DIPLOMARBEIT

Titel der Diplomarbeit

REPRODUCING KERNEL HILBERT SPACES

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Abstract

The reproducing kernel Hilbert space construction is a bijection or transform theory which associates a positive definite kernel with a Hilbert space of functions.

In this diploma thesis we will give an overview of the theory of reproducing kernel Hilbert spaces and its applications. For this purpose, we first collect all needed ingredients from functional and complex analysis in several variables (chapter 2 and 3) and then show the basic facts of reproducing kernel Hilbert spaces (chapter 4).
In the final chapter 5 we will specialize on one of the most important reproducing kernel Hilbert spaces resp. its kernel: Bergman spaces $A^2(\Omega)$ resp. the Bergman kernel $K(z, w)$, and show the connection to the Riemann mapping theorem and the Cauchy– Riemann differential equations.

Since the theory of reproducing kernel Hilbert spaces interacts with many subjects in mathematics, reproducing kernel Hilbert spaces and their kernels in particular is still an important field in mathematical research with fruitful applications. Especially, the calculation and estimations of kernels on different domains are essential goals.

Zusammenfassung

Hinter der Konstruktion von Hilberträumen mit reproduzierendem Kern verbirgt sich eine Theorie von Bijektionen bzw. Transformationen, die einen positiv definiten Kern mit einem Hilbertraum von Funktionen verbindet.

Das Ziel dieser Diplomarbeit ist es einen Überblick über die Theorie der Hilberträume mit reproduzierendem Kern und ihrer Anwendungen zu geben. Zu diesem Zweck ist die Arbeit in folgende Teile gegliedert.
Zuerst werden alle für die Theorie benötigten Grundlagen aus Funktionalanalysis sowie aus der Komplexen Analysis mehrerer Veränderlicher gesammelt (Kapitel 2 und 3), um dann in Kapitel 4 die wichtigsten Eigenschaften der Hilberträume mit reproduzierendem Kern zu beweisen.
Im letzten Kapitel 5 spezialisieren wir uns auf einen der wichtigsten Hilberträume mit reproduzierendem Kern bzw. dessen Kern: dem Bergmanraum $A^2(\Omega)$ bzw. dem Bergman-Kern $K(z, w)$. Abschließend zeigen wir noch den Zusammenhang zwischen dem Bergman - Kern und dem Riemannschen Abbildungssatz, sowie den Cauchy- Riemann'schen Differentialgleichungen.

Da sich die Theorie der Hilberträumen mit reproduzierendem Kern über viele mathematische Bereiche erstreckt, ist sie und ihre Kerne im Besonderen bis heute ein wichtiger Teil der mathematischer Forschung, die reichhaltige Anwendungen liefert. Ein besonderes Augenmerk wird dabei auf die Berechnung bzw. Abschätzung der Kerne auf verschiedenen Gebieten gelegt.

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"I dedicate this thesis to my parents who unremittingly supported me during my years of study. They made this work possible. "

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Chapter 0
Introduction

0.1 HISTORICAL OVERVIEW

The reproducing kernel was used for the first time at the beginning of the 20^{th} century by Stanislaw Zaremba in his work on boundary value problems for harmonic and biharmonic functions.
In 1907, he was the first who introduced, in a particular case, the kernel corresponding to a class of functions, and stated its reproducing property. However, he did not develop any general theory, nor did he give any particular name to the kernels he introduced.

In 1909, James Mercer examined functions which satisfy the reproducing property in the theory of integral equations developed by David Hilbert. Mercer called this functions "positive definite kernels".
He discovered the property

$$\sum_{i,j=1}^{n} K(y_i, y_j)\, \xi_i \cdot \xi_j \geq 0 \; (y_i \text{ any points, } \xi_i \in \mathbb{C}) \qquad (0.1.)$$

characterizing his kernels among all the continuous kernels of integral equations.

However, for a long time these results were not investigated. Then the idea of reproducing kernels appeared in the dissertations of three Berlin mathematicians Gabor Szegö (1921), Stefan Bergman (1922) and Salomon Bochner (1922).
In particular, Bergman introduced reproducing kernels in one and several complex variables for the class of harmonic and analytic respectively holomorphic functions and he called them "kernel functions".

In 1935, E.H.Moore introduced these kernels in the general analysis under the name of "positive Hermitian matrices" with a kind of generalization of integral equations. Moore considered kernels $K(x, y)$ defined on an abstract set E and characterized by the property (0.1.). He proved that to each positive Hermitian matrix there corresponds a class of functions forming what we now call a Hilbert space with scalar product $\langle f, g \rangle$ and in which the kernel has the reproducing property

$$f(y) = \langle f(x), K(x, y) \rangle \,. \qquad (0.2.)$$

In 1943, Nachman Aronszajn developed the general theory of reproducing kernels which contains, as particular cases, the Bergman kernel functions. In this theory a central role is played by the reproducing property of the kernel in respect to the class to which it belongs. The kernel is defined by its property . The simple fact was stressed that a reproducing kernel always possesses property (0.1.) characterisitc of positive Hermitian matrices (in the sense of E.H. Moore).

The original idea of Zaremba to apply the kernels to the solution of boundary value problems was developed by Stefan Bergman and Menahem Max Schiffer. In their investigations the kernel was proved to be a powerful tool for solving boundary value problems of partial differential equations of elliptic type. Moreover, by application of kernels to conformal mapping of multiply-connected domains, very beautiful results were obtained by Bergman, Schiffer and Garabedian. Quite recently,the connection was found between the Bergman kernel and the kernel introduced by Gabor Szegö.

Several important results ware achieved by the use of these kernels in the theory of one and several complex variables, in conformal mapping of simply- and multiply connected domains, in pseudo-conformal mappings, in the study of invariant Riemannian metrics and in other subjects. Meanwhile, in probability theory, the theory of positive definite kernels was used by A.N. Kolmogrov, E. Parzen and others.

Chapter I

Basic Notation

I.1 NOTATIONS

The aim of this chapter is to introduce some notations and conventions used throughout this thesis.

(a) $\mathbb{R}$ and $\mathbb{C}$ denote the field of real, respectively complex numbers.

(b) $\mathbb{Z}$ and $\mathbb{N}$ denote the integers, respectively nonnegative integers, while we use $\mathbb{N}^+$ for the positive integers.

(c) For $n \in \mathbb{N}^+$, the n - dimensional complex number space $\mathbb{C}^n: = \{z | z = (z_1, \dots, z_n), z_j \in \mathbb{C} \text{ for } 1 \le j \le n\}$ is the Cartesian product of n copies of $\mathbb{C}$ and carries the structure of an n - dimensional complex vector space.
The topology of $\mathbb{C}$ is identical with the one arising from the following identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Given $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, each coordinate z_j can be written as $z_j = x_j + iy_j$, with $x_j, y_j \in \mathbb{R}$ and i is the imaginary unit.The mapping

$$z \longmapsto (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$$

establishes an $\mathbb{R}$ - linear isomorphism between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$.

(d) $\subseteq$ means subset ($\subset$ will not be used).

(e) $U \subseteq (\mathbb{R}^n, \mathbb{C}^n)$ will always be an open and non-empty subset. $U^c = (\mathbb{R}^n, \mathbb{C}^n) \setminus U$ denotes the complement of U in $(\mathbb{R}^n, \mathbb{C}^n)$.

(f) Let $A \subseteq U$. We denote by A° the interior of A and by $\overline{A}^U$ the closure of A in U; if U is clear from the context, we will only write $\overline{A}$.

(g) $\Omega \subseteq (\mathbb{C}, \mathbb{C}^n)$ will denote a domain (a connected open set).

(h) Multi - index notation:
$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ will be called a multi - index of length (order) $|\alpha|: = \sum_{j=1}^{n} \alpha_j$.
For multi-indices α, β we define
- $\alpha + \beta: = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$
- $\beta \le \alpha \Leftrightarrow \beta_j \le \alpha_j \ \forall 1 \le j \le n$
- $\alpha - \beta: = (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$ if $\beta \le \alpha$.
- For $x \in \mathbb{R}^n$ we write $x^\alpha: = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
- $\alpha! = \alpha_1! \cdots \alpha_n!$
- $\binom{\alpha}{\beta}: = \frac{\alpha!}{\beta!\,(\alpha-\beta)!}$

Chapter 2
Hilbert Spaces

The aim of this chapter is to repeat basic definitions related to Hilbert spaces as well as to listen the main results of functional analysis that we will need in the following chapters.

2.1 Basic Definitions and Properties

Throughout this paragraph the scalar field of a vector space will be denoted by $\mathbb{K}$ and will be either the real number $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

2.1.1. Definition (Normed Vector Space):

(i) A norm on a vector space X is a function

$$\|\cdot\|: X \to \mathbb{R}^+ \quad \text{with} \quad x \longmapsto \|x\|$$

that satisfies, for all $x, y \in X$, $\alpha \in \mathbb{K}$:

(N1) $\|x\| \geq 0$ and $\|x\| = 0 \Longleftrightarrow x = 0$ (positive definite)

(N2) $\|\alpha x\| = |\alpha| \cdot \|x\|$ (homogeneous)

(N3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

(ii) We call the pair $(X, \| \, \|)$ a normed vector space. We use NVS as an abbreviation.

2.1.2. Remark:

(i) If necessary, we will denote the norm on the space X by $\| \, \|_X$.

(ii) Every NVS can be turned into a metric space via the definition: $d(x, y) := \|x - y\|$.
Consequently, terms like convergence, Cauchy sequence, boundedness, compactness, etc. are automatically defined.

2.1.3. Definition (Convergence):

Let $(x_n)_n$ be a sequence in a NVS $(X, \| \, \|)$.

(i) Limit:

$$x = \lim_{n \to \infty} x_n :\Longleftrightarrow \|x_n - x\| \to 0 \text{ in } \mathbb{R}, \text{ i.e.}$$
$$\Longleftrightarrow \forall \varepsilon > 0 \; \exists N(\varepsilon) \in \mathbb{N} \; \forall n \geq N : \|x_n - x\| < \varepsilon$$

(ii) Cauchy sequence: $(x_n)_n$ is called a Cauchy sequence :

$\Longleftrightarrow \forall \varepsilon > 0 \; \exists N(\varepsilon) \in \mathbb{N} \; \forall n, m \geq N : \|x_n - x_m\| < \varepsilon$

(iii) Series:

$$x = \sum_{n=1}^{\infty} x_n :\Longleftrightarrow x = \lim_{n\to\infty} \sum_{k=1}^{n} x_k$$

$$\sum_{n=1}^{\infty} x_n \text{ is absolute convergent} :\Longleftrightarrow \sum_{n=1}^{\infty} \|x_n\| < \infty$$

2.1.4. Definition (Banach Space):

A NVS $(X, \| \|)$ is called complete if every Cauchy sequence has a limit in X.
A Banach space ((B)-space) is a complete NVS.

2.1.5. Definition (Inner product space):

Let X be a vector space over $\mathbb{C}$.

(i) An inner product on X is a map

$$\langle .,. \rangle : X \times X \longrightarrow \mathbb{C}$$

that satisfies the following properties for all $x, y \in X,\ \alpha \in \mathbb{C}$:

(I1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)

(I2) $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$ (linearity in the first argument)

(I3) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ (positive definteness)

(ii) We call the pair $(X, \langle\ ,\ \rangle)$ an inner product space.

2.1.6. Remark:

Inner product spaces have a naturally defined norm based upon the inner product of the space itself: $\|x\| := \sqrt{\langle x, x \rangle}$

2.1.7. Proposition (Cauchy-Schwarz Inequality):

Lex X be an inner product space.
Then the Cauchy -Schwarz Inequality (CSI) holds:

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad \forall x, y \in X \tag{2.1.}$$

Proof:
Confirm [Has07].

■

2.1.8. Corollary (Triangle Inequality):

(2.1.) implies the triangle inequality

$$\|x+y\| \leq \|x\| + \|y\| \;\; \forall\, x, y \in X \tag{2.2.}$$

Proof:
Confirm [Has07].

■

2.1.9. Definition (Hilbert Space):

Let $(H, \langle\ ,\ \rangle)$ be an inner product space. H is called a Hilbert space ((H)-space) if it is complete, i.e. every Cauchy sequence in H is convergent in the sense of the topology induced by the norm $\|\cdot\| = \sqrt{\langle x, x\rangle}$ (see 2.1.6. Remark).

2.1.10. Example (Hilbert spaces):

(a) $\mathbb{R}^n$: Let $x, y \in \mathbb{R}^n$. Defining the standard inner product by

$$\langle x, y\rangle := \sum_{k=1}^{n} x_k \cdot y_k \tag{2.3.}$$

turns $\mathbb{R}^n$ into a Hilbert space.

(b) $\mathbb{C}^n$: Similarly, the standard (Hermitian) inner product on $\mathbb{C}^n$ for $z, w \in \mathbb{C}^n$ is defined by

$$\langle z, w\rangle := \sum_{k=1}^{n} z_k \cdot \overline{w_k} \tag{2.4.}$$

and turns $\mathbb{C}^n$ into a Hilbert space.

(c) ℓ^2: The space of square-summable sequences is defined by

$$\ell^2 := \left\{ \beta = (x_n)_n : x_n \in \mathbb{K}, \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$$

and with

$$\langle \beta, \eta\rangle := \sum_{k=1}^{\infty} x_n \cdot \overline{y_n} \tag{2.5.}$$

as inner product it is a Hilbert space.

(d) $\mathfrak{L}^2(\mu)$:

Let (X, μ) be a measure space equipped with a positive measure μ (further reading for terminology see [KT73]). Then we define the space of square-integrable functions by

$$\mathfrak{L}^2(\mu) := \left\{ f : X \to \mathbb{C} \,\middle|\, \|f\|_2^2 := \int_X |f|^2 < \infty \right\}$$

$\mathfrak{L}^2(\mu)$ is a Hilbert space together with the inner product defined by

$$\langle f, g \rangle := \int_X f(x)\overline{g(x)}\ d\mu(x) \tag{2.6.}$$

Note:

Actually, $\mathfrak{L}^2(\mu)$ is a Hilbert space of euqivalence classes of measurable functions via the equivalence relation

$$f \sim g :\Longleftrightarrow f = g\ \mu\text{-almost everywhere}$$

For more on that confirm [KT73].

2.1.11. Proposition (Subspaces):

Let H be a Hilbert space and $W \subseteq H$ a subspace.
W is a (H)-space if and only if W is closed, i.e. $(x_n)_n \in W, x_n \overset{\text{in } H}{\longrightarrow} x \Rightarrow x \in W$.

Proof:

We have to show: W complete $\Longleftrightarrow$ W closed.

($\Rightarrow$) Let $x_n \in W$, $\|x_n - x\|_H \to 0$. Then $(x_n)_n$ is a Cauchy sequence in H, and by the fact that W is complete, $(x_n)_n$ is a Cauchy sequence in W. Hence, by completeness $\exists y \in W : \|y - x_n\|_W \longrightarrow 0$
Besides, $x = y \in W$ holds:

$$\|x - y\|_W \leq \|x - x_n\|_W + \|x_n - y\|_W \longrightarrow 0$$

($\Leftarrow$) Let x_n be a Cauchy sequence in W , then it is a Cauchy sequence in H. Since H is complete, there $\exists x \in H$ such that $x = \lim_{n\to\infty} x_n$. Hence, because W is closed, $x \in W$, so W is complete.

■

2.1.12. Proposition (Parallelogram Identity):

In every Hilbert space H the following parallelogram identity holds $(u, v \in H)$:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \tag{2.7.}$$

Proof:

$$\begin{aligned}
&\|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
&= \langle u, u\rangle + \cancel{\langle v, u\rangle} + \cancel{\langle u, v\rangle} + \langle v, v\rangle + \langle u, u\rangle - \cancel{\langle u, v\rangle} - \cancel{\langle v, u\rangle} + \langle v, v\rangle \\
&= 2\|u\|^2 + 2\|v\|^2
\end{aligned}$$

■

2.2 Continuity and Dual Space

2.2.1. Definition (Operator):

(i) A continuous and linear map between NVS is called a continuous-linear operator.
(ii) If the target space is $\mathbb{K}$, we will say functional instead of operator.

2.2.2. Reminder (On Continuity):

A continuous operator $T: E \to F$ between two NVS fulfills one of the following equivalent conditions:

(i) $\forall x \in E\ \forall (x_n)_n \in E$ with $x_n \to x$ in $E \Rightarrow\ Tx_n \to Tx$ in F
(ii) $\forall x_0 \in E\ \forall \varepsilon > 0\ \exists \delta > 0 : \|x - x_0\|_E \le \delta \Rightarrow \|Tx - Tx_0\|_F \le \varepsilon$
(iii) $\forall O \subseteq F$ open $\Rightarrow T^{-1}(O) := \{ x \in E \mid Tx \in O \}$ is open in E

2.2.3. Theorem (Characterising Continuous Operators):

Let E, F be NVS, $T: E \to F$ linear. The following are equivalent:

(i) T is continuous at 0.
(ii) T is continuous everywhere in E.
(iii) T is uniformly continuous.

Proof:

<u>(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i):</u> is obvious.
<u>(i) $\Rightarrow$ (iii):</u>
Let $\varepsilon > 0 \overset{(i)}{\Longrightarrow} \exists \delta > 0 : T(B_\delta(0)) \subseteq B_\varepsilon(T(0)) \xlongequal{T \text{ linear}} B_\varepsilon(0)$.
Hence, $\|x - y\| \le \delta \Rightarrow\ x - y \in B_\delta(0) \Rightarrow T(x - y) \in B_\varepsilon(0)$
$\Rightarrow Tx - Ty \in B_\varepsilon(0) \Rightarrow \|Tx - Ty\| \le \varepsilon$ ■

Next we are going to study the relationship between continuous and bounded linear maps. To this end, the following definition:

2.2.4. Definition:

Let E, F be NVS.

(i) A subset $A \subseteq E$ is bounded, if it is absorbed by a multiple of the unit ball $B_{1,E}(0) := \{x \in E : \|x\|_E \le 1\}$, i.e. $\exists \alpha \in \mathbb{K}: A \subseteq \alpha \cdot B_{1,E}(0) := \{ \alpha x \mid x \in E \}$

(ii) A linear map $T\colon E \to F$ is called bounded, if the image of bounded sets are bounded.

2.2.5. Remark:

To show boundedness of T, it is enough to request that $T(B_{1,E}(0))$ is bounded:

$$\begin{aligned} A \subseteq E \text{ bounded} &\Rightarrow \exists \alpha \in \mathbb{K} : A \subseteq \alpha B_{1,E}(0) \\ &\Rightarrow T(A) \subseteq T(\alpha B_{1,E}) = \alpha \underbrace{T(B_{1,E}(0))}_{\subseteq \mu(B_{1,F}(0))} \subseteq (\alpha\mu)(B_{1,F}(0)) \end{aligned}$$

2.2.6. Theorem (Continuous vs. Bounded):

Let E, F be NVS, $T\colon E \to F$ linear. The following are equivalent

(i) T is continuous.

(ii) T is bounded.

(iii) $\sup\limits_{\|x\| \leq 1} \|Tx\| < \infty$.

Proof:

(i)$\Rightarrow$ (ii)

Choose $\varepsilon = 1 \overset{(i)}{\Longrightarrow} \exists \delta > 0\colon T(B_\delta(0)) \subseteq B_{1,F}(0)$. It is $B_{1,E}(0) = \frac{1}{\delta} B_\delta(0)$. Hence, $T(B_{1,E}(0)) \overset{T \text{ linear}}{=\!=\!=} \frac{1}{\delta} T(B_\delta(0)) \subseteq \frac{1}{\delta} B_{1,F}(0)$.

2.2.5. gives the desired property.

(ii) $\Rightarrow$ (iii)

By assumption, $\exists \mu : T(B_{1,E}(0)) \subseteq \mu B_{1,F}(0)$. Hence, for $\|x\|_E \leq 1$ we have:

$\|Tx\|_F \leq \mu \Rightarrow \sup \|Tx\|_F \leq \mu < \infty$

(iii) $\Rightarrow$ (i)

It is $\left\| T\frac{x}{\|x\|_E} \right\|_F \leq \sup\limits_{\|x\| \leq 1} \|Tx\| =: \alpha < \infty$, by assumption. So, $\|Tx\|_F \leq \alpha \|x\|_E \xrightarrow{T \text{ linear}}$ $T\left(B_{\frac{\varepsilon}{\alpha}(0)}\right) \subseteq B_\varepsilon(0)$, $\Rightarrow$ T is continuous at 0. ∎

We use 2.2.6 (iii) to define the norm for continuous operators.

2.2.7. Definition (Operator Norm):

Let E, F NVS, $T\colon E \to F$ linear operator.
The operator norm of T is defined by

$$\|T\|_{op} = \|T\| := \sup_{\|x\|\leq 1} \|Tx\|_F \tag{2.8.}$$

2.2.8. Remark (on the operator norm):

It can be shown that the following identities hold:

$$\begin{aligned}\|T\| := \sup_{\|x\|\leq 1} \|Tx\|_F &= \sup_{\|x\|=1} \|Tx\|_F \\ &= \sup_{y\neq 0} \frac{\|Ty\|}{\|y\|} \\ &= \inf\left\{M \in \mathbb{R}^+ : \|Tx\| \leq M\|x\| \ \forall x \in E\right\}\end{aligned}$$

2.2.9. Definition:

Let E, F be NVS. Then the space of continuous linear operators is defined by $L(E,F) := \{\, T\colon E \to F \mid \text{linear and continuous} \,\}$.

2.2.10. Remark (on $L(E,F)$):

(i) It can be shown that $(L(E,F), \|\ \|_{op})$ is a NVS and that $L(E,F)$ is complete if F is a Banach space (confirm [Wer07]).

(ii) Let $T \in L(E,F)$ and $S \in L(F,G)$, then $S \circ T \in L(E,G)$ and the following inequality holds: $\|S \circ T\| \leq \|S\| \cdot \|T\|$ (confirm [Wer07]).

2.2.11. Definition (Isomorphism and Isometries):

Let $T \in L(E,F)$.

(i) T is an isomorphism if T is bijectiv and T^{-1} continuous.

(ii) T is an isometric if $\forall x \in E$: $\|Tx\| = \|x\|$

(iii) E and F are called (isometrical) isomorph if there exists an (isometrical) isomorphism $T\colon E \to F$. We write $E \simeq F$ (resp. $E \cong F$) as an abbreviation.

2.2.12. Definition (Dual Space):

Let E be NVS over $\mathbb{K}$. The topological dual space E' of E is defined by

$$E' := L(E,\mathbb{K}) = \{\, f\colon E \to \mathbb{K} \mid \text{linear and continuous} \,\}$$

E' is always a Banach space via the norm $\|f\| = \sup\limits_{\|x\|=1} |f(x)|$

2.2.13. Proposition:

Let H, H_1, H_2 be Hilbert spaces $y \in H$ fix.
(i) The following mappings are continuous from H to $\mathbb{C}$:

$$x \mapsto \langle x, y\rangle \qquad\qquad x \mapsto \langle y, x\rangle \qquad\qquad x \mapsto \|x\|$$

(ii) A linear operator $T\colon H_1 \to H_2$ is continuous if and only if it is bounded, i.e.

$$\exists C > 0 \,(\text{constant}){:}\ \|Tx\| \leq C\|x\| \ \ \forall x \in H_1$$

Proof:

(i): Let $x_1, x_2 \in H$, then $|\langle x_1, y\rangle - \langle x_2, y\rangle| = |\langle x_1 - x_2, y\rangle| \overset{(2.1.)}{\leq} \|x_1 - x_2\|\|y\|$. By the inverse triangle inequality, we obtain $|\|x_1\| - \|x_2\|| \leq \|x_1 - x_2\|$. Hence, the mappings are uniformly continuous.

(ii): We have to show:
$T\colon H_1 \to H_2$ continuous $\Longleftrightarrow \exists C > 0 \ \|Tx\| \leq C\|x\| \ \forall x \in H_1$

($\Leftarrow$) Let $x_1, x_2 \in H_1$ fix, then T is Lipschitz continuous with constant $\delta := \frac{\varepsilon}{C} > 0$ $(\varepsilon > 0)$. In fact,

$$\|Tx_1 - Tx_2\| \underset{\text{T linear}}{=} \|T(x_1 - x_2)\| \underset{\text{assumpt.}}{\leq} C\|x_1 - x_2\|$$

($\Rightarrow$) Let T be continuous, i.e.

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x_1, x_2 \in H_1 : \|x_1 - x_2\| < \delta \Rightarrow \|Tx_1 - Tx_2\| < \varepsilon$$

Defining $y := x_1 - x_2$, implies $\|Ty\| < \varepsilon$, $\|y\| < \delta$. So, $\forall y \in H_1, y \neq 0$ it follows that

$$\|Ty\| = \frac{\|y\|}{\delta} \underbrace{\left\| T\left(\frac{\delta \cdot y}{\|y\|}\right)\right\|}_{\leq \varepsilon} \leq \frac{\varepsilon}{\delta}\|y\|$$

The desired result follows by setting $C := \frac{\varepsilon}{\delta}$ ■

2.3 ORTHOGONALITY, ORTHONORMALITY & PROJECTIONS

Throughout this section, H will denote a Hilbert space over $\mathbb{C}$, and E a convex subset of H.

$$(E \subseteq H \text{ is convex} :\iff \forall x, y \in E : tx + (1-t)y \in E \; \forall 0 \leq t \leq 1)$$

Orthogonality generalizes perpendicularity and thereby provides geometric insight in the general Hilbert space setting. Similarly, as in finite Euclidean vector space, we want to show that for each closed subspace $M \subseteq H$, the Hilbert space H can be decomposed as follows: $H = M \oplus M^{\perp}$, i.e. H is the direct sum of M and the orthogonal complement of$M$. Using this statement, allows us to characterize bounded linear functionals on a Hilbert space. This will be done by the so called "Riesz Lemma" or "Riesz representation theorem".

2.3.1. Definition (Orthogonal):

$x, y \in H$ and $A, B \subseteq H$.

(i) x is orthogonal to y, $x \perp y \iff \langle x, y \rangle = 0$

(ii) A is orthogonal to B, $A \perp B \iff \langle a, b \rangle = 0 \; \forall a \in A, \; \forall b \in B$

(iii) For $x \in H$ we define $x^{\perp} := \{y \in H : \langle x, y \rangle = 0\}$

(iv) For$A \subseteq H$ we define the orthogonal complement via
$A^{\perp} := \{y \in H : \langle a, y \rangle = 0 \; \forall a \in A\}$

2.3.2. Remark (simple consequences of 2.3.1.):

(i) Pythagoraen Theorem: $x \perp y \Rightarrow \|x\|^2 + \|y\|^2 = \|x+y\|^2$

(ii) $x^{\perp}$ is always a closed subspace of H, hence a Hilbert space.

(iii) $A^{\perp} = \bigcap_{x \in A} x^{\perp}$ is always a closed subspace of H, hence a Hilbert space ($A \subseteq H$ an arbitrary subset).

(iv) $\{0\}^{\perp} = H$, $\{H\}^{\perp} = 0$

(v) $A \subseteq B \Rightarrow B^{\perp} \subseteq A^{\perp}$

(vi) $A \subseteq A^{\perp\perp}$

(vii) $\left(A^{\perp}\right)^{\perp} = \overline{A}$

(viii) $A^{\perp} = A^{\perp\perp\perp} = \bar{A}^{\perp} = \bar{A}^{\perp}$

(ix) $(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$

(x) $A^{\perp} = (\text{span}A)^{\perp} = (\overline{\text{span}A})^{\perp}$

For proofs on this matter confirm [Ste08].

2.3.3. Theorem (Approximation):

Let H be a Hilbert space,E a non-empty, closed, convex subset of H.

(i) Then there exists a unique $x_0 \in E$ with minimal norm, i.e.

$$\exists! x_0 \in E: \ \|x_0\| = \min_{x \in E} \|x\|$$

(ii) For each $x \in H$ there is a unique $x_0 \in E$ that minimizes $\|x - x_0\|$ (best approximation), i.e.

$$\forall x \in H \ \ \exists! x_0 \in E: \ \|x - x_0\| = \min_{y \in E} \|y - x\|$$

Proof:

(i) <u>Existence :</u>

(a) Is $0 \in E$, then clearly x_0 must be 0 and there is nothing more to show.

(b) So, let $0 \notin E$ and $d := \inf\{\|x\| : x \in E\}$. Since E is convex, the mid-point $\frac{(x+y)}{2}$ of the line between$x$ and y lies in E. By definition of d, there exists a sequence $(y_n)_n$ in E such that $\|y_n\| \xrightarrow{n \to \infty} d$ in $\mathbb{R}$. So, by (2.7.) we obtain

$$0 \le \|y_n - y_m\|^2 = 2\|y_n\|^2 + 2\|y_m\|^2 - 4 \underbrace{\left\| \frac{y_n + y_m}{2} \right\|^2}_{\underset{\text{E convex}}{\ge} \ d^2},$$

hence,

$$0 \le \|y_n - y_m\|^2 = 2\underbrace{\|y_n\|^2}_{\xrightarrow{n \to \infty} d^2} + 2\underbrace{\|y_m\|^2}_{\xrightarrow{m \to \infty} d^2} - 4d^2 \qquad (*)$$

As m and n tend to infinity the right-hand side of $(*)$ tends to $2d^2 + 2d^2 - 4d^2 = 0$. Thus $(y_n)_n$ is a Cauchy sequence in H and converges to $x_0 \in H$, since H is complete,

i.e. $\exists x_0 \in H : x_0 = \lim\limits_{n\to\infty} y_n$.
Since E is closed, the limit x_0 is in E and by continuity of the map $x \mapsto \|x\|$ we obtain $\left\|\lim\limits_{n\to\infty} y_n\right\| = \lim\limits_{n\to\infty} \|y_n\| = \|x_0\| = d$

Uniqueness: Suppose $x_0' \in E$ with $\|x_0'\| = d$. Then, again by (2.7.) we obtain

$$\left\|x_0 - x_0'\right\|^2 = 2\left(\|x_0\|^2 + \|x_0'\|^2\right) - 4\left\|\frac{x_0 + x_0'}{2}\right\|$$
$$\leq 2(d^2 + d^2) - 4d^2 \leq 0, \text{hence } x_0 = x_0'$$

(ii) Apply (i) to $x - E$. (Note thereby that $x - E$ is closed and convex) ■

2.3.4. Theorem (Projection Theorem):

Let M be a closed subspace of a Hilbert space H. Then there exist unique mappings $P : H \to M$ and $Q : H \to M^\perp$ that satisfy

(1) $x = Px + Qx \ \forall x \in H$, i.e. $H = M \oplus M^\perp$

(2) Let $x \in M$. Then $Px = x$ and $Qx = 0$, hence $P^2 = P$.
Let $x \in M^\perp$.Then $Px = 0$ and $Qx = x$, hence $Q^2 = Q$.
(P and Q are idempotent).

(3) The distance of $x \in H$ from M is given by
$\min\{\|x - y\| : y \in M\} = \|x - Px\|$.

(4) For any $x \in H$ we have $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.

(5) P and Q are continuous linear and self-adjoint operators.

Proof:

Let $x \in H$ be arbitrary and consider $x + M := \{x + y : y \in M\}$. Then $x + M$ is a closed, non - empty and convex subset. Thus, by 2.3.3. (i) there exists an unique element $Qx \in x + M$ with minimal norm. If we define $Px := x - Qx$, we get since M is a subspace and $Qx \in x + M$: $Px \in M$.
It remains to show $Qx \in M^\perp$: Let $y \in M$ be arbitrary. Then we will show that $\langle Qx, y\rangle = 0$.

(a) If $y = 0$, this is clear.

(b) So, without loss of generality we may assume $\|y\| = 1$.
Let $\alpha \in \mathbb{C}$. Then by the minimality of Qx we obtain

$$\langle Qx, Qx\rangle = \|Qx\|^2 \le \|\underbrace{Qx-\alpha y}_{\in x+M}\|^2 = \langle Qx-\alpha y, Qx-\alpha y\rangle$$

$$\Rightarrow 0 \le \langle Qx, Qx\rangle \le \langle Qx, Qx\rangle - \alpha\langle y, Qx\rangle - \bar{\alpha}\langle Qx, y\rangle + \underbrace{\alpha\bar{\alpha}}_{=|\alpha|^2}\underbrace{\langle y, y\rangle}_{=1}$$

$$\Rightarrow 0 \le -\alpha\langle y, Qx\rangle - \bar{\alpha}\langle Qx, y\rangle + |\alpha|^2$$

Define $\alpha := \langle Qx, y\rangle$, then

$$\begin{aligned} 0 \le & -\langle Qx, y\rangle\langle y, Qx\rangle - \overline{\langle Qx, y\rangle}\langle Qx, y\rangle + |\langle Qx, y\rangle|^2 \\ = & -\langle Qx, y\rangle\overline{\langle Qx, y\rangle} - \overline{\langle Qx, y\rangle}\langle Qx, y\rangle + |\langle Qx, y\rangle|^2 \\ = & -2|\langle Qx, y\rangle|^2 + |\langle Qx, y\rangle|^2 = -|\langle Qx, y\rangle|^2, \end{aligned}$$

hence, $\langle Qx, y\rangle = 0$. This completes the argument.

<u>Uniqueness</u>

If $x = x_0 + x_1$ with $x_0 \in M$ and $x_1 \in M^\perp$ then writing $x = Px + Qx$ we have

$$\underbrace{Px - x_0}_{\in M} = \underbrace{Qx - x_1}_{\in M^\perp}$$

with the right-hand side being in $M^\perp$ and the left-hand side in M. Since, $M \cap M^\perp = \{0\}$ we conclude $Px = x_0$ and $Qx = x_1$. <u>Property (3)</u>

This follows from the definition of Q:
$\min\{\|x-y\| : y \in M\} = \|x - Px\| = \|Qx\|$.

<u>Property (4)</u>

This is an easy calculation using the fact that $\langle Px, Qx\rangle = 0\ \forall\, x \in H$:
$\|x\|^2 = \langle Px+Qx, Px+Qx\rangle = \langle Px, Px\rangle + \langle Px, Qx\rangle + \langle Qx, Px\rangle + \langle Qx, Qx\rangle = \|Px\|^2 + \|Qx\|^2$.

<u>Property (5)</u>

Let $x, y \in H$, $\alpha,\beta \in \mathbb{C}$.Then we have

$$\begin{aligned} P(\alpha x + \beta y) + Q(\alpha x + \beta y) &= \alpha x + \beta y \\ &= \alpha(Px + Qx) + \beta(Py + Qy) \\ &= (\alpha Px + \beta Py) + (\alpha Qx + \beta Qy) \end{aligned}$$

Moving the P terms to the left and the$Q$ terms to the right we get $P(\alpha x+\beta y)-(\alpha Px+\beta Py)=(\alpha Qx+\beta Qy)-Q(\alpha x+\beta y)$ The left side is in M and the right side is orthogonal to M. Therefore both sides must be 0. This implies that$P$ and Q are both linear.
Let $x, y \in H$ be arbitrary. By definition of Q we obtain $\|Q(x-y)\| = \min\{\|x-y+m\| : m \in M\} \le \|x-y\|$, hence Q is continuous. Since $P = I - Q$, I the identity, we conclude that P is continuous.
Let $x, y \in H$ be arbitrary.Then we have

$$\langle Px, y\rangle = \langle Px, Py + Qy\rangle = \langle Px, Py\rangle + \underbrace{\langle Px, Qy\rangle}_{=0} = \langle Px, Py\rangle$$

and

$$\langle x, Py\rangle = \langle Px + Qx, Py\rangle = \langle Px, Py\rangle + \underbrace{\langle Qx, Py\rangle}_{=0} = \langle Px, Py\rangle$$

hence $\langle x, Py\rangle = \langle Px, y\rangle$. So, $P = P^*$.

Analogously for Q.

Property (2)
Let $x \in M$, then $Px = x$. Indeed, $\underbrace{x - Px}_{\in M} = \underbrace{Qx}_{\in M^\perp} \Rightarrow x - Px = 0$. ∎

2.3.5. Remark:

(i) The mappings $P\colon H \to M$, $x \mapsto Px$ and $Q\colon H \to M^\perp$, $x \mapsto Qx$ are the orthogonal projections onto M resp. $M^\perp$.
(ii) The orthogonal projection onto M is (cf. 2.3.4 (3)) defined by $\|Px - x\| = \min_{y\in M} \|x-y\|$
(iii) Note that, $\mathrm{Im}(P) = M$ and $\mathrm{Ker}(P) = M^\perp$.

2.3.6. Theorem (Riesz – Frechét or Riesz representation theorem):

Let H be a Hilbert space and $L \in H'$, a continuous linear functional, i.e.

$$L\colon H \to \mathbb{C}$$
$$|Lx| \le C\|x\| \ (C > 0)$$

Then there exists a unique $y \in H$ such that $Lx = \langle x, y \rangle \;\; \forall x, y \in H$, with $\|L\| = \|y\|$.

Proof:

Let $M := \mathrm{Ker}(L) = \{x : Lx = 0\}$ denote the nullspace of L. Then M is a closed subspace of H, since L is linear and bounded.

(a) If $H = M$, then $L \equiv 0$, so we take $y = 0$.

(b) Suppose, $M \neq H$. By the projection theorem 2.3.4., it follows that $M^{\perp} \neq 0$.
Let $z \in M^{\perp}, z \neq 0$, then $Lz \neq 0$. Define $y := \alpha z$ with $\bar{\alpha} := \frac{Lz}{\|z\|^2}$ hence $y \in M^{\perp}$ and, since $\langle y, y \rangle = |\alpha|^2 \langle z, z \rangle = \frac{|Lz|^2}{\|z\|^2}$, we get

$$Ly = L(\alpha z) = \alpha L(z) = \frac{\overline{Lz}}{\|z\|^2} Lz = \frac{|Lz|^2}{\|z\|^2} = \langle y, y \rangle = |\alpha|^2 \langle z, z \rangle \qquad (*)$$

Let now be x an arbitrary element of H and we define

$$x' := x - \frac{Lx}{\langle y, y \rangle} y \qquad \text{and} \qquad x'' := \frac{Lx}{\langle y, y \rangle} y$$

Then from $(*)$ we see that $Lx' = Lx - \frac{Lx}{\langle y,y \rangle} Ly = Lx - \frac{Lx}{\langle y,y \rangle} \langle y, y \rangle = Lx - Lx = 0$, hence $x' \in M$ and $\langle x', y \rangle = 0$, since $y \in M^{\perp}$. Finally, it follows that

$$\begin{aligned} \langle x, y \rangle &= \left\langle x' + x'', y \right\rangle = \left\langle x', y \right\rangle + \left\langle x'', y \right\rangle = \left\langle x'', y \right\rangle = \left\langle \frac{Lx}{\langle y, y \rangle} y, y \right\rangle \\ &= \frac{Lx}{\langle y, y \rangle} \langle y, y \rangle = Lx \end{aligned}$$

and the existence result is proved.

To show uniqueness, suppose $\langle x, y \rangle = \langle x, y' \rangle \;\; \forall x \in H$,
then $\langle x, y - y' \rangle = 0$ and so $y - y' = 0$, hence $y = y'$.

It remains to show that $\|L\| = \|y\|$. The Cauchy – Schwarz inequality yields $\|L\| = \sup\limits_{\|x\|=1} |\langle x, y \rangle| \leq \|y\|$.
Choosing $\frac{y}{\|y\|}$ for x yields $\|L\| \geq \left| L\left(\frac{y}{\|y\|}\right) \right| = \frac{\langle y,y \rangle}{\|y\|} = \|y\|$.

Combining the two inequalities shows that $\|L\| = \|y\|$.

■

2.3.7. Remark:

From 2.3.6. Theorem we see that one can identify the dual space H' with the original Hilbert space H.

We now want to discuss bases in infinite dimensional Hilbert spaces.

2.3.8. Definition (Orthogonal and Orthonormal System):

Let H a Hilbert space and $S = \{ u_n \mid n \in \mathbb{N} \}$ a family of elements of H.

(i) S is called an orthogonal system/set (OGS) if $u_n \perp u_m \ \forall n \neq m$

(ii) S is called an orthonormal system/set (ONS) if

$$\langle u_n, u_m \rangle = \delta_{n,m} \ \forall n,m \ n \neq, \Longleftrightarrow \begin{cases} \langle u_n, u_m \rangle = 0 \ \forall n,m \ n \neq m \\ \|u_n\| = \sqrt{\langle u_n, u_n \rangle} = 1 \ \forall n \in \mathbb{N} \end{cases}$$

2.3.9. Remark:

(i) The difference between an OGS and ONS is that the vectors of an ONS are normed.

(ii) Every ONS is linearly independent:

$$\text{Let } \sum_{j=1}^{n} c_j u_j = 0 \Rightarrow 0 = \left\| \sum c_j u_j \right\|^2 = \sum_{i,j=1}^{n} \left\langle c_j u_j, c_i u_i \right\rangle$$
$$= \sum_j \left| c_j \right|^2 \Rightarrow c_j = 0 \ \forall j$$

2.3.10. Definition (ONB):

Let H be a Hilbert space, $S = \{u_n : n \in \mathbb{N}\}$ an ONS.
S is called complete if for any $x \in H$ there are scalars $c_n := \langle x, u_n \rangle$ such that $x = \sum_{n \in \mathbb{N}} c_n u_n$.
A complete ONS is also called an orthonormal basis (ONB). The numbers $c_n := \langle x, u_n \rangle$ are called the (generalised) Fourier coefficients of x with respect to the set S.

2.3.11. Theorem:

Let $S = \{u_n : n \in \mathbb{N}\}$ be an ONB for an Hilbert space H. For any x in the space, $x = \sum_{n\in\mathbb{N}} c_n u_n$, where c_n are uniquely determined by $c_n := \langle x, u_n\rangle$.

Proof:

By definition of a basis, there are numbers c_n such that $x = \sum_{n\in\mathbb{N}} c_n u_n$.

Then we have for any n : $\langle x, u_n\rangle = \left\langle \sum_{m\in\mathbb{N}} c_m u_m, u_n \right\rangle = \sum_{m\in\mathbb{N}} c_m \langle u_m, u_n\rangle$.

All terms in this series vanish by orthogonality except the term with $n = m$: $\langle x, u_n\rangle = c_n \underbrace{\langle u_n, u_n\rangle}_{=1} = c_n$.

■

2.3.12. Theorem (Bessel's inequality):

Let $\{u_\alpha : \alpha \in A\}$ be an ONS in a Hilbert space H, then for any x in the space

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2 \tag{2.9.}$$

where $c_\alpha = \langle x, u_\alpha\rangle$.

Proof:

(i) $\{u_m : m \in \mathbb{N}\}$ is a finite set, i.e. $\{u_1, \ldots, u_n\}$.

$$\begin{aligned}
0 &\leq \left\langle x - \sum_{r=1}^{n} \langle x, u_r\rangle u_r, x - \sum_{s=1}^{n} \langle x, u_s\rangle u_s \right\rangle \\
&= \langle x, x\rangle - \sum_{r=1}^{n} \langle x, u_r\rangle\langle u_r, x\rangle - \sum_{s=1}^{n} \overline{\langle x, u_s\rangle} \langle x, u_s\rangle + \sum_{r=1}^{n}\sum_{s=1}^{n} \langle x, u_r\rangle \overline{\langle x, u_s\rangle} \underbrace{\langle u_r, u_s\rangle}_{=\delta_{rs}} \\
&= \langle x, x\rangle - \sum_{r=1}^{n} \underbrace{\langle x, u_r\rangle\overline{\langle x, u_r\rangle}}_{|\langle x, u_r\rangle|^2} - \sum_{s=1}^{n} \overline{\langle x, u_s\rangle} \langle x, u_s\rangle + \sum_{r=1}^{n}\sum_{s=1}^{n} \langle x, u_r\rangle \overline{\langle x, u_s\rangle}\delta_{rs} \\
&= \langle x, x\rangle - \sum_{r=1}^{n} |\langle x, u_r\rangle|^2
\end{aligned}$$

$$\Rightarrow \sum_{r=1}^{n} |\langle x, u_r\rangle|^2 \leq \|x\|^2$$

(ii) Let $\{u_m : m \in \mathbb{N}\}$ be a countabe set.
Consider any partial sum $s_n = \sum_{j=1}^{n} \left|c_j\right|^2$ of the series $\sum_{j\in\mathbb{N}} \left|c_j\right|^2$. The argument given above for the finite cases shows that $s_n = \sum_{j=1}^{n} \left|c_j\right|^2 \le \|x\|^2$, hence the sequence is bounded above, and it is monotonic increasing, because the terms of the series are positive, hence converge by a basic theorem on monotonic bounded sequences, and its limit is $\le \|x\|^2$. ■

2.3.13. Theorem (Best Approximation):

Let $\{u_1, \dots, u_N\}$ be an ONS in a Hilbert space H. For any x, the numbers c_n which minimize $\left\| x - \sum_{n=1}^{N} c_n u_n \right\|$ are given by $c_n = \langle x, u_n \rangle \ \forall 1 \le n \le N$.

Proof:

Write $c_n = \langle x, u_n \rangle \cdot \|u_n\|^{-2} + d_n$. We shall show that best approximation is obtained by taking $d_n = 0$. We have

$$\left\| x - \sum_{n=1}^{N} c_n u_n \right\|^2 =$$

$$= \left\langle x - \sum_{n=1}^{N} \frac{\langle x, u_n \rangle}{\|u_n\|^2} u_n - \sum_{n=1}^{N} d_n u_n \,,\, x - \sum_{n=1}^{N} \frac{\langle x, u_n \rangle}{\|u_n\|^2} u_n - \sum_{n=1}^{N} d_n u_n \right\rangle$$

$$= \langle x, x \rangle - \sum_{n=1}^{N} \frac{|\langle x, u_n \rangle|^2}{\|u_n\|^2} \|u_n\|^2 + \sum_{n=1}^{N} |d_n|^2 \|u_n\|^2$$

after some similar calculation as in 2.3.12. Theorem. This obviously takes the smallest value when all the d_n are zero. ■

Bessel's inequality means that expansion coefficients die away faster than $n^{-\frac{1}{2}}$, roughly speaking, as $n \to \infty$. This inequality gives a criterion for deciding whether a given ONS is an ONB, see the following theorem:

2.3.14. Theorem (Characterising ONB):

Let H be a Hilbert space over $\mathbb{C}$, $\{ u_n : n \in \mathbb{N} \}$ an ONS. The following are equivalent:

(1) $\{ u_n : n \in \mathbb{N} \}$ is a complete ONS, thus an ONB.

(2) From $\langle x, u_n \rangle = 0 \, \forall n \in \mathbb{N}$ it follows that $x = 0$.

(3) The set S of all finite linear combinations $\sum\limits_{k=1}^{N} c_k u_k$ of elements from $\{ u_n : n \in \mathbb{N} \}$ is dense in H .

(4) Parseval's Relation: $\forall x, y \in H$ it is $\|x\|^2 = \sum\limits_{n \in \mathbb{N}} |\langle x, u_n \rangle|^2$

(5) $\forall x, y \in H$ it is $\langle x, y \rangle = \sum\limits_{n \in \mathbb{N}} \langle x, u_n \rangle \, \overline{\langle y, u_n \rangle}$

Proof:

$(1) \Rightarrow (2)$

Suppose $\langle x, u_n \rangle = 0 \; \forall n \in \mathbb{N}$. Since $\{ u_n : n \in \mathbb{N} \}$ is complete, we have

$x = \sum\limits_{n \in \mathbb{N}} \underbrace{\langle x, u_n \rangle}_{=0} u_n$, thus $x = 0$

$(2) \Rightarrow (3)$

Let $M = \bar{S}$, the closure of S, then M is a closed subspace. We have to show $M = H$.
Suppose to the contrary $M \neq H$. By 2.3.4., $M^\perp \neq \{0\}$, i.e. $\exists x \neq 0$, $x \in H$ such that $\langle x, u_n \rangle = 0 \; \forall n \in \mathbb{N}$, a contradiction to assumption (2).

$(3) \Rightarrow (4)$

Let $x \in H$ be fix and $\varepsilon > 0$. By assumption (3), it follows that $\exists u_1, \ldots, u_n \in \{ u_m : m \in \mathbb{N} \}$ and $\exists c_1, \ldots, c_n \in \mathbb{C} : \left\| x - \sum\limits_{k=1}^{n} c_k u_k \right\| < \varepsilon$.
Using 2.3.13., we obtain $\left\| x - \sum\limits_{k=1}^{n} \langle x, u_k \rangle u_k \right\| \leq \left\| x - \sum\limits_{k=1}^{n} c_k u_k \right\| < \varepsilon$,
i.e. $\|x - z\| < \varepsilon$, with
$z :=\sum\limits_{k=1}^{n} \langle x, u_k \rangle u_k$. Therefore, by the triangle inequality we get
$\|x\| = \big\|(x - z) + z\big\| \leq \|x - z\| + \|z\| < \varepsilon + \|z\|$ and

$$\begin{aligned}(\|x\| - \varepsilon)^2 \leq \|z\|^2 = \left\| \sum_{k=1}^{n} \langle x, u_k \rangle u_k \right\|^2 &\underset{u_k \text{ONS}}{=} \sum_{k=1}^{n} \|\langle x, u_k \rangle u_k\|^2 \\ &\leq \sum_{k \in \mathbb{N}} \|\langle x, u_k \rangle u_k\|^2\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $\|x\|^2 \leq \sum\limits_{k\in\mathbb{N}} \|\langle x, u_k\rangle u_k\|^2$.
Together with Bessel's inequality 2.3.12. we receive equality.

<u>(4) ⇒ (5)</u>
Similar calculations as in the proof of 2.3.12. Theorem resp. 2.3.13. Theorem show that

$$\left\| x - \sum_{r\in\mathbb{N}} \langle x, u_r\rangle \, u_r \right\|^2 = \|x\|^2 - \sum_{r\in\mathbb{N}} |\langle x, u_r\rangle|^2 \underset{(4)}{=} 0$$

So, we can write x as $x = \sum\limits_{r\in\mathbb{N}} \langle x, u_r\rangle \, u_r$.
Inserting the above formula for x into $\langle x, y\rangle$ yields

$$\begin{aligned}\langle x, y\rangle &= \left\langle \sum_{r\in\mathbb{N}} \langle x, u_r\rangle \, u_r, y \right\rangle = \sum_{r\in\mathbb{N}} \langle x, u_r\rangle \, \langle u_r, y\rangle \\ &= \sum_{r\in\mathbb{N}} \langle x, u_r\rangle \, \overline{\langle y, u_r\rangle}\end{aligned}$$

<u>(5) ⇒ (1)</u>
Suppose to the contrary, $\exists x \in H$ with $x \neq \sum\limits_{n\in\mathbb{N}} \langle x, u_n\rangle u_n$. Then,
$y = x - \sum\limits_{n\in\mathbb{N}} \langle x, u_n\rangle u_n \neq 0$.
$\Rightarrow \langle y, u_m\rangle = \langle x, u_m\rangle - \underbrace{\sum_{n\in\mathbb{N}} \langle x, u_n\rangle \underbrace{\langle u_n, u_m\rangle}_{=1 \text{ if } n=m}}_{=\langle x, u_m\rangle}$ $\forall m \in \mathbb{N}$, hence $\langle y, u_m\rangle = 0$, but
this is a contradiction to assumption (5), because
$0 \neq \langle y, y\rangle = \|y\|^2 \underset{(5)}{=} \sum\limits_{n\in\mathbb{N}} \underbrace{|\langle y, u_n\rangle|^2}_{=0} = 0$. ■

2.3.15. Definition:

A Hilbert space H is called separable if it contains a countable dense subset of H.

2.3.16. Theorem:

Let H be a Hilbert space. H is separable if and only if it has a countable ONB.

Proof:

($\Rightarrow$) Let H be separable, then $\exists A = \{x_1, x_2, \ldots\}$, a countable dense subset of H. Applying Gram – Schmidt orthonormalisation process to A, we obtain by 2.3.14. (3) a countable complete ONS.

($\Leftarrow$) Clear, just take only rational numbers as real and imaginary parts.

■

2.3.17. Remark:

(i) One can show, that from 2.3.16. follows that any separable, infinite-dimensional Hilbert space is isometric to ℓ^2.

(ii) It is also true, that every ONS B in a Hilbert space H is contained in a complete ONS of H, i.e. every Hilbert space has an ONB (confirm [Has07]).

Chapter 3

Holomorphic Functions in Several Complex Variables

Before we study the main results of holomorphic functions in several complex variables, we listen the basic facts in one complex variable.

3.1 Complex Analysis in One Variable

3.1.1. Definition (Holomorphic Functions):

Let $U \subseteq \mathbb{C}$ be an open set and $f\colon U \to \mathbb{C}$ a function.
f is called complex differentiable in $z_0 \in U$ if the limit $L := \lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. In this case L is called the derivative of f at z_0.

Notation:

$$f'(z_0) = \frac{df(z_0)}{dz} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If f is complex differentiable at every point z_0 in U, we say f is holomorphic on U. We say f is holomorphic at the point $z_0 \in U$, if it is holomorphic on some neighbourhood U_0 of z_0.

3.1.2. Remark:

Equivalent to the holomorphy of f are:

(1) Characterising via the Cauchy – Riemann differential equations:

f has continuous partial derivatives with respect to x and y at each point in U and satisfies

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} && \text{or shorter } u_x = v_y \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} && \text{or shorter } u_y = -v_y \end{aligned} \tag{3.1.}$$

the Cauchy – Riemann differential equations, whereupon we write for $f := u + iv$.
This is, by the Wirtinger calculus equivalent to $\overline{\partial} f = 0$, with $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$. See [Kra08] for more on this matter.

(2) Characterising via power series resp. Taylor series:

Locally, f is representable by a convergent power series which is equal to its Taylor series.

A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ around a fixed $z_0 \in \mathbb{C}$ is defined to be the limit of its partial sums $S_N(z) = \sum_{n=0}^{N} a_n(z-z_0)^n$, $(a_n \in \mathbb{C}, n \in \mathbb{N}_0)$.

Any given power series has a disk/ball of convergence and the radius of convergence is given, by the Cauchy - Hadamard theorem, as

$$r := \frac{1}{\left(\limsup_{n\to\infty}|a_n|\right)^{\frac{1}{n}}}. \tag{3.2.}$$

(3) Characterising via the complex line integral:

Cauchy - Integral Theorem :
If f is a holomorphic function on an open disc W in the complex plane, and if $\gamma\colon [a,b] \to W$ is a C^1-curve in W with $\gamma(a) = \gamma(b)$, then $\int_\gamma f(z)\, dz = 0$.

The expression on the left -hand side is called the complex line integral of f along γ and is defined as:

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\, \frac{d\gamma}{dt}(t)\, dt$$

The necessity of this condition is just the basic form of the Cauchy - Integral - Theorem, while Morera's Theorem shows that it is even sufficient.

Morera's Theorem: Let $U \subseteq \mathbb{C}$ be open, f continuous in U and all triangles $\Delta \subseteq U$ satisfy $\int_{b\Delta} f(z)\, dz = 0$. Then f is holomorphic on U.

3.1.3. Theorem (Cauchy Integral Formula (Basic Form)):

Let $\Omega \subseteq \mathbb{C}$ be a domain and suppose that $(r > 0)$, $\overline{D_r(z_0)} \subseteq \Omega$. Let $\gamma\colon [0,1] \to \mathbb{C}$ be the C^1 parametrization $\gamma(t) = z_0 + r\cos(2\pi t) + ir\sin(2\pi t)$

and f holomorphic in Ω. Then, for each $z \in D(z_0, r)$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi \tag{3.3.}$$

3.1.4. Remark:

For more general versions of the Cauchy Integral formula see [Kra08].

3.1.5. Theorem (Taylor series expansion):

Let $U \subseteq \mathbb{C}$ open and $a \in U, R > 0$ such that $D_R(a) \subseteq U$ and f holomorphic in U. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \tag{3.4.}$$

where the sum converges uniformly on all compact subsets of $D_R(a)$ and the following identity holds:

$$a_n = \frac{f^{(n)}(a)}{n!},\ n \in \mathbb{N}_0 \tag{3.5.}$$

a_n are called Taylor coefficients of f by expansion at a.

3.1.6. Theorem:

Let f be holomorphic in U, then $f^{(n)}$ is holomorphic in U $\forall n \in \mathbb{N}_0$.
f is therefore infinitely complex differentiable.

3.1.7. Theorem:

Assumption as in 3.1.5., $\gamma(t) = a + re^{it}$, $r < R, t,\ t \in [0, 2\pi]$. Then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - a)^{(n+1)}}\, d\xi \ (n \in \mathbb{N}_0) \tag{3.6.}$$

3.1.8. Remark:

For proofs of the previous theorems see [Kra08] resp. [Has05].

3.1.9. Theorem (Weierstraß' Convergence Theorem):

Let $U \subseteq \mathbb{C}$ be an open subset, f_n holomorphic on U, $n \in \mathbb{N}_0$. If the sequence $(f_n)_n$ converges uniformly on all compact sets K of U to a func-

tion f, i.e. $f_n \to f$ uniformly on K, then f is holomorphic on U.
Furthermore, $\forall k \in \mathbb{N}_0$: $f_n^{(k)}$ converges uniformly on compact sets to $f^{(k)}$.
In addition, if $f_n \in \mathcal{H}(U)$ is a Cauchy sequence in terms of uniformly convergence on compact subsets of U,i.e.

$$\forall K \subseteq U, \text{ compact}, \ \forall \varepsilon > 0 \, \exists N_{\varepsilon,K} > 0 \text{ such that}$$
$$\sup_{z \in K} |f_n - f_m| < \varepsilon \ \forall n, m > N_{\varepsilon,K}, \ \forall z \in K,$$

then there exists $f \in \mathcal{H}(U)$ such that $f_n \overset{n\to\infty}{\longrightarrow} f$ uniformly on compact sets of Ω.
$\mathcal{H}(U)$ is complete in terms of uniformly convergence on compact subsets of U.

3.1.10. Remark (The Space of Holomorphic Functions):

Let $U \subseteq \mathbb{C}$ be an open set. The vector space of all holomorphic functions on U, denoted by $\mathcal{H}(U)$, can be turned into a complete topological vector space. The topology can be described by a metric and coincides with the topology of uniform convergence on compact subsets of U :

It is known that there exits a sequence of compact subsets $K_n \subseteq U$ such that $K_n \subseteq K_{n+1}$, $\bigcup\limits_{n=1}^{\infty} K_n = U$ and such that if K is any compact subset of U, then $K \subseteq K_n$ for some n.
We now define a sequence of semi - norms by

$$\|f\|_n : = \sup_{z \in K_n} |f(z)|, \ f \in \mathcal{H}(U), n \in \mathbb{N}$$

and it yields that $\|f\|_n \leq \|f\|_{n+1}$. Now we can define the following

$$d(f,g) : = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\|f-g\|_n}{1+\|f-g\|_n} \ (f,g \in \mathcal{H}(U))$$

It can be shown that d is a metric, that the topology generated by d is independent of the choice of K_n and that convergence in d is the same as uniform convergence on compact subsets.

3.2 Holomorphic Functions of Several Complex Variables

In the following most of the results, definitions, theorems,etc. are taken from [Ebe07].

3.2.1 Preliminaries

Recall, that we defined for $n \in \mathbb{N}^+$ the n-dimensional complex vector space $\mathbb{C}^n: = \left\{ z = (z_1, \ldots, z_n) \,\middle|\, z_j \in \mathbb{C},\ j \in 1, \ldots, n \right\}$.
We write $z_j = x_j + iy_j$ for the decomposition of the coordinates z_j into real and imaginary parts. This gives a bijection

$$\begin{aligned} \mathbb{C}^n &\longrightarrow \mathbb{R}^{2n} \\ (z_1, \ldots, z_n) &\longmapsto (x_1, y_1, \ldots, x_n, y_n) \end{aligned}$$

even an isomorphism of vector spaces. We use this isomorphism to introduce a norm and thus a topology on $\mathbb{C}^n$. Since all norms on $\mathbb{R}^{2n}$ are equivalent, all norms on $\mathbb{C}^n$ define the same topology. We shall often consider the following norms:

♠ Euclidean norm: $\|z\| := \sqrt{\sum\limits_{j=1}^{n} \left|z_j\right|^2} = \sqrt{\sum\limits_{j=1}^{n} z_j\overline{z_j}} = \sqrt{\sum\limits_{j=1}^{n} (x_j^2 + y_j^2)}$

♠ maximum norm: $\|z\|_{max} : = \max\limits_{j=1,..,n} \left|z_j\right|$

3.2.1. Definition:

The (open) ball or disk of radius $r > 0$ and center $a \in \mathbb{C}^n$ is definded by $\mathbb{B}(a, r) = \mathbb{B}_r(a): = \{ z \in \mathbb{C}^n \mid \|z - a\| < r \}$.
$b\mathbb{B} = b\mathbb{B}_r(a): = \{ z \in \mathbb{C}^n \mid \|z - a\| = r \}$ is called the topological boundary.

3.2.2. Remark:

(i) We denote the topological boundary of a set $A \subseteq \mathbb{C}^n$ by bA, rather than the more commonly used ∂A , as the symbol generally has different meaning in complex analysis.

(ii) The collection of balls $\{\mathbb{B}(a, r) : r > 0, r \in \mathbb{Q}\}$ forms a countable neighbourhood basis at the point a for the topology of $\mathbb{C}^n$.

Often it is convenient to use another system of neighbourhoods:

3.2.3. Definition:

Let $r = (r_1, \ldots, r_n)$, $r_j \in \mathbb{R}^+$, $r_j > 0$ and $a \in \mathbb{C}^n$. Then

$$\mathbb{P}(a, r) := \mathbb{P}_r(a) := \left\{ z \in \mathbb{C}^n \,\middle|\, \left|z_j - a_j\right| < r_j,\ 1 \leq j \leq n \right\} \tag{3.7.}$$

is called the (open) polydisc or polycylinder around a with (poly-) radius r.
The distinguished boundary of $\mathbb{P}(a, r)$ is definded by

$$\mathbb{T}(a, r) := \mathbb{T}_r(a) := \left\{ z \in \mathbb{C}^n \,\middle|\, \left|z_j - a_j\right| = r_j,\ 1 \leq j \leq n \right\} \tag{3.8.}$$

an n-dimensional torus.

3.2.4. Definition:

Let $r = (r_1, \ldots, r_n)$, $r_j \in \mathbb{R}^+$, $r_j > 0$ and $\mathbb{T} := \mathbb{T}_r(0)$. Let $h \colon \mathbb{T} \to \mathbb{C}$ be a continuous function. We define

$$\int_{\mathbb{T}} h(\xi)\, d\xi := \int_{|\xi_n| = r_n} \cdots\cdots \int_{|\xi_1| = r_1} h(\xi_1, \ldots \xi_n)\, d\xi_1 \cdots d\xi_n \tag{3.9.}$$

where the last integrals are complex line integrals.

3.2.2 HOLOMORPHIC FUNCTIONS

In section 3.1. we stated the equivalent conditions of holomorphic functions in one complex variable. We now wish to generalize these characterizations to the n-dimensional case. In the following U denotes always an open subset of $\mathbb{C}^n$.

To define a holomorphic functions in several complex variables we first need to know something about power series in several complex variables.

3.2.5. Definition:

A formal power series around 0 is an expression

$$\sum_{\sigma_1, \ldots, \sigma_n = 0}^{\infty} a_{\sigma_1, \ldots, \sigma_n}\, z_1^{\sigma_1} \cdots z_n^{\sigma_n} \tag{3.10.}$$

with $a_{\sigma_1, \ldots, \sigma_n} \in \mathbb{C}$ and $\sigma_1, \ldots, \sigma_n \in \mathbb{N}_0$.

To simplify the formula, we use multi-indices σ(confirm chapter 1).

3.2.6. Definition:

Let $w \in \mathbb{C}^n$, $a_\sigma \in \mathbb{C}$, $|\sigma| \geq 0$. A formal power series centered at w is an expression $\sum_{\sigma=0}^{\infty} a_\sigma (z-w)^\sigma$.

We now want to introduce a concept of convergence. Here we must pay attention to the fact that the multi-indices can be ordered in many different ways.

3.2.7. Definition:

Let $z' \in \mathbb{C}^n$. The power series

$$\sum_{\sigma=0}^{\infty} a_\sigma (z-w)^\sigma \tag{3.11.}$$

is said to be convergent at the point z' if the series

$$\sum_{\sigma=0}^{\infty} a_\sigma (z'-w)^\sigma \tag{3.12.}$$

converges with respect to every (arbitrary) order of summation. From a theorem in analysis this is equivalent to the last series being absolutely convergent.

Now let U be an open subset of $\mathbb{C}^n$ and let $f_m\colon U \to \mathbb{C}$, $m = 0, 1, 2, \ldots,$ be a sequence of continuous functions. The series

$$\sum_{m=0}^{\infty} f_m \tag{3.13.}$$

is called normally convergent if the series

$$\sum_{m=0}^{\infty} \sup_{z \in K} |f_m(z)| \tag{3.14.}$$

converges for every compact subset $K \subseteq U$.

If $\sum\limits_{m=0}^{\infty} f_m$ converges normally, then the series converges absolutely and uniformly on every compact subset $K \subseteq U$.

3.2.8. Definition:

A power series $\sum\limits_{\sigma=0}^{\infty} a_\sigma(z-w)^\sigma$ is called normally convergent in U if it is normally convergent with respect to any, and therefore with respect to every, order of summation.
The individual terms $a_\sigma(z-w)^\sigma$ are to be understood as complex valued functions on U.

3.2.9. Proposition:

Let $w \in \mathbb{C}^n$ with $w_j \neq 0$ for $j = 1, \ldots, n$. If the power series $\sum\limits_{\sigma=0}^{\infty} a_\sigma z^\sigma$ converges at w, then it converges normally in the polycylinder

$$\mathbb{P} := \left\{ z \in \mathbb{C}^n | \; |z_j| < |w_j|, 1 \le j \le n \right\}$$

This proposition follows from the following result:

3.2.10. Lemma (Abel's Lemma):

Let $w \in \mathbb{C}^n$ with $w_j \neq 0$ for $j = 1, \ldots, n$. If the set $\{ a_\sigma w^\sigma \mid |\sigma| \geq 0 \}$ is bounded, then the power series $\sum\limits_{\sigma=0}^{\infty} a_\sigma z^\sigma$ converges normally in the polycylinder

$$\mathbb{P} := \left\{ z \in \mathbb{C}^n | \; |z_j| < |w_j|, 1 \le j \le n \right\}$$

Proof:

(a) Since the set $\{ a_\sigma w^\sigma \mid |\sigma| \geq 0 \}$ is bounded, there is an $M \in \mathbb{R}$ with $|a_\sigma w^\sigma| < M$ for all σ.
Now let $q \in \mathbb{R}$ with $0 < q < 1$.
Let $\mathbb{P}_{(q)} := \left\{ z \in \mathbb{C}^n | \left| z_j \right| < q \left| w_j \right|, 1 \le j \le n \right\}$
For $z \in \mathbb{P}_{(q)}$ we have

$$\begin{aligned} |a_\sigma z^\sigma| &= |a_\sigma||z_1|^{\sigma_1} \cdots |z_n|^{\sigma_n} < |a_\sigma||qw_1|^{\sigma_1} \cdots |qw_n|^{\sigma_n} = |a_\sigma w^\sigma| q^{|\sigma|} \\ &< Mq^{|\sigma|} \end{aligned}$$

and

$$\sum_{\sigma=0}^{\infty} M \cdot q^{|\sigma|} = M \cdot \sum_{\sigma=0}^{\infty} q^{\sigma_1+\ldots+\sigma_n} = M \cdot \left(\sum_{\sigma=0}^{\infty} q^{\sigma_1}\right) \cdots \left(\sum_{\sigma=0}^{\infty} q^{\sigma_n}\right)$$
$$= M \cdot \left(\frac{1}{1-q}\right)^n$$

Thus the power series converges normally in the polycylinder $\mathbb{P}_{(q)}$.

(b) Now let $K \subseteq \mathbb{P}$ be compact. The sets $\mathbb{P}_{(q)}$, $0 < q < 1$ form an open covering of $\mathbb{P}$, so also of K. Since K is compact, there exists a finite subcovering $K \subseteq \mathbb{P}_{(q_1)} \cup \cdots \cup \mathbb{P}_{(q_l)}$.
But $K \subseteq \mathbb{P}_{(q)}$ for $q = \max\limits_{j=1,\ldots,l} q_j$. By (a), $\sum\limits_{\sigma=0}^{\infty} a_\sigma z^\sigma$ converges uniformly and absolutely on K. This power series is therefore normally convergent on $\mathbb{P}$. ■

3.2.11. Proposition:

Let $r \in \mathbb{R}^n_{>0}$, $\mathbb{P} := \mathbb{P}_r(0) \subseteq \mathbb{C}^n$, a polycylinder around 0, $\mathbb{T} := \mathbb{T}_r(0)$ and let $h\colon \mathbb{T} \to \mathbb{C}$ be continuous. Then the function $f\colon \mathbb{P} \to \mathbb{C}$ with

$$f(z) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}} \frac{h(\xi)}{(\xi_1 - z_1)\cdots(\xi_n - z_n)}\, d\xi \tag{3.15.}$$

can be expanded as a convergent power series around 0 in the whole of $\mathbb{P}$.

Proof:

Let $r = (r_1, \ldots, r_n)$. Let $z \in \mathbb{P}$ be chosen fixed. Then $q_j := \frac{|z_j|}{r_j} < 1$ for $j = 1, \ldots, n$.
Now let $\xi \in \mathbb{T}$. Then

$$\frac{h(\xi)}{(\xi_1 - z_1)\cdots(\xi_n - z_n)} = \frac{h(\xi)}{\xi_1 \cdots \xi_n} \frac{1}{\left(1 - \frac{z_1}{\xi_1}\right)\cdots\left(1 - \frac{z_n}{\xi_n}\right)}$$
$$= \frac{h(\xi)}{\xi_1 \cdots \xi_n} \left(\sum_{\sigma_1=0}^{\infty} \left(\frac{z_1}{\xi_1}\right)^{\sigma_1}\right) \cdots \left(\sum_{\sigma_n=0}^{\infty} \left(\frac{z_n}{\xi_n}\right)^{\sigma_n}\right)$$
$$= \sum_{\sigma=0}^{\infty} \frac{h(\xi)}{\xi_1 \cdots \xi_n} \cdot \frac{1}{\xi^\sigma} \cdot z^\sigma$$

Since h is continuous and $\mathbb{T}$ is compact, the function

$$\xi \longmapsto \frac{h(\xi)}{\xi_1 \cdots \xi_n}$$

is bounded on $\mathbb{T}$; there is therefore an $M \in \mathbb{R}$ with

$$\left| \frac{h(\xi)}{\xi_1 \cdots \xi_n} \right| < M \text{ for all } \xi \in \mathbb{T} .$$

For fixed $z \in \mathbb{P}$ the sum $\sum_{\sigma=0}^{\infty} M \cdot q^{\sigma}$ is then a majorant on $\mathbb{T}$ for the series $\sum_{\sigma=0}^{\infty} \frac{h(\xi)}{\xi_1 \cdots \xi_n} \cdot \frac{1}{\xi_n} \cdot z^{\sigma}$.

This series therefore converges absolutely and uniformly on $\mathbb{T}$ and we may interchange summation and integration to get

$$\begin{aligned} f(z) &= \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}} \frac{h(\xi)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)}\, d\xi \\ &= \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}} \left(\sum_{\sigma=0}^{\infty} \frac{h(\xi) z^{\sigma}}{\xi_1^{\sigma_1+1} \cdots \xi_n^{\sigma_n+1}} \right) d\xi \\ &= \sum_{\sigma=0}^{\infty} z^{\sigma} \cdot \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}} \frac{h(\xi)}{\xi_1^{\sigma_1+1} \cdots \xi_n^{\sigma_n+1}}\, d\xi \\ &= \sum_{\sigma=0}^{\infty} a_{\sigma} z^{\sigma} \end{aligned}$$

with

$$a_{\sigma} = \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}} \frac{h(\xi)}{\xi_1^{\sigma_1+1} \cdots \xi_n^{\sigma_n+1}}\, d\xi$$

This last series converges for every $z \in \mathbb{P}$. ■

3.2.12. Definition (Holomorphic Function):

Let $U \subseteq \mathbb{C}^n$ be open. A function $f\colon U \to \mathbb{C}$ is called holomorphic (or analytic) in U if for every $w \in U$ there are an open neighbourhood V of w in U , $w \in V \subseteq U$, and a power series $\sum_{\sigma=0}^{\infty} a_{\sigma}(z-w)^{\sigma}$ that converges to

f on V for all $z \in V$.
The set of all functions holomorphic in U is denoted by $\mathcal{H}(U)$.

3.2.13. Remark:

Obviously every holomorphic function is continuous.

We next want to generalize Cauchy's integral formula for a disc in $\mathbb{C}$. Instead of a disc we consider a polycylinder.

3.2.14. Theorem (Cauchy's Integral Formula):

Let $U \subseteq \mathbb{C}^n$ be open, $r = (r_1, \ldots, r_n)$, $r_j > 0$, $r_j \in \mathbb{R}^+$,$(1 \leq j \leq n)$,
$\mathbb{P}\colon = \mathbb{P}(0)$ a polycylinder around 0 with $\overline{\mathbb{P}} \subseteq U$ and $\mathbb{T}\colon = \mathbb{T}(0)$.
If $f\colon U \to \mathbb{C}$ is holomorphic, then for any $z \in \mathbb{P}$ it holds that:

$$f(z) = \left(\frac{1}{2\pi i}\right) \int\limits_{\mathbb{T}} \frac{f(\xi)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)}\, d\xi \tag{3.16.}$$

Proof:

We prove the theorem by induction over the number of variables n.

(n=1):

For n=1 this is Cauchy's integral formula for the disc for a holomorphic function of one complex variable.

Induction hypothesis:

Suppose n>1, and that the theorem has been proved for (n-1)-variables.

Induction step (n-1)$\rightarrow$ n :

For fixed $(z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}$ with $|z_j| < r_j$ we consider the function $F\colon U_n \to \mathbb{C}$ with $F(\xi_n)\colon = f(z_1, \ldots, z_{n-1}, \xi_n)$ where $U_n\colon = \{\, \xi_n \in \mathbb{C} \mid (z_1, \ldots, z_{n-1}, \xi_n) \in U \,\}$.
Then F is holomorphic in U_n, since f is holomorphic on U. The circle $\{\, \xi_n \in \mathbb{C} \mid |\xi_n| = r_n \,\}$ is contained in U_n.
By Cauchy's integral formula for a holomorphic function of one complex variable we get

$$F(z_n) = \frac{1}{2\pi i} \int\limits_{|\xi_n| = r_n} \frac{F(\xi_n)}{\xi_n - z_n}\, d\xi_n \tag{$*$}$$

By the induction hypothesis (now ξ_n fixed and $z_1, \ldots, z_{n-1}$ are variable) we obtain

$$f(z_1, \ldots, z_{n-1}, \xi_n) = \left(\frac{1}{2\pi i}\right)^{n-1} \cdot I \qquad (**)$$

with $I = \int\limits_{|\xi_{n-1}|=r_{n-1}} \cdots \int\limits_{|\xi_1|=r_1} \frac{f(\xi_1, \ldots, \xi_{n-1}, \xi_n)}{(\xi_1 - z_1)\cdots(\xi_{n-1} - z_{n-1})} \, d\xi_1 \cdots d\xi_{n-1}$

If we substitute the formula $(**)$ into $(*)$ we get

$$f(z_1, .., z_{n-1}, z_n) = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|\xi_n|=r_n} \cdot\cdot \int\limits_{|\xi_1|=r_1} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1)\cdot\cdot(\xi_n - z_n)} \, d\xi_1 \cdots d\xi_n$$

as was to be shown. ∎

3.2.15. Lemma:

Let $U \subseteq \mathbb{C}^n$ be open and $f\colon U \to \mathbb{C}$ holomorphic. Furthermore, let $w \in U$ and let $\mathbb{P}\colon = \mathbb{P}_r(w)$ be a polycylinder around w with $\overline{\mathbb{P}} \subseteq U$, $\mathbb{T}\colon = \mathbb{T}_r(w)$. Then f can be expanded as a power series

$$f(z) = \sum_{\sigma=0}^{\infty} a_\sigma (z - w)^\sigma$$

in a neighbourhoodof w, with coefficients

$$a_\sigma = \left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}} \frac{f(\xi)}{(\xi_1 - w_1)^{\sigma_1+1} \ldots (\xi_n - w_n)^{\sigma_n+1}} \, d\xi.$$

Proof:

This follows from the definition of a holomorphic function (3.2.12.), Cauchy's integral formula (Proposition 3.2.14.) and then from the proof of Proposition 3.2.11. . ∎

3.2.16. Proposition (Weierstraß' Theorem):

Let $U \subseteq \mathbb{C}^n$ be open and let $f_m\colon U \to \mathbb{C}$, $m = 0, 1, 2, \ldots$, be holomorphic functions. If the sequence of functions$(f_m)_m$ converges normally to a function $f\colon U \to \mathbb{C}$, then f is also holomorphic.

Proof:

Let $w \in U$. We again assume that $w = 0$. We shall show that f is holomorphic at w.
So let $\mathbb{P}: = \mathbb{P}_r(w)$ be a polycylinder around w with $\overline{\mathbb{P}} \subseteq U$, $\mathbb{T}: = \mathbb{T}_r(w)$ and $z \in \mathbb{P}$. Then

$$\begin{aligned} f(z) &= \lim_{m\to\infty} f_m(z) \\ &= \lim_{m\to\infty} \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}} \frac{f_m(\xi)}{(\xi_1 - z_1)\cdots(\xi_n - z_n)}\, d\xi \quad \text{(Proposition 3.2.14.)} \\ &= (*). \end{aligned}$$

We now need to interchange integrating and taking limits. To justify this, we must show that the sequence of functions $\frac{f_m(\xi)}{(\xi_1 - z_1)\cdots(\xi_n - w_n)}$ converges uniformly to the function $\frac{f(\xi)}{(\xi_1 - z_1)\cdots(\xi_n - w_n)}$.

We put $g(\xi): = (\xi_1 - z_1)\cdots(\xi_n - w_n)$.
Then g is continuous and does not vanish on $\mathbb{T}$. Therefore $\frac{1}{g}$ is also continuous on $\mathbb{T}$, so $\frac{1}{g}$ is bounded because $\mathbb{T}$ is closed. Hence, there is an $M \in \mathbb{R}$ such that $\left|\frac{1}{g(\xi)}\right| < M$ for all $\xi \in \mathbb{T}$.
Then, on $\mathbb{T}$,

$$\left\|\frac{f_m}{g} - \frac{f}{g}\right\| = \left\|\frac{1}{g}\right\| \|f_m - f\| < M\|f_m - f\| \overset{m\to\infty}{\longrightarrow} 0$$

Therefore $\frac{f_m}{g}$ converges to $\frac{f}{g}$ on $\mathbb{T}$. Hence

$$\begin{aligned} (*) &= \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}} \lim_{m\to\infty} \frac{f_m(\xi)}{(\xi_1 - z_1)\cdots(\xi_n - z_n)}\, d\xi \\ &= \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}} \frac{f(\xi)}{(\xi_1 - z_1)\cdots(\xi_n - z_n)}\, d\xi \end{aligned}$$

Since all the f_m are continuous on $\mathbb{T}$, so too is f continuous on $\mathbb{T}$. It follows from Proposition 3.2.17. that f can be expanded around 0 in a power series that converges on all $\mathbb{P}$. ∎

3.2.17. Proposition:

Let $f(z): = \sum_{\sigma=0}^{\infty} a_\sigma z^\sigma$ be a formal power series that converges on a polycylinder $\mathbb{P}$ around 0. Then f is holomorphic in $\mathbb{P}$.

Proof:

By Proposition 3.2.9., the series $\sum\limits_{\sigma=0}^{\infty} a_\sigma z^\sigma$ is normally convergent in $\mathbb{P}$. We choose any ordering of the multi-indices. Let f_m be the m-th partial sum. Then f_m is holomorphic in $\mathbb{P}$ (even in all of $\mathbb{C}^n$) and the sequence of $(f_m)_m$ functions converges uniformly in $\mathbb{P}$. to the function f. By the Weierstraß' Theorem (Theorem 3.2.16.), f is holomorphic in $\mathbb{P}$. ∎

3.2.18. Proposition:

Let $U \subseteq \mathbb{C}^n$ be open and $f\colon U \to \mathbb{C}$ holomorphic. Then all the partial derivatives $f_{z_j}, 1 \leq j \leq n$, are also holomorpic in U.

If $\mathbb{P} \subseteq U$ is a polycylinder around 0 and $f(z) = \sum\limits_{\sigma=0}^{\infty} a_\sigma z^\sigma$ in $\mathbb{P}$, then

$f_{z_j}(z) = \sum\limits_{\sigma=0}^{\infty} a_\sigma \sigma_j\, z_1^{\sigma_1} \cdots z_j^{\sigma_j - 1} \cdots z_n^{\sigma_n}$ in $\mathbb{P}$.

Proof:

(a) First we show that the series $\sum\limits_{\sigma=0}^{\infty} a_\sigma \sigma_j z_1^{\sigma_1} \cdots z_j^{\sigma_j-1} \cdots z_n^{\sigma_n}$ converges.

Take $w \in \mathbb{P}$ with $w_j \neq 0$ for $j = 1, \ldots, n$, $0 < q < 1$ and $v\colon = q \cdot w$. Then we can estimate as follows:

$$\left| a_\sigma \sigma_j v_1^{\sigma_1} \cdots v_j^{\sigma_j - 1} \cdots v_n^{\sigma_n} \right| = \frac{\sigma_j}{\left| v_j \right|} \cdot |a_\sigma v^\sigma| = \frac{\sigma_j}{\left| v_j \right|} \cdot |a_\sigma w^\sigma| \cdot q^\sigma$$

Since $\sum\limits_{\sigma=0}^{\infty} a_\sigma w^\sigma$ converges, the set $\{\, |a_\sigma w^\sigma| \mid |\sigma| \geq 0 \,\}$ is bounded. Furthermore

$$\sum_{\sigma=0}^{\infty} \sigma_j \cdot q^{|\sigma|} = \left(\sum_{\sigma_1=0}^{\infty} q^{\sigma_1} \right) \cdots \left(\sum_{\sigma_j=0}^{\infty} \sigma_j q^{\sigma_j} \right) \cdots \left(\sum_{\sigma_n=0}^{\infty} q^{\sigma_n} \right).$$

Now $\sum\limits_{\sigma_k=0}^{\infty} q^{\sigma_k}$ is a geometric series for $k \neq j$. The series $\sum\limits_{\sigma_j=0}^{\infty} \sigma_j q^{\sigma_j}$ converges too. One has

$$\lim_{\sigma_j \to \infty} \frac{(\sigma_j + 1) q^{\sigma_j + 1}}{\sigma_j q^{\sigma_j}} = q \cdot \lim_{\sigma_j \to \infty} \frac{(\sigma_j + 1)}{\sigma_j} = q < 1$$

Form the ratio test it follows that this series also converges. Thus $\sum_{\sigma=0}^{\infty} \sigma_j \cdot q^{|\sigma|}$ converges , and in particular, the set $\left\{ \sigma_j q^{|\sigma|} \,\middle|\, |\sigma| \geq 0 \right\}$ is also bounded. From Abel's lemma it follows that the series

$$\sum_{\sigma=0}^{\infty} a_\sigma \sigma_j z_1^{\sigma_1} \cdots z_j^{\sigma_j - 1} \cdots z_n^{\sigma_n}$$

converges normally in $\mathbb{P}_\nu = \{ z \in \mathbb{C}^n \mid |z_k| < |\nu_k| , 1 \leq k \leq n \}$. Since $\mathbb{P}$ can be covered by polycylinders $\mathbb{P}_\nu$, the above series converges in $\mathbb{P}$ to a holomorphic function g_j.

(b) We now show that $g_j = f_{z_j}$. We put $h_\sigma(z) :\, = a_\sigma z^\sigma$. Then

$$f(z) = \sum_{\sigma=0}^{\infty} h_\sigma(z) \text{ and } g_j(z) = \sum_{\sigma=0}^{\infty} (h_\sigma)_{z_j}(z).$$

Next we consider

$$\int_0^{z_j} g_j\left(z_1, .., z_{j-1}, \xi, z_{j+1}, .., z_n\right) \, d\xi + f(z_1, .., 0, .., z_n)$$
$$= \int_0^{z_j} \left(\sum_{\sigma=0}^{\infty} (h_\sigma)_{z_j}(z) \left(z_1, .., z_{j-1}, \xi, z_{j+1}, .., z_n\right) \right) \, d\xi + h_\sigma(z_1, .., 0, .., z_n)$$
$$= (*).$$

We are dealing here with a conventional path integral; the path is the interval from 0 to z_j in the z_j -plane and must lie completely in $\mathbb{P}$. Since this path is a compact subset of $\mathbb{P}$, the series under the integral sign converges uniformly there. We can therefore interchange summation and integration to obtain

$$(*) = \sum_{\sigma=0}^{\infty} \left(\int_0^{z_j} (h_\sigma)_{z_j}(z) \left(z_1, .., z_{j-1}, \xi, z_{j+1}, .., z_n\right) d\xi + h_\sigma(z_1, \ldots, 0, .., z_n) \right)$$
$$= \sum_{\sigma=0}^{\infty} h_\sigma(z) = f(z).$$

Therefore, $f_{z_j} = g_j(z)$ for all $z \in \mathbb{P}$. ∎

Let $U \subseteq \mathbb{C}^n$ be open, $f\colon U \to \mathbb{C}$ a holomorphic function. By Proposition 3.2.18., f is arbitrarily often (partially) complex differentiable. For $w \in U$ and a multi-index σ we put

$$\partial^\sigma f(w) = \frac{\partial^{\sigma_1+\ldots+\sigma_n} f}{\partial z_1^{\sigma_1} \cdots \partial z_n^{\sigma_n}}\,(w).$$

Then

3.2.19. Proposition:

Let $U \subseteq \mathbb{C}^n$ be open, $f\colon U \to \mathbb{C}$ holomorphic. Then for every $w \in U$ there is a power series expansion

$$f(z) = \sum_{\sigma=0}^{\infty} a_\sigma (z-w)^\sigma$$

around w in a neighbourhood of w, with coefficients

$$a_\sigma = \frac{\partial^\sigma f(w)}{\sigma!}$$

Proof:

Let $\mathbb{P} = \{z \in \mathbb{C}^n \colon |z_j - w_j| < r_j, 1 \le j \le n\}$ be a polycylinder around w with $\overline{\mathbb{P}} \subseteq U$. Furthermore, let $\mathbb{T} := \mathbb{T}_r(w)$. By Corollary 3.2.15., there is a power series expansion $f(z) = \sum\limits_{\sigma=0}^{\infty} a_\sigma (z-w)^\sigma$ around w with

$$\begin{aligned}
a_\sigma &= \left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}} \frac{f(\xi)}{(\xi_1 - w_1)^{\sigma_1+1} \ldots (\xi_n - w_n)^{\sigma_n+1}}\, d\xi \\
&= \frac{1}{2\pi i} \int\limits_{|\xi_n|=r_n} \frac{1}{(\xi_n - w_n)^{\sigma_n+1}} \cdots \left[\frac{1}{2\pi i} \int\limits_{|\xi_1|=r_1} \frac{1}{(\xi_1 - w_1)^{\sigma_1+1}}\, d\xi_1\right] \cdots d\xi_n.
\end{aligned}$$

By Cauchy's integral formula for a single variable we now have

$$\begin{aligned}
a_\sigma &= \frac{1}{\sigma_1!}\frac{1}{2\pi i} \int\limits_{|\xi_n|=r_n} \frac{1}{(\xi_n - w_n)^{\sigma_n+1}} \cdot\cdot \left[\frac{1}{2\pi i} \int\limits_{|\xi_2|=r_2} \frac{1}{(\xi_2 - w_2)^{\sigma_2+1}} \frac{\partial^{\sigma_1} f(\xi')}{\partial z_1^{\sigma_1}}\, d\xi_2\right] \cdot\cdot\, d\xi_n \\
&= \ldots = \frac{\partial^\sigma f(w)}{\sigma!},
\end{aligned}$$

where $\xi' = (w_1, \xi_2, \ldots, \xi_n)$. ■

Many familiar results from the theory of functions of one variable can be generalized easily to functions of several complex variables.
The identity theorem is such an example. It is, however, not true that two holomorphic functions $f, g \colon \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}^n$ a domain, must agree if the set $\{ z \in \Omega \mid f(z) = g(z) \}$ has an accumulation point in Ω. Even for n=2 this is false in general:
Consider $\Omega \subseteq \mathbb{C}^2$, $f(z) \colon = z_1$, $g(z) \colon = z_1 z_2$. Then $\{ z \in \Omega \mid f(z) = g(z) \} = \left\{ z \in \mathbb{C}^2 \,\middle|\, z_1 = 0 \text{ or } z_2 = 1 \right\}$, but $f \neq g$ on $\mathbb{C}^2$.
However we do have the following results.

3.2.20. Proposition (Identity theorem for power series):

Let $\Omega \subseteq \mathbb{C}^n$ be a domain with $0 \in \Omega$ and let $\sum\limits_{\sigma=0}^{\infty} a_\sigma z^\sigma$ and $\sum\limits_{\sigma=0}^{\infty} b_\sigma z^\sigma$ be two power series convergent in Ω. If the set

$$\left\{ z \in \Omega \,\middle|\, \sum_{\sigma=0}^{\infty} a_\sigma z^\sigma = \sum_{\sigma=0}^{\infty} b_\sigma z^\sigma \right\}$$

contains an open neighbourhood V of 0, then $a_\sigma = b_\sigma$ for all σ .

Proof:

Let $f(z) \colon = \sum\limits_{\sigma=0}^{\infty} a_\sigma z^\sigma$ and $g(z) \colon = \sum\limits_{\sigma=0}^{\infty} b_\sigma z^\sigma$ for $z \in G$. The functions f and g are by Proposition 3.2.17., holomorphic in Ω and by hypothesis agree on the open neighbourhood V of 0.
Then their higher derivatives, which, by Proposition 3.2.19., exist, also agree. Repeated application of Proposition 3.2.18. leads to

$$a_\sigma = \frac{\partial^\sigma f(0)}{\sigma!} = \frac{\partial^\sigma g(0)}{\sigma!} = b_\sigma$$

for all σ. ■

3.2.21. Proposition (Identity theorem for holomorphic functions):

Let $\Omega \subseteq \mathbb{C}^n$ be a domain, $f, g \colon \Omega \to \mathbb{C}$ holomorphic. Let $U \subseteq \Omega$ be open and not empty, and assume $f|_U = g|_U$. Then $f = g$ on Ω.

Proof:

Let M be the interior of the set $\{ z \in \Omega \mid f(z) = g(z) \}$. Then M is an open subset of Ω, and $M \neq \emptyset$ since $U \subseteq M$. It is enough to show that M is relatively closed in G. It will follow that $M = G$, since G is connected.

So let $w \in \Omega \cap \overline{M}$. Take $r \in \mathbb{R}, r > 0$, so small that for the polycylinder $\mathbb{P}_\rho(w)$ around w with $\rho = (r, \dots, r)$ we have $\overline{\mathbb{P}_\rho(w)} \subseteq \Omega$. Since $w \in \overline{M}$, there exists a $w' \in M$ with $\left|w'_j - w_j\right| < \frac{r}{2}$ for $j = 1, \dots, n$. If we put $\frac{\rho}{2} = \left(\frac{r}{2}, \dots, \frac{r}{2}\right)$, then $w \in \mathbb{P}_{\frac{\rho}{2}}(w')$ and $\mathbb{P}_{\frac{\rho}{2}}(w') \subseteq \mathbb{P}_\rho(w') \subseteq \Omega$. Then f and g are holomorphic in $\mathbb{P}_{\frac{\rho}{2}}(w')$. Let $f(z) = \sum\limits_{\sigma=0}^{\infty} a_\sigma(z-w')^\sigma$ and $g(z) = \sum\limits_{\sigma=0}^{\infty} b_\sigma(z-w')^\sigma$ be the power series expansions of f, resp.g in $\mathbb{P}_{\frac{\rho}{2}}(w')$. These power series converge in $\mathbb{P}_{\frac{\rho}{2}}(w')$ and their values agree in a neighbourhood of w'.

By Proposition 3.2.20. $a_\sigma = b_\sigma$ for all σ. Thus $f(w) = g(w)$, and so $w \in M$. We have shown that M is relative closed in G as required. ■

From the definition we know that every holomorphic function f can be expanded as a power series, even as a Taylor series (cf. 3.2.15.) and one consequence from this fact is that the power series can be rearranged arbitrarily.

Thus, if the coordinates $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$ are given some fixed values $b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n$, then the power series can be rearranged as a convergent power series in $z_j - b_j$.

That is to say, a function $f \in \mathcal{H}(U)$ is holomorphic in each variable separately throughout U in the sense that $f(b_1, \dots, b_{j-1}, z_j, b_{j+1}, \dots, b_n)$ is holomorphic as a function of z_j for any fixed values b_j, whenever $(b_1, \dots, b_{j-1}, z_j, b_{j+1}, \dots, b_n) \in U$.

The converse is also true:

3.2.22. Lemma (Osgood's Lemma):

If a complex-valued function is continuous in an open set $U \subseteq \mathbb{C}^n$ and is holomorphic in each variable separately, then it is holomorphic in U.

Proof:

Choose any point $a \in U$ and a closed polydisc
$\overline{\mathbb{P}}(a,r)\colon \left\{z \in \mathbb{C}^n\colon \left|z_j - a_j\right| \leq r_j, 1 \leq j \leq n\right\} \subseteq U$. Fix any point $z \in \mathbb{P}(a,r)$.
By 3.2.14., we obtain

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int\limits_{S(a,r)} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1)\cdots(\xi_n - z_n)}\, d\xi_1 \cdots\, d\xi_n \qquad (**)$$

where $S(a,r)$ is the compact set defined by

$$S(a,r) = \left\{\xi \in \mathbb{C}^n\colon \left|\xi_j - a_j\right| = r_j, 1 \leq j \leq n\right\}$$

[We can use 3.2.10: Since f is holomorphic in each variable separately, it is holomorphic in an open neighbourhood of $\overline{\mathbb{P}}(a,r)$, a repeated application of the Cauchy – Integral formula for holomorphic functions of one variable leads to the formula

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|\xi_1 - a_1| = r_1} \cdots \int\limits_{|\xi_n - a_n| = r_n} f(\xi_1, \ldots, \xi_n) \frac{d\xi_n}{\xi_n - z_n} \frac{d\xi_1}{\xi_1 - z_1} \qquad (*)$$

for all $z \in \mathbb{P}(a,r)$.

For any fixed point z the integrand in $(*)$ is continuous on the compact domain of integration $S(a,r)$, so the iterated integral in $(*)$ can be replaced by the single multiple integral $(**)$ by Fubini's theorem.]

Furthermore, the series expansion

$$\frac{1}{(\xi_1 - z_1)\cdots(\xi_n - z_n)} = \sum_{\sigma_1, \ldots, \sigma_n = 0}^{\infty} \frac{(z_1 - a_1)^{\sigma_1} \cdots (z_n - a_n)^{\sigma_n}}{(\xi_1 - a_1)^{\sigma_1 + 1} \cdots (\xi_n - a_n)^{\sigma_n + 1}}$$

is absolutely and uniformly convergent for all points ξ on the domain of integration in $(**)$, and thus it can be integrated termwise.
So, upon substituting this series expansion into $(**)$ and interchanging the orders of summation and integration, it follows that $f(z)$ has a convergent

series expansion of the form

$$f(z) = \sum_{\sigma_1,\dots,\sigma_n=0}^{\infty} c_{\sigma_1,\dots,\sigma_n}(z_1 - a_1)^{\sigma_1} \cdots (z_n - a_n)^{\sigma_n}$$

in $\mathbb{P}(a, r)$, where

$$c_{\sigma_1,\dots,\sigma_n} = \left(\frac{1}{2\pi i}\right)^n \int\limits_{S(a,r)} \frac{f(\xi_1, \dots, \xi_n)}{(\xi_1 - a_1)^{\sigma_1+1} \cdots (\xi_n - a_n)^{\sigma_n+1}}\, d\xi$$

That completes the proof.

■

3.2.23. Remark:

The hypothesis that the function f be continuous in U is actually inessential, but this stronger theorem (see below) is surprisingly much more difficult.

Hartogs' Theorem:
If a complex - valued function is holomorphic in each variable separately in an open set $U \subseteq \mathbb{C}$, then it is holomorphic in U.
For detailed study on this matter see [Gun90, chapter 4, page 28].

One corollary which can be drawn from Osgood's Lemma is an extension of the familiar Cauchy - Riemann equations, as a criterion for holomorphy. As a convenient notation, introduce the first-order linear partial differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \text{ and } \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \qquad (\spadesuit)$$

where x_j,y_j are the underlying real coordinates in $\mathbb{C}^n$ and $z_j = x_j + iy_j$. It should perhaps be remarked that the left-hand sides in ($\spadesuit$) are defined by that equation, and have no separate meaning. However, note that

$$\frac{\partial}{\partial z_j} z_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(x_j + iy_j) = 1$$

and hence that

$$\frac{\partial}{\partial z_j} z_j^n = n z_j^{n-1}$$

therefore, when applied to holomorphic functions, the operator $\frac{\partial}{\partial z_j}$ coincides with the familiar complex derivative of a holomorphic function.

3.2.24. Theorem (Cauchy – Riemann Criterion):

Let $f\colon U \to \mathbb{C}$, $U \subseteq \mathbb{C}^n$ open, that is continuously differentable in the underlying real coordinates of $\mathbb{C}^n$. f is holomorphic in U if and only if it satisfies the system of partial differential equations

$$\frac{\partial}{\partial \overline{z_j}} f(z) = 0, \ j = 1, 2, \ldots, n \tag{3.17.}$$

Proof:

At any point of U, consider $f(z)$ as a function of the single variable z_j, holding the other variables constant. Decomposing f into its real and imaginary parts by writing $f(z) = u(z) + iv(z)$, note that

$$2\,\frac{\partial}{\partial \overline{z_j}} f(z) = \left(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j}\right) + i\left(\frac{\partial u}{\partial y_j} + \frac{\partial v}{\partial x_j}\right)$$

Therefore (3.17.) is equivalent to the classical Cauchy – Riemann equations for each variable separately; and, as is well –known, this in turn is equivalent to the function f being holomorphic in each variable separately. The desired theorem then follows immediately from Osgood's lemma. ∎

3.2.25. Remark:

For any open subset $U \subseteq \mathbb{C}^n$, $\mathcal{H}(U)$ is a Frechét – space (see [Gun90, chapter4 , page 31]).

3.3 Holomorphic Maps & the Iimplicit Function Theorem

Next we consider holomorphic maps.

3.3.1. Definition:

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. A map $f = (f_1, \ldots, f_n)\colon \Omega \to \mathbb{C}^n$, $f(z) = (f_1(z), \ldots, f_n(z))$, is called holomorphic if all the component functions f_μ are holomorphic in Ω.

Let $\Omega \subseteq \mathbb{C}^n$ be a domain, $f = (f_1, \ldots, f_n)\colon \Omega \to \mathbb{C}^n$ a holomorphic map. The matrix

$$J_{\mathbb{C}} f(w) \colon = \left(\frac{\partial f_\mu}{\partial z_j}(w)\right)_{j=1,\ldots,n}^{\mu=1,\ldots,n} = \frac{\partial(f_1(w), \ldots, f_n(w))}{\partial(z_1, \ldots, z_n)} \tag{3.18.}$$

is called the holomorphic functional matrix or the Jacobi matrix of f at $w \in \Omega$.

Next we put $(\mu, j = 1, \ldots, n)$

$$u_\mu \colon = \Re(f_\mu) \qquad v_\mu \colon = \Im(f_\mu)$$
$$x_j \colon = \Re(z_j) \qquad y_j \colon = \Im(z_j).$$

Then we have the Cauchy – Riemann differential equations:

$$\frac{\partial u_\mu}{\partial x_j} = \frac{\partial v_\mu}{\partial y_j} \qquad \frac{\partial u_\mu}{\partial y_j} = -\frac{\partial v_\mu}{\partial x_j}$$

Let us introduce the following abbreviated notation:

$$\frac{\partial u}{\partial x} \colon = \left(\frac{\partial u_\mu}{\partial x_j}\right)_{j=1,\ldots,n}^{\mu=1,\ldots,n} \qquad \frac{\partial u}{\partial y} \colon = \left(\frac{\partial u_\mu}{\partial y_j}\right)_{j=1,\ldots,n}^{\mu=1,\ldots,n}$$
$$\frac{\partial v}{\partial x} \colon = \left(\frac{\partial v_\mu}{\partial x_j}\right)_{j=1,\ldots,n}^{\mu=1,\ldots,n} \qquad \frac{\partial v}{\partial y} \colon = \left(\frac{\partial u_\mu}{\partial y_j}\right)_{j=1,\ldots,n}^{\mu=1,\ldots,n}$$

$$\frac{\partial(u,v)}{\partial(x,y)} \colon = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

The matrix $\frac{\partial(u,v)}{\partial(x,y)} = J_{\mathbb{R},f}$ is the (real) functional matrix of the real differentiable map $f \colon U \to \mathbb{R}^{2n}$ determined by f through the identification

$$\begin{aligned} \mathbb{C}^n &\to \mathbb{R}^{2n} \\ (z_1, \dots, z_n) &\mapsto (x_1, \dots, x_n, y_1, \dots, y_n),\ z_j = x_j + iy_j \end{aligned}$$

and the corresponding identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. From the Cauchy – Riemann differential equations, it follows that

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}.$$

Then,for the determinant of this matrix,

$$\det \frac{\partial(u,v)}{\partial(x,y)} = |\det J_{\mathbb{C}} f|^2.$$

Indeed,

$$\begin{aligned} \det \frac{\partial(u,v)}{\partial(x,y)} &= \det \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} && (1^{st} \text{ line} + i\, 2^{nd} \text{ line}) \\ &= \det \begin{pmatrix} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} & i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} && (\ 2^{nd} \text{ row} - i\, 1^{st} \text{ row}) \\ &= \det \begin{pmatrix} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x} \end{pmatrix} \\ &= |\det J_{\mathbb{C}} f|^2, \end{aligned}$$

where in the last step of this calculation we used the following representation of the derivative of f: We know by the Wirtinger calculus that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right). \qquad (\spadesuit)$$

On the other hand, we have for $f = u + iv$ that

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right), \qquad \frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

Inserting this into (♠) yields to

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\underbrace{\frac{\partial u}{\partial y}}_{=-\frac{\partial v}{\partial x}} + \underbrace{\frac{\partial v}{\partial y}}_{=\frac{\partial u}{\partial x}}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)$$

The above calculations prove the following proposition:

3.3.2. Proposition:

With the notation as in the above definition, we have $\det J_{\mathbb{R}} f = |\det J_{\mathbb{C}} f|^2$.

We next want to show an implicit function theorem for holomorphic maps. To this end we consider

$$\mathbb{C}^n \times \mathbb{C}^m = \{\, (z, w) \mid z = (z_1, \dots, z_n) \in \mathbb{C}^n,\ w = (w_1, \dots, w_m) \in \mathbb{C}^m \,\}$$

$U \subseteq \mathbb{C}^n \times \mathbb{C}^m$ an open subset, and $f\colon U \to \mathbb{C}^m$ a holomorphic map. Then given a point $c = (a, b) \in U$, the map $w \mapsto f(a, w)$ is holomorphic at b. We put

$$\frac{\partial f}{\partial w}(a,b) = \begin{pmatrix} \frac{\partial f_1}{\partial w_1}(a,b) & \cdots & \frac{\partial f_1}{\partial w_m}(a,b) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial w_1}(a,b) & \cdots & \frac{\partial f_m}{\partial w_m}(a,b) \end{pmatrix}$$

.

3.3.3. Proposition (Implicit Function Theorem):

Let $U \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be open and $f\colon U \to \mathbb{C}^m$ holomorphic. Furthermore, let $f(a, b) = 0$ and $\det\left(\frac{\partial f}{\partial w}\right)(a, b) \neq 0$. Then there are an open set $V \subseteq \mathbb{C}^n$ of a and a uniquely determined holomorphic map $\varphi\colon V \to \mathbb{C}^m$ such that for all $z \in V$

$$f(z, \varphi(z)) = 0$$

Proof:

We deduce this proposition from the implicit function theorem for real maps $f\colon U \to \mathbb{R}^{2m}$, $U \subseteq \mathbb{R}^{2n} \times \mathbb{R}^{2m}$. We treat f as a map $f\colon U \to \mathbb{R}^{2m}$.
For $(z, w) \in U$ we put

$$U_w\colon = \{\, z \in \mathbb{C}^n \mid (z, w) \in U \,\} \text{ and } U_z\colon = \{\, w \in \mathbb{C}^m \mid (z, w) \in U \,\}$$

Then the partial maps

$$\begin{aligned} f(z,\cdot)\colon U_z &\to \mathbb{R}^{2m} \text{ for fixed } z \\ w &\mapsto f(z,w) \\ f(\cdot,w)\colon U_w &\to \mathbb{R}^{2m} \text{ for fixed } w \\ z &\mapsto f(z,w) \end{aligned}$$

are also complex differentiable.
If we put $f = u + iv$ and $w = x + iy$ then, as we have seen above, $\det \frac{\partial(u,v)}{\partial(x,y)}(a,b) = \left|\det \frac{\partial f}{\partial w}(a,b)\right|^2 \neq 0$.

From the implicit function theorem of real analysis it follows that there are a neighbourhood V of a in $\mathbb{C}^n$ and a continuous real differentiable map $\varphi\colon V \to \mathbb{R}^{2m}$ such that for all $z \in V$, $f(z, \varphi(z)) = 0$. If we put $z = \alpha + i\beta$ then for the derivative $\varphi'(z)$ for all $z \in V$ we have

$$\varphi'(z) = -\left(\frac{\partial(u,v)}{\partial(x,y)}(z,\varphi(z))\right)^{-1} \circ \left(\frac{\partial(u,v)}{\partial(\alpha,\beta)}(z,\varphi(z))\right).$$

Since the partial maps $f(z,\cdot)$ and $f(\cdot,w)$ are complex differentiable, the Cauchy – Riemann differential equations are satisfied for these maps. It follows from the formula above that the Cauchy– Riemann differential equations are also satisfied for the map $\varphi\colon V \to \mathbb{R}^{2m}$. Therefore φ is holomorphic. ∎

3.3.4. Definition:

A holomorphic mapping $f\colon \Omega_1 \to \Omega_2$ of domains $\Omega_1 \subseteq \mathbb{C}^n$, $\Omega_2 \subseteq \mathbb{C}^m$ is said to be biholomorphic (or conformal) if it is one-to-one and onto and $\det J_{\mathbb{C},f}(z) \neq 0$ for every $z \in \Omega_1$, i.e. the inverse mapping f^{-1} is holomorphic.

3.4 Harmonic Functions

Holomorphic functions are closely related to harmonic functions.

3.4.1. Definition (Harmonic Function):

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $f\colon \Omega \to \mathbb{C}$ a C^2- function.
f is called harmonic if it is a solution of the Laplace equation:

$$\Delta f \equiv \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = 0 \text{ on } \Omega. \tag{3.19.}$$

Functions satisfying (3.19.) are automatically infinitely differentiable.
For details on this see for example [Eva10].

We briefly discuss the relationship between holomorphic and harmonic functions.

3.4.2. Proposition:

Suppose $f = u + i\,v$ is holomorphic in $U \subseteq \mathbb{C}$, open. Then u and v are harmonic in U, hence f is harmonic in Ω.

Proof:

Since f is holomorphic, we know from 3.1.6. that f is infinitely complex differentiable, and hence u and v are infinitely differentiable.
In particular, u and v have continuous second partial derivatives. By 3.1.2. Remark, u and v satisfy the Cauchy - Riemann Differential Equations $u_x = v_y$ and $u_y = -v_x$ in U. Hence in U,

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y \\ &= v_{yx} - v_{xy} \underset{\text{Theorem of Schwarz}}{=} v_{yx} - v_{yx} = 0 \end{aligned}$$

The proof for v satisfying the Laplace equation is completely analogous.

■

So, it follows now that a holomorphic function of several complex variables is harmonic since it is each holomorphic in each variable separately.

Harmonic functions satisfy the so-called mean value property which reads as follows:

3.4.3. Proposition:

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $u \in C^2(\Omega)$ harmonic in Ω, then

$$u(x) = \fint_{b\mathbb{B}(x,r)} u(y)\ d\sigma(y) = \fint_{\mathbb{B}(x,r)} u(y)\ d\lambda(y) \tag{3.20.}$$

$$\tag{3.21.}$$

for every ball $\overline{\mathbb{B}(x,r)} \subseteq \Omega$.

So, $u(x)$ equals both the average of u over the sphere $b\mathbb{B}(x,r)$ and the average of u over the entire ball $\mathbb{B}(x,r)$, provided $\overline{\mathbb{B}(x,r)} \subseteq \Omega$.

Proof:
For a proof of this proposition confirm [Eva10]. ■

3.4.4. Remark:

(i) <u>Averages:</u>

$$\fint_{\mathbb{B}(x,r)} u(y)\ d\lambda(y) = \frac{1}{V(B(x,r))} \int_{\mathbb{B}(x,r)} u(y)\ d\lambda(y),$$

where $V(B(x,r))$ denotes the volume of the ball $\mathbb{B}(x,r)$.

$$\fint_{b\mathbb{B}(x,r)} u(y) d\,\sigma(y) = \frac{1}{|\sigma(b\mathbb{B}(x,r))|} \int_{b\mathbb{B}(x,r)} u(y) d\,\sigma(y),$$

where σ is the (n-1) - dimensional surface measure of the sphere $b\mathbb{B}(x,r)$; $\sigma(b\mathbb{B}(x,r))$ denotes the surface area of the (n-1) - dimensional sphere $b\mathbb{B}(x,r)$ in $\mathbb{R}^n$.

(ii) Holomorphic functions satisfy the mean value property as well, since we already know that every holomorphic function is harmonic.

Chapter 4

Reproducing Kernel Hilbert Spaces

The theory of reproducing kernel Hilbert spaces (RKHS) interacts with many subjects in Mathematics, e.g. complex analysis, quantum mechanics, harmonic analysis, probability theory and statistics.
We will see that Reproducing Kernel Hilbert Spaces are spaces of functions with the nice property, that if a function f is close to a function g in the sense of the distance derived from the inner product, then the values $f(x)$ are close to the values $g(x)$. This property has consequences which are desirable in a wide variety of applications.
Among Hilbert Spaces of functions, RKHS are characterized by the property that the evaluation of functions at a fixed point x, $f \mapsto f(x)$ is a continuous mapping.
The reproducing kernel of such a space $\mathfrak{H}$ is a function of two variables $K(x, y)$ with the property that for fixed y, the function of $x \mapsto K(x, y)$ denoted by $\overline{k_y(x)}$ or $K(., y)$, belongs to $\mathfrak{H}$ and represent the evaluation function at the point x (this will be made precise in definition 4.2.1.).

In the following most of the results, definitions, theorems, etc. are taken from [BTA04].

4.1 Notation and Basic Definitions

Throughout this chapter 4, let X be a non- empty abstract set. Let $\mathfrak{H}$ be a Hilbert space of functions defined on X and taking their values in the set $\mathbb{C}$ of complex numbers.
$\mathfrak{H}$ is endowed with an inner product $\langle , \rangle_{\mathfrak{H}}$:

$$\begin{aligned} \mathfrak{H} \times \mathfrak{H} &\longrightarrow \mathbb{C} \\ (\varphi, \psi) &\longmapsto \langle \varphi, \psi \rangle_{\mathfrak{H}} \end{aligned}$$

and the associated norm $\|\cdot\|$:

$$\forall \varphi \in \mathfrak{H}: \ \|\varphi\|_{\mathfrak{H}} = \sqrt{\langle \varphi, \varphi \rangle_{\mathfrak{H}}}$$

4.1.1. Examples:

(1) Let $\mathfrak{H}$ be a finite dimensional complex vector space of functions with basis $f_1, \dots f_n$. Any vector of $\mathfrak{H}$ can be written in a unique way as a linear combination of $f_1, \dots f_n$.
Therefore an inner product $\langle , \rangle_{\mathfrak{H}}$ on $\mathfrak{H}$ is entirely defined by the numbers $g_{ij} = \left\langle f_i, f_j \right\rangle$ $1 \le i, j \le n$. If $v = \sum\limits_{i=1}^{n} v_i f_i$ and $w = \sum\limits_{j=1}^{n} w_j f_j$, then

$$\langle v, w \rangle_{\mathfrak{H}} = \left\langle \sum_{i=1}^{n} v_i f_i, \sum_{j=1}^{n} w_j f_j \right\rangle_{\mathfrak{H}} = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i \overline{w_j} g_{ij}$$

The matrix $G = \left(g_{ij}\right)_{i,j=1}^{n}$ is called the Gram matrix of the basis. G is Hermitian or self-adjoint, i.e. $G = G^*$, and positive definite ($v^* G v > 0$ whenever $v \neq 0$). A finite dimensional space endowed with any inner product is always complete and therefore it is a Hilbert space.

A particular finite dimensional space is the space $\mathcal{P}_r$ of polynomials of degree at most r. Let K_0 be a probability density on $\mathbb{R}$, that is a nonnegative integrable function satisfying

$$\int_{\mathbb{R}} K_0(x) \; d\lambda(x) = 1$$

Suppose that K_0 has finite moments up to order $2r (r \in \mathbb{N})$, i.e. the n-th moment ($n \in \mathbb{N}$) of a random variable X with probability density K_0 is the expectation E of X^n and is given by $E(X^n) = \int\limits_{-\infty}^{\infty} x^n K_0(x) dx$.
Then the space $\mathcal{P}_r$ is a Hilbert space with the inner product

$$\langle P, Q \rangle = \int_{\mathbb{R}} P(x) Q(x) K_0(x) \; d\lambda(x)$$

The Gram matrix of the canonical basis $1, x, x^2, \dots, x^r$ is the $(r+1) \times (r+1)$- matrix (g_{ij}) defined by

$$g_{ij} = \int_{\mathbb{R}} x^{i+j-2} K_0(x)\ d\lambda(x)\ (1 \le i, j \le r+1).$$

That is the Hankel matrix of moments of K_0 (for details see [BTA04]).

(2) Let $X = (a,b), -\infty \le a < b \le \infty$, and $\mathfrak{L}^2(a,b)$ be the set of complex measureable functions over (a,b) such that

$$\int_a^b |f(x)|^2\, d\lambda(x) < \infty$$

Identifying two functions f and g of $\mathfrak{L}^2(a,b)$, which are equal except on a set of Lebesgue measure equal to zero, we get a vector space $\mathfrak{L}^2(a,b)$ which is a Hilbert space with the inner product

$$\langle f, g\rangle_{\mathfrak{L}^2(a,b)} = \int_a^b f(x)\overline{g(x)}\ d\lambda(x)$$

(3) Let $X = (0,1)$ and $\mathfrak{H} = \left\{ \varphi \,\middle|\, \varphi(0) = 0, \varphi \text{ is absolutely continuous and } \varphi' \in \mathfrak{L}^2(0,1) \right\}$, where φ' is defined almost everywhere as the derivative of φ. $\mathfrak{H}$ is a Hilbert space with the inner product

$$\langle \varphi, \psi\rangle = \int_0^1 \varphi'\overline{\psi'}\ d\lambda$$

$\mathfrak{H}$ belongs to the class of Sobolev spaces(see [BTA04] or [Eva10]).

4.2 Definition, Uniqueness, Existence

4.2.1. Definition (Reproducing Kernel):

A function

$$\begin{aligned} K\colon X \times X &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto K(x, y) \end{aligned}$$

is a reproducing kernel of the Hilbert space $\mathfrak{H}$ if and only if

(1) For every y, $k_y(x) = K(x, y)$ as a function of x belongs to $\mathfrak{H}$, i.e. $\forall y \in X\colon K(., y) \in \mathfrak{H}$.

(2) $\forall y \in X \,\forall \varphi \in \mathfrak{H}$

$$\varphi(y) = \langle \varphi, k_y \rangle = \langle \varphi, K(., y) \rangle = \langle \varphi(x), K(x, y) \rangle \tag{4.1.}$$

This last condition is called "the reproducing property": the value of the function φ at the point y is reproduced by the inner product of φ with $k_y(\cdot)$.
So, applying (4.1.) to the function k_y at x we get

$$k_y(x) = \langle k_y, k_x \rangle \text{ for } x, y \in X \tag{4.2.}$$

and by (1) and (2),

$$K(x, y) = \langle k_y, k_x \rangle = \langle K(., y), K(., x) \rangle \text{ for } x, y \in X \tag{4.3.}$$

By the above relations, for $y \in X$ we obtain

$$\|k_y\| = \sqrt{\langle k_y, k_y \rangle} = \sqrt{K(y, y)} \tag{4.4.}$$

4.2.2. Definition (Reproducing Kernel Hilbert Space):

A Hilbert space $\mathfrak{H}$ of functions on a set X is called a reproducing kernel Hilbert space (abbreviated by RKHS) if there exists a reproducing kernel K of $\mathfrak{H}$, cf. 4.2.1. Definition.

The Hilbert space with reproducing kernel K is denoted by $\mathfrak{H}_K(X)$.
The corresponding norm will be denoted by $\|\cdot\|_{\mathfrak{H}_K}$ and the inner product by $\langle\cdot,\cdot\rangle_{\mathfrak{H}_K}$.

Let us examine the Hilbert spaces listed in 4.1.1. Examples.

4.2.3. Examples:

(1) Let$(e_1, e_2, \ldots, e_n)$ be an orthonormal basis in $\mathfrak{H}$ and define

$$K(x,y) = \sum_{i=1}^{n} e_i(x)\overline{e_i(y)}.$$

Then for any y in E,

$$K(.,y) = \sum_{i=1}^{n} \overline{e_i(y)} e_i(.)$$

belongs to $\mathfrak{H}$ and for any function

$$\varphi(.) = \sum_{i=1}^{n} a_i e_i(.)$$

in $\mathfrak{H}$, we have $\forall y \in X$

$$\langle \varphi, K(.,y)\rangle_{\mathfrak{H}} = \left\langle \sum_{i=1}^{n} a_i e_i(.), \sum_{j=1}^{n} \overline{e_j(y)} e_j(.) \right\rangle_{\mathfrak{H}} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \overline{\overline{e_j(y)}} \left\langle e_i, e_j \right\rangle_{\mathfrak{H}}$$
$$= \sum_{i=1}^{n} a_i e_i(y) = \varphi(y).$$

Any finite dimensional Hilbert space of functions has a reproducing kernel.

(2) $\mathfrak{H} = \mathfrak{L}^2(a,b)$ is rather a space of classes of functions than a space of functions, thus 4.2.1. Definition does not strictly apply in that case. However we could wonder whether there exists, for $t \in (a,b)$, a class of functions $K(.,t)$ such that $\forall \varphi \in \mathfrak{L}^2(a,b)$

$$\int_a^b \varphi \, \overline{K(.,t)} \, d\lambda = \varphi(t) \text{ a.s..} \tag{4.5.}$$

The answer is negative. A theorem of Yosida states that the identity is not an integral operator in $\mathfrak{L}^2(a,b)$, i.e. (4.5.) cannot be satisfied.

(3) $\mathfrak{H} = \left\{ \varphi \;\middle|\; \varphi(0) = 0, \varphi \text{ is absolutely continuous and } \varphi' \in \mathfrak{L}^2(0,1) \right\}$ has the reproducing kernel $K(x,y) = \min(x,y)$. Indeed, the weak derivative (see [Eva10] for example) of $\min(.,y)$ is the function $1_{(0,y)}$ and

$$\langle \varphi, K(.,y) \rangle_{\mathfrak{H}} = \int_0^y \varphi'(x) \, d\lambda(x) = \varphi(y).$$

4.2.4. Theorem (Uniqueness):

If a Hilbert space $\mathfrak{H}$ of functions on a set X admits a reproducing kernel then the reproducing kernel $K(x,y)$ is uniquely determined by the Hilbert space $\mathfrak{H}$.

Proof:

Let $K(x,y)$ be a reproducing kernel of $\mathfrak{H}$. Suppose that there exists another kernel $L(x,y)$ of $\mathfrak{H}$. Then for all $y \in X$, applying 4.2.1.(ii) for K and L we obtain

$$\begin{aligned}
0 \leq \|K(x,y) - L(x,y)\|^2 &= \langle K(x,y) - L(x,y), K(x,y) - L(x,y) \rangle \\
&= \langle K(x,y) - L(x,y), K(x,y) \rangle - \langle K(x,y) - L(x,y), L(x,y) \rangle \\
&= \langle K(x,y) - L(x,y), k_y(x)) \rangle - \langle K(x,y) - L(x,y), l_y(x)) \rangle \\
&= \langle k_y(x) - l_y(x)), k_y(x)) \rangle - \langle k_y(x) - l_y(x)), l_y(x)) \rangle \\
&= (k_y - l_y)(y) - (k_y - l_y)(y) = 0,
\end{aligned}$$

hence

$k_y = l_y$, i.e. $k_y(x) = l_y(x) \; \forall x \in X$.

This means $K(x,y) = L(x,y) \; \forall (x,y) \in X \times X$. ■

Riesz´s representation theorem 2.3.6. will provide the first characterization of RKHS.

4.2.5. Theorem (Existence):

For a Hilbert space $\mathfrak{H}$ of functions on X, there exists a reproducing kernel K for $\mathfrak{H}$ if and only if all the evaluation linear functionals

$$\begin{aligned} ev_y : \mathfrak{H} &\longrightarrow \mathbb{C} \ (y \in X) \\ \varphi &\longmapsto ev_y(\varphi) = \varphi(y) \end{aligned}$$

are bounded linear functionals, i.e. continuous on $\mathfrak{H}$.

Proof:

($\Rightarrow$) Suppose that K is <u>the</u> reproducing kernel for $\mathfrak{H}$, then for any $y \in X$, we have by the reproducing property
$\forall \varphi \in \mathfrak{H}\ \varphi(y) = ev_\varphi = \langle \varphi, K(.,y)\rangle_{\mathfrak{H}_K}$.
Thus, from the Cauchy-Schwarz inequality of the scalar product, we obtain
$|\varphi(y)| = |ev_y(\varphi)| = \left|\langle \varphi, K(.,y)\rangle_{\mathfrak{H}_K}\right| \overset{(2.1.)}{\leq} \|\varphi\|\|K(.,y)\| = \|\varphi\|\sqrt{K(y,y)}$,
hence the evaluation at y is a bounded linear functional,
i.e. continuous on $\mathfrak{H}$.

More over, for $\varphi = K(.,y)$ the upper bound is obtained so that the norm of the continuous linear functional ev_y is given by

$$\|ev_y\| = \sup_{\|\varphi\|\neq 0} \frac{|ev_y(\varphi)|}{\|\varphi\|} = \sqrt{K(y,y)}$$

($\Leftarrow$) Conversely from Riesz´s representation theorem, if the linear mapping $\mathfrak{H} \to \mathbb{C}$, $\varphi \mapsto ev_y(\varphi) = \varphi(y)$ is bounded, i.e. continuous, there exists $\forall y \in X$ a function $N_y(\cdot)$ in $\mathfrak{H}$ such that

$$\forall \varphi \in \mathfrak{H}:\ \langle \varphi, N_y\rangle = \varphi(y)$$

If we put k_y instead of N_y, then

$$\forall x \in X \text{ we get } k_y(x) = N_y(x).$$

Hence K is a reproducing kernel for $\mathfrak{H}$.

∎

4.2.6. Corollary:

Let $\mathfrak{H}$ be a Hilbert space of functions on a set X and let $\mathfrak{G}$ be a closed linear subspace of $\mathfrak{H}$. If $\mathfrak{H}$ has a reproducing kernel then $\mathfrak{G}$ has also a reproducing kernel.

Proof:

In this case if y is an arbitrary point in X the functional $ev_y\colon \varphi \mapsto \varphi(y)$ is continuous on $\mathfrak{G}$ and thus the assertion follows from the above theorem.

■

4.2.7. Theorem:

Every sequence of functions $(\varphi_n)_{n\geq 1} \in \mathfrak{H}_K(X)$ that converges strongly to a function $\varphi \in \mathfrak{H}_K(X)$, i.e. in the norm sense:

$$\forall \varepsilon > 0\, \exists N(\varepsilon) \forall n \geq N(\varepsilon)\colon\ \|\varphi_n - \varphi\|_{\mathfrak{H}_K(X)} < \varepsilon$$

converges also in the point- wise sense, that is

$$\lim_{n\to\infty} \varphi_n(y) = \varphi(y) \text{ for any point } y \in X.$$

Furthermore, this convergence is uniform on every subset of X on which $y \mapsto K(y,y)$ is bounded.

Proof:

For $y \in X$, using the reproducing property and the Cauchy-Schwarz inequality

$$\begin{aligned}
|\varphi_n(y) - \varphi(y)| &= |\langle \varphi_n, k_y\rangle - \langle \varphi, k_y\rangle| \\
&= |\langle \varphi_n - \varphi, k_y\rangle| \overset{(2.1.)}{\leq} \|\varphi_n - \varphi\|_{\mathfrak{H}_K(X)} \|k_y\|_{op} \\
&= \|\varphi_n - \varphi\|_{\mathfrak{H}K(X)} \sqrt{K(y,y)}
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \varphi_n(y) = \varphi(y)$ for any point $y \in X$.

Moreover, it is clear from the above inequality that this convergence is uniform on every subset of X on which $y \mapsto K(y,y)$ is bounded. ■

The aim of the following is to show that the set of positive definite functions and the set of reproducing kernels on $X \times X$ are identical. For this purpose a definition is needed.

4.2.8. Definition:

A function $K\colon X \times X \to \mathbb{C}$ is called Hermitian if for any finite set of points $\{y_1, \ldots, y_n\} \subseteq X$ and any complex numbers $a_1, \ldots a_n$ we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j}K(y_i, y_j) \in \mathbb{R}$$

and K is called a positive type function or a positive definite if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j}K(y_i, y_j) \geq 0 \tag{$*$}$$

It is worth noting that condition $(*)$ given in definition 4.2.8. is equivalent to the positive definiteness of the matrix $(K(y_i, y_j)_{1\leq i,j\leq n}$ or any choice of $n \in \mathbb{N}$ and $\{y_1, \ldots, y_n\} \subseteq X$.
In brief, sometimes we will denote this by $[K(x, y)] \geq 0$ on X, or equivalently, we will say that K is a positive definite matrix in the sense of E.H.Moore.

4.2.9. Examples (Examples of positive definite functions):

(1) Any constant non-negative function on $X \times X$ is of positive type since

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j} = \left|\sum_{i=1}^{n} a_i\right|^2 \in \mathbb{R}^+$$

(2) The delta function

$$(x, y) \longmapsto \delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

is of positive type.

Proof:

Let $n \geq 1$,$(a_1, \ldots, a_n) \in \mathbb{C}^n$,$(x_1, \ldots, x_n) \in X^n$ and $\{\alpha_1, \ldots, \alpha_p\}$ the set of different elements among $x_1, \ldots, x_n$. We can write

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \delta_{x_i, x_j} = \sum_{i=1}^{n} \sum_{x_j = x_i} a_i \overline{a_j} = \sum_{k=1}^{p} \sum_{x_j = x_i = \alpha_k} a_i \overline{a_j} = \sum_{k=1}^{p} \left| \sum_{x_i = \alpha_k} a_i \right|^2 \in \mathbb{R}^+$$

■

(3) The product αK of a positive type function K with a non - negative constant α is a positive type function.

4.2.10. Theorem:

The reproducing kernel $K(x, y)$ of a reproducing kernel Hilbert space $\mathfrak{H}$ is a positive matrix in the sense of E.H. Moore.

Proof:

We have

$$0 \leq \left\| \sum_{i=1}^{n} a_i k_{y_i} \right\|^2 = \left\langle \sum_{i=1}^{n} a_i k_{y_i}, \sum_{j=1}^{n} a_j k_{y_j} \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \left\langle k_{y_i}, k_{y_j} \right\rangle$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} K(y_i, y_j),$$

hence, $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_i \overline{a_j} K(y_i, y_j) \geq 0.$

■

4.2.11. Remark:

Given a reproducing kernel Hilbert space $\mathfrak{H}$ and its kernel $K(x, y)$ on $X \times X$, then for all $x, y \in X$ the following properties hold:

(1) $K(y, y) \geq 0 \; \forall y \in X$

(2) $K(x, y) = \overline{K(y, x)} \; \forall (x, y) \in X \times X$

(3) $\overline{K}$ is a positive type function.

(4) $|K(x, y)|^2 \leq K(x, x) K(y, y)$

Proof:

(1) is clear.

(2) $K(x,y) = \langle k_y, k_x \rangle = \overline{\langle k_x, k_y \rangle} = \overline{K(y,x)}$ resp.

(3) Taking the conjugate in $(*)$ in 4.2.8. one gets $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} \overline{a_i}\,\overline{\overline{a_j}}\,\overline{K(y_i,y_j)} \in \mathbb{R}^+$

(4) We use (2.1.) in $\mathfrak{H}$ and obtain

$$\begin{aligned}|K(x,y)|^2 = |\langle k_y, k_x \rangle|^2 &\leq \|k_y\|\,\|k_x\|\,\|k_y\|\,\|k_x\| \\ &= \|k_y\|^2\,\|k_x\|^2 = \langle k_y, k_y \rangle \langle k_x, k_x \rangle = K(y,y)K(x,x)\end{aligned}$$

which is the desired result.

■

4.2.12. Remark:

Same assumptions as in 4.2.10. .
Let $y_0 \in X$.Then the following properties are equivalent

(a) $K(y_0, y_0) = 0$

(b) $K(x, y_0) = 0 \; \forall x \in X$

(c) $\varphi(y_0) = 0 \; \forall \varphi \in \mathfrak{H}$

Proof:

$\underline{(a) \Rightarrow (b)}$

Let $K(y_0,y_0) = 0$. It follows by 4.2.10. (iv) that $|K(x,y_0)|^2 \leq K(x,x) \cdot 0$, hence $K(x,y_0) = 0 \; \forall x \in X$.

$\underline{(b) \Rightarrow (c)}$

Let $K(x,y_0) = 0 \; \forall x \in X$. By the reproducing property we obtain

$\varphi(y_0) = \langle \varphi, K(x,y_0) \rangle = \langle \varphi, 0 \rangle = 0 \; \forall \varphi$.

$\underline{(c) \Rightarrow (a)}$

Let $\varphi(y_0) = 0 \; \forall \varphi$. Taking $\varphi(x) = K(x,y_0)$ gives the desired property, since φ is in $\mathfrak{H}$ because the function $x \mapsto K(x,y_0)$ belongs to $\mathfrak{H}$, and $0 = \varphi(y_0) = K(y_0,y_0)$.

■

We are now in a position to state the main result of this paragraph, from which the converse of 4.2.10. theorem will appear as a consequence (see below).

4.2.13. Theorem:

Let $\mathfrak{H}_0$ be any subspace of $\mathbb{C}^X$, the space of complex functions on X, on which an inner product $\langle .,. \rangle_{\mathfrak{H}_0}$ is defined, with associated norm $\|\cdot\|_{\mathfrak{H}_0}$.

In order that there exists a Hilbert space $\mathfrak{H}$ such that

(a) $\mathfrak{H}_0 \subseteq \mathfrak{H} \subseteq \mathbb{C}^X$ and the topology defined on $\mathfrak{H}_0$ by the inner product $\langle .,. \rangle_{\mathfrak{H}_0}$ coincides with the topology induced on $\mathfrak{H}_0$ by $\mathfrak{H}$.

(b) $\mathfrak{H}$ has a reproducing kernel K.

it is necessary and sufficient that

(1) the evaluation functionals ev_y $(y\epsilon X)$ are continuous on $\mathfrak{H}_0$.

(2) any Cauchy sequence $(f_n)_n$ in $\mathfrak{H}_0$ converging pointwise to 0 converges also to 0 in the norm sense.

4.2.14. Remark:

(2) is equivalent to (2'): for any function f in $\mathfrak{H}_0$ and any Cauchy sequence $(f_n)_n$ in $\mathfrak{H}_0$ converging pointwise to f, $(f_n)_n$ converges also to f in the norm sense.

Proof:

Proof of 4.2.13.:

($\Rightarrow$) If $\mathfrak{H}$ does exist with conditions (a) and (b) satisfied, the evaluation functionals are continuous on $\mathfrak{H}$ (by Theorem 4.2.5.) and therefore on $\mathfrak{H}_0$.

Now, let $(f_n)_n$ be a Cauchy sequence in $\mathfrak{H}_0$ converging pointwise to 0. As $\mathfrak{H}$ is complete, $(f_n)_n$ converges in the norm sense to some $f \in \mathfrak{H}$. Thus we have

$$\forall y \in X \ f(y) = ev_y(f) = \lim_{n\to\infty} ev_y(f_n) = \lim_{n\to\infty} f_n(y) = 0$$

and $f = 0$.

($\Leftarrow$) Converse, suppose (1) and (2) hold. Define $\mathfrak{H}$ as being the set of functions f in $\mathbb{C}^X$ for which there exists a Cauchy sequence $(f_n)_n$ in $\mathfrak{H}_0$ converging pointwise to f,
$\mathfrak{H} = \left\{ f\colon X \to \mathbb{C} \,\middle|\, \exists (f_n)_n \text{C.S. in } \mathfrak{H}_0 \colon f_n(y) \overset{n\to\infty}{\longrightarrow} f(y)\ \forall y \in X \right\}$.
Obviously $\mathfrak{H}_0 \subseteq \mathfrak{H} \subseteq \mathbb{C}^X$.

The proof of 4.2.13. Theorem will be completed by Lemmas 4.2.15. to 4.2.20. below. Lemmas 4.2.15. and 4.2.16. enable us to extend to $\mathfrak{H}$ the inner product on $\mathfrak{H}_0$.
In Lemmas 4.2.15. to 4.2.20. we use the hypotheses and notation of 4.2.12 Theorem.

■

4.2.15. Lemma:

Let f and g belong to $\mathfrak{H}$. Let $(f_n)_n$ and $(g_n)_n$ be two Cauchy sequences in $\mathfrak{H}_0$ converging pointwise to f and g. Then the sequence $\langle f_n, g_n\rangle_{\mathfrak{H}_0}$ is convergent and its limit only depends on f and g.

Proof:

First recall that any Cauchy sequence is bounded $\forall n, m \in \mathbb{N}$,

$$\begin{aligned}\left|\langle f_n, g_n\rangle_{\mathfrak{H}_0} - \langle f_m, g_m\rangle_{\mathfrak{H}_0}\right| &= \left|\langle f_n - f_m, g_n\rangle_{\mathfrak{H}_0} - \langle f_m, g_n - g_m\rangle_{\mathfrak{H}_0}\right| \\ &\leq \underbrace{\|f_n - f_m\|_{\mathfrak{H}_0}}_{\to 0\ (\text{CS})} \|g_n\|_{\mathfrak{H}_0} + \|f_m\|_{\mathfrak{H}_0} \underbrace{\|g_n - g_m\|_{\mathfrak{H}_0}}_{\to 0\ (\text{CS})}\end{aligned}$$

by Cauchy-Schwarz inequality (2.1.). This shows that $(\langle f_n, g_n\rangle_{\mathfrak{H}_0})$ is a Cauchy sequence in $\mathbb{C}$ and therefore convergent. In the same way, if $(f'_n)_n$ and $(g'_n)_n$ are two other Cauchy sequences in $\mathfrak{H}_0$ converging pointwise respectively to f and g, we have $\forall n \in \mathbb{N}$,

$$\left|\langle f_n, g_n\rangle_{\mathfrak{H}_0} - \left\langle f'_n, g'_n\right\rangle_{\mathfrak{H}_0}\right| \leq \left\|f_n - f'_n\right\|_{\mathfrak{H}_0} \|g_n\|_{\mathfrak{H}_0} + \left\|f'_n\right\|_{\mathfrak{H}_0} \left\|g_n - g'_n\right\|_{\mathfrak{H}_0}.$$

$(f_n - f'_n)$ and $(g_n - g'_n)$ are Cauchy sequences in $\mathfrak{H}_0$ converging pointwise to 0. From assumption (2) they also converge to 0 in the norm sense. It follows that $\left(\langle f_n, g_n\rangle_{\mathfrak{H}_0}\right)$ and $\left(\langle f'_n, g'_n\rangle_{\mathfrak{H}_0}\right)$ have the same limit.

■

4.2.16. Lemma:

Suppose that $(f_n)_n$ is a Cauchy sequence in $\mathfrak{H}_0$ converging pointwise to f and that $\lim_{n\to\infty} \langle f_n, f_n\rangle_{\mathfrak{H}_0} = 0$ ($(f_n)_n$ tends to 0 in the norm sense).
Then $f = 0$.

Proof:

$\forall x \in X,\ f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} ev_x(f_n) = 0$, by assumption (1) in 4.2.13.. ∎

Thus we can define an inner product on $\mathfrak{H}$ by setting

$$\langle f, g\rangle_{\mathfrak{H}} := \lim_{n\to\infty} \langle f_n, g_n\rangle_{\mathfrak{H}_0}$$

where $(f_n)_n$ (resp. $(g_n)_n$) is a Cauchy sequence in $\mathfrak{H}_0$ converging pointwise to f (resp.g). The positivity, the Hermitian symmetry and the lineartiy of $\langle ., .\rangle_{\mathfrak{H}}$ in the first variable are clear because $\langle ., .\rangle_{\mathfrak{H}_0}$ has those properties.

Lemma 4.2.16. entails that $f = 0$ whenever $\langle f, f\rangle = 0$. Condition (a) in 4.2.13.Theorem is satisfied for the topology on $\mathfrak{H}$ associated with this inner product.

4.2.17. Lemma:

Let $f \in \mathfrak{H}$ and $(f_n)_n$ be a Cauchy sequence in $\mathfrak{H}_0$ converging pointwise to f. Then $(f_n)_n$ converges to f in the norm sense.

Proof:

Let $\varepsilon > 0$ and let $N(\varepsilon)$ be such that $\forall n, m \geq N(\varepsilon)$: $\|f_n - f_m\|_{\mathfrak{H}_0} < \varepsilon$.
Fix $n > N(\varepsilon)$. The sequence $(f_m - f_n)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{H}_0$ converging pointwise to $(f - f_n)$.
Therefore $\|f - f_n\|_{\mathfrak{H}} = \lim_{m\to\infty} \|f_m - f_n\|_{\mathfrak{H}_0} \leq \varepsilon$.
Thus $(f_n)_n$ converges to f in the norm sense. ∎

4.2.18. Corollary:

$\mathfrak{H}_0$ is dense in $\mathfrak{H}$.

Proof:

To show density we use the following characterization:

A subset $A \subseteq M$ of a metric (M, d) is dense
$\Leftrightarrow \forall \varepsilon > 0\, \forall x \in M\, \exists y \in A \colon d(x, y) < \varepsilon$.

By definition of $\mathfrak{H}$, for any $f \in \mathfrak{H}$ there exists a Cauchy sequence $(f_n)_n$ in $\mathfrak{H}_0$ converging pointwise to f. By Lemma 4.2.17.,$(f_n)_n$ converges to f in the norm sense. The corollary follows.

∎

4.2.19. Lemma:

The evaluation functionals are continuous on $\mathfrak{H}$.

Proof:

As the evaluation functionals are linear it suffices to show that they are continuous at 0. Let $x \in X$. The evaluation functionals ev_x is continuous on $\mathfrak{H}_0$ (assumption (1) in 4.2.13.Theorem). Fix $\varepsilon > 0$ and let η such that

$$\left(f \in \mathfrak{H}_0 \text{ and } \|f\|_{\mathfrak{H}_0} < \eta\right) \Rightarrow |f(x)| < \frac{\varepsilon}{2}.$$

For any function φ in $\mathfrak{H}$ with $\|\varphi\|_{\mathfrak{H}} < \frac{\eta}{2}$ there exists by Lemma 4.2.18. a function g in $\mathfrak{H}_0$ such that $|g(x) - \varphi(x)| < \frac{\varepsilon}{2}$ and $\|g - \varphi\|_{\mathfrak{H}} < \frac{\eta}{2}$.
This entails $\|g\|_{\mathfrak{H}_0} = \|g\|_{\mathfrak{H}} \leq \underbrace{\|g - \varphi\|_{\mathfrak{H}}}_{<\frac{\eta}{2}} + \underbrace{\|\varphi\|_{\mathfrak{H}}}_{<\frac{\eta}{2}} < \eta$,
hence, $|g(x)| < \frac{\varepsilon}{2}$ and $|\varphi(x)| < \varepsilon$.
[Indeed,
$|\varphi(x)| = |\varphi(x) - g(x) + g(x)| \leq \underbrace{|\varphi(x) - g(x)|}_{<\frac{\varepsilon}{2}} + \underbrace{|g(x)|}_{<\frac{\varepsilon}{2}} < \varepsilon$.]
Thus ev_x is continuous on $\mathfrak{H}$.

∎

4.2.20. Lemma:

$\mathfrak{H}$ is a reproducing kernel Hilbert space.

Proof:

In view of 4.2.19.Lemma it remains to prove that $\mathfrak{H}$ is complete.
Let $(f_n)_n$ be a Cauchy sequence in $\mathfrak{H}$ and let $x \in X$. As the evaluation functional ev_x is linear and continuous (4.2.19. Lemma), $(f_n(x))_n$ is a Cauchy sequence in $\mathbb{C}$ and thus converges to some $f(x)$. One has to prove that such defined f belongs to $\mathfrak{H}$.

Let $(\varepsilon_n)_n$ be any sequence of positive numbers tending to zero as n tends to ∞. As $\mathfrak{H}_0$ is dense in $\mathfrak{H}$, $\forall j \in \mathbb{N}\ \exists g_j \in \mathfrak{H}_0$ such that $\left\|f_j - g_j\right\|_{\mathfrak{H}} < \varepsilon_j$. From the inequalities

$$\left|g_j(x) - f(x)\right| \leq \left|g_j(x) - f_j(x)\right| + \left|f_j(x) - f(x)\right| \leq \underbrace{\left|ev_x(g_j - f_j)\right|}_{\to 0} + \underbrace{\left|f_j(x) - f(x)\right|}_{\to 0}$$

and from the properties of ev_x (4.2.19.Lemma) it follows that $(g_n(x))_n$ tends to $f(x)$ as n tends to ∞. We have

$$\begin{aligned}\left\|g_i - g_j\right\|_{\mathfrak{H}_0} = \left\|g_i - g_j\right\|_{\mathfrak{H}} &\leq \|g_i - f_i\|_{\mathfrak{H}} + \left\|f_i - f_j\right\|_{\mathfrak{H}} + \left\|f_j - g_j\right\|_{\mathfrak{H}} \\ &\leq \varepsilon_i + \varepsilon_j + \underbrace{\left\|f_i - f_j\right\|_{\mathfrak{H}}}_{\to 0 \text{ C.S. in } \mathfrak{H}}\end{aligned}$$

Thus $(g_n)_n$ is a Cauchy sequence in $\mathfrak{H}_0$ tending pointwise to f, and so $f \in \mathfrak{H}$. By 4.2.17.Lemma $(g_n)_n$ tends to f in the norm sense. Now,

$$\left\|f_i - f_j\right\|_{\mathfrak{H}} \leq \|f_i - g_i\|_{\mathfrak{H}} + \|g_i - f\|_{\mathfrak{H}}.$$

Therefore $(f_n)_n$ converges to f in the norm sense and $\mathfrak{H}$ is complete.

■

4.2.21. Remark:

As $\mathfrak{H}_0$ is dense in $\mathfrak{H}$, $\mathfrak{H}$ is isomorphic to the completion of $\mathfrak{H}_0$. It is the smallest Hilbert space of functions on X satisfying (a) in 4.2.13.Theorem. $\mathfrak{H}$ is called the functional completion of $\mathfrak{H}_0$.

Now we can prove the converse of 4.2.20.Lemma and give some properties of the Hilbert space having a given positive type function as reproducing kernel.

4.2.22. Theorem (Moore-Aronszajn Theorem):

Let K be a positive type function on $X \times X$. There exists only one Hilbert space $\mathfrak{H}$ of functions on X with K as reproducing kernel.
The subspace $\mathfrak{H}_0$ of $\mathfrak{H}$ spanned by the functions $(K(.,y)_{y \in X})$ is dense in $\mathfrak{H}$ and $\mathfrak{H}$ is the set of functions on X which are pointwise limits of Cauchy

sequences in $\mathfrak{H}_0$ with the inner product

$$\langle f, g\rangle_{\mathfrak{H}_0} = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\overline{\beta_j}K(y_j, x_i) \tag{4.6.}$$

where $f = \sum\limits_{i=1}^{n} \alpha_i K(., x_i)$ and $g = \sum\limits_{j=1}^{m} \beta_j K(., y_j)$.

Proof:

First remark that the complex number $\langle f, g\rangle$ defined by (4.1.) does not depend on the representations not necessarily unique of f and g: On the one hand, we have

$$\begin{aligned}\langle f, g\rangle_{\mathfrak{H}_0} &= \left\langle \sum_{i=1}^{n}\alpha_i K(., x_i), \sum_{j=1}^{m} \beta_j K(., y_j)\right\rangle_{\mathfrak{H}_0} = \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\overline{\beta_j}\left\langle K(., x_i), K(., y_j)\right\rangle_{\mathfrak{H}_0}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\overline{\beta_j}K(y_j, x_i) = \sum_{i=1}^{n}\alpha_i\underbrace{\sum_{j=1}^{n}\overline{\beta_j}K(y_j, x_i)}_{=\overline{g(x_i)}}\\ &= \sum_{i=1}^{n}\alpha_i\overline{g(x_i)}\end{aligned}$$

and on the other hand, it is

$$\langle f, g\rangle_{\mathfrak{H}_0} = \sum_{j=1}^{m}\overline{\beta_j}\underbrace{\sum_{i=1}^{n}\alpha_i K(y_j, x_i)}_{=f(y_j)} = \sum_{j=1}^{m}\bar{\beta}_j f(y_j),$$

This shows that $\langle f, g\rangle_{\mathfrak{H}_0}$ depends on f and g only through their values, and not on the representation of f and g in $\mathfrak{H}_0$.

Then, taking $f = \sum\limits_{i=1}^{n} \alpha_i K(., x_i)$ and $g = K(., x)$,we get

$$\langle f, K(., x)\rangle_{\mathfrak{H}_0} = \sum_{i=1}^{n}\alpha_i\overline{g(x_i)} = \sum_{i=1}^{n}\alpha_i K(x, x_i) = f(x).$$

Thus the inner product with $K(., x)$ "reproduces" the values of functions in

$\mathfrak{H}_0$, i.e. $\mathfrak{H}_0$ has the reproducing property. In particular

$$\|K(.,x)\|^2_{\mathfrak{H}_0} = \langle K(.,x), K(.,x)\rangle = K(x,x).$$

As K is a positive type function, $\langle .,. \rangle_{\mathfrak{H}_0}$ is a semi-positive Hermitian form on $\mathfrak{H}_0 \times \mathfrak{H}_0$, i.e. it is linear in the first variable and Hermitian, and, since K is positive definite, $\langle f, f\rangle_{\mathfrak{H}_0} \geq 0$.
Now, suppose that $\langle f, f\rangle_{\mathfrak{H}_0} = \|f\|_{\mathfrak{H}_0} = 0$. From the Cauchy-Schwarz inequality we have

$$\forall x \in X \;\; |f(x)| = \left|\langle f, K(.,x)\rangle_{\mathfrak{H}_0}\right| \leq \sqrt{\langle f, f\rangle}\sqrt{[K(x,x)]} = 0$$

and $f = 0$.
Let us consider $\mathfrak{H}_0$ endowed with the topology associated with the inner product $\langle .,. \rangle_{\mathfrak{H}_0}$ and check conditions (1) and (2) in 4.2.13.Theorem:
Let f and g in $\mathfrak{H}_0$, then by the reproducing property ($f(x) = \langle f, K(.,x)\rangle$, $g(x) = \langle g, K(.,x)\rangle$)

$$\forall x \in X \;\; |ev_x(f) - ev_x(g)| = \left|\langle f-g, K(.,x)\rangle_{\mathfrak{H}_0}\right| \leq \|f-g\|_{\mathfrak{H}_0}\sqrt{[K(x,x)]}.$$

Therefore the evaluation functionals are continuous on $\mathfrak{H}_0$ and (1) is satisfied.
Let us now check condition (2). Let $(f_n)_n$ be a Cauchy sequence (hence bounded) in $\mathfrak{H}_0$ converging pointwise to 0 and let $A > 0$ be an upper bound for $(\|f_n\|_{\mathfrak{H}_0})$. Let $\varepsilon > 0$ and $N(\varepsilon)$ such that

$$n > N(\varepsilon) \Rightarrow \left\|f_{N(\varepsilon)} - f_n\right\|_{\mathfrak{H}_0} < \frac{\varepsilon}{A}.$$

Fix $k, \alpha_1, \ldots, \alpha_k$ and $x_1, \ldots, x_k$ such that $f_{N(\varepsilon)} = \sum\limits_{i=1}^{k} \alpha_i K(.,x_i)$.
So, for $n > N(\varepsilon)$ we have

$$\begin{aligned}\|f_n\|^2_{\mathfrak{H}_0} &= \left\langle f_n - f_{N(\varepsilon)} + f_{N(\varepsilon)}, f_n\right\rangle_{\mathfrak{H}_0} = \left\langle f_n - f_{N(\varepsilon)}, f_n\right\rangle_{\mathfrak{H}_0} + \left\langle f_{N(\varepsilon)}, f_n\right\rangle_{\mathfrak{H}_0} \\ &\leq \underbrace{\left\|f_n - f_{N(\varepsilon)}\right\|}_{<\frac{\varepsilon}{A}}\underbrace{\|f_n\|}_{<A} + \left\langle f_{N(\varepsilon)}, f_n\right\rangle_{\mathfrak{H}_0} < \varepsilon + \sum_{i=1}^{k} \alpha_i f_n(x_i),\end{aligned}$$

hence, $\limsup_{n\to\infty} \|f_n\|^2 \leq \varepsilon$.

As ε is arbitrary this entails that $(f_n)_n$ converges to 0 in the norm sense.
We are now in a position to apply 4.2.13. Theorem to $\mathfrak{H}_0$:
There exists a Hilbert space $\mathfrak{H}$ of functions on X satisfying (a) and (b) in 4.2.13. Theorem. $\mathfrak{H}$ is the set of functions f for which there exists a Cauchy sequence $(f_n)_n$ in $\mathfrak{H}_0$ converging pointwise to f. From 4.2.17. Lemma such a sequence $(f_n)_n$ is also converging to f in the norm sense: $\mathfrak{H}_0$ is dense in $\mathfrak{H}$. Therefore $\mathfrak{H}$ is unique and

$$\begin{aligned}\forall x \in X \ f(x) &= \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \langle f_n, K(.,x)\rangle_{\mathfrak{H}_0} \\ &= \langle f, K(.,x)\rangle_{\mathfrak{H}}\,,\end{aligned}$$

then K is the reproducing kernel of $\mathfrak{H}$. ∎

Theorem 4.2.22. claims that a RKHS of functions on a set X is characterized by its kernel K on $X \times X$ and that the property for K of being a reproducing kernel is equivalent to the property of being a positive type function. For more on this matter and further characterizations see [BTA04].

Before we study other important properties of reproducing kernel Hilbert spaces, we list a few examples that can also be found in [BTA04] with further references.

4.2.23. Examples:

(1) A space of real sequences

$\mathfrak{H}$ is the space of real sequences $u = (u_k)_{k\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} u_i = 0 \text{ and } \sum_{k=0}^{\infty} \frac{(\Delta u)_k^2}{\theta^k} < \infty,$$

where $0 < \theta < 1$ is fixed. Δu denotes the sequence $(u_{k+1} - u_k)_{k\in\mathbb{N}}$. The corresponding norm respectively its reproducing kernel is given by

$$\|u\|_{\mathfrak{H}}^2 = \sum_{k=0}^{\infty} \frac{(\Delta u)_k^2}{\theta^k}$$

$$K(k,j) = \frac{\theta^{\max(k,j)}}{1-\theta}, (k,j) \in \mathbb{N}^2.$$

$\mathfrak{H}$ is a hilbertian subspace of the space $\ell^1(\mathbb{N})$ of absolutely summable complex sequences indexed by $\mathbb{N}$.
For details see [DJ83].

(2) Paley – Wiener space
$\mathfrak{H}$ is the space of $u \in \mathfrak{L}^2(\mathbb{R}, \lambda)$ such that the support of the Fourier transform of u is included in $[-a, a]$. Its inner product respectively its reproducing kernel is given by

$$\langle u, v \rangle_{\mathfrak{H}} = \int uv \, d\lambda$$
$$K(s,t) = \frac{\sin(a(t-s))}{a(t-s)}, (s,t) \in \mathbb{R}^2.$$

For details see [Lar70, pp. 911 – 921] resp. [Yao68, pp. 429 – 444].

(3) Space of the Dirichlet kernel
$\mathfrak{H}$ is the space of trigonometric polynomials $u(t) = \sum_{n=-N}^{N} a_n e^{int}$ with following norm and reproducing kernel:

$$\|u\|_{\mathfrak{H}}^2 = \int_{-\pi}^{\pi} u(t)^2 d\lambda(t)$$
$$K(s,t) = \frac{\sin\left(N + \frac{1}{2}\right)(t-s)}{2\pi \sin\left(\frac{(t-s)}{2}\right)}.$$

K is called Dirichlet kernel.
For details see [Nas91, pp. 363 – 390].

(4) Sobolev space $\mathfrak{H} = H^1(a,b)$
Let $\beta > 0, C > 0$, then we have the following norm and reproducing kernel:

$$\|u\|_{\mathfrak{H}}^2 = \frac{1}{2C}\left(u(a)^2 + u(b)^2\right) + \frac{1}{2\beta C}\int_a^b \left(u'(t)^2 + \beta^2 u(t)^2\right) d\lambda(t)$$

$$K(s,t) = Ce^{-\beta|t-s|}$$

This kernel is the covariance of the Ornstein – Uhlenbeck process. For details see [Att92].

(5) Fractional Brownian motion RKHS
$\mathfrak{H}$ is the space of functions u on $\mathbb{R}^d$ such that there exists a function $\psi \in \mathfrak{L}^2(\mathbb{R})$ and $0 < H < 1$ with

$$u(t) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \frac{e^{-it\xi} - 1}{C_H \|\xi\|^{\frac{d}{2}+H}} \mathfrak{F}\psi^*(\xi) d\lambda(\xi) :\, = I(\psi)(t),$$

where C_H is a positive constant and $\mathfrak{F}$ denotes the Fourier transform.

Norm and reproducing kernel are given by

$$\|u\|_{\mathfrak{H}}^2 = \|I(\psi)\|_{\mathfrak{H}}^2 = \|\psi\|_{\mathfrak{L}^2(\mathbb{R}^d)}^2$$

$$K(s,t) = \frac{1}{2}\left(\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H}\right)$$

This kernel is the covariance of the fractional Brownian motion on $\mathbb{R}^d$. For details see [Coh02].

4.3 ORTHOGONAL PROJECTIONS & REPRODUCING KERNELS

Our next aim is to prove the relationship between RKHS and orthogonal projections and the show that a reproducing kernel is given by

$$K(x,y) = \sum_{\alpha\in A} \varphi_\alpha(x)\overline{\varphi_\alpha(y)}, \tag{4.7.}$$

where $\{\varphi_\alpha \colon \alpha \in A\}$, A an arbitrary index set, is an ONB of a RKHS $\mathfrak{H}$.

4.3.1. Theorem:

Let $\mathfrak{H}$ be a RKHS on X and $\mathfrak{H}_0 \subseteq \mathfrak{H}$ is a closed subspace of $\mathfrak{H}$. Then the orthogonal projection $P_{\mathfrak{H}_0} \colon \mathfrak{H} \to \mathfrak{H}_0$ of $\mathfrak{H}$ onto $\mathfrak{H}_0$ is given by the formula:

$$(P_{\mathfrak{H}_0}\varphi)(y) = \langle \varphi, K_0(.,y)\rangle, \tag{4.8.}$$

where K_0 is the reproducing kernel of $\mathfrak{H}_0$.

Proof:

By 4.2.18.Theorem and 2.1.11. Proposition, $\mathfrak{H}_0$ is a RKHS and let K_0 the reproducing kernel of $\mathfrak{H}_0$. Let φ be arbitrary in $\mathfrak{H}$ and let $\varphi = \varphi_{\mathfrak{H}_0} + \varphi_{\mathfrak{H}_0^\perp}$ be the orthogonal decomposition of φ with $\varphi_{\mathfrak{H}_0} \in \mathfrak{H}_0$ and $\varphi_{\mathfrak{H}_0^\perp} \in \mathfrak{H}_0^\perp$ (cf.2.3.4.Theorem). Then we have by the reproducing property,

$$\begin{aligned}
\langle \varphi, K_0(.,y)\rangle_{\mathfrak{H}} &= \left\langle \varphi_{\mathfrak{H}_0} + \varphi_{\mathfrak{H}_0^\perp}, K_0(.,y)\right\rangle \\
&= \left\langle \varphi_{\mathfrak{H}_0}, K_0(.,y)\right\rangle + \left\langle \varphi_{\mathfrak{H}_0^\perp}, K_0(.,y)\right\rangle \\
&= \left\langle \varphi_{\mathfrak{H}_0}, K_0(.,y)\right\rangle = \varphi_{\mathfrak{H}_0}(y),
\end{aligned}$$

since the second term in the sum is 0, because $K_0(.,y)$ belongs to $\mathfrak{H}_0$ as a function of x.

∎

4.3.2. Corollary:

Let $\mathfrak{H}$ be a RKHS with kernel K and $\mathfrak{M}$ a closed subspace of $\mathfrak{H}$. Then the following relation holds:

$$K = K_{\mathfrak{M}} + K_{\mathfrak{M}^\perp},$$

where $K_{\mathfrak{M}}$ denotes the reproducing kernel of $\mathfrak{M}$ and $K_{\mathfrak{M}^\perp}$ the one of $\mathfrak{M}^\perp$.

Proof:

Let $\varphi \in \mathfrak{H}$ be arbitrary and $\varphi = \varphi_{\mathfrak{M}} + \varphi_{\mathfrak{M}^\perp}$ be the orthogonal decomposition of φ with $\varphi_{\mathfrak{M}} \in \mathfrak{M}$ and $\varphi_{\mathfrak{M}^\perp} \in \mathfrak{M}^\perp$.
Denote by $P_{\mathfrak{M}}\colon \mathfrak{H} \to \mathfrak{M}$ the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{M}$, and $Q_{\mathfrak{M}^\perp} : \mathfrak{H} \to \mathfrak{M}^\perp$ the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{M}^\perp$. Then we have by 2.3.4. Theorem,$\varphi = P_{\mathfrak{M}}\varphi + Q_{\mathfrak{M}^\perp}\varphi$. Thus by 4.3.1. we obtain

$$P_{\mathfrak{M}}\varphi(y) = \langle \varphi, K_{\mathfrak{M}}(.,y)\rangle \text{ and } Q_{\mathfrak{M}^\perp}\varphi(y) = \langle \varphi, K_{\mathfrak{M}^\perp}(.,y)\rangle .$$

Hence,we have by the reproducing property of K

$$\begin{aligned} \langle \varphi, K(.,y)\rangle = \varphi(y) &= P_{\mathfrak{M}}\varphi(y) + Q_{\mathfrak{M}^\perp}\varphi(y) \\ &= \langle \varphi, K_{\mathfrak{M}}(.,y)\rangle + \langle \varphi, K_{\mathfrak{M}^\perp}(.,y)\rangle \\ &= \langle \varphi, K_{\mathfrak{M}}(.,y) + K_{\mathfrak{M}^\perp}(.,y)\rangle , \end{aligned}$$

thus by the uniqueness of the kernel the claim follows. ∎

The following theorem is the converse to 4.3.1. .

4.3.3. Theorem:

Let $\mathfrak{H}$ be a RKHS with kernel K and $\mathfrak{H}_0$ a closed subspace of $\mathfrak{H}$. Then the reproducing kernel $K_{\mathfrak{H}_0}(x,y)$ for $\mathfrak{H}_0$ is given by

$$\begin{aligned} K_{\mathfrak{H}_0}(x,y) &= \left\langle P_{\mathfrak{H}_0}K(.,y), K(.,x)\right\rangle \\ &= \left\langle P_{\mathfrak{H}_0}k_y, k_x\right\rangle_{\mathfrak{H}} \underset{\text{rep. prop.}}{=} P_{\mathfrak{H}_0}(k_y)(x) \end{aligned} \tag{4.9.}$$

for the orthogonal projection $P_{\mathfrak{H}_0}$ from $\mathfrak{H}$ to $\mathfrak{H}_0$.

Proof:

By 4.3.2. Corollary , we can write k_y as

$$k_y = k_y = k_y^{\mathfrak{H}_0} + k_y^{\mathfrak{H}_0^\perp} = \underbrace{P_{\mathfrak{H}_0} k_y}_{\in \mathfrak{H}_0} + \underbrace{Q_{\mathfrak{H}_0^\perp} k_y}_{\in \mathfrak{H}_0^\perp} \text{ resp.}$$

$$K(x,y) = K_{\mathfrak{H}_0}(x,y) + K_{\mathfrak{H}_0^\perp}(x,y)$$

Hence,we get

$$\begin{aligned} K_{\mathfrak{H}_0}(x,y) &= \left\langle k_y^{\mathfrak{H}_0}, k_x^{\mathfrak{H}_0} \right\rangle = \left\langle P_{\mathfrak{H}_0} k_y, P_{\mathfrak{H}_0} k_x \right\rangle \\ &= \left\langle P_{\mathfrak{H}_0}^* P_{\mathfrak{H}_0} k_y, k_x \right\rangle \underset{P_{\mathfrak{H}_0}^* = P_{\mathfrak{H}_0}}{=} \left\langle P_{\mathfrak{H}_0}^2 k_y, k_x \right\rangle \\ &\underset{P_{\mathfrak{H}_0}^2 = P_{\mathfrak{H}_0}}{=} \left\langle P_{\mathfrak{H}_0}\left(K(.,y)\right), K(.,x) \right\rangle \\ &= P_{\mathfrak{H}_0}(K(.,y))(x). \end{aligned}$$

■

4.3.4. Lemma:

In a separable RKHS $\mathfrak{H}$ there is a countable set D_0 of finite linear combinations of functions $K(.,y)$, $y \in X$, which is dense in $\mathfrak{H}$.

Proof:

By 4.2.22. Theorem, the subspace $\mathfrak{H}_0$ of $\mathfrak{H}$ spanned by the function $K(.,y)$, $y \in X$ is dense in $\mathfrak{H}$.
Let $\{y_p : p \in \mathbb{N}\}$ be a countable set dense in $\mathfrak{H}$ and let n be a positive integer. As $\overline{\mathfrak{H}_0} = \mathfrak{H}$, for any $p \in \mathbb{N}$ there exists x_p^n in $\mathfrak{H}_0$ such that
$\left\| x_p^n - y_p \right\| < \frac{1}{n}$.

Then the countable set $D_0 = \bigcup\limits_{n>0} \left\{ x_p^n : p \in \mathbb{N} \right\} \subseteq \mathfrak{H}_0$ satisfies the requirement.
To see this, consider x in $\mathfrak{H}$, $\varepsilon > 0$ and $n > \frac{2}{\varepsilon}$. There exists p in $\mathbb{N}$ such that $\|x - y_p\| < \frac{\varepsilon}{2}$. Therefore, $\left\| y_p - x_p^n \right\| < \frac{1}{n} < \frac{\varepsilon}{2}$ and $\left\| x - x_p^n \right\| < \varepsilon$.
We conclude that D_0 is dense in $\mathfrak{H}$.

■

4.3.5. Theorem:

Let $\mathfrak{H}_K(X)$ be a separable RKHS and $\{\varphi_n\}_{n=1}^{\infty}$ an orthonormal basis in $\mathfrak{H}_K(X)$ and for any sequence $(\alpha_n)_{n=1}^{\infty}$ in ℓ^2 satisfying $\sqrt{\sum\limits_{n=1}^{\infty}|\alpha_n|^2} < \infty$,we have

$$\sum_{n=1}^{\infty}|\alpha_n||\varphi_n(x)| \leq \sqrt{K(x,x)}\sqrt{\sum_{n=1}^{\infty}|\alpha_n|^2}$$

Proof:

For fixed y, the Fourier coefficients of $K(x,y)$ for the system $\{\varphi_n\}_{n=1}^{\infty}$ are

$$\langle K(.,y),\varphi_n\rangle_{\mathfrak{H}_K(X)} = \overline{\langle \varphi_n, K(.,y)\rangle}_{\mathfrak{H}_K(X)} = \overline{\varphi_n(y)}$$

Consequently,

$$\sum_{n=1}^{\infty}|\varphi_n(y)|^2 = \sum_{n=1}^{\infty}|\langle K(.,y),\varphi_n\rangle| \overset{(2.9.)}{\leq} \|K(.,y)\|^2 = \langle K(.,y),K(.,y)\rangle = K(y,y)$$

Therefore

$$\sum_{n=1}^{\infty}|\alpha_n||\varphi_n(x)| \overset{(2.1.)}{\leq} \sqrt{\sum_{n=1}^{\infty}|\alpha_n|^2}\cdot\sqrt{\sum_{n=1}^{\infty}|\varphi_n(x)|^2} \leq \sqrt{K(x,x)}\sqrt{\sum_{n=1}^{\infty}|\alpha_n|^2}$$

■

Chapter 5

Bergman Kernel

In this chapter we will study the sesqui-analytic (sesqui-holomorphic) kernel: the Bergman Kernel.
Recall, that a function $K(z,w)$ from a domain $\Omega \times \Omega$ is called sesqui-holomorphic/sesqui-analytic, if it is holomorphic in z, i.e.
$z \mapsto K(z,w)$ is holomorphic $\forall w \in \Omega$
and conjugate holomorphic or antiholomorphic in w, i.e.
$w \mapsto K(z,w)$ is antiholomorphic $\forall z \in \Omega$.
After introducing the Bergman kernel on a domain $\Omega \subseteq \mathbb{C}^n$ we will compute the kernel explicitly for special domains and especially for the unit ball $\mathbb{D} \subseteq \mathbb{C}$ we will show the connection the Riemann mapping theorem. In the end we will discuss the relationship to the $\overline{\partial}$ - Neumann Problem in one complex variable.

5.1 Bergman space and kernel

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. We already know that since holomorphic functions are harmonic they satisfy the mean value property (cf.3.4.3. Proposition). Using the mean value property we show that estimates for holomorphic functions in $\mathfrak{L}^2(\Omega)$ imply estimates in the supremum norm. 5.1.3. Corollary then guarantees that $A^2(\Omega)$, the so called Bergman space, is a Hilbert space, and 5.1.4. Corollary shows that evaluation at a point is a continuous linear functional on $A^2(\Omega)$. These facts allow us to apply the Riesz representation theorem (see 2.3.6. Theorem). And thereby define the Bergman kernel function in this section.

5.1.1. Definition (Bergman Space):

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. Define the Bergman space

$$A^2(\Omega) := \left\{ f \text{ holomorphic on } \Omega \mid \underbrace{\left(\int_\Omega |f(z)|^2 \, d\lambda(z) \right)^{\frac{1}{2}}}_{\equiv \|f\|_{A^2(\Omega)}} < \infty \right\}, \tag{5.1.}$$

where λ denotes the Lebesgue measure of $\mathbb{C}^n$.

5.1.2. Proposition:

Let $K \subseteq \Omega$ be a compact subset of Ω . Then there exists a constant C_K , depending on K and n, such that $\forall f \in A^2(\Omega)$

$$\sup_{z\in K}|f(z)| \leq C_K(n)\|f\|_{A^2(\Omega)}$$

Proof:

Since K is compact, there exists an $r(K) = r > 0$ such that for any $z \in K, \mathbb{B}(z,r) \subseteq \Omega$.
Therefore, for each $z \in K$ and $f \in A^2(\Omega)$, the mean value property of harmonic functions (cf. 3.4.3. Proposition) implies that

$$\begin{aligned}
|f(z)| &\overset{3.4.3.}{=} \frac{1}{|V(\mathbb{B}(z,r))|}\left|\int\limits_{\mathbb{B}(z,r)} f(t)\ d\lambda(t)\right| \leq \frac{1}{|V(\mathbb{B}(z,r))|}\int\limits_{\mathbb{B}(z,r)} |f(t)|\ d\lambda(t) \\
&= \frac{1}{|V(\mathbb{B}(z,r))|}\int\limits_{\mathbb{B}(z,r)} 1\cdot|f(t)|\ d\lambda(t) \\
&\overset{(2.1.)}{\leq} \frac{1}{|V(\mathbb{B}(z,r))|}\underbrace{\left(\int\limits_{\mathbb{B}(z,r)} 1\ d\lambda(t)\right)^{\frac{1}{2}}}_{=\sqrt{V(\mathbb{B}(z,r))}}\underbrace{\left(\int\limits_{\mathbb{B}(z,r)} |f(t)|^2\ d\lambda(t)\right)^{\frac{1}{2}}}_{\leq\|f\|_{A^2(\Omega)}} \\
&\leq \underbrace{\frac{1}{\sqrt{V(\mathbb{B}(z,r))}}}_{\leq C(n)r^{-n}}\|f\|_{A^2(\Omega)} \leq C(n)r^{-n}\,\|f\|_{A^2(\Omega)} \equiv C_K(n)\,\|f\|_{A^2(\Omega)}
\end{aligned}$$

This proves the desired inequality. ■

5.1.3. Corollary:

$A^2(\Omega)$ is a closed subspace of $\mathfrak{L}^2(\Omega)$ and hence a Hilbert space with the inner product

$$\langle f,g\rangle_{A^2(\Omega)} = \int\limits_{\Omega} f(z)\overline{g(z)}\ d\lambda(z). \tag{5.2.}$$

Proof:

Clearly, $A^2(\Omega)$ is a subspace of $\mathfrak{L}^2(\Omega)$.

The issue is to show that it is closed. To do so, let $(f_j)_{j\geq 1}$ be a Cauchy sequence of holomorphic functions in $A^2(\Omega) \subseteq \mathfrak{L}^2(\Omega)$.

Applying, 5.1.2. Proposition, to $f_j - f_k$ we conclude that $(f_j)_{j\geq 1}$ converges uniformly on compact subsets of Ω . Indeed, let $K \subseteq \Omega$ compact and we obtain

$$\sup_{z\in K}\left|f_j(z) - f_k(z)\right| \underset{5.1.2.}{\leq} C_K \underbrace{\left\|f_j - f_k\right\|_{A^2(\Omega)}}_{\to 0} \quad \forall z \in K$$

By 3.2.16. Theorem, the limit function $f \in A^2(\Omega)$ exists and is holomorphic.

On the other hand, $(f_j)_j$converges to a limit in $\mathfrak{L}^2(\Omega)$, and since $\mathfrak{L}^2(\Omega)$ is complete there exists an $\mathfrak{L}^2(\Omega)$ - limit function g, i.e. $f_j \xrightarrow[j\to\infty]{\mathfrak{L}^2(\Omega)} g$.

So,finally we have to show that $f = g$. To do so, we use a standard argument from measure theory which can be found e.g. in [Rud91b] and reads as follows:

If $(f_n)_n$ is a Cauchy sequence in $\mathfrak{L}^2(\Omega)$ with limit g, then $(f_n)_n$ has a subsequence which converges pointwise almost everywhere to $g(x)$.

From this theorem equality of the $\mathfrak{L}^2$ - limit and A^2 -limit function follows, since g can be changed on zero sets because $\mathfrak{L}^2$ consists of equivalence classes and the inclusion $A^2 \subseteq \mathfrak{L}^2$ is viewed modulo them. ∎

5.1.4. Corollary:

Choose $z \in \Omega$ fix. Define a mapping

$$L_z \colon A^2(\Omega) \to \mathbb{C}$$
$$f \longmapsto f(z).$$

Then L_z is a bounded linear functional on $A^2(\Omega)$.

Proof:

Linearity:

$$L_z(f+g) = (f+g)(z) \underset{\text{addition in } \mathbb{C}}{=} f(z)+g(z) = L_z(f)+L_z(g) \ \forall f,g \in A^2(\Omega)$$
$$L_z(cf) = (cf)(z) \underset{\text{multip. in } \mathbb{C}}{=} cf(z) = cL_z(f) \ \forall f \in A^2(\Omega), \forall c \in \mathbb{C}$$

Boundedness:

Since the singleton $\{z\}$ is a compact subset of Ω , 5.1.2. yields
$|L_z(f)| = |f(z)| \leq C_{\{z\}}\|f\|_{A^2(\Omega)} < \infty$ hence, L_z is bounded. ∎

Since $A^2(\Omega)$ is a Hilbert space of functions on a domain Ω and by 5.1.4. the evaluation linear functionals are continuous, we know by 4.2.5. Theorem, that $A^2(\Omega)$ possesses a reproducing kernel, hence $A^2(\Omega)$ is a RKHS.

We may now apply the Riesz representation Theorem 2.3.6. to see that there is an element $k_w \in A^2(\Omega)$ such that the linear functional L_w is represented by an inner product with k_w . If $f \in A^2(\Omega)$, then for all $w \in \Omega$ we have:

$$f(w) = \langle f, k_w \rangle_{A^2(\Omega)} \tag{5.3.}$$

5.1.5. Definition (Bergman Kernel):

The Bergman kernel of $A^2(\Omega)$ is the function

$$\begin{aligned} K\colon \Omega \times \Omega &\to \mathbb{C} \\ K(z,w) &:= k_w(z), \forall (z,w) \in \Omega \times \Omega \end{aligned} \tag{5.4.}$$

5.1.6. Remark:

Usually, someone defines $K(z,w)$ by $\overline{k_z(w)}$, but to be consistent with the notation of chapter 4, we define it by $k_w(z)$. We will see below that this in fact is just the same.

5.1.7. Remark:

$K(z,w) = k_w(z)$ is indeed a reproducing kernel of $A^2(\Omega)$. To verify this, we show that the two properties of 4.2.1. are full-filled:

(ii) Let $w \in \Omega$ and $f \in A^2(\Omega)$

$$f(w) \underset{2.3.6.}{=} \langle f, k_w \rangle = \int_\Omega f(z)\overline{k_w(z)}\, d\lambda(z) = \int_\Omega f(z)\overline{K(z,w)}\, d\lambda(z)$$

(i) For $w \in \Omega$ fixed, k_w is in $A^2(\Omega)$ as a function of z .
Therefore, $z \mapsto k_w(z) = K(z,w)$ is holomorphic on Ω for fixed w.

5.1.8. Proposition:

The Bergman kernel $K(z,w)$ is conjugate symmetric, i.e. $K(z,w) = \overline{K(w,z)}$ resp. $k_w(z) = \overline{k_z(w)}$.

Proof:

We know this from 4.2.11. Remark (2), but we will show it here explicitly again, to get used to the formulas.
By its very definition for each fixed $z \in \Omega$,$\overline{K(.,z)} \in A^2(\Omega)$, i.e.

$$w \mapsto \overline{k_z(w)} = \overline{K(w,z)} \in A^2(\Omega)$$

Therefore,the reproducing property of the Bergman kernel gives

$$\begin{aligned}\overline{k_z(w)} &= \overline{\int_\Omega \overline{K(u,w)} k_z(u)\, d\lambda(u)} = \int_\Omega K(u,w)\overline{k_z(u)}\, d\lambda(u) \\ &= \int_\Omega K(u,w)\overline{K(u,z)}\, d\lambda(u) = \int_\Omega \overline{K(u,z)} k_w(u)\, d\lambda(u) = k_w(z),\end{aligned}$$

hence: $k_w(z) = \overline{k_z(w)}$ resp. $K(z,w) = \overline{K(w,z)}$. ■

From this, it follows that

$$z \mapsto k_w(z) = K(z,w) \text{ is a map in } A^2(\Omega)$$

i.e. the Bergman kernel is holomorphic in the first variable, and

$$w \mapsto \overline{k_z(w)} = k_w(z)$$

i.e. the Bergman kernel is anti-holomorphic in the second variable.

5.1.9. Proposition:

The Bergman kernel is uniquely determined by the properties that it is an element of $A^2(\Omega)$ in z, is conjugate symmetric and reproduces $A^2(\Omega)$.

Proof:

Let $K'(z,w)$ be another kernel of $A^2(\Omega)$ with the same properties. Then the following string of equalities holds:

$$\begin{aligned}
K(z,w) &= \int_\Omega K'(u,z)\overline{K(u,w)}\, d\lambda(u) = \underbrace{\int_\Omega \overline{\overline{K'(u,z)}K(u,w)\, d\lambda(u)}}_{=\overline{K(w,z)}} \\
&= \int_\Omega \overline{\overline{K(w,u)}K'(z,u)\, d\lambda(u)} = \int_\Omega \underbrace{K(w,u)\overline{K'(z,u)}}_{=K'(u,z)\overline{K(u,w)}}\, d\lambda(u) \\
&= K'(z,w)
\end{aligned}$$

■

As we have already seen in chapter 4, there is an intriguing formula connecting the Bergman kernel function with the notion of complete orthonormal system. We will use this to compute the Bergman kernel function for the unit ball in $\mathbb{C}$.

Since $\mathfrak{L}^2(\Omega)$ is a separable Hilbert space, then so is its subspace $A^2(\Omega)$. Thus there is a complete orthonormal basis $\{\varphi_j\}_{j=1}^\infty$ for $A^2(\Omega)$.

5.1.10. Theorem:

Let M be a compact subset of Ω, $\{\varphi_j\}_{j=1}^\infty$ an ONB of $A^2(\Omega)$. Then the series

$$\sum_{j=1}^\infty \varphi_j(z)\overline{\varphi_j(w)} \tag{5.5.}$$

sums uniformly on $M \times M$ to the Bergman kernel, i.e.

$$K(z,w) = \sum_{j=1}^\infty \varphi_j(z)\overline{\varphi_j(w)} \tag{5.6.}$$

Proof:

Suppose $f \in A^2(\Omega)$ Then, by orthogonal expansion ,$f = \sum_{\alpha=1}^{\infty} \langle f, \varphi_\alpha \rangle \varphi_\alpha$.

Since evaluation at z is continuous , it is $f(z) = \sum_{\alpha=1}^{\infty} \langle f, \varphi_\alpha \rangle \varphi_\alpha(z)$.

By the Riesz-Fischer Theorem 2.3.6. we obtain

$$\sup \left\{ \sqrt{\sum_{\alpha=1}^{\infty} |\varphi_\alpha(z)|^2} : z \in M \right\} = \sup \left\{ \| \varphi_\alpha(z)_{\alpha=1}^{\infty} \|_{\ell^2} : z \in M \right\}$$

$$= \sup \left\{ \left| \sum_{\alpha=1}^{\infty} b_\alpha \varphi_\alpha(z) \right| : z \in M, \left(\sum_{\alpha=1}^{\infty} |b_\alpha|^2 = 1 \right)^{\frac{1}{2}}, b_\alpha = \langle f, \varphi_\alpha \rangle \right\}$$

$$= \sup \left\{ |f(z)| : z \in M, \|f\|_{\mathfrak{H}_K} = 1 \right\} \underset{5.1.2.}{\leq} C(M) \qquad (*)$$

for some constant $C(M) > 0$ which depends only on M. For fixed $z \in M$, the sequence of complex numbers $\varphi_\alpha(z)$ is in ℓ^2.

This inequality $(*)$ yields the uniform convergence on compact sets of the diagonal (that is where $z = w$). Since the sequence $\varphi_\alpha(z)$ is square summable, we may use the (2.1.) on ℓ^2 to estimate

$$\left| \sum_{\alpha=1}^{\infty} \varphi_\alpha(z) \overline{\varphi_\alpha(w)} \right| \leq \underbrace{\sqrt{\sum_{\alpha=1}^{\infty} |\varphi_\alpha(z)|^2}}_{\leq C_{1,M}} \underbrace{\sqrt{\sum_{\alpha=1}^{\infty} |\varphi_\alpha(w)|^2}}_{\leq C_{2,M}}$$

Therefore uniform convergence on the diagonal implies uniform convergence on $\Omega \times \Omega$.

We have established the uniform convergence of the sum $\sum_{\alpha=1}^{\infty} \varphi_\alpha(z) \overline{\varphi_\alpha(w)}$ on compact subsets of $\Omega \times \Omega$, and convergence (in the first variable) in $A^2(\Omega)$, i.e. $\sum_{\alpha=1}^{\infty} \varphi_\alpha(z) \overline{\varphi_\alpha(w)} \in A^2(\Omega)$ as a function of w.

It remains to show: $K(z, w) = \sum_{\alpha=1}^{\infty} \varphi_\alpha(z) \overline{\varphi_\alpha(w)}$.

For each $f \in A^2(\Omega)$, the series converges in the Hilbert space $A^2(\Omega)$ and thus it converges uniformly on compact subsets of Ω (Hilbert space convergence dominates pointwise convergence).

In particular, if $w \in \Omega$

$$f(w) = \sum_{\alpha=1}^{\infty} \langle f, \varphi_\alpha \rangle \varphi_\alpha(w) = \left\langle f, \sum_{\alpha=1}^{\infty} \overline{\varphi_\alpha(w)} \varphi_\alpha \right\rangle .$$

Since $\sum\limits_{\alpha=1}^{\infty} \varphi_\alpha(w)\varphi_\alpha \in A^2(\Omega)$, the uniqueness of the Riesz representations using the reproducing property shows that

$$f(w) = \langle f, k_w \rangle = \left\langle f, \sum_{\alpha=1}^{\infty} \overline{\varphi_\alpha(w)} \varphi_\alpha \right\rangle \Rightarrow k_w(z) \quad = K(z, w) = \sum_{\alpha=1}^{\infty} \varphi_\alpha(z) \overline{\varphi_\alpha(w)}$$

■

5.1.11. Remark:

It is worth noting explicitly that the proof of the 5.1.10. Theorem resp. shows that $\sum\limits_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}$ equals the Bergman kernel $K(z, w)$, and that the representation is independent of the choice of the ONB $\{\varphi_j\}_{j=1}^{\infty}$ for $A^2(\Omega)$.

5.1.12. Remark:

Since $A^2(\Omega)$ is a closed subspace of $\mathfrak{L}^2(\Omega)$ it is worth looking at the orthogonal projection from $A^2(\Omega)$ onto $\mathfrak{L}^2(\Omega)$.

5.1.13. Theorem (Bergman Projection):

If Ω is a bounded domain in $\mathbb{C}^n$, then the mapping $P\colon f \mapsto \int\limits_{\Omega} \overline{K(z, .)} f(z)\, d\lambda(z)$ is the Hilbert space orthogonal projection of $\mathfrak{L}^2(\Omega)$ onto $A^2(\Omega)$.
This is called the Bergman projection.

Proof:

2.3.4. Theorem guarantees the existence, since $A^2(\Omega)$ is a closed subspace of $\mathfrak{L}^2(\Omega)$. By, 4.3.1.Theorem, the orthogonal projection of $\mathfrak{L}^2(\Omega)$ onto $A^2(\Omega)$ is given by the formula $Pf(w) = \langle f, K(., w) \rangle_{A^2(\Omega)}$, where K is the Bergman kernel of $A^2(\Omega)$. So, it follows that

$$Pf(w) = \int_\Omega f(z)\overline{K(z,w)}\, d\lambda(z).$$

Moreover, P is idempotent and self - adjoint:
idempotent:

$$P(Pf)(w) = Pf(w) \text{ since } Pf \text{ is in } A^2(\Omega).$$

self-adjoint:
Let $g \in A^2(\Omega)$. Then we have taking the inner product of Pf and g in $A^2(\Omega)$

$$\langle Pf, g\rangle = \langle f, P^* g\rangle .$$

On the other hand, it is

$$\begin{aligned}\int_\Omega\int_\Omega Pf(w)\overline{g(z)}d\lambda(w)d\lambda(z) &= \int_\Omega\int_\Omega K(z,w)f(w)\overline{g(z)}d\lambda(w)d\lambda(z)\\ &\underset{\text{Fubini}}{=} \int_\Omega f(w)\underbrace{\overline{\int_\Omega K(w,z)g(z)d\lambda(z)}}_{=Pg(w)}d\lambda(w)\\ &= \int_\Omega f(w)\overline{Pg(w)}d\lambda(w).\end{aligned}$$

So, it follows that

$$\langle Pf, g\rangle = \langle f, Pg\rangle .$$

Hence,

$$P = P^*$$

■

5.2 BERGMAN METRIC

The following result shows how the Bergman kernel function behaves under a biholomorphic map. In what follows we denote the Bergman kernel for a given domain Ω by K_Ω.

5.2.1. Theorem:

Let Ω_1, Ω_2 be domains in $\mathbb{C}^n$. Let $f\colon \Omega_1 \to \Omega_2$ be biholomorphic. Then

$$K_{\Omega_1}(z,w) = \det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), f(w)) \overline{\det J_{\mathbb{C}} f(w)} \tag{5.7.}$$

for all $z, w \in \Omega_1$.

Proof:

From 3.3.2. Proposition we know $\det J_{\mathbb{R}} f(z) = |\det J_{\mathbb{C}} f|^2$. Let $\phi \in A^2(\Omega_1)$. Then, by change of variable $(f(w) = \tilde{w})$.

$$\int_{\Omega_1} \det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), f(w)) \overline{\det J_{\mathbb{C}} f(w)} \phi(w) d\lambda(w) =$$

$$= \int_{\Omega_2} \det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), \tilde{w}) \overline{\det J_{\mathbb{C}} f^{-1}(\tilde{w})} \phi(f^{-1}(\tilde{w})) \det J_{\mathbb{R}} f^{-1}(\tilde{w}) d\lambda(\tilde{w})$$

By 3.3.2. Proposition, this simplifies to

$$\det J_{\mathbb{C}} f(z) \int_{\Omega_2} K_{\Omega_2}(f(z), \tilde{w}) \left\{ \left(\det J_{\mathbb{C}} f(f^{-1}(\tilde{w}))\right)^{-1} \phi(f^{-1}(\tilde{w})) \right\} d\lambda(\tilde{w}),$$

because

$$\begin{aligned} J_{\mathbb{R}} f^{-1}(\tilde{w}) &= \left|\det J_{\mathbb{C}} f^{-1}(\tilde{w})\right|^2 = \det J_{\mathbb{C}} f^{-1}(\tilde{w}) \cdot \overline{\det J_{\mathbb{C}} f^{-1}(\tilde{w})} \\ &= \det\left(J_{\mathbb{C}} f(f^{-1}(\tilde{w}))\right)^{-1} \cdot \overline{\det\left(J_{\mathbb{C}} f(f^{-1}(\tilde{w}))\right)^{-1}} \\ &= \frac{1}{\det J_{\mathbb{C}} f(f^{-1}(\tilde{w}))} \cdot \frac{1}{\overline{\det J_{\mathbb{C}} f(f^{-1}(\tilde{w}))}} \end{aligned}$$

By change of variables, the expression in braces is an element of $A^2(\Omega)$.

So, by the reproducing property of K_{Ω_2}, the last line equals:

$\det J_{\mathbb{C}}f(z)(J_{\mathbb{C}}f(z))^{-1}\phi(f^{-1}(f(z))) = \phi(z)$

Indeed,

$$\det J_{\mathbb{C}}f(z)\int_{\Omega_2} K_{\Omega_2}(f(z),\tilde{w})\left\{\left(\det J_{\mathbb{C}}f(f^{-1}(\tilde{w}))\right)^{-1}\phi(f^{-1}(\tilde{w}))\right\}\ d\lambda(\tilde{w}) =$$

$$= \det J_{\mathbb{C}}f(z)\left\langle \det\left(J_{\mathbb{C}}f(f^{-1}(\tilde{w}))\right)^{-1}\phi(f^{-1}(\tilde{w})), K_{\Omega_2}(f(z),\tilde{w})\right\rangle_{A^2(\Omega_2)} =$$

$$= \det J_{\mathbb{C}}f(z)\left\langle \det\left(J_{\mathbb{C}}f(f^{-1}(f(w)))\right)^{-1}\phi(f^{-1}(f(w))), K_{\Omega_2}(f(z),f(w))\right\rangle_{A^2(\Omega_2)}$$

$$= \det J_{\mathbb{C}}f(z)\left(\det J_{\mathbb{C}}f(f^{-1}(f(z)))\right)^{-1}\phi(f^{-1}(f(z)))$$

$$= \phi(z)$$

By the uniqueness of the Bergman kernel 5.1.9. Theorem, the theorem follows.

■

5.2.2. Corollary:

Let $f\colon \Omega_1 \to \Omega_2$ be a biholomorphic map between two domains Ω_1 and Ω_2 in $\mathbb{C}^n$, and let P_1, P_2 be the Bergman projection operators on Ω_1, Ω_2 respectively.Then $\forall g \in L^2(\Omega_2)$

$$P_1\left((\det J_{\mathbb{C}}f)\cdot(g\circ f)\right) = (\det J_{\mathbb{C}}f)\cdot(P_2(g)\circ f). \tag{5.8.}$$

Proof:

The proof follows directly from the transformation law of the Bergman kernel functions. For $g \in L^2(\Omega_2)$, $(\det J_{\mathbb{C}}f)\cdot(g\circ f) \in L^2(\Omega_1)$. Hence, from 5.2.1.Theorem,

$$P_1\left((\det J_{\mathbb{C}}f)\cdot(g\circ f)\right) \underset{5.1.13}{=} \int_{\Omega_1} K_{\Omega_1}(z,w)\det J_{\mathbb{C}}f(w)g(f(w))\ d\lambda(w)$$

$$\underset{5.2.1.}{=} \int_{\Omega_1} \det J_{\mathbb{C}}f(z)K_{\Omega_2}(f(z),f(w))\left|\det J_{\mathbb{C}}f(w)\right|^2 g(f(w))\ d\lambda(w)$$

$$= \det J_{\mathbb{C}} f(z) \int\limits_{\Omega_2} K_{\Omega_2}(f(z),\eta) g(\eta)\ d\lambda(\eta) \underset{5.1.13.}{=} \det J_{\mathbb{C}} f(z) (P_2(g) \circ f).$$

This proves the corollary. ∎

5.2.3. Proposition:

For $z \in M \subseteq \Omega$, M compact, it holds that $K_\Omega(z,z) > 0$.

Proof:

Now $K_\Omega(z,z) = \sum\limits_{j=1}^{\infty} \left|\phi_j(z)\right|^2 \geq 0$. If, in fact $K_\Omega(z,z) = 0$ for some z, then $\phi_j(z) = 0,\ \forall j$, hence $f(z) = 0$ for every $f \in A^2(\Omega)$. This is absurd. ∎

5.2.4. Definition:

For any $\Omega \subseteq \mathbb{C}^n$ we define a Hermitian metric on Ω by

$$g_{ij}(z) :=\frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K(z,z),\ z \in \Omega \tag{5.9.}$$

This means that the square of the length of a tangent vector $\zeta = (\zeta_1, \ldots, \zeta_n)$ at a point $z \in \Omega$ is given by

$$|\zeta|^2_{B,z} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}(z) \zeta_i \overline{\zeta_j} \tag{5.10.}$$

The metric g_{ij} is called the Bergman metric.

5.2.5. Remark:

(1) In a Hermitian metric $\{g_{ij}\}_{i,j=1}^{n}$ the length of a C^1-curve $\gamma\colon [0,1] \to \Omega$ is given by

$$\ell(\gamma) = \int\limits_0^1 \left|\frac{\partial \gamma}{\partial t}(t)\right|_{B,\gamma(t)}\ dt = \int\limits_0^1 \left(\sum_{i,j} g_{ij}(\gamma(t)) \gamma_i'(t) \overline{\gamma_j'(t)}\right)^{\frac{1}{2}}\ dt$$

(2) If p and q are points of Ω, then their distance in the metric is defined as $d_\Omega(p,q) = d_\Omega$:
$d_\Omega := \inf\left\{ \ell(\gamma) \;\middle|\; \text{all piecewise } C^1-\text{curves s.t. } \gamma(0) = p \text{ and } \gamma(1) = q \right\}$.
The distance d_Ω is called the Bergman distance.

(3) The Bergman metric is in fact a positive definite matrix at each point if Ω is a bounded domain (see [Kra08]).

More importantly, the distance d_Ω is invariant under biholomorphic mappings from Ω to another domain.

5.2.6. Proposition:

Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ be domains and let $f\colon \Omega_1 \to \Omega_2$ be a biholomorphic mapping.
Then f induces an isometry of Bergman metrics $|\zeta|_{B,z} = \left|\left(J^n_{\mathbb{C}} f\right)\zeta\right|_{B,f(z)}$ for all $z \in \Omega_1, \zeta \in \mathbb{C}^n$.
Equivalently, f induces an isometry for Bergman distances in the sense that $d_{\Omega_2}(f(p), f(q)) = d_{\Omega_1}(p,q)$.

Proof:

From the definition, it suffices to check:

$$\sum_{i,j} g^{\Omega_2}_{ij}(f(z))(J_{\mathbb{C}}f(z)w)_i\overline{(J_{\mathbb{C}}f(z)w)_j} = \sum_{i,j} g^{\Omega_1}_{ij}(z)w_i\overline{w_j} \tag{$*$}$$

for all $z \in \Omega_1, w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. But by 5.2.1.,

$$\begin{aligned} g^{\Omega_1}_{ij}(z) &= \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K_{\Omega_1}(z,z) \underset{5.2.1.}{=} \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log\left\{|\det J_{\mathbb{C}}f(z)|^2 \cdot K_{\Omega_2}(f(z), f(z))\right\} \\ &= \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K_{\Omega_2}(f(z), f(z)) + \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log\left\{|\det J_{\mathbb{C}}f(z)|^2\right\} \end{aligned}$$

Since $\log\left\{|\det J_{\mathbb{C}}f(z)|^2\right\}$ is locally $\log(\det J_{\mathbb{C}}f) + \log(\overline{\det J_{\mathbb{C}}f}) + C$, and hence is annihilated by the mixed second derivative. So, it follows that

$$g^{\Omega_1}_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K_{\Omega_2}(f(z), f(z))$$

and the right hand side of the last equation is nothing other than

$$\sum_{l,m} g^{\Omega_2}_{lm}(f(z)) \frac{\partial f_l(z)}{\partial z_i} \frac{\overline{\partial f_m(z)}}{\partial \overline{z_j}}$$

and $(*)$ follows, where we used the definition of $J_{\mathbb{C}}f$ (confirm (3.18.))in the last computation. ■

5.2.7. Proposition:

Let $M \subseteq \mathbb{C}^n$ be a compact. Let $z \in M$. Then

$$K(z,z) = \sup_{f \in A^2(M)} \frac{|f(z)|}{\|f\|^2_{A^2(M)}} = \sup_{\|f\|_{A^2(\Omega)}=1} |f(z)|^2 \tag{5.11.}$$

Proof:

Let $\{\phi_j\}_{j=1}^{\infty}$ be an ONB for $A^2(M)$.

$$K(z,z) = \sum_{j=1}^{\infty} \left|\phi_j(z)\right|^2 = \left(\sup_{\|\{a_j\}\|_{\ell^2}=1} \left| \sum_{j=1}^{\infty} \phi_j(z) a_j \right| \right)^2 = \sup_{\|f\|_{A^2(M)}=1} |f(z)|^2$$

This equals $\sup\limits_{f \in A^2(M)} \frac{|f(z)|^2}{\|f\|^2_{A^2(M)}}$. ■

5.2.8. Remark (Lu Qi – Keng Conjecture):

In 4.2.12. Remark we showed equivalent conditions for a reproducing kernel being zero. So it is worth looking at possible zeros of the Bergman kernel, since we have already noticed in 5.2.3. Proposition that $K_{\Omega}(z,z) > 0$ for all $z \in \Omega, \Omega$ bounded, i.e. the Bergman kernel function of the ball in $\mathbb{C}^n$ has no zeros along the diagonal.

Is it true that for every bounded domain the Bergman kernel has no zeros? Lu Qi – Keng raised this question in 1966 in his article "On Kaehler manifolds with constant curvature, Chinese Math. 8 (1966), 283 – 298 ". Subsequent authors, attributing a positive opinion to him, called the affirmative answer "the Lu Qi – Keng conjecture".

The Bergman kernel for the annulus was studied by M. Skwarczynski in "The distance in the theory of pseudo - conformal transformations and the Lu Qi -Keng conjecture, Proc. A.M.S. 22 (1969), 305 - 310" and was seen to vanish at some points. It is shown in N. Suita and A. Yamada "On the Lu Qi-Keng conjecture, Proc. A.M.S. 59 (1976), 222 - 224 " that if $\Omega \subseteq \mathbb{C}$ is a multiply connected domain with smooth boundary, then K_Ω must vanish - this is proved by an analysis of differentials on Riemann surfaces.
By using the fact that the Bergman kernel for a product domain is the product of the Bergman kernels, we may conclude that any domain in $\mathbb{C}^2$ of the form $H \times \Omega$, where H is multiply connected, has a Bergman kernel with zeros. It is known (R.E. Greene and S.G. Krantz, "Stability properties of the Bergman kernel and curvature properties of bounded domains, Recent Developments in Several Complex Variables, Princeton Univ. Press, Princeton, N.J., 1981) that a domain that is C^∞ sufficiently close to the ball in $\mathbb{C}^n$ has non-vanishing Bergman kernel.

Harold P. Boas showed 1996 in "The Lu Qi -Keng Conjecture Fails Generically, A.M.S., Volume 124, Number 7 " that the Lu Qi - Keng conjecture fails generically. He showed that the bounded domains of holomorphy in $\mathbb{C}^n$ whose Bergman kernel functions are zero -free form a nowhere dense subset (with respect to a variant of the Hausdorff distance) of all bounded domains of holomorphy.

5.3 Calculating the Bergman kernel

The Bergman kernel can almost never be calculated explicitly; unless the domain Ω has a great deal of symmetry – so that a useful orthonormal basis for $A^2(\Omega)$ can be determined – there are few techniques for determining K_Ω. Restrict attention to the unit ball $\mathbb{D} \subseteq \mathbb{C}$, i.e. $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$.
For simplicity we first calculate Bergman kernel for the unit ball $\mathbb{D} \subseteq \mathbb{C}$ and then generalize to the n-dimensional case.

5.3.1. Theorem:

The Bergman kernel of the open unit ball $\mathbb{D}$ is given by

$$K_{\mathbb{D}}(z, w) = \frac{1}{\pi}\frac{1}{(1 - \overline{w}z)^2} \tag{5.12.}$$

for $w, z \in \mathbb{D}$.

Proof:

We divide the proof into 3 steps.
Step 1:
For $n \in \mathbb{N}_0$, it is clear that the monomials z^n, are in $A^2(\mathbb{D})$, since they are holomorphic and their norm is less than infinity.

Claim: The functions $f_n(z)\colon = \sqrt{\frac{n+1}{\pi}}z^n$ from an orthonormal system in $A^2(\mathbb{D})$. Let $z^n, z^m \in A^2(\mathbb{D})$,$n, m \in \mathbb{N}_0$.Then we have

$$\begin{aligned}
\langle z^n, z^m\rangle &= \int_0^1\int_0^{2\pi} r^n e^{i\theta m} r^m e^{-i\theta m}\ d\theta r\ dr = \int_0^{2\pi} e^{i\theta(n-m)}\ d\theta \int_0^1 r^{n+m+1}\ dr \\
&= \int_0^{2\pi} e^{i\theta(n-m)}\ d\theta \frac{1}{2+n+m} = \begin{cases} \int_0^{2\pi} e^{i\theta 0}\ d\theta = \int_0^{2\pi} 1\ d\theta = 2\pi & m = n \\ \int_0^{2\pi} e^{i\theta(n-m)}\ d\theta \underset{(n-m)\in\mathbb{Z}}{=} 0 & m \neq n \end{cases} \\
&= \frac{2\pi}{n+m+2}\delta_{nm}, \text{i.e. } \langle z^n, z^m\rangle = \frac{2\pi}{n+m+2}\delta_{nm}
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle f_n, f_m \rangle &= \left\langle \sqrt{\frac{n+1}{\pi}} z^n, \sqrt{\frac{m+1}{\pi}} z^m \right\rangle \\
&= \sqrt{\frac{n+1}{\pi}} \sqrt{\frac{m+1}{\pi}} \langle z^n, z^m \rangle = \frac{\sqrt{n+1}\sqrt{m+1}}{\pi} \frac{2\pi}{n+m+2} \delta_{nm} \\
&= \begin{cases} \frac{(m+1)2}{2m+2} = 1 & n = m \\ 0 & n \neq m \end{cases}
\end{aligned}$$

hence, $\langle f_n, f_m \rangle = \delta_{nm}$, and so $\{f_n(z)\}_{n \geq 0}$ forms an orthonormal system of $A^2(\mathbb{D})$.

Step 2: Next we show that $\{f_n(z)\}_{n \geq 0}$ is total in $A^2(\mathbb{D})$.

Consider any function $f \in A^2(\mathbb{D})$. By the definition of the Bergman space, f is holomorphic in $\mathbb{D}$ and so we have the Taylor series expansion of f as $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ (cf. 3.1.2. Remark). Then we get

$$\begin{aligned}
\langle f, f_n \rangle &= \left\langle \sum_{k=0}^{\infty} a_k z^k, \sqrt{\frac{n+1}{\pi}} z^n \right\rangle = a_n \sqrt{\frac{n+1}{\pi}} \underbrace{\langle z^n, z^n \rangle}_{=1} \\
&= a_n \sqrt{\frac{n+1}{\pi}}
\end{aligned}$$

Therefore, if f is orthogonal to all of f_n , i.e. $\langle f, f_n \rangle = 0$, then $a_n = 0 \forall n \geq 0$. In other words, all Taylor coefficients of f vanish and that gives $f = 0$. Hence, $\{f_n\}_{n \geq 0}$ is total in $A^2(\mathbb{D})$.

So, $\{f_n\}_{n \geq 0}$ is an ONB of $A^2(\mathbb{D})$ and by 5.1.10. Theorem the Bergman kernel is given by

$$K_{\mathbb{D}}(z, w) = \sum_{n=0}^{\infty} f_n(z) \overline{f_n(w)}, \ \forall z, w \in \mathbb{D}$$

Step 3: The Bergman kernel for $\mathbb{D}$ is $K_{\mathbb{D}}(z, w) = \dfrac{1}{\pi} \dfrac{1}{(1 - \overline{w}z)^2}$.

Inserting the values of f_n and f_m into the above equation leads to

$$\begin{aligned}
K_{\mathbb{D}}(z,w) &= \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} = \sum_{n=0}^{\infty} \sqrt{\frac{n+1}{\pi}} z^n \sqrt{\frac{n+1}{\pi}} \overline{w}^n \\
&= \underbrace{\frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)(z\cdot\overline{w})^n}_{=\frac{d}{d\xi}\left(\sum_{n=0}^{\infty}\xi^n\right)\Big|_{\xi=z\overline{w}}} \underset{|z\cdot\overline{w}|<1}{=} \frac{1}{\pi}\frac{d}{d\xi}\left(\sum_{n=0}^{\infty}\xi^n\right)\Bigg|_{\xi=z\overline{w}} \\
&= \frac{1}{\pi}\frac{d}{d\xi}\left(\frac{1}{1-\xi}\right)\Bigg|_{\xi=z\overline{w}} = \frac{1}{\pi}\frac{1}{(1-\xi)^2}\Bigg|_{\xi=z\overline{w}} \\
&= \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2}
\end{aligned}$$

■

Now let $\mathbb{B} \subseteq \mathbb{C}^n$ be the ball. Similarly, to the proof of 5.3.1., step 1 ,one can show that the functions $\{ z^\alpha \mid \alpha$ is a multi -index $\}$ are each in $A^2(\mathbb{B})$ and are pairwise orthogonal (by the symmetry of the ball).
By the uniqueness of the power series expansion for an element of $A^2(\mathbb{B})$, the elements z^α form a complete orthogonal system on $\mathbb{B}$ (their closed linear span is $A^2(\mathbb{B})$). Setting

$$\gamma_\alpha :=\int_{\mathbb{B}} |z^\alpha| \; d\lambda(z),$$

we see that $\left\{ \frac{z_\alpha}{\sqrt{\gamma_\alpha}} : \alpha \text{ is a multi-index} \right\}$ is a complete ONS in $A^2(\mathbb{B})$.

Thus, by 4.3.8. Theorem,

$$K_{\mathbb{B}}(z,w) = \sum_\alpha \frac{z^\alpha \overline{w}^\alpha}{\gamma_\alpha}$$

So, if we want to calculate the Bergman kernel for the ball in a closed form, we need to calculate the γ_α 's . This requires some lemmas from real analysis. These lemmas will be formulated and proved on $\mathbb{R}^N$ and $\mathbb{B}_N :=\left\{x \in \mathbb{R}^N : |x| < 1\right\}$.

5.3.2. Lemma:

We have that $\int\limits_{\mathbb{R}^N} e^{-\pi|x|^2}\,dx = 1$.

Proof:

(N = 1): We have got to show that $\int\limits_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$.

To do so, we use the method of substitution and the fact that $\int\limits_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. By setting $y :\,= x\cdot\sqrt{\pi}$ we obtain

$$\int\limits_{-\infty}^{\infty} e^{-\pi\,x^2}\,dx \underset{\substack{dy=dx\sqrt{\pi}\\ x^2=\frac{y^2}{\pi}}}{=} \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} e^{-y^2}\,dy = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1.$$

(N-dimensional:) For the N-dimensional case, we write using Fubini´s formula

$$\int\limits_{\mathbb{R}^N} e^{-\pi|x|^2}\,dx = \int\limits_{\mathbb{R}} e^{-\pi x_1^2}\,dx_1\cdots\int\limits_{\mathbb{R}} e^{-\pi x_N^2}\,dx_N$$

and apply the one-dimensional result. ∎

Let σ be the unique rotationally invariant area measure on $S_{N-1} = b\mathbb{B}_N$ and let $\omega_{N-1} = \sigma(b\mathbb{B})$ (see [Kra08, Appendix II]).

5.3.3. Lemma:

We have

$$\omega_{N-1} = \frac{(2\pi)^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{5.13.}$$

where $\Gamma(x) = \int\limits_{0}^{\infty} t^{x-1}e^{-t}d\,t$ denotes Euler´s gamma function.

Proof:

Introducing polar coordinates we have

$$1 = \int_{\mathbb{R}^N} e^{-\pi|x|^2}\ dx = \int_{S_{N-1}} d\sigma \int_0^\infty e^{-\pi r^2} r^{N-1}\ dr$$

$$\text{or } \frac{1}{\omega_{N-1}} = \int_0^\infty e^{-\pi r^2} r^N \frac{dr}{r}$$

Letting $s = r^2$ in this last integral yields

$$\frac{1}{\omega_{N-1}} = \int_0^\infty e^{-\pi s} \sqrt{s}^N \frac{1}{2\sqrt{s}} \frac{1}{\sqrt{s}}\ ds = \frac{1}{2}\int_0^\infty e^{-\pi s} s^{\frac{N}{2}} \frac{1}{s}\ ds = \frac{1}{2}\int_0^\infty e^{-\pi s}\ s^{\frac{N}{2}-1}\ ds$$

$$\underset{t=s\pi}{=} \underbrace{\frac{\pi}{2\pi^{\frac{N}{2}} - 1}}_{=\frac{1}{(2\pi)^{\frac{N}{2}}}} \underbrace{\int_0^\infty e^{-t}\ t^{\frac{N}{2}-1}\ dt}_{=\Gamma\left(\frac{N}{2}\right)} = \frac{1}{2\pi^{\frac{N}{2}}} \Gamma\left(\frac{N}{2}\right)$$

■

Now we return to $\mathbb{B} \subseteq \mathbb{C}^n$. We set:
$\eta(k) := \int_{b\mathbb{B}} |z_1|^{2k}\ d\sigma$, and $N(k) := \int_{\mathbb{B}} |z_1|^{2k}\ d\lambda(z)$ $(k = 0, 1, ..)$.

5.3.4. Lemma:

We have

$$\eta(k) = \pi^n \frac{2(k!)}{(k+n-1)!} \text{ and } N(k) = \pi^n \frac{(k!)}{(k+n)!}$$

Proof:

Polar coordinates show the relationship between $\eta(k)$ and $N(k)$:
$\eta(k) = 2(k+n)N(k)$. So it is enough to calculate $N(k)$.
Let $z = (z_1, z_2, \ldots, z_n) = (z_1, z')$. We write

$$N(k) = \int\limits_{|z|<1} |z_1|^{2k} \, d\lambda(z) = \int\limits_{|z'|<1} \left(\int\limits_{|z_1| \le \sqrt{1-|z'|^2}} |z_1|^{2k} \, d\lambda(z_1) \right) d\lambda(z')$$

$$= \int\limits_{|z'|<1} \left(\int_0^{2\pi} \int_0^{\sqrt{1-|z'|^2}} r^{2k} r \, dr \, d\theta \right) d\lambda(z')$$

$$= 2\pi \int\limits_{|z'|<1} \left(\int_0^{\sqrt{1-|z'|^2}} r^{2k+1} \, dr \right) d\lambda(z')$$

$$= 2\pi \int\limits_{|z'|<1} \left. \frac{r^{2k+2}}{2k+2} \right]_0^{\sqrt{1-|z'|^2}} d\lambda(z') = 2\pi \int\limits_{|z'|<1} \frac{(1-|z'|^2)^{k+1}}{2k+2} \, d\lambda(z')$$

$$= \frac{\pi}{k+1} \int\limits_{|z'|<1} (1 - \left|z'\right|^2)^{k+1} \, d\lambda(z') = (*)$$

For further calculations, we use the relationship between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. If $z' \in \mathbb{C}^{n-1} \cong \mathbb{R}^{2n-2}$, then $\omega^{\mathbb{C}}_{n-1} = \omega^{\mathbb{R}}_{2n-3}$. Hence,

$$(*) = \frac{\pi}{k+1} \omega^{\mathbb{R}}_{2n-3} \int_0^1 (1-r^2)^{k+1} \, r^{2n-3} \, dr \underset{r^2=s}{=} \frac{\pi}{k+1} \omega^{\mathbb{R}}_{2n-3} \int_0^1 (1-s)^{k+1} \, s^{n-1} \frac{ds}{2s}$$

$$= \frac{\pi}{2(k+1)} \omega^{\mathbb{R}}_{2n-3} \int_0^1 (1-s)^{k+1} \, s^{n-2} \frac{ds}{s} = \frac{\pi}{2(k+1)} \omega^{\mathbb{R}}_{2n-3} \, \beta\left(n-1, k+2\right),$$

where β is the classical beta function of special function theory.

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt, \text{ for } \mathrm{Re}(x), \mathrm{Re}(y) > 0$$

The beta function has one of the following standard identities:

$$\beta(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

Using this identity we obtain from the above equality:

$$\begin{aligned} N(k) &= \frac{\pi}{2(k+1)}\omega^{\mathbb{R}}_{2n-3}\frac{\Gamma(n-1)\Gamma(k+2)}{\Gamma(n+k+1)} \\ &\underset{5.3.3(\triangle)}{=} \frac{\pi}{\cancel{2}(k+1)}\frac{\cancel{2}\pi^{n-1}}{\cancel{\Gamma(n-1)}}\frac{\cancel{\Gamma(n-1)}\Gamma(k+2)}{\Gamma(n+k+1)} = \frac{\pi^n\Gamma(k+2)}{(k+1)\Gamma(n+k+1)} \\ &\underset{(\diamondsuit)}{=} \frac{\pi^n k!}{(k+n)!} \end{aligned}$$

This is the desired result, where we used the following identities:

$$5.3.3(\triangle) = \omega^{\mathbb{R}}_{\underbrace{2n-2}_{N}-1} = \frac{2\pi^{\frac{2n-2}{2}}}{\Gamma\left(\frac{2n-2}{2}\right)} = \frac{2\pi^{n-1}}{\Gamma(n-1)}$$

$$(\diamondsuit): \Gamma(k+2) = (k+1)!,\ \Gamma(n+k+1) = (k+n)!$$

■

5.3.5. Lemma:

Let $z \in \mathbb{B} \subseteq \mathbb{C}^n$ and $0 < r < 1$. The symbol $\imath$ denotes the point $(1, 0, \ldots, 0)$. Then

$$K_{\mathbb{B}}(z, r\imath) = \frac{n!}{\pi^n}\frac{1}{(1-rz_1)^{n+1}}$$

Proof:

We still know

$$\begin{aligned} K_{\mathbb{B}}(z, r\imath) &= \sum_{\alpha}\frac{z^{\alpha}(r\imath)^{\alpha}}{\gamma_{\alpha}} \underset{\substack{\gamma_\alpha = \int_{\mathbb{B}}|z^\alpha|^2\, d\lambda(z) \\ N(k) = \int_{\mathbb{B}}|z_1|^{2k}\, d\lambda(z)}}{=} \sum_{k=0}^{\infty}\frac{(z_1 r)^k}{N(k)} \underset{5.3.4.}{=} \frac{1}{\pi^n}\sum_{k=0}^{\infty}(r\,z_1)^k\,\frac{(k+n)!}{k!} \\ &= \frac{n!}{\pi^n}\sum_{k=0}^{\infty}(r\,z_1)^k\underbrace{\frac{(k+n)!}{n!k!}}_{\binom{k+n}{n}} = \frac{n!}{\pi^n}\sum_{k=0}^{\infty}(rz_1)^k\binom{k+n}{n} = \frac{n!}{\pi^n}\frac{1}{(1-rz_1)^{n+1}} \end{aligned}$$

■

5.3.6. Theorem:

If $z, w \in \mathbb{B}$, then

$$K_{\mathbb{B}}(z,w) = \frac{n!}{\pi^n}\frac{1}{(1 - z \cdot \overline{w})^{n+1}}, \tag{5.14.}$$

where $z \cdot \overline{w} = z_1\overline{w_1} + z_2\overline{w_2} + \cdots z_n\overline{w_n}$.

Proof:

Let $z = r \cdot \tilde{z} \in \mathbb{B}$, where $r = |z|$ and $|\tilde{z}| = 1$. Also, fix $w \in \mathbb{B}$. Choose an unitary rotation ρ such that $\rho\tilde{z} = \imath$. Then,we get

$$\begin{aligned} K_{\mathbb{B}}(z,w) &= K_{\mathbb{B}}(r\tilde{z},w) = K_{\mathbb{B}}(r\rho^{-1}\imath,w) = K_{\mathbb{B}}(r\imath,\rho w) \underset{5.1.2.}{=} \overline{K_{\mathbb{B}}(\rho w, r\imath)} \\ &\underset{5.3.5.}{=} \frac{n!}{\pi^n}\frac{1}{(1 - r\overline{(\rho w)_1})^{n+1}} = \frac{n!}{\pi^n}\frac{1}{(1 - (r\imath)\overline{(\rho w)})^{n+1}} \\ &\underset{(\diamondsuit)}{=} \frac{n!}{\pi^n}\frac{1}{(1 - (r\rho^{-1}\imath)\overline{w})^{n+1}} = \frac{n!}{\pi^n}\frac{1}{(1 - z \cdot \overline{w})^{n+1}} \end{aligned}$$

($\diamondsuit$): Note that $(r\imath)\overline{(\rho w)}$ is in fact given by the inner product in $\mathbb{C}^n$ and so the further conversion is allowed since ρ is unitary ($\rho^* = \rho^{-1}$), i.e.

$$(r\imath)\overline{(\rho w)} = \langle r\imath, \rho w\rangle_{\mathbb{C}^n} = \left\langle \rho^*(r\imath), w\right\rangle_{\mathbb{C}^n} = \left\langle \rho^{-1}(r\imath), w\right\rangle_{\mathbb{C}^n} = (r\rho^{-1}\imath)\overline{w}$$

■

5.3.7. Proposition:

The Bergman metric for the ball $\mathbb{B} = \mathbb{B}(0,1) \subseteq \mathbb{C}^n$ is given by

$$g_{ij}(z) = \frac{n+1}{(1-|z|^2)^2}\left[\left(1-|z|^2\right)\delta_{ij} + \overline{z_i}z_j\right] \tag{5.15.}$$

Proof:

Since $K(z,z) = \dfrac{n!}{\pi^n}\dfrac{1}{(1-|z|^2)^{n+1}}$, we can compute

$$\log\left(\frac{n!}{\pi^n}\frac{1}{(1-|z|^2)^{n+1}}\right) = \log\left(\frac{n!}{\pi^n}\right) + \log\left(\frac{1}{(1-|z|^2)^{n+1}}\right) \text{ and so}$$

$$
\begin{aligned}
\frac{\partial}{\partial \overline{z_j}} \log\left(K(z,z)\right) &= 0 + \frac{\partial}{\partial \overline{z_j}} \log\left(\frac{1}{\left(1-|z|^2\right)^{n+1}}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial x_j} \log\left(\frac{1}{(1-|z|^2)^{n+1}}\right) + i\frac{1}{2}\frac{\partial}{\partial y_j} \log\left(\frac{1}{(1-|z|^2)^{n+1}}\right) \\
&= \frac{1}{2}\frac{1}{1-|z|^2}(-2)(n+1)x_j + i\frac{1}{2}\frac{1}{1-|z|^2}(-2)(n+1)y_j \\
&= \frac{n+1}{1-|z|^2}(x_j+iy_j) = \frac{n+1}{1-|z|^2}z_j
\end{aligned}
$$

Furthermore, we get

$$
\begin{aligned}
\frac{\partial}{\partial z_k}\frac{n+1}{1-|z|^2}z_j &= \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)\frac{n+1}{1-|z|^2}z_j \\
&= \frac{1}{2}\frac{\partial}{\partial x_k}\frac{n+1}{1-|z|^2}(x_j+iy_j) - i\frac{1}{2}\frac{\partial}{\partial y_k}\frac{n+1}{1-|z|^2}(x_j+iy_j) \\
&= \frac{1}{2}\frac{2(n+1)}{(1-|z|^2)^2}x_k z_j + \frac{1}{2}\frac{n+1}{1-|z|^2}\delta_{kj} - \\
&\quad - i\frac{1}{2}\frac{2(n+1)}{(1-|z|^2)^2}y_k z_j - i\frac{1}{2}\frac{n+1}{1-|z|^2}\delta_{kj} i \\
&= \frac{(n+1)}{(1-|z|^2)^2}\overline{z_k}z_j + \frac{n+1}{1-|z|^2}\delta_{kj} \\
&= \frac{(n+1)}{(1-|z|^2)^2}\left[\left(1-|z|^2\,\delta_{kj} + \overline{z_k}z_j\right)\right]
\end{aligned}
$$

■

5.3.8. Corollary:

The Bergman metric for the disc (i.e. the ball in dimension one) is

$$
g_{kj}(z) = \frac{2}{(1-|z|^2)^2} \quad (j=k=1) \tag{5.16.}
$$

This is the well-known Poincaré or Poincaré-Bergman metric.

5.3.9. Proposition:

The Bergman kernel for the polydisc $\mathbb{P}^n(0,1) \subseteq \mathbb{C}^n$ is the product

$$K(z,w) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{(1 - z_j \overline{w_j})^2} \tag{5.17.}$$

Proof:

By 5.3.1., the Bergman kernel for the disc $\mathbb{B}(0,1)$ is given by

$$K_{\mathbb{B}(0,1)}(z_1, w_1) = \frac{1}{\pi} \frac{1}{(1 - z_1 \overline{w_1})^2} \; (z_1, w_1 \in \mathbb{C})$$

Since the polydisc $\mathbb{P}^n(0,1)$ is nothing other than the cartesian product of discs, i.e. $\mathbb{P}^n(0,1) = \mathbb{B}(0,1) \times \cdots \times \mathbb{B}(0,1)$,we can calculate the Bergman kernel of $\mathbb{P}^n(0,1)$ by taking the product of the one-dimensional kernel and by the uniqueness of the Bergman kernel it is the one of $\mathbb{P}^n(0,1)$.

$$\begin{aligned} K_{\mathbb{P}^n(0,1)}(z,w) &= \frac{1}{\pi} \frac{1}{(1 - z_1 \overline{w_1})^2} \cdot \frac{1}{\pi} \frac{1}{(1 - z_2 \overline{w_2})^2} \cdots \frac{1}{\pi} \frac{1}{(1 - z_n \overline{w_n})^2} \\ &= \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{(1 - z_j \overline{w_j})^2}. \end{aligned}$$

■

5.4 Riemann Mapping Theorem & the Bergman Kernel

In this section we will show the relationship between the Riemann mapping theorem and the Bergman kernel of the unit disk $\mathbb{D} \subseteq \mathbb{C}$. One could say: They depend on each other.
For this purpose we first repeat the definition of a conformal map as well as the one of a simply connected domain in $\mathbb{C}$, state the Riemann mapping theorem and give a reference for a detailed proof.

5.4.1. Definition (Conformal Mapping):

Let U, V be open domains in $\mathbb{C}$, and $h \colon U \to V$ holomorphic.
h is called a conformal (or biholomorphic) mapping if it is one-to-one and onto. The fact that h is supposed to be injective implies that h' is nowhere zero on U. As a result, h^{-1} is also holormorphic.

5.4.2. Definition (Simply Connected):

A domain $U \subseteq \mathbb{C}$ is simply connected if any closed curve in U can be continuously deformed to a point.

5.4.3. Remark:

(1) Two open sets U and V in $\mathbb{C}$ are homeomorphic if there is a one-to-one, onto and continuous function $f \colon U \to V$ with $f^{-1} \colon V \to U$ also continuous. Such a function f is called a homeomorphism from U to V.

(2) There are lot of equivalent conditions to the definition of simply connected domains. We state here only a few. For the full list as well as a proof see [Has06, Satz 4.37, page 119].

$\Omega \subseteq \mathbb{C}$ is a simply connected domain $\Leftrightarrow \Omega$ is homeomorphic to $\mathbb{D}$
$\Leftrightarrow \overline{\mathbb{C}} \setminus \Omega$ is connected $\Leftrightarrow \forall f \in \mathcal{H}$ with $f(z) \neq 0, \forall z \in \Omega$exists a $h \in \mathcal{H}$ such that $h^2 = f$i.e. there exists a holomorphic square root.

5.4.4. Theorem (Riemann mapping theorem):

Formulation (1):
If Ω is an open subset, $\Omega \neq \mathbb{C}$, and if Ω is homeomorphic to $\mathbb{D}$, then Ω is conformally equivalent to$\mathbb{D}$.
That is, there is a holomorphic mapping $\phi\colon \Omega \to \mathbb{D}$ which is one-to-one and onto, such that $\phi(a) = 0$ for a point $a \in \Omega$.
Formulation (2):
If Ω is an open subset of $\mathbb{C}$, $\Omega \neq \mathbb{C}$, and if Ω is simply connected, then U is conformally equivalent to $\mathbb{D}$.

Proof:

For a proof and all technical ingredients see [Has06].

■

The Bergman space $A^2(\Omega)$ is an important tool in the study of conformal mappings. The reason is that the Bergman kernel function of a bounded simply connected domain is related to the Riemann mapping function as we will see in a moment.

5.4.5. Lemma:

Let Ω, $\Omega \neq \mathbb{C}$, be a bounded simply connected domain. Then a conformal mapping $\phi\colon \Omega \to \mathbb{D}$ can be recaptured from the Bergman kernel $K(z, w)$ of Ω.

Proof:

For fixed $a \in \Omega$ by the Riemann mapping theorem 5.4.4. there is a unique biholomorphic mapping $\phi\colon \Omega \to \mathbb{D}$, $\phi(z) = w$ such that
$\phi(a) = 0$ and $\phi'(a) > 0$ $(*)$.
For $w, z \in \Omega$ we know by 3.2.2. (transformation of the Bergman kernel)

$$K_\Omega(z, w) = \phi'(z) \cdot K\left(\phi(z), \phi(w)\right)_{\mathbb{D}} \cdot \overline{\phi'(w)}$$

Hence, by (5.12.) we get

$$K_\Omega(z, w) = \phi'(z) \cdot \frac{1}{\pi} \frac{1}{\left(1 - \phi(z)\,\overline{\phi(w)}\right)^2} \cdot \overline{\phi'(w)}.$$

Letting $w = a$ in this formula and using the facts $(*)$ lead to

$$K_\Omega(z,a) = \phi'(z) \cdot \frac{1}{\pi} \cdot \phi'(a) \tag{$**$}$$

hence,

$$\pi \cdot K_\Omega(z,a) = \phi'(z) \cdot \phi'(a)$$

Letting $z = a$ in this formula reveals that

$$\pi \cdot K_\Omega(a,a) = \Big(\phi'(a)\Big)^2.$$

Since $\phi'(a) > 0$, we may deduce that

$$\phi'(a) = \sqrt{\pi K_\Omega(a,a)}$$

Inserting into $\phi'(a)$ into $(**)$ gives the classical formula:

$$\phi'(z) = \sqrt{\frac{1}{K_\Omega(a,a)}} K_\Omega(z,a)$$

Then, if we integrate this equation we find the conformal mapping in terms of the Bergman kernel, i.e.

$$\phi(z) = \sqrt{\frac{1}{K_\Omega(a,a)}} \int_a^z K_\Omega(w,a)\ dw$$

This completes the proof. ∎

Thus, learning things about the Bergman kernel is going to yield information about conformal mappings.
We now turn the tables. Since, by the Riemann mapping theorem, each simply connected domain, which is not equal to $\mathbb{C}$, is mapped conformally onto the open unit disk $\mathbb{D}$, we can find the Bergman kernel for an arbitrary simply connected domain Ω in terms of the associated conformal mapping function, as we will see in the proof of the following theorem.

5.4.6. Theorem:

Let $\phi\colon \Omega \to \mathbb{D}$ be a conformal mapping function, $\Omega \neq \mathbb{C}$, Ω a simply connected domain. Then the Bergman kernel is given by

$$K_\Omega(z,w) = \frac{1}{\pi}\frac{\phi'(z)\cdot\overline{\phi'(w)}}{\left(1-\phi(z)\cdot\overline{\phi(w)}\right)^2} \text{ for } z,w \in \Omega \tag{5.18.}$$

Proof:

Let $f \in A^2(\mathbb{D})$. To show that the Bergman kernel can be described in terms of the Riemann mapping function ϕ we introduce a linear operator $H\colon A^2(\mathbb{D}) \to A^2(\Omega)$ and show that it is unitary. Using the unitarity of H will lead us to the stated formula of the $K_\Omega(z,w)$ above.
To do so, assign $f \longmapsto Hf$ where H is the linear mapping on Ω defined by

$$(Hf)(z) := f\left(\phi(z)\right)\cdot\phi'(z) \tag{$\diamondsuit$}$$

for $z \in \Omega$.

We are going to show that $H\colon A^2(\mathbb{D}) \to A^2(\Omega)$ is an isometric operator. Therefore, let $z = x+iy \in \Omega$ arbitrary. We already know that the Jacobian of ϕ is given by

$$\left|\phi'(z)\right|^2 = \det\begin{pmatrix} \frac{\partial \operatorname{Re}(\phi)}{\partial x} & \frac{\partial \operatorname{Re}(\phi)}{\partial y} \\ \frac{\partial \operatorname{Im}(\phi)}{\partial x} & \frac{\partial \operatorname{Im}(\phi)}{\partial y}\end{pmatrix}.$$

Then by the formula of change of variables we get by setting $\phi(z) = w, w = u+iv$,

$$\begin{aligned}\iint\limits_{\Omega} |(Hf)(z)|^2 \, dx\, dy &= \iint\limits_{\Omega} |f(\phi(z))|^2\cdot\left|\phi'(z)\right| \, dx\, dy \\ &= \iint\limits_{\mathbb{D}} |f(w)|^2 \, du\, dv, \text{ i.e.}\end{aligned}$$

$$\langle Hf, Hf\rangle_{A^2(\Omega)} = \langle f, f\rangle_{A^2(\mathbb{D})}$$

Similarly, take $g \in A^2(\Omega)$ and assign $g \longmapsto Lg$, where L is the linear mapping on $\mathbb{D}$ defined by

$$(Lg)(w) := g(\psi(w)) \cdot \psi'(w) \qquad (\diamondsuit\diamondsuit)$$

for $w \in \mathbb{D}$ and ψ is the inverse mapping of ϕ, i.e. $\psi(w) = z$. As above we conclude that $L\colon A^2(\Omega) \to A^2(\mathbb{D})$ is also an isometric operator. Since $\phi(\psi(w)) = \phi(z) = w$ and $\psi(\phi(z)) = \psi(w) = z$, we get combing $(\diamondsuit)$ and $(\diamondsuit\diamondsuit)$ that H and L are inverse to each other. Thus, since they are both isometric operators, H is an unitary one.

Now we are in the position to calculate the Bergman kernel $K_\Omega(z, w)$. Therefore, we fix $z \in \Omega$. Let $f \in A^2(\mathbb{D})$. Then we get, using the reproducing property of the Bergman kernels k^Ω and $k^\mathbb{D}$

$$\begin{aligned} f(\phi(z)) \cdot \phi'(z) = (Hf)(z) &= \left\langle Hf, k_z^\Omega \right\rangle_{A^2(\Omega)} \\ &\underset{\text{H unitary}}{=} \left\langle f, H^* k_z^\Omega \right\rangle_{A^2(\mathbb{D})} = \left\langle f, H^{-1} k_z^\Omega \right\rangle_{A^2(\mathbb{D})} \end{aligned}$$

and

$$\begin{aligned} f(\phi(z)) \cdot \phi'(z) &= \left\langle f, k_{\phi(z)}^\mathbb{D} \right\rangle_{A^2(\mathbb{D})} \cdot \phi'(z) \\ &= \left\langle f, \overline{\phi'(z)} \cdot k_{\phi(z)}^\mathbb{D} \right\rangle_{A^2(\mathbb{D})}. \end{aligned}$$

Hence, combining these formulas implies

$$H^{-1} k_z^\Omega = \overline{\phi'(z)} \cdot k_{\phi(z)}^\mathbb{D}$$

or equivalently by using the unitarity of H we get

$$k_z^\Omega = H\overline{\phi'(z)} \cdot k_{\phi(z)}^\mathbb{D}.$$

It remains to calculate $K_\Omega(z, w)$ by using the last formula and (5.12.) with unitarity of H. For $z, w \in \Omega$ we get

$$
\begin{aligned}
K_\Omega(z,w) &= \left\langle k^\Omega_w, k^\Omega_z \right\rangle_{A^2(\Omega)} = \left\langle H\overline{\phi'(z)}\cdot k^{\mathbb{D}}_{\phi(w)}, H\overline{\phi'(z)}\cdot k^{\mathbb{D}}_{\phi(z)} \right\rangle_{A^2(\Omega)} \\
&= \phi'(z)\cdot\overline{\phi'(w)} \left\langle k^{\mathbb{D}}_{\phi(w)}, k^{\mathbb{D}}_{\phi(z)} \right\rangle_{A^2(\mathbb{D})} = \phi'(z)\cdot\overline{\phi'(w)}\,\underbrace{K_{\mathbb{D}}(\phi(z),\phi(w))}_{=\frac{1}{\pi}\frac{1}{\left(1-\phi(z)\overline{\phi(w)}\right)^2}} \\
&= \frac{1}{\pi}\frac{\phi'(z)\cdot\overline{\phi'(w)}}{\left(1-\phi(z)\overline{\phi(w)}\right)^2}
\end{aligned}
$$

■

5.5 BERGMAN KERNEL OF THE FOCK SPACE

In this section we focus on a weighted $\mathfrak{L}^2$ space and calculate its Bergman kernel.

5.5.1. Definition:

Let $\Omega \subseteq \mathbb{R}^n, \mathbb{C}^n$ be a domain.

(1) A weight function or weighting function $\omega \colon \Omega \to \mathbb{R}^+$ is a non-negative measurable function.

(2) A weight is a positive measure such as $\omega(x)d\lambda(x)$, where $d\lambda(x)$ denotes the Lebesgue measure.

(3) Let $1 \leq p < \infty$. We define the weighted $\mathfrak{L}^p$-space $\mathfrak{L}^p(\Omega, \omega)$ as the space of all measurable functions u such that

$$\|u\|_{\mathfrak{L}^p(\Omega,\omega)} : = \left(\int\limits_{\Omega} |u(x)|^p \, \omega(x) d\lambda(x) \right)^{\frac{1}{p}} < \infty. \tag{5.19.}$$

We will focus on a weighted $\mathfrak{L}^2$ - space.

5.5.2. Definition (Fock space):

The Fock space $\mathfrak{F} = A^2\left(\mathbb{C}^n, e^{-|z|^2}\right)$ consists of holomorphic functions f on $\mathbb{C}^n$, i.e. entire functions on $\mathbb{C}^n$, that are square-integrable with respect to the weight function $e^{-|z|^2}$, i.e. the functions satisfy the following condition:

$$\mathfrak{F} = A^2\left(\mathbb{C}^n, e^{-|z|^2}\right) =$$

$$= \left\{ f \text{ holomorphic on } \mathbb{C}^n \colon \int\limits_{\mathbb{C}^n} |f(z)|^2 \cdot e^{-|z|^2} d\lambda(x) < \infty \right\}. \tag{5.20.}$$

The inner product on $\mathfrak{F}$ is defined as

$$\langle f, g \rangle_{\mathfrak{F}} : = \int\limits_{\mathbb{C}^n} f(z) \cdot \overline{g(z)} \omega_n d\lambda(z) \tag{5.21.}$$

with

$$\omega_n = \pi^{-n} \cdot e^{-z\cdot\bar{z}} \text{ and } z\cdot\bar{z} = \sum_{j=1}^{n} z_j \cdot \overline{z_j} = \sum_{j=1}^{n} x_j^2 + y_j^2.$$

If we define μ via $d\mu(z) := \omega_n d\lambda(z)$, we can write the inner product as

$$\langle f, g\rangle_{\mathfrak{F}} := \int_{\mathbb{C}^n} f(z) \cdot \overline{g(z)} d\mu(z). \qquad (5.22.)$$

From the definition is clear that f belongs to $\mathfrak{F}$ whenever $\langle f, f\rangle_{\mathfrak{F}}$. The corresponding norm is then given by $\|f\|_{\mathfrak{F}} = \sqrt{\langle f, f\rangle_{\mathfrak{F}}}$.

5.5.3. Theorem:

The Fock space $\mathfrak{F}$ is a Hilbert space.

Proof:
$\mathfrak{F}$ obvious is a subspace of $\mathfrak{L}^2\left(\mathbb{C}^n, e^{-|z|^2}\right)$. The only thing left to prove is that $\mathfrak{F}$ is closed. But this is an analogous proof as in 5.1.3. Corollary. ■

5.5.4. Theorem:

The monomials $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index, form an orthogonal basis in $\mathfrak{F}$ and

$$\|z^\alpha\|_{\mathfrak{F}}^2 = \alpha!$$

Proof:
First of all, it is clear from the definition of $\mathfrak{F}$ and the theory of complex analysis, that z^α is in $\mathfrak{F}$. Similarly as in the proof of 5.3.1. one shows that the monomials are orthogonal and that each function $f \in \mathfrak{F}$ orthogonal to all z^α is 0, hence $\{z^\alpha : \alpha \text{ a multi-index }\}$ is a basis.
Finally, we calculate the norm of the monomials z^α:

$$\begin{aligned}
\|z^\alpha\|^2 &= \int\limits_{\mathbb{C}^n} |z^\alpha|^2 \, e^{-|z|^2} \frac{1}{\pi^n} d\lambda(z) \\
&= \frac{1}{\pi^n} \cdot \int\limits_{\mathbb{C}} \cdots \int\limits_{\mathbb{C}} \left| z_1^{\alpha_1} \right|^{2\alpha_1} e^{-|z_1|^2} \cdot\cdot \left| z_n^{\alpha_n} \right|^{2\alpha_n} e^{-|z_n|^2} d\lambda(z_1) \cdot\cdot d\lambda(z_n) \\
&\underset{\text{Fubini}}{=} \frac{1}{\pi^n} \cdot \int\limits_{\mathbb{C}} \left| z_1^{\alpha_1} \right|^{2\alpha_1} e^{-|z_1|^2} d\lambda(z_1) \cdots \int\limits_{\mathbb{C}} \left| z_n^{\alpha_n} \right|^{2\alpha_n} e^{-|z_n|^2} d\lambda(z_n) = (\spadesuit)
\end{aligned}$$

Using polar coordinates, we get for $1 \leq m \leq n$:

$$\begin{aligned}
\frac{1}{\pi} \cdot \int\limits_{\mathbb{C}} z_m^{\alpha_m} \overline{z_m^{\alpha_m}} e^{-z_m \overline{z_m}} d\lambda(z_m) &= \frac{1}{\pi} \int\limits_0^{2\pi} \int\limits_0^{\infty} r^{\alpha_1} r^{\alpha_1} \underbrace{e^{i(m-m)\varphi}}_{=1} e^{-r^2} r dr d\varphi \\
&= \frac{1}{\pi} \int\limits_0^{2\pi} \int\limits_0^{\infty} r^{2\alpha_m} e^{-r^2} r dr d\varphi \\
&= \frac{1}{\pi} 2\pi \int\limits_0^{\infty} r^{2\alpha_m} e^{-r^2} r dr = 2 \int\limits_0^{\infty} r^{2\alpha_m} e^{-r^2} r dr \\
&\underset{\substack{r^2 = s \\ 2rdr = ds}}{=} \int\limits_0^{\infty} s^{\alpha_m} e^{-s} ds = \Gamma(\alpha_m + 1) = \alpha_m!
\end{aligned}$$

Hence,

$$(\spadesuit) = \alpha_1! \cdots \alpha_n! = \alpha!$$

■

5.5.5. Corollary:

The set $\left\{ u_\alpha(z) = \frac{z^\alpha}{\|z^\alpha\|}, z \in \mathbb{C}^n, \alpha \in \mathbb{N}_0^n \right\}$ forms an ONB of $\mathfrak{F}$.

Since $\mathfrak{F}$ is a closed subspace of the Hilbert space $\mathfrak{L}^2 \left(\mathbb{C}^n, e^{-|z|^2} \right)$ there exists an orthogonal projection

$$P_{\mathfrak{F}} \colon \mathfrak{L}^2 \left(\mathbb{C}^n, e^{-|z|^2} \right) \to \mathfrak{F} \left(\mathbb{C}^n, e^{-|z|^2} \right)$$

that coincides with the Bergman projection

$$P_B \colon \mathfrak{L}^2\left(\mathbb{C}^n, e^{-|z|^2}\right) \to A^2\left(\mathbb{C}^n, e^{-|z|^2}\right).$$

Hence, by 5.1.13. Theorem $P_{\mathfrak{F}}$ is given by

$$(P_{\mathfrak{F}}f)(z) = \int_{\mathbb{C}^n} K(z,w)f(w)d\mu(w)$$

for all $f \in \mathfrak{L}^2\left(\mathbb{C}^n, e^{-|z|^2}\right)$ and defines a function in the Fock space $\mathfrak{F}$.
So, if $f \in \mathfrak{F}$ then f has the following representation

$$f(z) = \int_{\mathbb{C}^n} K(z,w)f(w)d\mu(w)$$

with $d\mu(w) = \frac{1}{\pi^n}e^{-|w|^2}d\lambda(w)$ and $K(z,w)$ the Bergman kernel of $\mathfrak{F}$.

By, 5.1.10.Theorem $K(z,w)$ can be written as

$$K(z,w) = \sum_{k=1}^{\infty} \phi_k(z)\overline{\phi_k(w)}$$

where $\{\phi_k\}_{k=1}^{\infty}$ is an ONB for $\mathfrak{F}$.

Since $K(z,w)$ is holomorphic in z and anti-holomorphic in w and $K(z,w) = \overline{K(w,z)}$ holds, we can insert our specific ONB u_α and then the Bergman kernel of $\mathfrak{F}$ is of the following form:

$$\begin{aligned} K(z,w) &= \sum_{\alpha} \frac{z^\alpha}{\|z^\alpha\|} \cdot \frac{\overline{w}^\alpha}{\|w^\alpha\|} = \sum_{\alpha} \frac{z^\alpha}{\sqrt{\alpha!}} \cdot \frac{\overline{w}^\alpha}{\sqrt{\alpha!}} \\ &= \sum_{\alpha} \frac{(z \cdot \overline{w})^\alpha}{\alpha!} = e^{z \cdot \overline{w}} \end{aligned}$$

So, we proved the following proposition

5.5.6. Proposition:

The Bergman kernel of the Fock space is given by

$$K(z, w) = e^{z \cdot \overline{w}} \tag{5.23.}$$

In the following section we will focus on the Cauchy-Riemann equation ($\overline{\partial}$ -equation) and investigate the properties of canonical solution operator to $\overline{\partial}$, the solution with minimal $\mathfrak{L}^2$ -norm and show connections to the Bergman kernel.

5.6 BERGMAN KERNEL AND THE CANONICAL SOLUTION OPERATOR TO $\overline{\partial}$

The following contains a discussion of the solution operator to $\overline{\partial}$ restricted to holomorphic functions in one complex variable as well as a proof that the solution operator is a Hilbert- Schmidt operator.

Our aim is to solve the inhomogeneous Cauchy-Riemann equation

$$\frac{\partial u}{\partial \overline{z}} = g \text{ or } \overline{\partial} u = g \tag{5.24.}$$

where $g \in A^2(\mathbb{D})$ and $\frac{\partial}{\partial \overline{z}}$ is defined as in 3.1.2. Remark.
For this purpose let $S\colon A^2(\mathbb{D}) \to \mathfrak{L}^2(\mathbb{D})$ be defined by

$$S(g)(z) = \int\limits_{\mathbb{D}} K(z,w)g(w)\overline{(z-w)}d\lambda(w) \tag{5.25.}$$

where $K(z,w)$ denotes the Bergman kernel of $A^2(\mathbb{D})$.
We show that S is the canonical solution to $\overline{\partial}$.

Using the reproducing property of the Bergman kernel as well as the representation of the Bergman projection $P\colon \mathfrak{L}^2 \to A^2((D))$ (see 5.1.13. Theorem) leads to

$$S(g)(z) = \overline{z}g(z) - P(\tilde{g})(z) \tag{5.26.}$$

with $\tilde{g}(w) = \overline{w}g(w)$ and $\tilde{g} \in \mathfrak{L}^2(\mathbb{D})$.

5.6.1. Claim:

$S(g)$ is a solution to the inhomogeneous Cauchy –Riemann equation.

Proof:

$$\frac{\partial}{\partial \overline{z}} S(g)(z) = \frac{\partial}{\partial \overline{z}} \left(\overline{z}g(z) - P(\widetilde{g})(z)\right)$$

$$\underset{\substack{\text{product}\\\text{rule}}}{=} \underbrace{\frac{\partial}{\partial \overline{z}}\overline{z}}_{=1}\, g(z) + \overline{z}\cdot \underbrace{\frac{\partial}{\partial \overline{z}} g(z)}_{\substack{=0\\ \text{(holomorphy}\\ \text{of} g)}} - \underbrace{\frac{\partial}{\partial \overline{z}}(P(\widetilde{g})(z))}_{\substack{=0\\ \text{(holomorphy}\\ \text{of } P(\widetilde{g}))}}$$

$$= g(z)$$

hence, $\overline{\partial}S(g) = g$ and so $S(g)$ is a solution. ■

In addition we have $S(g) \perp A^2(\mathbb{D})$, because for arbitrary $f \in A^2(\mathbb{D})$ we get:

$$\langle Sg, f\rangle_{\mathfrak{L}^2} = \left\langle \widetilde{g} - P(\widetilde{g}), f\right\rangle = \langle \widetilde{g}, f\rangle - \left\langle P(\widetilde{g}), f\right\rangle \underset{P=P^*}{=} \langle \widetilde{g}, f\rangle - \langle \widetilde{g}, Pf\rangle$$

$$\underset{\substack{Pf=f\\ f\in A^2(\mathbb{D})}}{=} \langle \widetilde{g}, f\rangle - \langle \widetilde{g}, f\rangle = 0$$

5.6.2. Definition:

The operator $S\colon A^2(\mathbb{D}) \to \mathfrak{L}^2(\mathbb{D})$ is called to canonical solution operator to $\overline{\partial}$.

5.6.3. Proposition:

S is a compact operator.

Proof:

To show that S is a compact operator we consider the adjoint operator S^* and prove that S^*S is compact, which implies that S is compact (see [Has07, Satz 7. 9, page 52] for further details).

Let $g \in A^2(\mathbb{D})$ and $f \in \mathfrak{L}^2(\mathbb{D})$. Then we have

$$\begin{aligned}
\langle Sg, f\rangle_{\mathfrak{L}^2} &= \int_{\mathbb{D}} Sg(z)\overline{f(z)}d\lambda(z) \\
&\underset{(5.25.)}{=} \int_{\mathbb{D}} \left(\int_{\mathbb{D}} K(z,w)g(w)\overline{(z-w)}d\lambda(w) \right) \overline{f(z)}d\lambda(z) \\
&\underset{\text{Fubini}}{=} \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \underbrace{K(z,w)}_{=\overline{K(w,z)}} \overline{f(z)(z-w)}d\lambda(z) \right) g(w)d\lambda(w) \\
&= \int_{\mathbb{D}} g(w)\overline{\left(\int_{\mathbb{D}} K(w,z)(z-w)f(z)d\lambda(z) \right)}d\lambda(w) \\
&= \langle g, S^*f\rangle_{A^2} \qquad .
\end{aligned}$$

Hence,

$$S^*f(w) = \int_{\mathbb{D}} K(w,z)(z-w)f(z)d\lambda(z)$$

and S^*f is clearly holomorphic and $S^*\colon \mathfrak{L}^2(\mathbb{D}) \to A^2(\mathbb{D})$.

Now we set for $n \in \mathbb{N}_0$

$$c_n^2 \colon = \int_{\mathbb{D}} |z|^{2n}\, d\lambda(z) = \frac{\pi}{n+1} \tag{5.27.}$$

Indeed,

$$\int_{\mathbb{D}} |z|^{2n}\, d\lambda(z) = \int_{\mathbb{D}} \int_0^{2\pi} r^{2n} r dr d\theta = 2\pi \left. \frac{r^{n+2}}{2n+2} \right|_0^1 = \frac{2\pi}{2n+2} = \frac{\pi}{n+1}$$

Furthermore, we define the normalized monomials by

$$\phi_n(z) \colon = \frac{z^n}{c_n}, \ (n \in \mathbb{N}_0). \tag{5.28.}$$

Since $\{\phi_n(z)\colon n \in \mathbb{N}_0\}$ forms a complete ONS in $A^2(\mathbb{D})$ (confirm 5.3.1. Theorem) we can express the Bergman kernel in the following form

$$K(z,w) = \sum_{n=0}^{\infty} \phi_n(z) \cdot \overline{\phi_n(w)} = \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{c_n^2}. \tag{5.29.}$$

Next we compute the Bergman projection of $\widetilde{\phi_n}(w) = \overline{w}\frac{w^n}{c_n}$:

$$\begin{aligned}
P(\widetilde{\phi_n})(z) &\underset{5.1.13.}{=} \int_{\mathbb{D}} K(z,w)\widetilde{\phi_n}(w)d\lambda(w) \underset{(5.29.)}{=} \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{c_k^2}\,\overline{w}\,\frac{w^n}{c_n} d\lambda(w) \\
&\underset{(\spadesuit)}{=} \sum_{k=1}^{\infty} \frac{z^{k-1}}{c_k^2} \int_{\mathbb{D}} \frac{\overline{w}^k}{w}^n c_n d\lambda(w) \underset{(\diamondsuit)}{=} \frac{z^{n-1}}{c_{n-1}^2}\frac{c_n^2}{c_n} \\
&= \frac{c_n z^{n-1}}{c_{n-1}^2}
\end{aligned} \tag{$\heartsuit$}$$

($\spadesuit$) We can change integration and summation since we have convergence on compact subsets of $\mathbb{D}$.
($\diamondsuit$) Since we have an ONB, every integral over $\mathbb{D}$ is 0, except when $k = n$. Hence, we have

$$S(\phi_n)(z) \underset{\substack{(5.26.)\\(\heartsuit)}}{=} \overline{z}\phi_n(z) - \frac{c_n z^{n-1}}{c_{n-1}^2},\ n \in \mathbb{N}$$

Applying S^* leads to:

$$S^*S(\phi_n)(w) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}(z-w)\left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2}\right) d\lambda(z)$$

The last integral is computed in two steps:

(1) Multiplication by z:

$$\begin{aligned}
&\int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}\left(\frac{\overline{z}z^{n+1}}{c_n} - \frac{c_n}{c_{n-1}^2}\right) d\lambda(z) \\
&= \int_{\mathbb{D}} \frac{z^{n+1}}{c_n} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2} d\lambda(z) - \frac{c_n z^n}{c_{n-1}^2} \int_{\mathbb{D}} z^n \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} d\lambda(z)
\end{aligned}$$

$$\underset{z^{n+1}\overline{z}^{n+1}=|z|^{2n+2}}{\underset{\text{ONB}:k=n}{=}} \frac{w^n}{c_n^3}\underbrace{\int_{\mathbb{D}} |z|^{2n+2}\, d\lambda(z)}_{=c_{n+1}^2} - \frac{w^n}{c_{n-1}^2 c_n^2}\underbrace{\int_{\mathbb{D}} |z|^{2n}\, d\lambda(z)}_{=c_n^2}$$

$$= \left(\frac{c_{n+1}^2}{c_n^3} - \frac{c_n}{c_{n-1}^2}\right) w_n$$

(2) Multiplication by w:

$$w\int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{w^k\overline{z}^k}{c_k^2}\left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2}\right) d\lambda(z)$$

$$= w\int_{\mathbb{D}} \frac{z^n}{c_n}\sum_{k=0}^{\infty} \frac{w^k\overline{z}^{k+1}}{c_k^2} d\lambda(z) - w\int_{\mathbb{D}} \frac{c_n z^{n-1}}{c_{n-1}^2}\sum_{k=0}^{\infty}\frac{w^k z^k}{c_k^2} d\lambda(z)$$

$$= w\int_{\mathbb{D}} \frac{z^n}{c_n}\sum_{k=1}^{\infty} \frac{w^{k-1}\overline{z}^{k}}{c_{k-1}^2} d\lambda(z) - w\int_{\mathbb{D}} \frac{c_n z^{n-1}}{c_{n-1}^2}\sum_{k=1}^{\infty}\frac{w^{k-1} z^{k-1}}{c_{k-1}^2} d\lambda(z)$$

$$\underset{\text{ONB}:k=n}{=} w\int_{\mathbb{D}} \frac{z^n}{c_n}\frac{w^{n-1}\overline{z}^{n}}{c_{n-1}^2} d\lambda(z) - w\int_{\mathbb{D}} \frac{c_n z^{n-1}}{c_{n-1}^2}\frac{w^{n-1} z^{n-1}}{c_{n-1}^2} d\lambda(z)$$

$$= \frac{ww^{n-1}c_n}{c_n c_{n-1}^2}\int_{\mathbb{D}} \underbrace{z^n\overline{z}^n}_{=|z|^{2n}} d\lambda(z) - \frac{ww^{n-1}c_n}{c_{n-1}^2 c_{n-1}^2}\int_{\mathbb{D}} \underbrace{z^{n-1}\overline{z}^{n-1}}_{=|z|^{2n-2}} d\lambda(z)$$

$$= \frac{ww^{n-1}c_n}{c_n c_{n-1}^2}\underbrace{\int_{\mathbb{D}} z^n\overline{z}^n d\lambda(z)}_{=c_n^2} - \frac{ww^{n-1}c_n}{c_{n-1}^2 c_{n-1}^2}\underbrace{\int_{\mathbb{D}} z^{n-1}\overline{z}^{n-1} d\lambda(z)}_{=c_{n-1}^2}$$

$$= \frac{ww^{n-1}c_n}{c_{n-1}^2} - \frac{ww^{n-1}c_n}{c_{n-1}^2} = w\left(\frac{w^{n-1}c_n}{c_{n-1}^2} - \frac{w^{n-1}c_n}{c_{n-1}^2}\right) = 0$$

It follows that

$$S^*S(\phi_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\right)\phi_n(w)\ (n \geq 1). \tag{5.30.}$$

For $n = 0$, an analogous computation shows

$$S^*S(\phi_0)(w) = \frac{c_1^2}{c_0^2}\phi_0(w). \tag{5.31.}$$

Furthermore, let $\phi \in A^2(\mathbb{D})$. Then we can write ϕ by 2.3.11. Theorem as $\phi = \sum\limits_{k=0}^{\infty} \langle\phi, \phi_k\rangle_{A^2} \phi_k$. Applying S^*S leads to

$$\begin{aligned} S^*S\phi &= \sum_{k=0}^{\infty} \left\langle S^*S\phi, \phi_k \right\rangle \phi_k = \left\langle S^*S\phi, \phi_0 \right\rangle \phi_0 + \sum_{k=1}^{\infty} \left\langle S^*S\phi, \phi_k \right\rangle \phi_k \\ &\underset{\text{orthonormality}}{=} \frac{c_1^2}{c_0^2} \left\langle \phi, \phi_0 \right\rangle \phi_0 + \sum_{k=1}^{\infty} \left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right) \left\langle \phi, \phi_k \right\rangle \phi_k \end{aligned} \tag{5.32.}$$

Since

$$\begin{aligned} \frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} &\underset{(5.27.)}{=} \frac{\frac{\pi}{n+2}}{\frac{\pi}{n+1}} - \frac{\frac{\pi}{n+1}}{\frac{\pi}{n}} = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n-2)}{(n+2)(n+1)} \\ &= \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} \overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

it follows that S^*S is compact and S too. (For all ingredients for the penultimate conclusion confirm [Has07]). ■

In the proof of 5.6.3. Proposition we showed another important result, namely the spectral representation of S^*S:

5.6.4. Proposition:

Let $S\colon A^2(\mathbb{D}) \to \mathfrak{L}^2(\mathbb{D})$ be the canonical solution operator for $\overline{\partial}$ and $(\phi_k)_{k\in\mathbb{N}_0}$ the normalized monomials.Then

$$S^*S\phi = \frac{c_1^2}{c_0^2} \left\langle \phi, \phi_0 \right\rangle \phi_0 + \sum_{k=1}^{\infty} \left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right) \left\langle \phi, \phi_k \right\rangle \phi_k \tag{5.33.}$$

5.6.5. Proposition:

The canonical solution operator $S\colon A^2(\mathbb{D}) \to \mathfrak{L}^2(\mathbb{D})$ is even Hilbert-Schmidt.

Proof:

By [Has07, Chapter 8] it is enough to show that the function $N\colon (z,w) \mapsto K(z,w)\overline{(z-w)}$ belongs to $\mathfrak{L}^2(\mathbb{D}\times\mathbb{D})$. So, we have to prove that

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|\overline{z}-\overline{w}|^2}{|1-z\overline{w}|^4} d\lambda(z) d\lambda(w) < \infty,$$

since the Bergman kernel of $\mathbb{D}$ is given by $K(z,w) = \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2}$.

First of all, the following inequality holds for $z, w \in \mathbb{D}$

$$|z-w| \leq |1-z\overline{w}| \qquad (\clubsuit)$$

Indeed,

$$\begin{aligned}
&|z|^2 < 1\\
\Leftrightarrow &|z|^2\left(1-|w|^2\right) \leq 1-|w|^2 \Leftrightarrow |z|^2 - |z|^2\cdot|w|^2 \leq 1-|w|^2\\
\Leftrightarrow &|z|^2+|w|^2 \leq 1+|z|^2\cdot|w|^2 \Leftrightarrow |z|^2 - z\overline{w} - w\overline{z} + |w|^2 \leq 1+|z|^2\cdot|w|^2 - z\overline{w} - w\overline{z}\\
\Leftrightarrow &(z-w)\overline{(z-w)} \leq (1-\overline{z}w)(1-z\overline{w}) \Leftrightarrow |z-w|^2 \leq |1-z\overline{w}|^2
\end{aligned}$$

Hence,

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|\overline{z}-\overline{w}|^2}{|1-z\overline{w}|^4} d\lambda(z) d\lambda(w) \overset{\clubsuit}{\leq} \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{1}{|1-z\overline{w}|^2} d\lambda(z) d\lambda(w).$$

Introducing polar coordinates $z = re^{i\alpha}$ and $w = se^{i\varphi}$ leads to $|1-z\overline{w}|^2 = (1-rse^{i(\alpha-\varphi)})\cdot(1-rse^{-i(\alpha-\varphi)}) = 1-2rs\cos(\alpha-\varphi)+r^2s^2$. Inserting this into the last integral, we get

$$\begin{aligned}
\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{1}{|1-z\overline{w}|^2} d\lambda(z) d\lambda(w) &= \int_0^1\int_0^1\int_0^{2\pi}\int_0^{2\pi} \frac{rs}{1-2rs\cos(\alpha-\varphi)+r^2s^2} d\alpha\, d\varphi\, dr\, ds\\
&= \int_0^1\int_0^1\int_0^{2\pi}\int_0^{2\pi} \underbrace{\frac{1-r^2s^2}{1-2rs\cos(\alpha-\varphi)+r^2s^2}}_{=\text{Poisson kernel}} \frac{rs}{1-r^2s^2} d\alpha\, d\varphi\, dr\, ds \qquad (\spadesuit)
\end{aligned}$$

From complex analysis we know that integration of the Poisson kernel with respect to α yields

$$\int_0^{2\pi} \frac{1-\rho^2}{2\rho\cos(\alpha-\phi)+\rho^2}\,d\alpha = 2\pi$$

Therefore we have by setting $\rho = rs$

$$(\spadesuit) = (2\pi)^2 \int_0^1\int_0^1 \frac{rs}{1-r^2 2s^2}\,dr\,ds \underset{\substack{u=1-r^2s^2\\ du=-2rs^2dr}}{=} -(2\pi)^2 \int_0^1 \left[\int_1^{1-s^2} \frac{rs}{u2rs}\,du\right] ds$$

$$= -(2\pi)^2 \int_0^1 \frac{1}{2s}\left[\int_1^{1-s^2} \frac{1}{u}\,du\right] ds = -(2\pi)^2 \int_0^1 \frac{1}{2s}\log(1-s^2)\,ds$$

$$\underset{\substack{v=1-s^2\\ dv=-2s\,ds}}{=} (2\pi)^2 \int_1^0 \frac{\log(v)}{4s^2}\,dv \underset{s^2=1-v}{=} \frac{(2\pi)^2}{4}\underbrace{\int_1^0 \frac{\log(v)}{1-v}\,dv}_{=\frac{\pi^2}{6}}$$

$$= \frac{(2\pi)^2}{4}\frac{\pi^2}{6} = \frac{\pi^4}{6} < \infty,$$

i.e.

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|\overline{z}-\overline{w}|^2}{|1-z\overline{w}|^4}\,d\lambda(z)\,d\lambda(w) < \frac{\pi^4}{6} < \infty$$

■

5.6.6. Remark:

In the case of several variables the corresponding operator S^*S is more complicated. To generalize the above results, we need to introduce complex differential forms. For a detailed study on this matter confirm [Has01].

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