DIPLOMARBEIT

Titel der Diplomarbeit

The classical and distributional Denjoy integral

angestrebter akademischer Grad

Magister der Naturwissenschaften (Mag. rer. nat.)

Verfasser:	Clemens Sämann
Matrikel-Nummer:	0648728
Studienrichtung:	A 405 Mathematik
Betreuer:	Ao. Univ.-Prof. Dr. Günther Hörmann

Wien, am 23. September 2011

Abstract

The main goal in this thesis is to develop an integration theory where every derivative of a function $F: [a,b] \to \mathbb{R}$ is integrable and the fundamental theorem of calculus (FTC) holds, i.e., $\int_a^x F' = F(x) - F(a)$ for all $x \in [a,b]$.
First in Chapter 0 we show how the Riemann and Lebesgue integrals fail to have this property. Then in Chapter 1 we discuss one solution of this problem, namely the classical Denjoy integral. It relies on ideas from measure theory and is a direct generalization of the Lebesgue integral. Every primitive of a Lebesgue integrable function is absolutely continuous (AC) and one can define Lebesgue integrability in terms of these primitives. For the classical Denjoy integral we mimic this by generalizing the notion of absolute continuity and defining the classical Denjoy integral in a descriptive way in terms of the primitives. There is a second solution to the problem of integrating all derivatives and the FTC - the distributional Denjoy integral, which is discussed in Chapter 2. It uses the theory of distributions (generalized functions) and the integral is also defined in terms of primitives. In this case the primitives are continuous functions with limits at infinity (or just continuous functions in the case of bounded intervals) and the differentiation is understood in the distributional sense. Finally in Chapter 3 we discuss further properties, applications and generalizations of the classical and distributional Denjoy integral.

Zusammenfassung

Das Ziel dieser Diplomarbeit ist es eine Integrationstheorie zu entwickeln, in der alle Ableitungen von Funktionen $F : [a,b] \to \mathbb{R}$ integrierbar sind und der Hauptsatz der Differential- und Integralrechnung (HDI) gilt, d.h., $\int_a^x F' = F(x) - F(a)$ für alle $x \in [a,b]$.
Als erstes werden wir in Kapitel 0 zeigen, dass das Riemann und Lebesgue Integral diese Eigenschaft nicht haben. Dann in Kapitel 1 präsentieren wir eine Lösung dieses Problems, nämlich das klassische Denjoy Integral. Es beruht auf Ideen aus der Maßtheorie und ist eine direkte Verallgemeinerung des Lebesgue Integrals. Jede Stammfunktion einer Lebesgue integrierbaren Funktion ist absolut stetig (AC) und man kann Lebesgue Integrierbarkeit in Form von diesen Stammfunktionen definieren. Wir imitieren dies indem wir den Begriff der absoluten Stetigkeit verallgemeinern und so das klassische Denjoy Integral in einer beschreibenden Form durch die Stammfunktionen definieren. Es gibt eine zweite Lösung zu dem Problem alle Ableitungen zu integrieren so dass der HDI gilt - das distributionelle Denjoy integral. Es wird in Kapitel 2 besprochen. Es baut auf der Distributionentheorie (verallgemeinerte Funktionen) auf und wird auch in Form von Stammfunktionen definiert. In diesem Fall sind die Stammfunktionen stetige Funktionen mit Grenzwerten bei unendlich (oder einfach nur stetige Funktionen, falls wir nur beschränkte Intervalle betrachten) und die Differentiation ist im distributionellem Sinne zu verstehen. Im Kapitel 3 besprechen wir schlussendlich weitere Eigenschaften, Anwendungen und Verallgemeinerungen des klassischen und distributionellen Denjoy Integrals.

Acknowledgment

First of all I would like to thank my advisor Günther Hörmann for his steady support and for always finding time for me, despite of his many commitments. He gave me a lot of freedom to work on my own but never left me alone with my problems. Furthermore I appreciated his rigorous accuracy when he was reading my thesis and his many comments on it. At last I have to express how much I enjoyed working with him and that I value him as an advisor and good friend.

Secondly I want thank all members of the DIANA group for their inspiring teaching, which I had the pleasure to enjoy right from my first semester at the University of Vienna.

At this point I would like to thank my family for their support and for making it possible to study mathematics. Especially I am very grateful to my parents, Beata and Karl, for trusting in me and for sharing my enthusiasm. Moreover I appreciated the keen interest my sister Carla has shown in my studies and for her several helpful comments about the language and style of my thesis. Finally I want to express how important the encouragement, optimism and trust of my partner in life Veronika were for me. I am also indebted for her help with many aspects of this thesis.

Thank you!

Contents

0. Introduction

In this chapter we will observe a common deficit of the Riemann and Lebesgue integral. We will show that some derivatives are not integrable and so motivate why it is desirable to develop new integration theories. The classical Denjoy integral was developed by Arnaud Denjoy (1884-1974) around 1912. So we will start with the classical Denjoy integral in Chapter 1 and then in Chapter 2 we will see a completely different approach which leads to similar results but with much less effort.

0.1. Motivation

0.1.1. Fundamental Theorem of Calculus (FTC)

At a first glance it seems that differentiation and integration are 'inverse' operations. For example in a lecture by Isaac Barrow (1630-1677)[1] the idea is mentioned that "finding tangents is inverse to quadrature (calculation of areas)". A more explicit example dates back to 1686 when Leibniz stated $\frac{d}{dx} \int f(x)dx = f(x)$. See [Wuß08, p.453] and [Sti01, p.159].

If we take a closer look at the relationship between differentiation and integration we need to be more precise. So we state the classical results for the Riemann and Lebesgue integral below. Proofs can be found in [KS04]. (Actually these are only parts of the FTC).

0.1.1 Theorem (FTC-Riemann)
Let $F : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and F' Riemann-integrable then

$$\int_a^x F' = F(x) - F(a) \quad \forall x \in [a, b] \ .$$

0.1.2 Theorem (FTC-Lebesgue)
Let $F : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and F' bounded then

$$\int_a^x F' = F(x) - F(a) \quad \forall x \in [a, b] \ . \tag{0.1}$$

In Theorem 0.1.1 the condition cannot be dropped that F' is Riemann-integrable and in Theorem 0.1.2 that F' is bounded. Otherwise the theorems are no longer true as the next example shows.

[1] Isaac Barrow was a professor at the University of Cambridge and teacher of Newton. From 1662-1669 he was the first occupier of the Lucasian chair at Cambridge. See [Wuß08, p.452].

0.1.2. A motivating example

0.1.3 Example

Consider the function $F: \ [0,1] \to \mathbb{R}, \quad F(x) := \begin{cases} 0 & x = 0, \\ x^2 \sin(\frac{\pi}{x^2}) & 0 < x \leq 1. \end{cases}$

Then F *is differentiable everywhere on* $[0,1]$ *with derivative* $F' : [0,1] \to \mathbb{R}$ *given by*

$F'(x) = \begin{cases} 0 & x = 0, \\ 2x \sin(\frac{\pi}{x^2}) - \frac{2\pi}{x} \cos(\frac{\pi}{x^2}) & 0 < x \leq 1. \end{cases}$

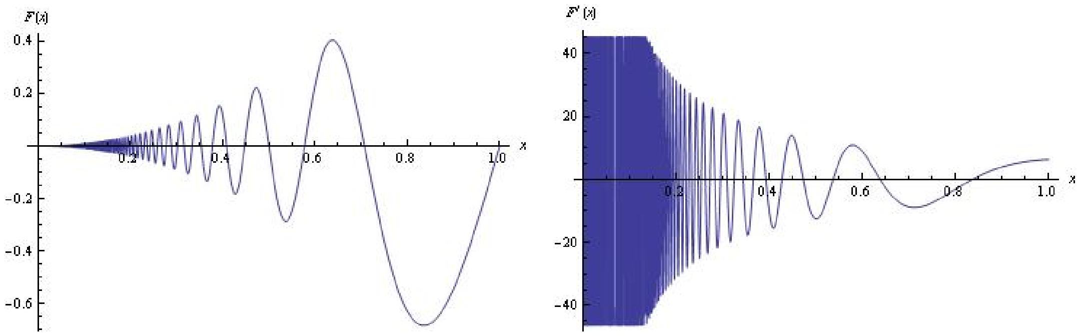

Figure 0.1.: F(x) and F'(x)

Clearly F' *is unbounded on* $[0,1]$ *and hence not Riemann integrable. Now we show that* F' *is not Lebesgue integrable:*
For $0 < a < b \leq 1$ *the derivative* F' *is continuous on* $[a,b]$ *and hence Riemann integrable on* $[a,b]$*. Therefore*

$$\int_a^b F' = b^2 \sin(\frac{\pi}{b^2}) - a^2 \sin(\frac{\pi}{a^2}) \ .$$

Now we set $a_n := \frac{\sqrt{2}}{\sqrt{4n+1}}$, $b_n := \frac{\sqrt{2}}{\sqrt{4n}}$ *then the intervals* $[a_n, b_n]$ *are pairwise disjoint and in* $[0,1]$ *for* $n \geq 1$*. Additionally we have* $F(a_n) = a_n^2$ *and* $F(b_n) = 0$*. So we get*

$$\int_0^1 |F'| \geq \sum_{n=1}^{\infty} \int_{a_n}^{b_n} |F'| \geq \sum_{n=1}^{\infty} \frac{2}{4n+1} = \infty \ .$$

The last equality follows from the fact that the sum is a general harmonic series. So F' *is not Lebesgue integrable. On the other hand it is not hard to see that F' is improperly Riemann integrable. In the next Chapter we will examine the relationship between improper Riemann integrability and Denjoy integrability in Subsection 1.4.2.*

0.1.4 Remark

We want a FTC of the form:
Let $F: \ [a,b] \to \mathbb{R}$ *differentiable on* $[a,b]$ *then*

$$\int_a^x F' = F(x) - F(a) \quad \forall x \in [a,b] \ .$$

This leads us directly to the classical and distributional Denjoy integral.

0.2. Basic notation and conventions

1. **Intervals**
   $[a, b]$ will always be a closed interval in $\mathbb{R}$ with $a < b$.
   Two intervals $[a, b]$, $[c, d]$ are non-overlapping if $|[a, b] \cap [c, d]| \leq 1$ i.e. if they have at most one point of intersection.
   Every finite collection of non-overlapping intervals in $[a, b]$ denoted by $([a_n, b_n])_{n=1}^N$ is assumed to be ordered e.g. $a \leq a_1 < b_1 \leq a_2 \ldots \leq a_N < b_N \leq b$.

2. **Lebesgue measure**
   μ^* denotes the Lebesgue outer measure and μ denotes the Lebesgue measure on $\mathbb{R}$.
   A condition holds almost everywhere (a.e.) on a set E if the subset of E where the condition does not hold has Lebesgue measure zero. A condition holds nearly everywhere (n.e.) on a set E if the subset of E where the condition does not hold is countable.

3. **Function spaces**
   Let $\Omega \subseteq \mathbb{R}$ be arbitrary and $k \in \mathbb{N} \cup \{\infty\}$.
   By $\mathcal{B}(\Omega)$ we denote the set of all bounded real-valued functions on Ω and by $\mathcal{C}^{(k)}(\Omega)$ we denote the set of all k-times continuously differentiable real-valued functions on Ω. For simplicity we write $\mathcal{C}(\Omega)$ instead of $\mathcal{C}^{(0)}(\Omega)$. Moreover we denote by $\mathcal{R}([a, b])$ the Riemann integrable functions on the interval $[a, b]$.

4. **Miscellaneous**
   We will use the abbreviation **WLOG** for ”without loss of generality”.
   We will often abbreviate the left-hand side of an equation or expression by **LHS** respectively the right-hand side by **RHS**.

1. The classical Denjoy integral

In this chapter we will develop the theory of the classical Denjoy integral. First we introduce the concepts of absolute continuity (AC) and bounded variation (BV) for functions $F: [a,b] \to \mathbb{R}$ and prove some basic properties. Then in the second section we examine differentiability of BV and AC functions, the integrability of their derivatives and the FTC for the Lebesgue integral and thereby providing a new approach to the Lebesgue integral. In the third section we generalize the concept of absolute continuity and we establish similar results in this new and broader setting. Then, finally, in the fourth and last section we are in a position to define the classical Denjoy integral and prove the FTC in this setting. In this part of the work we will follow mainly the books [Gor94] and [KS04].

1.1. Bounded variation and absolute continuity

Now we will define absolute continuity and bounded variation on an interval. Later on we will define what these two notions mean for arbitrary Lebesgue measurable subsets of an interval. For now it is important that we understand these basic concepts very well because on our way to the classical Denjoy integral we need more complex notions of bounded variation and absolute continuity as we will see in Section 1.3. So in this section we will collect a lot of useful facts about absolutely continuous functions and functions of bounded variation and examine many examples to get accustomed to these concepts.

1.1.1 Definition
Let $F: [a,b] \to \mathbb{R}$.

1. *F is called absolutely continuous (AC) on $[a,b]$:⇔ $\forall \epsilon > 0\ \exists \delta > 0$ such that*

$$\sum_{n=1}^{N} |F(b_n) - F(a_n)| < \epsilon \tag{1.1}$$

whenever $([a_n, b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a,b]$ such that $\sum_{n=1}^{N} (b_n - a_n) < \delta$.

2. *$V(F,[a,b]) := \sup \{\sum_{n=1}^{N} |F(b_n) - F(a_n)| : N \in \mathbb{N}$, $([a_n, b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a,b]$ $\}$ is called the variation of F on $[a,b]$.*
F is of bounded variation (BV) on $[a,b]$:⇔ $V(F,[a,b]) < \infty$.

Other equivalent definitions are possible, see for example for BV [KS04, p.172] where it is assumed that $([a_n, b_n])_{n=1}^{N}$ is a partition of $[a,b]$ or a really different definition would be: F is BV if and only if $F = F_1 - F_2$ where F_1, F_2 are monotonically increasing, see [Gor94, p.52, Thm. 4.4] or Proposition 1.1.7. For AC another definition would be: F is AC if and only if F is differentiable a.e. and the derivative is Lebesgue-integrable and equation (0.1) holds. See [Gor94, p.61, Thm. 4.15] and later on in Section 1.2 we will prove this equivalence in Theorem 1.2.16.

1.1.1. Basic results and examples

A simple consequence from the definition is that every AC function is continuous. To get used to these definitions we will prove this as well as other easy facts about AC and BV functions.

1.1.2 Example
Here we collect some basic results and examples which will be useful throughout this section.

1. *Let $F: [a,b] \to \mathbb{R}$ be AC on $[a,b]$. Then F is continuous on $[a,b]$ and since $[a,b]$ is closed and bounded (compact) F is uniformly continuous on $[a,b]$.*

   Proof: Let $\epsilon > 0$ and $x \in [a,b]$. There exists $\delta > 0$ such that $\sum_{n=1}^{N} |F(b_n) - F(a_n)| < \epsilon$ whenever $([a_n,b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^{N}(b_n - a_n) < \delta$. So let $y \in [a,b]$ with $|x-y| < \delta$ then $|F(x) - F(y)| < \epsilon$ and hence F is continuous at x and since x was arbitrary in $[a,b]$, F is continuous on $[a,b]$. □

   The converse is not true: there are uniformly continuous functions which are not AC. See Example 3.

2. *Let $F: [a,b] \to \mathbb{R}$ be monotone. Then $V(F,[a,b]) = |F(b) - F(a)| < \infty$. So every monotone function is of bounded variation on a compact interval.*

   Proof: WLOG let F be monotonically increasing and let $([a_n,b_n])_{n=1}^{N}$ be a finite collection of non-overlapping intervals in $[a,b]$. Now we define $x_{2n-1} := a_n$ and $x_{2n} := b_n$ $(1 \leq n \leq N)$. Then

   $$\sum_{n=1}^{N} |F(b_n) - F(a_n)| \leq \sum_{n=1}^{2N-1} |F(x_{n+1}) - F(x_n)| = F(x_{2N}) - F(x_1) = F(b_N) - F(a_1) \leq F(b) - F(a)\ .$$

   So $V(F,[a,b]) = F(b) - F(a)$. This proves the result for the case where F is monotonically increasing. Analogously if F is monotonically decreasing we get that $V(F,[a,b]) = F(a) - F(b)$. □

3. *The function* $F: [0,1] \to \mathbb{R}, \quad F(x) := \begin{cases} 0 & x = 0, \\ x\sin(\frac{\pi}{x}) & 0 < x \leq 1, \end{cases}$
   is not AC, see [Gor94, p.50]. It is also not BV: we modify the proof from [Gor94, p.50] and so we define $a_n := \frac{2}{4n+1}$, $b_n := \frac{2}{4n}$ for $n \in \mathbb{N}$, then $F(a_n) = a_n$ and $F(b_n) = 0$ and

additionally $\forall N \in \mathbb{N}$ $([a_n, b_n])_{n=1}^N$ *is a finite collection of non-overlapping intervals in* $[0,1]$*. Now we calculate*

$$\sum_{n=1}^{N} |F(b_n) - F(a_n)| = \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} \frac{2}{4n+1} \to \infty \ (N \to \infty) \ .$$

Here we used again the divergence of the general harmonic series as in Example 0.1.3. So $V(F,[0,1]) = \infty$*. But* F *is continuous on* $[0,1]$ *and hence uniformly continuous. Compare this result with Example 1 which states that every AC function is uniformly continuous.*

4. *The function* $F: [0,1] \to \mathbb{R}$ *from Example 0.1.3 given by* $F(x) := \begin{cases} 0 & x = 0, \\ x^2 \sin(\frac{\pi}{x^2}) & 0 < x \leq 1, \end{cases}$ *is continuous and differentiable everywhere on* $[0,1]$ *but not BV and not AC. An argument very similar to that in Example 3 works here.*

5. *The function* $F: [0,1] \to \mathbb{R}$, $F(x) := \begin{cases} 0 & x = 0, \\ x^2 \sin(\frac{\pi}{x}) & 0 < x \leq 1, \end{cases}$ *is continuous, differentiable everywhere on* $[0,1]$*, AC and BV (cf. [Gor94, p.284,Exercise 4.2]).*

6. *Let* $G: [0,1] \to \mathbb{R}$, $G(x) := \begin{cases} 0 & 0 \leq x \leq \frac{1}{2}, \\ 1 & \frac{1}{2} < x \leq 1, \end{cases}$ *then* G *is not AC (not even continuous), but BV (with* $V(G,[0,1]) = 1$*) and differentiable almost everywhere on* $[0,1]$*. So there are functions which are BV but not AC, on the other hand every AC function is BV. See Lemma 1.1.5,2.*

7. *The characteristic function* $\chi_{\mathbb{Q}}$ *is not BV on* $[0,1]$*:*
For $n \in \mathbb{N}$ *let* $a_n \in]\frac{1}{n+1}, \frac{1}{n}[\setminus\mathbb{Q}$*. Then* $\forall N \in \mathbb{N}$ *we get that* $([a_n, \frac{1}{n}])_{n=1}^N$ *is a finite collection of non-overlapping intervals in* $[0,1]$ *and so:*

$$\sum_{n=1}^{N} |\chi_{\mathbb{Q}}(\frac{1}{n}) - \chi_{\mathbb{Q}}(a_n)| = \sum_{n=1}^{N} 1 = N$$

Therefore $V(\chi_{\mathbb{Q}}, [0,1]) = \infty$*. Of course it is also not AC since it is not continuous. So there are functions which are bounded but not BV. On the other hand every BV function is bounded. See Lemma 1.1.5,1.*

1.1.2. Further properties of BV and AC functions

In this subsection we will prove some useful results which deal with the actual computation of the variation of a function and how the variation behaves on subintervals. Furthermore we establish the relationship between bounded, BV and AC functions. First we will show that the variation has an additive behavior when splitting the domain into subintervals.

1.1.3 Lemma
Let $F: [a,b] \to \mathbb{R}$ *and* $c \in]a,b[$*. Then* $V(F,[a,b]) = V(F,[a,c]) + V(F,[c,b])$*.*

Proof: The proof requires only to closely inspect the definition of the variation. We will split it into two parts: LHS less than or equal to the RHS and then LHS $\geq$ RHS.

$\leq$ Let $([a_n, b_n])_{n=1}^N$ be a finite collection of non-overlapping intervals in $[a, b]$ then there exists a minimal $M \in \{1, \ldots, N\}$ such that $a_M \geq c$ or $a_M < c$ and $b_M \geq c$. Then in the first case ($a_M \geq c$) nothing is to do. In the second case ($a_M < c$ and $b_M \geq c$) we replace $[a_M, b_M]$ by $[a_M, c]$ and $[c, b_M]$. In both cases the first set of intervals is a finite collection of non-overlapping intervals in $[a, c]$ and the second set is a finite collection of non-overlapping intervals in $[c, b]$. So we get for the first case (and for the second case it is essentially the same):

$$\sum_{n=1}^{N} |F(b_n) - F(a_n)| \leq \sum_{n=1}^{M-1} |F(b_n) - F(a_n)| + \sum_{n=M}^{N} |F(b_n) - F(a_n)| \leq V(F, [a, c]) + V(F, [c, b]) \ .$$

Then taking the supremum over all finite collections of non-overlapping intervals in $[a, b]$ proves that $V(F, [a, b]) \leq V(F, [a, c]) + V(F, [c, b])$.

$\geq$ Let $([a_n, b_n])_{n=1}^M$ be a finite collection of non-overlapping intervals in $[a, c]$ and let $([a_n, b_n])_{n=M+1}^N$ be a finite collection of non-overlapping intervals in $[c, b]$ then $([a_n, b_n])_{n=1}^N$ is a finite collection of non-overlapping intervals in $[a, b]$ and so:

$$\sum_{n=1}^{M} |F(b_n) - F(a_n)| \ + \sum_{n=M+1}^{N} |F(b_n) - F(a_n)| = \sum_{n=1}^{N} |F(b_n) - F(a_n)| \leq V(F, [a, b]) \ .$$

Then taking the supremum as in the first part of the proof yields the desired result.

□

From the above lemma we get that if F is BV on $[a, b]$ then F is BV on every subinterval of $[a, b]$ and on the other hand if F is BV on $[a, c]$ and $[c, b]$ then F is BV on $[a, b]$. We also get a practical way to calculate the variation of 'nice' functions. See the following example.

1.1.4 Example

Let $F: \ [a, b] \to \mathbb{R}$ be continuous and having a finite number of local extrema, say $a = a_0 < a_1 < \ldots < a_N = b$. Then the variation of F is given by $V(F, [a, b]) = \sum_{n=1}^N |F(a_n) - F(a_{n-1})|$. We only need to show that $F|_{[a_{i-1}, a_i]}$ is monotone for $(1 \leq n \leq N)$ because then our claim would follow from Example 1.1.2,2 and Lemma 1.1.3. So assume WLOG that $F: \ [a, b] \to \mathbb{R}$ is continuous with local extrema only in a and b. Since $[a, b]$ is compact F attains its global maximum and global minimum on this interval. WLOG we suppose that the global minimum is in a and the global maximum is in b (otherwise consider $-F$ instead of F). Now our claim is that F is monotonically increasing on $[a, b]$. We will show this by contradiction: assume there exists $a \leq x_1 < x_2 \leq b$ with $F(x_1) > F(x_2)$. Now set $g := F|_{[x_1, b]}$ and $m := \min_{x \in [x_1, b]} g(x)$ which will be attained in x_0 i.e. $F(x_0) = g(x_0) = m$. Since the global maximum is at b we get $F(x_0) = m \leq F(x_2) < F(x_1) \leq F(b)$. From this inequality we can conclude that $x_1 < x_0 < b$ and so F has a local minimum in x_0 which is clearly a contradiction to our assumption that F has only one local minimum at a.

For example the function $F:\ [0,3]\to\mathbb{R}$ *given by* $F(x):=(x-1)^2$ *has variation 5 since it is monotonically decreasing on* $[0,1]$ *and monotonically increasing on* $[1,3]$ *and* $F(0)=1$, $F(1)=0$ *and* $F(3)=4$.

Now we will prove the easy fact that every BV function is bounded and the more important result that every AC function is BV.

1.1.5 Lemma (AC⇒BV⇒bounded)
Let $F:\ [a,b]\to\mathbb{R}$.

1. *If F is BV then F is bounded.*
2. *If F is AC then F is BV.*

Proof:

1. Let F be BV on $[a,b]$ and let $x,x_0\in[a,b]$. Then by the triangle inequality $|F(x)|\leq|F(x)-F(x_0)|+|F(x_0)|\leq V(F,[a,b])+|F(x_0)|<\infty$. So F is bounded on $[a,b]$.

2. Let F be AC on $[a,b]$ and set $\epsilon:=1$ in the definition of absolute continuity. Then $\exists\delta>0$ such that
$$\sum_{n=1}^{N}|F(b_n)-F(a_n)|<1, \tag{1.2}$$
whenever $([a_n,b_n])_{n=1}^N$ is a finite collection of non-overlapping intervals in $[a,b]$ satisfying $\sum_{n=1}^N(b_n-a_n)<\delta$. So let $M\in\mathbb{N}$ such that $\frac{b-a}{M}<\delta$ and set $a_n:=a+n\frac{b-a}{M}$ for $n=0,1,\ldots,M$. Then $([a_n,a_{n+1}])_{n=0}^{M-1}$ is a finite collection of non-overlapping intervals in $[a,b]$ and by Lemma 1.1.3 and by Equation (1.2) we get
$$V(F,[a,b])=\sum_{n=0}^{M-1}V(F,[a_n,a_{n+1}])\overset{(1.2)}{\leq}\sum_{n=0}^{M-1}1=M<\infty\ .$$
So F is BV on $[a,b]$.

□

From the proof of the first part of the above lemma we get the following inequality just by taking the supremum over all $x\in[a,b]$:
$$\|F\|_\infty\leq|F(x_0)|+V(F,[a,b])\quad\forall x_0\in[a,b]\ . \tag{1.3}$$

This inequality holds for all functions $F:\ [a,b]\to\mathbb{R}$.
From the results so far one could be tempted to speculate whether if a function is continuous and BV then it is AC. That is not true as the Cantor function (see for example [Gor94, p.14, Thm. 1.21 and p.284f. Exc. 4.3] shows. The Cantor function is a continuous and monotonically increasing (and hence BV) function, yet not AC. It is an important counterexample in other respects too. For example it is differentiable a.e. but the FTC does not hold.

1.1.6 Example

If F is Lipschitz continuous on $[a,b]$ with Lipschitz constant $L > 0$ then F is AC. This can be seen very easily: Let $\epsilon > 0$ and choose $0 < \delta < \frac{\epsilon}{L}$ then $\sum_{n=1}^{N} |F(b_n) - F(a_n)| \leq L \sum_{n=1}^{N} (b_n - a_n) \leq L\delta < \epsilon$ whenever $([a_n, b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^{N} (b_n - a_n) < \delta$. The converse is not true as the example $F(x) := \sqrt{x}$ on $[0,1]$ shows. It is a useful exercise to show that F is AC but not Lipschitz continuous. We start with showing that F is not Lipschitz continuous on $[0,1]$. Assume there is a $L > 0$ such that $\forall x, y \in [0,1]$: $|\sqrt{x} - \sqrt{y}| \leq L|x-y|$, then we set $y := 0$ and choose $0 < x < \frac{1}{L^2}$. Now from our Lipschitz estimate we get with these values of x, y that $|\sqrt{x} - \sqrt{y}| = \sqrt{x} \leq L|x| = Lx$. Dividing by L and $\sqrt{x}$ gives $\frac{1}{L} \leq \sqrt{x}$, since all numbers are positive we can square this equation and get $\frac{1}{L^2} \leq x$ which is clearly a contradiction to our choice of x. So F is not Lipschitz continuous on $[0,1]$.

Now we show that F is AC on $[0,1]$ (cf. [Gor94, p.286, Exercise 4.5]). Let $0 < \alpha < 1$ and $\epsilon > 0$, choose $\delta > 0$ such that $\delta < \frac{\epsilon}{2}\sqrt{\alpha}$ and let $([a_n, b_n])_{n=1}^{N}$ be a finite collection of non-overlapping intervals in $[\alpha, 1]$ with $\sum_{n=1}^{N} (b_n - a_n) < \delta$. Then $\sum_{n=1}^{N} |F(b_n) - F(a_n)| = \sum_{n=1}^{N} |\sqrt{b_n} - \sqrt{a_n}| = \sum_{n=1}^{N} \frac{b_n - a_n}{\sqrt{b_n} + \sqrt{a_n}} \leq \sum_{n=1}^{N} \frac{b_n - a_n}{\sqrt{a_n}} \leq \frac{1}{\sqrt{\alpha}} \sum_{n=1}^{N} (b_n - a_n) < \frac{\delta}{\sqrt{\alpha}} < \frac{\epsilon}{2}$. Now let $\epsilon > 0$ and choose $M \in \mathbb{N}$ such that $\frac{1}{M} < \frac{\epsilon}{2}$, set $\alpha := \frac{1}{M^2}$ and choose δ as above i.e. $\delta < \frac{\epsilon}{2}\sqrt{\alpha} = \frac{\epsilon}{2}\frac{1}{M} < \frac{\epsilon^2}{4}$. Then let $([a_n, b_n])_{n=1}^{N}$ be a finite collection of non-overlapping intervals in $[0,1]$ and so we calculate: $\sum_{n=1}^{N} |F(b_n) - F(a_n)| = \sum_{b_n \leq \frac{1}{M^2}} |F(b_n) - F(a_n)| + \sum_{b_n > \frac{1}{M^2}} |F(b_n) - F(a_n)| \leq |F(\frac{1}{M^2}) - F(0)| + \frac{\epsilon}{2} = \frac{1}{M} + \frac{\epsilon}{2} < \epsilon$. Here we have used the above calculation for the interval $[\frac{1}{M^2}, 1]$ and the fact that F is monotone on $[0, \frac{1}{M^2}]$ and the same calculation as in the proof of 1.1.2,2.

Intuitively it is clear that the variation of a function increases if we increase the size of the interval on which we calculate the variation i.e. $x \mapsto V(F, [a,x])$ is monotonically increasing, where $a < x \leq b$. This is not hard to see: Let $a < x < y \leq b$ and let $([a_n, b_n])_{n=1}^{N}$ be a finite collection of non-overlapping intervals in $[a,x]$ then $([a_n, b_n])_{n=1}^{N}$ is of course a finite collection of non-overlapping intervals in $[a,y]$ too. Thus $\sum_{n=1}^{N} |F(b_n) - F(a_n)| \leq V(F, [a,y])$ and so by taking the supremum over all finite collections of non-overlapping intervals in $[a,x]$ we get that $V(F, [a,x]) \leq V(F, [a,y])$. If we take this observation a little further we obtain an equivalent characterization of BV, as stated in the following proposition.

1.1.7 Proposition (Jordan decomposition)

Let $F : [a,b] \to \mathbb{R}$ then F is BV if and only if there exist $F_1, F_2 : [a,b] \to \mathbb{R}$ monotonically increasing such that $F = F_1 - F_2$. The representation of F as $F = F_1 - F_2$ is called a Jordan decomposition of F.

Proof:

($\Rightarrow$) Define $F_1(x) := V(F, [a,x])$ then F_1 is monotonically increasing as observed in the calculation before the proposition and set $F_2(x) := F_1(x) - F(x)$. Now it is clear that $F = F_1 - F_2$. It remains to show that F_2 is monotonically increasing: let $a \leq x < y \leq b$ then from Lemma 1.1.3 we get $F(y) - F(x) \leq |F(y) - F(x)| \leq V(F, [x,y]) = V(F, [a,y]) - V(F, [a,x]) = F_1(y) - F_1(x)$. So $F_2(x) = F_1(x) - F(x) \leq F_1(y) - F(y) = F_2(y)$. This proves the first part.

⇐ From Example 1.1.2,2 we know that F_1, F_2 are BV and it is very easy to see that the difference of two BV functions is BV again. A more general proof will be given in Lemma 1.1.8,1.

□

A Jordan decomposition of a BV function is not unique. For example if $F : [0,1] \to \mathbb{R}$ is the identity on $[0,1]$ and we set $F_1(x) := V(F,[0,x])$ for $x \in [0,1]$ as in the proof above then we know from Example 1.1.2,2 that $F_1(x) = |F(x) - F(0)| = F(x)\ \forall x \in [0,1]$, so $F_2 = 0$. Therefore F is its own Jordan decomposition but it is not hard to see that $F(x) = 2x - x\ (x \in [0,1])$ is also a Jordan decomposition of F. We will call $F = F_1 - F_2$ standard Jordan decomposition of F if $F_1 = V([a,.])$ and $F_2 = F_1 - F$.

1.1.3. Spaces of BV and AC functions

In this subsection we will examine the structure of the spaces of all BV or AC functions on an interval. As it turns out they are Banach spaces when endowed with appropriate norms but this result has to wait until we know more about these functions. First we will show below that they are vector spaces which follows easily from the definitions. Another interesting property of BV and AC functions is that they are closed under pointwise multiplication, though not closed under composition.

1.1.8 Lemma (Vector space structure)
Let $F, G : [a,b] \to \mathbb{R}$ and let $\alpha, \beta \in \mathbb{R}$.

1. *Then*
$$V(\alpha F + \beta G, [a,b]) \le |\alpha| V(F,[a,b]) + |\beta| V(G,[a,b])\ . \tag{1.4}$$
So linear combinations of BV functions are BV.

2. *If F and G are AC on $[a,b]$ then $\alpha F + \beta G$ is AC on $[a,b]$.*

Proof: Let $([a_n,b_n])_{n=1}^N$ be a finite collection of non-overlapping intervals in $[a,b]$ then

$$\sum_{n=1}^{N} |\alpha F(b_n) + \beta G(b_n) - (\alpha F(a_n) + \beta G(a_n))| \le$$
$$\sum_{n=1}^{N} (|\alpha|\ |F(b_n) - F(a_n)| + |\beta|\ |G(b_n) - G(a_n)|) =$$
$$|\alpha| \sum_{n=1}^{N} |F(b_n) - F(a_n)| + |\beta| \sum_{n=1}^{N} |G(b_n) - G(a_n)| =: \Delta\ .$$

1. From the definition it follows that $\Delta \le |\alpha|\ V(F,[a,b]) + |\beta|\ V(G,[a,b])$. So by taking the supremum over all finite collections of non-overlapping intervals in $[a,b]$ we get the result.

2. WLOG assume that both α and β are nonzero (the other cases are clear). Let $\epsilon > 0$ then $\exists \delta_1 > 0$ such that $\sum_{n=1}^{M} |F(y_n) - F(x_n)| < \frac{\epsilon}{2|\alpha|}$ whenever $([x_n,y_n])_{n=1}^{M}$ is a finite

collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^{M}(y_n - x_n) < \delta_1$ and $\exists \delta_2 > 0$ such that $\sum_{n=1}^{L}|G(y_n) - G(x_n)| < \frac{\epsilon}{2|\beta|}$ whenever $([x_n, y_n])_{n=1}^{L}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^{L}(y_n - x_n) < \delta_2$. Now it follows from the definition that $\Delta < \epsilon$ when $\sum_{n=1}^{N}(b_n - a_n) < \min(\delta_1, \delta_2)$. So $\alpha F + \beta G$ is AC on $[a,b]$.

□

From the calculation in the above proof we also see that $V(\alpha F, [a,b]) = |\alpha| V(F, [a,b])$. This will be useful when proving norm properties of $V(\ .\ , [a,b])$. Moreover we observe that for $\alpha \geq 0$ and $\beta = 0$ in the first part of the above lemma we get equality in formula (1.4) and if we take $F = G = \chi_{\mathbb{Q}}$, $\alpha = 1$ and $\beta = -1$ then the LHS gives zero and the RHS side gives ∞. The above lemma justifies the following definition of the spaces of BV and AC functions.

1.1.9 Definition (Vector spaces of BV and AC functions)

1. *The real vector space $BV([a,b]) := \{F :\ [a,b] \to \mathbb{R} : F$ is BV$\}$ is called the* *space of functions of bounded variation on $[a,b]$.*
2. *The real vector space $AC([a,b]) := \{F :\ [a,b] \to \mathbb{R} : F$ is AC$\}$ is called the* *space of absolutely continuous functions on $[a,b]$.*

From Lemma 1.1.5,2 we know that $AC([a,b]) \subseteq BV([a,b])$. Now we want to define a norm on these spaces such that they are complete, i.e. Banach spaces. To prove this result we need more knowledge of BV and AC functions. It is also not a good idea to prove that the BV-norm is a norm right now, since it involves the L^1-norm and we have not yet shown the integrability of BV functions. On the other hand it is clear that an AC function is Lebesgue integrable since it is continuous and we are only studying functions on a bounded interval. We will show in Theorem 1.2.18 that $\|.\|_{BV}$ defines a norm on $BV([a,b])$ but we give the expression of the norm already here:

1.1.10 Definition (Norms on $BV([a,b])$ and $AC([a,b])$)
For $F \in BV([a,b])$ we define $\|F\|_{BV} := \|F\|_1 + V(F, [a,b])$.

In the following lemma we show that $BV([a,b])$ and $AC([a,b])$ are closed under pointwise multiplication of functions.

1.1.11 Lemma (Multiplication of BV and AC functions)
Let $F, G :\ [a,b] \to \mathbb{R}$.

1. *If F and G are BV then $F \cdot G$ is BV.*
2. *If F and G are AC then $F \cdot G$ is AC.*

Proof: From Lemma 1.1.5,1 we know that every BV and so every AC function is bounded, so in both cases exist $M_F > 0$ and $M_G > 0$ such that $|F(x)| \leq M_F$ and $|G(x)| \leq M_G$ $\forall x \in [a,b]$. Let $([a_n, b_n])_{n=1}^{N}$ be a finite collection of non-overlapping intervals in $[a,b]$. Then by the triangle inequality we get for $1 \leq n \leq N$:

$$|F(b_n)G(b_n) - F(a_n)G(a_n)| \leq |F(b_n)G(b_n) - F(a_n)G(b_n)| + |F(a_n)G(b_n) - F(a_n)G(a_n)| =$$
$$|G(b_n)||F(b_n) - F(a_n)| + |F(a_n)||G(b_n) - G(a_n)| \leq M_G|F(b_n) - F(a_n)| + M_F|G(b_n) - G(a_n)|$$

and this yields

$$\sum_{n=1}^{N} |F(b_n)G(b_n) - F(a_n)G(a_n)| \leq M_G \sum_{n=1}^{N} |F(b_n) - G(a_n)| + M_F \sum_{n=1}^{N} |G(b_n) - G(a_n)| =: \Delta \ .$$

1. From the definition we get that $\Delta \leq M_G V(F,[a,b]) + M_F V(G,[a,b])$. Then by taking the supremum over all finite collection of non-overlapping intervals in $[a,b]$ we get that $F \cdot G$ is BV if F and G are BV.

2. Now suppose F and G are AC then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $\sum_{n=1}^{M} |F(y_n) - F(x_n)| < \frac{\epsilon}{2M_F}$ whenever $([x_n,y_n])_{n=1}^{M}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^{M} (y_n - x_n) < \delta_1$ and $\sum_{n=1}^{M} |G(y_n) - G(x_n)| < \frac{\epsilon}{2M_G}$ whenever $([x_n,y_n])_{n=1}^{M}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^{M} (y_n - x_n) < \delta_2$. So similarly as above we get from the definition that $\Delta < \epsilon$ if $([a_n,b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^{N} (b_n - a_n) < \min(\delta_1,\delta_2)$. This shows that $F \cdot G$ is AC.

□

In the above proof we can choose $M_F := \sup_{x\in[a,b]} |F(x)| = \|F\|_\infty$ respectively $M_G := \|G\|_\infty$. So we get the following useful formula:

$$V(F \cdot G,[a,b]) \leq \|G\|_\infty V(F,[a,b]) + \|F\|_\infty V(G,[a,b]) \ . \tag{1.5}$$

As we mentioned in the beginning of this subsection $BV([a,b])$ and $AC([a,b])$ are not closed under the composition of functions. To see this it suffices to show that the composition of two AC functions is not BV. This will settle both cases. From the reverse triangle inequality it follows easily that if F is AC then $|F|$ is AC. So we take $F(x) := \sqrt{x}$ and $G(x) := \begin{cases} 0 & x = 0, \\ |x^2 \sin(\frac{\pi}{x})| & 0 < x \leq 1, \end{cases}$ both on the interval $[0,1]$. Then we know that F is AC from our calculation in Example 1.1.6 and we know that G is AC since it is just the absolute value of the AC function from Example 1.1.2, 5. Then we get for the composition $H(x) := (F \circ G)(x) = \begin{cases} 0 & x = 0, \\ x\sqrt{|\sin(\frac{\pi}{x})|} & 0 < x \leq 1. \end{cases}$ The proof that H is not BV works just as in Example 1.1.2,3. Take again $a_n := \frac{2}{4n+1}$, $b_n := \frac{2}{4n}$ for $n \in \mathbb{N}$ which then yields $H(a_n) = a_n$ and $H(b_n) = 0$ and so we get the same expression in the sum. By the same reasoning as in Example 1.1.2,3 we conclude that H is not BV.

1.1.4. Summary

Here we will give a brief summary of what we have achieved so far.

Let $LC([a,b])$ denote the Lipschitz continuous functions and $UC([a,b])$ the uniformly continuous functions on the interval $[a,b]$, then with our notation we have proven:

$$LC([a,b]) \underset{1.1.6}{\overset{1.1.6}{\subsetneq}} AC([a,b]) \underset{1.1.2,6}{\overset{1.1.5,2}{\subsetneq}} BV([a,b]) \underset{1.1.2,7}{\overset{1.1.5,1}{\subsetneq}} \mathcal{B}([a,b]), \tag{1.6}$$

$$LC([a,b]) \underset{1.1.6}{\overset{1.1.6}{\subsetneq}} AC([a,b]) \underset{1.1.2,3}{\overset{1.1.2,1}{\subsetneq}} \mathcal{C}([a,b]) \overset{(\star)}{=} UC([a,b]), \tag{1.7}$$

$$AC([a,b]) \underset{\text{before } 1.1.6}{\subsetneq} \mathcal{C}([a,b]) \cap BV([a,b]). \tag{1.8}$$

$(\star)$: See for example [AE06, p.273, III Thm. 3.13].

The reference above the subset symbol gives the result where we have proven the subset relationship and the reference below gives the result where we have shown that these are proper subsets i.e. the sets are not equal.

We also summarize these results in a table. In this table the columns are given and we have shown the results in the rows. That means for example in column 2, row 3: a continuous function is in general not AC.

	Lipschitz	continuous	AC	BV	bounded
$\Rightarrow$ Lipschitz		n	n	n	n
$\Rightarrow$ continuous	y		y	n	n
$\Rightarrow$ AC	y	n		n	n
$\Rightarrow$ BV	y	n	y		n
$\Rightarrow$ bounded	y	y	y	y	

Table 1.1.: Relationship between AC, BV and other concepts

Also we have proven that the variation is additive i.e. $V(F,[a,b]) = V(F,[a,c]) + V(F,[c,b])$ for $a < c < b$ (see Lemma 1.1.3) and that F is BV if and only if it can be expressed as the difference of two monotonically increasing functions (see Proposition 1.1.7). Then in Lemma 1.1.8 we have shown that $BV([a,b])$ and $AC([a,b])$ are real vector spaces and the variation satisfies $V(\alpha F + \beta G,[a,b]) \leq |\alpha|V(F,[a,b]) + |\beta|V(G,[a,b])$. Also $BV([a,b])$ and $AC([a,b])$ are closed under pointwise multiplication (see Lemma 1.1.11) with the estimate of the variation of the product: $V(F \cdot G,[a,b]) \leq \|G\|_\infty V(F,[a,b]) + \|F\|_\infty V(G,[a,b])$. On the other hand we observed that the composition of two BV or AC functions need not be BV respectively AC again.

Our next goal is to examine differentiability and integrability of BV and AC functions since we want to get a FTC in the form of Remark 0.1.4. As we will see we need to consider generalizations of BV and AC functions to achieve this goal. This will be the main part of our efforts in the next sections.

1.2. Differentiation and integration of BV and AC functions

After we have established basic facts about BV and AC functions in the previous section, we will now investigate analytical properties of BV and AC functions, like differentiability and integrability. We will see that every BV function (and hence every AC function) is differentiable almost everywhere. This result yields directly that every BV function is Lebesgue integrable, but we are more interested in the integrability of the derivative of a BV function. It will be a major result to show that the derivative of a BV function is Lebesgue integrable and for AC functions the FTC holds. As a by-product we get an alternative definition of Lebesgue-integrability via AC functions (the descriptive definition of the Lebesgue integral). This definition can be generalized and will lead us to the classical Denjoy integral. We will conclude this section by proving that $BV([a,b])$ and $AC([a,b])$ are Banach spaces.

1.2.1. Vitali Covering Lemma

In this subsection we will develop a main tool, the Vitali Covering Lemma, which will be used to prove that a monotone function is differentiable almost everywhere and therefore every BV (and hence also AC) function as well. The idea of a Vitali cover is that for every point (in the given set) we can find arbitrary small intervals containing that point. So from a topological point of view this is nothing new but the Vitali Covering Lemma will give us new knowledge of the properties of a Vitali Cover.

1.2.1 Definition (Vitali Cover)
Let $E \subseteq \mathbb{R}$. A collection of intervals $\mathcal{I}$ is called a Vitali cover (VC) of E if and only if $\forall x \in E$ $\forall \epsilon > 0$ $\exists I \in \mathcal{I}$ such that $x \in I$ and $\mu(I) < \epsilon$.

1.2.2 Example
Here we give some basic examples. The first one is just to illustrate the definition but the following examples are important because we will use them later on in various proofs.

1. *Let $E = [0,1]$ and set $P := [0,1] \cap \mathbb{Q}$ then $\mathcal{I} := \left\{[p - \frac{1}{n}, p + \frac{1}{n}] : p \in P, n \in \mathbb{N}\right\}$ is a (countable) Vitali cover for $[0,1]$. This follows easily since $\mathbb{Q}$ is dense in $\mathbb{R}$: let $x \in [0,1]$ and $\epsilon > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ and since P is dense in $[0,1]$ there exists $p \in P$ with $|p - x| < \frac{1}{2n}$. This is equivalent to $x \in [p - \frac{1}{2n}, p + \frac{1}{2n}] \in \mathcal{I}$ and $\mu([p - \frac{1}{2n}, p + \frac{1}{2n}]) = \frac{1}{n} < \epsilon$. So $\mathcal{I}$ is a Vitali cover of $[0,1]$.*

2. *Let E be some arbitrary subset of $\mathbb{R}$. If for every $x \in E$ we choose a strictly decreasing sequence $(y_n^x)_n$ converging to x (for example with $x < y_n^x < x + \frac{1}{n}$) then the set $\mathcal{I} := \{[x, y_n^x] : x \in E, n \in \mathbb{N}\}$ is a VC of E. To show this let $\epsilon > 0$ and $x \in E$. Since $y_n^x \to x$ $(n \to \infty)$ there exists $n \in \mathbb{N}$ such that $|y_n^x - x| < \epsilon$, so $x \in [x, y_n^x] \in \mathcal{I}$ with $\mu([x, y_n^x]) = y_n^x - x < \epsilon$. So $\mathcal{I}$ is a VC of E.*

3. *If $\mathcal{I}$ is a VC of E, so is $\tilde{\mathcal{I}} := \left\{\bar{I} : I \in \mathcal{I}\right\}$ since $I \subseteq \bar{I}$ and $\mu(\bar{I}) = \mu(I)$ $\forall I \in \mathcal{I}$ (where $\bar{I}$ denotes the closure of I).*

4. *If $\mathcal{I}$ is a VC of E and $\mu^*(E) < \infty$ then there exists an open set O such that $E \subseteq O$, $\mu(O) < \infty$ and $\tilde{\mathcal{I}} := \{I \in \mathcal{I} : I \subseteq O\}$ is also a VC of E.*

   Proof: From the definition of the Lebesgue outer measure we get the existence of an open set O with $\mu(O) < \infty$ and $E \subseteq O$. Define $\tilde{\mathcal{I}}$ as above, and let $x \in E$, $\epsilon > 0$. Then there exists $I \in \mathcal{I}$ with $x \in I$ and $\mu(I) < \epsilon$. If $I \subseteq O$ the proof is complete, if not we can decrease the length of the interval. WLOG assume that there is an open interval J contained in O such that the connected component of x is contained in J. Say $J =]u, v[$ and define $m := \min(x - u, v - x)$. Since x is in the interior of J, $m > 0$. We can find an interval $\tilde{I} \in \mathcal{I}$ such that $x \in \tilde{I}$ and $\mu(\tilde{I}) < \frac{m}{2}$. Then clearly $\tilde{I} \subseteq O$ and so $\tilde{I} \in \tilde{\mathcal{I}}$. □

The idea of the Vitali Covering Lemma is that if we are given a set in $\mathbb{R}$ with finite Lebesgue outer measure we can cover the set almost everywhere with countably many intervals in the VC, i.e. the uncovered part will have measure zero. Additionally (and this will be the part most useful to us) we can decrease the size of the uncovered part as much as we want with a finite number of intervals in the VC. At first sight the VCL does not look like a powerful tool but as we will see it is nevertheless the main ingredient in the proof of the (a.e.-)differentiability of a monotone function.

1.2.3 Lemma (Vitali Covering Lemma - VCL)
Let $E \subseteq \mathbb{R}$ with Lebesgue outer measure $\mu^*(E) < \infty$. If $\mathcal{I}$ is a VC of E then $\forall \epsilon > 0$ exists a finite collection $\{I_n \in \mathcal{I} : 1 \leq n \leq N\}$ of disjoint intervals in $\mathcal{I}$ such that $\mu^*(E \backslash \bigcup_{n=1}^{N} I_n) < \epsilon$. In addition there exists a sequence $(I_n)_n$ of disjoint intervals in $\mathcal{I}$ such that $\mu^*(E \backslash \bigcup_{n=1}^{\infty} I_n) = 0$. (Cf. [Gor94, Lem. 4.6, p.52-54]).

Proof: From Examples 1.2.2,3 and 4 we know that we can assume WLOG that every interval in $\mathcal{I}$ is closed and contained in an open set O with $\mu(O) < \infty$. We will construct a sequence of disjoint intervals $(I_n)_n$ in $\mathcal{I}$. Let $\epsilon > 0$ and take any $I_1 \in \mathcal{I}$. If $E \subseteq I_1$ the proof is complete since $\mu^*(E \backslash I_1) = 0 < \epsilon$, if not we continue this construction. Now suppose $I_1, \ldots, I_N$ are disjoint intervals in $\mathcal{I}$. If $E \subseteq \bigcup_{n=1}^{N} I_n$ we are done, if not let $\mathcal{I}_N := \left\{I \in \mathcal{I} : I \cap (\bigcup_{n=1}^{N} I_n) = \emptyset\right\}$ and $\alpha_N := \sup\{\mu(I) : I \in \mathcal{I}_N\}$. Since every I_n $(1 \leq n \leq N)$ is closed, $\bigcup_{n=1}^{N} I_n$ is closed and from the assumption we get the existence of a $y \in E \backslash (\bigcup_{n=1}^{N} I_n)$. So $\beta := \mathrm{dist}(y, \bigcup_{n=1}^{N} I_n) > 0$ and because $\mathcal{I}$ is a VC of E there exists an interval $I \in \mathcal{I}$ such that $y \in I$ and $\mu(I) < \frac{\beta}{2}$. Then clearly $I \cap (\bigcup_{n=1}^{N} I_n) = \emptyset$, so $\mathcal{I}_N \neq \emptyset$ and $\alpha_N > 0$. It is clear from the definition of α_N that there exists an interval $I_{N+1} \in \mathcal{I}_N$ such that

$$\mu(I_{N+1}) > \frac{\alpha_N}{2} \; . \tag{1.9}$$

Continue this process until either $E \subseteq \bigcup_{n=1}^{M} I_n$ for some $M \in \mathbb{N}$ or a sequence $(I_n)_n$ of disjoint intervals in $\mathcal{I}$ is constructed. In the second case we get, since the intervals are disjoint and every interval is contained in the open set O, that $\sum_{n=1}^{\infty} \mu(I_n) = \mu(\bigcup_{n=1}^{\infty} I_n) \leq \mu(O) < \infty$. From this we get that

$$\lim_{n \to \infty} \mu(I_n) = 0 \tag{1.10}$$

and hence there exists an $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \mu(I_n) < \frac{\epsilon}{5} . \tag{1.11}$$

Now we define $A := E \setminus \bigcup_{n=1}^{N} I_n$ and we will show that $\mu^*(A) < \epsilon$. For $n > N$ let J_n be the interval with the same center as I_n but $\mu(J_n) = 5\mu(I_n)$. From equation (1.11) we get that $\mu(\bigcup_{n=N+1}^{\infty} J_n) \leq \sum_{n=N+1}^{\infty} \mu(J_n) = 5 \sum_{n=N+1}^{\infty} \mu(I_n) < \epsilon$. So it suffices to show that $A \subseteq \bigcup_{n=N+1}^{\infty} J_n$. Let $x \in A$, then as above there exists an interval $I_x \in \mathcal{I}_N$ such that $x \in I_x$. We claim that $I_x \cap I_n \neq \emptyset$ for some $n > N$. Assume that $I_x \cap I_n = \emptyset \ \forall n > N$ then $I_x \in \mathcal{I}_n \ \forall n > N$ and so $\alpha_n \geq \mu(I_x) \ \forall n > N$. From equation (1.10) we conclude $0 \leq \lim_{n\to\infty} \alpha_n \leq 2 \lim_{n\to\infty} \mu(I_{n+1}) = 0$ and so $0 < \mu(I_x) \leq \alpha_n \to 0 \ (n \to \infty)$, a contradiction. Now let $M := \min \{n \in \mathbb{N} : I_x \cap I_n \neq \emptyset\}$, which exists by the previous argument and M has to be greater then N since $I_x \in \mathcal{I}_N$. From the definition of M it is clear that $I_x \in \mathcal{I}_{M-1}$. From our equation (1.9) we get that $\mu(I_x) \leq \alpha_{M-1} < 2\mu(I_M)$. Let c be the center of the interval I_M, then we estimate the distance from x to c: $|x - c| \leq \mu(I_x) + \frac{1}{2}\mu(I_M) < \frac{5}{2}\mu(I_M)$ and so $x \in J_M$. This proves that $A = E \setminus \bigcup_{n=1}^{N} I_n \subseteq \bigcup_{n=N+1}^{\infty} J_n$ and from this we get our desired result: $\mu^*(A) = \mu^*(E \setminus \bigcup_{n=1}^{N} I_n) \leq \mu(\bigcup_{n=N+1}^{\infty} J_n) \leq \sum_{n=N+1}^{\infty} J_n < \epsilon$. Since $\epsilon > 0$ was arbitrary this yields $\mu^*(A) = 0$. □

We will use the VCL in a more convenient form which we state below as a corollary.

1.2.4 Corollary
Let $E \subseteq [a,b]$, $\mathcal{I}$ a VC of E then $\forall \epsilon > 0$ exists a finite collection $\{I_n \in \mathcal{I} : 1 \leq n \leq N\}$ of disjoint intervals in $\mathcal{I}$ such that $\sum_{n=1}^{N} \mu(I_n) > \mu^*(E) - \epsilon$.

Proof: Since $E \subseteq [a,b]$, the outer measure of E is finite. So let $\epsilon > 0$ then we get from the VCL that there exists a finite collection $\{I_n \in \mathcal{I} : 1 \leq n \leq N\}$ of disjoint intervals in $\mathcal{I}$ such that $\mu^*(E \setminus \bigcup_{n=1}^{N} I_n) < \epsilon$. Now we set $B := \bigcup_{n=1}^{N} I_n$ and since B is a union of intervals it is measurable and so we deduce: $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^C) = \mu^*(E \cap B) + \mu^*(E \setminus B) < \mu^*(B) + \epsilon$. □

For some applications of the Vitali Covering Lemma we have to wait until Proposition 1.2.7. First we need the concept of lower and upper derivatives to study the differentiability of monotone functions.

1.2.2. Differentiation

Now we have the main tool to prove that a monotone function is differentiable almost everywhere but first we need the simple concept of upper and lower derivatives. Although it is standard material in analysis courses we will shortly discuss this concept here just to unify notation and conventions. Afterwards we can prove our major result in this subsection and consequently we will see that BV functions are differentiable almost everywhere.

1.2.5 Definition (Upper and lower derivatives)
Let $F: [a,b] \to \mathbb{R}$ and $x_0 \in [a,b]$. The upper derivative and the lower derivative of F at x_0 are

defined by

$$\overline{D}F(x_0) := \limsup_{x\to x_0, x\neq x_0} \frac{F(x)-F(x_0)}{x-x_0}, \tag{1.12}$$

$$\underline{D}F(x_0) := \liminf_{x\to x_0, x\neq x_0} \frac{F(x)-F(x_0)}{x-x_0}\ . \tag{1.13}$$

It is clear that $-\infty \leq \underline{D}F(x_0) \leq \overline{D}F(x_0) \leq \infty \quad \forall x_0 \in [a,b]$. It is also not hard to show that if F is monotonically increasing then $0 \leq \underline{D}F(x_0)\ \forall x_0 \in [a,b]$ and similarly if F is monotonically decreasing then $\overline{D}F(x_0) \leq 0\ \forall x_0 \in [a,b]$. A function $F: \ [a,b] \to \mathbb{R}$ is differentiable at a point $x_0 \in [a,b]$ if and only if $-\infty < \underline{D}F(x_0) = \overline{D}F(x_0) < \infty$. See for example [AE06, p.184, II Thm. 5.7]. It is easier to prove that every monotone function is differentiable almost everywhere using upper and lower derivatives than verifying the definition of differentiability. Now we give an example to see how the upper and lower derivative tell us whether the function is differentiable at a given point.

1.2.6 Example
Let F be the function from Example 1.1.2,3. Then $\overline{D}F(0) = 1$ and $\underline{D}F(0) = -1$: Let $\delta > 0$ and $0 < y < \delta$, then $\frac{F(y)-F(0)}{y} = \sin(\frac{\pi}{y}) \leq 1$. We can find $N \in \mathbb{N}$ such that $y := \frac{2}{4N+1} < \delta$ which yields $\frac{F(y)-F(0)}{y} = 1$, this shows that $\overline{D}F(0) = 1$. Similarly we can show that $\underline{D}F(0) = -1$. It is not hard to find a function F with $\overline{D}F(0) = \infty$ and $\underline{D}F(0) = -\infty$. For example $F(x) := \begin{cases} 0 & x = 0, \\ \sin(\frac{\pi}{x}) & 0 < x \leq 1. \end{cases}$

The proof of the following proposition is divided into two parts. First we will show that the upper and lower derivatives are finite a.e. and then we will determine that the upper and lower derivatives are equal almost everywhere.

1.2.7 Proposition (Differentiability of monotone functions)
Every monotone function $F: \ [a,b] \to \mathbb{R}$ is differentiable almost everywhere. (Cf. [Gor94, p.55f., Lem. 4.8 and Thm. 4.9] and [KS04, p.200f., Thm. 4.99])

Proof: We will assume that F is monotonically increasing, the other case is analogous. Our first goal is to show that $\overline{D}F(x) < \infty$ almost everywhere (since F is monotonically increasing we know that $0 \leq \underline{D}F(x)\ \forall x \in [a,b]$, so we can rule out $\underline{D}F(x) = -\infty$ beforehand). Now we define $A := \{x \in [a,b] : \overline{D}F(x) = \infty\}$ and we will show that $\mu^*(A) = 0$. We will prove this claim by contradiction, so we assume that $\mu^*(A) =: \alpha > 0$. Choose $C > 0$ such that $\frac{F(b)-F(a)}{C} < \frac{\alpha}{2}$, then from the definition of A and the upper derivative we get that for every $x \in A$ and every $n \in \mathbb{N}, n \geq 1$ there is a $y_n^x \in [a,b]$ such that $|y_n^x - x| < \frac{1}{n}$ and $\frac{F(y_n^x)-F(x)}{y_n^x - x} > C$. WLOG we can assume that $y_n^x \searrow x\ (n \to \infty)$ and so by Example 1.2.2,2 we get that $\mathcal{I} := \{[x, y_n^x] : x \in A, n \in \mathbb{N}\}$ is a Vitali Cover of A. Then by the Vitali Covering Lemma(1.2.3) we know that for every $\epsilon > 0$ there exists a finite collection of disjoint intervals in $\mathcal{I}$, say $([x_n, y_n])_{n=1}^N$ with $0 < \mu^*(A) = \alpha < \sum_{n=1}^N (y_n - x_n) + \epsilon$. Now set $\epsilon := \frac{\alpha}{2}$, then $\frac{\alpha}{2} < \sum_{n=1}^N (y_n - x_n)$. We calculate $\sum_{n=1}^N (F(y_n) - F(x_n)) \geq C \sum_{n=1}^N (y_n - x_n) > \frac{C\alpha}{2} > F(b) - F(a)$. This is a contradiction, since from Example 1.1.2,2 we know that $\sum_{n=1}^N (F(y_n) - F(x_n)) \leq V(F, [a,b]) = F(b) - F(a)$.
Our second step is to show that $\underline{D}F(x) = \overline{D}F(x)$ almost everywhere. Again we will prove

this claim by contradiction. So we define $B := \{x \in [a,b] : \underline{D}F(x) < \overline{D}F(x)\}$ and we assume that $\mu^*(B) > 0$. For $p, q \in \mathbb{Q}$ with $p < q$ define $B_{p,q} := \{x \in B : \underline{D}F(x) < p < q < \overline{D}F(x)\}$. It is clear that $B = \bigcup_{p,q\in\mathbb{Q}, p<q} B_{p,q}$, so it suffices to show that $\mu^*(B_{p,q}) = 0$ for every pair $p, q \in \mathbb{Q}$ with $p < q$. Now assume there exist $p, q \in \mathbb{Q}$, $p < q$ with $\mu^*(B_{p,q}) := \beta_{p,q} > 0$ and let $\epsilon > 0$. Then there is an open set $O \supseteq B_{p,q}$ with $\mu^*(O \backslash B_{p,q}) < \epsilon$ (see for example [Gor94, p.8, Thm. 1.12]). Then $\mu(O) \leq \mu^*(O \cap B_{p,q}) + \mu^*(O \backslash B_{p,q}) < \mu^*(B_{p,q}) + \epsilon = \beta_{p,q} + \epsilon$, so

$$\mu(O) < \beta_{p,q} + \epsilon. \tag{1.14}$$

Then, similarly as in the first step, for all $x \in B_{p,q}$ and every $n \in \mathbb{N}, n \geq 1$ there is a y_n^x such that $|y_n^x - x| < \frac{1}{n}$ and $\frac{F(y_n^x)-F(x)}{y_n^x - x} < p$. Again WLOG we can assume that $y_n^x \searrow x (n \to \infty)$ and so once more by Example 1.2.2,2 we get that $\mathcal{I}_{p,q} := \{[x, y_n^x] : x \in B_{p,q}, n \in \mathbb{N}\}$ is a Vitali Cover of $B_{p,q}$. By the VCL we get that there exists a finite collection of disjoint intervals $([x_n, y_n])_{n=1}^N$ in $\mathcal{I}_{p,q}$ such that $\mu^*(B_{p,q}) = \beta_{p,q} < \sum_{n=1}^N (y_n - x_n) + \epsilon$. From inequality (1.14) and the above application of the VCL we get

$$\sum_{n=1}^{N} (F(y_n) - F(x_n)) < p \sum_{n=1}^{N} (y_n - x_n) < p\mu(O) < p(\beta_{p,q} + \epsilon). \tag{1.15}$$

This inequality is only the first half of the work. We need an analogous inequality derived from the upper derivative. So we define $C_{p,q} := B_{p,q} \cap (\bigcup_{n=1}^N ([x_n, y_n]))$ then

$$\mu^*(C_{p,q}) > \beta_{p,q} - \epsilon, \tag{1.16}$$

since $\beta_{p,q} = \mu^*(B_{p,q}) \leq \underbrace{\mu^*(B_{p,q} \cap (\bigcup_{n=1}^{N} ([x_n, y_n])))}_{C_{p,q}} + \underbrace{\mu^*(B_{p,q} \backslash \bigcup_{n=1}^{N} ([x_n, y_n]))}_{<\epsilon \text{ (by the VCL)}} < \mu^*(C_{p,q}) + \epsilon$. A third time we construct a Vitali Cover: for every $u \in C_{p,q}$ and every $n \in \mathbb{N}, n \geq 1$ there exists a v_n^u such that $|v_n^u - u| < \frac{1}{n}$ and $\frac{F(v_n^u)-F(u)}{v_n^u - u} > q$. Again WLOG we can assume that $v_n^u \searrow u (n \to \infty)$ and once more by Example 1.2.2,2 we get that $\mathcal{J}_{p,q} := \{[u, v_n^u] : u \in C_{p,q}, n \in \mathbb{N}\}$ is a Vitali Cover of $C_{p,q}$. Then if we modify the argument from Example 1.2.2,4 we get that $\mathcal{L}_{p,q} := \{[u, v_m^u] \in \mathcal{J}_{p,q} : \exists n \in \mathbb{N}, 1 \leq n \leq N, [u, v_m^u] \subseteq [x_n, y_n]\}$ is also a Vitali Cover of $C_{p,q}$. For the last time we apply the VCL to get the existence of a finite collection of disjoint intervals in $\mathcal{L}_{p,q}$, say $([u_m, v_m])_{m=1}^M$ with $\sum_{m=1}^M (v_m - u_m) > \mu^*(C_{p,q}) - \epsilon > \beta_{p,q} - 2\epsilon$, where we have used Equation (1.16) in the last inequality. Now from this last inequality and the construction of $([u_m, v_m])_{m=1}^M$ we get

$$\sum_{m=1}^{M} (F(v_m) - F(u_m)) > q \sum_{m=1}^{M} (v_m - u_m) > q(\beta_{p,q} - 2\epsilon). \tag{1.17}$$

Now we want to combine Equation (1.15) and Equation (1.17). To achieve this we need the observation that for every $1 \leq m \leq M$ there is a $n \in \{1, \ldots, N\}$ such that $[u_m, v_m] \subseteq [x_n, y_n]$ and since F is monotonically increasing we get $\sum_{m=1}^M (F(v_m) - F(u_m)) \leq \sum_{n=1}^N (F(y_n) - F(x_n))$. With this inequality we get from Equations (1.15) and (1.17) that $q(\beta_{p,q} - 2\epsilon) < \sum_{m=1}^M (F(v_m) - F(u_m)) \leq \sum_{n=1}^N (F(y_n) - F(x_n)) < p(\beta_{p,q} + \epsilon)$. Since $\epsilon > 0$ was arbitrary we conclude that $q\beta_{p,q} \leq p\beta_{p,q}$ and since $\beta_{p,q} > 0$ we deduce $q \leq p$. We assumed $p < q$ so this a contradiction and $\mu^*(B_{p,q}) = 0$ for all $p, q \in \mathbb{Q}$ with $p < q$. Furthermore $\mu^*(B) = 0$ because $B = \bigcup_{p,q\in\mathbb{Q},p<q} B_{p,q}$, which finishes our proof. □

With this prerequisites it is not hard to see that every BV function is differentiable almost everywhere. Since it is an important result, we will state it as a proposition.

1.2.8 Proposition (Differentiability of BV functions)
Every BV function is differentiable almost everywhere.

Proof: We have already proved all necessary steps, so we just collect the relevant statements. First from Proposition 1.1.7 (Jordan decomposition) we know that every BV function can be expressed as a difference of two monotonically increasing functions. So let $F = F_1 - F_2$ and from the above proposition (1.2.7) we get that the functions F_1 and F_2 are differentiable almost everywhere. It is clear that F is differentiable a.e. since the union of two set of measure zero has measure zero. □

It is also important to note that of course every AC function is differentiable a.e., since every AC function is also BV (see Lemma 1.1.5,2). Let $F : [a, b] \to \mathbb{R}$ be BV then F' exists a.e., so we will adopt the convention that we will set $F'(x) := 0$ if the derivative of F at x does not exist. In this way we obtain a function $F' : [a, b] \to \mathbb{R}$ and we can state that the derivative of a monotonically increasing (resp. decreasing) function is nonnegative (resp. nonpositive). We can easily see that the derivative of a BV function need not be BV again. Take for example the function $F(x) := \sqrt{x}$ from Example 1.1.6, then $F'(x) = \frac{1}{2\sqrt{x}}$ for $x \neq 0$ and $F'(0) = 0$ by convention. It is not BV since it is not bounded.
In the next subsection we will find that the derivative of an AC (actually even BV) function is Lebesgue integrable and the primitive of a Lebesgue integrable function is AC. In this respect absolute continuity will be a more important concept than bounded variation.

1.2.3. Integration and FTC

Let $F : [a, b] \to \mathbb{R}$ be BV, from Proposition 1.2.8 we can deduce that F is also continuous a.e. and from Lemma 1.1.5,1 we know that F is bounded. If a function is bounded then it is Riemann integrable if and only if it is continuous a.e., see Theorem 3.15 [Gor94, p.39] and so it is Lebesgue integrable (see for example [KS04, p.112, Thm. 3.103]). Actually we are more interested in the Lebesgue integrability of F' since we want to prove the FTC for the Lebesgue integral (cf. Equation (0.1)). So we start by investigating the Lebesgue integrability of a BV function. Then we will show that the primitive of a Lebesgue integrable function is AC and the FTC holds for AC functions. As a corollary we get the descriptive definition of the Lebesgue integral, which opens up the possibility of generalizing it. This in turn will lead us (after still some work) to the (descriptive) definition of the classical Denjoy integral.
The major part of this subsection is about the Lebesgue integral, although many results can be found in standard books about Lebesgue integration theory but often the emphasis is not on AC functions. So we will prove here some well known results but with the

theory of BV and AC functions we have developed so far. Additionally this will give us some understanding of the underlying concepts, which will be useful when we are lead to generalize our concepts to get to the classical Denjoy integral.

1.2.9 Lemma
Let $F: [a,b] \to \mathbb{R}$.

1. *If* F *is differentiable a.e. on* $[a,b]$ *then* F' *is measurable.*

2. *If* F *is monotonically increasing, then* F' *is Lebesgue integrable and*

$$\int_a^b F' \le F(b) - F(a). \tag{1.18}$$

(Cf. [Gor94, p.20, Thm. 2.8] and [Gor94, p.57, Thm. 4.10]).

Proof: We define

$$G(x) := \begin{cases} F(x) & x \in [a,b], \\ F(b) & x \in]b, b+1], \end{cases} \qquad \text{and for } n \in \mathbb{N} \;\; g_n(x) := \frac{G(x+\frac{1}{n}) - G(x)}{\frac{1}{n}} = n(G(x+\tfrac{1}{n}) - G(x)).$$

Then it is clear that $g_n \to F'$ $(n \to \infty)$ a.e. on $[a,b]$.

1. Since F is measurable (it is continuous a.e. on $[a,b]$) each g_n is measurable and so it follows from [Gor94, p.20, Cor. 2.7] that F' is measurable as the pointwise limit of measurable functions.

2. From Proposition 1.2.7 we know that F is differentiable a.e. and hence F' is measurable. Since F is monotonically increasing so is G and hence $g_n \ge 0$. To show integrability of F' it suffices to show that $\int_a^b F' < \infty$ since $F' \ge 0$. By Fatou's Lemma (see for example [Gor94, p.43, Lem. 3.20]) we get that $\int_a^b F' \le \liminf_{n\to\infty} \int_a^b g_n$. Since G is continuous a.e. and monotonically increasing (hence bounded) it is Riemann integrable (cf. note at the beginning of this subsection). Therefore we can use the usual transformation formula to get $\int_a^b g_n = n \int_a^b (G(x+\frac{1}{n}) - G(x))dx = n(\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} G - \int_a^b G) = n(\int_b^{b+\frac{1}{n}} F(b) - \int_a^{a+\frac{1}{n}} F) \le F(b) - F(a) < \infty$, where we have additionally used that $F(x) \ge F(a)$ $\forall x \in [a, a+\frac{1}{n}]$. So F' is Lebesgue integrable and the estimate has been verified.

□

Let F be BV and $F = F_1 - F_2$ its standard Jordan decomposition (1.1.7) then $F' = F_1' - F_2'$ a.e. on $[a,b]$. If we apply the second part of the above lemma and observe that $F_2' \ge 0$ then we get that F' is also Lebesgue integrable and $\int_a^b F' = \int_a^b F_1' - \int_a^b F_2' \le \int_a^b F_1' \le F_1(b) - F_1(a) = V(F,[a,b]) - V(F,[a,a]) = V(F,[a,b])$. This yields the useful formula:

$$\int_a^b F' \le V(F,[a,b]) \; . \tag{1.19}$$

We will see in Lemma 1.2.13 that equality holds in Equation (1.18) for AC functions but in general not for BV functions. Take for example G from 1.1.2,6 then $G' = 0$ a.e. on $[0,1]$ and so $\int_0^1 G' = 0 < 1 = G(1) - G(0)$.
Now we show that a primitive of a bounded and measurable function is AC and the derivative agrees with the integrand almost everywhere. This result will be used to prove the more general result in 1.2.11.

1.2.10 Lemma
Let $f:\ [a,b] \to \mathbb{R}$ be bounded and measurable and define $F(x) := \int_a^x f$ for $x \in [a,b]$. Then

1. *F is AC and*
2. *$F' = f$ almost everywhere on $[a,b]$.*

(Cf. [Gor94, p.57, Lem. 4.11]).

Proof: Let $M > 0$ be a bound for f.

1. Let $\epsilon > 0$, choose $\delta < \frac{\epsilon}{M}$ and let $([a_n, b_n])_{n=1}^N$ be a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^N (b_n - a_n) < \delta$. Then
$$\sum_{n=1}^N |F(b_n) - F(a_n)| = \sum_{n=1}^N |\int_a^{b_n} f - \int_a^{a_n} f| = \sum_{n=1}^N |\int_{a_n}^{b_n} f| \le \sum_{n=1}^N \int_{a_n}^{b_n} |f| \le M \sum_{n=1}^N (b_n - a_n) < \epsilon.$$

2. From the first part we know that F is AC and so by Proposition 1.2.7 F is differentiable almost everywhere on $[a,b]$. Now we define G and g_n as in the proof of Lemma 1.2.9, then again $g_n \to F'$ $(n \to \infty)$ a.e. on $[a,b]$. Also every g_n is Lebesgue integrable since F is continuous and they are uniformly bounded by M: $|g_n(x)| = n|G(x+\frac{1}{n}) - G(x)| = n|\int_x^{x+\frac{1}{n}} f| \le M$. So we can apply the Bounded Convergence Theorem (see for example [Gor94, p.34, Thm. 3.9]) to get that F' is Lebesgue integrable and
$$\int_a^x F' = \lim_{n\to\infty} \int_a^x g_n \quad \forall x \in [a,b]. \tag{1.20}$$
Since F is continuous on $[a,b]$, it is Riemann integrable and so we can use the FTC for the Riemann integral. Let $x \in [a,b]$ then:
$$\begin{aligned} F(x) = \frac{d}{dx}\int_a^x F &= \lim_{n\to\infty} n(\int_a^{x+\frac{1}{n}} F - \int_a^x F) = \lim_{n\to\infty} n(\int_a^{x+\frac{1}{n}} F - \int_a^x F) - \lim_{n\to\infty} n(\int_a^{a+\frac{1}{n}} F) \\ &= \lim_{n\to\infty} n(\int_{a+\frac{1}{n}}^{x+\frac{1}{n}} F - \int_a^x F) = \lim_{n\to\infty} n(\int_a^x (G(y+\frac{1}{n}) - G(y))dy) = \lim_{n\to\infty}(\int_a^x g_n) \overset{(1.20)}{=} \int_a^x F'\ . \end{aligned}$$
So in the end $\int_a^x F' = F(x)$ for all $x \in [a,b]$ and now we can conclude that $F' = f$ a.e. on $[a,b]$ because $\int_a^b (F' - f) = \int_a^b F' - \int_a^b f = F(b) - F(b) = 0$. This finishes the proof.

□

Now, with the help of the above lemma, we are able to prove the general result about primitives of Lebesgue integrable functions.

1.2.11 Theorem (Primitives of Lebesgue integrable functions)
Let $f: [a,b] \to \mathbb{R}$ be Lebesgue integrable and define $F(x) := \int_a^x f$ for $x \in [a,b]$. Then

1. *F is AC and*
2. *$F' = f$ almost everywhere on $[a,b]$.*

(Cf. [Gor94, p.58, Thm. 4.12]).

Proof:

1. We know that if f is Lebesgue integrable then $\forall \epsilon > 0\ \exists \delta > 0$ such that $\int_E |f| < \epsilon$ whenever $E \subseteq [a,b]$ is a measurable subset with $\mu(E) < \delta$ (see for example [Gor94, p.46, Thm. 3.26]). So let $\epsilon > 0$ and choose $\delta > 0$ as to satisfy the above property. Now let $([a_n,b_n])_{n=1}^N$ be a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^N (b_n - a_n) < \delta$ and set $E := \bigcup_{n=1}^N [a_n,b_n]$. Then we obtain $\sum_{n=1}^N |F(b_n) - F(a_n)| \leq \sum_{n=1}^N \int_{a_n}^{b_n} |f| = \int_E |f| < \epsilon$. This proves that F is AC.
2. We suppose first that $f \geq 0$ because this assumption will make the proof substantially easier and afterwards we can prove the general result with the usual decomposition into positive and negative part of a function. With the assumption $f \geq 0$ it can be seen directly that F is monotonically increasing. Let $x, y \in [a,b]$ with $x \leq y$, then $F(y) = \int_a^y f = \int_a^x f + \int_x^y f \geq \int_a^x f = F(x)$. This yields $F' \geq 0$. Now we define $f_n(x) := \min(f(x), n)$ $(x \in [a,b], n \in \mathbb{N})$. It is not hard to check that $f_n \to f$ $(n \to \infty)$ pointwise (choose $n > f(x)$) and that $(f_n)_n$ is monotonically increasing ($\min(f(x),n) \leq \min(f(x),m)$ for $n < m$). Additionally it is clear that $f - f_n \geq 0$, therefore $G_n(x) := F(x) - \int_a^x f_n = \int_a^x (f - f_n)$ is monotonically increasing, as a calculation, similar to the one where we checked the monotonicity of F, shows. We can conclude that G_n is bounded and measurable and so we can apply Lemma 1.2.10. This lemma tells us that G_n is AC and $0 \leq G_n' = F' - f_n$, consequently $F' - f_n \geq 0$ for all $n \in \mathbb{N}$. In the limit $F' - f \geq 0$ and hence by Lemma 1.2.9,2 we get $0 \leq \int_a^b (F' - f) = \int_a^b F' - \int_a^b f \leq F(b) - F(a) - \int_a^b f = 0$. Therefore $F' = f$ a.e. on $[a,b]$ and so we are finished with this case.
   Now let f be an arbitrary integrable function then we write $f = f^+ - f^-$, where $f^+, f^- \geq 0$ and so $F(x) = \int_a^x f^+ - \int_a^x f^-$ $(x \in [a,b])$. By the above result $F' = f^+ - f^- = f$ a.e. on $[a,b]$. This finishes the proof.

□

At this point we know that if we start with a Lebesgue integrable function f and define a primitive F as before then $F' = f$ almost everywhere. Now it is natural to ask if this primitive is unique if we additionally assume that $F(a) = 0$. As we will see it is and to prove this we need the following lemma, which states that, similarly as in classical calculus, we can deduce from $G' = 0$ a.e. that G is constant, if G is AC. For BV functions this is not true as the simple Example 1.1.2,6 shows.

1.2.12 Lemma
Let $F: [a,b] \to \mathbb{R}$ be AC and $F' = 0$ a.e. on $[a,b]$ then F is constant. (Cf. [Gor94, p.60, Thm. 4.13])

Proof: We will prove that $F(a) = F(c)$ for all $c \in]a, b]$. So let $c \in]a, b]$ and define $E := \{x \in [a, c[\colon F'(x) = 0\}$. In this definition we really mean that the derivative of F at x should exists and not $F'(x) = 0$ by convention. Additionally $\mu(]a, c[\backslash E) = 0$ since $F' = 0$ a.e. on $[a, b]$ and hence also on $[a, c]$. Let $\epsilon > 0$ and $\eta > 0$ then there exists a $\delta > 0$ such that $\sum_{n=1}^{N} |F(b_n) - F(a_n)| < \epsilon$ whenever $([a_n, b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a, b]$ with $\sum_{n=1}^{N} (b_n - a_n) < \delta$ (F is AC). With this prerequisites we can define $\mathcal{I} := \left\{[x, y] : x \in E,\ x < y < c \text{ and } |\frac{F(y)-F(x)}{y-x}| < \eta\right\}$. It is not difficult to check that $\mathcal{I}$ is a VC of E. To this end let $\alpha > 0$ and $x \in E$, so $F'(x) = 0$ and consequently there is an $h \in \mathbb{R}$, $0 < h < \alpha$ with the property that $|\frac{F(x+h)-F(x)}{h}| < \eta$. Set $y := x + h$ and upon possibly decreasing h such that $y = x + h < c$ is ensured. Therefore $x \in [x, x + h] \in \mathcal{I}$ and $\mu([x, x+h]) = h < \alpha$, thus $\mathcal{I}$ is a VC of E. From the VCL (Lemma 1.2.3) we get the existence of a finite collection of disjoint intervals $([x_n, y_n])_{n=1}^{N}$ in $\mathcal{I}$ with $\mu(E \backslash \bigcup_{n=1}^{N} ([x_n, y_n])) < \delta$. WLOG we can assume that $a < x_1 < y_1 \leq x_2 < \ldots \leq x_N < y_N < c$. Now we want to estimate $\mu(]a, c[\backslash \bigcup_{n=1}^{N} ([x_n, y_n]))$. For abbreviation set $B := \bigcup_{n=1}^{N} ([x_n, y_n])$ and $C :=]a, c[$. We know $\mu(E\backslash B) < \delta$, $\mu(C\backslash E) = 0$ and $B \subseteq C$. We can write $C\backslash B$ as $C\backslash B = (C\backslash(E \cup B)) \cup E\backslash B$. Therefore $\mu(C\backslash B) \leq \mu(C\backslash E) + \mu(E\backslash B) < \delta$. On the other hand

$$\delta > \mu(]a, c[\backslash \bigcup_{n=1}^{N} ([x_n, y_n])) = (x_1 - a) + \sum_{n=1}^{N-1} (x_{n+1} - y_n) + (c - y_N). \tag{1.21}$$

At this point we are able to calculate with the help of the triangle inequality

$$|F(c) - F(a)| \leq \underbrace{|F(x_1) - F(a)| + \sum_{n=1}^{N-1} |F(x_{n+1}) - F(y_n)| + |F(c) - F(y_N)|}_{<\epsilon} + \underbrace{\sum_{n=1}^{N} |F(y_n) - F(x_n)|}_{<\eta \sum_{n=1}^{N} (y_n - x_n)}$$

$$< \eta \sum_{n=1}^{N} (y_n - x_n) + \epsilon < \eta(c - a) + \epsilon.$$

In the first inequality we used Equation (1.21), which tells us that $\{[a, x_1], [y_1, x_2], \ldots, [y_{N-1}, x_N], [y_N, c]\}$ is a finite collection of non-overlapping intervals with measure less than δ and the second inequality is a direct consequence of the construction of $([x_n, y_n])_{n=1}^{N}$. Since $\epsilon > 0$ and $\eta > 0$ were arbitrary we can deduce that $|F(c) - F(a)| = 0$, so F is constant. □

By the above lemma we now can change our point of view and start with an AC function F and integrate its derivative. Then the following lemma tells us that the function generated differs only by a constant from F.

1.2.13 Lemma

Let $F : [a, b] \to \mathbb{R}$ be AC then F' is Lebesgue integrable on $[a, b]$ and for all $x \in [a, b]$ $\int_a^x F' = F(x) - F(a)$ holds.

Proof: From the remark after Lemma 1.2.8 we conclude that F' is Lebesgue integrable on $[a, b]$ and hence we can define $G(x) := \int_a^x F'$ $(x \in [a, b])$. Then from Theorem 1.2.11 we know that G is AC and $G' = F'$ a.e. on $[a, b]$. Since $F - G$ is AC (Lemma 1.1.8,2) we can deduce that $F - G$ is constant (by Lemma 1.2.12). So let $c \in \mathbb{R}$ with $F(x) = G(x) + c$ then by setting $x = a$ we deduce that $c = F(a)$ and so $G(x) = \int_a^x F' = F(x) - F(a)$. □

Now we have collected all results to prove the FTC for the Lebesgue integral.

1.2.14 Theorem (FTC for the Lebesgue integral)

1. *Let $f : [a, b] \to \mathbb{R}$ be Lebesgue integrable on $[a, b]$ then $F(x) := \int_a^x f$ is differentiable a.e. and $F' = f$ a.e. on $[a, b]$.*
2. *Let $F : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and F' bounded then $\int_a^x F' = F(x) - F(a)$ for all $x \in [a, b]$.*

Proof:

1. We just have to combine Theorem 1.2.11 and Proposition 1.2.7 to conclude that F is differentiable almost everywhere and $F' = f$ a.e. on $[a, b]$.
2. Since F' is bounded and measurable, F' is Lebesgue integrable (see for example [Gor94, p.32, Thm. 3.7]). Also F is Lipschitz continuous because F is differentiable everywhere and F' bounded. So by Example 1.1.6 we get that F is AC and now we can apply Lemma 1.2.13 to get the desired result.

□

Note that in the second part of the FTC we could alternatively assume that F is only differentiable a.e. but then we have to assume that F is AC too (and then differentiability would follow already from AC).
With the above results we can give an alternative definition of the Lebesgue integrability of a function $f : [a, b] \to \mathbb{R}$ (compare this with the usual definition, for example in [Gor94, p.40, Def. 3.16 and p.41, Def. 3.18] or [KS04, p.99, Def. 3.74 and p.103, Def. 3.83]). This definition is more suitable for the generalizations we have in mind than the usual definition. Later on we will see that this will be one of the most important analogies between the Lebesgue integral and the classical Denjoy integral.

1.2.15 Corollary (Descriptive definition of the Lebesgue integral)
Let $f : [a, b] \to \mathbb{R}$, then f is Lebesgue integrable if and only if there exists an $F \in AC([a, b])$ such that $F' = f$ almost everywhere on $[a, b]$.

Proof:

$\Rightarrow$ If we define $F(x) := \int_a^x f$ then by Theorem 1.2.11 we have found an AC primitive.

$\Leftarrow$ From Lemma 1.2.13 we know that F' is Lebesgue integrable and since $F' = f$ a.e. on $[a, b]$, f is Lebesgue integrable because it is a.e. equal to a Lebesgue integrable function (see for example [AE01, p.94, X Lem. 2.15]).

□

As mentioned in the beginning of Section 1.1 after Definition 1.1.1, we could have started with an alternative definition of AC. So we will state it here and prove its equivalence although this will not be important for the development of our theory.

1.2.16 Corollary (Equivalent definition of AC)
Let $F: [a,b] \to \mathbb{R}$ then F is AC if and only if F is differentiable almost everywhere, the derivative is Lebesgue integrable and $\int_a^x F' = F(x) - F(a) \quad \forall x \in [a,b]$ holds.

Proof:

$\Rightarrow$ From the remark after Proposition 1.2.7 we know that F is differentiable a.e. on $[a,b]$ and from Lemma 1.2.13 we get the other part of the claim.

$\Leftarrow$ Since F' is Lebesgue integrable we are able to define $G(x) := \int_a^x F'$ for $x \in [a,b]$. Then from Theorem 1.2.11 we get that G is AC and consequently F also, because $F(x) = G(x) + F(a)$ $(x \in [a,b])$, where the RHS is an AC function (1.1.8,2).

□

Another corollary of the FTC is an easier way to calculate the variation of an AC function.

1.2.17 Corollary
If $F: [a,b] \to \mathbb{R}$ is AC then the variation of F is given by

$$V(F,[a,b]) = \int_a^b |F'|. \tag{1.22}$$

Proof:

$\leq$ Let $([a_n,b_n])_{n=1}^N$ be a finite collection of non-overlapping intervals in $[a,b]$ then by the FTC we get: $\sum_{n=1}^N |F(b_n) - F(a_n)| = \sum_{n=1}^N |\int_{a_n}^{b_n} F'| \leq \sum_{n=1}^N \int_{a_n}^{b_n} |F'| \leq \int_a^b |F'|$. Taking the supremum over all finite collections of non-overlapping intervals in $[a,b]$ yields $V(F,[a,b]) \leq \int_a^b |F'|$.

$\geq$ We observed before that if F is AC then also $|F|$ is AC and by the reverse triangle inequality we get $V(|F|,[a,b]) \leq V(F,[a,b])$. Since $G(x) := |F(x)|$ is differentiable at x if $F'(x)$ exists and $F(x) \neq 0$ we get that $G' = F'$ a.e. on $[a,b]$. So we can apply Equation (1.19) for $G = |F|$: $\int_a^b |F'| = \int_a^b |F|' \leq V(|F|,[a,b]) \leq V(F,[a,b])$. This finishes our proof.

□

We defined a norm on $BV([a,b])$ respectively on $AC([a,b])$ in Definition 1.1.10 but we had to postpone the proof that it really is a norm until now. Since now we know that a BV function is Lebesgue integrable (cf. argument at the beginning of this subsection). By restricting this norm to $AC([a,b])$ we get of course also a norm for AC functions.

1.2.18 Theorem (Norm on $BV([a,b])$)
$BV([a,b])$ is a normed space with norm $\|.\|_{BV} := \|.\|_1 + V(.,[a,b])$.

Proof:

1. First we want to prove that $\|.\|_{BV}$ is positive definite. It is clear that $\|0\|_{BV} = 0$, therefore we just have to prove that the null function is the only function of bounded variation with norm zero. So let $F \in BV([a,b])$ with $0 = \|F\|_{BV} = \|F\|_1 + V(F,[a,b])$. From the properties of the Lebesgue integral we know that this implies that $F = 0$ a.e. on $[a,b]$ (see for example [AE01, p. 96, X Cor. 2.19]). Now we have to show that $F = 0$ everywhere on $[a,b]$. To this end assume $F \neq 0$. Consequently there exists an $x_0 \in [a,b]$ such that $F(x_0) \neq 0$ and there exists an $x_1 \in [a,b]$ with $F(x_1) = 0$. We get a contradiction because $0 = V(F,[a,b]) \geq |F(x_0) - F(x_1)| = |F(x_0)| > 0$. So $F = 0$ everywhere on $[a,b]$.

2. Now we want to show that $\|\alpha F\|_{BV} = |\alpha|\|F\|_{BV}$ for all $\alpha \in \mathbb{R}$ and $F \in BV([a,b])$ and $\|F+G\|_{BV} \leq \|F\|_{BV} + \|G\|_{BV}$ for all $F,G \in BV([a,b])$. Both statements follow from the fact that $\|.\|_1$ is a seminorm (see for example [AE01, p.90, X Cor. 2.9]) and from Equation (1.4).

□

Our goal was to find a norm on $BV([a,b])$ respectively $AC([a,b])$ such that they are complete. First we observe that $BV([a,b])$ respectively $AC([a,b])$ are not complete with respect to the supremum-norm. Consider the sequence of functions
$F_n(x) := \begin{cases} 0 & 0 \leq x \leq \frac{1}{n}, \\ x\sin(\frac{\pi}{x}) & \frac{1}{n} < x \leq 1, \end{cases}$ for $n \in \mathbb{N}\backslash\{0\}$. Then every F_n is AC, see [Gor94, p.287 Exc. 4.7] and it is clear that $(F_n)_n$ converges to the function from Example 1.1.2,3 with respect to $\|.\|_\infty$ and this function is not BV. So nor $BV([a,b])$ neither $AC([a,b])$ are complete when equipped with the supremum-norm. Now we will give a proof that $BV([a,b])$ is indeed complete with the BV-norm and then we will show that $AC([a,b])$ is a closed subspace of $BV([a,b])$ and hence it is itself a Banach space.

1.2.19 Theorem
$(BV([a,b]), \|.\|_{BV})$ is a Banach space.

Proof: Let $(f_n)_n$ be a Cauchy sequence with respect to $\|.\|_{BV}$, then it is clear that $(f_n)_n$ is also a Cauchy sequence with respect to $\|.\|_1$. Since $L^1([a,b])$ is complete there exists a $f \in L^1([a,b])$ such that $f_n \to f$ $(n \to \infty)$ with respect to $\|.\|_1$. A convergent sequence in $L^1([a,b])$ has a subsequence that converges pointwise almost everywhere (see [AE01, p.95, X Thm. 2.18]). So let $(f_{n_k})_k$ be such a subsequence with $f_{n_k} \to f$ $(k \to \infty)$ a.e. on $[a,b]$. For simplicity of the notation we define $g_k := f_{n_k}$ and we fix $x_0 \in [a,b]$ such that $g_k(x_0) \to f(x_0)$ $(k \to \infty)$. From inequality (1.3) we get that

$$\|g_m - g_n\|_\infty \leq V(g_m - g_n, [a,b]) + |g_m(x_0) - g_n(x_0)| \quad \forall m,n \in \mathbb{N}. \tag{1.23}$$

Now let $\epsilon > 0$ and choose $N_1 \in \mathbb{N}$ such that $V(g_m - g_n,[a,b]) < \frac{\epsilon}{2}$ for all $m,n \geq N_1$. Additionally choose $N_2 \in \mathbb{N}$ such that $|(g_m(x_0) - g_n(x_0)| < \frac{\epsilon}{2}$ for all $m,n \geq N_2$. Here we have used the Cauchy sequence property for $(f_n)_n$ (with respect to $\|.\|_{BV}$) and for $(g_n(x_0))_n$ (in $\mathbb{R}$). Then set $N := \max(N_1,N_2)$ and let $m,n \geq N$. By (1.23) we get $\|g_m - g_n\|_\infty < \epsilon$, so $(g_n)_n$ is a Cauchy sequence with respect to $\|.\|_\infty$. Since $\mathcal{B}([a,b])$ is a Banach space there exists a $g \in \mathcal{B}([a,b])$ such that $g_n \to g$ $(n \to \infty)$ with respect to $\|.\|_\infty$. Therefore $(g_n)_n$ is a convergent subsequence of the Cauchy sequence $(f_n)_n$, so $(f_n)_n$ converges also to g in $\mathcal{B}([a,b])$ (see [AE06, p.188, II Satz 6.3]), but it is not hard to see that in fact $g = f$ and $f_n \to f$ $(n \to \infty)$ in $L^1([a,b])$. It remains to show that f is a function of bounded variation and that $f_n \to f$ $(n \to \infty)$ with respect to $\|.\|_{BV}$. First we will show that $(f_n)_n$ converges to f in the BV-norm. From the convergence of $(f_n)_n$ to f in $\mathcal{B}([a,b])$ we easily get the following: for all $\epsilon > 0$, for all $k \in \mathbb{N}$ exists a $M_k \in \mathbb{N}$ such that

$$\|f - f_m\|_\infty < \frac{\epsilon}{k} \quad \forall m \geq M_k. \tag{1.24}$$

Now let $\epsilon > 0$ and let $([a_i, b_i])_{i=1}^k$ be a collection of non-overlapping intervals in $[a,b]$, then choose M_{8k} as in (1.24) and choose $N_1 \in \mathbb{N}$ such that $\|f_m - f_n\|_{BV} < \frac{\epsilon}{4}$ for all $m, n \geq N_1$, this is possible because $(f_n)_n$ is a $\|.\|_{BV}$-Cauchy sequence. Additionally, since $f_n \to f$ $(n \to \infty)$ in $L^1([a,b])$, choose $N_2 \in \mathbb{N}$ such that $\|f - f_m\|_1 < \frac{\epsilon}{2}$ for all $m \geq N_2$. Then set $N := \max(M_{8k}, N_1, N_2)$ and let $m, n \geq N$. Then the triangle inequality yields

$$\sum_{i=1}^k |(f - f_n)(b_i) - (f - f_n)(a_i)| \leq \sum_{i=1}^k \underbrace{|(f - f_m)(b_i) - (f - f_m)(a_i)|}_{\leq 2\|f - f_m\|_\infty < \frac{\epsilon}{4k}} +$$
$$\underbrace{\sum_{i=1}^k |(f_m - f_n)(b_i) - (f_m - f_n)(a_i)|}_{\leq V(f_m - f_n, [a,b]) \leq \|f_m - f_n\|_{BV} < \frac{\epsilon}{4}} < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

Here we observe that the RHS is independent of the collection $([a_i, b_i])_{i=1}^k$, so by taking the supremum over all finite collections of non-overlapping intervals in $[a,b]$ we get $V(f - f_n, [a,b]) \leq \frac{\epsilon}{2}$ $\forall n \geq N$. Therefore $\|f - f_n\|_{BV} = V(f - f_n, [a,b]) + \|f - f_n\|_1 < \epsilon$ $\forall n \geq N$. Now it is clear that $f \in BV([a,b])$: let $\epsilon > 0$ then there exists an $N \in \mathbb{N}$ such that $\|f - f_N\|_{BV} < \epsilon$ and so by (1.4) we conclude that $V(f, [a,b]) \leq V(f - f_N, [a,b]) + V(f_N, [a,b]) < \infty$. So $f \in BV([a,b])$ and $f \to f_n$ $(n \to \infty)$ in the BV-norm. This finishes our proof. □

1.2.20 Theorem

$(AC([a,b]), \|.\|_{BV})$ *is a closed subspace of* $(BV([a,b]), \|.\|_{BV})$*, hence it is a Banach space itself.*

Proof: Let $f_n \to f$ $(n \to \infty)$, with respect to $\|.\|_{BV}$ and $f_n \in AC([a,b])$ for all $n \in \mathbb{N}$. Then from the previous theorem we know that $f \in BV([a,b])$. So it remains to show that $f \in AC([a,b])$. To this end let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $V(f - f_n, [a,b]) \leq \|f - f_n\|_{BV} < \frac{\epsilon}{2}$ for all $n \geq N$. Now let $\delta > 0$ such that $\sum_{i=1}^k |f_N(b_i) - f_N(a_i)| < \frac{\epsilon}{2}$ whenever $([a_i, b_i])_{i=1}^k$ is a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{i=1}^k (b_i - a_i) < \delta$ (since $f_N \in AC([a,b])$). Then we calculate:

$$\sum_{i=1}^k |f(b_i) - f(a_i)| \leq \sum_{i=1}^k |f(b_i) - f_N(b_i) - f(a_i) + f_N(a_i)| + \sum_{i=1}^k |f_N(b_i) - f_N(a_i)|$$
$$\leq V(f - f_N, [a,b]) + \sum_{i=1}^k |f_N(b_i) - f_N(a_i)| < \epsilon.$$

Where $([a_i, b_i])_{i=1}^k$ is a finite collection of non-overlapping intervals in $[a, b]$ with $\sum_{i=1}^k (b_i - a_i) < \delta$. So f is in $AC([a, b])$ and hence $AC([a, b])$ is a closed subspace of $BV([a, b])$ and therefore a Banach space. □

It is possible to define two other norms on $BV([a, b])$, namely $\|f\|_e := |f(a)| + V(f, [a, b])$ (e for endpoint) and $\|f\|_\infty^{BV} := \|f\|_\infty + V(f, [a, b])$. $BV([a, b])$ is complete with respect to both norms. For $\|.\|_e$ see [Heu82, p.37f.], for $\|.\|_\infty^{BV}$ an analogous (but much simpler) proof works as for $\|.\|_{BV}$. The norms $\|.\|_{BV}$, $\|.\|_e$ and $\|.\|_\infty^{BV}$ are equivalent as can be seen by the following argument. First we want to show that $\|.\|_\infty^{BV}$ is stronger than the other two norms i.e. there exist constants $C_1, C_2 > 0$ such that $\|f\|_{BV} \leq C_1 \|f\|_\infty^{BV}$ and $\|f\|_e \leq C_2 \|f\|_\infty^{BV}$ for all $f \in BV([a, b])$. It is clear that $\|f\|_1 \leq (b - a)\|f\|_\infty$ for all $f \in BV([a, b])$. So we can choose $C_1 := \max(1, b - a)$ and since $|f(a)| \leq \|f\|_\infty$ for all $f \in BV([a, b])$ we can choose $C_2 := 1$. Now we can conclude that all three norms are equivalent because all three spaces are complete (cf. A.1).

1.2.4. Summary

As in the previous section we will give at this point a short overview of the results we got so far. First we used the Vitali Covering Lemma to show that a monotone function is differentiable almost everywhere. Then from the Jordan decomposition of a BV function it was clear that BV and AC functions are also differentiable almost everywhere. Afterwards we observed that BV and AC functions are Lebesgue integrable and then we shifted our attention to the integrability of the derivative of BV and AC functions. It was not hard to show that the derivative is again Lebesgue integrable and so we could study primitives of the derivative of a BV or AC function. For AC functions we concluded our efforts with the FTC and related results like the descriptive definition of the Lebesgue integral and an equivalent condition to AC. We will summarize these results in the following table, where the columns are given and we have shown the results in the rows. That means for example in column 2, row 2: every BV function is differentiable almost everywhere.

	monotone	BV	AC	reference
⇒ differentiable (everywhere)	n	n	n	1.1.2,6; AC: $F(x) := \|x\|$ on $[-1, 1]$
⇒ differentiable a.e.	y	y	y	1.2.7 and remark afterwards
⇒ Lebesgue integrable	y	y	y	at the beginning of 1.2.3
⇒ derivative L-integrable	y	y	y	1.2.9,2 and remark afterwards
⇒ FTC holds	n	n	y	example after 1.2.9; AC: 1.2.14

Table 1.2.: Important properties of monotone, BV and AC functions

1.3. Generalizations of AC

There are three ways to generalize the concepts of bounded variation and absolute continuity. We can restrict our attention to an arbitrary subset of the interval $[a, b]$ or we can replace the absolute value of the difference of two function values by a more elaborate construction (the oscillation) or we can decompose the set into subsets and require only BV respectively AC on every subset. All generalizations yield together $2^3 = 8$ concepts. For the sake of simplicity we will not discuss all eight concepts and hence our results will not be stated in the most general form but it will suffice for our purposes. For an introduction and development of all eight concepts see [Gor94, Chapter 6, p. 89-105].
We will proceed as in the previous sections, that is, first we will establish some basic properties of the oscillation and of the weak and strong variation. Then we are able to define the concepts of BV$_*$, AC$_*$ and ACG$_*$ functions and we will investigate their relationship as in the previous section. As before an AC$_*$ function is also BV$_*$ and these functions are closed under linear combinations, so we get three vector spaces: $BV_*([a, b], E)$, $AC_*([a, b], E)$ and $ACG_*([a, b], E)$. Unlike in the case of BV and AC functions, they can not in general be equipped with a natural norm. The third part is concerned with the differentiability of BV$_*$ functions. A major result will be that every BV$_*$ function is differentiable a.e. on the given set and hence also AC$_*$ and ACG$_*$ functions have this property. Additionally we get a characterization of ACG$_*$ functions, namely that a continuous and nearly everywhere differentiable function on $[a, b]$ is ACG$_*$ on $[a, b]$. The last result is very important for the uniqueness of a Denjoy primitive. It states that an ACG$_*$ function F with $F' = 0$ a.e. is constant.

1.3.1. Oscillation and variation

At first we will replace the absolute value of the difference of F at the endpoints of an interval in $[a, b]$ by the supremum over all absolute values of differences in that interval. This supremum is called oscillation and we state a few basic properties. Our next step is then to generalize the concept of variation from the whole interval $[a, b]$ to an arbitrary subset of $[a, b]$. Together with the oscillation this gives us the weak and the strong variation of a function on an arbitrary subset of $[a, b]$. If we specialize again to $[a, b]$ then the weak and the strong variation become the variation from Section 1.1 and very similar to these results we will prove some basic properties of the weak and strong variation.

1.3.1 Definition (Oscillation)
Let $F : [a, b] \to \mathbb{R}$ *and let* $a \leq c < d \leq b$. *Then*

$$\omega(F, [c, d]) := sup\,\{|F(y) - F(x)| : c \leq x < y \leq d\,\} \tag{1.25}$$

is called the <u>*oscillation of* F *on* $[c, d]$</u>.

We saw in Lemma 1.1.3 that the variation behaves additively when splitting the interval into two parts. The oscillation on the other hand behaves only subadditively i.e.

$$\omega(F,[c,d]) \leq \omega(F,[c,z]) + \omega(F,[z,d]) \ \forall z \in]c,d[\ . \tag{1.26}$$

Moreover we have the following inequality between the oscillation and the variation of F on $[c,d]$.

$$|F(d)-F(c)| \leq \omega(F,[c,d]) \leq V(F,[c,d]) \tag{1.27}$$

We have an additional way to calculate the oscillation of a function on an interval:

$$\omega(F,[c,d]) = \sup\{F(t) : t \in [c,d]\} - \inf\{F(t) : t \in [c,d]\} \ . \tag{1.28}$$

These three facts follow easily from the definition. The following lemma will be used later to establish that all BV$_*$, AC$_*$ respectively ACG$_*$ functions form a vector space and for the integration by parts formula.

1.3.2 Lemma
Let $F,G : [a,b] \to \mathbb{R}$, $\alpha,\beta \in \mathbb{R}$ and $a \leq c < d \leq b$, then

1.
$$\omega(\alpha F + \beta G,[c,d]) \leq |\alpha|\omega(F,[c,d]) + |\beta|\omega(G,[c,d]) \quad \textit{and} \tag{1.29}$$

2.
$$\omega(F\cdot G,[c,d]) \leq \|F\|_\infty \,\omega(G,[c,d]) + \|G\|_\infty \,\omega(F,[c,d]) \ . \tag{1.30}$$

Proof: Let $c \leq x < y \leq d$.

1. By the triangle inequality we get $|\alpha F(y) + \beta G(y) - \alpha F(x) - \beta G(x)| \leq |\alpha||F(y)-F(x)| + |\beta||G(y)-G(x)| \leq |\alpha|\omega(F,[c,d]) + |\beta|\omega(G,[c,d])$.
2. Again by the triangle inequality we get $|F(y)\cdot G(y) - F(x)\cdot G(x)| \leq |F(y)||G(y)-G(x)| + |G(x)||F(y)-F(x)| \leq \|F\|_\infty \,\omega(G,[c,d]) + \|G\|_\infty \,\omega(F,[c,d])$.

Now by taking the supremum the results follow. □

The inequalities (1.26),(1.27) and (1.29) cannot be improved as the following example shows.

1.3.3 Example
Here we give two basic examples to illustrate that in the inequalities (1.26),(1.27) *and* (1.29) *there can be equality or a less than relationship.*

1. *Define* $F : [0,1] \to \mathbb{R}$ *by* $F(x) := \begin{cases} 0 & x \in [0,1]\setminus\left\{\frac14,\frac34\right\}, \\ 1 & x \in \left\{\frac14,\frac34\right\}. \end{cases}$
   Then $\omega(F,[0,\frac12]) + \omega(F,[\frac12,1]) = 1+1 = 2 > 1 = \omega(F,[0,1])$*. Moreover* $|F(1)-F(0)| = 0 < \omega(F,[0,1]) = 1 < V(F,[0,1]) = 4$, $\omega(F-F,[0,1]) = 0 < 2 = 2\,\omega(F,[0,1])$ *and* $\omega(0\cdot F,[0,1]) = 0 < 1 = \omega(F,[0,1]) + \omega(0,[0,1])$.

2. *Let* $F: [a,b] \to \mathbb{R}$ *be monotone. Then we get from Equation* (1.27) *that* $\omega(F,[c,d]) = |F(b) - F(a)|$ *since* $V(F,[c,d]) = |F(b) - F(a)|$ *(see 1.1.2,2). So we get equality in the inequalities* (1.26) *and* (1.27)*.*

At this point we are able to replace the absolute value of the difference of two function values by the oscillation. This yields the now familiar concept of variation but now it comes in two flavors: the weak and the strong variation.

1.3.4 Definition (Weak and strong variation)
Let $F: [a,b] \to \mathbb{R}$ *and* $E \subseteq [a,b]$*.*

1. *The weak variation of* F *on* E *is defined by*

$$V(F,E) := \sup\left\{\sum_{n=1}^{N} |F(b_n) - F(a_n)| \ : \ N \in \mathbb{N}, ([a_n,b_n])_{n=1}^{N} \text{ is a finite collection of non-overlapping intervals in } [a,b] \text{ with } a_n, b_n \in E \text{ for } n = 1,\dots,N\right\}.$$

2. *The strong variation of* F *on* E *is defined by*

$$V_*(F,E) := \sup\left\{\sum_{n=1}^{N} \omega(F,[a_n,b_n]) \ : \ N \in \mathbb{N}, ([a_n,b_n])_{n=1}^{N} \text{ is a finite collection of non-overlapping intervals in } [a,b] \text{ with } a_n, b_n \in E \text{ for } n = 1,\dots,N\right\}.$$

3. *For convenience we define* $V(F,\emptyset) := V_*(F,\emptyset) := 0$ *and also if* E *has only one element we set the weak and the strong variation to be zero (since the set of which we want to take the supremum will be empty).*

There are some similarities between the variation from Section 1.1 and the weak respectively strong variation, especially if E is a closed interval, but there are major differences in general.

1.3.5 Remark
Let $F: [a,b] \to \mathbb{R}$ *and* $E \subseteq [a,b]$*.*

1. *It follows from the Inequality in* (1.27) *that*

$$V(F,E) \le V_*(F,E) \le V(F,[a,b]) \tag{1.31}$$

since $\sum_{n=1}^{N} |F(b_n) - F(a_n)| \le \sum_{n=1}^{N} \omega(F,[a_n,b_n]) \le V_*(F,E)$ *for all finite collections of non-overlapping intervals* $([a_n,b_n])_{n=1}^{N}$ *in* $[a,b]$ *with* $a_n, b_n \in E$ *(*$n = 1,\dots,N$*). Then by taking the supremum over all such collections the first inequality follows. For the second inequality let* $([a_n,b_n])_{n=1}^{N}$ *be a finite collection of non-overlapping intervals in* $[a,b]$ *with* $a_n, b_n \in E$ *(*$n = 1,\dots,N$*). Then* $\sum_{n=1}^{N} \omega(F,[a_n,b_n]) \le \sum_{n=1}^{N} V(F,[a_n,b_n]) \le V(F,[a,b])$ *by Lemma 1.1.3. Taking the supremum over all such finite collections yields* $V_*(F,E) \le V(F,[a,b])$*.*

2. *If $E = [a,b]$ then $V(F,[a,b]) = V_*(F,[a,b])$ and it coincides with the variation from Section 1.1. This follows from the above inequality ((1.31)).*

3. *The weak variation depends only on the values of F on E whereas the strong variation depends on the values of F on an interval containing E. To illustrate this behavior let $F := \chi_{\mathbb{Q}}$ on $[0,1]$ and $E := [0,1] \cap \mathbb{Q}$. Then of course $F|_E = 1$ and so $V(F,E) = 0$ but $V_*(F,E) = \infty$. To see this, observe that for every interval $[c,d]$ in $[0,1]$ with $c,d \in \mathbb{Q}$ one can find $c \leq x < y \leq d$ with $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ and therefore $\omega(F,[c,d]) = 1$. So for every finite collection of non-overlapping intervals $([a_n,b_n])_{n=1}^N$ in $[0,1]$ with $a_n, b_n \in \mathbb{Q}$ $(i = 1,\ldots,N)$ we get that $\sum_{n=1}^N \omega(F,[a_n,b_n]) = N$.*

4. *To summarize the above remark: it is not hard to show that F is constant on E if and only if $V(F,E) = 0$ and F is constant on $]c,d[$ if and only if $V_*(F,E) = 0$, where $c := \inf E$, $d := \sup E$.*

We will illustrate the statements from the previous remark by calculating the weak and the strong variation explicitly for some (nice) functions.

1.3.6 Example

1. *Let $F:\ [a,b] \to \mathbb{R}$ be constant. Then $V(F,E) = V_*(F,E) = 0$ for all $E \subseteq [a,b]$. We have seen in Remark 1.3.5,3 that if F is only constant on E then $V(F,E) = 0$ but $V_*(F,E)$ could be any non-negative value.*

2. *Let F be the function from Example 1.3.3,1 and $E := [0,1] \backslash \{\frac{1}{4}, \frac{3}{4}\}$. Then $V(F,E) = 0$ but $V_*(F,E) = 2$ and $V(F,[0,1]) = 4$.*

The weak and the strong variation (respectively the summands in their definition) play a analogous role as in the definition of BV and AC in Section (1.1). Therefore we need to understand their behavior well and so we conclude this subsection by proving two basic properties of the strong variation. The analogous properties hold for the weak variation as well but they will not be needed in our development of the classical Denjoy integral.

1.3.7 Lemma

Let $F:\ [a,b] \to \mathbb{R}$, $A, B, E \subseteq [a,b]$ and $c \in E$. Then

1. $$V_*(F,A) + V_*(F,B) \leq V_*(F,A \cup B), \tag{1.32}$$

2. $$V_*(F,E) = V_*(F,E \cap [a,c]) + V_*(F,E \cap [c,b]). \tag{1.33}$$

Proof:

1. Let $([a_n,b_n])_{n=1}^M$ be a finite collection of non-overlapping intervals in $[a,b]$ with endpoints in A and let $([a_n,b_n])_{n=M+1}^N$ be a finite collection of non-overlapping intervals in $[a,b]$ with endpoints in B, then we set $x_{2n-1} := a_n$ and $x_{2n} := b_n$ for $n = 1,\ldots,N$. Then clearly $([x_n,x_{n+1}])_{n=1}^{2N-1}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $x_n \in A \cup B$ for $n = 1,\ldots,2N$. Consequently $\sum_{n=1}^N \omega(F,[a_n,b_n]) \leq \sum_{n=1}^{2N-1} \omega(F,[x_n,x_{n+1}]) \leq V_*(F,A \cup B)$. Taking the supremum yields $V_*(F,A) + V_*(F,B) \leq V_*(F,A \cup B)$.

2. $\leq$ Let $([a_n, b_n])_{n=1}^N$ be a finite collection of non-overlapping intervals in $[a, b]$ with $a_n, b_n \in E$ for $(n = 1, \dots, N)$. Let $n_0 \in \{1, \dots, N\}$ be minimal such that either ($b_{n_0-1} \leq c$ and $a_{n_0} \geq c$) or ($b_{n_0} \geq c$ and $a_{n_0} < c$). If there is no such n_0 then there is nothing to do since our collection is completely in $[a, c]$ or $[c, b]$. So, maybe upon inserting c into the appropriate collection, we get that $\sum_{n=1}^N \omega(F, [a_n, b_n]) \leq V_*(F, E \cap [a, c]) + V_*(F, E \cap [c, b])$ since the first intervals give a collection in $[a, c]$ with endpoints in $E \cap [a, c]$ and the second intervals give a collection in $[c, b]$ with endpoints in $E \cap [c, b]$. So by taking the supremum over all finite collections of non-overlapping intervals in $[a, b]$ with endpoints in E we get that $V_*(F, E) \leq V_*(F, E \cap [a, c]) + V_*(F, E \cap [c, b])$.

   $\geq$ Set $A := E \cap [a, c]$ and $B := E \cap [c, b]$ then (1.32) yields $V_*(F, E \cap [a, c]) + V_*(F, E \cap [c, b]) \leq V_*(F, E)$.

□

Of course by induction we get from the above lemma that

$$V_*(F, E) = \sum_{n=0}^{N-1} V_*(F, E \cap [e_n, e_{n+1}]) \tag{1.34}$$

where $e_1 < \ldots < e_{N-1}$ in E and $e_0 := a$, $e_N := b$.
The following example shows that we cannot drop the condition that $c \in E$ in the above lemma and that in general we can not get equality in (1.32).

1.3.8 Example

1. *Let F be the function from Example 1.3.3,1 and $E := \{\frac{1}{4}, \frac{3}{4}\}$. Then $V_*(F, E) = 2$ but $V_*(F, E \cap [0, \frac{1}{2}]) = V_*(F, \{\frac{1}{4}\}) = 0$ and similarly $V_*(F, E \cap [\frac{1}{2}, 1]) = 0$.*

2. *Let $F : [-1, 1] \to \mathbb{R}$ given by $F(x) := \begin{cases} 0 & x = 0, \\ 1/x & 0 \neq x \in [-1, 1], \end{cases}$ and define $A := \{-1\}$, $B := \{1\}$ then $V_*(F, A) + V_*(F, B) = 0 < \infty = V_*(F, A \cup B)$.*

1.3.2. Strongly bounded variation and (generalized) strong absolute continuity

At this point we are able to define the new concepts we were aiming at, namely: weakly respectively strongly bounded variation and (generalized) strong absolute continuity. Again very similar to Section 1.1 we will prove some basic properties and we will see that some results hold true in the new setting. Of course we will also discuss the differences to the more basic concepts BV and AC.

1.3.9 Definition
Let $F : [a, b] \to \mathbb{R}$, $E \subseteq [a, b]$ and $c := \inf E$, $d := \sup E$.

1. *F is of weakly bounded variation (BV) on E $:\Leftrightarrow V(F, E) < \infty$.*

2. *F is of strongly bounded variation (BV_*) on E* $:\Leftrightarrow V_*(F,E) < \infty$.

3. *F is called strongly absolutely continuous (AC_*) on E* $:\Leftrightarrow$ *F is bounded on $[c,d]$ and $\forall \epsilon > 0\ \exists \delta > 0$ such that $\sum_{n=1}^{N} \omega(F,[a_n,b_n]) < \epsilon$ whenever $([a_n,b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $\sum_{n=1}^{N}(b_n - a_n) < \delta$ and $a_n, b_n \in E$ for $n = 1,\dots,N$.*

4. *F is called generalized strongly absolutely continuous (ACG_*) on E* $:\Leftrightarrow$ *$F|_E$ is continuous and E can be written as countable union of sets E_n i.e. $E = \bigcup_{n=1}^{\infty} E_n$ such that F is AC_* on E_n for all $n \in \mathbb{N}, n \geq 1$.*

As stated in the beginning of this section we could have introduced more concepts but these four concepts will be the most important to our theory. It should be clear how these definitions should look like since for example we could have defined AC on a given subset E in $[a,b]$ where we just replace the oscillation by the absolute value. The next step would be to write the set E as a countable union of sets on which the function is BV, BV_* or AC. All in all this yields the additional concepts of AC, ACG, BVG and BVG_*. See for example [Gor94, Chapter 6, p. 89-105] for a full introduction.
There are some basic relationships between BV, BV_*, AC_* and ACG_* introduced here and BV, AC from Section 1.1.

1.3.10 Remark

1. *From inequality* (1.31) *we get that BV_* implies BV and from Remark 1.3.5,2 we see that in the case $E = [a,b]$ the concepts BV with BV_* and AC with AC_* coincide and are also the same as in Section 1.1.*

2. *It is not hard to see (and very similar to 1.1.2,1) that if F is AC_* on E then F is continuous on E. Therefore AC_* implies ACG_*, since we need not to decompose E at all. As we will see in Example 1.3.11,2 the converse is of course not true.*

3. *Note that we have to add the condition that F is bounded on $[c,d]$ for AC_*, since otherwise BV_* and AC_* would not be related at all. See Example 1.3.11,4.*

To summarize the above remark: every BV_* function is BV and every AC_* function is ACG_*. The converse is not true, which will be elaborated by the following example.

1.3.11 Example

1. *As we have seen in Remark 1.3.5,3 the function $\chi_{\mathbb{Q}}$ is BV on $[0,1] \cap \mathbb{Q}$ but not BV_*. For the same reason this function is not AC_*. Nevertheless it is ACG_*. To show this let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of the rationals in $[0,1]$, then define $E_n := \{q_n\}$ for $n \in \mathbb{N}$. Now it is trivial that $\chi_{\mathbb{Q}}$ is AC_* on each E_n.*

2. *To give an example of a function which is not ACG_* we just have to modify the above example appropriately. Now define $E := [0,1]\backslash\mathbb{Q}$ then χ_E is not ACG_* on E. Assume it is then there are $(E_n)_{n=1}^{\infty}$ such that $E = \bigcup_{n=1}^{\infty} E_n$ and χ_E is AC_* on each E_n. Since E is uncountable there must be an uncountable E_n and hence there are $c, d \in E_n$ with $c < d$ and therefore $\omega(\chi_E,[c,d]) = 1$ (as in 1.3.5,3) and so χ_E cannot be AC_* (not even BV_*) on this E_n, a contradiction.*

3. *The function from Example 1.1.2,6 shows that there are functions which are BV (or BV$_*$) but not AC (or AC$_*$).*

4. *This example illustrates why we have to add the condition in the definition of AC$_*$ that F must be bounded on $[c,d]$. Assume for now that we drop this condition and define $F:\ [0,1]\to\mathbb{R}$ by*

$$F(x) := \begin{cases} 0 & 0\le x<\frac{1}{4} \text{ or } \frac{3}{4}<x\le 1 \text{ or } x=\frac{1}{2}, \\ |x-\frac{1}{2}|^{-1}-4 & \frac{1}{4}\le x\le\frac{3}{4} \text{ and } x\neq\frac{1}{2}, \end{cases}$$

   and $E := \{\frac{1}{4},\frac{3}{4}\}$. Then clearly $V_*(F,E)=\omega(F,[\frac{1}{4},\frac{3}{4}])=\infty$ since F is unbounded on this interval. So F is not BV$_*$ on E but F would be AC$_*$. Let $\epsilon>0$ and choose $\delta<\frac{1}{2}$. Since there are no finite collections of non-overlapping intervals in $[0,1]$ with endpoints in E and the sum of the lengths less then $\frac{1}{2}$ (there is only one interval of length $\frac{1}{2}$) there is nothing to show and so F would be AC$_*$. Note that in [Gor94, p.90, Def. 6.1(d)] this condition is missing but in the reprint of 1997 this was corrected. Also in [CD89, p.14, Def. in 6] this condition is missing.

5. *An ACG$_*$ function can be unbounded on the given set. Take $E :=]0,1]$ and define $F:\ [0,1]\to\mathbb{R}$ by $F(x):=\frac{1}{x}$ for $x\in]0,1]$ and $F(0):=0$, then clearly F is AC$_*$ on each interval $[\frac{1}{n},1]$ for $n\in\mathbb{N}, n\ge 1$ and so F is an unbounded ACG$_*$ function on $]0,1]$.*

Very similar to the result in Section 1.1 a BV$_*$ function on E is bounded, but now only on the minimal closed interval containing E. Also if a function is BV$_*$, AC$_*$ or ACG$_*$ on a set E then it is so on any subset of E. Moreover the product of an ACG$_*$ function with an AC function is ACG$_*$.

1.3.12 Proposition
Let $F:\ [a,b]\to\mathbb{R}$ and $E\subseteq[a,b]$.

1. *Let $a\le c<d\le b$. Then F is bounded on $[c,d]$ if and only if $\omega(F,[c,d])<\infty$.*

2. *If F is BV$_*$ on E (nonempty), then F is bounded on $[c,d]$ where $c:=\inf E$, $d:=\sup E$.*

3. *If F is AC$_*$ (respectively BV$_*$) on E and $A\subseteq E$, then F is also AC$_*$ (respectively BV$_*$) on A.*

4. *If F is ACG$_*$ on E and $A\subseteq E$, then F is also ACG$_*$ on A.*

5. *If F is ACG$_*$ on $[a,b]$ and G is AC on $[a,b]$, then $F\cdot G$ is ACG$_*$ on $[a,b]$.*

Proof:

1. $\Rightarrow$ Say F is bounded by $C>0$ and let $c\le x<y\le d$, then by the triangle inequality we get that $|F(x)-F(y)|\le|F(x)|+|F(y)|\le 2C$. Taking the supremum over all $c\le x<y\le d$ yields $\omega(F,[c,d])\le 2C<\infty$.

   $\Leftarrow$ By hypothesis there is an $C>0$ such that $C\ge|F(y)-F(x)|\ge|F(x)|-|F(y)|$ for all $x,y\in[c,d]$ by the reverse triangle inequality. Now we set $y:=d$ and so $|F(x)|\le C+|F(d)|$ for all $x\in[c,d]$. So F is bounded on $[c,d]$.

2. The case where E has only one element is clear, so WLOG we can assume that $|E| \geq 2$ and we fix $e \in E$. Now let $x \in]c, d[$ then there exists $e' \in E$ with either $e \leq x \leq e'$ or $e' \leq x \leq e$. WLOG we can assume the former and so by the triangle inequality we get that $|F(x)| \leq |F(x) - F(e)| + |F(e)| \leq \omega(F, [e, e']) + |F(e)| \leq V_*(F, E) + |F(e)|$ and so F is bounded on $[c, d]$.

3. We will prove the case where F is AC$_*$ on E, the BV$_*$ case works analogously (and is easier). Let $\epsilon > 0$, then there is a $\delta > 0$ such that $\sum_{n=1}^{N} \omega(F, [a_n, b_n]) < \epsilon$ whenever $([a_n, b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a, b]$ with endpoints in E and $\sum_{n=1}^{N} (b_n - a_n) < \delta$. Now let $([a_n, b_n])_{n=1}^{N}$ be a finite collection of non-overlapping intervals in $[a, b]$ with endpoints in A and $\sum_{n=1}^{N} (b_n - a_n) < \delta$. This collection is also a collection with endpoints in E, therefore we get that $\sum_{n=1}^{N} \omega(F, [a_n, b_n]) < \epsilon$, as required.

4. Write $E = \bigcup_{n=1}^{\infty} E_n$ such that F is AC$_*$ on each E_n ($n \in \mathbb{N}, n \geq 1$). Now we define $A_n := E_n \cap A \subseteq E_n$ and so F is AC$_*$ on each A_n ($n \in \mathbb{N}, n \geq 1$) by 3. This concludes the proof since $\bigcup_{n=1}^{\infty} A_n = E \cap A = A$ and $F|_A$ is continuous on A.

5. WLOG assume that $F \neq 0$, $G \neq 0$ and we write $[a, b] = \bigcup_{m=1}^{\infty} E_m$ such that F is AC$_*$ on each E_m. Fix $m \in \mathbb{N}, m \geq 1$ and let $\epsilon > 0$. Choose $\delta_1 > 0$ such that $\sum_{n=1}^{N} \omega(F, [a_n, b_n]) < \frac{\epsilon}{2\|G\|_\infty}$, whenever $([a_n, b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a, b]$ with $a_n, b_n \in E_m$ for $n = 1, \ldots, N$ and $\sum_{n=1}^{N} (b_n - a_n) < \delta_1$. Choose $\delta_2 > 0$ such that $\sum_{n=1}^{N} \omega(G, [a_n, b_n]) < \frac{\epsilon}{2\|F\|_\infty}$, whenever $([a_n, b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a, b]$ with $\sum_{n=1}^{N} (b_n - a_n) < \delta_2$. This is possible since in this case we are working on a compact interval and so AC equals AC$_*$. Now let $0 < \delta \leq \min(\delta_1, \delta_2)$ and let $([a_n, b_n])_{n=1}^{N}$ be a finite collection of non-overlapping intervals in $[a, b]$, then

$$\sum_{n=1}^{N} \omega(F \cdot G, [a_n, b_n]) \overset{(1.30)}{\leq} \|F\|_\infty \sum_{n=1}^{N} \omega(G, [a_n, b_n]) + \|G\|_\infty \sum_{n=1}^{N} \omega(F, [a_n, b_n]) < \epsilon \ .$$

   This shows that $F \cdot G$ is AC$_*$ on E_m. Since m was arbitrary and $F \cdot G$ is continuous on $[a, b]$ we conclude that $F \cdot G$ is ACG$_*$ on $[a, b]$.

□

As in Lemma 1.1.5,2 we want to show that every AC$_*$ function is also BV$_*$ on the given set. Unfortunately this is much harder to prove than in the simpler case where the set is the whole interval. The problem is that if a function is AC$_*$ on a set, we can estimate this function only on small intervals but we do not know if there are any such intervals at all. This problem led us also to add the extra condition to the definition of AC$_*$ that the function has to be bounded on the minimal closed interval containing E.

1.3.13 Proposition (AC$_*$ $\Rightarrow$ BV$_*$)
If $F: [a, b] \to \mathbb{R}$ is AC$_*$ on $E \subseteq [a, b]$ nonempty then F is BV$_*$ on E. (Cf. [Gor94, Thm. 6.2.(b), p.90])

Proof: Let $c := \inf E$, $d := \sup E$ and since F is AC_* on E we know that F is bounded on $[c, d]$, say $|F| \leq C$, where (for simplicity) $C > \frac{1}{2}$. Therefore $\omega(F, [x, y]) \leq 2C$ for all $x, y \in E$ with $x < y$. Now corresponding to $\epsilon = 1$ there is a $\delta > 0$ such that $\sum_{n=1}^{N} \omega(F, [a_n, b_n]) < 1$ whenever $([a_n, b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a, b]$ with $a_n, b_n \in E$ for $n = 1, \ldots, N$ and $\sum_{n=1}^{N} (b_n - a_n) < \delta$. For $e \in E$ define $B(e) :=]e - \frac{\delta}{2}, e + \frac{\delta}{2}[\cap [a, b]$. Then there are two cases:

<u>First case:</u> $\bigcup_{e \in E} B(e) \supseteq [c, d]$

Since $[c, d]$ is compact there are $e_1 < e_2 < \ldots < e_M \in E$ such that $B(e_1) \cup \ldots \cup B(e_M) \supseteq [c, d]$. Now because $c \in B(e_1)$ and $e_1 - c < \frac{\delta}{2}$ all finite collections of non-overlapping intervals $([a_n, b_n])_{n=1}^{N}$ in $[a, e_1]$ with endpoints in E satisfy $\sum_{n=1}^{N} (b_n - a_n) < \delta$. Therefore $V_*(F, E \cap\ [a, e_1]) \leq 1$. Similarly $V_*(F, E \cap [e_M, b]) \leq 1$ since $d \in B(e_M)$. Additionally since $e_{i+1} - e_i < \delta$ it is clear that $V_*(F, E \cap [e_i, e_{i+1}]) \leq 1$ for $i = 1, \ldots, (M - 1)$. Now set $e_0 := a$ and $e_{M+1} := b$, then if we put all together we can conclude by Equation (1.34) that

$$V_*(F, E) = \sum_{i=0}^{M} V_*(F, E \cap [e_i, e_{i+1}]) \leq M + 1 < \infty \, .$$

This proves the first case.

<u>Second case:</u> $\bigcup_{e \in E} B(e) \not\supseteq [c, d]$

i.e. there is an $z \in [c, d] \backslash \bigcup_{e \in E} B(e)$. So clearly all $e \in E$ satisfy $|z - e| \geq \frac{\delta}{2}$, this means that $]z - \frac{\delta}{2}, z + \frac{\delta}{2}[\subseteq [c, d] \backslash E$. We can extend this interval, $]z - \frac{\delta}{2}, z + \frac{\delta}{2}[$, to a maximal interval completely in $[c, d] \backslash E$ containing $]z - \frac{\delta}{2}, z + \frac{\delta}{2}[$. There can be only finitely many such disjoint intervals (since each such interval has length greater then or equal to δ). So WLOG we can assume that there is only one such interval. Now we define $A := E \cap [c, z - \frac{\delta}{2}[$ and $B := E \cap]z + \frac{\delta}{2}, d]$. These sets are not empty since (by the definition of c, d as supremum respectively infimum) arbitrary close to them we can find elements of E. This allows as to define $s := \sup A$ and $i := \inf B$. At this point we can choose $e_1 \in A$ such that $s - \frac{\delta}{2} < e_1$ and $e_2 \in B$ such that $i + \frac{\delta}{2} > e_2$. These two points, e_1 and e_2, have distance at least δ. Now we want to estimate $V_*(F, E \cap [e_1, e_2])$. To this end let $([a_n, b_n])_{n=1}^{N}$ be a finite collection of non-overlapping intervals in $[e_1, e_2]$ with $a_n, b_n \in E \cap [e_1, e_2]$ for $(n = 1, \ldots, N)$. From the disjointness of these intervals we get that there can be at most one interval $[a_j, b_j]$ with $b_j - a_j \geq \delta$ $(j \in \{1, \ldots, N\})$. So we calculate

$$\sum_{n=1}^{N} \omega(F, [a_n, b_n]) = \underbrace{\sum_{n=1, n \neq j}^{N} (\omega(F, [a_n, b_n]))}_{<1} + \underbrace{\omega(F, [a_j, b_j])}_{\leq 2C} < 1 + 2C \, .$$

In the first term we have used that the total sum of the lengths is less than δ, since $s - e_1 < \frac{\delta}{2}$ respectively $e_2 - i < \frac{\delta}{2}$. So by taking the supremum over all finite collections of non-overlapping intervals with endpoints in $E \cap [e_1, e_2]$ we conclude that $V_*(F, E \cap [e_1, e_2]) \leq 1 + 2C < \infty$. At this point we will estimate the strong variation on the whole interval. We assumed that we have only one interval not covered by the intervals $B(e)$ $(e \in E)$. So as in the first case we can cover the remainder of $[c, d]$ by finitely many intervals $B(e)$ $(e \in E)$. This means that there are $g_1 < g_2 < \ldots < g_K < e_1$ in E with $g_{i+1} - g_i < \delta$ $(i = 1, \ldots, K - 1)$,

$g_1 - a < \delta$ and $e_1 - g_K < \delta$ and there are $e_2 < h_1 < h_2 \ldots < h_L$ in E with $h_{i+1} - h_i < \delta$ $(i = 1, \ldots, L-1)$, $h_1 - e_2 < \delta$ and $b - h_L < \delta$ (of course there can be other cases but this case is exemplary and it should be clear from this how to handle the other cases). For the other intervals (except $[e_1, e_2]$) we argument as in the first case, each of these other subintervals has length less then δ and so the strong variation is less then or equal to one. Now set $g_0 := a, g_{K+1} := e_1$ and $h_0 := e_2, h_{L+1} := b$ and we can conclude again by equation (1.34) that

$$V_*(F,E) = \sum_{i=0}^{K} \underbrace{V_*(F, E \cap [g_i, g_{i+1}])}_{\leq 1} + \underbrace{V_*(F, E \cap [e_1, e_2])}_{\leq 1+2C} + \sum_{i=0}^{L} \underbrace{V_*(F, E \cap [h_i, h_{i+1}])}_{\leq 1} \leq K+2C+L+3 < \infty \, .$$

This proves this case and finishes our proof. □

For technical reasons it is much simpler to work with closed sets, since for example if $E \subseteq [a,b]$ is closed and nonempty then $c, d \in E$ where $c := \inf E$, $d := \sup E$ or $]c, d[\backslash E$ can be written as a countable union of open and disjoint intervals. Fortunately if a function F is BV_* on a set E, then it is also BV_* on its closure $\overline{E}$.

1.3.14 Lemma

Let $F: [a,b] \to \mathbb{R}$ be BV_* on $E \subseteq [a,b]$ nonempty, then F is also BV_* on $\overline{E}$. (Cf. [CD89, Thm. 14, p.10f.] respectively [Gor94, Thm. 6.2c), p. 91])

Proof: Let $c := \inf E \geq a$ and $d := \sup E \leq b$. WLOG we can assume $|E| \geq 2$ (the case $|E| = 1$ is clear) and therefore $c < d$. Since F is BV_* on E we know that F is bounded on $[c,d]$, say by M. This yields $\omega(F, [x,y]) \leq 2M$ for all $c \leq x < y \leq d$. Define $x_{2n-1} := a_n$, $x_{2n} := b_n$ for $n = 1, \ldots, N$. Since $\omega(F, [x_1, x_2]) \leq 2M$ and $\omega(F, [x_{2N-1}, x_{2N}]) \leq 2M$ we can WLOG suppose that $N \geq 2$. If for some $n \in \{3, \ldots, 2N-1\}$: $E \cap [x_{n-1}, x_n] = \emptyset$, then $E \cap [x_{n-2}, x_{n-1}] \neq \emptyset$ and $E \cap [x_n, x_{n+1}] \neq \emptyset$, since in this case there exist sequences $(e_i)_i$, $(e'_i)_i$ of elements in E such that $e_i \nearrow x_{n-1}$ and $e'_i \searrow x_n$ for $i \to \infty$. Now we set $\Pi_1 := \{2 \leq n \leq 2N : [x_{n-1}, x_n] \cap E \neq \emptyset\}$ respectively $\Pi_2 := \{2, \ldots, 2N\} \backslash \Pi_1$. For convenience we enumerate Π_1 as $\{1, \ldots, L\}$. We assumed $|E| \geq 2$, so $|\Pi_1| \geq 2$ as well. For $n \in \Pi_1$ choose $v_n \in E \cap [x_{n-1}, x_n]$ and so for all $j \in \Pi_2$ there exists a unique $n_j \in \Pi_1$ such that $[x_{n-1}, x_n] \subseteq [v_{n_j - 1}, v_{n_j}]$, hence $\omega(F, [x_{n-1}, x_n]) \leq \omega(F, [v_{n_j-1}, v_{n_j}])$. From the construction of the intervals $[v_{n-1}, v_n]$ and Inequality (1.26) we get for $2 \leq n \leq L-1$

$$\omega(F, [x_{n-1}, x_n]) \leq \omega(F, [x_{n-1}, v_n]) + \omega(F, [v_n, x_n]) \leq \omega(F, [v_{n-1}, v_n]) + \omega(F, x[v_n, v_{n+1}]) \, .$$

The above inequality yields all together

$$\begin{aligned} \sum_{n \in \Pi_1} \omega(F, [x_{n-1}, x_n]) \;&\leq\; \omega(F, [x_1, x_2]) + 2 \sum_{n=2}^{L-1} (\omega(F, [v_{n-1}, v_n])) + \omega(F, [x_{2N-1}, x_{2N}]) \\ &\leq 4M + 2 \sum_{n=2}^{L-1} \omega(F, [v_{n-1}, v_n]) \, . \end{aligned}$$

At this point we can calculate

$$\sum_{n=1}^{N} \omega(F,[a_n,b_n]) \leq \sum_{n=2}^{2N} \omega(F,[x_{n-1},x_n]) = \sum_{n\in\Pi_1} \omega(F,[x_{n-1},x_n]) + \sum_{j\in\Pi_2} \omega(F,[x_{j-1},x_j]) \leq$$
$$4M + 2\sum_{n=2}^{L-1} \omega(F,[v_{n-1},v_n]) + \sum_{j\in\Pi_2} \omega(F,[v_{n_j-1},v_{n_j}]) \leq 4M + 3V_*(F,E)\ .$$

So by taking the supremum over all finite collections of non-overlapping intervals in $[a,b]$ with endpoints in $\overline{E}$ we conclude that $V_*(F,\overline{E}) \leq 4M + 3V_*(F,E) < \infty$ and so F is BV$_*$ on $\overline{E}$. □

It is very hard to show that a given function is not ACG$_*$ because there are so many ways to choose the decomposition of the given set. So we are looking for a necessary condition of a function to be ACG$_*$. As it turns out every ACG$_*$ function maps sets of measure zero to sets of measure zero. This property will also be very useful when we will prove that an ACG$_*$ function with derivative a.e. equal to zero is constant.

1.3.15 Lemma
Let $F:\ [a,b] \to \mathbb{R}$ be ACG$_*$ on $E \subseteq [a,b]$ nonempty. Then F maps sets of measure zero to sets of measure zero i.e. for $A \subseteq E$, $\mu(A) = 0$, $\mu(F(A)) = 0$ holds. (Cf. [Gor94, Thm. 6.12, p.97])

Proof: Write $E = \bigcup_{m=1}^{\infty} E_m$ such that F is AC$_*$ on each E_m and fix $m \in \mathbb{N}, m \geq 1$. We will show the claim first on E_m, so let $A \subseteq E_m$ with $\mu(A) = 0$. Additionally let $\epsilon > 0$ and since F is AC$_*$ on E_m there is a $\delta > 0$ such that $\sum_{n=1}^{N} \omega(F,[a_n,b_n]) < \epsilon$ whenever $([a_n,b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $a_n, b_n \in E_m$ for $n = 1,\ldots,N$ and $\sum_{n=1}^{N}(b_n - a_n) < \delta$. Since A has measure zero we can choose open intervals $(I_k)_{k=1}^{\infty}$ such that $A \cap I_k \neq \emptyset$ for $k \in \mathbb{N}, k \geq 1$ and $A \subseteq \bigcup_{k=1}^{\infty} I_k$ with $\sum_{k=1}^{\infty} \mu(I_k) < \delta$. Now let $M \in \mathbb{N}, M \geq 1$ and calculate

$$\sum_{k=1}^{M} \mu^*(F(A\cap I_k)) \leq \sum_{k=1}^{M} (\sup F(A\cap I_k) - \inf F(A\cap I_k)) \ \leq$$
$$\sum_{k=1}^{M} (\sup F(E_m\cap I_k) - \inf F(E_m\cap I_k)) \ \leq$$
$$\sum_{k=1}^{M} V_*(F, E_m\cap I_k) \leq V_*(F, E_m \cap \bigcup_{k=1}^{M} I_k) \leq \epsilon$$

where we have used (1.32) in the second to the last inequality and the fact that all finite collections of non-overlapping intervals with endpoints in $E_m \cap \bigcup_{k=1}^{M} I_k$ have length less then δ. Therefore we conclude that $\mu^*(F(A)) = \mu^*(\bigcup_{k=1}^{\infty} F(A\cap I_k)) \leq \sum_{k=1}^{\infty} \mu(F(A\cap I_k)) \leq \epsilon$, hence $\mu^*(F(A)) = 0$. Now let $B \subseteq E$ be such that $\mu(B) = 0$ then we know from the first part that $\mu(F(B\cap E_m)) = 0$ since $\mu(B\cap E_m) = 0$ for all $m \in \mathbb{N}, m \geq 1$. Therefore we get that $\mu^*(F(B)) \leq \sum_{m=1}^{\infty} \mu^*(F(B\cap E_m)) = 0$. So F maps sets of measure zero to sets of measure zero. □

Additionally the proof of the above lemma shows that also every AC$_*$ functions maps sets of measure zero to sets of measure zero. This follows also from the fact that every AC$_*$ function is ACG$_*$. Further the above lemma tells us that the Cantor function F can not be ACG$_*$ since $F(C) = [0, 1]$ (where C is the Cantor set) and so $\mu(F(C)) = 1$ but $\mu(C) = 0$. Moreover this lemma will be important when we want to prove that an ACG$_*$ function is increasing if the lower derivative is greater than or equal to zero a.e. on the given interval (see 1.3.23).

1.3.3. Spaces of BV$_*$, AC$_*$ and ACG$_*$ functions

In this subsection we will establish that all BV$_*$, AC$_*$ respectively ACG$_*$ functions on a given set form a real vector space. We will also try to define a natural topology on these spaces but this can only be achieved for special cases. To be more precise we can define norms analogous to the norms in 1.1.10 for the BV$_*$ and AC$_*$ functions only if the infimum and the supremum belong to the given set. For the ACG$_*$ functions it is still more complicated and in this case we are able to define a norm only if the given set is the whole interval $[a, b]$. In these cases the BV$_*$ and AC$_*$ form a Banach space very similar to 1.2.19 respectively 1.2.20 but the ACG$_*$ functions do not.

1.3.16 Lemma (Vector space structure)
Let $F, G :\ [a, b] \to \mathbb{R}$, let $\alpha, \beta \in \mathbb{R}$ and $E \subseteq [a, b]$.

1. *Then*
$$V_*(\alpha F + \beta G, E) \leq |\alpha| V_*(F, E) + |\beta| V_*(G, E)\ . \tag{1.35}$$
So linear combinations of BV$_*$ functions are BV$_*$ and additionally
$$V_*(\alpha F, E) = |\alpha| V_*(F, E)\ . \tag{1.36}$$

2. *If F and G are AC_* on E then $\alpha F + \beta G$ is AC_* on E.*

3. *If F and G are ACG_* on E then $\alpha F + \beta G$ is ACG_* on E.*

(Cf. Lemma 1.1.8).

Proof: Let $([a_n, b_n])_{n=1}^N$ be a finite collection of non-overlapping intervals in $[a, b]$ with $a_n, b_n \in E$ for $n = 1, \ldots, N$ then by Inequality (1.29) we get that

$$\sum_{n=1}^{N} \omega(\alpha F + \beta G, [a_n, b_n]) \leq |\alpha| \sum_{n=1}^{N} \omega(F, [a_n, b_n]) + |\beta| \sum_{n=1}^{N} \omega(G, [a_n, b_n]) =: \Delta\ .$$

1. From the definition it follows that $\Delta \leq |\alpha|\ V_*(F, E) + |\beta|\ V_*(G, E)$. So by taking the supremum over all finite collections of non-overlapping intervals in $[a, b]$ with end-points in E we get the result.

2. WLOG assume that both α and β are nonzero (the other cases are clear). Let $\epsilon > 0$ then $\exists \delta_1 > 0$ such that $\sum_{n=1}^{M} \omega(F, [x_n, y_n]) < \frac{\epsilon}{2|\alpha|}$ whenever $([x_n, y_n])_{n=1}^M$ is a finite collection of non-overlapping intervals in $[a, b]$ with endpoints in E and $\sum_{n=1}^{M} (y_n - x_n) < \delta_1$

and $\exists\delta_2 > 0$ such that $\sum_{n=1}^{L} \omega(G, [x_n, y_n]) < \frac{\epsilon}{2|\beta|}$ whenever $([x_n, y_n])_{n=1}^{L}$ is a finite collection of non-overlapping intervals in $[a, b]$ with endpoints in E and $\sum_{n=1}^{L} (y_n - x_n) < \delta_2$. Now from the definition we get that $\Delta < \epsilon$ when $\sum_{n=1}^{N} (b_n - a_n) < \min(\delta_1, \delta_2)$. So $\alpha F + \beta G$ is AC$_*$ on E.

3. Write $\bigcup_{n=1}^{\infty} A_n = E = \bigcup_{n=1}^{\infty} B_n$ such that F is AC$_*$ on A_n and G is AC$_*$ on B_n for each $n \in \mathbb{N}, n \geq 1$. Then from Proposition 1.3.12,3 we get that F and G are AC_* on $A_n \cap B_k$ for all $n, k \in \mathbb{N}\backslash\{0\}$. Now from part 2 of this lemma we get that $\alpha F + \beta G$ is AC$_*$ on $A_n \cap B_k$ for all $n, k \in \mathbb{N}\backslash\{0\}$. So we can write $E = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_n \cap B_k$ which is (after an appropriate enumeration) the desired decomposition of E (of course some $A_n \cap B_k$ will be empty), so $\alpha F + \beta G$ is ACG$_*$ on E.

□

The preceding lemma allows us to define the real vector spaces consisting of the BV$_*$, AC$_*$ respectively ACG$_*$ functions a given set.

1.3.17 Definition (Vector spaces of BV$_*$, AC$_*$ and ACG$_*$ functions)
Let $E \subseteq [a, b]$.

1. *The real vector space $BV_*([a, b], E) := \{F : [a, b] \to \mathbb{R} : F$ is BV$_*$ on $E\}$ is called the space of functions of strong bounded variation on E.*

2. *The real vector space $AC_*([a, b], E) := \{F : [a, b] \to \mathbb{R} : F$ is AC$_*$ on $E\}$ is called the space of strongly absolutely continuous functions on E.*

3. *The real vector space $ACG_*([a, b], E) := \{F : [a, b] \to \mathbb{R} : F$ is ACG$_*$ on $E\}$ is called the space of generalized strongly absolutely continuous functions on E. For convenience we write $ACG_*([a, b]) := ACG_*([a, b], [a, b])$.*

Similarly to 1.1.10 we want to define a norm on these spaces such that they are complete i.e. $BV_*([a, b], E)$, $AC_*([a, b], E)$ and $ACG_*([a, b], E)$ become Banach spaces. Unfortunately we cannot easily adapt the norms from 1.1.10 to achieve this goal. We will start this part with a discussion of the problems arising and we give (partial) solutions. We start with $F \in BV_*([a, b], E)$. Unlike Theorem 1.2.18 we cannot define $\|F\|_{BV_*} := \|F\|_1 + V_*(F, E)$, since F need not be Lebesgue integrable on the whole interval $[a, b]$. For example take $E := [1, 2]$ and $a := 0, b := 2$ then F is BV on $[1, 2]$ and hence Lebesgue integrable on $[1, 2]$ (cf. Subsection 1.2.3) but F could be any function on $[0, 1[$. So in general F is Lebesgue integrable only on E and therefore we have to modify $\|F\|_{BV_*}$. We can try to restrict the L^1-norm to $[c, d]$ where $c := \inf E$, $d := \sup E$ but then $\|.\|_{BV_*}$ is not positive definite. For similar reasons $\|F\|_e := |F(a)| + V_*(F, E)$ does not work (in general $a \notin E$) and also $\|F\|_\infty^{BV_*} := \|F\|_\infty + V_*(F, E)$ does not work (in the above example F can be unbounded on $[0, 1[$). The problem is that we get only information about F on $[c, d]$ where $c := \inf E$ and $d := \sup E$ but of course it is kind of unnatural to investigate such functions which are defined on a larger interval than $[c, d]$. So if we now assume that $a, b \in E$ then the problem simplifies and we can define all three norms. Observe that it does not suffice to assume

that $a = c$ and $b = d$, since the norms would not be positive definite, because a function F with norm equal to zero could be zero on $[0, 1[$ and $F(1) = 1$ (where $E := [0, 1[$).
We summarize this in the following theorem.

1.3.18 Theorem (BV_* and AC_* are Banach spaces)
Let $E \subseteq [a, b]$ and $a, b \in E$, then $BV_*([a, b], E)$ is a Banach space with respect to any of the three norms: $\|.\|_{BV_*}$, $\|.\|_e$ and $\|.\|_\infty^{BV_*}$. Moreover $AC_*([a, b], E)$ is a closed subspace.

Proof: We will omit the proof since it is completely analogous to theorems 1.2.18, 1.2.19, 1.2.20 and the discussion afterwards. We just remark that from Proposition 1.3.12,2 we know that every $F \in BV([a, b], E)$ is bounded on $[a, b]$, hence the definitions of $\|.\|_{BV_*}$, $\|.\|_e$ and $\|.\|_\infty^{BV_*}$ make sense. Now everything follows as in Section 1.2, we just have to replace the weak by the strong variation and use some properties of the oscillation. □

Now we investigate the structure of $ACG_*([a, b], E)$. The first observation is, that it is not a subspace of $BV_*([a, b], E)$ (by Example 1.3.11,1) and so we need a different approach. We cannot define a norm in a natural way. For example $V_*(F, E)$ could be infinite or F could be unbounded on $[a, b]\backslash E$. Even if we assume that $a, b \in E$ as before the situation does not improve, since in a sense countable sets do not matter for ACG_* functions. If we further assume that $E = [a, b]$, then $ACG_*([a, b], [a, b])$ is a subspace of $\mathcal{C}([a, b])$ and so it is normed by the supremum-norm $\|.\|_\infty$. The downside is that with this norm the space is not complete. A counterexample is the Cantor function (again). The Cantor function F is a uniform limit of AC (and hence ACG_*, since $E = [a, b]$) functions, but it is not ACG_*. See the remark after Lemma 1.3.15 and see also Theorem 1.20 and 1.21 in [Gor94, p.13f.]. The supremum-norm in this context will be also called Alexiewicz norm, which will be discussed further in the following section.
Another approach is to use the information that if $F \in ACG_*([a, b])$, then $[a, b]$ can be written as a countable union of sets on which F is AC_*. To do this fix an increasing sequence of closed sets $\mathcal{E} := (E_n)_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty E_n = [a, b]$ and define $ACG_*([a, b], \mathcal{E}) := \{F \in ACG_*([a, b]) : V_*(F, E_n) < \infty \, \forall n \in \mathbb{N}, n \geq 1\}$. From Lemma 1.3.16 it follows that $ACG_*([a, b], \mathcal{E})$ is a subspace of $ACG_*([a, b], [a, b])$. Now we can define seminorms $p_n(F) := V_*(F, E_n)$ for $n \in \mathbb{N}, n \geq 1$ and $p_0(F) := |F(a)|$. Then from (1.36) we conclude that these are really seminorms and so they define a locally convex topology on $ACG_*([a, b], \mathcal{E})$ (cf. [Heu82, Ex. 83.3, p.323f.]). It can be shown that equipped with this topology $ACG_*([a, b], \mathcal{E})$ is complete and metrizable. Since the topology is generated by countably many seminorms, the metrizability is clear and for the completeness see Theorem 3.1 [Tho99, p.717f.]. At this point we can observe that for any increasing sequence of closed sets $\mathcal{E}$ with $\bigcup_{n=1}^\infty E_n = [a, b]$ we get a canonical injection $i_\mathcal{E}$ from $ACG_*([a, b], \mathcal{E})$ into $ACG_*([a, b])$. Therefore we can define the topology on $ACG_*([a, b])$ to be the finest locally convex topology such that all canonical injections $i_\mathcal{E} : ACG_*([a, b], \mathcal{E}) \to ACG_*([a, b])$ are continuous. Surprisingly this topology is the same as the topology generated just by the Alexiewicz norm (see Theorem 4.1, [Tho99, p.723-725]). So in the end we may as well just consider the incomplete normed space $(ACG_*([a, b]), \|.\|_\infty)$. We will come back to this topic in the following section, since, as it will turn out, the ACG_* functions F on $[a, b]$ with $F(a) = 0$ are the primitives of Denjoy integrable functions.

1.3.4. Differentiation

Not surprisingly a BV$_*$ (or AC$_*$, ACG$_*$) function is differentiable a.e. on the given subset of $[a, b]$ and also accordingly to the results in Section 1.2 there are conditions that ensure that a function is ACG$_*$ (for example differentiability n.e.). Then our next step is to prove some basic properties we will need in the following section to define the classical Denjoy integral.

1.3.19 Proposition (Differentiability of BV$_*$ functions)
If $F : [a, b] \to \mathbb{R}$ is BV$_*$ on $E \subseteq [a, b]$ nonempty and closed then F is differentiable a.e. on E. (Cf. [Gor94, Thm. 6.18, p.100]).

Proof: Define $c := \inf E \geq a$ and $d := \sup E \leq b$. Let $]c, d[\backslash E = \bigcup_{k=1}^{\infty} I_k$ the intervals contiguous to E in $[c, d]$ and $I_k =]x_k, y_k[$ $(k = 1, \ldots, \infty)$. Then we define functions $m, M : [c, d] \to \mathbb{R}$ by

$$m(x) := \begin{cases} F(x) & x \in E, \\ \inf\{F(t) : t \in \overline{I_k}\} & x \in I_k, \end{cases} \quad \text{and} \quad M(x) := \begin{cases} F(x) & x \in E, \\ \sup\{F(t) : t \in \overline{I_k}\} & x \in I_k. \end{cases}$$

Now we claim that m and M are BV on $[c, d]$. First we will show that m is BV on $[c, d]$. So let $([a_n, b_n])_{n=1}^{N}$ be a finite collection of non-overlapping intervals in $[c, d]$ and define $c_n := m(b_n) - m(a_n)$ for $(n = 1, \ldots, N)$. By the definition of m we have to distinguish four cases corresponding to the different cases where the endpoints are. Fix $n \in \{1, \ldots, N\}$.
First case: $a_n, b_n \in E$
By Equation (1.27) we get that $|c_n| \leq \omega(F, [a_n, b_n])$.
Second case: $a_n, b_n \in I_k$ for an $k \in \mathbb{N}$
In this case $c_n = 0$.
Third case: $a_n \in I_k$, $b_n \in E$ or $a_n \in E$,$b_n \in I_k$ for an $k \in \mathbb{N}$
WLOG we can assume the former and so $c_n = F(b_n) - \inf\{F(t) : t \in \overline{I_k}\}$. If $c_n \geq 0$ then $|c_n| = c_n = F(b_n) - \inf\{F(t) : t \in \overline{I_k}\} = \sup\{F(b_n) - F(t) : t \in \overline{I_k}\} \leq \omega(F, [x_k, b_n])$. If $c_n < 0$ then $|c_n| = -c_n = \inf\{F(t) : t \in \overline{I_k}\} - F(b_n) = \inf\{F(t) - F(b_n) : t \in \overline{I_k}\} \leq \sup\{F(t) - F(b_n) : t \in \overline{I_k}\} \leq \omega(F, [x_k, b_n])$.
Fourth case: $a_n \in I_k$ and $b_n \in I_l$ for $k, l \in \mathbb{N}, k \neq l$
If $c_n \geq 0$ then $|c_n| = c_n = \inf\{F(t) : t \in \overline{I_l}\} - \inf\{F(s) : s \in \overline{I_k}\} \leq \sup\{F(t) : t \in \overline{I_l}\} - \inf\{F(s) : s \in \overline{I_k}\} \leq \sup\{F(t) : t \in \overline{I_k} \cup \overline{I_l}\} - \inf\{F(s) : s \in \overline{I_k} \cup \overline{I_k}\} = \omega(F, [x_k, y_l])$ by Equation (1.28). Similarly if $c_n < 0$ then $|c_n| = -c_n \leq \omega(F, [x_k, y_l])$.
So in all four cases we get that either $c_n = 0$ or $|c_n| \leq \omega(F, [e_1, e_2])$ where $e_1, e_2 \in E$.
At this point we need this collection of intervals to be non-overlapping. Unfortunately this is not the case, so we have to split them into five collections of non-overlapping intervals. The first collection consists of the intervals from the first case, these are non-overlapping by assumption. The second collection consists of the intervals from the third case which are in the form $[x_k, b_n]$ with $b_n \in E$. Now assume that $z \in]x_k, b_n[\cap]x_l, b_m[$ with $m \neq n$. Since $]a_n, b_n[\cap]a_m, b_m[= \emptyset$ it follows that $z \in]x_k, y_k[\cap]x_l, y_l[$, hence $k = l$. This yields a contradiction since either $x_k \leq a_n < b_n \leq a_m < b_m \leq y_k$ or $x_k \leq a_m < b_m \leq a_n < b_n \leq y_k$ but in both cases $I_k \cap E \neq \emptyset$, a contradiction. The third collection consists of the intervals from the third case

which are in the form $[a_n, y_k]$ with $a_n \in E$. As for the second collection it can be shown that it consists of non-overlapping intervals. The collection of intervals from the fourth case could be overlapping, since if we take two such intervals say $[x_n, y_m]$ and $[x_k, y_l]$ with $n \neq m, k \neq l$ then for example $m = k$ i.e. y_m and x_k lie in same interval I_k. Fortunately there could be at most two such points for every interval I_i. So if we split this collection appropriately into two collections we get two finite collections of non-overlapping intervals with endpoints in E.

Now if we put everything together we conclude that $\sum_{n=1}^{N} |c_n| \leq 5\ V_*(F, E)$. Taking the supremum over all finite collections of non-overlapping intervals in $[c, d]$ yields $V(m, [c, d]) \leq 5\ V_*(F, E) < \infty$. This proves that m is BV on $[c, d]$. Similarly (tedious) one can show that M is BV on $[c, d]$. Now we can conclude that m and M are differentiable a.e. on $[c, d]$ by Proposition 1.2.8. At this point we define the set

$$H := \{x \in E : \exists \text{ sequences } (y_n)_n, (z_n)_n \text{ in } E \text{ s.t. } y_n \nearrow x, z_n \searrow x\ (n \to \infty) \text{ and } m'(x), M'(x) \text{ exist}\}.$$

Our first claim is that $\mu(E \backslash H) = 0$ but it is a general fact that the set of points of any subset of $\mathbb{R}$, which are not two-sided limit points of this set, is countable (see for example [Gor94, Exc. 6.12, p.308]. Our second and last claim is that F is differentiable at all points of H. So let $x \in H$, therefore x is a two-sided limit point and since $m|_E = F|_E = M|_E$ we deduce that

$$M'(x) = \lim_{y \to x, y \in E} \frac{M(y) - M(x)}{y - x} = \lim_{y \to x, y \in E} \frac{m(y) - m(x)}{y - x} = m'(x).$$

Additionally observe that $m \leq F \leq M$ on $[c, d]$. Now let $y \in]x, d[$ and consequently

$$\underbrace{\frac{m(y) - m(x)}{y - x}}_{\to m'(x)\ (y \searrow x)} \leq \frac{F(y) - F(x)}{y - x} \leq \underbrace{\frac{M(y) - M(x)}{y - x}}_{\to M'(x)\ (y \searrow x)}.$$

This time we let $y \in]c, x[$ and thus

$$\underbrace{\frac{M(y) - M(x)}{y - x}}_{\to M'(x)\ (y \nearrow x)} \leq \frac{F(y) - F(x)}{y - x} \leq \underbrace{\frac{m(y) - m(x)}{y - x}}_{\to m'(x)\ (y \nearrow x)}.$$

This calculation shows that F is differentiable at x with $m'(x) = F'(x) = M'(x)$. □

From this proposition we get easily that also every ACG$_*$ function is differentiable a.e. on the given set.

1.3.20 Corollary

If $F : [a, b] \to \mathbb{R}$ is ACG$_*$ on $E \subseteq [a, b]$ nonempty then F is differentiable a.e. on E. (Cf. [Gor94, Cor. 6.19, p. 100]).

Proof: Write $E = \bigcup_{n=1}^{\infty} E_n$ such that F is AC$_*$ on each E_n, hence F is also BV$_*$ on each E_n. Then by Corollary 1.3.14 we know that F is also BV$_*$ on each $\overline{E_n}$ and so by Proposition 1.3.19 differentiable a.e. on $\overline{E_n}$ for all $n \in \mathbb{N}, n \geq 1$. Therefore F is differentiable a.e. on $\bigcup_{n=1}^{\infty} \overline{E_n} \supseteq E$ and hence differentiable a.e. on E. □

At this point we want to characterize ACG$_*$ functions in terms of differentiability. The following theorem allows us to recognize functions which have finite lower and upper derivative nearly everywhere to be ACG$_*$.

1.3.21 Theorem (Differentiability nearly everywhere implies ACG$_*$)
Let $F:\ [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$ nonempty. If $-\infty < \underline{D}F(x) \leq \overline{D}F(x) < \infty$ nearly everywhere on E, then E can be written as countable union of sets, i.e. $E = \bigcup_{n=1}^{\infty} E_n$, such that F is AC$_*$ on each E_n. If additionally $F|_E$ is continuous on E then F is ACG$_*$ on E. (Cf. [Gor94, Thm. 6.22, p.103]).

Proof: Define $A := \{x \in E : -\infty < \underline{D}F(x) \leq \overline{D}F(x) < \infty\}$ and $B := E \setminus A$. By assumption B is countable and therefore F is AC$_*$ on each singleton $\{b\}$ where $b \in B$. So it remains to show that $A = \bigcup_{n=1}^{\infty} E_n$ such that F is AC$_*$ on each E_n. Let $n \in \mathbb{N}\setminus\{0\}$ and define $A_n := \{x \in A : y \in [x, x+\frac{1}{n}]$ implies $|F(y) - F(x)| \leq n(y-x)\}$. It is clear that $A = \bigcup_{n=1}^{\infty} A_n$ because the upper and lower derivatives of F at all elements of A are finite. Now we define for $i \in \mathbb{Z}$ $A_n^i := A_n \cap]\frac{i}{n}, \frac{i+1}{n}]$. If $u, v \in A_n^i$ with $u < v$ and $x, y \in [u,v]$ with $x < y$, then $|F(y) - F(x)| \leq |F(y) - F(u)| + |F(u) - F(x)| \leq n(y-u) + n(x-u) \leq 2n(v-u)$. Then by taking the supremum over all $x, y \in [u,v]$ with $x < y$ we conclude that $\omega(F, [u,v]) \leq 2n(v-u)$. From this we easily see that F is AC$_*$ on A_n^i for all $n \in \mathbb{N}\setminus\{0\}$ and $i \in \mathbb{Z}$. We can write E as $E = \bigcup_{n=1}^{N} \bigcup_{i=-\infty}^{\infty} A_n^i \cup \bigcup_{n=1}^{\infty} \{b_n\}$ where $b_1, b_2, \ldots$ is an enumeration of B. This proves the first part of the claim and of course F is ACG$_*$ on E if $F|_E$ is continuous on E. □

From now on we want to restrict our attention to the special case $E = [a,b]$ and ACG$_*$ functions on $[a,b]$. These functions will be the primitives of Denjoy integrable functions, as we will see in the following section. Now we state an immediate consequence of Theorem 1.3.21, since that result will be used later on extensively.

1.3.22 Corollary
Let $F:\ [a,b] \to \mathbb{R}$ continuous on $[a,b]$ and differentiable n.e. on $[a,b]$ then F is ACG$_*$ on $[a,b]$.

Proof: This follows directly from Theorem 1.3.21 with the special case $E = [a,b]$. □

Now we are aiming at the uniqueness of primitives of Denjoy integrable functions. To this end we will give a sufficient condition of an ACG$_*$ function to be monotonically increasing. Not surprisingly it is monotonically increasing if the derivative is greater or equal than zero almost everywhere.

1.3.23 Theorem
Let $F:\ [a,b] \to \mathbb{R}$ be ACG$_*$ on $[a,b]$ such that $\underline{D}F \geq 0$ a.e. on $[a,b]$, then F is monotonically increasing on $[a,b]$. (Cf. [Gor94, Lem. 6.24 and Thm. 6.25, p.103f.] or [Bru78, Thm. 4.1, p. 189]).

Proof: Since F is ACG$_*$ on $[a,b]$, F is continuous on $[a,b]$. Let $\epsilon > 0$ and define $G:\ [a,b] \to \mathbb{R}$ by $G(x) := F(x) + \epsilon x$. Then G is also ACG$_*$ on $[a,b]$ since a linear combination of two ACG$_*$ functions is ACG$_*$ again (see 1.3.16, 3). Now we define the set $A := \{x \in [a,b] : \underline{D}G(x) \leq 0\}$ and since $\underline{D}G(x) = \underline{D}F(x) + \epsilon > 0$ a.e. on $[a,b]$, the set A has measure zero. Now we can apply Lemma 1.3.15 to get that $\mu(G(A)) = 0$. In particular $G(A)$ does not contain any intervals. At this point we claim that G is monotonically increasing on $[a,b]$. We will prove this claim by contradiction, so assume there are $u,v \in [a,b]$ such that $u < v$ and $G(u) > G(v)$. There is a point $y_0 \in]G(v), G(u)[\setminus G(A)$, since $G(A)$ contains no intervals. Define $x_0 := \sup\{x \in [u,v] : G(x) \geq y_0\}$. Now by the definition of y_0, $x_0 \in]u,v[$ and by the definition of x_0, $G(x) \leq y_0$ for all $x \in [x_0, v]$. Furthermore by the continuity of G we get that $G(x_0) \geq y_0$ and so $G(x_0) = y_0$. This yields that $\underline{D}G(x_0) \leq 0$, since $G(x) \leq G(x_0)$ for all $x \in [x_0, v]$, hence $x_0 \in A$. A contradiction to the definition of x_0, so G is monotonically increasing on $[a,b]$ and therefore F too because $\epsilon > 0$ was arbitrary. □

A direct consequence is the following proposition, which establishes the uniqueness of the primitive of a Denjoy integrable function. We had an analogous result for the primitive of a Lebesgue integrable function (which is AC) in 1.2.12.

1.3.24 Proposition
Let $F : [a,b] \to \mathbb{R}$ be ACG$_*$ on $[a,b]$. If $F' = 0$ a.e. on $[a,b]$ then F is constant. (Cf. [Gor94, Cor. 6.26, p. 104]).

Proof: Now we can apply Theorem 1.3.23 to F and $-F$ to conclude that F is monotonically increasing and monotonically decreasing at the same time, hence constant. □

1.3.5. Summary

We conclude this section by summarizing the main topics in this part of the thesis. First we introduced the concept of oscillation and stated some basic results. With the help of the oscillation we defined the notion of weak and strong variation. Similar to Section 1.1 we established elementary properties of the (strong) variation. Second we defined BV, BV$_*$, AC$_*$ and ACG$_*$ and investigated their relationship with another and the concepts from the first section of this thesis. A major result was that every AC$_*$ function is also BV$_*$, which required substantially more work than the analogous result for BV and AC functions. The next topic was to establish the vector space structure of these functions and equipping these spaces with a topology. The last topic was the differentiability of these functions and related results. As it turned out every BV$_*$ and hence also every AC$_*$ and ACG$_*$ function is differentiable a.e. on the given set. Moreover we proved that a continuous function which has finite lower and upper derivatives n.e. on a closed interval is ACG$_*$ on this interval. We ended this section by determining that an ACG$_*$ function with derivate equal to zero a.e. is constant.
Below we summarize important properties of BV$_*$, AC$_*$ and ACG$_*$ in a table, where the columns are given and we have shown the results in the rows. That means for example in column 3, row 3: in general an ACG$_*$ function is not AC$_*$. Additionally it should be understood that all functions are defined on $[a,b]$, $E \subseteq [a,b]$ is the given set and $c := \inf E$, $d := \sup E$ (E nonempty).

	BV_*	AC_*	ACG_*	reference
$\Rightarrow$ bounded on $[c, d]$	y	y	n	1.3.12,2; 1.3.11,5
$\Rightarrow$ BV_*		y	n	1.3.13; 1.3.11,1
$\Rightarrow$ AC_*	n		n	1.3.10,3; 1.3.11,1
$\Rightarrow$ ACG_*	n	y		1.1.2,6; 1.3.10,2
$\Rightarrow$ differentiable a.e.	y	y	y	1.3.19, 1.3.20
$\Rightarrow$ maps zero sets to zero sets	n	y	y	1.3.15 and afterwards

Table 1.3.: Important properties of BV_*, AC_* and ACG_* functions

1.4. The classical Denjoy integral

In this section we can finally give the definition of the classical Denjoy integral. Now is the time to harvest and we will see that almost all results follow from work done in Sections 1.1 through 1.3. After the definition we will prove the most elementary properties of the classical Denjoy integral and investigate its relationship with the Riemann and Lebesgue integral. Subsection 1.4.2 is especially devoted to examine the relationship between the improper Riemann integral and the classical Denjoy integral. There we will show that every improper Riemann integrable function is classically Denjoy integrable. In the last subsection we will discuss briefly the topology of the classical Denjoy integrable functions. The topology is given by the Alexiewicz norm, which is essentially given by the supremum-norm of the definite ACG$_*$ primitives.

1.4.1. Definition of the classical Denjoy integral

After we have done a lot of work in Sections 1.1 through 1.3 we are now able to give the definition of classical Denjoy integrability. Then we will establish that with this definition we have achieved our main goal: every derivative is classically Denjoy integrable and the FTC holds. Furthermore we will prove some basic facts about the new integral but as we will see these results follow easily from results in Section 1.3.

1.4.1 Definition (Classical Denjoy integrability)
Let $f : [a, b] \to \mathbb{R}$ and $E \subseteq [a, b]$ measurable.

1. *The function f is said to be classically Denjoy integrable if there exists an ACG$_*$ function F on $[a, b]$ (i.e. $F \in ACG_*([a, b])$) such that $F' = f$ a.e. on $[a, b]$. In this case such an F is called an indefinite integral or just a(n) (indefinite) primitive for f.*

2. *Moreover f is said to be classically Denjoy integrable on E if $f\chi_E$ is classically Denjoy integrable, where χ_E denotes the characteristic function of the set E.*

3. *We write $\mathcal{D}([a, b])$ respectively $\mathcal{D}([a, b], E)$ for the sets of classically Denjoy integrable functions on $[a, b]$ respectively E. We will see in Proposition 1.4.5,1 that these sets are vector spaces.*

For any classically Denjoy integrable function f there are infinitely many indefinite integrals for f. To see this, note that $F + c$, where F is an indefinite integral for f and $c \in \mathbb{R}$, is an indefinite integral for f again (because $(F + c)' = F' = f$ and Lemma 1.3.16,3). So if we further assume that $F(a) = 0$ for the indefinite integral, then there is only one such primitive. This follows from Proposition 1.3.24 since if F, G are two indefinite integrals for f, then $(F - G)' = f - f = 0$, hence $F - G$ is constant. Therefore we can define the notion of a definite integral.

1.4.2 Definition (Definite classical Denjoy integral)
Let $F : [a, b] \to \mathbb{R}$ be an indefinite integral of the classically Denjoy integrable function $f : [a, b] \to \mathbb{R}$. When F satisfies $F(a) = 0$, then F is called a definite integral or just an

integral for f. We will write $D\int_a^x f := F(x)$ or just $\int_a^x f$ when there is no ambiguity between the classical Denjoy integral and the Lebesgue or Riemann integral.

The classical Denjoy integral behaves as nicely as the Lebesgue (respectively the Riemann) integral when dealing with subintervals.

1.4.3 Proposition
Let $f : [a, b] \to \mathbb{R}$.

1. *If f is classically Denjoy integrable on $[a, b]$, then it is also classically Denjoy integrable on every subinterval of $[a, b]$.*
2. *Let $c \in]a, b[$ and assume that f is classically Denjoy integrable on $[a, c]$ and $[c, b]$, then f is classically Denjoy integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.*

(Cf. [Gor94, Thm. 7.3, p.109] or [CD89, Thm. 29, p.28]).

Proof:

1. By the definition of classical Denjoy integrability there is an $F \in ACG_*([a, b])$ such that $F' = f$ a.e. on $[a, b]$. Now let $a \le c < d \le b$, then by Proposition 1.3.12,4 we know that F is also ACG$_*$ on $[c, d]$ and since $F' = f$ a.e. on $[c, d]$ we conclude that f is classically Denjoy integrable on $[c, d]$.
2. Since $f \in \mathcal{D}([a, c])$ and $f \in \mathcal{D}([c, b])$ there are $F_1 \in ACG_*([a, c])$ and $F_2 \in ACG_*([c, b])$ such that $F_1' = f$ a.e. on $[a, c]$ and $F_2' = f$ a.e. on $[c, b]$. Now we define $F(x) := \begin{cases} F_1(x) + F_2(c) - F_1(c) & a \le x \le c, \\ F_2(x) & c < x \le b. \end{cases}$ Then F is continuous on $[a, b]$ since $F(c) = F_2(c) = \lim_{x \searrow c} F(x)$ and because we can decompose $[a, c]$ and $[c, b]$ separately such that F is AC$_*$ on each component, it is clear that F is ACG$_*$ on $[a, b]$. So $F \in \mathcal{D}([a, b])$. Furthermore $\int_a^b f = F(b) - F(a) = F_2(b) - F_1(a) - F_2(c) + F_1(c) = F_1(c) - F_1(a) + F_2(b) - F_2(c) = \int_a^c f + \int_c^b f$.

□

Now we can show that every derivative is classically Denjoy integrable and the FTC holds, thus we have achieved our main goal. Moreover the assumptions can be weakened to requiring only that the function is differentiable nearly everywhere on the given interval and that it is continuous.

1.4.4 Theorem (FTC)
Let $F : [a, b] \to \mathbb{R}$ be continuous and differentiable nearly everywhere on $[a, b]$, then F' is classically Denjoy integrable on $[a, b]$ and $\int_a^x F' = F(x) - F(a)$. (Cf. [Gor94, Thm. 7.2, p.108]).

Proof: By Corollary 1.3.22 we know that $F \in ACG_*([a, b])$ and hence $F' \in \mathcal{D}([a, b])$. Now by Proposition 1.4.3 we get that $F' \in \mathcal{D}([a, x])$ for all $x \in [a, b]$ and by the definition of the definite Denjoy integral $\int_a^x F' = F(x) - F(a)$. □

The FTC gives us immediately many examples of classical Denjoy integrable functions. Every derivative of a continuous and nearly everywhere differentiable function on an interval is classical Denjoy integrable. Especially we can now integrate the derivative of the function F from Example 0.1.3 and $\int_0^x F' = F(x)$ for all $x \in [0,1]$. One easy example of a function which is not classically Denjoy integrable is the function $f(x) = \frac{1}{x}$ for $x \in]0,1]$ and $f(0) = 0$.
The classical Denjoy integral has all the properties one expects of an integral. Throughout this subsection it suffices to prove these properties for the case where we integrate over an interval $[a,b]$ since the general case where we integrate over a measurable subset $E \subseteq [a,b]$ follows easily from this special case and $\int_E f = \int_a^b f\chi_E$.

1.4.5 Proposition
Let $f, g \in \mathcal{D}([a,b])$.

1. *For all $\lambda, \mu \in \mathbb{R}$ the function $\lambda f + \mu g$ is in $\mathcal{D}([a,b])$. So $\mathcal{D}([a,b])$ respectively $\mathcal{D}([a,b],E)$ (for $E \subseteq [a,b]$ measurable) are vector spaces. Additionally $\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g$.*
2. *If $f \leq g$ (respectively $f = g$) a.e. on $[a,b]$, then $\int_a^b f \leq \int_a^b g$ (respectively $\int_a^b f = \int_a^b g$).*
3. *If $h : [a,b] \to \mathbb{R}$ and $f = h$ a.e. on $[a,b]$ then $h \in \mathcal{D}([a,b])$ and $\int_a^b f = \int_a^b h$.*

(Cf. [Gor94, Thm. 7.4 and 7.5, p.109] or [CD89, Thm. 33 and 34, p. 29]).

Proof: Since $f, g \in \mathcal{D}([a,b])$ there are $F, G \in ACG_*([a,b])$ such that $F' = f$ and $G' = g$ a.e. on $[a,b]$.

1. Define the function $H := \lambda F + \mu G$, then $H \in ACG_*([a,b])$ by Lemma 1.3.16,3 and $H' = \lambda F' + \mu G' = \lambda f + \mu g$, so $\lambda f + \mu g \in \mathcal{D}([a,b])$. Moreover $\int_a^b (\lambda f + \mu g) = H(b) - H(a) = \lambda F(b) + \mu G(b) - \lambda F(a) - \mu G(a) = \lambda(F(b) - F(a)) + \mu(G(b) - G(a)) = \lambda \int_a^b f + \mu \int_a^b g$.
2. Define the function $K := G - F$, then $K \in ACG_*([a,b])$ again by Lemma 1.3.16,3. Additionally $K' = G' - F' = g - f \geq 0$ and so by Theorem 1.3.23 K is monotonically increasing. That is for all $x, y \in [a,b]$ with $x < y$ we have that $K(x) \leq K(y)$, which can be written as $F(y) - F(x) \leq G(y) - G(x)$. This yields that $\int_a^b f = F(b) - F(a) \leq G(b) - G(a) = \int_a^b g$.
3. By assumption we have that $F' = f = h$ a.e. on $[a,b]$ and since $F \in ACG_*([a,b])$ we conclude that $h \in \mathcal{D}([a,b])$ and $\int_a^b h = F(b) - F(a) = \int_a^b f$.

□

The following proposition is an analogue of Theorem 1.2.11 which was about the primitives of Lebesgue integrable functions. Note that of course the primitive will not be AC in general.

1.4.6 Proposition
Let $f \in \mathcal{D}([a,b])$ and define $F(x) := \int_a^x f$ $(x \in [a,b])$.

1. *Then F is continuous on $[a,b]$ (more specifically it is ACG$_*$ on $[a,b]$).*

2. *Moreover F is differentiable a.e. on $[a,b]$ and $F' = f$ a.e. on $[a,b]$.*

3. *Additionally f is measurable on $[a,b]$.*

(Cf. [Gor94, Thm. 7.6, p. 109] or [CD89, Thm. 32, p.28f.]).

Proof: Since $f \in \mathcal{D}([a,b])$ there is a $G \in ACG_*([a,b])$ such that $G' = f$ a.e. on $[a,b]$. Therefore F and G differ only by a constant ($G(x) - G(a) = \int_a^x f = F(x)$), so $F = G - G(a)$ and hence $F \in ACG_*([a,b])$ by Lemma 1.3.16,3.

1. From the definition of ACG$_*$ we know that F has to be continuous on $[a,b]$.

2. By Corollary 1.3.20 F is differentiable a.e. on $[a,b]$ and $F' = G' = f$ by the definition of the primitive of f.

3. From Lemma 1.2.9, 1 we know that F' is measurable and since $F' = f$ a.e. on $[a,b]$ we get that f is measurable. See for example [Gor94, Thm. 2.3, p.18].

□

Now we investigate the relationship between the Lebesgue integral and the classical Denjoy integral. Every Lebesgue integrable function is also classically Denjoy integrable and their integrals agree. Moreover there are essentially three cases when a classically Denjoy integrable function is Lebesgue integrable as we will see in the following theorem.

1.4.7 Theorem
Let $f \in \mathcal{D}([a,b])$.

1. *If f is bounded on $[a,b]$ or*

2. *if $f \geq 0$ (respectively $f \leq 0$) on $[a,b]$ or*

3. *if $f \in \mathcal{D}([a,b], E)$ for all measurable subsets $E \subseteq [a,b]$,*

then in all three cases f is Lebesgue integrable on $[a,b]$.

4. *If $g : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$ then $g \in \mathcal{D}([a,b])$ and $L\int_a^b g = D\int_a^b g$.*

(Cf. [Gor94, Thm. 7.7, p. 109] or [CD89, Thm. 31 and 35, p.28f.]).

Proof: Since $f \in \mathcal{D}([a,b])$ there is an $F \in ACG_*([a,b])$ such that $F' = f$ a.e. on $[a,b]$.

1. In this case f is bounded and by 1.4.6,3 it is also measurable and so Lebesgue integrable on $[a,b]$ (cf. [Gor94, Thm. 3.7, p.32f.]).

2. In this case $F' = f \geq 0$ and so F is monotonically increasing by Theorem 1.3.23. Therefore F' is Lebesgue integrable by 1.2.9,2 and hence f is also Lebesgue integrable on $[a,b]$ since $F' = f$ a.e. on $[a,b]$ (cf. [Gor94, Thm. 3.17(d), p.40]). In the case where $f \leq 0$ just replace f by $-f \geq 0$.

3. Now define the sets $A := \{x \in [a, b] : f(x) \geq 0\}$ and $B := [a, b] \backslash A$. It is clear that A and B are measurable subsets of $[a, b]$ and so f is classically Denjoy integrable on A and B by assumption. Therefore the positive part f^+ respectively the negative part f^- of f are classically Denjoy integrable on $[a, b]$. Since $f^+, f^- \geq 0$ we can conclude that f^+, f^- are Lebesgue integrable on $[a, b]$ by part 2. All in all we get that $f = f^+ - f^-$ is Lebesgue integrable on $[a, b]$.

4. In this case g is Lebesgue integrable on $[a, b]$ and so we can define a primitive by $G(x) := L \int_a^x g$. From Theorem 1.2.11 we know that G is AC on $[a, b]$ and $G' = g$ a.e on $[a, b]$. Now it is clear that G is also ACG$_*$ on $[a, b]$ since it is AC$_*$(=AC) on $[a, b]$. Therefore $L \int_a^b g = G(b) = D \int_a^b g$.

□

1.4.8 Remark (Difference between the classical Denjoy integral and the Lebesgue integral) *From Theorem 1.4.7,4 we know that every Lebesgue integrable function is classically Denjoy integrable. Now what do we know if $f \in \mathcal{D}([a, b])$ but f is not Lebesgue integrable? The first observation is that the classical Denjoy integral is a non-absolute integral i.e. we cannot conclude from $f \in \mathcal{D}([a, b])$ that $|f| \in \mathcal{D}([a, b])$. The other implication holds since if $|f| \in \mathcal{D}([a, b])$ we get from Theorem 1.4.7,2 that $|f|$ is Lebesgue integrable and hence also f. Thus f is classically Denjoy integrable by 1.4.7,4. So assume that $f \in \mathcal{D}([a, b])$ but f is not Lebesgue integrable on $[a, b]$ then $|f| \notin \mathcal{D}([a, b])$ since otherwise by Theorem 1.4.7,2 we would deduce that f is Lebesgue integrable. Note that the Lebesgue integral is an absolute integral (i.e. f is Lebesgue integrable if and only if $|f|$ is) and the Riemann integral is a non-absolute integral. Our second observation is that if f is classically Denjoy integrable but not Lebesgue integrable on $[a, b]$ then there has to be a measurable subset of $[a, b]$ such that f is not classically Denjoy integrable on this subset. This follows from Theorem 1.4.7,3. Cf. [Gor94, Remark before Thm. 7.8, p.110]. See also Example 1.4.14.*

Finally we prove the integration by parts formula in the case where one function is classically Denjoy integrable and the other function is AC. This result will be used in Subsection 3.1.1.

1.4.9 Theorem (Integration by parts)
Let $f \in \mathcal{D}([a, b])$, $F(x) := \int_a^x f$ $(x \in [a, b])$ its primitive and let $G : [a, b] \to \mathbb{R}$ be AC on $[a, b]$. Then $f \cdot G$ is classically Denjoy integrable on $[a, b]$ and

$$D \int_a^b (f \cdot G) = F(b)G(b) - L \int_a^b (F \cdot G') . \tag{1.37}$$

Note that the integral on the LHS is a classical Denjoy integral (indicated by 'D') and the integral on the RHS is a Lebesgue integral (indicated by 'L'). (Cf. [Gor94, Thm. 12.6, p.185]).

Proof: From Proposition 1.4.6,1 we know that F is ACG$_*$ on $[a, b]$ and so by Proposition 1.3.12,5 we get that $F \cdot G$ is ACG$_*$ on $[a, b]$. This yields that $(FG)'$ is classically Denjoy integrable on $[a, b]$ (by the FTC). Moreover $F \cdot G'$ is Lebesgue integrable over $[a, b]$ since F is continuous on $[a, b]$ (and hence bounded on $[a, b]$) and G' is Lebesgue integrable by

Lemma 1.2.13. Additionally by Theorem 1.4.7,4 we know that $F \cdot G'$ is also classically Denjoy integrable. Now we observe that $f \cdot G = (F \cdot G)' - F \cdot G'$ a.e. on $[a,b]$ which yields that $f \cdot G \in \mathcal{D}([a,b])$ since the RHS is a classical Denjoy integrable function by Proposition 1.4.5,1 and 3. Finally we calculate using the FTC, Proposition 1.3.12,5 and $F(a) = 0$ that

$$\int_a^b (f \cdot G) = \int_a^b (F \cdot G)' - \int_a^b (F \cdot G') = F(b)G(b) - F(a)G(a) - L\int_a^b (F \cdot G') .$$

□

If we are focusing more on the primitive rather than on the integrable function we get the following corollary.

1.4.10 Corollary

Let $F \in ACG_*([a,b])$ and $G: [a,b] \to \mathbb{R}$ be AC on $[a,b]$, then $F' \cdot G \in \mathcal{D}([a,b])$ and

$$\int_a^b (F' \cdot G) = F(b)G(b) - F(a)G(a) - \int_a^b (F \cdot G') \tag{1.38}$$

holds.

Proof: By the FTC we know that $F' \in \mathcal{D}([a,b])$ and $H := F - F(a) \in ACG_*([a,b])$ is its primitive. Now we can use integration by parts (Equation (1.37)): $\int_a^b (F' \cdot G) = H(b)G(b) - \int_a^b (H \cdot G') = (F(b) - F(a))G(b) - \int_a^b ((F - F(a)) \cdot G') = F(b)G(b) - F(a)G(b) - \int_a^b (F \cdot G') + F(a)\int_a^b G' = F(b)G(b) - F(a)G(a) - \int_a^b (F \cdot G')$. □

1.4.2. Improper Riemann integral and the classical Denjoy integral

In this subsection we investigate the relationship between the classical Denjoy integral and the improper Riemann integral. We will see that we can think of the classical Denjoy integral as kind of 'improper Lebesgue integral'. First we define improper Riemann integrability for the exemplary case where the singularity is at the right endpoint of the interval. The cases where the singularity is at the left endpoint or in the interior work analogously. For simplicity we state and prove all results in this subsection only for the former case. Our main result in this subsection is that every improper Riemann integrable function is classically Denjoy integrable and if a classically Denjoy integrable function $f: [a,b] \to \mathbb{R}$ is Riemann integrable on every subinterval $[a,c]$ for $a < c < b$ then it is improperly Riemann integrable over $[a,b]$ and in both cases the integrals agree. Cf. [KS04, Subsection 2.7, p.42-46] for a discussion of the improper Riemann integral.

1.4.11 Definition

Let $f: [a,b] \to \mathbb{R}$ be such that f is Riemann integrable on every subinterval $[a,c]$ ($a < c < b$). Then f is said to be improperly Riemann integrable over $[a,b]$ if $\lim_{c \nearrow b} \int_a^c f$ exists and we define $\int_a^b f := \lim_{c \nearrow b} \int_a^c f$. We denote all improper Riemann integrable functions on the interval $[a,b]$ by $\mathcal{IR}([a,b])$.

The following example illustrates the difference between proper and improper Riemann integrability.

1.4.12 Example

Define the function $f: [0,1] \to \mathbb{R}$ *by* $f(x) := \begin{cases} \frac{1}{\sqrt{1-x}} & 0 \le x < 1, \\ 0 & x = 1. \end{cases}$ *Then* f *is not Riemann integrable in the proper sense since it is not bounded on* $[0,1]$ *but it is improperly Riemann integrable. Let* $0 \le c < 1$ *then* $\int_0^c f = 2 - \sqrt{1-c}$, *hence* $\lim_{c \nearrow 1} \int_0^c f = 2$. *So* $\int_0^1 f = 2$.

Without further efforts we can prove our main result about the relationship of improper Riemann integrability and classical Denjoy integrability.

1.4.13 Proposition (Improper Riemann integrability versus classical Denjoy integrability) *Let* $f: [a,b] \to \mathbb{R}$.

1. *If* f *is improperly Riemann integrable over* $[a,b]$, *then* f *is also classically Denjoy integrable on* $[a,b]$ *and* $R \int_a^b f = D \int_a^b f$.
2. *If* $f \in \mathcal{D}([a,b])$ *and* f *is Riemann integrable on every interval* $[a,c]$ *where* $a < c < b$, *then* f *is improperly Riemann integrable over* $[a,b]$ *and* $R \int_a^b f = D \int_a^b f$.

Proof:

1. Define the function $F: [a,b] \to \mathbb{R}$ by $F(x) := \begin{cases} \int_a^x f & a \le x < b, \\ \lim_{c \nearrow b} \int_a^c f & x = b. \end{cases}$
   Then it is not hard to see that F is ACG$_*$ on $[a,b]$. We can write $[a,b] = \bigcup_{n=1}^{\infty} [a, b - \frac{1}{n}] \cup \{b\}$ and note that f is Riemann integrable on each interval $[a, b - \frac{1}{n}]$ ($n \in \mathbb{N}, n \ge 1$) by assumption. Therefore f is also Lebesgue integrable on each interval $[a, b - \frac{1}{n}]$ and the integrals over these intervals are equal (see [KS04, Thm. 3.103, p.112f.]). This shows that F is AC$_*$ (equals AC in this case) on these intervals (and trivially on $\{b\}$) and $F' = f$ a.e. on $[a,b]$ by Theorem 1.2.11. Furthermore F is continuous on $[a,b]$ since $F(x) \to F(b)$ for ($x \nearrow b$).

2. Since $f \in \mathcal{D}([a,b])$ there is an $F \in ACG_*([a,b])$ such that $F' = f$ a.e. on $[a,b]$ and $F(a) = 0$. Now let $a < c < b$ then f is Riemann integrable on $[a,c]$ and hence Lebesgue integrable on $[a,c]$ (as in the first part). By Theorem 1.4.7,4 we conclude that $f \in \mathcal{D}([a,c])$ and therefore there exists a $F_c \in ACG_*([a,c])$ with $F_c' = f$ a.e. on $[a,c]$, $F_c(a) = 0$ and $R \int_a^c f = F_c(c)$. From Proposition 1.3.12,4 we know that F is also ACG$_*$ on $[a,c]$ and hence by the uniqueness of the definite Denjoy integral (1.3.24) we get that $F|_{[a,c]} = F_c$. At this point we have to show that $\lim_{c \nearrow b} \int_a^x f$ exists. The only reasonable candidate for this limit is $F(b)$. From the continuity of F at b we know that for $\epsilon > 0$ there is a $\delta > 0$ such that $|F(c) - F(b)| < \epsilon$ when $c \in]b - \delta, b[$. Let $c \in]b - \delta, b[$ then, since $\int_a^c f = F_c(c) = F(c)$, we get that $|\int_a^c f - F(b)| < \epsilon$. So all in all f is improperly Riemann integrable with $R \int_a^b f = \lim_{c \nearrow b} \int_a^c f = F(b) = D \int_a^b f$.

□

To illustrate the above result we give an example of a function which is not Lebesgue integrable but improperly Riemann integrable and hence classically Denjoy integrable.

1.4.14 Example
Set $I_n :=]1-2^{-n}, 1-2^{-n+1}[$ *for* $n \in \mathbb{N}, n \geq 1$*, then* $(I_n)_n$ *is a collection of non-overlapping intervals in* $[0,1]$ *and we define the function* $f: [0,1] \to \mathbb{R}$ *by* $f(x) := \begin{cases} 2^n(-1)^{n+1}\frac{1}{n} & x \in I_n, \\ 0 & \text{otherwise.} \end{cases}$
Then f *is not Lebesgue integrable since* $|f|$ *is not Lebesgue integrable on* $[0,1]$*:* $\int_0^1 |f| = \sum_{n=1}^{\infty} \int_{I_n} |f| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ *. On the other hand* f *is improperly Riemann integrable on* $[0,1]$*:* $\int_0^1 f = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = \log(2)$ *and hence by Proposition 1.4.13,1 it is classically Denjoy integrable on* $[0,1]$ *with* $D\int_0^1 f = R\int_0^1 f = \log(2)$*.*

The following result can be understood as saying that there is no 'improper' classical Denjoy integral. This should mean that if we define a concept analogous to the improper Riemann integral by just replacing 'Riemann' with 'classical Denjoy' then we get no new concept since all the functions will be classically Denjoy integrable in the proper sense.

1.4.15 Proposition
Let $f: [a,b] \to \mathbb{R}$ *such that* $f \in \mathcal{D}([a,c])$ *for all* $a < c < b$*. If the limit* $\lim_{c \nearrow b} \int_a^c f$ *exists then* $f \in \mathcal{D}([a,b])$ *and* $\int_a^b f = \lim_{c \nearrow b} \int_a^c f$*. (Cf. [Gor94, Thm. 7.11, p. 111]).*

Proof: The proof is very similar to the proof of 1.4.13,1. Let $a < c < b$, then by assumption there is a $F_c \in ACG_*([a,c])$ such that $F_c' = f$ a.e. on $[a,c]$ and $F_c(a) = 0$ (and so $\int_a^c f = F_c(c)$). By the uniqueness of the definite classical Denjoy integral (1.3.24) we can conclude that for $a < c < d < b$ the restriction of F_d to $[a,c]$ agrees with F_c i.e. $F_c = F_d|_{[a,c]}$. Now we define the function $F(x) := \begin{cases} 0 & x = a, \\ F_x(x) & a < x < b, \\ \lim_{c \nearrow b} \int_a^c f & x = b. \end{cases}$ We will show that $F \in ACG_*([a,b])$ and $F' = f$ a.e. on $[a,b]$ because this will prove that $f \in \mathcal{D}([a,b])$. For $n \in \mathbb{N}, n \geq 1$, define $c_n := b - \frac{1}{n}$. By assumption $F_{c_n} \in \mathcal{D}([a,c_n])$ and so F is continuous on $[a,c_n]$ and $[a,c_n] = \bigcup_{m=1}^{\infty} E_m^n$ where F_{c_n} is AC$_*$ on each E_m^n. At this point we can write $[a,b] = \bigcup_{n,m=1}^{\infty} E_m^n \cup \{b\}$ and F is AC$_*$ on each E_m^n from the definition of F. Since every F_{c_n} is continuous on $[a,c_n]$ and from the restriction property we conclude that F is continuous on $[a,b[$. It remains to show that F is continuous at b. From $\int_a^c f = F_c(c)$ we conclude that $\lim_{c \nearrow b} F(c) = \lim_{c \nearrow b} F_c(c) = \lim_{c \nearrow b} \int_a^c f = F(b)$ and hence F is continuous at b. Once again by the restriction property we deduce that $F' = f$ a.e. on $[a,b]$ and so $\int_a^b f = F(b) = \lim_{c \nearrow b} \int_a^c f$. □

1.4.3. The Alexiewicz norm and a topology on $\mathcal{D}([a,b])$

To get a norm on $\mathcal{D}([a,b])$ (respectively $\mathcal{D}([a,b],E)$ for a measurable subset E of $[a,b]$) we need to identify functions which are equal a.e. on $[a,b]$ (respectively a.e. on E). This identification yields the spaces $\mathcal{D}([a,b])/\sim$ (respectively $\mathcal{D}([a,b],E)/\sim$) which can be equipped with a canonical norm. For a similar discussion see the articles [Tho99] and [Ost83].

1.4.16 Definition (The Alexiewicz norm)
For $f \in \mathcal{D}([a,b])/\sim$ *define the* *Alexiewicz norm* *of* f *by*

$$\|f\|_A := \sup_{x \in [a,b]} \left| \int_a^x f \right| . \tag{1.39}$$

For $E \subseteq [a,b]$ measurable and $f \in \mathcal{D}([a,b],E)/\sim$ we define

$$\|f\|_{A,E} := \sup_{x\in[a,b]} \left| \int_a^x f\chi_E \right| . \tag{1.40}$$

The following proposition establishes a close connection between the supremum-norm on $ACG_*([a,b])$ and the Alexiewicz norm on $\mathcal{D}([a,b])/\sim$.

1.4.17 Proposition (Isometric isomorphism of a subset of $ACG_*([a,b])$ and $\mathcal{D}([a,b])/\sim$)
Define $A := \{F \in ACG_*([a,b]) : F(a) = 0\}$, then the normed vector spaces $(A, \|.\|_\infty)$ and $(\mathcal{D}([a,b])/\sim, \|.\|_A)$ are isometrically isomorphic.

Proof: Not surprisingly an isomorphism is given by differentiation. So we define $\Phi : A \to \mathcal{D}([a,b])$ by $\Phi(F) := F'$. From the definition of the classical Denjoy integral and the properties of ACG$_*$ functions we get that the definition makes sense and that Φ is linear. To prove injectivity we assume that $\Phi(F) = 0$ a.e. on $[a,b]$ i.e. $F' = 0$ a.e. on $[a,b]$, hence F is constant by Proposition 1.3.24 and from $F(a) = 0$ we conclude that $F = 0$, so Φ is injective. To show that Φ is also surjective let $f \in \mathcal{D}([a,b])$ and define $F : [a,b] \to \mathbb{R}$ by $F(x) := \int_a^x f$ then by Proposition 1.4.6 we know that $F \in ACG_*([a,b])$, $F' = f$ a.e. on $[a,b]$ and $F(a) = 0$. So $\Phi(F) = F' = f$. This shows that Φ is an isomorphism. Now we establish that Φ is also an isometry. Let $F \in A$, by the FTC (1.4.4) we get that $\int_a^x F' = F(x)$ for all $x \in [a,b]$ and hence $\|\Phi(F)\|_A = \|F'\|_A = \sup_{x\in[a,b]} |\int_a^x F'| = \sup_{x\in[a,b]} |F(x)| = \|F\|_\infty$. $\square$

1.4.18 Remark
At this point we wonder what the relation between $ACG_*([a,b],E)$ and $\mathcal{D}([a,b],E)$ is. We get a very similar result to the above proposition. Let $E \subseteq [a,b]$ be measurable and let $f \in \mathcal{D}([a,b],E)$, then from the definition follows that $f\chi_E \in \mathcal{D}([a,b])$ and so there is an $F \in ACG_*([a,b])$ such that $F(a) = 0$ and $F' = f\chi_E$ a.e. on $[a,b]$, especially $F' = f$ a.e. on E. By Proposition 1.3.12,4 we know that $F \in ACG_*([a,b],E)$. Therefore we are lead to define the subset $A := \{F \in ACG_*([a,b],E) : F$ is continuous on $[a,b],\ F(a) = 0\}$ of $ACG_*([a,b],E)$. Since all functions in A are continuous we also see that A is a subspace of $\mathcal{C}([a,b])$ and so it is normed by the supremum-norm. Then we get as in the above proposition that $(A, \|.\|_\infty)$ is isometrically isomorphic to $(\mathcal{D}([a,b],E)/\sim, \|.\|_{A,E})$, where the isomorphism is again given by differentiation.

1.4.19 Remark ($\mathcal{D}([a,b])/\sim$ is incomplete)
In Subsection 1.3.3 we have seen that $ACG_*([a,b])$ is incomplete and for the same reasons the subspace $\{F \in ACG_*([a,b]) : F(a) = 0\}$ is incomplete too. Since $\mathcal{D}([a,b])/\sim$ is isometrically isomorphic to this subspace it is also incomplete (see Appendix, Lemma A.2).

1.4.4. Summary

Let us finish this section by giving a brief overview of the results we established in this part of the thesis. At first we finally gave the definition of the classical Denjoy integral. The next step was to show that with this new integral we are able to integrate every derivative and that the FTC holds. So we achieved our main goal for this chapter. Afterwards we proved elementary properties of the integral (including an integration by parts formula) and investigated the relationships between the Riemann, Lebesgue, improper Riemann and classical Denjoy integral. Below we will summarize these observations with subset relations and in a table. The last topic in this section was the topology of the classical Denjoy integrable functions. The topology is given by the Alexiewicz norm and we established the close connection to the supremum-norm of ACG_* functions.

Denote by $\mathcal{R}([a,b])$ respectively by $\mathcal{IR}([a,b])$ the Riemann respectively the improper Riemann integrable functions on $[a,b]$. We established the following relationships:

$$\mathcal{R}([a,b]) \underset{1.1.2,7}{\overset{(\star)}{\subsetneq}} L^1([a,b]) \underset{1.4.14}{\overset{1.4.7,4}{\subsetneq}} \mathcal{D}([a,b]), \tag{1.41}$$

$$\mathcal{R}([a,b]) \underset{1.4.12}{\overset{(\star\star)}{\subsetneq}} \mathcal{IR}([a,b]) \underset{1.1.2,7}{\overset{1.4.13,1}{\subsetneq}} \mathcal{D}([a,b]). \tag{1.42}$$

($\star$): See for example [KS04, Thm. 3.103, p.112-113]. ($\star\star$): Directly from the definition.

The reference above the subset symbol gives the result where we have proven the subset relationship and the reference below gives the result where we have shown that these are proper subsets i.e. the sets are not equal.

We also summarize these results in a table. In this table the columns are given and we have shown the results in the rows. That means for example in column 3, row 4: every improperly Riemann integrable function is also classically Denjoy integrable.

	Riemann	Lebesgue	improper Riemann	classical Denjoy
⇒ Riemann		n	n	n
⇒ Lebesgue	y		n	n
⇒ improper Riemann	n	n		n
⇒ classical Denjoy	y	y	y	

Table 1.4.: Relationship between Riemann, improper Riemann, Lebesgue and classical Denjoy integrable functions

2. The distributional Denjoy integral

In this chapter we present an alternative approach to the problem of integrating all derivatives and the FTC. We saw in Chapter 1 that this problem can be solved but it was technically challenging and often tedious. So it is desirable to examine a very different approach. For this new way we need the theory of distributions, so we give a brief introduction in the first section of this chapter. Unlike for the classical Denjoy integral we do not need a lot of preliminaries and so in the second and last section we can directly give the definition of the distributional Denjoy integral.
Note that throughout Section 2.1 all integrals are understood as Lebesgue integrals.

2.1. Distributions

In this section we give a brief introduction to the theory of distributions. We will establish only the main results we will need for the distributional Denjoy integral and so we do not want to give a complete discussion of this topic. Moreover we restrict ourselves to $\mathbb{R}$ since this is all we need. Of course many results can be easily generalized to higher dimensions. In this introduction to the theory of distributions we follow mainly the lecture [HS09]. For a full introduction see the book [DK10].
Throughout this section Ω denotes an open subset in $\mathbb{R}$.

2.1.1. Test functions

Before we can define distribution we need to know the objects on which the distributions act. Unlike classical functions, distributions act on functions rather than on numbers. These functions are called test functions and this entire subsection is devoted to a description of these and their properties.
We start this subsection by defining the support of a function. The support is essentially the closure of the set where the function does not vanish.

2.1.1 Definition (Support)
Let $f : \Omega \to \mathbb{R}$, then the support of f is defined by

$$supp(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}^{\Omega}, \tag{2.1}$$

where the closure is understood with respect to Ω.

Now that we have the concept of support we can give the definition of a test function.

2.1.2 Definition (Test functions)
The vector space of test functions is defined as $\mathcal{C}_c^\infty(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) : \mathit{supp}(f) \text{ compact}\}$*. For* $K \subseteq \Omega$ *compact we define* $\mathcal{C}_c^\infty(K) := \{f \in \mathcal{C}_c^\infty(\Omega) : \mathit{supp}(f) \subseteq K\}$*.*

Here we list some important examples of test functions, which are used extensively in distribution theory.

2.1.3 Example (Bump functions and mollifer)

1. *This example is used to construct more complicated or higher dimensional test functions. Define* $\psi_0 : \mathbb{R} \to \mathbb{R}$ *by* $\psi_0(x) := \begin{cases} \exp(-\frac{1}{1-x^2}) & |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$
   Then $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ *and* $\mathit{supp}(\psi_0) = [-1, 1]$*. The function* ψ_0 *is called a bump function.*
2. *For* $\epsilon \in]0, 1]$ *we define the function* $\psi_\epsilon \in \mathcal{C}_c^\infty(\mathbb{R})$ *by* $\psi_\epsilon(x) := \psi_0(\frac{x}{\epsilon})/(\epsilon \int_{\mathbb{R}} \psi_0)$*, where* ψ_0 *is the bump function from 1. Then clearly* $\int_{\mathbb{R}} \psi_\epsilon = 1$ *and* $\mathit{supp}(\psi_\epsilon) = [-\epsilon, \epsilon]$*. The function* ψ_ϵ *is called a mollifier.*

As in the case of real functions, where we need to understand convergence of real numbers to understand continuity of real functions, we need to understand convergence of test functions to understand continuity of distributions.

2.1.4 Definition (Convergence and Cauchy sequences)
Let $(\phi_n)_n$ *be a sequence in* $\mathcal{C}_c^\infty(\Omega)$ *and* $\phi \in \mathcal{C}_c^\infty(\Omega)$*.*

1. *The sequence* $(\phi_n)_n$ *converges to* ϕ *in* $\mathcal{C}_c^\infty(\Omega)$ *i.e.* $\phi_n \to \phi$ $(n \to \infty)$ $:\Leftrightarrow$
   a) *There is a compact set* $K \subseteq \Omega$ *such that* $\mathit{supp}(\phi_n) \subseteq K$ $\forall n \in \mathbb{N}$ *and* $\mathit{supp}(\phi) \subseteq K$*.*
   b) *For all* $k \in \mathbb{N}$ *we have that* $\phi_n^{(k)} \to \phi^{(k)}$ $(n \to \infty)$ *uniformly on* K*.*
2. *The sequence* $(\phi_n)_n$ *is called a Cauchy sequence if*
   a) *There is a compact set* $K \subseteq \Omega$ *such that* $\mathit{supp}(\phi_n) \subseteq K$ $\forall n \in \mathbb{N}$*.*
   b) *For all* $k \in \mathbb{N}$*, for all* $\epsilon > 0$ *there is a* $N \in \mathbb{N}$ *such that* $\|\phi_n^{(k)} - \phi_m^{(k)}\|_{\infty,K} < \epsilon$ *for* $m, n \geq N$*.*
3. *Let* $K \subseteq \Omega$ *be compact and let* $(\phi_n)_n$ *and* ϕ *in* $\mathcal{C}_c^\infty(K)$*. Then* $(\phi_n)_n$ *converges to* ϕ *for* $(n \to \infty)$ *in* $\mathcal{C}_c^\infty(K)$ *if* $\phi_n^{(k)} \to \phi^{(k)}$ $(n \to \infty)$ *uniformly on* K *for all* $k \in \mathbb{N}$*.*
4. *All definitions here can be extended analogously to nets of the form* $(\phi_\epsilon)_{\epsilon \in]0,1]}$ *in* $\mathcal{C}_c^\infty(\Omega)$*.*

2.1.2. Distributions on $\mathbb{R}$

At this point we are able to define distributions since we now know the basic properties of test functions. Moreover we will prove some basic results and give important examples.

2.1.5 Definition (Distributions)
A distribution u *on* Ω *is a linear functional on* $\mathcal{C}_c^\infty(\Omega)$ *which is sequentially continuous i.e.* $u : \mathcal{C}_c^\infty(\Omega) \to \mathbb{C}$ *linear such that* $u(\phi_n) \to 0$ $(n \to \infty)$ *for all sequences of test functions* $(\phi_n)_n$ *in* $\mathcal{C}_c^\infty(\Omega)$ *with* $\phi_n \to 0$ $(n \to \infty)$ *in* $\mathcal{C}_c^\infty(\Omega)$*. We write* $\langle u, \phi \rangle$ *instead of* $u(\phi)$*. Moreover we write*

$\mathcal{C}_c^\infty(\Omega)'$ for the vector space of all distributions on Ω. (This notation is justified since the distributions are just the linear and continuous functionals on $\mathcal{C}_c^\infty(\Omega)$, but that will not be important for us.)

We can characterize continuity of a linear functional on $\mathcal{C}_c^\infty(\Omega)$ by an estimate in terms of specific seminorms.

2.1.6 Proposition (A criterion for continuity)
Let $u : \mathcal{C}_c^\infty(\Omega) \to \mathbb{C}$ be linear, then $u \in \mathcal{C}_c^\infty(\Omega)'$ if and only if $\forall K \subset \Omega$ compact $\exists C > 0$, $\exists m \in \mathbb{N}$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{k=0}^{m} \|\phi^{(k)}\|_{\infty,K} \quad \forall \phi \in \mathcal{C}_c^\infty(K)\ . \tag{2.2}$$

(Cf. [DK10, Thm. 3.8, p.38f.]).

Proof:

$\Rightarrow$ Proof by contradiction: we assume that there is a $u \in \mathcal{C}_c^\infty(\Omega)'$ such that Equation (2.2) does not hold, i.e., there is a compact set $K \subseteq \Omega$ such that $\forall C > 0$, $\forall m \in \mathbb{N}$ there is a $\phi \in \mathcal{C}_c^\infty(K)$ with $|\langle u, \phi \rangle| > C \sum_{k=0}^{m} \|\phi^{(k)}\|_{\infty,K}$. Now for $m \in \mathbb{N}$ we choose $C = m$ and so we generate a sequence of functions $(\phi_m)_m$ with

$$|\langle u, \phi_m \rangle| > m \sum_{k=0}^{m} \|\phi_m^{(k)}\|_{\infty,K}. \tag{2.3}$$

It is clear that $\phi_m \neq 0$ for all $m \in \mathbb{N}$ and hence we can define a new sequence of functions by $\psi_m := \phi_m / (m \sum_{k=0}^{m} \|\phi_m^{(k)}\|_{\infty,K})$. Then $\mathrm{supp}(\psi_m) = \mathrm{supp}(\phi_m)$, hence $\psi_m \in \mathcal{C}_c^\infty(K)$ for all $m \in \mathbb{N}$. Let $l \in \mathbb{N}$ and choose $m \in \mathbb{N}, m \geq l$, then $\|\psi_m^{(l)}\|_{\infty,K} = \|\phi_m^{(l)}\|_{\infty,K} / (m \sum_{k=0}^{m} \|\phi_m^{(k)}\|_{\infty,K}) \leq \frac{1}{m} \to 0$ for $m \to \infty$ and so $\psi_m \to 0$ in $\mathcal{C}_c^\infty(K) \subseteq \mathcal{C}_c^\infty(\Omega)$ for $m \to \infty$. At this point we use the sequential continuity of u to conclude that $\langle u, \psi_m \rangle \to 0$ for $m \to \infty$. But by Inequality (2.3)

$$|\langle u, \psi_m \rangle| = \frac{|\langle u, \phi_m \rangle|}{m \sum_{k=0}^{m} \|\phi_m^{(k)}\|_{\infty,K}} \geq 1 \quad \forall m \in \mathbb{N},$$

which is a contradiction to $\langle u, \psi_m \rangle \to 0$ $(m \to \infty)$.

$\Leftarrow$ Let $(\phi_n)_n$ be a sequence in $\mathcal{C}_c^\infty(\Omega)$ with $\phi_n \to 0$ in $\mathcal{C}_c^\infty(\Omega)$ for $n \to \infty$ and $\mathrm{supp}(\phi_n) \subseteq K$ for all $n \in \mathbb{N}$, where $K \subseteq \Omega$ is compact. Then by Inequality (2.2) we get that
$|\langle u, \phi_n \rangle| \leq C \sum_{k=0}^{m} \|\phi_n^{(k)}\|_{\infty,K} \to 0$ $(n \to \infty)$, hence $u \in \mathcal{C}_c^\infty(\Omega)'$.

□

Many classical functions can be regarded as distributions (regular distributions) but there are also distributions which are not classical functions. Below we give examples of both kinds.

2.1.7 Example

1. *The Delta distribution is defined as follows: let $x_0 \in \mathbb{R}$, then we define $\langle \delta_{x_0}, \phi \rangle := \phi(x_0)$ $(\phi \in \mathcal{C}_c^\infty(\mathbb{R}))$. It is a distribution on $\mathbb{R}$ since it is clearly linear and for $K \subseteq \mathbb{R}$ compact we get: $|\langle \delta_{x_0}, \phi \rangle| = |\phi(x_0)| \leq \|\phi\|_{\infty,K}$ for all $\phi \in \mathcal{C}_c^\infty(K)$. So we can choose $C := 1$ and $m := 0$ in (2.2) and hence $\delta_{x_0} \in \mathcal{C}_c^\infty(\mathbb{R})'$.*

2. *Let $f \in L^1_{loc}(\Omega)$, i.e., f is measurable and for all $K \subseteq \Omega$ compact $\|f\|_{1,K} = \int_K |f| < \infty$. Then we can define a distribution u_f by $\langle u_f, \phi \rangle := \int_\Omega f\phi$ $(\phi \in \mathcal{C}_c^\infty(\Omega))$. It is linear by the linearity of the Lebesgue integral and it is continuous because we can show that the seminorm estimate holds. Let $K \subseteq \Omega$ be compact then for all $\phi \in \mathcal{C}_c^\infty(K)$ we get $|\langle u_f, \phi \rangle| \leq \int_\Omega |f||\phi| = \int_K |f||\phi| \leq \|\phi\|_{\infty,K} \int_K |f| = \|f\|_{1,K}\|\phi\|_{\infty,K}$, hence we can choose $C := \|f\|_{1,K}$ and $m := 0$ in (2.2). The distribution u_f is called a regular distribution. This example shows that $L^1_{loc}(\Omega) \subseteq \mathcal{C}_c^\infty(\Omega)$ where we identify f with u_f. It can be shown that the mapping $f \mapsto u_f$ is linear and injective (see for example [DK10, Lem. 3.6 and Rem. 3.7, p. 36f.].*

3. *Another interesting distribution is the Heaviside function (at $x_0 \in \mathbb{R}$). It is defined by $H_{x_0} : \mathbb{R} \to \mathbb{R}$, $H_{x_0}(x) := \begin{cases} 0 & x \leq x_0, \\ 1 & x > x_0. \end{cases}$ The convention is that we write H_{x_0} instead of $u_{H_{x_0}}$ for this regular distribution. Its action on a test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ is given by $\langle H_{x_0}, \phi \rangle = \int_\mathbb{R} H_{x_0}\phi = \int_{x_0}^\infty \phi$.*

4. *The delta distribution δ_{x_0} is not a regular distribution. WLOG we can assume that $x_0 = 0$. Suppose that there is an $f \in L^1_{loc}(\mathbb{R})$ such that $\delta_0 = u_f$. Then let ψ_0 be as in Example 2.1.3,1 and we define $\rho_\epsilon \in \mathcal{C}_c^\infty(\mathbb{R})$ by $\rho_\epsilon(x) := \psi_0(\frac{x}{\epsilon})$ for $\epsilon \in]0,1]$. We observe that $\rho_\epsilon(0) = \psi_0(0) = \exp(-1) > 0$ and that $\|\rho_\epsilon\|_\infty = \|\psi_0\|_\infty$ for all $\epsilon \in]0,1]$. Therefore $0 < |\psi_0(0)| = |\rho_\epsilon(0)| = \langle \delta_0, \rho_\epsilon \rangle| = |\langle u_f, \rho_\epsilon \rangle| = |\int_{-\epsilon}^\epsilon f\rho_\epsilon| \leq \|\rho_\epsilon\|_\infty \int_{-\epsilon}^\epsilon |f| = \|\psi_0\|_\infty \||f|\|_{1,[-\epsilon,\epsilon]} \to 0$ $(\epsilon \to 0)$, which is clearly a contradiction.*

The delta distribution can be understood as a limit of mollifiers. To make this precise we need a notion of convergence for distributions.

2.1.8 Definition (Convergence of distributions)
Let $(u_n)_n$ be a sequence in $\mathcal{C}_c^\infty(\Omega)'$ and $u \in \mathcal{C}_c^\infty(\Omega)'$. We say that u_n converges to u in $\mathcal{C}_c^\infty(\Omega)'$, i.e., $u_n \to u$ $(n \to \infty)$, if $\langle u_n, \phi \rangle \to \langle u, \phi \rangle$ $(n \to \infty)$ for all $\phi \in \mathcal{C}_c^\infty(\Omega)$. Analogously for nets of the form $(u_\epsilon)_{\epsilon \in]0,1]}$ in $\mathcal{C}_c^\infty(\Omega)'$.

2.1.9 Example (The delta distribution as limit of mollifier functions)
Let $(\psi_\epsilon)_{\epsilon \in]0,1]}$ be as in Example 2.1.3,2, then $\psi_\epsilon \to \delta$ for $\epsilon \to 0$ in $\mathcal{C}_c^\infty(\mathbb{R})'$. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, then we calculate

$$\langle \psi_\epsilon, \phi \rangle - \langle \delta_0, \phi \rangle = \frac{1}{\epsilon \int_\mathbb{R} \psi_0} \int_\mathbb{R} \psi_0(\frac{x}{\epsilon})\phi(x)dx - \phi(0) \underbrace{\frac{1}{\epsilon \int_\mathbb{R} \psi_0} \int_\mathbb{R} \psi_0(\frac{x}{\epsilon})dx}_{=1} =$$

$$\frac{1}{\epsilon \int_\mathbb{R} \psi_0} \int_\mathbb{R} \psi_0(\frac{x}{\epsilon})(\phi(x) - \phi(0))dx \stackrel{y=\frac{x}{\epsilon}}{=} \frac{1}{\int_\mathbb{R} \psi_0} \int_\mathbb{R} \psi_0(y) \underbrace{(\phi(\epsilon y) - \phi(0))}_{\to 0} dy \to 0 \ (\epsilon \to 0) .$$

In this calculation we used the substitution $y = \frac{x}{\epsilon}$ and that $\phi(\epsilon y) - \phi(0)$ converges to zero uniformly on supp$(\psi_0) = [-1, 1]$. This shows that $\langle \psi_\epsilon, \phi \rangle \to \langle \delta_0, \phi \rangle$ for $\epsilon \to 0$ for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ i.e. $\psi_\epsilon \to \delta_0$ in $\mathcal{C}_c^\infty(\mathbb{R})'$ for $\epsilon \to 0$.

2.1.3. Differentiation of distributions

Unlike classical functions, every distribution is differentiable and the derivative is a distribution again. In this subsection we define the distributional derivative and prove some elementary properties. Moreover we show that if the distributional derivative is zero, then the distribution is a constant function. This will be used to establish the uniqueness of the distributional Denjoy primitive.

2.1.10 Definition
Let $u \in \mathcal{C}_c^\infty(\Omega)'$ and $k \in \mathbb{N}$, then we define the distributional derivative of order k, $u^{(k)} \in \mathcal{C}_c^\infty(\Omega)'$, by

$$\langle u^{(k)}, \phi \rangle := (-1)^k \langle u, \phi^{(k)} \rangle \quad (\phi \in \mathcal{C}_c^\infty(\Omega)) . \tag{2.4}$$

For $k = 1$ we write u' instead of $u^{(1)}$.

It is not hard to see that the mapping $u \mapsto u^{(k)}$ is a linear continuous function from $\mathcal{C}_c^\infty(\Omega)'$ to $\mathcal{C}_c^\infty(\Omega)'$ (for a $k \in \mathbb{N}$). Therefore the distributional derivative has nicer properties than the classical derivative since, for example, limits and differentiation can be interchanged unconditionally.

2.1.11 Proposition (Properties of the distributional derivative)
Let $k \in \mathbb{N}$, then the mapping $(.)^{(k)} : \mathcal{C}_c^\infty(\Omega)' \to \mathcal{C}_c^\infty(\Omega)'$ is linear and continuous. To be more precise: let $u, v \in \mathcal{C}_c^\infty(\Omega)'$, $(u_n)_n$ a sequence in $\mathcal{C}_c^\infty(\Omega)'$ and $\lambda \in \mathbb{R}$.

1. *Then $u^{(k)}$ is in $\mathcal{C}_c^\infty(\Omega)'$.*
2. *The mapping is linear i.e. $(u + v)^{(k)} = u^{(k)} + v^{(k)}$ respectively $(\lambda u)^{(k)} = \lambda u^{(k)}$.*
3. *Moreover the mapping is continuous (differentiation and limits can be interchanged) i.e. $u_n \to \infty$ $(n \to \infty)$ in $\mathcal{C}_c^\infty(\Omega)'$ implies that $u_n^{(k)} \to u^{(k)}$ for $n \to \infty$ in $\mathcal{C}_c^\infty(\Omega)'$.*

Proof:

1. The linearity of $u^{(k)}$ follows easily from the linearity of u and for the sequentially continuity let $(\phi_n)_n$ be a sequence in $\mathcal{C}_c^\infty(\Omega)$ with $\phi_n \to 0$ for $n \to \infty$. Then we see that $\langle u^{(k)}, \phi_n \rangle = (-1)^k \langle u, \phi_n^{(k)} \rangle \to 0$ $(n \to \infty)$ by the sequential continuity of u and the fact that $\phi_n^{(k)} \to 0$ for $n \to \infty$ in $\mathcal{C}_c^\infty(\Omega)$.
2. This is clear from the definition of the distributional derivative.
3. Let $\phi \in \mathcal{C}_c^\infty(\Omega)$, then $\langle u_n^{(k)}, \phi \rangle = (-1)^k \langle u_n, \phi^{(k)} \rangle \to (-1)^k \langle u, \phi^{(k)} \rangle = \langle u^{(k)}, \phi \rangle$ for $j \to \infty$ because $u_n \to u$ in $\mathcal{C}_c^\infty(\Omega)'$ for $n \to \infty$.

□

Below we give a basic example and the reason why the distributional derivative is defined as in Equation (2.4).

2.1.12 Example

1. *The delta distribution is the derivative of the Heavyside function. Let* $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$*, then* $\langle H', \phi\rangle = -\langle H, \phi'\rangle = -\int_{x_0}^\infty \phi' = \phi(x_0) = \langle \delta_{x_0}, \phi\rangle$.
2. *The reason behind the definition of the distributional derivative as in Equation* (2.4) *is the fact that for continuously differentiable functions it is irrelevant if we first differentiate and then regard the derivative as regular distribution or that we regard the function as regular distribution and differentiate it in* $\mathcal{C}_c^\infty(\Omega)'$*. In both cases the resulting distribution should be the same. To give precise meaning to that let* $f \in \mathcal{C}^1(\mathbb{R})$*, then* $(u_f)' = u_{f'}$*: for* $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ *we get from the integration by parts formula that* $\langle u_{f'}, \phi\rangle = \int_\mathbb{R} f'\phi = -\int_\mathbb{R} f\phi' + f\phi|_{-\infty}^\infty = -\langle u_f, \phi'\rangle = \langle u_f', \phi\rangle$*. Here we used that* $\phi|_{-\infty}^\infty = 0$.

Now we want to look at the differences between classical differentiation and distributional differentiation.

2.1.13 Example (Differences to classical results)
Yet again the Cantor function provides us with an interesting example. The pointwise classical derivative of F *is zero a.e. on* $[0,1]$ *but the distributional derivative of* F *is not zero. We write* D_{distr} *for the distributional derivative to distinguish it from the classical pointwise derivative. Then for* $\phi \in \mathcal{C}_c^\infty(]0,1[)$ *we get that* $\langle D_{\text{distr}}F, \phi\rangle = -\langle F, \phi'\rangle = -\int_0^1 F\phi'$*. This shows that* $D_{\text{distr}}F \neq 0$.

Our last goal in this subsection is to prove that if the derivative of a distribution is zero, then the distribution is a constant function. First we need a short lemma to do this and from now on let $I =]a,b[\subseteq \mathbb{R}$ be an open interval with $-\infty \leq a < b \leq \infty$.

2.1.14 Lemma
Let $\psi \in \mathcal{C}_c^\infty(I)$ *then there is an unique* $\phi \in \mathcal{C}_c^\infty(I)$ *such that* $\phi' = \psi$ *if and only if* $\int_I \psi = 0$.

Proof:

(⇒) We just calculate $\int_I \psi = \int_I \phi' = \phi(x)|_{x=a}^{x=b} = 0$ since ϕ has compact support in I.

(⇐) We define $\phi(x) := \int_a^x \psi + C$ for $C \in \mathbb{R}$, then $\phi \in \mathcal{C}^\infty(I)$ and $\phi' = \psi$ by the FTC. Since ψ has compact support there is a $R > 0$ such that supp$(\psi) \subseteq [-R, R]$. For $|x| > R$ we get that $\phi(x) = C$ and hence for ϕ to be in $\mathcal{C}_c^\infty(I)$ we need that $C = 0$.

□

With the lemma above we can now prove the main theorem of this subsection.

2.1.15 Theorem
Let $u \in \mathcal{C}_c^\infty(I)'$ *and assume that* $u' = 0$*, then* u *is a constant function. (Cf. [DK10, Thm. 4.3, p.47]).*

Proof: Let $\psi_0 \in \mathcal{C}_c^\infty(I)$ such that $\int_I \psi_0 = 1$ and set $C := \langle u, \psi_0 \rangle$. That the distributional derivative of u is zero means that

$$0 = \langle u', \phi \rangle = -\langle u, \phi' \rangle \quad \forall \phi \in \mathcal{C}_c^\infty(I) \ . \tag{2.5}$$

Now let $\phi \in \mathcal{C}_c^\infty(I)$ and set $\zeta := \phi - (\int_I \phi)\psi_0$. It is clear that $\zeta \in \mathcal{C}_c^\infty(I)$ and

$$\int_I \zeta = \int_I \phi - \int_I \phi \underbrace{\int_I \psi_0}_{=1} = 0 \ .$$

By Lemma 2.1.14 we know that there is a unique $\xi \in \mathcal{C}_c^\infty(I)$ such that $\xi' = \zeta$ and hence by Equation (2.5) we get that $\langle u, \zeta \rangle = \langle u, \xi' \rangle = 0$. At this point we write $\phi = \zeta + (\int_I \phi)\psi_0$, and calculate

$$\langle u, \phi \rangle = \underbrace{\langle u, \zeta \rangle}_{=0} + \langle u, (\int_I \phi)\psi_0 \rangle = (\int_I \phi) \underbrace{\langle u, \psi_0 \rangle}_{=C} = \int_I C\phi = \langle C, \phi \rangle \ .$$

Therefore u is a classical function and $u = C$. $\square$

2.2. The distributional Denjoy integral

In this section we will define the distributional Denjoy integral. The definition will be a descriptive one, like for the classical Denjoy integral, so the focus will lie on the primitives. Fortunately the primitives will be just continuous functions (with limits at infinity in the case of unbounded intervals). This simplifies the theory a lot, compared to the theory of the classical Denjoy integral. Only the use of distributions adds more complexity but we will see that we can prove similar results with really less effort than for the classical Denjoy integral. Especially the FTC follows more or less directly from the definition without previous work. Of course one could object that this approach to the FTC is cheating, since we changed the notion of differentiation. Nevertheless it is an interesting approach and we will see the differences to the approach via the classical Denjoy integral and also the differences coming from the different notions of differentiation in some examples. The first subsection is devoted to the definition of the distributional Denjoy integral, the FTC and the basic properties of the new integral. The second subsection establishes a useful topology on the distributionally Denjoy integrable functions via the Alexiewicz norm. This subsection is very similar to Subsection 1.4.3. The third and last subsection examines the relationship between the distributional Denjoy integral and the (proper and improper) Riemann integral, the Lebesgue integral and the classical Denjoy integral. Especially in the first subsection we follow the article [Tal07].

2.2.1. Definition of the distributional Denjoy integral and the FTC

We start this subsection by giving the definition of the primitives of distributionally Denjoy integrable functions, which are just continuous functions on $\mathbb{R}$ with limits at infinity. Then we can directly define distributional Denjoy integrability and the distributional Denjoy integral over $\mathbb{R}$ and bounded intervals. The next topic is an alternative definition for bounded intervals and the relationship with the original definition. After that we prove some basic properties like additivity of the integral and the vector space structure of the integrable distributions. The FTC and some examples conclude this subsection.

2.2.1 Definition (Continuous functions with limits at infinity)
We write $\mathcal{C}^{\lim} := \{F \in \mathcal{C}(\mathbb{R}) : \exists \lim_{x\to-\infty} F(x)$ *and* $\exists \lim_{x\to\infty} F(x)\}$ *for the vector space of continuous functions with limits at infinity. For* $F \in \mathcal{C}^{\lim}$ *we define* $F(\pm\infty) := \lim_{x\to\pm\infty} F(x)$ *and we set* $\mathcal{C}_0^{\lim} := \{F \in \mathcal{C}^{\lim} : F(-\infty) = 0\}$*.*

Just to clarify the above definition we list a few examples.

2.2.2 Example
The function $F(x) = x$ *is continuous on* $\mathbb{R}$ *but has no limits at infinity, so* $F \notin \mathcal{C}^{\lim}$*. An example of a function in* $\mathcal{C}^{\lim}$ *is* $\arctan$ *with* $\arctan(\pm\infty) = \pm\frac{\pi}{2}$*, hence* $\arctan + \frac{\pi}{2} \in \mathcal{C}_0^{\lim}$*. Another example is the bump function* ψ *from Example 2.1.3, 1. Since* ψ *has compact support,* $\psi(-\infty) = 0$ *and hence* $\psi \in \mathcal{C}_0^{\lim}$*.*

First we define distributional Denjoy integrability on $\mathbb{R}$, later on we define it also for bounded intervals.

2.2.3 Definition (Distributional Denjoy integrability)
Let $u \in \mathcal{C}_c^\infty(\mathbb{R})'$, then u is called distributionally Denjoy integrable if there is an $F \in \mathcal{C}_0^{lim}$ such that $F' = u$, where the derivative is understood in the distributional sense. The continuous function F is called the distributional Denjoy primitive of f. We write $\mathcal{D}_d(\mathbb{R})$ for all distributionally Denjoy integrable functions on $\mathbb{R}$ (in contrast to $\mathcal{D}$ for the the classically Denjoy integrable functions).

Note that there are several ingredients in the above definition. The following remark tries to clarify the situation.

2.2.4 Remark
Let $u \in \mathcal{D}_d(\mathbb{R})$ and F a primitive of u. Then $F' = u$ means that for all test functions $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ we have that $\langle u, \phi\rangle = \langle F', \phi\rangle = -\langle F, \phi'\rangle = -\int_{-\infty}^{\infty} F\phi'$ by the definition of the distributional derivative. Note that the integral $\int_{-\infty}^{\infty} F\phi'$ is a Riemann integral since F and ϕ' are continuous and ϕ' has compact support.

Similarly to the Riemann, Lebesgue and classical Denjoy integral the distributional Denjoy primitive is unique up to a constant. So we required right from the beginning that the limit at minus infinity of the primitives is zero to get this uniqueness result.

2.2.5 Lemma (Uniqueness of the distributional Denjoy primitive)
Let $u \in \mathcal{D}_d(\mathbb{R})$ then its distributional Denjoy primitive is unique.

Proof: Let $u \in \mathcal{C}_c^\infty(\mathbb{R})'$ and $F_1, F_2 \in \mathcal{C}_0^{lim}$ two distributional Denjoy primitives of u, then $(F_1 - F_2)' = u - u = 0$ by Proposition 2.1.11,2 and so by Theorem 2.1.15 we conclude that $F_1 - F_2$ is constant. Moreover since $F_1(-\infty) = F_2(-\infty) = 0$ we conclude that $F_1 = F_2$. □

From the above lemma we know that the primitive is unique and so we can define the distributional Denjoy integral over $\mathbb{R}$ and bounded intervals.

2.2.6 Definition (Distributional Denjoy integral)
Let $u \in \mathcal{D}_d(\mathbb{R})$ and $F \in \mathcal{C}_0^{\lim}$ its distributional Denjoy primitive.

1. *We define the distributional Denjoy integral of u as*
$$\int_{-\infty}^{\infty} u := F(\infty) \ . \tag{2.6}$$

2. *Let $-\infty \le a < b \le \infty$, then the distributional Denjoy integral of u over $[a, b]$ is defined as*
$$\int_a^b u := F(b) - F(a) \ . \tag{2.7}$$

There is another approach to define the distributional Denjoy integral on bounded intervals.

2.2.7 Definition (Alternative definition)
Let $\Omega =]a, b[$ be an open bounded interval. Then $u \in \mathcal{C}_c^\infty(\Omega)'$ is called distributionally Denjoy integrable on $\overline{\Omega}$ if there is an $F \in \mathcal{C}(\overline{\Omega})$ such that $F' = u$ in the distributional sense and

$F(a) = 0$. The distributional Denjoy integral of u on $\overline{\Omega}$ is defined by $\int_a^b u := F(b)$. We write $\mathcal{D}_d(\Omega)$ for all distributionally Denjoy integrable functions on $\overline{\Omega}$.

Note that as in Lemma 2.2.5 we get from Theorem 2.1.15 that the primitive $F \in \mathcal{C}(\overline{\Omega})$ of u is unique.
Now we want to clarify the relationship between the original and alternative definition.

2.2.8 Lemma (Relationship between the two definitions)
Let $-\infty < a < b < \infty$.

1. *If $u \in \mathcal{D}_d(\mathbb{R})$, then u is distributionally Denjoy integrable on $[a, b]$.*
2. *If $v \in \mathcal{D}_d([a, b])$, then there is a $\tilde{v} \in \mathcal{D}_d(\mathbb{R})$ such that $\tilde{v}|_{\mathcal{C}_c^\infty(]a,b[)} = v$.*

Furthermore in both cases the integrals over $[a, b]$ agree. (Cf. [Tal07, p.6]).

Proof:

1. Let $F \in \mathcal{C}_0^{\lim}$ be the primitive of u and set $A := F - F(a) \in \mathcal{C}(\overline{\Omega})$. Then clearly $A(a) = 0$ and we know that $u \in \mathcal{C}_c^\infty(\Omega)'$ since $\mathcal{C}_c^\infty(\Omega) \subseteq \mathcal{C}_c^\infty(\mathbb{R})$. Therefore it suffices to show that $A' = u$ in the distributional sense. So let $\phi \in \mathcal{C}_c^\infty(\Omega)$, then
$$\langle A', \phi \rangle = -\langle A, \phi' \rangle = -\int_a^b F\phi' + F(a) \underbrace{\int_a^b \phi'}_{=0} = -\int_{-\infty}^\infty F\phi' = -\langle F, \phi' \rangle = \langle F', \phi \rangle \underset{\phi \in \mathcal{C}_c^\infty(\mathbb{R})}{\overset{F'=u}{=}} \langle u, \phi \rangle \ .$$
So $A' = u$ and therefore $\int_a^b u = A(b) = F(b) - F(a)$ which is the same as Equation (2.7) gives.

2. Let $G \in \mathcal{C}([a, b])$ be the primitive of v i.e. $G' = v$ on $\mathcal{C}_c^\infty(]a, b[)$ and $G(a) = 0$. Now we define the function
$$K : \mathbb{R} \to \mathbb{R} \quad \text{by} \quad K(x) := \begin{cases} 0 & x \in]-\infty, a], \\ G(x) & x \in]a, b], \\ G(b) & x \in]b, \infty[. \end{cases}$$
Clearly $G \in \mathcal{C}_0^{\lim}$ and $\tilde{v} := G' \in \mathcal{D}_d(\mathbb{R})$ (cf. FTC, Theorem 2.2.12,2). To show that $\tilde{v} = v$ on $\mathcal{C}_c^\infty(]a, b[)$, let $\phi \in \mathcal{C}_c^\infty(]a, b[)$ then $\langle \tilde{v}, \phi \rangle = -\int_{-\infty}^\infty K\phi' = -\int_{-\infty}^a 0 \cdot \phi' - \int_a^b G\phi' - \int_b^\infty G(b)\phi' = -\langle G, \phi' \rangle + G(b)\phi(b) = \langle G', \phi \rangle = \langle v, \phi \rangle$, where we used that $\phi(b) = 0$ since ϕ has compact support in $]a, b[$ and that $G' = v$ on $\mathcal{C}_c^\infty(\Omega)$. The integral as in Equation (2.7) gives $\int_a^b \tilde{v} = K(b) - K(a) = G(b)$, which is the same as the alternative definition gives.

$\square$

Not surprisingly the distributional Denjoy integral is linear and so the integrable distributions form a vector space.

2.2.9 Lemma ($\mathcal{D}_d(\mathbb{R})$ respectively $\mathcal{D}_d(\Omega)$ is a vector space)
Let $\Omega \subseteq \mathbb{R}$ be an open bounded interval or $\Omega = \mathbb{R}$. Then $\mathcal{D}_d(\Omega)$ is a vector space. Furthermore for $u, v \in \mathcal{D}_d(\Omega)$ and $\lambda, \mu \in \mathbb{R}$ we get that $\int_\Omega (\lambda u + \mu v) = \lambda \int_\Omega u + \mu \int_\Omega v$.

Proof: Let $F, G \in (\overline{\Omega})$ respectively $F, G \in \mathcal{C}_0^{\lim}$ be the primitives of u and v. Then by Proposition 2.1.11,2 we get that $(\lambda F + \mu G)' = \lambda u + \mu v$ in the distributional sense and hence $\lambda u + \mu v \in \mathcal{D}_d(\Omega)$. Moreover for $b := \sup \Omega \in \mathbb{R} \cup \{\infty\}$ we get that $\int_\Omega (\lambda u + \mu v) = (\lambda F + \mu G)(b) = \lambda F(b) + \mu G(b) = \lambda \int_\Omega u + \mu \int_\Omega v$. □

The distributional Denjoy integral over $\mathbb{R}$ has the usual properties concerning subintervals, but surprisingly we will see that in the case of bounded intervals the situation changes.

2.2.10 Lemma (Additivity of the distributional Denjoy integral)
Let $u \in \mathcal{D}_d(\mathbb{R})$ and let $-\infty \leq a < c < b \leq \infty$, then

$$\int_a^c u + \int_c^b u = \int_a^b u \ . \tag{2.8}$$

Proof: Let $F \in \mathcal{C}_0^{\lim}$ be the primitive of u, then by the definition of the integral over $[a, c]$ respectively $[c, b]$ we get that $\int_a^c u + \int_c^b u = F(c) - F(a) + F(b) - F(c) = F(b) - F(a) = \int_a^b u$. □

The above lemma shows that if u is distributionally Denjoy integrable on $\mathbb{R}$ it is distributionally Denjoy integrable on every subinterval. Moreover if u is distributionally Denjoy integrable on $[a, b]$ then it follows easily from the definition that u is also distributionally Denjoy integrable on every subinterval of $[a, b]$. The following example shows that the converse statement is not true.

2.2.11 Example
A common property of integrals is that if a function is integrable on two contiguous intervals then it is integrable on the whole interval and the sum of the integrals over the two subinterval equals the integral of the whole interval. Surprisingly this is not true for the distributional Denjoy integral. To be more precise it would be desirable to have the following property of the distributional Denjoy integral. Let $-\infty < a < c < b < \infty$ and let $u \in \mathcal{C}_c^\infty(]a, b[)'$. If u is distributionally Denjoy integrable on $[a, c]$ and $[c, b]$, then u is also distributionally Denjoy integrable on $[a, b]$ and Equation (2.8) holds.
To see that this is in general not true we define $u \in \mathcal{C}_c^\infty(]0, 2[)'$ by $u := \delta_1$ i.e. $\langle u, \phi \rangle = \phi(1)$ for all $\phi \in \mathcal{C}_c^\infty(]0, 2[)$. Moreover $u|_{\mathcal{C}_c^\infty(]0,1[)} = u|_{\mathcal{C}_c^\infty(]1,2[)} = 0$ since for these test functions we have that $\phi(1) = 0$. This yields that u is distributionally Denjoy integrable over $[0, 1]$ and $[1, 2]$ because we can use the zero function as primitive. Additionally $\int_0^1 u = \int_1^2 u = 0$ but u is not distributionally Denjoy integrable over $[0, 2]$, since the delta distribution is never distributionally Denjoy integrable over any interval. See Example 2.2.14, 2.

At this point we establish the FTC in this new setting and we will see in the proof that we need only one previous result. Specifically we only need the lemma about the uniqueness of the primitive besides from the definition of the distributional Denjoy integral.

2.2.12 Theorem (FTC)

1. *Let $u \in \mathcal{D}_d(\mathbb{R})$ and define $F(x) := \int_{-\infty}^x u \ (x \in \mathbb{R})$, then $F \in \mathcal{C}_0^{\lim}$ and $F' = u$ in the distributional sense.*

2. *Let $G \in \mathcal{C}^{\lim}$, then $G' \in \mathcal{D}_d(\mathbb{R})$ and $\int_{-\infty}^{x} G' = G(x) - G(-\infty)\ \forall x \in \mathbb{R}$.*

Proof:

1. Since $u \in \mathcal{D}_d(\mathbb{R})$ there is a unique $K \in \mathcal{C}_0^{\lim}$ such that $K' = u$ and so $K(x) = \int_{-\infty}^{x} u = F(x)$ for all $x \in \mathbb{R}$, hence $K = F \in \mathcal{C}_0^{\lim}$ and $F' = K' = u$.

2. Since G has limits at infinity we get that $G - G(-\infty) \in \mathcal{C}_0^{\lim}$. So it suffices to show that $(G - G(-\infty))' = G'$ in the distributional sense. Let $\phi \in \mathcal{C}_c^{\infty}(\mathbb{R})$ then $\langle (G - G(-\infty))', \phi \rangle = -\int_{-\infty}^{\infty} (G - G(-\infty))\phi' = -\int_{-\infty}^{\infty} G\phi' + G(-\infty)\int_{-\infty}^{\infty} \phi' = \langle G', \phi \rangle$. In the last equality we used that ϕ has compact support. This holds for all $\phi \in \mathcal{C}_c^{\infty}(\mathbb{R})$ and therefore we get that $G' \in \mathcal{D}_d(\mathbb{R})$ and $\int_{-\infty}^{x} G' = G(x) - G(-\infty)\ \forall x \in \mathbb{R}$.

□

Of course there is also a FTC on bounded intervals, which can be proved analogously to the above FTC on $\mathbb{R}$.

2.2.13 Remark (FTC on bounded intervals)
Let $a, b \in \mathbb{R}$ with $a < b$.

1. *Let $u \in \mathcal{D}_d([a,b])$ and define $F(x) := \int_a^x u$ $(x \in [a,b])$, then $F \in \mathcal{C}([a,b])$ and $F' = u$ in the distributional sense.*

2. *Let $G \in \mathcal{C}([a,b])$, then $G' \in \mathcal{D}_d([a,b])$ and $\int_a^x G' = G(x) - G(a)\ \forall x \in [a,b]$.*

Here we give some basic examples. For more examples see Subsection 2.2.3, where we will discuss the relationship of the distributional Denjoy integral to other integrals.

2.2.14 Example

1. *From the FTC we get that every distributional derivative of a continuous function with limits at infinity is a distributionally Denjoy integrable function and that the integral is given by this function (possibly minus a constant).*

2. *The delta distribution δ_{x_0} is not distributionally Denjoy integrable for any $x_0 \in \mathbb{R}$. We know that $H_{x_0}' = \delta_{x_0}$, where H_{x_0} is the Heaviside function at $x_0 \in \mathbb{R}$. Assume that $\delta_{x_0} \in \mathcal{D}_d(\mathbb{R})$, then there is an $F \in \mathcal{C}_0^{\lim}$ such that $F' = \delta_{x_0}$, therefore $(F - H_{x_0})' = 0$ and so by Theorem 2.1.15 we deduce that $F - H_{x_0}$ is constant. Moreover $F = H_{x_0}$, since $F(-\infty) = 0$ and $H_{x_0}(x) = 0$ for $x \leq x_0$. This is a contradiction to $H_{x_0} \notin \mathcal{C}(\mathbb{R})$. Therefore δ_{x_0} is not distributionally Denjoy integrable. Furthermore it is easy to see that δ_{x_0} is also not distributionally Denjoy integrable over any bounded interval I, with x_0 in the interior of I, for the same reasons.*

3. *Let $x_0 \in \mathbb{R}$, then the Heaviside function at x_0 is distributionally Denjoy integrable over a bounded interval $[a,b]$ if $x_0 \leq a$ or $x_0 \geq b$ (in this case $H_{x_0} = 0$ on $\mathcal{C}_c^{\infty}(]a,b[)$ and so the statement is trivially true). But it is not distributionally Denjoy integrable over $\mathbb{R}$. To see this we define the function $F : \mathbb{R} \to \mathbb{R}$ by setting F to zero on $]-\infty, a]$ and $F(x) = x - a$ for $x > a$, then $F \in \mathcal{C}(\mathbb{R})$ but $F \notin \mathcal{C}_0^{\lim}$ since $\lim_{x\to\infty} F(x) = \infty$. Now*

we show that $F' = H_{x_0}$ on $\mathcal{C}_c^\infty(]a,b[)$. Let $\phi \in \mathcal{C}_c^\infty(]a,b[)$ and so $\langle F', \phi\rangle = -\int_a^b F\phi' = -\int_a^b (x-x_0)\phi'(x)dx = \int_a^b \phi + x_0(\phi(b)-\phi(a)) = \int_a^b \phi = \langle H_{x_0}, \phi\rangle$, since $\phi(a)=\phi(b)=0$. This shows that $H_{x_0} \in \mathcal{D}_d([a,b])$. We assume that $H_{x_0} \in \mathcal{D}_d(\mathbb{R})$, i.e., there is a $G \in \mathcal{C}_0^{\lim}$ such that $G' = H_{x_0}$ on $\mathcal{C}_c^\infty(\mathbb{R})$. For all $b > a \geq x_0$ we know that $G - G(a) \in \mathcal{C}([a,b])$ and by the uniqueness of the distributional Denjoy integral we get that $F|_{[a,b]} = (G-G(a))|_{[a,b]}$ and hence $\lim_{x\to\infty} G(x) = \lim_{x\to\infty} F(x) = \infty$, a contradiction to $G \in \mathcal{C}_0^{\lim}$. Note that actually we examine two different distributions since first we view H_{x_0} as a distribution in $\mathcal{C}_c^\infty(]a,b[)'$ and then $H_{x_0} \in \mathcal{C}_c^\infty(\mathbb{R})'$.

2.2.2. Topology on the space of distributionally Denjoy integrable functions

Similarly as in Subsection 1.4.3 we now define the Alexiewicz norm for distributionally Denjoy integrable functions. First we observe that the space $\mathcal{C}_0^{\lim}$ is a Banach space with respect to the supremum norm and then we establish an isometry between $\mathcal{C}_0^{\lim}$ and $\mathcal{D}_d(\mathbb{R})$.

2.2.15 Lemma ($\mathcal{C}_0^{\lim}$ is a Banach space)
The space of distributional Denjoy primitives $\mathcal{C}_0^{\lim}$ is a Banach space with respect to the supremum norm $\|.\|_\infty$.

Proof: First we show that $\mathcal{C}_0^{\lim}$ is a subspace of $\mathcal{C}_b(\mathbb{R})$ (continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}$) and so a normed space with respect to $\|.\|_\infty$. Let $F \in \mathcal{C}_0^{\lim}$, then we know that for $\epsilon > 0$ there are $x_0, x_1 \in \mathbb{R}$ such that $\forall x < x_0$ respectively $x' > x_1$ we get that $|F(x)| < \epsilon$ and $|F(x') - F(\infty)| < \epsilon$. Now choose $\epsilon = 1$ and $x_0, x_1 \in \mathbb{R}$ correspondingly. Then for $x \in \mathbb{R}$ arbitrary we get that $|F(x)| < 1$ if $x < x_0$ and $|F(x)| \leq |F(x) - F(\infty)| + |F(\infty)| < 1 + |F(\infty)|$ if $x > x_1$. Moreover F is bounded on the compact interval $[x_0, x_1]$, say by $C > 0$, since it is continuous. Therefore $\|F\|_\infty \leq \max(1 + |F(\infty)|, C)$ and $F \in \mathcal{C}_b(\mathbb{R})$. So $\mathcal{C}_0^{\lim}$ is a normed linear space.
Our second step is to show that $\mathcal{C}_0^{\lim}$ is complete. It suffices to show that $\mathcal{C}_0^{\lim}$ is closed in $\mathcal{C}_b(\mathbb{R})$ since $(\mathcal{C}_b(\mathbb{R}), \|.\|_\infty)$ is a Banach space (cf. [AE06, Thm. 2.6.(i), p.391]). Let $(F_n)_n$ be a sequence in $\mathcal{C}_0^{\lim}$ which converges to $F \in \mathcal{C}_b(\mathbb{R})$ with respect to $\|.\|_\infty$. We have to show that $F \in \mathcal{C}_0^{\lim}$. At this point we claim that $(F_n(\infty))_n$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges. Let $\epsilon > 0$, then for all $k \in \mathbb{N}$, there is a $x_k \in \mathbb{R}$ such that $|F_k(\infty) - F_k(x)| < \frac{\epsilon}{3}$ for all $x \in \mathbb{R}$ with $x > x_k$. Furthermore there is a $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have that $\|F_n - F_m\|_\infty < \frac{\epsilon}{3}$ (because $(F_n)_n$ converges and hence it is a Cauchy sequence). Now let $n, m \geq N$ and $x > \max(x_n, x_m)$, then we calculate

$$|F_n(\infty) - F_m(\infty)| \leq \underbrace{|F_n(\infty) - F_n(x)|}_{<\frac{\epsilon}{3}} + \underbrace{|F_n(x) - F_m(x)|}_{\leq \|F_n - F_m\|_\infty < \frac{\epsilon}{3}} + \underbrace{|F_m(x) - F_m(\infty)|}_{<\frac{\epsilon}{3}} < \epsilon\ .$$

This shows that the limit $\lim_{n\to\infty} F_n(\infty) =: F(\infty)$ exists. It remains to show that $\lim_{x\to\infty} F(x) = F(\infty)$. So let $\epsilon > 0$, then there is a $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $|F_n(\infty) - F(\infty)| < \frac{\epsilon}{3}$. Additionally we know that for all $n \in \mathbb{N}$, there exists a $x_n \in \mathbb{R}$ such that $x \in \mathbb{R}, x \geq x_n$ implies $|F_n(x) - F_n(\infty)| < \frac{\epsilon}{3}$. Moreover there is a $N_2 \in \mathbb{N}$ with the property that for all $n \geq N_2 :\ \|F - F_n\|_\infty < \frac{\epsilon}{3}$. All in all for $n \geq \max(N_1, N_2)$ and $x > x_n$ we get that $|F(x) - F(\infty)| \leq |F(x) - F_n(x)| + |F_n(x) - F_n(\infty)| + |F_n(\infty) - F(\infty)| < \epsilon$. Similarly one can show that $\lim_{x\to-\infty} F(x) = 0$ and hence $F \in \mathcal{C}_0^{\lim}$. Therefore $\mathcal{C}_0^{\lim}$ is closed in $(\mathcal{C}_b(\mathbb{R}), \|.\|_\infty)$ and hence a Banach space. $\square$

At this point we introduce the Alexiewicz norm for distributionally Denjoy integrable functions. We will see in Subsection 2.2.3 that there is no ambiguity with the Alexiewicz norm for classical Denjoy integrable functions, since in this case the norms agree.

2.2.16 Definition (Alexiewicz norm)
Let $u \in \mathcal{D}_d(\mathbb{R})$ or $u \in \mathcal{D}_d([a,b])$, then we define the Alexiewicz norm of u by $\|u\|_A := \|F\|_\infty$, where F is the distributional Denjoy primitive of u and the supremum is taken over $\mathbb{R}$ respectively $[a,b]$.

2.2.17 Remark
It is easy to see that $\mathcal{D}_d(\mathbb{R})$ can be normed by the Alexiewicz norm since almost all required properties follow from the properties of the supremum norm respectively the distributional derivative. Only the positive definiteness of $\|.\|_A$ needs a little justification. Assume that for $u \in \mathcal{D}_d(\mathbb{R})$ the norm is zero i.e. $\|u\|_A = 0$. This means for the primitive $F \in \mathcal{C}_0^{\lim}$ that $0 = \|u\|_A = \|F\|_\infty$ and hence $F = 0$. Furthermore $u = F' = 0$ in $\mathcal{D}_d(\mathbb{R})$.

Very similarly to Proposition 1.4.17 we now show that the distributional Denjoy primitives are isometrically isomorphic to the integrable distributions and hence they are a Banach space.

2.2.18 Theorem (Isometric isomorphism of $\mathcal{C}_0^{\lim}$ and $\mathcal{D}_d(\mathbb{R})$)
The Banach space $(\mathcal{C}_0^{\lim}, \|.\|_\infty)$ is isometrically isomorphic to $(\mathcal{D}_d(\mathbb{R}), \|.\|_A)$ i.e. the distributionally Denjoy integrable functions on $\mathbb{R}$ form a Banach space. (Cf. [Tal07, Thm. 2, p.8f.] and Proposition 1.4.17).

Proof: As in the proof of Proposition 1.4.17 the isomorphism is given by differentiation. So we define $\Phi : \mathcal{C}_0^{\lim} \to \mathcal{D}_d(\mathbb{R})$ by $\Phi(F) := F'$, where the derivative is understood in the distributional sense. From the FTC (2.2.12,2) we know that F' is distributionally Denjoy integrable and hence $\Phi(F) \in \mathcal{D}_d(\mathbb{R})$ for all $F \in \mathcal{C}_0^{\lim}$. Moreover Φ is linear since the distributional derivative is. Assume that $\Phi(F) = 0$ for a $F \in \mathcal{C}_0^{\lim}$ i.e. $F' = 0$. From Theorem 2.1.15 we know that F has to be constant and hence $F = 0$ because $\lim_{x\to-\infty} F(x) = 0$. This shows that Φ is injective. The surjectivity of Φ follows from the second part of the FTC. Let $f \in \mathcal{D}_d(\mathbb{R})$ and define $F(x) := \int_{-\infty}^x f$, then from the FTC (2.2.12,1) we know that $F \in \mathcal{C}_0^{\lim}$ and $F' = f$. Therefore $\Phi(F) = F' = f$ and so Φ is an isomorphism. It remains to show that Φ is an isometry. Let $F \in \mathcal{C}_0^{\lim}$, then $\|\Phi(F)\|_A = \|F'\|_A = \|F\|_\infty$ since F is the distributional Denjoy primitive of F'. $\square$

2.2.19 Remark
Similarly to the above theorem one can show that the distributional Denjoy integrable functions on a bounded interval $[a,b]$ are isometrically isomorphic to the continuous functions on $[a,b]$ which are zero at a. To be more precise we define $C := \{F \in \mathcal{C}([a,b]) : F(a) = 0\}$, then $(C, \|.\|_\infty) \cong (\mathcal{D}_d([a,b]), \|.\|_A)$ isometrically.

2.2.3. Relation to other integrals

In this subsection we will establish the relationship between the distributional Denjoy integral and all other integrals we came across in this thesis. To be more specific we will show that the distributional Denjoy integral contains the proper and improper Riemann integral, the Lebesgue integral and the classical Denjoy integral. Moreover in all these case the integrals agree. Furthermore we will give some useful examples which show how the change of the notion of differentiability affects the integral. We conclude this subsection by showing that the distributionally Denjoy integrable functions are the completion of the classical Denjoy integrable functions with respect to the Alexiewicz norm.

2.2.20 Proposition
The distributional Denjoy integral contains the classical Denjoy integral and hence the proper and improper Riemann integral and the Lebesgue integral. Furthermore in all these cases the integrals agree.

Proof: Let f be classically Denjoy integrable over $[a, b]$, i.e., $f \in \mathcal{D}([a, b])$. Then there is an $F \in ACG_*([a, b])$ such that $F' = f$ a.e. on $[a, b]$ and $F(a) = 0$. Now we have to observe that the pointwise relation $F' = f$ a.e. on $[a, b]$ implies that $F' = f$ also in the distributional sense. Let $\phi \in \mathcal{C}_c^\infty(]a, b[)$ then by the integration by parts formula ((1.38), ϕ is AC on $[a, b]$) we get that

$$\langle F', \phi \rangle = -\int_a^b F\phi' \overset{(1.38)}{=} \int_a^b F'\phi \overset{F'=f \text{ a.e.}}{=} \int_a^b f\phi = \langle f, \phi \rangle \ .$$

Moreover since F is continuous on $[a, b]$ we get that $F \in \mathcal{C}([a, b])$, hence by the definition (2.2.7) we get that f is distributionally Denjoy integrable over $[a, b]$ and the integral is given by $\int_a^b f = F(b)$. This value is the same as one would get from the definition of the classical Denjoy integral. From Section 1.4 we know that the classical Denjoy integral contains the proper and improper Riemann integral and the Lebesgue integral. So by the above argument the distributional Denjoy integral contains these integrals as well and the integrals agree in each of these cases. □

So every classical Denjoy integrable function is also distributionally Denjoy integrable and their integrals agree, but what exactly is the difference between the classical and distributional Denjoy integral? The following examples should give a satisfactory answer to this question. See also Examples 1 to 5 in [Tal07, p.7f.].

2.2.21 Example
We write D_{distr} for the distributional derivative to distinguish it from the classical pointwise derivative.

1. *Again the Cantor function $F : [0, 1] \to \mathbb{R}$ gives an interesting example. It is continuous, monotonically increasing (and hence BV) and differentiable a.e. with $F' = 0$ pointwise a.e. on $[0, 1]$ (cf. note before 1.1.6). So by the FTC for bounded intervals (Remark 2.2.13,2) we know that $D_{\text{distr}}F$ is distributionally Denjoy integrable over $[0, 1]$ with integral $\int_0^x D_{\text{distr}}F = F(x) \neq 0$ for all $x \in]0, 1]$ but since $F' = 0$ pointwise a.e. on $[0, 1]$ the Lebesgue integral of F' gives zero over every subinterval of $[0, 1]$.*

So in this case the function $D_{\text{distr}}F$ is distributionally Denjoy integrable and the pointwise derivative F' is Lebesgue integrable but their integrals do not agree. Note that this is not a contradiction to Proposition 2.2.20 since actually we integrate two different objects. On the one hand we integrate the distributional derivative of F and on the other hand we integrate the 'zero function', since pointwise $F' = 0$ a.e. on $[0, 1]$.

2. *To illustrate another interesting fact about the distributional Denjoy integral let $F : [a, b] \to \mathbb{R}$ be a continuous function which is nowhere differentiable i.e. $F'(x)$ does not exist for any $x \in [a, b]$. For example see [GO03, Ex. 2.21, p.29 and Ex. 3.8, p.38f.]. Although the derivative of F does not exist pointwise, it exists as a distribution, since every distribution is differentiable. Moreover again from the FTC for bounded intervals we get that $D_{\text{distr}}F \in \mathcal{D}_d([a, b])$ and $\int_a^x D_{\text{distr}}F = F(x) - F(a)$ for all $x \in [a, b]$. So we see there are distributionally Denjoy integrable functions which are not functions in the classical (pointwise) sense. This is a major difference between the classical and distributional Denjoy integral.*

Another way to look at the distributionally Denjoy integrable functions is to view them as the completion of the classical Denjoy integrable functions with respect to the Alexiewicz norm.

2.2.22 Proposition
The closure of $\mathcal{D}([a, b])/\sim$ is $\mathcal{D}_d([a, b])$. To be more precise let $(f_n)_n$ be a sequence of classical Denjoy integrable functions, where we identify functions which are equal a.e. on $[a, b]$ i.e. $f_n \in \mathcal{D}([a, b])/\sim$ for all $n \in \mathbb{N}$. If $(f_n)_n$ is a Cauchy sequence with respect to the Alexiewicz norm $\|.\|_A$, then $(f_n)_n$ converges to a distributionally Denjoy integrable function $f \in \mathcal{D}_d([a, b])$ with respect to $\|.\|_A$. Additionally for $f \in \mathcal{D}_d([a, b])$ there exists a sequence of classically Denjoy integrable functions which converge to f with respect to $\|.\|_A$.

Proof: By the isometry given in Proposition 1.4.17 we get that the primitives $(F_n)_n$ of the functions $(f_n)_n$ are a Cauchy sequence with respect to $\|.\|_\infty$. Since $\mathcal{C}([a, b])$ is a Banach space, there exists a continuous function F on $[a, b]$ with $F(a) = 0$ such that $\|F - F_n\|_\infty \to 0$ for $n \to \infty$. From Proposition 2.2.20 we know that the functions $(f_n)_n$ are also distributionally Denjoy integrable and their distributional Denjoy primitives are just the functions $(F_n)_n$. So the classical Alexiewicz norm of the $(f_n)_n$ agrees with the distributional Alexiewicz norm of the $(f_n)_n$. Furthermore from the FTC we know that $F' \in \mathcal{D}_d([a, b])$ and now we claim that $(f_n)_n$ converges to F' with respect to the Alexiewicz norm. We calculate $\|F' - f_n\|_A = \|F - F_n\|_\infty \to 0$ for $n \to \infty$. This shows that the closure of $\mathcal{D}([a, b])/\sim$ is contained in $\mathcal{D}_d([a, b])$. It remains to show that $\mathcal{D}_d([a, b])$ is actually the closure of $\mathcal{D}([a, b])/\sim$. Let $f \in \mathcal{D}_d([a, b])$ and $F \in \mathcal{C}$ its primitive. By the Weierstrass approximation theorem (cf. [AE06, Cor. 4.9, p.416]) we know that F can be uniformly approximated by polynomials. So there is a sequence of polynomials $(F_n)_n$ such that $\|F_n - F\|_\infty \to 0$ for $n \to \infty$. It is not hard to see that $F_n(a) = 0$ for all $n \in \mathbb{N}$. From Corollary 1.3.22 we know that each F_n is ACG$_*$ on $[a, b]$ since it is differentiable everywhere on $[a, b]$. Now the derivatives of these functions, $(F_n')_n$, are classically Denjoy integrable and they converge to f with respect to the Alexiewicz norm: $\|F_n' - f\|_A = \|F_n - F\|_\infty \to 0$ for $n \to \infty$. This finishes the proof. □

2.2.4. Summary

To conclude the second chapter we give an overview of the results we proved in this section. First we defined distributional Denjoy integrability and the distributional Denjoy integral and proved some basic properties. Then we showed that the FTC holds in this new setting and gave some examples. The second part was about the topology of the distributionally Denjoy integrable functions. We proved that they are a Banach space and are isometrically isomorphic to their primitives, which are continuous functions with limits at infinity. The third and last part was about the relationship of the distributional Denjoy integral to the Riemann, Lebesgue and classical Denjoy integral. These results are summarized below and this is essentially an extended and updated version of the results in the summary of Section 1.4.

$$\mathcal{R}([a,b]) \underset{1.1.2,7}{\overset{(\star)}{\subsetneq}} L^1([a,b]) \underset{1.4.14}{\overset{1.4.7,4}{\subsetneq}} \mathcal{D}([a,b]) \underset{2.2.21,2}{\overset{2.2.20}{\subsetneq}} \mathcal{D}_d([a,b]) \tag{2.9}$$

$$\mathcal{R}([a,b]) \underset{1.4.12}{\overset{(\star\star)}{\subsetneq}} \mathcal{IR}([a,b]) \underset{1.1.2,7}{\overset{1.4.13,1}{\subsetneq}} \mathcal{D}([a,b]) \underset{2.2.21,2}{\overset{2.2.20}{\subsetneq}} \mathcal{D}_d([a,b]) \tag{2.10}$$

($\star$): See for example [KS04, Thm. 3.103, p.112f.]. ($\star\star$): Directly from the definition.

The reference above the subset symbol gives the result where we have proven the subset relationship and the reference below gives the result where we have shown that these are proper subsets i.e. the sets are not equal.

We also summarize these results in a table. In this table the columns are given and we have shown the results in the rows. That means for example in column 4, row 2: in general a classically Denjoy integrable function is not Lebesgue integrable.

	Riemann	Lebesgue	im. Riemann	c. Denjoy	d. Denjoy
⇒ Riemann		n	n	n	n
⇒ Lebesgue	y		n	n	n
⇒ improper Riemann	n	n		n	n
⇒ classically Denjoy	y	y	y		n
⇒ distributionally Denjoy	y	y	y	y	

Table 2.1.: Relationship between Riemann, improper Riemann, Lebesgue, classically Denjoy and distributionally Denjoy integrable functions

3. Outlook on further threads

Up to now we developed the theory of the classical and distributional Denjoy integral so far as to establish elementary properties and to show that every derivative is integrable and the FTC holds. So in this chapter we sketch briefly what further properties these integrals have and what else can be done. First we discuss two applications of the classical Denjoy integral. Second we describe two alternative approaches to the problem of integrating all derivatives and the FTC, which turn out to be equivalent to the classical Denjoy integral. Finally we discuss further properties and possible generalizations of the classical and distributional Denjoy integral.

3.1. Applications

In this section we show how the classical Denjoy integral can be applied to problems in the fields of analytical number theory and ordinary differential equations. Additionally we briefly discuss why the distributional Denjoy integral is not well suited for these applications.

3.1.1. Analytical number theory

In this subsection we show how the classical Denjoy integral can be used to extend a result from analytical number theory. We will extend the Abel summation theorem and its corollary, the Euler summation theorem, to ACG_* functions. The standard formulation of these two results requires the function to be continuously differentiable, so that the FTC and integration by parts can be used. In the end we will briefly sketch why the distributional Denjoy integral cannot be used in this case too. In this subsection all integrals are classical Denjoy integrals unless stated otherwise.
We start by giving a few basic definitions from analytical number theory. For an introduction to this topic see [Brue95].

3.1.1 Definition

1. *The floor function* $\lfloor . \rfloor : \mathbb{R} \to \mathbb{Z}$ *is defined by* $\lfloor x \rfloor := \max\{k \in \mathbb{Z}, k \leq x\}$ $(x \in \mathbb{R})$. *In other words the floor function takes a real number and gives the integer part of this number.*
2. *Any function* $\alpha : \mathbb{N} \to \mathbb{C}$ *is called an arithmetic function.*
3. *Let* $G \subseteq \mathbb{C}$, $f : G \to \mathbb{C}$ *and* $g : G \to [0, \infty[$. *We say that* f *is a Big O of* g, *written* $f = O(g)$, *if there is a constant* $C \in \mathbb{R}$ *such that* $|f(z)| \leq Cg(z)$ *for all* $z \in G$.

4. *For $k \in \mathbb{N}\backslash\{0\}$ and $s \in \mathbb{R}$ we define the* <u>generalized harmonic number</u> *by* $H(k,s) := \sum_{n=1}^{k} \frac{1}{n^s}$.
5. *The Euler-Mascheroni constant γ is defined by $\gamma := 1 - \int_1^\infty \frac{t-\lfloor t \rfloor}{t^2} dt$ and its numerical value is $\gamma = 0.57721\ldots$.*

At this point we state and prove the Abel summation theorem for ACG$_*$ functions. It relates sums to integrals, which may be easier to compute or to estimate. After we have proven the Abel and Euler summation theorems we will give two examples and see how they can be used the give an approximation to certain sums.
The following theorem is a generalization of [Brue95, Lem. 1.1.3, p.9] where the function F is assumed to be continuously differentiable.

3.1.2 Theorem (Abel summation theorem)
Let $a, b \in \mathbb{R}$ with $0 < a < b-1$, let $F \in ACG_*([a,b])$ and α be an arithmetic function. Define the function $A : \mathbb{R} \to \mathbb{R}$ by

$$A(x) := \begin{cases} \sum_{n \le b} (\alpha(n)F(n)) & x \in \mathbb{R}, x \ge 1, \\ 0 & x \in \mathbb{R}, x < 1. \end{cases} \tag{3.1}$$

Then

$$\sum_{a<n\le b} (\alpha(n)F(n)) = A(b)F(b) - A(a)F(a) - \int_a^b A\ F' \ . \tag{3.2}$$

Proof: First we observe that A is constant on $[n, n+1[$ for all $n \in \mathbb{N}$, therefore AF' is classically Denjoy integrable on $[a,b]$ by Proposition 1.4.3, 2 and Proposition 1.4.5,1. Define $k := \lfloor b \rfloor$ and $m := \lfloor a \rfloor$, then clearly $A(b) = A(k)$ and $A(a) = A(m)$. Moreover since A is constant on $[k,b]$ respectively $[a, m+1[$ we get that

$$A(b)F(b) - \int_k^b AF' = A(b)F(b) - A(b)\int_k^b F' \overset{\text{FTC}}{=} A(k)F(k) \tag{3.3}$$

and similarly we get

$$A(a)F(a) + \int_a^{m+1} AF' = A(m)F(m+1) \ . \tag{3.4}$$

Additionally α can be written as

$$\alpha(n) = A(n) - A(n-1)\ \forall n \in \mathbb{N}, n \ge 1 \ . \tag{3.5}$$

Finally we can put everything together:

$$\sum_{a<n\le b} \alpha(n)F(n) = \sum_{n=m+1}^{k} \alpha(n)F(n) \overset{(3.5)}{=} = \sum_{n=m+1}^{k} (A(n) - A(n-1))F(n) =$$
$$\sum_{n=m+1}^{k} A(n)F(n) - \sum_{n=m+1}^{k} A(n-1)F(n) \overset{n\mapsto n-1}{=} = \sum_{n=m+1}^{k} A(n)F(n) - \sum_{n=m}^{k-1} A(n)F(n+1) =$$
$$A(k)F(k) - A(m)F(m+1) + \sum_{n=m+1}^{k-1} A(n)(F(n) - F(n+1)) \overset{\text{FTC}}{=}$$

$$A(k)F(k) - A(m)F(m+1) - \sum_{n=m+1}^{k-1} A(n) \int_n^{n+1} F' =$$
$$A(k)F(k) - A(m)F(m+1) - \sum_{n=m+1}^{k-1} \int_n^{n+1} AF' = A(k)F(k) - A(m)F(m+1) - \int_{m+1}^{k} AF' \overset{(3.3),(3.4)}{=}$$
$$A(b)F(b) - \int_k^b AF' - A(a)F(a) - \int_a^{m+1} AF' - \int_{m+1}^{k} AF' = A(b)F(b) - A(a)F(a) - \int_a^b AF' \ .$$

□

The Euler summation theorem is basically a corollary of the Abel summation theorem. We only have to use integration by parts for classically Denjoy integrable functions (cf. Theorem 1.4.9). Again it is a generalization to ACG$_*$ functions of the standard Euler summation theorem (cf. [Brue95, Formula (4.5), p.122]).

3.1.3 Theorem (Euler summation theorem)
Let $a, b \in \mathbb{R}$ *with* $0 < a < b-1$ *and let* $F \in ACG_*([a,b])$*. Then*

$$\sum_{a<n\leq b} F(n) = F(b)(\lfloor b \rfloor - b) - F(a)(\lfloor a \rfloor - a) + \int_a^b F + \int_a^b (t - \lfloor t \rfloor)F'(t)dt \ . \qquad (3.6)$$

Proof: First we have to note that the function $G(t) := t - \lfloor t \rfloor$ $(t \in \mathbb{R})$ is AC on every interval $[n, n+1[$ for all $n \in \mathbb{N}$ since on $[n, n+1[$ we have that $G(t) = t - n$. Hence by Theorem 1.4.9 GF' is classically Denjoy integrable over $[n, n+1[$ and so by Proposition 1.4.3, 2 and Proposition 1.4.5,1 we conclude that GF' is classically Denjoy integrable over $[a, b]$. Now define the arithmetic function α by $\alpha(n) := 1$ for all $n \in \mathbb{N}$, then the function A from Equation (3.1) simplifies to $A(x) = \lfloor x \rfloor$ for all $x \in \mathbb{R}, x \geq 0$. Now we can use the Abel theorem (3.1.2) and get

$$\sum_{a<n\leq b} F(n) = \sum_{a<n\leq b} \alpha(n)F(n) \overset{\text{Abel}}{=} A(b)F(b) - A(a)F(a) - \int_a^b AF' =$$
$$\lfloor b \rfloor F(b) - \lfloor a \rfloor F(a) - \int_a^b \lfloor t \rfloor F'(t)dt =$$
$$\lfloor b \rfloor F(b) - \lfloor a \rfloor F(a) - \int_a^b \lfloor t \rfloor F'(t)dt + \int_a^b tF'(t)dt - \int_a^b tF'(t)dt \overset{\text{integration by parts}}{\underset{(1.38)}{=}}$$
$$f \lfloor b \rfloor F(b) - \lfloor a \rfloor F(a) - \int_a^b \lfloor t \rfloor F'(t)dt + \int_a^b tF'(t)dt - F(b)b + F(a)a + \int_a^b F =$$
$$F(b)(\lfloor b \rfloor - b) - F(a)(\lfloor a \rfloor - a) + \int_a^b (t - \lfloor t \rfloor)F'(t)dt + \int_a^b F \ .$$

□

Here we give two examples to see how the Euler theorem can be applied to approximate a harmonic series or to give a different expression of a certain sum. The first example is very important in analytical number theory and the prototypical example of the use of the Euler summation theorem in this field. The second example uses our extension of the Euler theorem to ACG$_*$ functions. Moreover the function used in the second example is improperly Riemann integrable and hence it seems possible one could get to this result without the use of the classical Denjoy integral. So we could have given an example, where the function is classically Denjoy integrable but neither (improperly) Riemann or Lebesgue integrable. However these functions seem to yield nothing interesting from a number theoretic point of view. Notwithstanding these objections our extension of the Euler respectively Abel summation theorem allows us to apply them to a broader class of functions and the question of whether this yields something interesting can only be answered from within number theory.

3.1.4 Example

1. *This example is one of the most basic results in analytical number theory, where one uses the Euler summation theorem to prove it. Let* $b \in \mathbb{N}, b \geq 1$*, then*

$$H(b,1) = \sum_{n \leq b} \frac{1}{n} = \log(b) + \gamma + O(\frac{1}{b}) \ ,$$

   where γ *is the Euler-Mascheroni constant (cf. Definition 3.1.1,5). To get this formula, take* $F(t) := \frac{1}{t}$ *and* $a := 1$ *in the Euler theorem. Then*

$$\sum_{n \leq b} \frac{1}{n} = 1 + \sum_{1<n \leq b} \frac{1}{n} \overset{\text{Euler}}{=} 1 + \int_1^b \frac{1}{t} dt - \int_1^b \frac{t - \lfloor t \rfloor}{t^2} dt + \frac{\lfloor b \rfloor - b}{b} =$$
$$\log(b) + \underbrace{1 - \int_1^\infty \frac{t - \lfloor t \rfloor}{t^2} dt}_{=\gamma} + \int_b^\infty \frac{t - \lfloor t \rfloor}{t^2} dt + O(\frac{1}{b}) = \log(b) + \gamma + O(\frac{1}{b}) \ .$$

   In this calculation we used that $\int_c^\infty \frac{t-\lfloor t \rfloor}{t^2} dt \leq \int_c^\infty \frac{1}{t^2} dt = \frac{1}{c}$ *for* $c = 1$ *respectively* $c = b$*. (Cf. [Brue95, p.12]).*

2. *In this example we apply the Euler theorem to an improperly Riemann integrable hence classically Denjoy integrable function, which is not continuously differentiable. Thus we take advantage of our extension of the Euler Theorem to ACG*$_*$ *functions. Generalized harmonic numbers are well known in analytical number theory, see for example [GKP89, p.263 ff.]. With the help of the Euler theorem we can now express the sum* $\sum_{n=1}^b (n\sqrt{n+1})$ *in terms of generalized harmonic numbers. Let* $b \in \mathbb{N}, b > 1$ *and* $f : [0,b] \to \mathbb{R}$ *be given by* $f(x) := \begin{cases} \frac{1}{\sqrt{x}} & x \in]0,b], \\ 0 & x = 0. \end{cases}$ *Then we define* $F(x) := \int_0^x f = 2\sqrt{x}$ *for* $x \in [0,b]$ *and we know that* F *is ACG*$_*$ *on* $[0,b]$ *by Proposition 1.4.6, 1. So by the*

Euler theorem we get

$$2\ H(b,-\frac{1}{2}) = \sum_{n=1}^{b} F(n) \overset{\text{Euler}}{=} \int_0^b F + \int_0^b (t-\lfloor t\rfloor)f(t)dt = \frac{4\sqrt{b^3}}{3} + \int_0^b \frac{t}{\sqrt{t}}dt - \int_0^b \frac{\lfloor t\rfloor}{\sqrt{t}}dt =$$
$$2\sqrt{b^3} - \sum_{n=1}^{b-1}\int_n^{n+1}\frac{n}{\sqrt{t}}dt = 2\sqrt{b^3} - \sum_{n=1}^{b-1} n\int_n^{n+1}\frac{1}{\sqrt{t}}dt = 2\sqrt{b^3} - \sum_{n=1}^{b-1} 2n(\sqrt{n+1}-\sqrt{n}) =$$
$$2(\sqrt{b^3} - \sum_{n=1}^{b-1}(n\sqrt{n+1}) + H(b-1,-\frac{1}{2}))\ .$$

This yields

$$\sum_{n=1}^{b-1}(n\sqrt{n+1}) = 2(\sqrt{b^3} - H(b,-\frac{1}{2})) + H(b-1,-\frac{3}{2})\ .$$

Note that we could have used the Euler theorem for the (continuously differentiable) function $F(x) := x\sqrt{x+1}$ but that would not give a simpler expression of this sum since it would involve higher terms. To be more precise we would get

$$\sum_{n=1}^{b}(n\sqrt{n+1}) = b^2\sqrt{b+1} - \sum_{n=1}^{b-1} n((n+1)\sqrt{n+2} - n\sqrt{n+1})\ .$$

3.1.5 Remark
It seems that the distributional Denjoy integral cannot be used for analogous statements like the Abel or Euler summation theorem. First we do not have an analogue of Proposition 1.4.3, 2, as was shown in Example 2.2.11. Second we do not have a integration by parts formula. One can prove an integration by parts formula for the distributional Denjoy integral, see for example [Tal07, Sec. 5, p. 9-12]. But note that in this case even the multiplication of the two functions is a problem.

3.1.2. Ordinary differential equations

Another application of the classical Denjoy integral is to first order ordinary differential equations (ODEs). In the theory of ODEs one key step in proving the basic existence result (the Picard-Lindelf theorem, cf. [AE08, Thm. 8.14, p.244f.]) is to reformulate the ODE as an integral equation. With the classical Denjoy integral we can achieve the same for a broader class of functions.

3.1.6 Proposition
Let $f : [a,b]\times[c,d] \to \mathbb{R}$ with $-\infty \le c < d \le \infty$, let $x :\ [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ (one-sided derivatives at a respectively b) and let $x_0 \in \mathbb{R}$. Then the following are equivalent:

1. *The function x is satisfies $x([a,b]) \subseteq\ [c,d]$, $x(a) = x_0$ and*

$$x'(t) = f(t,x(t)) \quad \textit{holds for almost all } t\in[a,b]\ . \tag{3.7}$$

2. *The function $t \mapsto f(t, x(t))$ is classically Denjoy integrable on $[a, b]$ and x satisfies*

$$x(t) = x_0 + \int_a^t f(s, x(s))ds \quad (t \in [a, b]) \ . \tag{3.8}$$

Moreover if there is such a function x satisfying one of the above equations, it is unique.

Proof: Define the function $g : [a, b] \to \mathbb{R}$ by $g(t) := f(t, x(t))$ $(t \in [a, b])$.

($\Rightarrow$) Since x is differentiable (and hence continuous) on $[a, b]$ it is ACG$_*$ on $[a, b]$ by Corollary 1.3.22. So x' is classically Denjoy integrable on $[a, b]$ (by the FTC 1.4.4) and hence by Proposition 1.4.5, 3, the function g is classically Denjoy integrable on $[a, b]$ because $x' = g$ a.e. on $[a, b]$. Therefore

$$\int_a^t g = \int_a^t x' = x(t) - x(a) = x(t) - x_0 \ .$$

($\Leftarrow$) Now $x(t) = x_0 + \int_a^t g$, where g is classically Denjoy integrable on $[a, b]$. By Propositions 1.4.6,2 and 1.4.5,1 we know that $x' = g$ a.e. on $[a, b]$. Moreover $x(a) = x_0$ and $x([a, b]) \subseteq [c, d]$ since otherwise the formula (3.8) would not make sense.

Finally let $y : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and assume that x, y satisfy Equation (3.7) or (3.8) and $x(a) = y(a) = x_0$. Then we know that x and y are ACG$_*$ on $[a, b]$ with $x' = y' = g$ a.e. on $[a, b]$, hence by Proposition 1.3.24 $x - y$ is constant. Therefore from $x(a) = x_0 = y(a)$ we conclude that $x = y$. □

Note that in the common formulation the RHS of the ODE (in our case the function f) is assumed to be continuous, but in our formulation we also allow for discontinuous RHS. Yet the RHS cannot be arbitrary since it has to be the derivative of a continuous function (if a solution exists at all).

Also observe that we could adjust the conditions of the above theorem by imposing different conditions on the function x. We could also have assumed that x needs to be continuous on $[a, b]$ and only differentiable nearly everywhere on $[a, b]$. Then the equivalence would hold too. On the other hand if we had assumed only that x is differentiable a.e. on $[a, b]$, then we would have only the implication $2.) \Rightarrow 1.)$. Moreover if we had assumed that x is differentiable n.e. and continuous on $[a, b]$ in the first case, then we would have only the implication $1.) \Rightarrow 2.)$, since from the second case we could deduce only that x is differentiable a.e. on $[a, b]$.

With this brief exposition here we are trying to show how the classical Denjoy integral can be used in the theory of ordinary differential equations. Of course one could go further and ask about existence of solutions and so on, but this would go beyond the scope of this thesis. We finally remark that this kind of problem lead Jaroslav Kurzweil to his Henstock-Kurzweil integral (he called it generalized Riemann integral) in his theory of generalized ODEs. See [Sch89] for a survey of this field (especially Thm. 2, p.60-61).

At last we discuss what happens if we use the distributional Denjoy integral in this case. First we have to clarify what notion of derivative we use in Equation (3.7). Since Example

2.2.21,2 showed us that there are pointwise nowhere differentiable functions which are primitives of distributionally integrable functions, we have to use the distributional derivative. Then the LHS of (3.7) is a distribution and not a classical function. So we see that in the case of the distributional Denjoy integral, this problem has no obvious adequate formulation.

3.2. Alternative equivalent approaches

In this section we briefly sketch two different approaches to the problem of integrating all derivatives and the FTC. First the Henstock-Kurzweil integral and then the Perron integral. As it turns out these two integrals are equivalent to the classical Denjoy integral. That means that every classical Denjoy integrable function is also integrable with respect to these two notions of integrability and vice versa. Moreover the integrals give the same values and of course share the same properties. Although some properties are harder respectively easier to prove for different integrals depending on how the property relates to the definition of the integral. For example the FTC in its most general form (assuming only differentiability nearly everywhere) is quite hard to prove for the Perron integral, but for the classical Denjoy integral it follows essentially from the definition of the integral.
The equivalence of these three integrals is discussed and proved in [Gor94, Chap. 11, p.169-179].

3.2.1. The Henstock-Kurzweil integral

As mentioned in Subsection 3.1.2 the Henstock-Kurzweil (HK) integral was introduced as a generalization of the Riemann integral by Jaroslav Kurzweil around 1950 in his work on ordinary differential equations. Around 1960 Ralph Henstock started to work on an integration theory for the HK-integral. What they did was to generalize the Riemann sums in the definition of the Riemann integral.
A Riemann sum is a sum of the form

$$\sum_{n=1}^{N} f(c_n)(b_n - a_n) \ ,$$

where $f: \ [a,b] \to \mathbb{R}$ and $([a_n,b_n])_{n=1}^{N}$ is a finite collection of non-overlapping intervals in $[a,b]$ with $a_1 = a, b_N = b$ and $c_n \in [a_n,b_n]$ for $n = 1,\dots,N$. Then we let the lengths of the intervals go to zero, to get the Riemann integral of f on $[a,b]$ (if it exists). One can see that we have no restriction on the points c_n other than to be in the interval $[a_n,b_n]$. Now the idea is that for different points c_n we have different maximal lengths of these intervals. This leads us to the notion of tagged intervals subordinate to a positive function.

3.2.1 Definition (Tagged intervals and partitions)
Let $\delta : [a,b] \to]0,\infty[$*.*

1. *We call* $(x,[c,d])$ *a* <u>*tagged interval*</u> *if* $[c,d] \subseteq [a,b]$ *and* $x \in [c,d]$*. The tagged interval* $(x,[c,d])$ *is called* <u>*subordinate*</u> *to* δ *if* $[c,d] \subseteq]x-\delta(x), x+\delta(x)[$*.*

2. *Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. Then $\mathcal{P}$ is called subordinate to δ if every $(x_i, [c_i, d_i])$ is subordinate to δ for $i = 1, \ldots, n$. Moreover if $\mathcal{P}$ is subordinate to δ and $[a, b] = \bigcup_{i=1}^{n} [c_i, d_i]$, then $\mathcal{P}$ is called a tagged partition of $[a, b]$.*

(Cf. [Gor94, Def. 9.1, p.138] respectively [KS04, Def. 4.7, p.139]).

For the definition of the Henstock-Kurzweil integral we only need a small observation. Namely that for every $\delta : [a, b] \to]0, \infty[$ there is a tagged partition of $[a, b]$ subordinate to δ. See for example [Gor94, Lem. 9.2, p.139; Exc. 9.1, p.319 for the proof].

3.2.2 Definition
Let $f : [a, b] \to \mathbb{R}$, then f is Henstock-Kurzweil integrable on $[a, b]$ if there exists $L \in \mathbb{R}$ such that for all $\epsilon > 0$ there exists a $\delta : [a, b] \to]0, \infty[$ with $|\sum_{i=1}^{n} f(x_i)(d_i - c_i) - L| < \epsilon$, whenever $((x_i, [c_i, d_i])_{i=1}^{n}$ is a tagged partition of $[a, b]$ subordinate to δ. The number L is the Henstock-Kurzweil integral on $[a, b]$ and will be denoted by $\int_a^b f$. (Cf. [Gor94, Def. 9.3, p.140] respectively [KS04, Def. 4.12, p.141]).

With this definition it is much harder to prove the FTC than it was for the classical Denjoy integral. The advantage here is that we can easily give the definition and it is very intuitive. The FTC for the Henstock-Kurzweil integral can be found in [Gor94, Thm. 9.6, p. 141f.] and [KS04, Thm. 4.24, p.148f.]. Of course this integral has the same properties as the classical Denjoy integral. For a full development of this theory see [Gor94, Chap. 9, p.137-156] or [KS04, Chap. 4, p. 133- 221].

3.2.2. The Perron integral

The last integral we are discussing in this thesis is the Perron integral (by Oskar Perron, around 1914). Its definition is rather simple compared to the machinery used to define the classical Denjoy integral. We just start by saying what a major respectively a minor function is and then we can define the Perron integral.
In the following definition we are using upper and lower derivatives as given in Definition 1.2.5.

3.2.3 Definition (Major and minor functions)
Let $f : [a, b] \to \mathbb{R} \cup \{\pm\infty\}$. A function $U : [a, b] \to \mathbb{R}$ is called a major function of f on $[a, b]$ if $\underline{D}U(x) > -\infty$ and $\underline{D}U(x) \geq f(x)$ for all $x \in [a, b]$. A function $V : [a, b] \to \mathbb{R}$ is called a minor function of f on $[a, b]$ if $\overline{D}V(x) < \infty$ and $\overline{D}V(x) \leq f(x)$ for all $x \in [a, b]$. (Cf. [Gor94, Def. 8.1, p.121]).

Now we can give the definition of the Perron integral.

3.2.4 Definition
Let $f : [a, b] \to \mathbb{R} \cup \{\pm\infty\}$, then f is Perron integrable on $[a, b]$ if there are major and minor

functions of f on $[a,b]$ and

$$\inf\{U(b)-U(a) : U \text{ is a major function of } f \text{ on } [a,b]\}$$
$$=$$
$$\sup\{V(b)-V(a) : U \text{ is a minor function of } f \text{ on } [a,b]\} \ .$$

This value is the Perron integral of f on $[a,b]$ and will be denoted by $\int_a^b f$. (Cf. [Gor94, Def. 8.5, p.123]).

It takes only a few brief calculations that the Perron integral is always finite and that we have the following equivalence.

3.2.5 Theorem
Let $f : [a,b] \to \mathbb{R} \cup \{\pm\infty\}$, then f is Perron integrable on $[a,b]$ if and only if for all $\epsilon > 0$ there is a major function U of f on $[a,b]$ and a minor function V of f on $[a,b]$ such that $U(b)-U(a)-V(b)+V(a) < \epsilon$. (Cf. [Gor94, Thm. 8.6, p. 123]).

Note that this generalizes the following integrability condition for the Lebesgue integral.

3.2.6 Theorem
Let $f : [a,b] \to \mathbb{R} \cup \{\pm\infty\}$, then f is Lebesgue integrable on $[a,b]$ if and only if for all $\epsilon > 0$ there is an AC major function U of f on $[a,b]$ and an AC minor function V of f on $[a,b]$ such that $U(b)-U(a)-V(b)+V(a) < \epsilon$. (Cf. [Gor94, Thm. 8.2, p. 121f.]).

So we easily see that every Lebesgue integrable function is also Perron integrable. Moreover since a derivative of a differentiable function $F : [a,b] \to \mathbb{R}$ is always a major and minor function of this function F, we get that every derivative is Perron integrable (cf. [Gor94, Thm. 8.7, p.123]). The drawback here is that this argument works only if the function F is differentiable everywhere on $[a,b]$, but for the classical Denjoy integral we had the result that F needs to be only differentiable nearly everywhere on $[a,b]$. To prove this stronger statement needs a lot more work than the previous results together (cf. [Gor94, Thm. 8.26, p.134]).
For further properties of the Perron integral see [Gor94, Chap. 8, p. 121-136].

3.3. Outlook

In this final section we briefly discuss what topics in the theory of the classical and distributional Denjoy integral we have not touched so far. We list several topics beginning with important properties we have not proven and finishing with possible generalizations of the Denjoy integral.

3.3.1. Further properties

A desired property of an integral is a change of variables formula (or a substitution rule). For the HK-integral one can find two related results in [YV00, Thm. 2.7.5, p.52f. and

Thm. 2.7.8, p.53f.]. Since the classical Denjoy integral is equivalent to the HK-integral it shares this property. For the distributional Denjoy integral see [Tal07, Chap.6, p.64f.].
Also one often needs results about convergence of integrals and interchanging limits with integration. For the classical Denjoy integral see [CD89, Sec. 1.12, p. 37-44] or [Gor94, Chap. 13, especially p.204-208] and for the distributional Denjoy integral see [Tal07, Chap. 7, p.65-70].
We proved in Theorem 1.4.9 an integration by parts formula for the classical Denjoy integral. As mentioned in Subsection 3.1.1 an analogous result for the distributional Denjoy integral is much harder to achieve. Nevertheless see [Tal07, Chap. 5, p.59-64] for a discussion of this topic, where also a Hölder like inequality is proven.
In [CD89, Chap. 3, p.185-225] a theory of Fourier-Denjoy series is developed, an analogue of the classical Fourier theory, where one uses the classical Denjoy integral instead of the Lebesgue integral. In [Tal07, Chap. 8, p.70-72] Talvila gives an analogue of the Poisson integral and Laplace transform for the distributional Denjoy integral.
Of course there are many more interesting properties of the classical and distributional Denjoy integral, but we restricted ourselves here to main properties which have an analogue in classical (Lebesgue) integration theory.

3.3.2. Generalizations

One obvious generalization of the classical Denjoy integral is to include functions whose domains are unbounded intervals in $\mathbb{R}$. This was done for the HK-integral in [KS04, Sec. 4.4, p. 154-162].
Another generalization of the classical Denjoy integral is the wide Denjoy integral, which uses ACG functions (instead of ACG_* functions) as primitives and the derivative is understood as approximate derivative. In [CD89, Chap. 1, p. 1-63] the classical and wide Denjoy integral are introduced simultaneously. The wide Denjoy integral includes the classical Denjoy integral and the distributional Denjoy integral includes the wide Denjoy integral (cf. [CD89, Thm. 30, p. 28] respectively [Tal07, Ex. 1.3, p.56f.]).
One more possibility of generalization is to consider functions with values in $\mathbb{R}^n$ (for $n \in \mathbb{N}, n > 1$) or Banach space valued functions. For an extension of the classical Denjoy integral to functions $[a, b] \to \mathbb{R}^2$, the so called double Denjoy integral, see [CD89, Chap. 2, p.64-184]. Moreover for an extension of the classical Denjoy integral respectively the HK-integral to functions with values in a Banach space see [Sch05] respectively [Gor89]. For a brief discussion of the distributional Denjoy integral in $\mathbb{R}^n$ see [Tal07, Sec. 12.2, p.79f.]. Of course one can also consider functions with a more general domain than an one-dimensional interval. See [Yee89, Chap. 5, Sec. 21, p.127-136] for the case where the domain is an n-dimensional (axis-aligned) cube and for a more general introduction see [Mul87].

A. Appendix

Here we list some results which are not very important for the theory of the Denjoy integral but are used nevertheless in this thesis.

A.1 Lemma

Let E be a vector space and $\|.\|_1$, $\|.\|_2$ two norms on E such that $(E, \|.\|_1)$ and $(E, \|.\|_2)$ are complete i.e. Banach spaces. If there exits $\mu > 0$ such that $\|x\|_2 \leq \mu\|x\|_1$ $\forall x \in E$ ($\|.\|_1$ is stronger than $\|.\|_2$) then the two norms are equivalent i.e. $\exists \lambda > 0$ such that $\|x\|_1 \leq \lambda\|x\|_2$ $\forall x \in E$. (Cf. [Wer07, Cor. IV.3.5, p.153]).

Proof: First we observe that $\|x\|_2 \leq \mu\|x\|_1$ $\forall x \in E$ means that $id_1 : (E, \|.\|_1) \to (E, \|.\|_2)$ is continuous and also it is of course injective. So by the open mapping principle (see for example [Heu82, p.140, §32 Thm. 32.2]) $id_1^{-1} = id_2 : (E, \|.\|_2) \to (E, \|.\|_1)$ is also continuous i.e. there exists a constant $\lambda > 0$ such that $\|x\|_1 = \|id_2(x)\|_1 \leq \lambda\|x\|_2$ $\forall x \in E$, hence $\|.\|_1 \sim \|.\|_2$. □

A.2 Lemma

Let $(E, \|.\|_E), (F, \|.\|_F)$ be two isometrically isomorphic normed vector spaces, then E is complete if and only if F is.

Proof: First we assume that $(E, \|.\|_E)$ is complete. Let $\Phi : F \to E$ be an isometry and let $(x_n)_n$ be a Cauchy sequence in F with respect to $\|.\|_F$ i.e. $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ such that for $\forall m, n \in \mathbb{N}$ with $m, n \geq N$ we have that $\|x_n - x_m\|_F < \epsilon$. Then $(\Phi(x_n))_n)$ is a Cauchy sequence in E with respect to $\|.\|_E$, because $\|\Phi(x_n) - \Phi(x_m)\|_E = \|\Phi(x_n - x_m)\|_E = \|x_n - x_m\|_F$. So there is an $e \in E$ such that $\|e - \Phi(x_n)\|_E \to 0$ $(n \to \infty)$. We claim that $x_n \to \Phi^{-1}(e)$ $(n \to \infty)$ with respect to $\|.\|_F$. This follows from the fact that Φ^{-1} is also an isometry and so $\|\Phi^{-1}(e) - x_n\|_F = \|\Phi^{-1}(e) - \Phi^{-1}(\Phi(x_n))\|_F = \|e - \Phi(x_n)\|_E \to 0$ $(n \to \infty)$. Thus $(F, \|.\|_F)$ is complete. To show the other implication just interchange the roles of E and F. □

References

[AE01] Herbert Amann and Joachim Escher, Analysis III, 1 ed., Birkhäuser, 2001.

[AE06] ______, Analysis I, 3 ed., Birkhäuser, 2006.

[AE08] ______, Analysis II, 2 ed., Birkhäuser, 2008.

[Brue95] Jörg Brüdern, Einführung in die analytische Zahlentheorie (Springer-Lehrbuch) (German Edition), 1 ed., Springer, 1995.

[Bru78] Andrew M Bruckner, Differentiation of Real Functions (Lecture Notes in Mathematics), Springer, 1978.

[CD89] V. G. Celidze and A.G. Dzvarseisvili, Theory of the Denjoy Integral and Some Applications (Series in Real Analysis Vol, 3), World Scientific Publishing Company, 1989.

[DK10] J.J. Duistermaat and J.A.C. Kolk, Distributions: Theory and Applications, 1st edition. ed., Birkhäuser Boston, 2010.

[GKP89] Ronald L. Graham, Donald Knuth, and Oren Patashnik, Concrete Mathematics: A Foundation for Computer Science, Addison-Wesley Pub (Sd), 1989.

[GO03] Bernard R. Gelbaum and John M. H. Olmsted, Counterexamples in Analysis (Dover Books on Mathematics), Dover Publications, 2003.

[Gor89] Russell A. Gordon, The Denjoy extension of the Bochner, Pettis and Dunford integrals, Studia Mathematica **92** (1989), 73–91.

[Gor94] ______, The Integrals of Lebesgue, Denjoy, Perron, and Henstock, American Mathematical Society, 1994.

[Heu82] Harro G. Heuser, Functional analysis, John Wiley & Sons Ltd, 1982.

[HS09] Günther Hörmann and Roland Steinbauer, Distributionentheorie, Lecture, Summerterm 2009, University of Vienna.

[KS04] Douglas S. Kurtz and Charles W. Swartz, Theories of Integration: The Integrals of Riemann, Lebesgue, Henstock-Kurzweil, and Mcshane (Series in Real Analysis), World Scientific Publishing Company, 2004.

[Mul87] P Muldowney, General theory of integration in function spaces, including Wiener and Feynman integration (Pitman Research Notes in Mathematics Series), Wiley, 1987.

[Ost83] K. Ostaszewski, Topology for the spaces of Denjoy integrable functions, Real Analysis Exchange **9** (1983), 79–85.

[Sch89] Štefan Schwabik, Generalized ordinary differential equations - a survey, Jaroslav Kurzweil (ed.): Equadiff 7, Proceedings of the 7th Czechoslovak Conference on Differential Equations and Their Applications held in Prague (1989), 59–70.

[Sch05] ______, Topics in Banach Space Integration (Series in Real Analysis), World Scientific Publishing Company, 2005.

[Sti01] John Stillwell, Mathematics and its history, 2nd ed., Springer, 2001.

[Tal07] E. Talvila, The distributional Denjoy integral, Real Analysis Exchange **33** (2007), no. 1, 51–84.

[Tho99] B.S. Thomson, The space of Denjoy-Perron integrable functions, Real Analysis Exchange **25** (1999), no. 2, 711–726.

[Wer07] Dirk Werner, Funktionalanalysis (Springer-Lehrbuch) (German Edition), 6. ed., Springer, 2007.

[Wuß08] Hans Wußing, 6000 Jahre Mathematik: Eine kulturgeschichtliche Zeitreise - 1. Von den Anfängen bis Leibniz und Newton (Vom Zählstein zum Computer) German Edition), 1 ed., Springer, 2008.

[Yee89] Lee Peng Yee, Henstock Integration, Lanzhou Lectures On Henstock Integration (Series in Real Analysis), World Scientific Publishing Company, 1989.

[YV00] Lee Peng Yee and Rudolf Vyborny, Integral: An Easy Approach after Kurzweil and Henstock (Australian Mathematical Society Lecture Series), Cambridge University Press, 2000.

Curriculum Vitae

Personal information

Name:	Clemens Sämann
Date of birth:	28th October, 1986
Birth location:	Vienna, Austria
Nationality:	Austria

Education and professional experience

June 2006:	A-Levels (Matura) at HTL Spengergasse Wien with distinction
July 2006 - Jan. 2007:	Military service
Jan. 2007 - Feb. 2008:	Software developer at Genlan (part time)
March 2007 - Oct. 2011:	Mathematics at the University of Vienna
Winter semester 2009/10:	Exchange semester at the University of Edinburgh (UK)
Winter semester 2010/11:	Tutor for "Mathematik für Lebensmittel- und Biotechnologen" at the University of Natural Resources and Life Sciences and Tutor for "Lineare Algebra für PhysikerInnen" at the University of Vienna, Faculty of Physics

SACHERSCHLIESSUNG

der

Fachbereichsbibliothek Mathematik, Statistik, Informatik der Universität Wien

Klassifikation

Aufstellungsort: ☒ HoA ☐ HoD

MSC2000: 26A39 – 26A45 – 26A46 – 46F10

BK: 31.41

Schlagwortketten nach RSWK

Denjoy-Integration

Kontrollvermerk der Fachbibliothek:

20. Sep. 2011

Datum

Unterschrift