# DISSERTATION

Titel der Dissertation

# Pseudodifferential Operators, Wireless Communications and Sampling Theorems


Verfasser

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# Abstract

One objective of this thesis is to combine methods from sampling theory and time-frequency analysis in order to study contemporary problems in the theory of pseudodifferential operators and its applications to wireless communications. Another objective is to provide a rigorous mathematical justification for assumptions which are often made about wireless communication systems.

We present reconstruction formulas for the symbol of the pseudodifferential operator that use one or finitely many side diagonals of the channel matrix. We use these reconstruction formulas to derive impossibility results. We prove that the channel matrix cannot be strictly diagonal if the underlying Gabor system is a frame for $\boldsymbol{L}^2(\mathbb{R})$. Furthermore, we prove that if one or finitely many side diagonals of the channel matrix vanish, then the channel is equal to zero. We also prove a uniqueness theorem for pseudodifferential operators assuming that the underlying Gabor system is generated by the Gaussian function.

We also consider a discretized transmission model. We investigate properties of the discrete-time model, and how properties of the continuous-time model transfer to the discrete-time model. The main observation in this context is that if the continuous-time channel matrix satisfies a certain off-diagonal decay condition, then the discrete-time channel matrix satisfies an analogous off-diagonal decay condition.

# Zusammenfassung

Ein Ziel dieser Dissertation ist es, Methoden aus dem Gebiet des regulären Abtastens und der Zeit-Frequenz Analyse zu kombinieren um aktuelle Probleme in der Theorie der Pseudodifferentialoperatoren und deren Anwendungen in der drahtlosen Kommunikation zu untersuchen. Ein weiteres Ziel ist es, eine mathematische Rechtfertigung für Annahmen, die im Zusammenhang mit drahtlosen Kommunikationssystemen gemacht werden, zu liefern.

Wir präsentieren Rekonstruktionsformeln für das Symbol des Pseudodifferentialoperators, die eine, beziehungsweise endlich viele, Nebendiagonalen der Kanalmatrix verwenden. Mit Hilfe dieser Rekonstruktionsformeln formulieren wir Unmöglichkeitsresultate. Wir zeigen, dass die Kanalmatrix keine Diagonalmatrix sein kann, wenn das zugrundeliegende Gabor System einen Frame für $\boldsymbol{L}^2(\mathbb{R})$ bildet. Wir zeigen auch, dass, wenn eine oder endlich viele Nebendiagonalen der Kanalmatrix verschwinden, der Kanal konstant Null ist. Wir beweisen auch ein Eindeutigkeitsresultat für Pseudodifferentialoperatoren unter der Annahme, dass das zugrundeliegende Gabor System von der Gauß-Funktion erzeugt wird.

Wir betrachten weiters ein diskretes Übertragungsmodell. Wir untersuchen Eigenschaften des diskreten Modells und wie sich Eigenschaften des kontinuierlichen Modells auf das diskrete Modell übertragen. Die wesentliche Erkenntnis dabei ist, dass, wenn die kontinuierliche Kanalmatrix eine bestimmte Bedingung bezüglich Abklingverhalten entlang der Nebendiagonalen erfüllt, dann erfüllt auch die diskrete Kanalmatrix eine analoge Bedingung bezüglich Abklingverhalten entlang der Nebendiagonalen.

# Contents

# Chapter 1

# Introduction

## 1.1 Overview

One objective of this thesis is to combine methods from sampling theory and time-frequency analysis to deal with contemporary problems in the theory of pseudodifferential operators and its applications to wireless communications. Another objective is to consider common assumptions which are often made about wireless communication systems and provide a rigorous mathematical justification.

It is generally recognized in the engineering community that wireless communication channels can be conveniently modeled as pseudodifferential operators. Realistic wireless channels have a fairly complicated structure due to the presence of multipaths and the Doppler effect. We consider a special class of operators, namely operators which possess a symbol with compactly supported Fourier transform. A special case of such operators is the class of underspread operators, which are defined in terms of the size of the support of the symbol.

To estimate wireless communcation channels, it is customary to use pilot symbols. This means that at certain locations known at the receiver, specific symbols are placed instead of user data. In the spirit of sampling theory, we can interpret this approach as sampling of a bandlimited symbol at the pilot positions. If we assume that the Fourier transform of the symbol is compactly supported, then it is possible to reconstruct the symbol of the pseudodifferential operator from the receive signal and pilot information. The first part of this thesis contains results of this type. Specifically, we show that it is possible to reconstruct the symbol from the main diagonal, from a side diagonal, or from finitely many side diagonals of the channel matrix. We further propose an iterative method for joint channel estimation and equalization of the receive signal which is based on such a reconstruction formula. Several channel models, which are frequently used in wireless communications, involve point scatterers, which are viewed as distributions. This motivates distributional versions of the reconstruction theorems.

Our next objective is to study the structure of the channel matrix with respect to the underlying Gabor system. We investigate under what conditions diagonal channel matrices are necessarily equal to zero. We also derive reconstruction formulas for the symbol of the operator from the entries located on a finite set of diagonals. As an immediate consequence, we obtain uniqueness theorems. One important consequence of these uniqueness theorems is that the channel matrix cannot be diagonal. We further develop a uniqueness theorem for a Gabor system generated by the Gaussian function.

In the engineering community it is generally accepted that discrete-time transmission models inherit properties like off-diagonal decay from the continuous-time model. From a mathematical viewpoint this is absolutely not trivial, and has to be justified. Therefore, we consider a discrete-time transmission model [48, 65] and describe how the discrete-time model is derived from the continuous-time model. The main result in this context is that the off-diagonal decay property of the continuous-time model carries over to the discrete-time model. Futhermore, we investigate properties of the discrete-time model, and establish a connection between the discrete-time model and pseudodifferential operators acting on $\boldsymbol{\ell}^2(\mathbb{Z})$.

## 1.2 Previous Work

The goal of channel estimation is to compute approximate entries of the channel matrix or other parameters that determine the channel matrix. For channel estimation with a multicarrier system it is common to use pilot-aided estimation. This means that known symbols (pilot symbols) are sent at known positions to estimate the channel at the receiver. Methods that do not use pilot symbols are called blind estimation methods, but we do not study such methods in this thesis. Wireless channels that only cause multipath propagation of the transmit signal are modeled with a convolution operator. In this case, the corresponding pseudodifferential operator reduces to a Fourier multiplier. Since such channels are diagonal in the frequency domain, they are called frequency selective channels. Pilot-aided estimation for frequency selective channels is well known [7]. When there is a moving reflector in the channel, or the transmitter or the receiver are moving, the wireless channel is not frequency selective any more. Such wireless channels are called doubly selective, and we work with these channels throughout this thesis.

A special class of wireless channels are channels that correspond to underspread operators. The notion of underspread operators was studied by W. Kozek [36, 37, 38]. Since that time, applications of underspread operators to wireless communications have been investigated by the engineering community [5, 11, 39, 46]. Our work is also related to identifcation and reconstruction of operators with band-limited Kohn-Nirenberg symbols presented in [32, 40, 51]. In this context, a great deal is known about operators with symbols in $\boldsymbol{L}^2(\mathbb{R}^{2d})$. By Pool's theorem [53], an operator with a symbol in $\boldsymbol{L}^2(\mathbb{R}^{2d})$ is a Hilbert-Schmidt operator [40, 44, 52]. Having a symbol in $\boldsymbol{L}^2(\mathbb{R}^{2d})$ is quite a strict assumption, e.g., it excludes distortion-free channels and time-invariant channels. Consequently, we also need to consider more general symbol classes, like modulation spaces and tempered distributions.

The theory of pseudodifferential operators is mainly developed in the context of function spaces. However, pseudodifferential operators have also an interesting connection to wireless communications and time-frequency analysis [21, 22, 62]. Of special interest are pseudodifferential operators with symbols in a modulation space [14, 23, 26, 66, 67]. The modulation space $\boldsymbol{M}^{\infty,1}(\mathbb{R}^{2d})$, also called the Sjöstrand class, is especially relevant to this thesis, since the corresponding channel matrix satisfies certain off-diagonal decay conditions [21, 24, 28, 59, 60].

## 1.3 Contributions

In this thesis we study relationships among sampling theory, pseudodifferential operators and channel estimation. We propose several reconstruction formulas for the symbol of a pseudodifferential operator and propose how they can be used for channel estimation and equalization of the receive signal. We also provide impossibility results for the channel matrix with respect to the underlying Gabor system. Furthermore, we consider a discrete-time transmission model and investigate its properties and its relationship to the continuous-time model.

The main contributions of this thesis can be summarized as follows.

- We derive a reconstruction formula for the symbol of a pseudodifferential operator with compactly supported spreading function from the main diagonal of the operator (Section 4.1). This reconstruction formula is based on the assumption that the short-time Fourier transform of the window function $g$, which generates the Gabor system, does not vanish on the support of the spreading function.

- We derive the reconstruction formulas for symbols in $\boldsymbol{L}^2(\mathbb{R}^{2d})$, $\boldsymbol{L}^p(\mathbb{R}^{2d})$, $1 < p < \infty$ and $\mathcal{S}'(\mathbb{R}^{2d})$.

- We derive reconstruction formulas for the symbol of a pseudodifferential operator from a side diagonal and from finitely many diagonals (Section 4.3 and Section 4.4).

- We propose an iterative method for joint channel estimation and equalization of the receive signal that uses such a reconstruction formula. (Section 4.5).

- We prove that the channel matrix cannot be diagonal if the underlying Gabor system is a frame (Chapter 5).

- We prove that the channel matrix cannot vanish identically on one side diagonal, or on finitely many diagonals simultaneously (Chapter 5). These results imply that a non-trivial channel matrix cannot be diagonal.

- We prove that if the channel matrix is identically zero, and the underlying Gabor system is generated by the Gaussian function, then the operator is identically zero (Chapter 5).

- We investigate properties of a discrete-time transmission model and establish a connection between the discrete-time model and pseudodifferential operators acting on $\boldsymbol{\ell}^2(\mathbb{Z})$ (Section 6.3).

- We prove that if the continuous-time symbol $\sigma$ belongs to the Sjöstrand class $\boldsymbol{M}^{\infty,1}_{v\circ j^{-1}}(\mathbb{R}^2)$, then the discrete symbol $\sigma_d$ belongs to the class $\boldsymbol{M}^{\infty,1}_{(v\otimes 1)\circ j^{-1}}(\mathbb{Z}\times\mathbb{T})$ (Section 6.3).

- We provide a condition under which the discrete-time model inherits causality from the continuous-time model (Section 6.4).

# Chapter 2

# Mathematical Preliminaries

## 2.1 Fourier Transform

Here we collect the definition and some basic properties of the Fourier transform. For more details and proofs see [12] and references therein.

The *Fourier transform* of a function $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx, \quad \xi \in \mathbb{R}^d. \tag{2.1}$$

If we want to emphasize the action of the Fourier transform as a linear operator on a function space, then we write $\mathcal{F}f$ instead of $\hat{f}$.

The *lemma of Riemann-Lebesgue* provides some information about the decay properties of the Fourier transform.

**Lemma 2.1.1** (Riemann-Lebesgue)**.** *Let* $f \in \boldsymbol{L}^1(\mathbb{R}^d)$*. Then* $\hat{f}$ *is uniformly continuous and* $\lim_{|\xi| \to \infty} |\hat{f}(\xi)| = 0$*.*

The following mapping properties follow from Lemma 2.1.1:

$$\mathcal{F} : \boldsymbol{L}^1(\mathbb{R}^d) \to \boldsymbol{C}_0(\mathbb{R}^d), \tag{2.2}$$

where $\boldsymbol{C}_0(\mathbb{R}^d)$ is the Banach space of continuous functions vanishing at infinity.

The image of $\boldsymbol{L}^1(\mathbb{R}^d)$ under the Fourier transform is a subalgebra of $\boldsymbol{C}_0(\mathbb{R}^d)$ with pointwise multiplication. This algebra is called the *Fourier algebra* and denoted by $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

Another fundamental property of the Fourier transform is *Plancherel's theorem.*

**Theorem 2.1.2** (Plancherel)**.** *If $f \in \boldsymbol{L}^1 \cap \boldsymbol{L}^2(\mathbb{R}^d)$ then*

$$\|f\|_2 = \|\hat{f}\|_2. \tag{2.3}$$

A consequence of Theorem 2.1.2 is that the Fourier transform extends to a unitary operator on $\boldsymbol{L}^2$ and satisfies *Parseval's formula*

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle, \tag{2.4}$$

for all $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$. In the field of signal analysis, Plancherel's theorem states that the Fourier transform preserves the energy of a signal.

When dealing with functions that depend on two variables, we can also define a *partial Fourier transform.*

**Definition 2.1.3.** Let $F$ be a function on $\mathbb{R}^{2d}$. The *partial Fourier transform* in the first variable $\mathcal{F}_1$ is defined as

$$\mathcal{F}_1 F(x, \xi) = \int_{\mathbb{R}^d} F(t, \xi) e^{-2\pi i x \cdot t} dt. \tag{2.5}$$

The *partial Fourier transform* in the second variable $\mathcal{F}_2$ is defined in a similar fashion.

A very important property of the Fourier transform is its action on a convolution product.

The convolution of two functions $f$ and $g$ is defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(t) g(x - t) dt. \tag{2.6}$$

We have

$$\widehat{(f * g)} = \hat{f} \cdot \hat{g}. \tag{2.7}$$

and

$$\widehat{(f \cdot g)} = \hat{f} * \hat{g}. \tag{2.8}$$

**Example 2.1.4.** If $\varphi_\alpha = e^{-\pi x^2/\alpha}$ is the Gaussian function with the scale parameter $\alpha > 0$ on $\mathbb{R}^d$, then [21, p.17]

$$\widehat{\varphi_\alpha}(\xi) = \alpha^{d/2}\varphi_{1/\alpha}(\xi). \tag{2.9}$$

We observe that the Fourier transform of a Gaussian function is again a Gaussian function whose scale is the reciprocal of the original scale. For $\alpha = 1$, we get $(e^{-\pi x^2})\hat{}(\xi) = e^{-\pi\xi^2}$.

Next we state the Poisson Summation formula for rectangular lattices [21, p.16].

**Definition 2.1.5.** For a given lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, the lattice $\Lambda^\perp = \frac{1}{a}\mathbb{Z}^d \times \frac{1}{b}\mathbb{Z}^d$ is called the dual lattice and the lattice $\Lambda^\circ = \frac{1}{b}\mathbb{Z}^d \times \frac{1}{a}\mathbb{Z}^d$ is called the adjoint lattice.

**Theorem 2.1.6.** *Let $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$. If for some $\varepsilon > 0$ and $C > 0$ we have $|f(z)| \le C(1+|z|)^{-2d-\varepsilon}$ and $|\hat{f}(\xi)| \le C(1+|\xi|)^{-2d-\varepsilon}$, then*

$$\sum_{\lambda\in\Lambda} f(z+\lambda) = \frac{1}{(ab)^d}\sum_{\eta\in\Lambda^\perp}\hat{f}(\eta)e^{2\pi i\eta\cdot z}. \tag{2.10}$$

*The identity holds pointwise for all $z \in \mathbb{R}^{2d}$, and both sums converge absolutely for all $z \in \mathbb{R}^{2d}$.*

The Poisson summation formula holds also in $\boldsymbol{L}^p(\mathbb{R}^d), 1 < p < \infty$.

**Theorem 2.1.7.** *Let $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$. If $f \in \boldsymbol{L}^p(\mathbb{R}^d), 1 < p < \infty$, is compactly supported, then*

$$\sum_{\lambda\in\Lambda} f(z+\lambda) = \frac{1}{(ab)^d}\sum_{\eta\in\Lambda^\perp}\hat{f}(\eta)e^{2\pi i\eta\cdot z}. \tag{2.11}$$

*The series on the left hand side is finite on $\mathbb{T}^d$, the series on the right hand side converges in $\boldsymbol{L}^p(\mathbb{T}^d)$, where $\mathbb{T}^d = \mathbb{R}^d/\Lambda$.*

*Proof.* Since $f$ is compactly supported, the sum on the left runs over a finite index set, and belongs to $\boldsymbol{L}^p(\mathbb{T}^d)$. Moreover, $f$ is in $\boldsymbol{L}^1(\mathbb{R}^d)$, and the sum is also in $\boldsymbol{L}^1(\mathbb{T}^d)$. We compute the Fourier coefficients of $\mathcal{P}f(z) = \sum_{\lambda\in\Lambda} f(z+\lambda)$ on the torus $\mathbb{T}^d$:

$$\begin{aligned}
(\mathcal{P}f)\hat{}(\eta) &= \int_{\mathbb{T}^d} \sum_{\lambda\in\Lambda} e^{-2\pi i\eta\cdot x} f(z+\lambda)dz && (2.12)\\
&= \sum_{\lambda\in\Lambda} \int_{\mathbb{T}^d} e^{-2\pi i\eta\cdot x} f(z+\lambda)dz && (2.13)\\
&= \int_{\mathbb{R}^d} e^{-2\pi i\eta\cdot x} f(z)dz && (2.14)\\
&= \hat{f}(\eta). && (2.15)
\end{aligned}$$

It follows from the classical $\boldsymbol{L}^p$-theory of Fourier series on $\boldsymbol{L}^p(\mathbb{T}^d)$ that the sum on the right hand side of Equation (2.11) converges in $\boldsymbol{L}^p(\mathbb{T}^d)$ to $\mathcal{P}f$. $\square$

## 2.2 Time-Frequency Shifts

We collect some basic definitions of time-frequency analysis. More details can be found in [21]. The two fundamental operations are the operations of translation and modulation

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i\xi\cdot t} f(t), \tag{2.16}$$

for $t, x, \xi \in \mathbb{R}^d$.

**Definition 2.2.1.** The *time-frequency shift operator* $\pi$ is defined as

$$\pi(z)f(t) = M_\xi T_x f(t) = e^{2\pi i\xi\cdot t} f(t-x), \tag{2.17}$$

for $t \in \mathbb{R}^d$ and $z = (x, \xi) \in \mathbb{R}^{2d}$. This operator combines a translation and a modulation.

We could also consider a shift-operator of the form $T_x M_\xi$. It is important to define which form is used because of the following *commutation relation*

$$T_x M_\xi = e^{-2\pi i\xi\cdot x} M_\xi T_x. \tag{2.18}$$

Time-frequency shifts have the following basic properties. They are isometries on $\boldsymbol{L}^p(\mathbb{R}^d)$ for $1 \le p \le \infty$, that is

$$\|M_\xi T_x f\|_p = \|f\|_p. \tag{2.19}$$

Applying the Fourier transform gives

$$\widehat{T_x f} = M_{-x}\hat{f} \quad \text{and} \quad \widehat{M_\xi f} = T_\xi \hat{f}. \tag{2.20}$$

This leads to us to a fundamental formula of time-frequency analysis

$$\widehat{T_x M_\xi f} = M_{-x} T_\xi \hat{f}. \tag{2.21}$$

Let the *involution* be defined by

$$f^*(x) = \overline{f(-x)} \tag{2.22}$$

and the *reflection* be defined by

$$\mathcal{I}f(x) = f(-x). \tag{2.23}$$

The involution and the reflection are related in the following way

$$\widehat{f^*} = \overline{\hat{f}} \quad \text{and} \quad \widehat{\mathcal{I}f} = \mathcal{I}\hat{f}. \tag{2.24}$$

## 2.3 Short-Time Fourier Transform

In this section we define the short-time Fourier transform and investigate some of its properties. The short-time Fourier transform is a convenient tool for a joint time-frequency analysis of functions or signals.

**Definition 2.3.1.** The *short-time Fourier transform* (STFT) of a function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ with respect to a window $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is defined as

$$V_g f(x,\xi) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i \xi \cdot t} dt, \tag{2.25}$$

for $x, \xi \in \mathbb{R}^d$.

The STFT is also well-defined if $f$ is a tempered distribution and $g$ is a Schwartz function, see Section 2.6 for the corresponding definitions. The domain of the STFT can even be extended to the case when both $f$ and $g$ are tempered distributions. Then $V_g f$ is also a well-defined tempered distribution [21, p.41].

The STFT can be written in several equivalent forms.

**Lemma 2.3.2.** *If $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$, then the STFT is uniformly continuous on $\mathbb{R}^{2d}$ and*

$$\begin{aligned} V_g f(x,\xi) &= \langle f, M_\xi T_x g\rangle && (2.26)\\ &= \langle \hat{f}, T_\xi M_{-x}\hat{g}\rangle && (2.27)\\ &= e^{-2\pi i\xi\cdot x} V_{\hat{g}}\hat{f}(\xi,-x) && (2.28)\\ &= e^{-\pi i \xi\cdot x}\int_{\mathbb{R}^d} f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)} e^{-2\pi i\xi\cdot t}dt. && (2.29)\end{aligned}$$

The proof of this Lemma is done by straightforward computations. The uniform continuity of the STFT follows from corresponding properties of the operators $T_x$ and $M_\xi$ [21, Lemma 3.1.1].

Formula (2.28) is often referred to as the *fundamental identity of time-frequency analysis*. We observe that, in this case, the Fourier transform performs a rotation of the time-frequency plane by an angle of $\frac{\pi}{2}$.

As an example, we compute the STFT of the Gaussian function explicitly. We use the STFT of the Gaussian function again in Chapter 5.

**Example 2.3.3.** Let $\varphi_\alpha(x) = e^{-\pi x^2/\alpha}$ be the Gaussian function with the scale parameter $\alpha > 0$ on $\mathbb{R}^d$. We use Equation (2.9) to compute

$$\begin{aligned} \langle \varphi_\alpha, M_\xi T_x \varphi_\alpha\rangle &= \int_{\mathbb{R}^d} e^{-\pi t^2/\alpha} e^{-\pi(t-x)^2/\alpha} e^{-2\pi i\xi t} dt && (2.30)\\ &= e^{-\pi x^2/(2\alpha)}\int_{\mathbb{R}^d} e^{-2\pi(t-x/2)^2/\alpha} e^{-2\pi i \xi t}dt && (2.31)\\ &= \varphi_{2\alpha}(x)(T_{\frac{x}{2}}\varphi_{\frac{\alpha}{2}})\hat{\ }(\xi) && (2.32)\\ &= e^{-\pi i x\xi}\left(\frac{\alpha}{2}\right)^{\frac{d}{2}}\varphi_{2\alpha}(x)\varphi_{\frac{2}{\alpha}}(\xi). && (2.33)\end{aligned}$$

combining this with Equation (2.26), we obtain

$$\begin{aligned} V_{\varphi_\alpha}\varphi_\alpha(x,\xi) &= \langle \varphi_\alpha, M_\xi T_x \varphi_\alpha \rangle &(2.34)\\ &= \left(\frac{\alpha}{2}\right)^{\frac{d}{2}} e^{-\pi i x\xi} e^{-\pi x^2/2\alpha} e^{-\pi \xi^2 \alpha/2}. &(2.35) \end{aligned}$$

We observe that the STFT of a Gaussian function has a simple expression in terms of Gaussian functions.

We use the following property of the STFT, see [28], which follows immediately from the definition of the STFT and the commutation relations for time-frequency shifts (2.18).

**Lemma 2.3.4.** *For $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$, the STFT satisfies the following property.*

$$V_{M_\zeta T_v g}(M_\eta T_u f)(x,\xi) = e^{-2\pi i(\xi-\eta)\cdot u} e^{2\pi i \zeta\cdot(x-u)} T_{(u-v,\eta-\zeta)} V_g f(x,\xi), \qquad (2.36)$$

*for $u, v, \eta, \zeta, x, \xi \in \mathbb{R}^d$.*

## 2.4 Frames and Riesz Bases

The notion of a basis is often too restrictive. One problem in communication engineering is that bases are very sensitive to a loss of information because the basis coefficients are unique. Therefore, one introduces more redundancy. This motivates the more general concept of frames which is used to describe redundant information. The concept of frames was introduced by Duffin and Schaeffer already in 1952 [9].

**Definition 2.4.1.** A sequence $\{f_i : i \in I\}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if and only if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2. \qquad (2.37)$$

The two constants $A$ and $B$ are called *frame bounds*. If $A = B$, then the sequence $\{f_i : i \in I\}$ is called a *tight frame*.

We immediately observe that the concept of frames generalizes the concept of (orthonormal) bases, because the definition of an (orthonormal) basis is simply a special case of Definition 2.4.1. Namely, an orthonormal basis is a tight frame with a normalized window function and frame bounds $A = B = 1$. Similarly, the union of two orthonormal bases is a tight frame with a normalized window function and frame bounds $A = B = 2$.

Next we introduce some important operators which are studied for a better understanding of frames and reconstruction methods.

Let $\{f_i : i \in I\} \subseteq \mathcal{H}$. The *coefficient operator* or *analysis operator* $C$ is defined by

$$Cf = \{\langle f, f_i\rangle : i \in I\}. \tag{2.38}$$

The *reconstruction operator* or *synthesis operator* $D$ is defined by

$$Dc = \sum_{i \in I} c_i f_i \in \mathcal{H}, \tag{2.39}$$

for a finite sequence $c = (c_i)_{i \in I}$. The *frame operator* $S$ is defined on $\mathcal{H}$ by

$$Sf = \sum_{i \in I} \langle f, f_i\rangle f_i. \tag{2.40}$$

The next question is how to reconstruct $f$ from the frame coefficients.

**Proposition 2.4.2.** *If $\{f_i : i \in I\}$ is a frame for $\mathcal{H}$ with frame bounds $A, B > 0$, then the operator $S$ maps $\mathcal{H}$ onto $\mathcal{H}$ and is a positive invertible operator.*

**Corollary 2.4.3.** *If $\{f_i : i \in I\}$ is a frame with frame bounds $A, B > 0$, then $\{S^{-1} f_i : i \in I\}$ is a frame with frame bounds $B^{-1}, A^{-1} > 0$.*

**Definition 2.4.4.** The frame $\{S^{-1} f_i : i \in I\}$ is called the *dual frame* of $\{f_i : i \in I\}$.

**Definition 2.4.5.** Let $\{a_i\}_{i \in I}$ be a sequence in a Banach space. A series $\sum_{i \in I} a_i$ is *unconditionally convergent* if $\sum_{i \in I} a_{\tau(i)}$ converges for every permutation $\tau$ of $I$.

**Theorem 2.4.6.** *If $\{f_i : i \in I\}$ is a frame, then every $f \in \mathcal{H}$ has non-orthogonal expansions*

$$f = \sum_{i \in I} \langle f, S^{-1} f_i\rangle f_i \tag{2.41}$$

*and*

$$f = \sum_{i \in I} \langle f, f_i \rangle S^{-1} f_i \tag{2.42}$$

*where both sums converge unconditionally in $\mathcal{H}$.*

For tight frames and orthonormal bases, the two expansions in Theorem 2.4.6 coincide. The frame coefficients are in general not unique. This motivates the definition of Riesz sequences and Riesz bases.

**Definition 2.4.7.** A sequence $\{f_i : i \in I\}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz sequence*, if and only if there exist constants $A', B' > 0$ such that the inequalities

$$A' \|c\|_2 \leq \| \sum_{i \in I} c_i f_i \|_2 \leq B' \|c\|_2 \tag{2.43}$$

hold for all finite sequences $c = (c_i)_{i \in I}$. For a Riesz sequence, the coefficients in the frame expansions (2.41) and (2.42) are unique. A Riesz sequence is called a *Riesz basis* for $\mathcal{H}$, if $\text{span}\{f_i : i \in I\} = \mathcal{H}$.

### 2.4.1 Gabor Frames

A special case of the general frame theory in Section 2.4 are the so-called *Gabor frames*. We fix a non-zero window function $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and a lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d \subseteq \mathbb{R}^{2d}$ with the lattice parameters $a, b > 0$. The parameters $a$ and $b$ are called the time-shift and the frequency-shift parameter, respectively. The set of all time-frequency shifts of $g$ parametrized by the lattice $\Lambda$,

$$\mathcal{G}(g, a, b) = \mathcal{G}(g, \Lambda) = \{\pi(\lambda) g : \lambda \in \Lambda\}, \tag{2.44}$$

is called a *Gabor system*. If such a Gabor system is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$, it is called a *Gabor frame*. The Gabor frame operator has the form

$$Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g. \tag{2.45}$$

If we want to emphasize the dependence on the window $g$ in (2.45), we write $S_{g,g}$ instead of $S$.

**Proposition 2.4.8.** *If $\mathcal{G}(g,a,b)$ is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$, then there exists a dual window $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$, such that the dual frame of $\mathcal{G}(g,a,b)$ is $\mathcal{G}(\gamma,a,b)$. Every $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ can be represented as*

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma \tag{2.46}$$

$$= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda)g. \tag{2.47}$$

*The representation (2.46) or, equivalently, (2.47) is called the Gabor expansion of $f \in \boldsymbol{L}^2(\mathbb{R}^d)$.*

In general there are infinitely many dual windows for a given window $g \in \boldsymbol{L}^2(\mathbb{R}^d)$. One special choice is the so-called *canonical dual window* $\gamma = S^{-1}g$.

**Corollary 2.4.9.** *If $\mathcal{G}(g,a,b)$ is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$ with dual window $\gamma = S^{-1}g \in \boldsymbol{L}^2(\mathbb{R}^d)$, then the inverse frame operator is given by*

$$S_{g,g}^{-1} f = S_{\gamma,\gamma} f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda)\gamma. \tag{2.48}$$

All dual windows are characterized by the so-called *Wexler-Raz biorthogonality relations.*

**Theorem 2.4.10.** *If $D_g$ and $D_\gamma$ are bounded on $\boldsymbol{\ell}^2(\mathbb{Z}^{2d})$, then the following conditions are equivalent:*

1. $S_{g,\gamma} = S_{\gamma,g} = I$ *on* $\boldsymbol{L}^2(\mathbb{R}^d)$.
2. $(ab)^{-d} \left\langle \gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g \right\rangle = \delta_{l0}\delta_{n0}$ *for* $l, n \in \mathbb{Z}^d$.

Another important result is the *Ron-Shen duality principle.*

**Theorem 2.4.11.** *Let $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and $a, b > 0$. The Gabor system $\mathcal{G}(g,a,b)$ is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$ if and only if $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is a Riesz basis for its closed linear span.*

Next we state results concerning the density of Gabor frames. We start with a necessary condition on the lattice parameters $a$ and $b$.

**Corollary 2.4.12.** *If $\mathcal{G}(g, a, b)$ is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$, then $ab \leq 1$.*

When $ab = 1$, we have the following result.

**Corollary 2.4.13.** *(i) The Gabor system $\mathcal{G}(g, a, b)$ is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$ and $ab = 1$ if and only if $\mathcal{G}(g, a, b)$ is a Riesz basis for $\boldsymbol{L}^2(\mathbb{R}^d)$.*

*(ii) The Gabor system $\mathcal{G}(g, a, b)$ is an orthonormal basis for $\boldsymbol{L}^2(\mathbb{R}^d)$ if and only if $\mathcal{G}(g, a, b)$ is a tight frame, $\|g\|_2 = 1$ and $ab = 1$.*

From Corollaries 2.4.12 and 2.4.13, we conclude that we can distinguish different cases for the product of the lattice parameters $a$ and $b$. It is customary in the field of signal analysis to speak about

- oversampling when $ab < 1$,
- critical sampling when $ab = 1$, and
- undersampling when $ab > 1$.

## 2.5 Weights

In general, a weight function is just a non-negative, locally integrable function on $\mathbb{R}^{2d}$. In time-frequency analysis, weight functions describe the decay or growth of functions. We need the following types of weights.

**Definition 2.5.1.** A weight function $v$ on $\mathbb{R}^{2d}$ is called *submultiplicative*, if

$$v(z_1 + z_2) \leq v(z_1)v(z_2), \quad \text{for all } z_1, z_2 \in \mathbb{R}^{2d}. \tag{2.49}$$

A weight function $m$ on $\mathbb{R}^{2d}$ is called *$v$-moderate* if there exists a constant $C > 0$ such that

$$m(z_1 + z_2) \leq Cv(z_1)m(z_2), \quad \text{for all } z_1, z_2 \in \mathbb{R}^{2d}. \tag{2.50}$$

Two weights $m_1, m_2$ are equivalent if there exists a constant $C > 0$ such that

$$C^{-1}m_1(z) \leq m_2(z) \leq Cm_1(z), \quad \text{for all } z \in \mathbb{R}^{2d}. \tag{2.51}$$

A common example of weight functions are *polynomial weights* on $\mathbb{R}^{2d}$. They are of the form

$$v_s(z) = (1 + |z|)^s, \tag{2.52}$$

with $z = (x, \xi) \in \mathbb{R}^{2d}$ and $s \geq 0$.

A specially interesting class of weights are those that satisfy the *Gelfand-Raikov-Shilov* condition (GRS)

$$\lim_{n \to \infty} v(nz)^{1/n} = 1 \tag{2.53}$$

for all $z \in \mathbb{R}^{2d}$.

**Definition 2.5.2.** A weight function $v$ is called an *admissible weight* if it satisfies the following properties:

1. $v$ is continuous, even and normalized so that $v(0) = 1$.
2. $v$ is submultiplicative.
3. $v$ satisfies the GRS condition.

## 2.6 Distributions

We need some well known definitions and results from distribution theory. More detailed information and proofs of the results can be found in [18, 33, 61].

**Definition 2.6.1.** Let $X$ be an open set in $\mathbb{R}^d$. The space $\mathcal{D}(X)$ consists of all infinitely differentiable functions $\varphi : X \to \mathbb{R}^d$ with compact support contained in $X$.

The vector space $\mathcal{D}'(X)$, the dual space of $\mathcal{D}(X)$, is the space of all distributions on $X$.

The *Schwartz class* is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual space, the space of *tempered distributions*, is denoted by $\mathcal{S}'(\mathbb{R}^d)$.

**Definition 2.6.2.** Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $X \subset \mathbb{R}^d$ be an open set. The *restriction* $\mathcal{R}u$ of $u$ to $X$ is defined as

$$\langle \mathcal{R}u, \varphi \rangle = \langle u, \varphi \rangle, \tag{2.54}$$

for $\varphi \in \mathcal{D}(X)$.

The following theorem can be found in [33, Theorem 2.2.1].

**Theorem 2.6.3.** *Let $X$ be an open subset of $\mathbb{R}^d$. If $u \in \mathcal{D}'(X)$ and every point in $X$ has a neighborhood to which the restriction of $u$ is zero, then $u$ is zero.*

Next we state the Poisson summation formula for distributions.

**Theorem 2.6.4.** *[18, Theorem 8.5.1] Let $\delta$ be the Dirac distribution at the origin. The following identity holds in $\mathcal{S}'(\mathbb{R}^d)$:*

$$\sum_{j \in \mathbb{Z}^d} T_j \delta = \sum_{j \in \mathbb{Z}^d} e^{2\pi i j \cdot x}. \tag{2.55}$$

We also state a version of the Poisson summation formula for compactly supported distributions.

**Corollary 2.6.5.** *[18, Corollary 8.5.1] If $u$ is a compactly supported distribution, then*

$$\sum_{j \in \mathbb{Z}^d} T_j u = \sum_{j \in \mathbb{Z}^d} \hat{u}(j) e^{2\pi i j \cdot x}. \tag{2.56}$$

In the proof of Theorem 5.1.1 we need the following version of the Poisson summation formula.

**Corollary 2.6.6.** *Let $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ be a lattice. If $u$ is a compactly supported distribution, then*

$$\sum_{\lambda \in \Lambda} T_\lambda u = \frac{1}{(ab)^d} \sum_{\nu \in \Lambda^\perp} \hat{u}(\nu) e^{2\pi i \nu \cdot x}. \tag{2.57}$$

We also use the following property of distributions [18, Theorem 10.2.1].

**Theorem 2.6.7.** *The Fourier transform of a distribution with compact support on $\mathbb{R}^d$ is an analytic function on $\mathbb{C}^d$.*

## 2.7 Modulation Spaces

In this section, we introduce modulation spaces and state some of their elementary properties. We first define mixed-norm spaces on $\mathbb{R}^{2d}$.

**Definition 2.7.1.** Let $1 \le p, q < \infty$ and let $m$ be a weight function on $\mathbb{R}^{2d}$. The weighted mixed-norm space $\boldsymbol{L}^{p,q}_m$ is defined as the space of all measurable functions (in the Lebesgue sense) on $\mathbb{R}^{2d}$ such that the norm

$$\|F\|_{\boldsymbol{L}^{p,q}_m} := \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x,\xi)|^p m(x,\xi)^p dx \right)^{\frac{q}{p}} d\xi \right)^{\frac{1}{q}} \tag{2.58}$$

is finite. If $p = \infty$ or $q = \infty$, then the corresponding norm has to be replaced by the esssential supremum as usual.

The idea of weighted mixed-norm spaces is simply to take a weighted $\boldsymbol{L}^p$-norm with respect to $x$ and a weighted $\boldsymbol{L}^q$-norm with respect to $\xi$. If $p = q$ then $\boldsymbol{L}^{p,q}_m = \boldsymbol{L}^p_m$.

Weighted mixed-norm spaces retain some properties of ordinary $\boldsymbol{L}^p$-spaces.

**Lemma 2.7.2.** *If $m$ is a $v$-moderate weight and $1 \le p, q \le \infty$, then*

- *$\boldsymbol{L}^{p,q}_m(\mathbb{R}^{2d})$ is a Banach space.*
- *$\boldsymbol{L}^{p,q}_m$ is invariant under translations $T_z, z \in \mathbb{R}^{2d}$, and*
$$\|T_z F\|_{\boldsymbol{L}^{p,q}_m} \le C v(z) \|F\|_{\boldsymbol{L}^{p,q}_m} \tag{2.59}$$
- *Duality: If $p, q < \infty$, then $(\boldsymbol{L}^{p,q}_m)^* = \boldsymbol{L}^{p',q'}_m$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

Now we are able to define *modulation spaces* [14, 21].

**Definition 2.7.3.** Let $1 \le p, q \le \infty$. Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ and a $v$-moderate weight $m$ of polynomial growth on $\mathbb{R}^{2d}$. The *modulation space* $\boldsymbol{M}^{p,q}_m(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{\boldsymbol{M}^{p,q}_m} := \|V_g f\|_{\boldsymbol{L}^{p,q}_m} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x,\xi)|^p m(x,\xi)^p dx \right)^{\frac{q}{p}} d\xi \right)^{\frac{1}{q}} \tag{2.60}$$

is finite.

Again we write $\boldsymbol{M}^p_m$ if $p = q$ and $\boldsymbol{M}^{p,q}$ if the weight $m(z) \equiv 1$ on $\mathbb{R}^{2d}$. As for weighted mixed-norm spaces we have the following properties

**Lemma 2.7.4.** *Let $m$ be a $v$-moderate weight and $1 \le p, q \le \infty$. Then*

- $\boldsymbol{M}^{p,q}_m(\mathbb{R}^{2d})$ *is a Banach space.*
- $\boldsymbol{M}^{p,q}_m$ *is invariant under time-frequency shifts and*

$$\|T_x M_\xi f\|_{\boldsymbol{M}^{p,q}_m} \le C v(x,\xi) \|f\|_{\boldsymbol{M}^{p,q}_m} \tag{2.61}$$

- *Duality: If $p, q < \infty$, then $(\boldsymbol{M}^{p,q}_m)^* = \boldsymbol{M}^{p',q'}_{1/m}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

Some modulation spaces coincide with other well-known function spaces. Let $g \in \mathcal{S}(\mathbb{R}^d)$.

1. $\boldsymbol{M}^2(\mathbb{R}^d) = \boldsymbol{L}^2(\mathbb{R}^d)$.
2. If $m(x,\xi) = m(x)$, then $\boldsymbol{M}^2_m(\mathbb{R}^d) = \boldsymbol{L}^2_m(\mathbb{R}^d)$.
3. If $m(x,\omega) = m(\omega)$, then $\boldsymbol{M}^2_m(\mathbb{R}^d) = \mathcal{F}\boldsymbol{L}^2_m(\mathbb{R}^d)$.
4. $\boldsymbol{M}^1(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$, which is known as *Feichtinger's algebra* [13].

## 2.8 Pseudodifferential Operators

In this section, we define two well known forms of pseudodifferential operators and state some of their properties.

Up to now, our focus has been on the analysis of functions or signals. Now we study linear operators acting on functions. In mathematics, one uses several forms to represent and investigate linear operators. One might represent an operator as an (infinite) matrix with respect to a basis, or as an integral operator with a (distributional) kernel. The most interesting form for us is to express an operator as a superposition of time-frequency shifts. In this case, we speak of pseudodifferential operators. Pseudodifferential operators have initially been a part of the theory of partial differential equations. However, they can also be viewed as a part of time-frequency analysis.

Before we define pseudodifferential operators, we first define the Rihaczek distribution and state some of its properties.

**Definition 2.8.1.** The *Rihaczek distribution* of two functions $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is defined as

$$R(f,g)(x,\xi) = f(x)\overline{\hat{g}(\xi)}e^{-2\pi i\xi\cdot x}, \tag{2.62}$$

for $x, \xi \in \mathbb{R}^d$.

**Proposition 2.8.2.** *Let $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$. The Rihaczek distribution has the following properties [28].*

1. *Let $\lambda = (\lambda_1, \lambda_2), \mu = (\mu_1, \mu_2) \in \mathbb{R}^{2d}$. If $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$, then*
$$R(\pi(\lambda)f, \pi(\mu)g) = e^{2\pi i(\lambda_1-\mu_1)\cdot\mu_2}\, M_{j(\mu-\lambda)}T_{(\lambda_1,\mu_2)}R(f,g), \tag{2.63}$$
*where $j(\lambda) = j(\lambda_1, \lambda_2) = (\lambda_2, -\lambda_1)$.*
2. *$\widehat{R(f,g)} = \mathcal{U}V_g f$, where $\mathcal{U}F(\xi, x) = F(-x, \xi)$ and $(\xi, x) \in \mathbb{R}^{2d}$.*

Now we can define pseudodifferential operators.

The *Kohn-Nirenberg transform* of a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is the operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ defined by

$$\langle \sigma^{KN} f, g\rangle = \langle \sigma, R(g,f)\rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{2.64}$$

If $\sigma$ is a (measurable) function on $\mathbb{R}^{2d}$, then the Kohn-Nirenberg transform can written as

$$\sigma^{KN} f(x) = \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\eta, u) M_\eta T_{-u} f(x)\, du d\eta. \tag{2.65}$$

Alternatively, we can also write

$$\sigma^{KN} f(x) = \int_{\mathbb{R}^d} \sigma(x,\xi)\, \hat{f}(\xi)\, e^{2\pi i x\cdot\xi} d\xi. \tag{2.66}$$

We call $\sigma$ the *Kohn-Nirenberg symbol* and $\hat{\sigma}$ the *spreading function* of the operator $\sigma^{KN}$, since it describes how much the function $f$ is "spread out" in time and frequency due to a delay spread and a Doppler spread, and we adopt this terminology.

We say that the symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ of a pseudodifferential operator belongs to the modulation space $\boldsymbol{M}_v^{\infty,1}(\mathbb{R}^{2d})$, also known as the Sjöstrand class [24, 28, 59, 60], if its STFT satisfies

$$\int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |\langle \sigma, M_\zeta T_z \Phi\rangle| v(\zeta) d\zeta < \infty \tag{2.67}$$

for a window function $\Phi(x,\xi),(x,\xi) \in \mathbb{R}^{2d}$. Note that that the STFT of a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is a function on $\mathbb{R}^{4d}$. It is often convenient to consider symbols in the Sjöstrand class, because of the almost diagonalization property, see Theorem 2.8.3, and because the corresponding pesudodifferential operators are bounded on all the modulation spaces, in particular on $\boldsymbol{L}^2(\mathbb{R}^d)$.

**Theorem 2.8.3** ([24, 28])**.** *Let $g \in \boldsymbol{M}_v^1(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^{2d}$ a lattice, assume that $\mathcal{G}(g,\Lambda)$ is a tight Gabor frame for $\boldsymbol{L}^2(\mathbb{R}^d)$ and $j(x,\xi) = (-\xi, x)$, for $(x,\xi) \in \mathbb{R}^{2d}$. Then for $\sigma \in \boldsymbol{M}^\infty(\mathbb{R}^{2d})$ the following properties are equivalent:*

*(i)* $\sigma \in \boldsymbol{M}^{\infty,1}_{v\circ j^{-1}}(\mathbb{R}^{2d})$.

*(ii) There exists a function $H \in \boldsymbol{L}_v^1(\mathbb{R}^{2d})$ such that*

$$|\langle \sigma^{KN}\pi(z)g, \pi(y)g\rangle| \le H(y-z), \tag{2.68}$$

*for all $y,z \in \mathbb{R}^{2d}$.*

*(iii) There exists a sequence $h \in \boldsymbol{\ell}_v^1(\Lambda)$ such that*

$$|\langle \sigma^{KN}\pi(\lambda)g, \pi(\mu)g\rangle| \le h(\mu-\lambda), \tag{2.69}$$

*for all $\lambda,\mu \in \Lambda$.*

*Remark.* The equivalence $(i) \Leftrightarrow (ii)$ requires only that $g \in \boldsymbol{M}_v^1(\mathbb{R}^d)$ without any restriction. The implication $(ii) \Rightarrow (iii)$ holds for arbitrary Gabor systems with $g \in \boldsymbol{M}_v^1(\mathbb{R}^d)$. Only the implication $(iii) \Rightarrow (ii)$ requires that the Gabor system is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$.

Theorem 2.8.3 also holds for locally compact abelian groups[28, Theorem 4.3]. More detailed information on pseudodifferential operators and the Kohn-Nirenberg transform can be found in [15, 17, 21, 22, 26].

## 2.9 The Sampling Theorem

In digital signal processing, one studies a discrete representation of the signal $f$ obtained by sampling $f$ on a discrete set. A question arises whether and how $f$ can be reconstructed from its sampled values. Thus the main objective of sampling theory is to formulate conditions under which it is possible to recover particular classes of functions from given sets of discrete samples. A second goal is to develop explicit reconstruction schemes for the analysis and processing of digital signals.

Since in general infinitely many functions can have the same sampled values, it is necessary to impose some a priori conditions on $f$. One common assumption is that $f$ is *bandlimited*.

**Definition 2.9.1.** For a fixed $\Omega > 0$, the space $B_\Omega$ of *bandlimited functions* with highest frequency $\Omega$ is defined as

$$B_\Omega = \{f \in \boldsymbol{L}^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subseteq [-\Omega, \Omega]\}. \tag{2.70}$$

Now we can formulate the classical Whittaker-Kotelnikov-Shannon (WKS) sampling theorem [6, 30, 34, 41, 50].

**Theorem 2.9.2.** *Every function $f \in B_\Omega$ can be recovered from the set of samples $\{f(\frac{k}{2\Omega}) : k \in \mathbb{Z}\}$ by the formula*

$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2\Omega}\right) \operatorname{sinc}(2\Omega x - k), \tag{2.71}$$

*where* $\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}$, *and the series converges uniformly and absolutely.*

Theorem 2.9.2 also holds for bandlimited functions in $\boldsymbol{L}^p(\mathbb{R}), 1 \leq p < \infty$. For bandlimited functions in $\boldsymbol{L}^p(\mathbb{R}), 1 < p < \infty$, the sum converges also in the $\boldsymbol{L}^p$-sense [3].

The sampling rate of $2\Omega$ is called the *Nyquist rate*. One can show [41, 43], that the Nyquist rate is the smallest possible sampling rate to reconstruct $f$ uniquely. It is also possible to perform sampling with a higher sampling rate (oversampling). In practice, oversampling helps to reduce noise and improve stability.

To extend the sampling theorem to bandlimited functions in $\boldsymbol{L}^2(\mathbb{R}^d)$ sampled at a rectangular lattice, one defines

$$\operatorname{sinc}(x) := \prod_{j=1}^{d} \operatorname{sinc}(x_j), \tag{2.72}$$

where $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, as a sampling kernel [30]. This kernel is used in Theorems 4.1.5 and 4.3.5.

There are several other variations of the sampling theorem which deal with generalized sampling kernels or sampling in shift invariant spaces [6, 30, 34, 41, 42, 43, 64]. A more general setting is that of irregular sampling where the sampling points are irregularly distributed [1, 2]. In this thesis, we only consider uniform sampling.

### 2.9.1 Generalized Sampling

In this section, we present a generalization of the sampling theorem developed by Papoulis [41, 49, 50]. With this generalization, it is possible to reconstruct a function from $n$ filtered versions. More precisely, given $n$ functions $k_j$, $1 \le j \le n$, every function $f \in B_\Omega$ can be reconstructed from samples of the functions $g_j = f * k_j, 1 \le j \le n$. Each of the functions $g_j$ is sampled at $1/n^{th}$ of the original sampling rate. Define $\Omega_n = \Omega/n$ and $T_n = 1/(2\Omega_n)$. The goal is to reconstruct $f$ from the sample set

$$\{(f * k_j)(mT_n), 1 \le j \le n, m \in \mathbb{Z}\}. \tag{2.73}$$

We have the following theorem.

**Theorem 2.9.3.** *Let $k_j$, $1 \le j \le n$ be $n$ functions in $\boldsymbol{L}^1(\mathbb{R})$ such that for* *every $t \in \mathbb{R}$ and $u \in [\Omega - \Omega_n, \Omega]$, the linear system*

$$2\Omega_n \sum_{j=1}^{n} H_j(u,t)\widehat{k_j}(u - 2v\Omega_n) = e^{-2\pi i v t/\Omega_n}, \tag{2.74}$$

*$0 \le v \le n-1$, has a unique solution for the quantities $H_j(u,t), 1 \le j \le n$. If*

$$h_j(t) = \int_{[\Omega-\Omega_n,\Omega]} H_j(u,t)e^{2\pi i u t}du, \tag{2.75}$$

*then every function $f \in B_\Omega$ can be reconstructed from the sampled values $(f * k_j)(mT_n)$ by the formula*

$$f(t) = \sum_{j=1}^{n} \sum_{m=-\infty}^{\infty} (f * k_j)(mT_n)h_j(t - mT_n). \tag{2.76}$$

## 2.10 Vandermonde Matrix

**Definition 2.10.1.** A *Vandermonde matrix* is a matrix with the terms of a geometric progression in each row, i.e., an $m \times n$ matrix

$$V(x_1, x_2, \ldots, x_m) = \begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \ldots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \ldots & x_m^{n-1} \end{pmatrix}, \tag{2.77}$$

or

$$V_{i,j} = x_i^{j-1} \tag{2.78}$$

for all indices $1 \le i \le m$, $1 \le j \le n$.

**Lemma 2.10.2.** *[20, p.1078] The determinant of a square Vandermonde matrix $V$ can be written as*

$$\det(V) = \prod_{1 \le i < j \le n} (x_j - x_i). \tag{2.79}$$

The following corollary follows immediately.

**Corollary 2.10.3.** *A Vandermonde matrix is invertible if and only if $x_i \neq x_j$ for all $i \neq j$.*

# Chapter 3

# Mathematical Model of Wireless Transmission

## 3.1 Wireless Channel Model

Our description of wireless channels and the transmission setup follows [21, Chapter 14] and [50, 62]. We consider a time-varying wireless channel, which occurs, e.g., when the transmitter and the receiver are in motion relative to each other. It is also possible that there are moving reflectors in the channel. The presence of scatterers in the wireless channel results in multiple versions of the transmitted signal, which give rise to multipath propagation, see Figure 3.1. Multipath propagation leads to time dispersion, since the transmission pulses are spread out in time. This delay spread can cause intersymbol interference (ISI), i.e., interference between successive symbols. Furthermore, a relative motion between the transmitter and the receiver, or moving reflectors in the channel, result in a Doppler spread due to different Doppler shifts of each of the multipath components. This Doppler spread causes frequency dispersion, which can lead to inter carrier interference (ICI), i.e., interference between transmission pulses. In this section, we informally present a mathematical model of wireless channels.

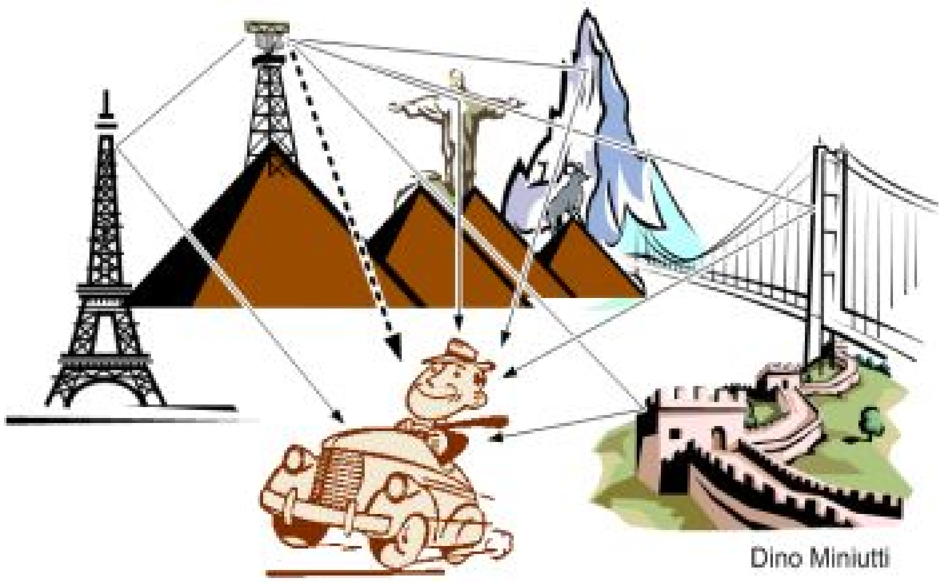


Figure 3.1: Wireless Channel [source: www.kn-s.dlr.de].

### 3.1.1 Time-Invariant Channel

Let us consider the case of a linear time-invariant communication channel $H$, that is when the transmitter and the receiver are stationary. Linear means that

$$H(c_1 f_1 + c_2 f_2) = c_1 H f_1 + c_2 H f_2 \tag{3.1}$$

for all inputs $f_1, f_2$ and coefficients $c_1, c_2 \in \mathbb{C}$.

A time-invariant system commutes with time shifts

$$H(T_x f) = T_x H f \tag{3.2}$$

for all $x \in \mathbb{R}^d$. That is, a shift of the input results in an equal shift of the output.

It is well known that a linear time-invariant channel $H$ can be expressed as [44, 62]

$$y(t) = (Hx)(t) = (h * x)(t) = \int_{\mathbb{R}^d} h(t-s)x(s)ds, \tag{3.3}$$

where $h$ is the *impulse response* of the wireless channel, i.e., $h(t)$ is the response at time $t$ to a unit pulse (a $\delta$-distribution) transmitted at time 0. Here $x$ denotes the transmitted signal and the received signal is denoted by $y$. In (3.3) and in the rest of this thesis we ignore additive noise.

The *transfer function* $\hat{h}$ is determined as the response to a "pure frequency" $e^{2\pi i\xi\cdot t}$. We get

$$H(e^{2\pi i\xi\cdot s})(t) = \int_{\mathbb{R}^d} h(s)e^{2\pi i\xi\cdot(s-t)}ds = \hat{h}(\xi)e^{2\pi i\xi\cdot t}. \tag{3.4}$$

In other words, the "pure frequencies" are "eigenfunctions" of the time-invariant channel $H$ with eigenvalues $\hat{h}(\xi)$.

Considering a baseband communication system, that is

$$\hat{x}(\xi) \in [-\Omega, \Omega]^d, \quad \text{for some } \Omega > 0, \tag{3.5}$$

we can express (3.3) equivalently as

$$y(t) = \int_{-\Omega}^{\Omega} \hat{h}(\xi)\hat{x}(\xi)e^{2\pi i\xi\cdot t}d\xi. \tag{3.6}$$

Thus multipath propagation has the effect of attenuating different frequency components of the transmitted signal differently, which is also referred to as *frequency selective fading.*

In the language of mathematics, a linear time-invariant channel $H$ is a convolution operator or a Fourier multiplier of the form

$$Hx = h * x = \mathcal{F}^{-1}(\hat{h}\ \hat{x}). \tag{3.7}$$

### 3.1.2 Time-Varying Channel

If either the transmitter or the receiver is moving or the motion takes place in the channel, then the relative location of reflectors in the transmission path is varying with time and so is the impulse response $h$. Thus we have to model the impulse response by a time-dependent family $\{h_t\}$. We call such a channel a *time-varying channel*. The input-output relation can then be represented by the *linear time-varying system*

$$y(t) = (Hx)(t) = \int_{\mathbb{R}^d} h_t(s)x(t-s)ds, \tag{3.8}$$

where $h_t$ is the impulse response at time $t$. In contrast to a time-invariant system where $h_0 = h_t$ for all $t$, the impulse response is now modeled by a time dependent family of functions $h_t$. By interpreting $h_t$ as a function of two variables, i.e., $h_t(s) = h(t,s)$ and renaming variables, we can rewrite (3.8) as

$$y(t) = (Hx)(t) = \int_{\mathbb{R}^d} h(t, t-s)x(s)ds. \tag{3.9}$$

Equation (3.9) is an analog of (3.3) with $h(t)$ replaced by the time-varying impulse response $h(t,s)$. We now consider the analog of (3.6)

$$(Hx)(t) = \int_{\mathbb{R}^d} \sigma(t,\xi)\hat{x}(\xi)e^{2\pi i\xi\cdot t}d\xi. \tag{3.10}$$

Here $\sigma$ can be interpreted as *time-varying transfer function*. In this formulation the operator $H = \sigma^{KN}$ becomes a pseudodifferential operator with a Kohn-Nirenberg symbol $\sigma$.

## 3.2 Transmission Setup

In the previous section, we described the time-varying wireless channel and how it affects the transmission of signals. Now it is time to have a closer look at the signal $x$ itself and how the receiver may extract the transmitted information from the received signal $y = Hx$.

We consider a *multicarrier communication system*. This means that the available transmission bandwidth is not occupied by a single transmission pulse $g$, but by a set of transmission pulses $\{g_l\}_{l=0}^{N-1}$ , where the index $l$ is usually related to the carrier frequency of $g_l$. A typical choice for $g_l$ is

$$g_l(t) = g(t)e^{2\pi ilbt}, \tag{3.11}$$

where $g$ is some prototype transmission pulse with well-localized Fourier transform, and $0 < b \in \mathbb{R}$ is referred to as the carrier separation. In this setup, multiple complex exponentials are used as information bearing carriers.

The mathematical model for the transmit signal $x$ at time $t \in \mathbb{R}^d$, is given by

$$x(t) = \sum_{k \in \mathbb{Z}} \sum_{l=0}^{N-1} c_{k,l} g(t - ka) e^{2\pi i lbt}, \tag{3.12}$$

where $a > 0$ is called the symbol period, and $\{c_{k,l}\}_{l=0}^{N-1}$ denotes the data symbol at time $k \in \mathbb{Z}$ and subcarrier $l = 0, 1, \ldots, N-1$. Here, the word data symbol has to be distinguished from the symbol $\sigma$ of an operator. It is easy to see that we can attempt to recover the discrete data $c_{k,l}$ only if the set of functions $\{g_{k,l}\}$ are linearly independent.

If we let $N \to \infty$ and consider an infinite number of subcarriers, then the set of functions $\{g_{k,l}\}_{k,l \in \mathbb{Z}}$ corresponds to a Gabor system $\mathcal{G}(g, a, b)$ as in (2.44), generated by the elementary pulse $g$.

To get a discrete representation of the receive signal $y$, we compute

$$d_\lambda = \langle y, \pi(\lambda) g \rangle. \tag{3.13}$$

From this we get

$$\begin{aligned} d_\lambda &= \langle \sigma^{KN} x, \pi(\lambda) g \rangle && (3.14) \\ &= \langle \sigma^{KN} \sum_{\mu \in \Lambda} c_\mu \pi(\mu) g, \pi(\lambda) g \rangle && (3.15) \\ &= \sum_{\mu \in \Lambda} c_\mu \langle \sigma^{KN} \pi(\mu) g, \pi(\lambda) g \rangle. && (3.16) \end{aligned}$$

Motivated by (3.16), we define the matrix $H(\sigma)$ associated to the symbol $\sigma$ with respect to the Gabor system $\mathcal{G}(g, a, b)$ by

$$H(\sigma)_{\lambda,\mu} = \langle \sigma^{KN} \pi(\mu) g, \pi(\lambda) g \rangle, \quad \lambda, \mu \in \Lambda. \tag{3.17}$$

If it is clear which symbol is used, we write $H$ instead of $H(\sigma)$. We use the letter $H$ for the continuous and the discrete representation of a time-varying wireless channel, since it should be clear from the context which representation is used.

### 3.2.1 Pilot Arrangement

In this section, we explain how we estimate some diagonal entries of the channel matrix. We use a pilot-aided approach, that is we designate some of the sub-carriers in some of the OFDM symbols to carry known symbols. These symbols are called *pilot symbols*. Specifically, fix two positive integers $K$ and $L$ and assume that every $K$-th entry in the time direction and every $L$-th entry in the frequency direction is known, so that the pilot symbols are placed on a sublattice $\Lambda_p = aK\mathbb{Z}^d \times bL\mathbb{Z}^d \subseteq \Lambda$. Several pilot arrangements are known [7, 16, 63]. Two commonly used pilot arrangements are the block-type arrangement in Figure 3.2 and the comb-type arrangement in Figure 3.3. The block-type arrangement corresponds to $K > 1$ and $L = 1$, while the comb-type arrangement corresponds to $K = 1$ and $L > 1$.

Since the receive signal $y$ and the pilot symbols are known, we can estimate the diagonal entries of $H$ corresponding to the lattice $\Lambda_p$ as follows

$$\tilde{H}_{\mu,\mu} = \frac{d_\mu}{c_\mu}, \quad \mu \in \Lambda_p. \tag{3.18}$$

We only get an estimate of the diagonal entries of $H$, because $\tilde{H}$ in general contains information from neighbouring symbols due to the doubly selective channel.

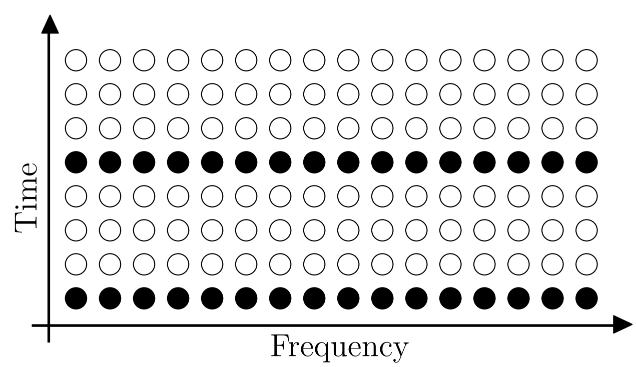


Figure 3.2: An illustration of the block-type pilot arrangement ('$\circ$' represents data symbols and '$\bullet$' represents pilot symbols).

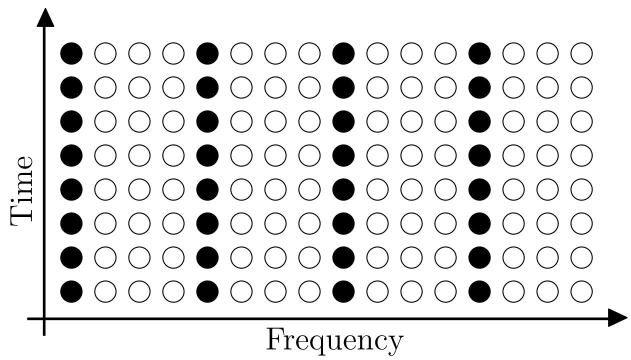


Figure 3.3: An illustration of the comb-type pilot arrangement ('$\circ$' represents data symbols and '$\bullet$' represents pilot symbols).

# Chapter 4

# Reconstruction of the Symbol

In this chapter, we investigate the relation between properties of the operator and properties of the corresponding channel matrix. The goal is to reconstruct the symbol of the operator from specfic entries of the channel matrix.

We derive a reconstruction formula for the symbol of a pseudodifferential operator with compactly supported spreading function from the main diagonal, from a side diagonal, and from finitely many diagonals of the operator. These reconstruction formulas are based on the assumption that the short-time Fourier transform of the window function $g$, which generates the Gabor system, does not vanish on the support of the spreading function. It is not sufficient, to study the reconstruction formulas only for $\boldsymbol{L}^p$ spaces, because commonly used models, like point scatterers, can, from a mathematical viewpoint, only be modeled in the setting of tempered distributions. Therefore, we prove the reconstruction formulas for symbols in $\boldsymbol{L}^2, \boldsymbol{L}^p, 1 < p < \infty$, and $\mathcal{S}'$. We also propose an iterative method for joint channel estimation and equalization of the receive signal that uses such a reconstruction formula.

Our results can be applied in the context of channel estimation and equalization of the receive signal. If we have an estimate for entries of the channel matrix, then we can use the reconstruction formulas to derive an estimate of the whole channel matrix, in other words, an estimate of the operator.

## 4.1 Reconstruction from the Main Diagonal

We assume that we know all elements of the main diagonal of the channel matrix $H$. Our goal is to reconstruct the symbol $\sigma$ using only these entries.

Let $(c_\lambda)_{\lambda\in\Lambda}$ be the matrix of data symbols over the lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$. These data symbols are in general not known, and should be transmitted over the wireless channel. Let $(d_\lambda)_{\lambda\in\Lambda}$ be the matrix of received symbols.

We derive an expression for the diagonal entries in terms of the Rihaczek distribution of $g$.

**Lemma 4.1.1.** *Let $\sigma \in \boldsymbol{L}^p(\mathbb{R}^{2d}), 1 < p < \infty$, $\hat{\sigma}$ compactly supported, $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ and $\hat{g} \in \boldsymbol{L}^1(\mathbb{R}^d)$. The diagonal entries of $H$ can be written as follows*

$$H_{\lambda,\lambda} = \langle \sigma^{KN}\pi(\lambda)g, \pi(\lambda)g\rangle = \sigma * R(g,g)^*(\lambda), \quad \lambda \in \Lambda, \tag{4.1}$$

*where $f^*(x) = \overline{f(-x)}$.*

*Proof.* Using the definition of the Rihaczek distribution (2.62), we get

$$\|R(g,g)^*\|_1 = \|g \otimes \hat{g}\|_1 = \|g\|_1 \cdot \|\hat{g}\|_1 < \infty. \tag{4.2}$$

Since $\sigma \in \boldsymbol{L}^p(\mathbb{R}^{2d})$, the convolution $\sigma * R(g,g)^*$ is well-defined in the $\boldsymbol{L}^p$-sense.

From the intertwining property of the Rihaczek distribution (2.63), we have

$$R(\pi(\lambda)g, \pi(\lambda)g)(x,\xi) = R(g,g)(x-\lambda_1, \xi-\lambda_2). \tag{4.3}$$

Combining the definition of the Kohn-Nirenberg transform (2.64) and Equation (4.3), we obtain

$$\begin{aligned}
\langle \sigma^{KN}\pi(\lambda)g, \pi(\lambda)g\rangle &= \langle \sigma, R(\pi(\lambda)g, \pi(\lambda)g)\rangle & (4.4)\\
&= \int_{\mathbb{R}^{2d}} \sigma(x,\xi)\overline{R(\pi(\lambda)g, \pi(\lambda)g)}(x,\xi)\,dx d\xi & (4.5)\\
&= \int_{\mathbb{R}^{2d}} \sigma(x,\xi)\overline{R(g,g)}(x-\lambda_1, \xi-\lambda_2)\,dx d\xi & (4.6)\\
&= \int_{\mathbb{R}^{2d}} \sigma(x,\xi)R(g,g)^*(\lambda_1 - x, \lambda_2-\xi)\,dx d\xi & (4.7)\\
&= \sigma * R(g,g)^*(\lambda). & (4.8)
\end{aligned}$$

In general, Equation (4.8) is valid for almost every $\lambda \in \mathbb{R}^{2d}$. However, $\sigma * R(g,g)^*$ is an analytic function, since its Fourier transform $\hat{\sigma} \cdot \overline{\mathcal{U}V_g g}$, where $\mathcal{U}F(\xi, x) = F(-x, \xi)$, is compactly supported, see Theorem 2.6.7. Therefore, Equation (4.8) is valid for every $\lambda \in \mathbb{R}^{2d}$. □

Lemma 4.1.1 provides a concise and useful representation of the diagonal elements of $H$ as the convolution of the symbol $\sigma$ with a specific quadratic functional of $g$.

A similar result to Lemma 4.1.1 can be obtained when $\sigma$ is a tempered distribution.

**Lemma 4.1.2.** *Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and $g \in \mathcal{S}(\mathbb{R}^d)$. The diagonal entries of $H$ can be written as follows*

$$H_{\lambda,\lambda} = \langle \sigma^{KN} \pi(\lambda) g, \pi(\lambda) g \rangle = \sigma * R(g,g)^*(\lambda), \quad \lambda \in \Lambda, \tag{4.9}$$

*where $f^*(x) = \overline{f(-x)}$.*

In the following reconstruction formula, we assume that $\hat{\sigma}$ has a compact support contained in the fundamental domain of the lattice $\Lambda^\perp = \frac{1}{a}\mathbb{Z}^d \times \frac{1}{b}\mathbb{Z}^d$.

Now we prove the main theorem on the reconstruction of $\sigma \in \boldsymbol{L}^2(\mathbb{R}^{2d})$.

**Theorem 4.1.3.** *Let $\sigma \in \boldsymbol{L}^2(\mathbb{R}^{2d})$, $\operatorname{supp} \hat{\sigma} \subseteq Q = [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$, $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ and $\hat{g} \in \boldsymbol{L}^1(\mathbb{R}^d)$. If $\overline{\mathcal{U}V_g g}$ does not vanish on $Q$, then*

$$\sigma = \frac{1}{ab} \sum_{\lambda \in \Lambda} H_{\lambda,\lambda} T_\lambda \mathcal{F}^{-1} \left( \frac{\chi_Q}{\overline{\mathcal{U}V_g g}} \right), \tag{4.10}$$

*where $H_{\lambda,\lambda}$ are the matrix coefficients of $\sigma^{KN}$ w.r.t. the lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, $\chi$ denotes the indicator function, $\mathcal{U}F(\xi, x) = F(-x, \xi)$ and the sum converges in the $\boldsymbol{L}^2$-sense.*

*Proof.* Our goal is to reconstruct $\sigma$ from the entries of the main diagonal of the channel matrix $H$ with respect to the lattice $\Lambda$ in the form

$$\sigma = \sum_{\lambda \in \Lambda} H_{\lambda,\lambda} T_\lambda h, \tag{4.11}$$

where the function $h$ is to be determined.

Our proof follows the classical proof of the sampling theorem [6, 41, 50]. The idea is as follows. We construct the periodization of $\hat{\sigma} \cdot \overline{\mathcal{U}V_g g}$ using the Poisson summation formula. Then we multiply the periodization by $\hat{h} = \frac{\chi_Q}{ab\,\overline{\mathcal{U}V_g g}}$ to get $\hat{\sigma}$, and then take the inverse Fourier transform.

We use the fact that $\mathcal{F}^{-1} = \mathcal{I}\mathcal{F}$, where $\mathcal{I}$ is the reflection operator, and $\widehat{R(g,g)} = \overline{\mathcal{U}V_g g}$, see Proposition 2.8.2, to compute

$$\begin{aligned} (\hat{\sigma} \cdot \overline{\mathcal{U}V_g g})(\omega - \eta) &= \mathcal{I}\mathcal{F}(\sigma * R(g,g)^*)(\eta - \omega) && (4.12) \\ &= \mathcal{F}^{-1}(\sigma * R(g,g)^*)(\eta - \omega) && (4.13) \end{aligned}$$

for all $\omega \in \mathbb{R}^{2d}, \eta \in \Lambda^\perp$.

Now we apply the Poisson summation formula (2.11) to $f = \mathcal{I}(\hat{\sigma} \cdot \overline{\mathcal{U}V_g g})$. This function is compactly supported, and in $\boldsymbol{L}^2(\mathbb{R}^{2d})$, since $\overline{\mathcal{U}V_g g}$ is bounded on $Q$. Moreover, $\hat{f} = \sigma * R(g,g)^*$. Note that the roles of the lattices $\Lambda$ and $\Lambda^\perp$ are interchanged and $\omega$ corresponds to $-z$. We have

$$ab \sum_{\eta \in \Lambda^\perp} (\hat{\sigma} \cdot \overline{\mathcal{U}V_g g})(\omega - \eta) = \sum_{\lambda \in \Lambda} (\sigma * R(g,g)^*)(\lambda) e^{-2\pi i \lambda \cdot \omega}. \tag{4.14}$$

Let

$$\hat{h}(\omega) := \frac{\chi_Q(\omega)}{ab \cdot \overline{\mathcal{U}V_g g}(\omega)}. \tag{4.15}$$

We claim that $\hat{\sigma}$ can be expressed as the left hand sum in Equation (4.14) multiplied by $\hat{h}$ as follows

$$\hat{\sigma}(\omega) = ab \cdot \hat{\sigma}(\omega) \cdot \overline{\mathcal{U}V_g g}(\omega) \cdot \hat{h}(\omega) + ab \sum_{\substack{\eta \in \Lambda^\perp \\ \eta \neq 0}} (\hat{\sigma} \cdot \overline{\mathcal{U}V_g g})(\omega - \eta) \hat{h}(\omega). \tag{4.16}$$

We consider now $\omega \in Q$. For every $\eta \in \Lambda^\perp, \eta \neq 0$, $\hat{\sigma}(\omega - \eta) = 0$, since $\omega$ and $\omega - \eta$ cannot lie simultaneously in $Q$. Thus the summation term in Equation (4.16) vanishes and Equation (4.16) follows from Equation (4.15).

For $\omega \notin Q$, both sides of Equation (4.16) vanish because of the definition of $\hat{h}$ and because $\operatorname{supp} \hat{\sigma} \subseteq Q$.

Combining Equations (4.16) and (4.14) gives

$$\hat{\sigma}(\omega) = \sum_{\lambda\in\Lambda} (\sigma * R(g,g)^*)(\lambda) e^{-2\pi i\lambda\cdot\omega} \hat{h}(\omega). \tag{4.17}$$

Next we apply the inverse Fourier transform to Equation (4.17) and the proposed reconstruction formula follows.

Since $\overline{\mathcal{U}V_g g}$ is continuous and does not vanish on $Q$, $\frac{\chi_Q}{\overline{\mathcal{U}V_g g}}$ is bounded and $h \in \boldsymbol{L}^2(\mathbb{R}^{2d})$. Therefore, the preceding algebraic manipulations are justified in the $\boldsymbol{L}^2$-sense. In particular, the series in Equation (4.10) converges in the $\boldsymbol{L}^2$-sense. □

Theorem 4.1.3 has a distributional counterpart, which is obtained by applying the Poisson summation formula (2.57) to the compactly supported distribution $u = \hat{\sigma} \cdot \overline{\mathcal{U}V_g g}$.

**Theorem 4.1.4.** *Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\operatorname{supp}\hat{\sigma} \subseteq Q_\varepsilon = [-\frac{1}{2a}+\varepsilon, \frac{1}{2a}-\varepsilon]^d \times [-\frac{1}{2b}+\varepsilon, \frac{1}{2b}-\varepsilon]^d$ for some $\varepsilon > 0$. Furthermore, let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ be such that $\varphi = 1$ on $Q_\varepsilon$ and $\operatorname{supp}\varphi \subseteq [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$. If $g \in \mathcal{S}(\mathbb{R}^d)$ and $\overline{\mathcal{U}V_g g}$ does not vanish on $\operatorname{supp}\varphi$, then*

$$\sigma = \frac{1}{ab} \sum_{\lambda\in\Lambda} H_{\lambda,\lambda} T_\lambda \mathcal{F}^{-1}\left(\frac{\varphi}{\overline{\mathcal{U}V_g g}}\right), \tag{4.18}$$

*where $H_{\lambda,\lambda}$ are the matrix coefficients of $\sigma^{KN}$ w.r.t. the lattice $\Lambda = a\mathbb{Z}\times b\mathbb{Z}$, $\mathcal{U}F(\xi,x) = F(-x,\xi)$ and the sum converges in the distributional sense.*

The proof of Theorem 4.1.4 is similar to the proof of Theorem 4.1.3. We only have to replace the characteristic function in Equation (4.15) by a properly chosen smooth function $\varphi$. In order to avoid aliasing introduced by $\varphi$, we have to restrict the support of $\hat{\sigma}$ to the rectangle $[-\frac{1}{2a}+\varepsilon, \frac{1}{2a}-\varepsilon]^d \times [-\frac{1}{2b}+\varepsilon, \frac{1}{2b}-\varepsilon]^d$.

Next we state the reconstruction formula for $\sigma \in \boldsymbol{L}^p(\mathbb{R}^{2d}), 1 < p < \infty$. In the following theorem we use the multidimensional sinc function as defined in Equation (2.72).

**Theorem 4.1.5.** *Let $\sigma \in \boldsymbol{L}^p(\mathbb{R}^{2d}), 1 < p < \infty$, and $\operatorname{supp}\hat{\sigma} \subseteq Q = [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$. Furthermore, let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ be such that $\varphi = 1$ on $Q$, $g \in$*

$\boldsymbol{L}^1(\mathbb{R}^d)$ *and* $\hat{g} \in \boldsymbol{L}^1(\mathbb{R}^d)$. *If* $\overline{\mathcal{U}V_g g}$ *does not vanish on* $\operatorname{supp}\varphi$, *then*

$$\sigma = \frac{1}{ab} \sum_{\lambda \in \Lambda} H_{\lambda,\lambda} T_\lambda(\operatorname{sinc} * K), \tag{4.19}$$

*where* $H_{\lambda,\lambda}$ *are the matrix coefficients of* $\sigma^{KN}$ *w.r.t. the lattice* $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, $K = \mathcal{F}^{-1}\left(\frac{\varphi}{\overline{\mathcal{U}V_g g}}\right)$, $\mathcal{U}F(\xi, x) = F(-x, \xi)$, *and the sum converges absolutely, uniformly and in the* $\boldsymbol{L}^p$*-sense.*

*Proof.* The idea of the proof is as follows. Since $\sigma \in \boldsymbol{L}^p(\mathbb{R}^{2d})$ and $R(g,g)^* \in \boldsymbol{L}^1(\mathbb{R}^{2d})$, because $g$ and $\hat{g}$ are in $\boldsymbol{L}^1(\mathbb{R}^d)$, it follows that $\sigma * R(g,g)^* \in \boldsymbol{L}^p(\mathbb{R}^{2d})$. We consider the Shannon reconstruction formula for $\sigma * R(g,g)^*$, which converges absolutely, uniformly and in the $\boldsymbol{L}^p$-sense, see [3, 6]. We note that the Fourier transform of $\sigma * R(g,g)^*$ is also supported in $[-\frac{1}{2a}, \frac{1}{2a}] \times [-\frac{1}{2b}, \frac{1}{2b}]$. Then we convolve $\sigma * R(g,g)^*$ with a properly chosen function $K$ to achieve a reconstruction formula for $\sigma$.

We recall from Lemma 4.1.1 that $H_{\lambda,\lambda} = \sigma * R(g,g)^*(\lambda)$ and write [3]

$$\sigma * R(g,g)^* = \frac{1}{ab} \sum_{\lambda \in \Lambda} H_{\lambda,\lambda} T_\lambda \operatorname{sinc}, \tag{4.20}$$

where the sum converges absolutely, uniformly and in the $\boldsymbol{L}^p$-sense.

Next we define

$$K = \mathcal{F}^{-1}\left(\frac{\varphi}{\overline{\mathcal{U}V_g g}}\right). \tag{4.21}$$

Since $R(g,g)^* \in \boldsymbol{L}^1(\mathbb{R}^{2d})$, we have that $\overline{\mathcal{U}V_g g} = \widehat{R(g,g)}^* \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^{2d})$. Since $\overline{\mathcal{U}V_g g} \neq 0$ on $\operatorname{supp}\varphi$, we apply the Wiener-Lévy theorem, see [54, Theorem 1.3.1], to conclude that there exists a function $\psi \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^{2d})$ such that $\psi = \frac{1}{\overline{\mathcal{U}V_g g}}$ on $\operatorname{supp}\varphi$. Since $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, $\varphi$ is also in $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^{2d})$ and we have $\varphi\,\psi = \frac{\varphi}{\overline{\mathcal{U}V_g g}} \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^{2d})$. Thus we conclude that $K \in \boldsymbol{L}^1(\mathbb{R}^{2d})$. Now we claim that

$$\sigma * R(g,g)^* * K = \sigma. \tag{4.22}$$

To prove this, we consider the Fourier transform of Equation (4.22). We get

$$\begin{aligned} \hat{\sigma} \cdot \widehat{R(g,g)}^* \cdot \hat{K} &= \hat{\sigma} \cdot \overline{\mathcal{U}V_g g} \cdot \overline{\mathcal{U}V_g g}^{-1} \cdot \varphi && (4.23) \\ &= \hat{\sigma}, && (4.24) \end{aligned}$$

because $\varphi = 1$ on $\operatorname{supp}\hat{\sigma}$.

Now we combine Equations (4.22) and (4.20) to compute

$$\begin{aligned} \sigma &= \sigma * R(g,g)^* * K && (4.25) \\ &= \frac{1}{ab}\sum_{\lambda\in\Lambda} H_{\lambda,\lambda} T_\lambda(\operatorname{sinc} * K). && (4.26) \end{aligned}$$

Since we convolve $\sigma * R(g,g)^* \in \boldsymbol{L}^p(\mathbb{R}^{2d})$ with $K \in \boldsymbol{L}^1(\mathbb{R}^{2d})$, the series in Equation (4.26) inherits the convergence properties from Equation (4.20). $\square$

In Theorem 4.1.3, we reconstruct the symbol $\sigma$ from the entries of the main diagonal of the channel matrix $H$. In practice, pilot symbols are used in wireless communication systems. This means that, assuming a distortion free channel, only the elements corresponding to the "pilot"-lattice $\Lambda_p$ (see Section 3.2.1) of the main diagonal of $H$ are used for the reconstruction. However, the symbol can also be reconstructed from the lattice $\Lambda_p$ as long as the support of $\hat{\sigma}$ lies inside the fundamental domain of the lattice $\Lambda_p^\perp$.

Our approach is applicable to the class of *underspread operators*, which is often used to model wireless channels [37]. An operator $\sigma^{KN}$ is called *underspread*, if the support of $\hat{\sigma}$ is contained in a rectangular region of area less than 1 which is symmetric around the origin. Denoting by $S$ the support of $\hat{\sigma}$, we have

$$S := \operatorname{supp}\hat{\sigma} \subseteq [-\nu_0,\nu_0]^d \times [-\tau_0,\tau_0]^d \qquad (4.27)$$

with

$$4\tau_0\nu_0 < 1. \qquad (4.28)$$

The reconstruction formula for underspread operators follows as a special case of Theorem 4.1.3 when we additionally assume that $ab > 1$.

## 4.2 Zero-Free Regions of the STFT

In Theorem 4.1.3 we use the assumption that the short-time Fourier transform does not vanish on the support of $\hat{\sigma}$. In this section we show the existence of a condition on the window function $g$ under which the STFT does not vanish on a certain region around the origin. To derive such conditions, we use the Taylor expansion and assume that $d = 1$, that is $x, \xi \in \mathbb{R}$.

The MacLaurin series of the short-time Fourier transform up to $n$-th order terms is given by [27]

$$V_g f(x,\xi) = \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{(-2\pi i\xi)^k}{k!} \frac{(-x)^{m-k}}{(m-k)!} \langle f, X^k D^{m-k} g\rangle, \tag{4.29}$$

where $Xg(t) = tg(t)$ and $Dg(t) = \frac{\partial}{\partial t} g(t)$ for a continuously differentiable function $g$.

Next we consider the MacLaurin series of $V_g g$ up to first order terms and assume that $\|g\|_2^2 = 1$. This gives

$$\begin{aligned} V_g g(x,\xi) &= \sum_{m=0}^{1} \sum_{k=0}^{m} \frac{(-2\pi i\xi)^k}{k!} \frac{(-x)^{m-k}}{(m-k)!} \langle g, X^k D^{m-k} g\rangle + E(x,\xi) && (4.30) \\ &= 1 - x\langle g, Dg\rangle - 2\pi i\xi\langle g, Xg\rangle + E(x,\xi), && (4.31) \end{aligned}$$

where the Lagrange form of the error term $E(x,\xi)$ is given by

$$E(x,\xi) = \sum_{k+l=2} R_{k,l}(x,\xi) x^{l-k} \xi^k. \tag{4.32}$$

Setting $Q(x,\xi) = 1 - x\langle g, Dg\rangle - 2\pi i\xi\langle g, Xg\rangle$, we can write

$$V_g g(x,\xi) = Q(x,\xi) + E(x,\xi). \tag{4.33}$$

The coefficients of the error term are bounded by [20, Section 0.317]

$$\begin{aligned} |R_{0,2}(x,\xi)| &\le \frac{1}{2} \sup |\langle g, \pi(x_0,\xi_0) D^2 g\rangle|, && (4.34) \\ |R_{1,1}(x,\xi)| &\le 2\pi \sup |\langle g, \pi(x_0,\xi_0) XDg\rangle|, && (4.35) \\ |R_{2,0}(x,\xi)| &\le 2\pi^2 \sup |\langle g, \pi(x_0,\xi_0) X^2 g\rangle|, && (4.36) \end{aligned}$$

where the supremum is taken over all $|x_0| \le |x|$ and $|\xi_0| \le |\xi|$. Since we use these coefficients and estimates frequently in the following, we define

$$\begin{aligned} j &= |\langle g, Dg\rangle|, && (4.37) \\ k &= 2\pi |\langle g, Xg\rangle|, && (4.38) \\ l &= \frac{1}{2} \sup |\langle g, \pi(x_0,\xi_0) D^2 g\rangle|, && (4.39) \\ m &= 2\pi^2 \sup |\langle g, \pi(x_0,\xi_0) X^2 g\rangle|, && (4.40) \\ n &= 2\pi \sup |\langle g, \pi(x_0,\xi_0) XDg\rangle|. && (4.41) \end{aligned}$$

The following two inequalities hold

$$Q(x,\xi) \geq 1 - |x|j - |\xi|k \tag{4.42}$$
$$E(x,\xi) \geq -x^2 l - \xi^2 m - |x\xi|n. \tag{4.43}$$

From (4.33) we conclude that the short-time Fourier transform is non zero for all $x, \xi$ which satisfy the following inequality obtained by adding (4.42) and (4.43)

$$x^2 l + \xi^2 m + |x\xi|n + |x|j + |\xi|k < 1. \tag{4.44}$$

In Theorem 4.1.3 we assume that the support of $\hat{\sigma}$ is contained in a rectangular region around the origin, that is

$$S = \operatorname{supp}\hat{\sigma} \subseteq \left[-\frac{1}{2a}, \frac{1}{2a}\right] \times \left[-\frac{1}{2b}, \frac{1}{2b}\right]. \tag{4.45}$$

To ensure that the STFT does not vanish on $S$, the region $S$ has to be contained in the planar region defined by (4.44). This means that if the point $\left(\frac{1}{2a}, \frac{1}{2b}\right)$ satisfies

$$\left(\frac{1}{2a}\right)^2 l + \left(\frac{1}{2b}\right)^2 m + \frac{n}{4ab} + \frac{j}{2a} + \frac{k}{2b} < 1, \tag{4.46}$$

then $S$ is contained in the planar region (4.44).

## 4.3 Reconstruction from One Side Diagonal

Before we consider the reconstruction from finitely many diagonals, we present one special case, namely the reconstruction from one side diagonal. Let us therefore consider the side diagonal which corresponds to $\nu \in \Lambda$. That is, we consider the entry

$$H_{\lambda,\lambda+\nu} = \langle \sigma^{KN} \pi(\lambda+\nu)g, \pi(\lambda)g \rangle \tag{4.47}$$
$$= \langle \sigma, R(\pi(\lambda)g, \pi(\lambda+\nu)g) \rangle. \tag{4.48}$$

of the channel matrix $H$.

We have the relation

$$\pi(\lambda+\nu)g = e^{2\pi i \nu_2 \cdot \lambda_1} \pi(\lambda)\pi(\nu)g \tag{4.49}$$

which gives

$$\begin{aligned} R\left(\pi(\lambda)g,\pi(\lambda+\nu)g\right) &= R\left(\pi(\lambda)g, e^{2\pi i\nu_2\cdot\lambda_1}\pi(\lambda)\pi(\nu)g\right) && (4.50)\\ &= T_\lambda R\left(g, e^{2\pi i\nu_2\cdot\lambda_1}\pi(\nu)g\right) && (4.51)\\ &= e^{-2\pi i\nu_2\cdot\lambda_1}T_\lambda R\left(g,\pi(\nu)g\right). && (4.52)\end{aligned}$$

From the intertwining property of the Rihaczek distribution (2.63) we get

$$R(g,\pi(\nu)g) = e^{-2\pi i\nu_1\cdot\nu_2}M_{(-\nu_2,\nu_1)}T_{(0,\nu_2)}R(g,g). \qquad (4.53)$$

Now we have the following relation

$$R\left(\pi(\lambda)g,\pi(\lambda+\nu)g\right) = e^{-2\pi i\nu_2\cdot\lambda_1}e^{-2\pi i\nu_1\cdot\nu_2}T_\lambda M_{(-\nu_2,\nu_1)}T_{(0,\nu_2)}R(g,g). \qquad (4.54)$$

The next step is to interchange translation and modulation in Equation (4.54). We get

$$R\left(\pi(\lambda)g,\pi(\lambda+\nu)g\right) = e^{-2\pi i\nu_2\cdot\lambda_1}T_{(\lambda_1,\lambda_2+\nu_2)}M_{(-\nu_2,\nu_1)}R(g,g). \qquad (4.55)$$

The following lemma provides an expression for $H_{\lambda\lambda+\nu}$ in terms of the Rihaczek distribution.

**Lemma 4.3.1.** *Let $\sigma\in \boldsymbol{L}^p(\mathbb{R}^{2d}), 1<p<\infty$, $\hat\sigma$ compactly supported, $g\in \boldsymbol{L}^1(\mathbb{R}^d)$ and $\hat g\in \boldsymbol{L}^1(\mathbb{R}^d)$. The side diagonal entries of $H$ can be written as follows*

$$\begin{aligned} H_{\lambda,\lambda+\nu} &= \langle \sigma^{KN}\pi(\lambda+\nu)g,\pi(\lambda)g\rangle && (4.56)\\ &= c_{\lambda\nu}\left(\sigma * T_{(0,\nu_2)}M_{(-\nu_2,\nu_1)}R(g,g)^*\right)(\lambda), && (4.57)\end{aligned}$$

*where $c_{\lambda\nu} = e^{2\pi i\nu_2\cdot\lambda_1}$ and $f^*(x) = \overline{f(-x)}$.*

*Proof.* From Lemma 4.1.1, we already know that the convolution in Equation (4.57) is well defined in the $\boldsymbol{L}^p$-sense. Consider equation (4.48)

$$\begin{aligned} H_{\lambda,\lambda+\nu} &= \langle \sigma^{KN}\pi(\lambda+\nu)g,\pi(\lambda)g\rangle && (4.58)\\ &= \langle \sigma, R(\pi(\lambda)g,\pi(\lambda+\nu)g)\rangle. && (4.59)\end{aligned}$$

Now we apply equation (4.55) and compute

$$\begin{aligned} H_{\lambda,\lambda+\nu} &= \int_{\mathbb{R}^{2d}}\sigma(x,\xi)\overline{R(\pi(\lambda)g,\pi(\lambda+\nu)g)}(x,\xi)dxd\xi && (4.60)\\ &= c_{\lambda\nu}\int_{\mathbb{R}^{2d}}\sigma(x,\xi)T_{(\lambda_1,\lambda_2+\nu_2)}M_{(\nu_2,-\nu_1)}\overline{R(g,g)}(x,\xi)dxd\xi && (4.61)\end{aligned}$$

for all $\lambda \in \Lambda$, fixed $\nu \in \Lambda$ and $c_{\lambda\nu} = e^{2\pi i \nu_2 \cdot \lambda_1}$. Next we interchange translation and modulation in Equation (4.61) and proceed as in the proof of Lemma 4.1.1 to get

$$H_{\lambda,\lambda+\nu} = c_{\lambda\nu} \left(\sigma * T_{(0,\nu_2)} M_{(\nu_2,-\nu_1)} R(g,g)^*\right)(\lambda) \tag{4.62}$$

for all $\lambda \in \Lambda$, fixed $\nu \in \Lambda$ and $c_{\lambda\nu} = e^{2\pi i \nu_2 \cdot \lambda_1}$. Due to Theorem 2.6.7, the equality is again valid for every $\lambda \in \mathbb{R}^{2d}$. □

Again, we formulate a similar result to Lemma 4.3.1 when $\sigma$ is a tempered distribution.

**Lemma 4.3.2.** *Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and $g \in \mathcal{S}(\mathbb{R}^d)$. The side diagonal entries of $H$ can be written as follows*

$$\begin{aligned} H_{\lambda,\lambda+\nu} &= \langle \sigma^{KN} \pi(\lambda+\nu) g, \pi(\lambda) g \rangle & (4.63)\\ &= c_{\lambda\nu} \left(\sigma * T_{(0,\nu_2)} M_{(-\nu_2,\nu_1)} R(g,g)^*\right)(\lambda), & (4.64)\end{aligned}$$

*where $c_{\lambda\nu} = e^{2\pi i \nu_2 \cdot \lambda_1}$ and $f^*(x) = \overline{f(-x)}$.*

As in Section 4.1 we use the representation (4.57) of the side diagonal entries of $H$ to get a reconstruction formula for the symbol $\sigma$.

**Theorem 4.3.3.** *Let $\sigma \in \boldsymbol{L}^2(\mathbb{R}^{2d})$, $\operatorname{supp} \hat{\sigma} \subseteq Q = [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$, $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ and $\hat{g} \in \boldsymbol{L}^1(\mathbb{R}^d)$. If $T_{(\nu_2,-\nu_1)}\overline{\mathcal{U}V_g g}$ does not vanish on $Q$, then*

$$\sigma = \frac{1}{ab} \sum_{\lambda \in \Lambda} \frac{1}{c_{\lambda\nu}} H_{\lambda,\lambda+\nu} T_\lambda \mathcal{F}^{-1} \left( \frac{\chi_Q}{M_{(0,-\nu_2)} T_{(\nu_2,-\nu_1)} \overline{\mathcal{U}V_g g}} \right), \tag{4.65}$$

*where $H_{\lambda,\lambda+\nu}$ are the matrix coefficients of $\sigma^{KN}$ w.r.t. the lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, $c_{\lambda\nu} = e^{2\pi i \nu_2 \cdot \lambda_1}$, $\chi$ denotes the indicator function and $\mathcal{U}F(\xi, x) = F(-x, \xi)$ and the sum converges in the $\boldsymbol{L}^2$-sense.*

The proof of Theorem 4.3.3 is similar to the proof of Theorem 4.1.3.

Using Lemma 4.3.2, we obtain the following theorem.

**Theorem 4.3.4.** *Let* $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, $\operatorname{supp}\hat{\sigma} \subseteq Q_\varepsilon = [-\frac{1}{2a}+\varepsilon, \frac{1}{2a}-\varepsilon]^d \times [-\frac{1}{2b}+\varepsilon, \frac{1}{2b}-\varepsilon]$. *Furthermore, let* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ *be such that* $\varphi = 1$ *on* $\operatorname{supp}\hat{\sigma}$ *and* $\operatorname{supp}\varphi \subseteq [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$. *If* $g \in \mathcal{S}(\mathbb{R}^d)$ *and* $T_{(\nu_2,-\nu_1)}\overline{\mathcal{U}V_g g}$ *does not vanish on* $\operatorname{supp}\varphi$*, then*

$$\sigma = \frac{1}{ab}\sum_{\lambda\in\Lambda}\frac{1}{c_{\lambda\nu}}H_{\lambda,\lambda+\nu}T_\lambda\mathcal{F}^{-1}\left(\frac{\varphi}{M_{(0,-\nu_2)}T_{(\nu_2,-\nu_1)}\overline{\mathcal{U}V_g g}}\right), \tag{4.66}$$

*where* $H_{\lambda,\lambda+\nu}$ *are the matrix coefficients of* $\sigma^{KN}$ *w.r.t. the lattice* $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, $c_{\lambda\nu} = e^{2\pi i\nu_2\cdot\lambda_1}$, $\chi$ *denotes the indicator function and* $\mathcal{U}F(\xi,x) = F(-x,\xi)$ *and the sum converges in the distributional sense.*

The reconstruction from a side diagonal also holds for $\sigma \in \boldsymbol{L}^p(\mathbb{R}^{2d}), 1 < p < \infty$. Again, we use the multidimensional sinc function as defined in Equation (2.72).

**Theorem 4.3.5.** *Let* $\sigma \in \boldsymbol{L}^p(\mathbb{R}^{2d}), 1 < p < \infty$ *and* $\operatorname{supp}\hat{\sigma} \subseteq [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$. *Furthermore, let* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ *be such that* $\varphi = 1$ *on* $\operatorname{supp}\hat{\sigma}$, $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ *and* $\hat{g} \in \boldsymbol{L}^1(\mathbb{R}^d)$. *If* $T_{(\nu_2,-\nu_1)}\overline{\mathcal{U}V_g g}$ *does not vanish on* $\operatorname{supp}\varphi$*, then*

$$\sigma = \frac{1}{ab}\sum_{\lambda\in\Lambda}\frac{1}{c_{\lambda\nu}}H_{\lambda,\lambda+\nu}T_\lambda(\operatorname{sinc} * K), \tag{4.67}$$

*where* $H_{\lambda,\lambda+\nu}$ *are the matrix coefficients of* $\sigma^{KN}$ *w.r.t. the lattice* $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, $K = \mathcal{F}^{-1}\left(\frac{\varphi}{M_{(0,-\nu_2)}T_{(\nu_2,-\nu_1)}\overline{\mathcal{U}V_g g}}\right)$, $c_{\lambda\nu} = e^{2\pi i\nu_2\cdot\lambda_1}$, $\chi$ *denotes the indicator function and* $\mathcal{U}F(\xi,x) = F(-x,\xi)$ *and the sum converges absolutely, uniformly and in the* $\boldsymbol{L}^p$*-sense.*

The proof of Theorem 4.3.5 is similar to the proof of Theorem 4.1.5.

## 4.4 Reconstruction from Finitely Many Diagonals

In this section we assume that the entries of finitely many diagonals of the channel matrix $H$ are known. From Lemma 4.1.1 and Lemma 4.3.1 we derive expressions for the (side) diagonal elements of $H$. In the following we want to use finitely many diagonals to reconstruct the symbol $\sigma$.

Assume that we know the values of

$$H_{\lambda,\lambda+\nu_l} = (\sigma * F_{\nu_l})(\lambda), \quad \lambda, \nu_l \in \Lambda, 1 \leq l \leq n, \tag{4.68}$$

where each function $F_{\nu_l}$ corresponds to a specific diagonal. For the main diagonal, that is $\nu_l = (0,0)$, we have $F(\lambda) = R(g,g)^*(\lambda)$ and for the side diagonal corresponding to $\nu_l = (\nu_{l,1}, \nu_{l,2})$, we have

$$F_{\nu_l}(\lambda) = T_{(0,\nu_{l,2})} M_{(-\nu_{l,2},\nu_{l,1})} R(g,g)^*(\lambda). \tag{4.69}$$

*Remark* 4.4.1. In Theorem 2.9.3 we assume that the signal $f \in B_\Omega$ is reconstructed from $n$ filtered versions $g_l = f * k_l, 1 \leq l \leq n$. In this case, every function $g_l$ is sampled at $1/n^{th}$ of the original sampling rate. In the following theorem, we sample the functions $g_l$ at the original sampling rate. Therefore, we can assume that the support of $f$ is $n$ times as large as in Theorem 2.9.3.

Now we prove the following theorem.

**Theorem 4.4.2.** *Let $\sigma \in \boldsymbol{L}^2(\mathbb{R}^{2d})$, $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ and $\hat{g} \in \boldsymbol{L}^1(\mathbb{R}^d)$. If $\operatorname{supp} \hat{\sigma} \subseteq \bigcup_{j=1}^n (Q - \eta_j)$, where $Q = [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$ and $\det(\widehat{F_{\nu_l}}(\omega - \eta_j))_{1\leq l,j \leq n} \neq 0$ for $\omega \in \operatorname{supp} \hat{\sigma}, \eta_j \in \Lambda^\perp, 1 \leq j \leq n$, then the symbol $\sigma$ can be reconstructed uniquely from the given $n$ diagonals.*

*Proof.* Let $\omega = (\zeta, u) \in \mathbb{R}^{2d}$. By assumption we know the values

$$(\sigma * F_{\nu_l})(\lambda) = d_{\lambda,l}, \tag{4.70}$$

for $\lambda \in \Lambda, 1 \leq l \leq n$. We consider the corresponding Fourier series

$$\sum_{l=1}^n \sum_{\lambda \in \Lambda} (\sigma * F_{\nu_l})(\lambda) e^{2\pi i \lambda \cdot \omega} = \sum_{l=1}^n \sum_{\lambda \in \Lambda} d_{\lambda,l} e^{2\pi i \lambda \cdot \omega}. \tag{4.71}$$

Now we apply the Poisson summation formula (2.11) to the left hand side of Equation (4.71) and get

$$ab \sum_{l=1}^n \sum_{\eta \in \Lambda^\perp} \hat{\sigma}(\omega - \eta) \widehat{F_{\nu_l}}(\omega - \eta) = D_{\lambda,l}(\omega), \quad \omega \in \mathbb{R}^{2d}, \tag{4.72}$$

where $D_{\lambda,l}(\omega) = \sum_{l=1}^{n} \sum_{\lambda\in\Lambda} d_{\lambda,l} e^{2\pi i\lambda\cdot\omega}$.

Since by assumption

$$\operatorname{supp}\hat{\sigma} \subseteq \bigcup_{j=1}^{n} (Q - \eta_j), \tag{4.73}$$

it follows that (4.72) reduces to

$$ab \sum_{l=1}^{n} \sum_{j=1}^{n} \hat{\sigma}(\omega - \eta_j) \widehat{F_{\nu_l}}(\omega - \eta_j) = D_{\lambda,l}(\omega), \quad \omega \in Q. \tag{4.74}$$

From linear algebra we know that the system of equations (4.74) is solvable uniquely if and only if

$$\det(\widehat{F_{\nu_l}}(\omega - \eta_j))_{1\le l,j\le n} \neq 0 \quad \text{for } \omega \in Q. \tag{4.75}$$

In other words, the determinant of the matrix

$$\begin{pmatrix} M_{(0,-\nu_{1,2})} T_{(\nu_{1,2},-\nu_{1,1})} \overline{\mathcal{U}V_g g}(\omega - \eta_1) & \dots & M_{(0,-\nu_{1,2})} T_{(\nu_{1,2},-\nu_{1,1})} \overline{\mathcal{U}V_g g}(\omega - \eta_n) \\ M_{(0,-\nu_{2,2})} T_{(\nu_{2,2},-\nu_{2,1})} \overline{\mathcal{U}V_g g}(\omega - \eta_1) & \dots & M_{(0,-\nu_{2,2})} T_{(\nu_{2,2},-\nu_{2,1})} \overline{\mathcal{U}V_g g}(\omega - \eta_n) \\ \vdots & \ddots & \vdots \\ M_{(0,-\nu_{n,2})} T_{(\nu_{n,2},-\nu_{n,1})} \overline{\mathcal{U}V_g g}(\omega - \eta_1) & \dots & M_{(0,-\nu_{n,2})} T_{(\nu_{n,2},-\nu_{n,1})} \overline{\mathcal{U}V_g g}(\omega - \eta_n) \end{pmatrix} \tag{4.76}$$

needs to be nonzero for every $\omega \in Q$. □

We observe that it is sufficient to compute the determinant of a finite matrix constructed for a rectangle $Q$ to check if a given symbol $\sigma$ can be reconstructed uniquely from finitely many diagonals. In other words, to check if a symbol, which is defined on all of $\mathbb{R}^{2d}$, can be reconstructed, one only has to consider a finite matrix corresponding to a finite rectangle.

In the proof of Theorem 5.1.1, we explicitly compute the elements of the matrix appearing in Theorem 4.4.2 for the Gaussian function $g$.

We also propose a formula for reconstruction of the symbol from finitely many diagonals. The following theorem is a special case of Theorem 2.9.3.

**Theorem 4.4.3.** *Let* $\sigma \in \boldsymbol{L}^2(\mathbb{R}^{2d})$, $\operatorname{supp}\hat{\sigma} \subseteq \bigcup_{j=1}^{n}(Q - \eta_j)$ *with* $Q = [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$, $\eta_j \in \Lambda^\perp$ *and* $g \in \boldsymbol{L}^2(\mathbb{R}^d)$. *Furthermore let* $F_{\nu_l}$,

*$1 \leq l \leq n$ be $n$ functions in $\boldsymbol{L}^1(\mathbb{R}^{2d})$. If the set of equations*

$$\frac{1}{ab} \sum_{l=1}^{n} H_l(\omega, z) \widehat{F_{\nu_l}}(\omega - \eta_j) = e^{-2\pi i \eta_j \cdot z}, \tag{4.77}$$

*$1 \leq j \leq n$, has a unique solution for the quantities $H_l(\omega, z), 1 \leq l \leq n$ for every $z \in \mathbb{R}^{2d}$ and $\omega \in Q$, and*

$$h_l(z) = \int_Q H_l(\omega, z) e^{2\pi i \omega \cdot z} d\omega, \tag{4.78}$$

*then $\sigma$ can be reconstructed from the sample values $(\sigma * F_{\nu_l})(\lambda)$ by the formula*

$$\sigma(z) = \sum_{l=1}^{n} \sum_{\lambda \in \Lambda} (\sigma * F_{\nu_l})(\lambda)\, T_\lambda h_l(z), \tag{4.79}$$

*and the sum in Equation (4.79) converges in the $\boldsymbol{L}^2$-sense.*

The proof of Theorem 4.4.3 can be found in [41, 49, 50].

## 4.5 Proposal for Iterative Channel Estimation

In this section, we propose an iterative method for channel estimation and ISI/ICI equalization. This method is based on the reconstruction formula (4.10) of Theorem 4.1.3, and on an equalization method called *single-tap equalization* of the receive signal.

We assume that the transmitter transmits the vector

$$c_\lambda^{(trans)} = \begin{cases} \text{pilot symbol}, & \lambda \in \Lambda_p \\ \text{data symbol}, & \text{otherwise.} \end{cases} \tag{4.80}$$

Our initial approximation $c^{(0)}$ to the vector $c^{(trans)}$ is defined as follows

$$c_\lambda^{(0)} = \begin{cases} \text{pilot symbol}, & \lambda \in \Lambda_p \\ 0, & \text{otherwise.} \end{cases} \tag{4.81}$$

Our iterative method is initialized by a straightforward estimation of the diagonal entries of $H$ at the pilot positions as in (3.18).

0. $H^{(1)}_{\mu\mu} = \frac{d_\mu}{c^{(0)}_\mu}, \quad \mu \in \Lambda_p.$

Now we describe the $i$-th iteration.

1. Reconstruct $\sigma^{(i)}$ based on $H^{(i)}_{\mu\mu}, \mu \in \Lambda_p$ using the reconstruction formula (4.10).
2. Compute the $i$-th approximation $H^{(i)}$ of the whole channel matrix using $\sigma^{(i)}$.
3. Compute the $i$-th approximation of the vector $c^{(trans)}$,
$$c^{(i)} = (H^{(i)})^{-1} d. \tag{4.82}$$
4. Now we perform ISI/ICI equalization and obtain a new approximation of the diagonal entries of $H$ at the pilot positions. That is, for $\mu \in \Lambda_p$ we compute
$$H^{(i+1)}_{\mu\mu} c^{(i)}_\mu = d_\mu - \sum_{\lambda \in \Lambda:\, \lambda \neq \mu} H^{(i)}_{\mu\lambda} c^{(i)}_\lambda. \tag{4.83}$$

Now the $i$-th iteration of the algorithm is finished and we use $H^{(i+1)}_{\mu\mu}$ in the $(i+1)$-th iteration to get the next approximation $\sigma^{(i+1)}$ of the symbol $\sigma$.

The goal of this method is to improve the quality of channel estimation and equalization of the receive signal. With the reconstruction formula used in this method, it is possible to compute the interference terms between neighbouring symbols and adjacent frequency carriers. This additional information can now be used to equalize the receive signal. The most important open question in this context is the numerical stability of the proposed method.

# Chapter 5

# Impossibility Results

In this chapter, we present several impossibility results for the channel matrix with respect to the structure of the underlying Gabor system.

One result states that if the channel matrix is identically zero, and the underlying Gabor system is generated by the Gaussian function, then the operator is identically zero. Another result shows that the channel matrix cannot be diagonal if the underlying Gabor system is a frame.

Other impossibility results follow from the reconstruction formulas in Chapter 4. Namely that the channel matrix cannot vanish identically on one side diagonal, or on finitely many diagonals simultaneously. These results further imply that a non-trivial channel matrix cannot be diagonal, which is a commonly made assumption in the engineering community [10, 11, 47, 58]. However, we prove that this assumption is not satisfied from a mathematical viewpoint. A common way to deal with this problem is to assume some off-diagonal decay conditions for the channel matrix. But again, one has to be careful with off-diagonal decay conditions, as we will show in Section 5.3.

## 5.1 Impossibility Result for Gabor Systems

For notational simplicity, we state the following result in dimension $d = 1$. We need the following derivations.

Let $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ be a lattice, let $g \in \boldsymbol{L}^2(\mathbb{R})$ be such that $\mathcal{G}(g, a, b)$ is a Gabor system in $\boldsymbol{L}^2(\mathbb{R})$, and let us assume that

$$H_{\lambda\mu} = \langle \sigma^{KN}\pi(\mu)g, \pi(\lambda)g\rangle = 0, \quad \forall \lambda, \mu \in \Lambda. \tag{5.1}$$

We write $\lambda = (\lambda_1, \lambda_2), \mu = (\mu_1, \mu_2) \in \mathbb{R}^2$. From the definition of $\sigma^{KN}$, see Equation (2.65), we have

$$\langle \sigma^{KN}\pi(\mu)g, \pi(\lambda)g\rangle = \langle \hat{\sigma}, \mathcal{U}V_{\pi(\mu)g}\pi(\lambda)g\rangle, \tag{5.2}$$

where $\mathcal{U}F(\xi, x) = F(-x, \xi)$. Using Lemma 2.3.4, we compute

$$\begin{aligned} V_{\pi(\mu)g}\pi(\lambda)g(-x,\xi) &= e^{-2\pi i(\xi-\lambda_2)\lambda_1}e^{2\pi i\mu_2(-x-\lambda_1)}T_{\lambda-\mu}V_g g(-x,\xi) && (5.3)\\ &= e^{2\pi i\lambda_2\lambda_1}e^{-2\pi i\mu_2\lambda_1}M_{(\mu_2,-\lambda_1)}T_{\lambda-\mu}V_g g(-x,\xi), && (5.4)\end{aligned}$$

and conclude that (5.1) is equivalent to

$$\langle \hat{\sigma}, M_{(\mu_2,-\lambda_1)}T_{\lambda-\mu}G\rangle = 0, \quad \forall \lambda, \mu \in \Lambda, \tag{5.5}$$

where $G = \mathcal{U}V_g g$.

**Theorem 5.1.1.** *Let $\sigma \in \mathcal{S}'(\mathbb{R}^2)$, $\varphi(x) = e^{-\pi x^2}$ be the Gaussian function, $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ a lattice, and $\mathcal{G}(\varphi, a, b)$ be a Gabor system. If $\hat{\sigma}$ is compactly supported and*

$$H_{\lambda\mu} = \langle \sigma^{KN}\pi(\mu)\varphi, \pi(\lambda)\varphi\rangle = 0, \quad \forall \lambda, \mu \in \Lambda, \tag{5.6}$$

*then $\sigma^{KN}$ is identically zero.*

*Proof.* Combining Equations (5.5) and (5.6), we obtain

$$\langle \hat{\sigma}, M_\mu T_\lambda G\rangle = 0, \quad \forall \lambda \in \Lambda, \mu \in \Lambda' = b\mathbb{Z} \times a\mathbb{Z}, \tag{5.7}$$

where $G = \mathcal{U}V_\varphi\varphi$.

From Example 2.3.3 we know that

$$G(\xi, x) = \mathcal{U}V_\varphi\varphi(\xi, x) = V_\varphi\varphi(-x, \xi) = \frac{1}{\sqrt{2}}\, e^{-\frac{\pi}{2}\xi^2 - \frac{\pi}{2}x^2 + \pi i \xi x}. \tag{5.8}$$

Setting $\mu = (\mu_1, \mu_2) \in \Lambda'$, $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, $z = (\xi, x) \in \mathbb{R}^2$, we expand the right hand side of the inner product in (5.7).

$$\begin{aligned} M_\mu T_\lambda G &= e^{2\pi i z\cdot\mu} G(z-\lambda) && (5.9)\\ &= \frac{1}{\sqrt{2}}\, e^{2\pi i(\xi\mu_1 + x\mu_2) + \pi i(\xi-\lambda_1)(x-\lambda_2) - \frac{\pi}{2}(\xi-\lambda_1)^2 - \frac{\pi}{2}(x-\lambda_2)^2}. && (5.10) \end{aligned}$$

The expression in the exponent in Equation (5.10) can be written as follows

$$\begin{aligned} &2\pi i(\xi\mu_1 + x\mu_2) + \pi i(\xi-\lambda_1)(x-\lambda_2) - \frac{\pi}{2}(\xi-\lambda_1)^2 - \frac{\pi}{2}(x-\lambda_2)^2 = \\ &\quad = \pi i\lambda_1\lambda_2 - \frac{\pi}{2}\lambda_1^2 - \frac{\pi}{2}\lambda_2^2 - \frac{\pi}{2}\xi^2 - \frac{\pi}{2}x^2 + \pi i\xi x - \\ &\quad - \pi i\xi\lambda_2 - \pi i x\lambda_1 + \pi\xi\lambda_1 + \pi x\lambda_2 + 2\pi i(\xi\mu_1 + x\mu_2). \end{aligned} \tag{5.11}$$

Now we collect the linear coefficients that depend on $\lambda$ and define

$$F_\lambda(\xi, x) = e^{-\pi i\xi\lambda_2 - \pi i x\lambda_1 + \pi\xi\lambda_1 + \pi x\lambda_2} \tag{5.12}$$

and

$$\Phi(\xi, x) = e^{-\frac{\pi}{2}\xi^2 - \frac{\pi}{2}x^2 + \pi i\xi x}. \tag{5.13}$$

Substituting Equations (5.12) and (5.13) into Equation (5.7) and dropping all constant terms gives

$$\langle \hat{\sigma}, M_\mu \Phi F_\lambda \rangle = 0 \tag{5.14}$$

for all $\lambda \in \Lambda$ and for all $\mu \in \Lambda'$. Next we define

$$\rho = \hat{\sigma}\, \overline{\Phi}. \tag{5.15}$$

We set $S := \operatorname{supp}\hat{\sigma}$, and observe that $\operatorname{supp}\rho = S$. Now we substitute Equation (5.15) into (5.14) and get

$$\langle \rho, M_\mu F_\lambda \rangle = 0, \tag{5.16}$$

for all $\lambda \in \Lambda$ and for all $\mu \in \Lambda'$.

The idea of the remainder of the proof is as follows. For each $\lambda$, the periodization of $\rho F_\lambda$ vanishes due to Equation (5.16) and the Poisson summation formula for compactly supported distributions (2.57). We restrict the periodization to a specific open set and divide out $F_\lambda$ to get a system of equations for the restrictions of $\rho$. Then we use the invertibility of a Vandermonde matrix for some selected $\lambda$'s to conclude that all restrictions of $\rho$ vanish. Finally, we conclude that $\rho$ vanishes using a partition of unity.

By the Poisson summation formula for compactly supported distributions (2.57), Equation (5.16) is equivalent to

$$\mathscr{P}_\lambda := \sum_{\nu \in \Lambda^\circ} T_\nu(\rho F_\lambda) = 0, \quad \forall \lambda \in \Lambda, \tag{5.17}$$

where $\Lambda^\circ = \frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z}$ is the adjoint lattice as defined in Definition 2.1.5. Since $S$ is compact, there is a positive integer $L$ such that

$$S = \operatorname{supp} \rho \subset R := \left(\frac{-L+1}{b}, \frac{L-1}{b}\right) \times \left(\frac{-L+1}{a}, \frac{L-1}{a}\right). \tag{5.18}$$

We define $Q = \left(-\frac{1}{b}, \frac{1}{b}\right) \times \left(-\frac{1}{a}, \frac{1}{a}\right)$ and $Q_{j,k} = T_{(\tau_j, \zeta_k)} Q$, where $\tau_j = \frac{j}{b}, \zeta_k = \frac{k}{a}$ for $-L \le j, k \le L$. Clearly

$$R \subset \left(\frac{-L}{b}, \frac{L}{b}\right) \times \left(\frac{-L}{a}, \frac{L}{a}\right) = \bigcup_{-L+1 \le j,k \le L-1} Q_{j,k}. \tag{5.19}$$

Next we consider the restriction of the periodization $\mathscr{P}_\lambda$ to $Q$,

$$\mathscr{P}_\lambda|_Q = \sum_{\nu \in \Lambda^\circ} T_\nu(\rho F_\lambda)|_Q = 0, \quad \forall \lambda \in \Lambda. \tag{5.20}$$

Since $R$ contains $\operatorname{supp} \rho$, Equation (5.20) reduces to

$$\mathscr{P}_\lambda|_Q = \sum_{j=-L}^{L} \sum_{k=-L}^{L} T_{(\tau_j, \zeta_k)}(\rho F_\lambda)|_Q = 0 \tag{5.21}$$

for all $\lambda \in \Lambda$. In Equation (5.21) we sum from $-L$ to $L$, because this covers all shifts of $Q$ which possibly intersect with $\operatorname{supp} \rho$ according to Equation (5.19). We can write

$$\mathscr{P}_\lambda|_Q = \sum_{j=-L}^{L} \sum_{k=-L}^{L} T_{(\tau_j, \zeta_k)} \rho|_Q \cdot T_{(\tau_j, \zeta_k)} F_\lambda|_Q = 0 \tag{5.22}$$

for all $\lambda \in \Lambda$. Now we define

$$\rho_{j,k} := T_{(\tau_j,\zeta_k)}\rho|_Q, \tag{5.23}$$

and compute

$$F_\lambda(\xi - \tau_j, x - \zeta_k) = e^{-\pi i(\xi-\tau_j)\lambda_2 - \pi i(x-\zeta_k)\lambda_1 + \pi(\xi-\tau_j)\lambda_1 + \pi(x-\zeta_k)\lambda_2}. \tag{5.24}$$

In view of Equation (5.24), we observe that we can divide Equation (5.22) by the exponentials in $x$ and $\xi$ to get

$$\sum_{j=-L}^{L}\sum_{k=-L}^{L} \rho_{j,k}\; F_\lambda(-\tau_j, -\zeta_k) = 0 \tag{5.25}$$

for all $\lambda \in \Lambda$, where

$$F_\lambda(-\tau_j, -\zeta_k) = e^{\lambda_1(\pi i\zeta_k - \pi\tau_j)} e^{\lambda_2(\pi i\tau_j - \pi\zeta_k)}. \tag{5.26}$$

Now we substitute the explicit forms of $\lambda = (al_1, bl_2)$ and $\tau_j = \frac{j}{b}, \zeta_k = \frac{k}{a}$, as defined before, to write

$$F_\lambda(-\tau_j, -\zeta_k) = e^{l_1(\pi ik - \pi j\frac{a}{b})} e^{l_2(\pi ij - \pi k\frac{b}{a})}. \tag{5.27}$$

We observe that in Equation (5.27) the numbers $e^{-\pi j\frac{a}{b}}$, $-L \le j \le L$ are distinct, but the numbers $e^{\pi ik} = (-1)^k$ need not be distinct. This can happen, if and only if two $k$'s differ by an even integer. Therefore, we consider the even $k$'s and the odd $k$'s separately. We get

$$\begin{aligned}&\sum_{j=-L}^{L}\sum_{k \text{ even}} \rho_{j,k}\; \left(e^{-\pi j\frac{a}{b}}\right)^{l_1} e^{l_2(\pi ij - \pi k\frac{b}{a})} + \\ &+ \sum_{j=-L}^{L}\sum_{k \text{ odd}} \rho_{j,k}\; \left(-e^{-\pi j\frac{a}{b}}\right)^{l_1} e^{l_2(\pi ij - \pi k\frac{b}{a})} = 0.\end{aligned} \tag{5.28}$$

For a fixed $l_2$, we choose $l_1 = 0, 1, \ldots, 4L+1$, and get a Vandermonde system generated by $4L+2$ distinct elements $\pm e^{-\pi j\frac{a}{b}}$ with $-L \le j \le L$. By the invertibility of the Vandermonde matrix

$$V\left(\{\pm e^{-\pi j\frac{a}{b}}\}_{-L\le j\le L}\right), \tag{5.29}$$

see Corollary 2.10.3, we conclude that for every $l_2$ and for every $j$

$$\sum_{k \text{ even}} \rho_{j,k} \, e^{l_2(\pi i j - \pi k \frac{b}{a})} = 0 \tag{5.30}$$

and

$$\sum_{k \text{ odd}} \rho_{j,k} \, e^{l_2(\pi i j - \pi k \frac{b}{a})} = 0. \tag{5.31}$$

We divide Equations (5.30) and (5.31) by $e^{l_2 \pi i j}$ and add them to obtain

$$\sum_{k=-L}^{L} \rho_{j,k} \, \left(e^{-\pi k \frac{b}{a}}\right)^{l_2} = 0 \tag{5.32}$$

for every $l_2$ and for every $j$.

We note that all the numbers $e^{-\pi k \frac{b}{a}}, -L \leq k \leq L$, are distinct, so in a similar fashion one can conclude that $\rho_{j,k} = 0$ for $-L \leq j, k \leq L$. To this end, one creates an invertible Vandermonde system by setting $l_2 = 0, \ldots, 2L$ in Equation (5.32). From Theorem 2.6.3 it follows that $\rho = 0$ and from this we conclude that $\hat{\sigma} = 0$. □

Theorem 5.1.1 can be viewed as a uniqueness theorem for pseudodifferential operators. It states that the coefficients of the matrix representation determine the operator uniquely. It also follows that if two operators possess the same matrix representation, then they are equal.

In comparison to Theorem 5.2.1, Theorem 5.1.1 does not require that the Gabor system is a frame and it requires no restrictions on the lattice parameters. This means that the product $ab$ can be arbitrarily large.

The following impossibility results follow from the reconstruction formulas in Chapter 4. We present the results in their distributional versions, because these are most relevant for the engineering community, e.g., to model point scatterers.

**Theorem 5.1.2.** *Let* $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and* $\operatorname{supp} \hat{\sigma} \subseteq Q_\varepsilon = [-\frac{1}{2a} + \varepsilon, \frac{1}{2a} - \varepsilon]^d \times [-\frac{1}{2b} + \varepsilon, \frac{1}{2b} - \varepsilon]^d$ *for some* $\varepsilon > 0$. *Furthermore, let* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ *be such that* $\varphi = 1$ *on* $Q_\varepsilon$ *and* $\operatorname{supp} \varphi \subseteq [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$. *If* $g \in \mathcal{S}(\mathbb{R}^d)$, $\overline{\mathcal{U}V_g g}$ *does not vanish on* $\operatorname{supp} \varphi$ *and* $H_{\lambda\lambda} = 0$ *for all* $\lambda \in \Lambda$, *then* $\sigma \equiv 0$.

Theorem 5.1.2 follows immediately from Theorem 4.1.4.

**Theorem 5.1.3.** *Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, $\operatorname{supp}\hat{\sigma} \subseteq Q_\varepsilon = [-\frac{1}{2a}+\varepsilon, \frac{1}{2a}-\varepsilon]^d \times [-\frac{1}{2b}+\varepsilon, \frac{1}{2b}-\varepsilon]$. Furthermore, let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ be such that $\varphi = 1$ on $\operatorname{supp}\hat{\sigma}$ and $\operatorname{supp}\varphi \subseteq [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$. If $g \in \mathcal{S}(\mathbb{R}^d)$, $T_{(\nu_2,-\nu_1)}\overline{\mathcal{U}V_g g}$ does not vanish on $\operatorname{supp}\varphi$ , then and $H_{\lambda\lambda} = 0$ for all $\lambda \in \Lambda$, then $\sigma \equiv 0$.*

Theorem 5.1.3 follows immediately from Theorem 4.3.4.

Similar versions of Theorems 5.1.2 and 5.1.3 can be obtained for $\sigma \in \boldsymbol{L}^2(\mathbb{R}^{2d})$ and $\sigma \in \boldsymbol{L}^p(\mathbb{R}^{2d}), 1 < p < \infty$.

**Theorem 5.1.4.** *Let $\sigma \in \boldsymbol{L}^2(\mathbb{R}^{2d})$, $\operatorname{supp}\hat{\sigma} \subseteq \bigcup_{j=1}^n (Q - \eta_j)$ with $Q = [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d$, $\eta_j \in \Lambda^\perp$ and $g \in \boldsymbol{L}^2(\mathbb{R}^d)$. If $H_{\lambda,\lambda+\nu_l} = 0$ for all $\lambda \in \Lambda$ and fixed $\nu_l \in \Lambda, 1 \leq l \leq n$, then $\sigma \equiv 0$.*

*Proof.* By assumption, $H_{\lambda,\lambda+\nu_l} = 0$ for all $\lambda \in \Lambda$ and fixed $\nu_l \in \Lambda, 1 \leq l \leq n$. It follows from the reconstruction formula (4.79) of Theorem 4.4.3 that the symbol $\sigma$ is identically zero. $\square$

## 5.2 Impossibility Result for Gabor Frames

For the following theorem, we assume that $\mathcal{G}(g, a, b)$ is a Gabor frame for $\boldsymbol{L}^2(\mathbb{R}^d)$.

**Theorem 5.2.1.** *Let $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ be a lattice with $ab < 1$ and $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ such that $\mathcal{G}(g, a, b)$ is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$. Let $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ be a lattice with $ab < 1$. If*

$$\langle \sigma^{KN}\pi(\mu)g, \pi(\lambda)g\rangle = d_\mu \delta_{\lambda\mu}, \quad \forall \lambda, \mu \in \Lambda, \tag{5.33}$$

*then $\sigma^{KN} \equiv 0$.*

*Proof.* It follows from (5.33), that $\sigma^{KN}\pi(\mu)g$ is orthogonal to the linear span of the set $\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}, \forall \mu \in \Lambda$. That is

$$\sigma^{KN}\pi(\mu)g \perp \operatorname{span}\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}, \quad \forall \mu \in \Lambda. \tag{5.34}$$

Since $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$, according to [9, Lemma IX], there are two possibilities for the set $\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}$. Either

- $\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}$ is incomplete in $\boldsymbol{L}^2(\mathbb{R}^d)$, or
- $\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}$ is again a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$.

If the set $\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}$ is incomplete in $\boldsymbol{L}^2(\mathbb{R}^d)$, then the set $\{\pi(\lambda)g : \lambda \in \Lambda\}$ has to be a Riesz basis for $\boldsymbol{L}^2(\mathbb{R}^d)$. Otherwise it would not be possible to have an incomplete set after removing one single element. It follows from [21, Corollary 7.5.2], that in this case, $ab = 1$. This contradicts the assumption $ab < 1$.

Therefore the set $\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}$ is again a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$. This implies that the orthogonal complement of $\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}$ has to be equal to the set $\{0\}$. That is,

$$\sigma^{KN}\pi(\mu)g \perp \operatorname{span}\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}, \quad \forall \mu \in \Lambda, \tag{5.35}$$

is only possible if $\sigma^{KN}\pi(\mu)g \equiv 0$, $\forall \mu \in \Lambda$ and therefore $\sigma^{KN} \equiv 0$. □

## 5.3 Counterexample to Decay in Frequency

An assumption commonly made in the theory of wireless comunications is that the entries of the channel matrix decay arbitrarily fast away from the main diagonal or that the channel matrix is banded in the frequency direction, see [10, 11, 47, 55, 58] and references therein. In this section, we give an example of a channel matrix whose entries do not decay in the frequency direction when one uses a rectangular window. This means that, in this case, the entries of the channel matrix do not decay arbitrarily fast with respect to the frequency. It follows that one has to use other windows instead of the rectangular function. Such pulses are especially designed to produce off-diagonal decay of the channel matrix and are called "shaped pulses" [4, 29, 31, 39, 57].

The results in this section are stated in dimension $d = 1$.

**Example 5.3.1.** Let $\sigma \equiv 1$, that is, the corresponding channel is the identity operator, and $g = \chi_{[-\frac{1}{2},\frac{1}{2}]}$. If $w_1 = z_1$ and $z_2 = 0$, then the entries of the channel matrix satisfy

$$\limsup_{|w_2|\to\infty} |\langle \sigma^{KN}\pi(z)g, \pi(w)g\rangle|\,|w_2| = \frac{1}{\pi} > 0. \tag{5.36}$$

Inequality (5.36) follows from the following computations.

We use the definition of $\sigma^{KN}$, Equation (2.66), and compute

$$\begin{aligned}\langle \sigma^{KN}\pi(z)g, \pi(w)g\rangle &= \langle \pi(z)g, \pi(w)g\rangle \\ &= \tilde{c}_{w,z}\int_{\mathbb{R}} g(x+w_1-z_1)\,\overline{g(x)}\,e^{-2\pi i x(w_2-z_2)}dx\end{aligned} \tag{5.37}$$

where $\tilde{c}_{w,z} = e^{-2\pi i w_1(w_2-z_2)}$.

Setting $w_1 = z_1$ and $z_2 = 0$, we get

$$\int_{\mathbb{R}} g(x)\,\overline{g(x)}\,e^{-2\pi i x w_2}dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i x w_2}dx \tag{5.38}$$

$$= \operatorname{sinc} w_2. \tag{5.39}$$

Inequality (5.36) follows from the following

$$\limsup_{|w_2|\to\infty} |\langle \sigma^{KN}\pi(z)g, \pi(w)g\rangle|\,|w_2| = \limsup_{|w_2|\to\infty} \frac{|\sin \pi w_2|}{\pi} = \frac{1}{\pi}. \tag{5.40}$$

In the following example, we consider a spreading function $\hat{\sigma}$ which is a finite linear combination of Dirac delta's. Such spreading functions are often used to model point scatterers.

**Example 5.3.2.** Let $\hat{\sigma} = \sum_{j=1}^{n} c_j\delta_{\nu_j}$. If the $\nu_{j,1}$'s are distinct and different from 1, $g = \chi_{[-\frac{1}{2},\frac{1}{2}]}$ and $w_1 = z_1 = z_2 = 0$, then the entries of the channel matrix satisfy

$$\limsup_{|w_2|\to\infty} |\langle \sigma^{KN}\pi(z)g, \pi(w)g\rangle|\,|w_2| \geq \frac{1}{2\pi}\left(\sum_{j=1}^{n}|c_j|^2\right)^{\frac{1}{2}} > 0. \tag{5.41}$$

To prove this claim, we need the following well known lemma which is stated here for readers convenience. More results of this type can be found in [35] in the context of almost periodic functions.

**Lemma 5.3.3.** *If $f(x) = \sum_{j=1}^{n} c_j e^{i\lambda_j x}$, where the $\lambda_j$'s are distinct real numbers and the $c_j$'s are arbitrary coefficients, then*

*1.*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^2 dx = \sum_{j=1}^{n} |c_j|^2, \tag{5.42}$$

*2.*

$$\limsup_{|x|\to\infty} |f(x)| \geq \left(\sum_{j=1}^{n} |c_j|^2\right)^{\frac{1}{2}}. \tag{5.43}$$

*Proof.* *1.* We compute

$$|f(x)|^2 = \sum_{1\leq j,k\leq n} c_j\, e^{i\lambda_j x}\, \overline{c_k}\, e^{-i\lambda_k x} \tag{5.44}$$

$$= \sum_{j=1}^{n} |c_j|^2 + \sum_{\substack{1\leq j,k\leq n\\ j\neq k}} c_j\, \overline{c_k}\, e^{i(\lambda_j-\lambda_k)x}. \tag{5.45}$$

Since the $\lambda_j$'s are distinct, we have:

$$\frac{1}{2T}\int_{-T}^{T} |f(x)|^2 dx = \sum_{j=1}^{n} |c_j|^2 + \frac{1}{2T}\sum_{\substack{1\leq j,k\leq n\\ j\neq k}} c_j\, \overline{c_k} \int_{-T}^{T} e^{i(\lambda_j-\lambda_k)x} dx \tag{5.46}$$

$$= \sum_{j=1}^{n} |c_j|^2 + \frac{1}{2T}\sum_{\substack{1\leq j,k\leq n\\ j\neq k}} c_j\, \overline{c_k}\, \frac{2\sin(\lambda_j-\lambda_k)T}{\lambda_j-\lambda_k}. \tag{5.47}$$

The second term in Equation (5.47) goes to zero when $T\to\infty$.

*2.* Clearly,

$$\limsup_{|x|\to\infty} |f(x)| \geq \lim_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T} |f(x)|^2 dx\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} |c_j|^2\right)^{\frac{1}{2}}. \tag{5.48}$$

□

We compute

$$\langle \sigma^{KN}\pi(z)g, \pi(w)g\rangle = \sum_{j=1}^{n} c_j \langle \pi(\nu_j)\pi(z)g, \pi(w)g\rangle, \tag{5.49}$$

for all $w, z \in \mathbb{R}^2$ and $\nu_j = (-\nu_{j,1}, \nu_{j,2}) \in \mathbb{R}^2$, and consider

$$\begin{aligned}
&\sum_{j=1}^{n} c_j \langle \pi(\nu_j)\pi(z)g, \pi(w)g\rangle = \\
&\qquad = \sum_{j=1}^{n} c_j \int_{\mathbb{R}} M_{\nu_{j,2}} T_{-\nu_{j,1}} M_{z_2} T_{z_1} g(t)\, \overline{M_{-w_2} T_{w_1} g(t)} dt \\
&\qquad = \sum_{j=1}^{n} c_j \int_{\mathbb{R}} e^{-2\pi i \nu_{j,1} z_2} M_{(\nu_{j,2}+z_2)} T_{(-\nu_{j,1}+z_1)} g(t)\, \overline{M_{-w_2} T_{w_1} g(t)} dt \\
&\qquad = \sum_{j=1}^{n} c_j\, e^{-2\pi i \nu_{j,1} z_2} \int_{\mathbb{R}} g(t+\nu_{j,1}-z_1)\, g(t-z_1)\, e^{-2\pi i t(w_2-\nu_{j,2}-z_2)} dt.
\end{aligned} \tag{5.50}$$

Setting $w_1 = z_1 = z_2 = 0$ and assuming that $-1 < \nu_{j,1} < 0$, we get

$$\begin{aligned}
&\sum_{j=1}^{n} c_j \int_{\mathbb{R}} g(t+\nu_{j,1})\, g(t)\, e^{-2\pi i t(w_2-\nu_{j,2})} dt = \\
&\qquad = \sum_{j=1}^{n} c_j \int_{-\frac{1}{2}}^{\frac{1}{2}} \chi_{[-\frac{1}{2}-\nu_{j,1}, \frac{1}{2}-\nu_{j,1}]}\, e^{-2\pi i t(w_2-\nu_{j,2})} dt \qquad (5.51) \\
&\qquad = \sum_{j=1}^{n} c_j \left[ -\frac{1}{2\pi i(w_2-\nu_{j,2})} \left( e^{-2\pi i(\frac{1}{2}-\nu_{j,1})(w_2-\nu_{j,2})} - e^{\pi i(w_2-\nu_{j,2})} \right) \right].
\end{aligned}$$

Next we consider

$$\begin{aligned}
&\sum_{j=1}^{n} c_j \left( e^{-2\pi i(\frac{1}{2}-\nu_{j,1})(w_2-\nu_{j,2})} - e^{\pi i(w_2-\nu_{j,2})} \right) \\
&\qquad = \sum_{j=1}^{n} c_j\, e^{\pi i(1-2\nu_{j,1})\nu_{j,2}}\, e^{-\pi i(1-2\nu_{j,1})w_2} - \sum_{j=1}^{n} c_j\, e^{-\pi i \nu_{j,2}}\, e^{\pi i w_2}.
\end{aligned} \tag{5.52}$$

Lemma 5.3.3 can be applied to the expression in Equation (5.52), because the exponents $-\pi i(1-2\nu_{j,1}), 1 \leq j \leq n$, and $\pi i$ are distinct. This follows from the fact that the $\nu_{j,1}$'s are all distinct, and $\nu_{j,1} \neq 1$ for all $1 \leq j \leq n$.

To summarize, we have

$$\begin{aligned}
&\limsup_{|w_2|\to\infty} |\langle \sigma^{KN}\pi(z)g, \pi(w)g\rangle||w_2| = \\
&\qquad = \limsup_{|w_2|\to\infty} \frac{1}{2\pi} \left| \left( \sum_{j=1}^{n} c_j\, e^{\pi i \nu_{j,2}}\, e^{-\pi i (1-2\nu_{j,1}) w_2} - \sum_{j=1}^{n} c_j\, e^{-\pi i \nu_{j,2}(1-2\nu_{j,1})}\, e^{\pi i w_2} \right) \right| \\
&\qquad \geq \frac{1}{2\pi} \left( \sum_{j=1}^{n} |c_j|^2 + \left| \sum_{j=1}^{n} c_j\, e^{-\pi i \nu_{j,2}(1-2\nu_{j,1})} \right|^2 \right)^{\frac{1}{2}} \\
&\qquad \geq \frac{1}{2\pi} \left( \sum_{j=1}^{n} |c_j|^2 \right)^{\frac{1}{2}},
\end{aligned} \tag{5.53}$$

for $w_1 = z_1 = z_2 = 0$.

# Chapter 6

# Discrete-Time Model

From a mathematical viewpoint it is absolutely not obvious that discrete-time transmission models inherit properties like off-diagonal decay from the continuous-time model. Therefore, we consider a discrete-time transmission model [48, 65] and describe how the discrete-time model is derived from the continuous-time model. We also investigate properties of the discrete-time model, and establish a connection between the discrete-time model and discrete pseudodifferential operators.

The discretized model and its relation to the continuous time model is discussed in Sections 6.1 and 6.2. In Section 6.3 we investigate properties of the discretized model. The results in this chapter are presented in dimensioon $d = 1$.

## 6.1 Preliminaries

In this section, we collect definitions and concepts which we need for development of the discrete channel model in Section 6.2. We define the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Functions defined on $\mathbb{T}$ are considered 1-periodic on $\mathbb{R}$.

The *discrete-time Fourier transform* of a sequence $(c_n)_{n\in\mathbb{Z}}$ of complex numbers is defined by

$$\hat{c}(\xi) = \sum_{n=-\infty}^{\infty} c_n \, e^{-2\pi i n \xi} \tag{6.1}$$

where $\xi \in \mathbb{T}$. Its inverse transform is given by

$$c_n = \int_{\mathbb{T}} \hat{c}(\xi) e^{2\pi i n \xi} d\xi. \tag{6.2}$$

We need the definition of the Schwartz class and tempered distributions on $\mathbb{Z}$ [56].

**Definition 6.1.1.** Let $\mathcal{S}(\mathbb{Z})$ denote the space of rapidly decreasing functions $\varphi : \mathbb{Z} \to \mathbb{C}$. That is, $\varphi \in \mathcal{S}(\mathbb{Z})$ if and only if for every $M \in \mathbb{R}$ there exists a constant $C_{\varphi,M}$ such that $|\varphi(m)| \le C_{\varphi,M}(1 + |m|^2)^{-M/2}$ holds for all $m \in \mathbb{Z}$. The topology on $\mathcal{S}(\mathbb{Z})$ is given by the seminorms $p_k$, where $k \in \mathbb{N}_0$ and $p_k := \sup_{m \in \mathbb{Z}^d}(1+|m|^2)^{k/2}|\varphi(m)|$. The continuous linear functionals on $\mathcal{S}(\mathbb{Z})$ are of the form

$$\varphi \mapsto \langle u, \varphi \rangle := \sum_{m \in \mathbb{Z}} u(m)\varphi(m), \tag{6.3}$$

where the function $u : \mathbb{Z} \to \mathbb{C}$ grows at most polynomially at infinity. Such distributions $u : \mathbb{Z} \to \mathbb{C}$ form the space $\mathcal{S}'(\mathbb{Z})$.

We also define the Schwartz class on the product space $\mathbb{Z} \times \mathbb{T}$.

**Definition 6.1.2.** The space $\mathcal{S}(\mathbb{Z}\times\mathbb{T})$ consists of all functions $f \in C^\infty(\mathbb{Z}\times\mathbb{T})$ such that the seminorm

$$\|f\|_{\alpha,\beta} = \sup_{n \in \mathbb{Z}, x \in \mathbb{T}} (1 + |n|)^\alpha D_x^\beta f(n, x) \tag{6.4}$$

is finite for all $\alpha, \beta$. The dual space of $\mathcal{S}(\mathbb{Z}\times\mathbb{T})$ is called the space of tempered distributions on $\mathbb{Z} \times \mathbb{T}$ and denoted by $\mathcal{S}'(\mathbb{Z} \times \mathbb{T})$.

**Definition 6.1.3.** Let $\sigma_d \in \mathcal{S}'(\mathbb{Z} \times \mathbb{T})$. The mapping $\sigma_d \to \sigma_d^{KN}$ is defined as

$$\langle \sigma_d^{KN} c, d \rangle = \langle \sigma_d, R(d, c) \rangle, \tag{6.5}$$

where $c, d \in \mathcal{S}(\mathbb{Z})$ and $R(d, c)(m, \xi) = d_m \hat{c}(\xi) e^{-2\pi i m \xi} \in \mathcal{S}(\mathbb{Z} \times \mathbb{T})$ is the discrete Rihaczek distribution. We call $\sigma_d$ the (discrete) Kohn-Nirenberg symbol.

*Remark* 6.1.4. The sequence $c \in \mathcal{S}(\mathbb{Z})$ if and only if $\hat{c} \in C^\infty(\mathbb{T})$. This follows from [35, Theorem 4.4].

*Remark* 6.1.5. Due to the duality of $\mathbb{Z}$ and $\mathbb{T}$, $\hat{\sigma}_d \in \mathcal{S}'(\mathbb{T} \times \mathbb{Z})$ whenever $\sigma_d \in \mathcal{S}'(\mathbb{Z} \times \mathbb{T})$.

If $\sigma_d$ is an integrable function, then the discrete Kohn-Nirenberg transform can be defined as follows.

$$\left(\sigma_d^{KN} c\right)_m = \int_{\mathbb{T}} \sigma_d(m, \xi)\ \hat{c}(\xi)\ e^{2\pi i m \xi} d\xi. \tag{6.6}$$

If $\sigma_d \in \boldsymbol{L}^1(\mathbb{Z} \times \mathbb{T})$ and $\hat{\sigma}_d \in \boldsymbol{L}^1(\mathbb{T} \times \mathbb{Z})$, then we can write $\sigma_d^{KN}$ as a superposition of time-frequency shifts. Specifically,

$$\left(\sigma_d^{KN} c\right)_m = \int_{\mathbb{T}} \sum_{u \in \mathbb{Z}} \hat{\sigma}_d(\omega, u)\ M_\omega T_{-u} c_m d\omega. \tag{6.7}$$

This follows from the following well-known computation. Since $\sigma_d \in \boldsymbol{L}^1(\mathbb{Z} \times \mathbb{T})$, we use Equation (6.1) and (6.6) to get

$$\begin{aligned} \left(\sigma_d^{KN} c\right)_m &= \int_{\mathbb{T}} \sigma_d(m, \xi)\ \hat{c}(\xi)\ e^{2\pi i m \xi} d\xi & (6.8) \\ &= \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} \sigma_d(m, \xi)\ c_n\ e^{-2\pi i (n-m) \xi} d\xi. & (6.9) \end{aligned}$$

We observe that Equation (6.9) can be written as

$$\left(\sigma_d^{KN} c\right)_m = \sum_{n \in \mathbb{Z}} \mathcal{F}_2 \sigma_d(m, n-m) c_n. \tag{6.10}$$

In order to get a representation in terms of $\hat{\sigma}_d$, we rewrite Equation (6.10) as follows

$$\begin{aligned} \left(\sigma_d^{KN} c\right)_m &= \sum_{n \in \mathbb{Z}} \int_{\mathbb{T}} \mathcal{F}_1 \mathcal{F}_2 \sigma_d(\omega, n-m)\ c_n\ e^{2\pi i \omega m} d\omega & (6.11) \\ &= \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} \mathcal{F}_1 \mathcal{F}_2 \sigma_d(\omega, n-m)\ c_n\ e^{2\pi i \omega m} d\omega & (6.12) \\ &= \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} \hat{\sigma}_d(\omega, n-m)\ c_n\ e^{2\pi i \omega m} d\omega. & (6.13) \end{aligned}$$

Going from Equation (6.10) to (6.11) and interchanging summation and integration are valid since $\hat{\sigma}_d \in \boldsymbol{L}^1(\mathbb{T} \times \mathbb{Z})$. Next we substitute $u = n - m$ into Equation (6.13) to get

$$\left(\sigma_d^{KN} c\right)_m = \int_{\mathbb{T}} \sum_{u \in \mathbb{Z}} \hat{\sigma}_d(\omega, u)\ c_{m+u}\ e^{2\pi i \omega m} d\omega \tag{6.14}$$

$$= \int_{\mathbb{T}} \sum_{u \in \mathbb{Z}} \hat{\sigma}_d(\omega, u)\ M_\omega T_{-u}\ c_m d\omega. \tag{6.15}$$

Equation (6.7) is called the *spreading representation*, and $\hat{\sigma}_d$ is called the (discrete) spreading function, in analogy to Equation (2.65).

Since $\hat{\sigma}_d \in \boldsymbol{L}^1(\mathbb{T} \times \mathbb{Z})$, Equation (6.11) gives rise to:

$$\left(\sigma_d^{KN} c\right)_m = \sum_{n \in \mathbb{Z}} c_n \int_{\mathbb{T}} \hat{\sigma}_d(\omega, n - m)\ c_n\ e^{2\pi i \omega m} d\omega \tag{6.16}$$

$$= \sum_{n \in \mathbb{Z}} c_n\ \mathcal{F}_1^{-1} \hat{\sigma}_d(m, n - m). \tag{6.17}$$

We need the following notation concerning the bandwidth of a matrix [19]. Recall, that for a fixed integer $N \geq 0$, a matrix $A = (A_{m,n})$, $m, n \in \mathbb{Z}$, is called *banded with bandwidth* $2N + 1$, if $A_{m,n} = 0$ for $|n - m| > N$. A matrix $A = (A_{m,n})$, $m, n \in \mathbb{Z}$ has *lower bandwidth* $L$ if $A_{m,n} = 0$ whenever $m > n + L$ and *upper bandwidth* $U$ if $A_{m,n} = 0$ whenever $n > m + U$.

We also need the definition of a convolution dominated matrix [25, p.202].

**Definition 6.1.6.** Let $v$ be an admissible weight. A matrix $A = A_{m,n}, m, n \in \mathbb{Z}$, is called *convolution dominated*, in short notation $A \in \mathscr{C}_v$, if there exists a sequence $a \in \boldsymbol{\ell}_v^1$, such that

$$|A_{m,n}| \leq a(m - n), \quad m, n \in \mathbb{Z}. \tag{6.18}$$

We define the norm of $A \in \mathscr{C}_v$ by

$$\|A\|_{\mathscr{C}_v} := \sum_{n \in \mathbb{Z}} \sup_{m \in \mathbb{Z}} |A_{m,m-n}| v(n). \tag{6.19}$$

Let us have a closer look on the class $\mathscr{C}_v$. Consider the action of the matrix $A \in \mathscr{C}_v$ on a sequence $c$.

$$|(Ac)(m)| = |\sum_{n\in\mathbb{Z}} A_{m,n} c_n| \leq \sum_{n\in\mathbb{Z}} a(m-n)|c_n| = (a * |c|)(m). \tag{6.20}$$

Thus $A$ is dominated pointwise by a convolution. This explains the notation.

## 6.2 System Model

In this section, we consider a discretized transmission model [48, 65]. Since the channel is a physical object, and therefore continuous, a discrete sequence has to be transmitted over a continuous-time channel. Therefore, we need to consider how the discretization and interpolation steps affect the properties of the transmission models. We write $s[n]$ for a discrete time signal and $s(t)$ for a continuous time signal.

In the following, we consider the finite group $(N^{-1}\mathbb{Z})/\mathbb{Z}$ as a subset of the torus $\mathbb{T}$.

**Definition 6.2.1.** Let $g \in \boldsymbol{\ell}^1(\mathbb{Z}), a \in \mathbb{Z}$ and $\Lambda = a\mathbb{Z} \times (N^{-1}\mathbb{Z})/\mathbb{Z} \subset \mathbb{Z} \times \mathbb{T}$ be a lattice. A *discrete time-frequency shift* with respect to the lattice $\Lambda$ is defined as

$$g_{k,l}[n] = g[n-ak]e^{2\pi i \frac{nl}{N}}, \quad (ak, l/N) \in \Lambda. \tag{6.21}$$

We proceed with formal algebraic derivations. Let us assume that $\{g_{k,l}, k \in \mathbb{Z}, l = 0, 1, \ldots, N-1\}$ is a Gabor system. The mathematical model for the discrete-time transmit signal is

$$s[n] = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{k,l}\; g_{k,l}[n], \tag{6.22}$$

where $c_{k,l}$ are the data symbols. Since the signal $s[n]$ has to be transmitted over a continuous-time channel, we have to convert it into a continuous-time signal. Hence

$$s(t) = \sum_{n=-\infty}^{\infty} s[n]\; T_{\tilde{a}n} f_1(t), \tag{6.23}$$

where $\tilde{a} \in \mathbb{R}$ and $f_1(t)$ is an interpolating function. Now the signal $s(t)$ is transmitted, and the receive signal has the form

$$r(t) = \sigma^{KN} s(t). \tag{6.24}$$

The receive signal is converted into a discrete-time signal as follows

$$r[m] = \int_{-\infty}^{\infty} r(t)\ T_{\tilde{a}m}\overline{f_2(t)}dt, \tag{6.25}$$

where $f_2(t)$ is a sampling function. To finally compute the receive data symbols, we compute

$$d_{k,l} = \langle r, g_{k,l} \rangle. \tag{6.26}$$

Let us now consider the discrete-time receive signal $r[m]$. Combining Equations (6.23), (6.24) and (6.25), we get

$$\begin{aligned} r[m] &= \int_{-\infty}^{\infty} \left(\sigma^{KN} s\right)(t)\ T_{\tilde{a}m}\overline{f_2(t)}dt & (6.27)\\ &= \int_{-\infty}^{\infty} \left(\sigma^{KN} \sum_{n=-\infty}^{\infty} s[n]\ T_{\tilde{a}n} f_1\right)(t)\ T_{\tilde{a}m}\overline{f_2(t)}dt & (6.28)\\ &= \sum_{n=-\infty}^{\infty} s[n] \int_{-\infty}^{\infty} \left(\sigma^{KN}\ T_{\tilde{a}n} f_1\right)(t)\ T_{\tilde{a}m}\overline{f_2(t)}dt & (6.29)\\ &= \sum_{n=-\infty}^{\infty} s[n] \langle \sigma^{KN}\ T_{\tilde{a}n} f_1, T_{\tilde{a}m} f_2 \rangle. & (6.30) \end{aligned}$$

We define the *discretized channel* as

$$k[m,n] := \langle \sigma^{KN}\ T_{\tilde{a}n} f_1, T_{\tilde{a}m} f_2 \rangle. \tag{6.31}$$

Combining Equations (6.31) and (6.30) we obtain the following proposition.

**Proposition 6.2.2** (Transmit-receive relation)**.** *The discrete-time transmit signal $s$ and the receive signal $r$ satisfy*

$$r[m] = \sum_{n=-\infty}^{\infty} s[n]\ k[m,n] \tag{6.32}$$

*where $k$ is given by Equation (6.31).*

Comparing Equations (6.17) and (6.32) we arrive at the following proposition.

**Proposition 6.2.3.** *For the particular system model described in this section, the discrete Kohn-Nirenberg symbol is given by:*

$$k[m,n] = \mathcal{F}_1^{-1}\hat{\sigma}_d(m, n-m). \tag{6.33}$$

Next we look at the received data symbols.

$$\begin{aligned}
d_{k,l} &= \langle r, g_{k,l}\rangle && (6.34)\\
&= \sum_{m=-\infty}^{\infty} r[m]\ \overline{g_{k,l}}[m] && (6.35)\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} k[m,n]\ s[n]\ \overline{g_{k,l}}[m] && (6.36)\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} k[m,n] \sum_{k'=-\infty}^{\infty}\sum_{l'=0}^{N-1} c_{k',l'}\ g_{k',l'}[n]\ \overline{g_{k,l}}[m] && (6.37)\\
&= \sum_{k'=-\infty}^{\infty}\sum_{l'=0}^{N-1} c_{k',l'} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} k[m,n]\ g_{k',l'}[n]\ \overline{g_{k,l}}[m]. && (6.38)
\end{aligned}$$

Consequently, we define the discrete-time channel matrix as

$$H_{k,l,k',l'} := \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} k[m,n]\ g_{k',l'}[n]\ \overline{g_{k,l}}[m] = \langle \sigma_d^{KN} g_{k',l'}, g_{k,l}\rangle. \tag{6.39}$$

**Proposition 6.2.4.** *The transmit-receive relation in terms of data symbols is given by:*

$$d_{k,l} = \sum_{k'=-\infty}^{\infty}\sum_{l'=0}^{N-1} c_{k',l'}\ H_{k,l,k',l'}. \tag{6.40}$$

## 6.3 Properties of the Discrete-Time Model

In this section, we investigate properties of the discrete Kohn-Nirenberg symbol. Our main result is that if the continuous-time symbol $\sigma$ belongs to the class $\boldsymbol{M}^{\infty,1}_{v\circ j^{-1}}(\mathbb{R}^2)$, then the discrete-time symbol $\sigma_d$ belongs to the class $\boldsymbol{M}^{\infty,1}_{(v\otimes 1)\circ j^{-1}}(\mathbb{Z}\times\mathbb{T})$. The proof of the main theorem is divided into several steps. Moreover, we investigate properties of the system model of Section 6.2 and establish a connection to convolution dominated matrices.

In Section 6.2 we introduced the matrix $k[m,n] = \langle \sigma^{KN} T_{\tilde{a}n} f_1, T_{\tilde{a}m} f_2\rangle$. This matrix is crucial for the investigation of the discrete-time model, because it establishes a connection between the continuous-time and the discrete-time model.

In the previous chapters, one important assumption is that the support of the continuous-time spreading function is compact. In this context, we prove the following results.

**Theorem 6.3.1.** *Let $\hat{\sigma} \in \mathcal{S}'(\mathbb{R}^2)$, $f_1, f_2 \in \mathcal{S}(\mathbb{R})$. If $\operatorname{supp}\hat{\sigma} \subseteq \mathbb{R}\times[-\tau, 0]$, $\operatorname{supp} f_1 \subseteq [A_1, B_1]$, and $\operatorname{supp} f_2 \subseteq [A_2, B_2]$, then the matrix $k[m,n]$ is banded with upper bandwidth $(B_2 - A_1)/a$ and lower bandwidth $(-\tau + A_2 - B_1)/a$.*

*Proof.* We recall the definition of the pseudodifferential operator $\sigma^{KN}$, Equation (2.65), and compute

$$\langle \sigma^{KN} T_{an} f_1, T_{am} f_2\rangle = \int_{\mathbb{R}^2} \hat{\sigma}(\eta, u)\langle M_\eta T_{-u} T_{an} f_1, T_{am} f_2\rangle du d\eta. \tag{6.41}$$

Since by assumption $\operatorname{supp} f_1 \subseteq [A_1, B_1]$ and $\operatorname{supp} f_2 \subseteq [A_2, B_2]$, we conclude that $\operatorname{supp} T_{-u}T_{an} f_1 \subseteq [an - u + A_1, an - u + B_1]$ and $\operatorname{supp} T_{am} f_2 \subseteq [am + A_2, am + B_2]$. If the expression in Equation (6.41) is non-zero, then the supports intersect on a set of measure zero and, consequently,

$$[an - u + A_1, an - u + B_1] \cap [am + A_2, am + B_2] \neq \emptyset. \tag{6.42}$$

Equation (6.42) states that either

$$an - u + A_1 \leq am + B_2, \tag{6.43}$$

or

$$am + A_2 \leq an - u + B_1, \tag{6.44}$$

which can be written as

$$u + A_2 - B_1 \leq a(n - m) \leq u + B_2 - A_1. \tag{6.45}$$

By assumption, we have $-\tau \leq u \leq 0$ so that

$$\frac{-\tau + A_2 - B_1}{a} \leq n - m \leq \frac{B_2 - A_1}{a}. \tag{6.46}$$

It follows that $k[m,n]$ is a banded matrix. $\square$

Theorem 6.3.1 implies the following corollary.

**Corollary 6.3.2.** *Let $\hat{\sigma} \in \mathcal{S}'(\mathbb{R}^2)$, $f_1, f_2 \in \mathcal{S}(\mathbb{R})$. If $\operatorname{supp}\hat{\sigma}$ is compact and $f_1$ and $f_2$ have finite support, then the matrix $k[m,n]$ is banded.*

Next we prove a theorem that provides a condition for the bandedness of the matrix $k[m,n]$ which depends on the support of the discrete spreading function.

**Theorem 6.3.3.** *Let $\sigma_d \in \mathcal{S}'(\mathbb{Z} \times \mathbb{T})$. The matrix $k[m,n]$ is banded with bandwidth $2N+1$ if and only if $\operatorname{supp}\hat{\sigma}_d \subseteq \mathbb{T} \times \{-N, -N+1, \ldots, N-1, N\}$.*

*Proof.* From Proposition 6.2.3 we know that

$$k[m,n] = \mathcal{F}_1^{-1}\hat{\sigma}_d(m, n-m). \tag{6.47}$$

Since $\hat{\sigma}_d$ has support in the second variable in the set $\{-N, -N+1, \ldots, N-1, N\}$, it follows that $k[m,n] = 0$ if $|n-m| > N$.

Let us now assume that $k[m,n]$ is a banded matrix with $k[m,n] = 0$ for $|n-m| > N$. From Proposition 6.2.3 we conclude that

$$k[m,n] = \mathcal{F}_2\sigma_d(m, n-m) \tag{6.48}$$

Since $\hat{\sigma}_d(\omega, u) = \mathcal{F}_1\mathcal{F}_2\sigma_d(\omega, u)$, with $\omega \in \mathbb{T}, u \in \mathbb{Z}$, we conclude that

$$\hat{\sigma}_d(\omega, u) = 0, \quad \text{if } |u| > N. \tag{6.49}$$

It follows that $\operatorname{supp}\hat{\sigma}_d \subseteq \mathbb{T} \times \{-N, -N+1, \ldots, N-1, N\}$. $\square$

The proof of Theorem 6.3.3 implies the following corollaries.

**Corollary 6.3.4.** *Let $\sigma_d \in \mathcal{S}'(\mathbb{Z}\times\mathbb{T})$. The matrix $k[m,n]$ is lower triangular if and only if $\operatorname{supp}\hat{\sigma}_d \subseteq \mathbb{T} \times \{-N, -N+1, \ldots, 0\}$.*

**Corollary 6.3.5.** *Let $\sigma_d \in \mathcal{S}'(\mathbb{Z} \times \mathbb{T})$. The matrix $k[m,n]$ is banded if and only if $\operatorname{supp} \hat{\sigma}_d$ is compact.*

In Theorem 6.3.3, we assume that the support of the discrete spreading function is compact. If we additionally assume that the window sequence $g$ has a finite support, then we have the following theorem.

**Theorem 6.3.6.** *Let $\sigma_d \in \mathcal{S}'(\mathbb{Z} \times \mathbb{T})$. If $\operatorname{supp} \hat{\sigma}_d \subseteq [0, \frac{1}{M}] \times \{-N, -N + 1 \ldots, 0\}$ and $\operatorname{supp} g \subseteq \{0, 1, \ldots, A\}$, then the channel matrix $H_{k,l,k',l'}$ is banded with respect to the time variable.*

The proof is similar to the proof of Theorem 6.3.1.

Theorem 6.3.6 implies the following.

**Corollary 6.3.7.** *Let $\sigma_d \in \mathcal{S}'(\mathbb{Z} \times \mathbb{T})$. If $\operatorname{supp} \hat{\sigma}_d$ is compact and the sequence $g$ has finite support, then the discrete channel matrix $H_{k,l,k',l'}$ is banded with respect to the time variable.*

In the following, we assume that $v$ is an admissible weight, see Definition 2.5.2. Now we state the main theorem of this section.

**Theorem 6.3.8.** *Let $f_1, f_2 \in \boldsymbol{M}^1_v(\mathbb{R})$ be system model functions, and let a sequence $g \in \boldsymbol{\ell}^1_v(\mathbb{Z})$ generate a tight frame for $\boldsymbol{\ell}^2(\mathbb{Z})$ with respect to the lattice $\Lambda = a\mathbb{Z} \times (N^{-1}\mathbb{Z})/\mathbb{Z} \subseteq \mathbb{Z} \times \mathbb{T}$. If $\sigma$ belongs to the class $\boldsymbol{M}^{\infty,1}_{v \circ j^{-1}}(\mathbb{R}^2)$, then $\sigma_d$ belongs to the class $\boldsymbol{M}^{\infty,1}_{(v \otimes 1) \circ j^{-1}}(\mathbb{Z} \times \mathbb{T})$.*

The proof is divided into several lemmas.

For the investigation of the system model of Section 6.2, we need the following lemma which follows from thw weighted version of Theorem 2.8.3 [28, Theorem 4.3].

**Lemma 6.3.9.** *Let $f_1, f_2 \in \boldsymbol{M}^1_v(\mathbb{R})$. If $\sigma$ belongs to the class $\boldsymbol{M}^{\infty,1}_{v \circ j^{-1}}(\mathbb{R}^2)$, then there exists a sequence $\tilde{k} \in \boldsymbol{\ell}^1_v(\Lambda)$ such that*

$$|\langle \sigma^{KN} T_{\tilde{a}n} f_1, T_{\tilde{a}m} f_2 \rangle| \leq \tilde{k}(am - an). \tag{6.50}$$

Now we prove the following property of the discrete channel matrix.

**Lemma 6.3.10.** *Let $a \in \mathbb{Z}$ and $\Lambda = a\mathbb{Z} \times (N^{-1}\mathbb{Z})/\mathbb{Z} \subseteq \mathbb{Z} \times \mathbb{T}$ be a lattice. If $|k[m,n]| \le \tilde{k}(am - an)$, with $\tilde{k} \in \boldsymbol{\ell}^1_v(\Lambda)$, and $g \in \boldsymbol{\ell}^1_v(\mathbb{Z})$, then $|H_{k,l,k',l'}| \le \kappa(k-k')$ with $\kappa \in \boldsymbol{\ell}^1_v(\mathbb{Z})$.*

*Proof.* We use Lemma 6.3.9 and get

$$
\begin{aligned}
|H_{k,l,k',l'}| &= \left| \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} k[m,n]\; g_{k',l'}[n]\; \overline{g_{k,l}}[m] \right| && (6.51)\\
&\le \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left| k[m,n]\; T_{ak'}g[n]\; \overline{T_{ak}g}[m] \right| && (6.52)\\
&\le \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{k}(m-n)\;\; |g[n-ak']\; \overline{g}[m-ak]| && (6.53)\\
&= \sum_{m=-\infty}^{\infty} \left(\tilde{k} * |g|\right)(m-ak')\;\; |\overline{g}[m-ak]|\,. && (6.54)
\end{aligned}
$$

Now we make the substitution $s = ak - m$ and get

$$
\begin{aligned}
|H_{k,l,k',l'}| &\le \sum_{s=-\infty}^{\infty} \left(\tilde{k} * |g|\right)(ak - ak' - s)\;\; |\overline{g}[-s]| && (6.55)\\
&= \left(\tilde{k} * |g| * |g^*|\right)(a(k-k')) := \kappa(k-k') && (6.56)
\end{aligned}
$$

with $g^*[s] = \overline{g}[-s]$. Since by assumption $\tilde{k} \in \boldsymbol{\ell}^1_v(\Lambda)$, $g \in \boldsymbol{\ell}^1_v(\mathbb{Z})$ we have $\kappa \in \boldsymbol{\ell}^1_v(\mathbb{Z})$ and the proof is finished. □

We also need the following result which is a special case of Theorem 2.8.3 for discrete pseudodifferential operators [24, 28].

**Lemma 6.3.11.** *Let $g \in \boldsymbol{M}^1_v(\mathbb{Z})$, $\Lambda = a\mathbb{Z} \times (N^{-1}\mathbb{Z})/\mathbb{Z} \subseteq \mathbb{Z} \times \mathbb{T}$ be a lattice and assume that $\{g_{k,l}, k \in \mathbb{Z}, l = 0, 1, \ldots, N-1\}$ is a tight frame for $\boldsymbol{\ell}^2(\mathbb{Z})$. If there exists a $\tilde{H} \in \boldsymbol{\ell}^1_{v\otimes 1}(\Lambda)$ such that $|H_{k,l,k',l'}| \le \tilde{H}(k-k', l-l')$, then $\sigma_d$ belongs to the class $\boldsymbol{M}^{\infty,1}_{(v\otimes 1)\circ j^{-1}}(\mathbb{Z} \times \mathbb{T})$.*

Now we can proof Theorem 6.3.8.

*Proof of Theorem 6.3.8.* Let $f_1, f_2 \in \boldsymbol{M}^1_v(\mathbb{R})$ be system model functions. Since $\sigma$ belongs to the class $\boldsymbol{M}^{\infty,1}_{v\circ j^{-1}}(\mathbb{R}^2)$, we conclude from Lemma 6.3.9 that there exists a sequence $\tilde{k} \in \boldsymbol{\ell}^1_v(\Lambda)$ such that

$$|\langle \sigma^{KN} T_{\tilde{a}n} f_1, T_{\tilde{a}m} f_2 \rangle| \leq \tilde{k}(am - an). \tag{6.57}$$

Since $g \in \boldsymbol{\ell}^1_v(\mathbb{Z})$ we conclude from Lemma 6.3.10 that $|H_{k,l,k',l'}| \leq \kappa(k - k')$ with $\kappa \in \boldsymbol{\ell}^1_v(\mathbb{Z})$. We define $\tilde{H}(k, l) = \kappa(k), \forall k \in \mathbb{Z}, l = 0, \ldots, N-1$. Since $\kappa \in \boldsymbol{\ell}^1_v(\mathbb{Z})$, $\tilde{H} \in \boldsymbol{\ell}^1_{v\otimes 1}(\mathbb{Z} \times \mathbb{T})$ and the claim now follows from Lemma 6.3.11.

□

Next we state a theorem that provides a connection between the discrete Kohn-Nirenberg symbol and convolution dominated matrices.

Lemma 6.3.9 together with Definition 6.1.6 implies that the matrix $k[m, n] \in \mathscr{C}_v$. This leads us to the following theorem.

**Theorem 6.3.12.** *[28] The matrix $k[m, n]$ in in $\mathcal{C}_v$ if and only if $\mathcal{T}_2\sigma_d$ belongs to the class $\boldsymbol{M}^{\infty,1}_{(v\otimes 1)\circ j^{-1}}(\mathbb{Z} \times \mathbb{T})$, where $\mathcal{T}_2\sigma_d(x, \xi) := \sigma_d(x, -\xi)$.*

## 6.4 Causality

Causality is a natural assumption in several scientific fields like physics or probability theory. In the context of wireless communications, causality means that channels can only spread symbols into the future. In other words, the channel output can only depend on the current and on previous inputs.

In causal wireless communication systems the received signals can only interact with the current and the previously sent signals. However, a common assumption is that the support of the spreading function is contained in a rectangle centered around the origin in time and in frequency [38, 40]. If we assume causality, then the proper assumption is that the rectangle containing the support of the spreading function $\hat{\sigma}$ is symmetric w.r.t. the origin in the frequency direction, but asymmetric in the time direction. In mathematical terms, causality means that $\operatorname{supp} \hat{\sigma} \subseteq [-\nu, \nu] \times [0, \tau]$, [45, 47].

Note that for our specific approach, we need to consider causality in two different settings. Causality in the continuous setting means that

$$\operatorname{supp} \hat{\sigma} \subseteq [-\nu, \nu] \times [0, \tau] \tag{6.58}$$

and the system model functions $f_1$ and $f_2$ have finite support.

In the discrete setting, causality means that

$$\operatorname{supp} \hat{\sigma}_d \subseteq \mathbb{T} \times \{0, 1, \ldots, N\}. \tag{6.59}$$

Therefore, we need to analyze how these two assumptions interact.

For the following derivations, we need to consider $-t$ instead of $t$ due to the definition of the Kohn-Nirenberg transform (2.65).

Now we show the following. Assuming for the continuous-time model that $\operatorname{supp} \hat{\sigma} \subseteq \mathbb{R} \times [-\tau, 0]$, $\operatorname{supp} f_1 \subseteq [A_1, B_1]$ and $\operatorname{supp} f_2 \subseteq [A_2, B_2]$, we claim that the discrete-time model inherits the causality from the continuous-time model if and only if $B_2 - A_1 \leq 0$.

From Theorem 6.3.1 we conclude that, if $\operatorname{supp} \hat{\sigma} \subseteq \mathbb{R} \times [-\tau, 0]$, $\operatorname{supp} f_1 \subseteq [A_1, B_1]$ and $\operatorname{supp} f_2 \subseteq [A_2, B_2]$, then the matrix $k[m, n]$ has the upper bandwidth equal to $(B_2 - A_1)/a$. One can show that the bound is strict. Loosely speaking, the boundary is obtained for spreading functions $\hat{\sigma}$ which are not vanishing at zero.

If we assume that $B_2 - A_1 \leq 0$, then it follows that $k[m, n]$ is lower triangular. Next we conclude from Corollary 6.3.4 that if $k[m, n]$ is lower triangular, then $\operatorname{supp} \hat{\sigma}_d \subseteq \mathbb{T} \times \{-N, -N + 1, \ldots, 0\}$, which means that the discrete-time model is causal.

To summarize, we conclude from Theorem 6.3.1 and Corollary 6.3.4 that the discrete-time model inherits the causality from the continuous-time model if $B_2 - A_1 \leq 0$.

This condition can be interpreted as follows. Since $\operatorname{supp} f_1 \subseteq [A_1, B_1]$ and $\operatorname{supp} f_2 \subseteq [A_2, B_2]$, the condition $B_2 \leq A_1$ means that the support of the sampling function $f_2$ precedes the support of the interpolating function $f_1$ with respect to time.

# Bibliography


[1] A. Aldroubi and K. Gröchenig. Nonuniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.*, 43(4):585–620, 2001.

[2] J. J. Benedetto and W. Heller. Irregular sampling and the theory of frames. I. *Note di Matematica*, 10(1):103–125, 1990.

[3] H. Boche and U. Mönich. There exists no globally uniformly convergent reconstruction for the Paley-Wiener space $PW_\pi^1$ of bandlimited bunctions sampled at Nyquist rate. *IEEE Trans. Sig. Proc.*, 56(7):3170–3179, 2008.

[4] H. Bölcskei, P. Duhamel, and R. Hleiss. Design of pulse shaping OFDM/OQAM systems for high data-rate transmission over wireless channels. In *Proc. of the 1999 IEEE Int. Conf. on Communications (ICC '99)*, volume 1, pages 559–564, jun 1999.

[5] H. Bölcskei, R. Koetter, and S. Mallik. Coding and modulation for underspread fading channels. In *IEEE International Symposium on Information Theory (ISIT) 2002*, page 358, jun 2002.

[6] P. L. Butzer, W. Splettstösser, and R. L. Stens. The sampling theorem and linear prediction in signal analysis. *Jber.d.Dt.Math.-Verein.*, 90:1–70, 1988.

[7] S. Colieri, M. Ergen, A. Puri, and A. Bahai. A study of channel estimation in OFDM systems. In *Vehicular Technology Conference, 2002. Proceedings. VTC 2002-Fall. 2002 IEEE 56th*, volume 2, pages 894–898, Sept. 2002.

[8] S. Das. *Mathematical Methods for Wireless Channel Estimation and Equalization*. PhD thesis, University of Vienna, 2009.

[9] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Am. Math. Soc.*, 72:341–366, 1952.

[10] G. Durisi, V. I. Morgenshtern, H. Bölcskei, U. G. Schuster, and S. Shamai (Shitz). Information theory of underspread WSSUS channels. In F. Hlawatsch and G. Matz, editors, *Wireless Communications over Rapidly Time-Varying Channels*. Elsevier Academic Press, 2011.

[11] G. Durisi, U. Schuster, H. Bölcskei, and S. Shamai (Shitz). Noncoherent Capacity of Underspread Fading Channels. *IEEE Transactions on Information Theory*, 56(1):367–395, 2010.

[12] H. Dym and H. P. McKean. *Fourier Series and Integrals. (Paperback Ed.)*. Academic Press, 1975.

[13] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92:269–289, 1981.

[14] H. G. Feichtinger. Modulation spaces of locally compact Abelian groups. In R. Radha, M. Krishna, and S. Thangavelu, editors, *Proc. Internat. Conf. on Wavelets and Applications*, pages 1–56. New Delhi Allied Publishers, 2003.

[15] H. G. Feichtinger, B. Helffer, M. P. Lamoureux, N. Lerner, and J. Toft. *Pseudo-Differential Operators. Quantization and Signals. CIME lecture notes*. Springer, 2006.

[16] P. Fertl and G. Matz. Multi-user channel estimation in OFDMA uplink systems based on irregular sampling and reduced pilot overhead. In *Proc. of the IEEE Conference on Acoustics, Speech and Signal Processing (ICASSP 2007)*, Honolulu, Hawaii, USA, April 2007.

[17] G. B. Folland. *Harmonic Analysis in Phase Space*. Princeton University Press, Princeton, N.J., 1989.

[18] F. G. Friedlander. *Introduction to the Theory of Distributions. With Additional Material by M. Joshi.* Cambridge University Press, Cambridge, 2nd ed. edition, 1998.

[19] G. H. Golub and C. F. van Loan. *Matrix Computations. 2nd ed.* The Johns Hopkins University Press, Baltimore, 1989.

[20] I. S. Gradsteyn and I. M. Ryzhik. *Table of Integrals, Series and Products. 7th ed.* Elsevier, 2007.

[21] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.

[22] K. Gröchenig. A pedestrian's approach to pseudodifferential operators. In C. Heil, editor, *Harmonic Analysis and Applications in Honor of John J. Benedetto's 65th Birthday*, pages 139–170, Boston, 2006. Birkhauser.

[23] K. Gröchenig. Composition and spectral invariance of pseudodifferential operators on modulation spaces. *J. Anal. Math.*, 98:65–82, 2006.

[24] K. Gröchenig. Time-frequency analysis of Sjöstrand's class. *Revista Mat. Iberoam.*, 22(2):703–724, 2006.

[25] K. Gröchenig. Wiener's Lemma: Theme and Variations. An Introduction to Spectral Invariance. In *Four Short Courses on Harmonic Analysis.* Birkhäuser Boston, 2010.

[26] K. Gröchenig and C. Heil. Modulation spaces and pseudodifferential operators. *Integral Equations Operator Theory*, 34(4):439–457, 1999.

[27] K. Gröchenig and M. Neuhauser. An uncertainty principle for the time-frequency plane. 2009. preprint.

[28] K. Gröchenig and T. Strohmer. Pseudodifferential operators on locally compact abelian groups and Sjöstrand's symbol class. *J. Reine Angew. Math.*, 613:121–146, 2007.

[29] M. Hampejs, Svac P., G. Tauböck, K. Gröchenig, F. Hlawatsch, and G. Matz. Sequential LSQR-based ICI equalization and decision-feedback ISI cancellation in pulse-shaped multicarrier systems. In *Proc. IEEE SPAWC09*, pages 1 – 5, 2009.

[30] J. R. Higgins. Five short stories about the cardinal series. *Bull. Am. Math. Soc., New Ser.*, 12:45–89, 1985.

[31] F. Hlawatsch, G. Matz, and D. Schafhuber. Pulse-shaping OFDM/BFDM systems for time-varying channels: ISI/ICI analysis, optimal pulse design, and efficient implementation. In *Proc. IEEE PIMRC-02*, pages 1012–1016, Lisbon, Portugal, September 2002.

[32] Y. Hong and G. Pfander. Irregular and multi-channel sampling of operators. *Appl. Comp. Harm. Anal*, 29(2):214–231, 2010.

[33] L. Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis.* Classics in Mathematics, Reprint of the 2nd Edition 1990. Springer, 2003.

[34] A. J. Jerri. The Shannon sampling theorem - its various extensions and applications: a tutorial review. *Proc. IEEE*, 65(11):1565–1596, 1977.

[35] Y. Katznelson. *An Introduction to Harmonic Analysis.* Cambridge University Press, 2003.

[36] W. Kozek. Adaptation of Weyl-Heisenberg frames to underspread environments. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 323–352. Birkhäuser Boston, Boston, 1997.

[37] W. Kozek. *Matched Weyl-Heisenberg Expansions of Nonstationary Environments.* PhD thesis, University of Technology Vienna, 1997.

[38] W. Kozek. On the transfer function calculus for underspread LTV channels. *ISP*, 45(1):219–223, January 1997.

[39] W. Kozek and A. Molisch. Nonorthogonal pulseshapes for multicarrier communications in doubly dispersive channels. *Selected Areas in Communications, IEEE Journal on*, 16(8):1579–1589, Oct 1998.

[40] W. Kozek and G. E. Pfander. Identification of operators with bandlimited symbols. *SIAM J. Math. Anal.*, 37(3):867–888, 2006.

[41] R. J. Marks II. *Introduction to Shannon Sampling and Interpolation Theory.* Springer Texts in Electrical Engineering. Springer-Verlag, New York etc., 1991.

[42] R. J. Marks II. *Advanced Topics in Shannon Sampling and Interpolation Theory.* Springer Texts in Electrical Engineering. Springer, New York, NY, 1993.

[43] F. Marvasti. *Nonuniform Sampling. Theory and Practice. With 1 CD-ROM.* Kluwer Academic Publishers, Dordrecht, 2001.

[44] G. Matz. *A time-frequency calculus for time-varying systems and non-stationary processes with applications*. PhD thesis, Vienna University of Technology, 2000.

[45] G. Matz, K. Gröchenig, F. Hlawatsch, A. Klotz, G. Tauböck, and A. Skupch. Advanced mathematical models for the design and optimization of low-interference wireless multicarrier systems. 2006.

[46] G. Matz and F. Hlawatsch. Time-frequency transfer function calculus (symbolic calculus) of linear time-varying systems (linear operators) based on a generalized underspread theory. *J. Math. Phys.*, 39(8):4041–4070, 1998.

[47] G. Matz, D. Schafhuber, K. Gröchenig, M. Hartmann, and F. Hlawatsch. Analysis, optimization, and implementation of low-interference wireless multicarrier systems. *IEEE Trans. Wireless Comm.*, 6(4):1–11, 2007.

[48] G. Matz, G. Tauböck, M. Hampejs, P. Svac, F. Hlawatsch, and K. Gröchenig. Low-complexity ICI/ISI equalization in doubly dispersive multicarrier systems using a decision-feedback LSQR algorithm. *IEEE Trans. Sig. Proc.*, 59(5):2432–2436, 2011.

[49] A. Papoulis. Generalized sampling expansion. *IEEE Trans. Circuits and Systems*, 24(11):652 – 654, 1977.

[50] A. Papoulis. *Signal Analysis.* McGraw-Hill Book Company, New York, 1977.

[51] G. Pfander and D. F. Walnut. Measurement of time-varying multiple-input multiple-output channels. *Appl. Comput. Harm. Anal.*, 24(3):393–401, 2008.

[52] G. E. Pfander and D. F. Walnut. Measurement of time-variant channels. *IEEE Trans. Inform. Theory*, 52(11):4808–4820, November 2006.

[53] J. C. T. Pool. Mathematical aspects of the Weyl correspondence. *J. Math. Phys.*, 7:66–76, 1966.

[54] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups.* Clarendon Press, Oxford, 1968.

[55] L. Rugini, P. Banelli, and G. Leus. Simple Equalization of Time-Varying Channels for OFDM. *IEEE Communications Letters*, 9(7):619–621, 2005.

[56] M. Ruzhansky and V. Turunen. Quantization of pseudo-differential operators on the torus. *Journal Fourier Anal. Appl.*, 16:943–982, 2010.

[57] P. Schniter. On doubly dispersive channel estimation for pilot-aided pulse-shaped multicarrier modulation. *Proc. Conf. on Information Sciences and Systems*, 2006.

[58] P. Schniter, S.-J. Hwang, S. Das, and A. P. Kannu. Equalization of Time-Varying Channels. In F. Hlawatsch and G. Matz, editors, *Wireless Communications over Rapidly Time-Varying Channels*. Elsevier Academic Press, 2011.

[59] J. Sjöstrand. An algebra of pseudodifferential operators. *Math. Res. Lett.*, 1(2):185–192, 1994.

[60] J. Sjöstrand. Wiener type algebras of pseudodifferential operators. 1995.

[61] R. S. Strichartz. *A Guide to Distribution Theory and Fourier Transforms.* World Scientific, NJ, 2003.

[62] T. Strohmer. Pseudodifferential operators and Banach algebras in mobile communications. *Appl. Comput. Harmon. Anal.*, 20(2):237–249, March 2006.

[63] T. Strohmer and S. Beaver. Optimal OFDM system design for time-frequency dispersive channels. *IEEE Trans. Commun.*, 51(7):1111–1122, 2003.

[64] T. Strohmer and J. Tanner. Implementations of Shannon's sampling theorem, a time-frequency approach. *Sampl. Theory Signal Image Process.*, 4(1):1–17, 2005.

[65] G. Tauböck, F. Hlawatsch, D. Eiwen, and H. Rauhut. Compressive estimation of doubly selective channels in multicarrier systems: leakage effects and sparsity-enhancing processing. *IEEE J. Sel. Top. Sig. Process.*, 4(2):255–271, 2010.

[66] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I. *J. Funct. Anal.*, 207(2):399–429, 2004.

[67] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. II. *Ann. Global Anal. Geom.*, 26(1):73–106, 2004.

# Curriculum Vitæ


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