{M}}\text{-cof } \Box(I_{\mathcal{S}}\Box J_{\mathcal{S}})\text{-cof } \subseteq cI_{\mathcal{M}}\text{-cof } \Box J_{\mathcal{S}}\text{-cof}$$

and therefore the desired inclusion:

$$(cI_{\mathcal{M}}\Box I_{\mathcal{S}})\text{-cof}\,\Box J_{\mathcal{S}}\text{-cof} \ \subseteq (cI_{\mathcal{M}}\Box I_{\mathcal{S}}\Box J_{\mathcal{S}})\text{-cof} \ \subseteq (cI_{\mathcal{M}}\Box J_{\mathcal{S}})\text{-cof}\ .$$

By Proposition 3.3.6 and by part 1, $(cI_{\mathcal{M}}\Box J_{\mathcal{S}})$-cof are trivial cofibrations in the realization model category.

□

3.5 The Realization Model Category is Quillen Equivalent to the Original Model Category

Above, we have shown that the realization model category exists, provided it satisfies the assumptions of the Theorem 3.3.8. Before we harvest after all the hard work, we need the last two results.

If for any morphism f in a category, f has a property as soon as its image Ff under a functor F has it, the functor is said to $reflect$ this property.

Lemma 3.5.1. *In the realization model category $\mathcal{M}^{\Delta^{op}}$, a map between fibrant objects is a weak equivalence if and only if it is an objectwise weak equivalence. In particular the functor ev_0 reflects the weak equivalences between fibrant objects.*

Proof. The trivial fibrations in the realization model category are the I-inj and they are the Reedy trivial fibrations. Accordingt to the *Ken Brown's lemma*, see [DHKS04, 14.5.] a map between fibrant objects is a weak equivalence if it is an objectwise weak equivalence. Moreover, since the fibrant objects are the simplicial resolutions, if it evaluation $ev_0(f)$ is a weak equivalence, then the map f is a weak equivalence. □

Proposition 3.5.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two model categories, let $\tilde{C}$ be a functorial cofibrant approximation and let $\hat{F}$ be a functorial fibrant approximation. If $L\colon \mathcal{M} \to \mathcal{N} : R$ is a Quillen pair, then the following are equivalent:*

1. *The Quillen pair (L, R) is a Quillen equivalence.*
2. *The left Quillen functor L reflects weak equivalences between cofibrant objects and the map $L\tilde{C}RX \to X$ is a weak equivalence for every fibrant object X.*
3. *The right Quillen functor R reflects weak equivalences between fibrant objects and the map $A \to R\hat{F}LA$ is a weak equivalence for every cofibrant object A.*

Proof. We refer the reader to [Hov99, Corollary 1.3.16]. □

The next theorem gives us the long-awaited result and its proof acquits us of the promise to find a Quillen equivalent simplicial model category.

Theorem 3.5.3. *Let $\mathcal{M}$ be a model category and $\mathcal{M}^{\Delta^{op}}$ the realization model category. The pair (c, ev_0) of adjoint functors gives rise to a Quillen equivalence of $\mathcal{M}$ and the realization model category on $\mathcal{M}^{\Delta^{op}}$*

$$c\colon \mathcal{M} \longleftrightarrow \mathcal{M}^{\Delta^{op}} \colon ev_0.$$

Proof. According to Proposition 3.5.2 it is sufficient to verify that the right Quillen functor ev_0 reflects weak equivalences between fibrant objects and the map $A \to ev_0\hat{F}cA$ is a weak equivalence for every cofibrant object A and a functorial fibrant approximation $\hat{F}$. The first requirement is the statement in Lemma 3.5.1. For the second requirement, we assume that $\hat{F}$ is a functorial Reedy fibrant approximation. Then $\hat{F}cA$ is the simplicial resolution of the object A. Hence, it is a fibrant approximation in the realization model category. It follows that $ev_0\hat{F}cA$ is a fibrant approximation of A in $\mathcal{M}$ and the map in question is a weak equivalence. □

Summary

This diploma thesis consists of three chapters. In the first chapter we recall all the important terminologies of the theory of model categories which we will need later. We introduce the notation of a homotopical category as defined in [DHKS04], then we give the definition of a model category in such a way that the class of homotopical model categories includes all model categories. We discuss the small object argument and the cofibrantly generated model categories. Thereafter we see that for a small category $\mathcal{C}$ and a cofibrantly generated model category $\mathcal{M}$, the functor category $\mathcal{M}^{\mathcal{C}}$ admits a cofibrantly generated model structure, called the Bousfield-Kan model model structure. We mention that even for an arbitrary model category $\mathcal{M}$ there always exists a Reedy model structure on $\mathcal{M}^{\mathcal{C}}$, if the indexing category is a Reedy category. This model structure allow us to define the cosimplicial and simplicial resolutions, which are need for introducing the homotopy function complexes. At the end of this chapter, the definitions and important properties of the left proper celluar model categories and the left proper combinatorial model categories are given. All these terminologies are standard, most of the definitions are taken from the following sources: [Hir03], [Hov99], [DHK97] and [DHKS04].

The second chapter is devoted to the proof of the existence a simplicial replacement of a model category which is either left proper and cellular or left proper and combinatorial which are introduced by Hirschhorn and Smith (see [Hir03] and [Smi]). At the beginning we define simplicial structures, then we discuss the properties of a Reedy model category with a cartegorical simplicial action. As a preparation for the proof we recall the notation of a left Bousfield localization. For the rest of the chapter we follow the idea presented in Dugger's paper [Dug01a] and verify that the Reedy model structure becomes simplicial by localizing it. The only noteworthy difference in the proofs is that we do not use the Bousfield-Kan model structure as a stepping stone.

For a better understanding of the fibrations in the simplicial replacement we found in the previous chapter, we present a constructive proof as in [RSS01] in Chapter 3. The proofs here are all due to the authors of [RSS01], we just try to explain them. We define a canonical model categories as a model category with reasonable properties. It is general enough to contain the simplicial replacements of Chapter 2. 3.1.3 provides the uniqueness of the canonical model structure, hence giving a constructive proof of a simplicial replacement is reduced to constructing a canonical model category. In fact, we construct a realization model category which is a canonical model category where the fibrations have a special form. It seems that a stronger definition of a cofibrantly generated model category is necessary for proving the existence of a realization model category. We motivate the definition of weak equivalences as well as the definitions of the sets of generating cofibrations and trivial cofibrations of a realization model category respectively. After introducing the Realization Axiom, we verify the conditions in the Recognition Lemma. Once the

model structure is established by the Recognition Lemma, we reveal that the realization model category is even a simplicial one. In the last section of this chapter we show that the realization model category is Quillen equivalent to the original model category, hence a realization model category is a simplicial replacement.

Zusammenfassung

Diese Diplomarbeit besteht aus drei Kapiteln. Im ersten Kapitel wiederholen wir die wichtigsten Begriffe in der Theorie der Modellkategorie, die wir später verwenden werden. Wie in [DHKS04] führen wir zuerst die homotopischen Kategorien ein, dann definieren wir eine Modellkategorie, sodass die Klasse der homotopischen Kategorien alle Modellkategorien beinhaltet. Weiters werden Themen wie das Kleine-Objekt-Argument und die kofaser-erzeugte Modellkategorien behandelt. Wir werden sehen, dass es für eine kleine Kategorie $\mathcal{C}$ und eine kofaser-erzeugte Modelkategorie $\mathcal{M}$ immer eine kofaser-erzeugte Modellstruktur auf der Funktorkategorie $\mathcal{M}^{\mathcal{C}}$, die sogenannte Bousfield-Kan Modellstruktur, existiert. Es sei auch erwähnt, dass es für jede beliebigen Modellkategorie $\mathcal{M}$ eine sogenannte Reedy Modellstruktur auf $\mathcal{M}^{\mathcal{C}}$ existiert, falls die Indexkategorie $\mathcal{C}$ eine Reedy Kategorie ist. Diese Modellstruktur erlaubt uns kosimpliziale und simpliziale Auflösungen zu definieren, diese werden wiederum werden für die Definition der Homotopiefunktionenkomplexen benötigt. Am Ende des Kapitels besprechen wir dann die wichtigsten Eigenschaften der links eigentlichen und zellulären bzw. der links eigentlichen und kombinatorischen Modellkategorien. Die Definitionen und Eigenschaften der Objekten sind meist den Büchern [Hir03], [Hov99], [DHK97] und [DHKS04] entnommen.

Im zweiten Kapitel widmen wir uns dem Existenzbeweis eines simplizialen Ersatzes für eine Modellkategorie, die entweder links eigentlich und zellulär oder links eigentlich und kombinatorisch ist. Diese zwei wichtigen Klassen von Modellkategorien wurden Hirschhorn (vgl. [Hir03]) und Smith (vgl. [Smi]) eingeführt. Wir beginnen das Kapitel mit der Definition einer simplizialen Struktur, dann besprechen wir die Eigenschaften einer Reedy Modellstruktur, ausgestattet mit einer katgoriell simplizialen Wirkung. Nach einer Wiederholung der linken Bousfieldlokalisierung, folgen wir Dugger [Dug01a] und verifizieren, dass die Reedy Modellstruktur durch den Prozess der Lokalisierung simplizial wird. Der einzige nennenswerte Unterschied zum Beweis von Dugger ist, dass wir im Beweis die Bousfield-Kan Modellstruktur nicht verwenden werden.

Um das Verhalten der Faserungen in dem durch Lokalisierung gefundenen simplizialen Ersatz besser zu verstehen, geben wir einen konstruktiven Beweis für die Existenz in Kapitel 3. Der Beweis stammt von den Autoren von [RSS01], wir motivieren und erklären lediglich ihn. Wir definieren zuerst die kanonische Modellkategorie als eine Modellkategorie mit gewissen vernünftigen Eigenschaften. Sie ist allgemein genug um die lokalisierten Reedy Modellkategorien zu beinhalten. Da die Eindeutigkeit einer kanonischen Modellkategorie im Theorem 3.1.3. bewiesen wird, genügt es einen konstruktiven Beweis für eine kanonische Modellkategorie zu geben. Genauer bedeutet dies, dass wir eine spezielle Klasse von kanonischen Modellkategorien konsturieren, in der die Faserungen explizit beschrieben werden. Diese Modellkategorien werden Realisierungsmodellkategorie genannt. Wie sich herausstellen wird, benötigen wir für deren Ex-

istenz eine stärkere Definition der kofaser-erzeugten Modellkategorien. Außerdem motivieren wir die Definition der schwachen Equivalenzen und der Mengen der erzeugenden Kofaserungen bzw. trivialen Kofaserungen in der Realizierungsmodellkategorien. Nach der Einführung des sogenannten Realizierungsaxioms, verifizieren wir die geforderten Bedingungen in dem Recognition Lemma. Das Lemma liefert die Modellstruktur, welche, wie wir zeigen werden, sogar simplizial ist. Im letzten Abschnitt wird bewiesen, dass die Realizierungsmodelkategorie Quillen equivalent zu der ursprünglichen Modellkategorie ist, somit haben wir haben einen simplizialen Ersatz explizit konstruiert.

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