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Abstract

This work deals with the extension of the classical non-relativistic radiation hydrodynamic equations to their relativistic form.
Starting with a recapitulation of some basic knowledge of special relativity (chapter 1), a covariant formalism of the quantities of radiation hydrodynamics is derived (chapter 2). Based on these fundamentals the equations of relativistic radiation hydrodynamics (RRHD) are obtained in their conservative form (chapter 3) and hence ready for the discretization and their implementation in the RHD-code (chapter 4). Chapter 4 also deals with the calculation of derivatives with MATHEMATICA and shows some alternative ways (to the original version established by Matthias Kittel) to deal with them.
It must also be noted that the theoretical part of this work is mostly based on the work of Mihalas ([MWM84]) and Castor ([Cas04]).

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CHAPTER 1

Basic Principles of Special Relativity

This first chapter gives a brief introduction into the fundamental principles of special relativity ([MWM84], [d'I95], [Pad00]), in order to get the derivations of the equations of relativistic radiation hydrodynamics (RRHD) easily done.
Based on known quantities such as energy, specific intensity, emission and absorption coefficients the covariant form of the radiation transport equation is derived later on. Therefore the physical framework for the subsequent formalism will be defined within these considerations.

The starting point of this work are two questions that might be of interest. First of all: Why relativistic?
The answer to this is quite clear as there are many astrophysical processes in which gases reach high velocities. Such radiation hydrodynamic processes are for example supernova explosions, accretion on compact objects, cosmic expansion, gamma-ray-bursts, jets and so on. Under such high velocities relativistic effects arise, e.g. abberation, dopplershifts, time dilatation, Lorentz contraction.
The second question is: And why exactly special relativistic?
The simplest argument for this is: It is easier to calculate (analytical and numerical).
On the other hand, there are two types of gamma-ray-bursts: those which occur during merging processes of two neutron stars to black holes ('short and hard' breakout) and those which occur during a collapse / supernova explosion of a red giant ('long and soft' breakout, collapsar model). The latter form the basis of the physical model that was applied and of the subsequent calculations. Therefore the physical framework is set in a flat space time. This is done because the interest is in the radiation outbreak, hence the core-collapse is ignored and there is only the self-gravitation of the red giant itself to deal with. Certainly, if the core collapse is also taken into account, curved space time must be assumed and therefore general relativistic effects must be considered.

1.1 The Lorentz Transformation

Based on two inertial systems which move with velocity v relative to each other along the z axis, one can switch between those systems by applying a Lorentz transformation:

$$x' = x \tag{1.1.1}$$
$$y' = y \tag{1.1.2}$$
$$z' = \gamma(z - vt) \tag{1.1.3}$$
$$t' = \gamma(t - \beta\frac{z}{c}) \tag{1.1.4}$$

where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ and $\beta = \frac{v}{c}$. As is immediately seen, for small velocities $\frac{v}{c} \ll 1$ this reduces to the Galilean transformation.
Because of the fact that the Lorentz transformation is a four-dimensional transformation, notation is defined as follows for future use: $\mathbf{x}$ is a four-vector with its components $x^0 \equiv ct$, $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$. Alternatively $\mathbf{x} \equiv x^\alpha$, where $\alpha = (0, 1, 2, 3)$. Following the notation that Greek indices have a range from $0-3$ and Latin indices lie between $1-3$.

The Lorentz transformation can be written in matrix notation $x^{'\alpha} = L^{'\alpha}_{\beta} x^\beta$ where $L^{'\alpha}_{\beta}$ is nothing else than the Jacobi matrix $\left(\frac{\partial x^{'\alpha}}{\partial x^\beta}\right) = L^{'\alpha}_{\beta}$.
Furthermore, it is known that the Lorentz transformation is symmetric, which means that $\mathbf{L}' \equiv \mathbf{L}$. With this knowledge it is clear that the inverse Lorentz transformation can be written as

$$\mathbf{x} = \mathbf{L}^{-1}\mathbf{x}' \tag{1.1.5}$$

or alternatively

$$\left(\frac{\partial x^\alpha}{\partial x^{\beta'}}\right) = L^\alpha_{\beta'} \tag{1.1.6}$$

$\mathbf{L}$ and $\mathbf{L}^{-1}$ is written out

$$L = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{1.1.7}$$

$$L^{-1} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{1.1.8}$$

The relation between $\mathbf{L}$ and $\mathbf{L}^{-1}$ is given by:

$$L^{\alpha}_{\gamma'} L^{\gamma'}_{\beta} = \delta^{\alpha}_{\beta'} \tag{1.1.9}$$

$$L^{\alpha'}_{\gamma} L^{\gamma}_{\beta'} = \delta^{\alpha'}_{\beta} \tag{1.1.10}$$

1.2 The Metric

As described above it is sufficient to use the standard metric with the line element

$$ds^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{1.2.1}$$

and

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2.2}$$

The transformation between the coordinate systems implies also the transformation of the metric

$$\eta'_{\alpha\beta} = \frac{\partial x^{\epsilon}}{\partial x'^{\alpha}} \frac{\partial x^{\zeta}}{\partial x'^{\beta}} \eta_{\epsilon\zeta} = L^{\epsilon}_{'\alpha} L^{\zeta}_{'\beta} \eta_{\epsilon\zeta} \tag{1.2.3}$$

Because of the fact that ds^2 is invariant under Lorentz transformation it is known that $\eta'_{\alpha\beta} = \eta_{\alpha\beta}$. Furthermore a metric is used which satisfies

$$L^{\epsilon}_{'\alpha} L^{\zeta}_{'\beta} \eta_{\epsilon\zeta} = \eta_{\alpha\beta'} \tag{1.2.4}$$

The transformation behaviour of the metrik gives the general transformation behaviour of four vectors (i. e. second order covariant tensors) too:

$$T'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} T^{\mu\nu} = L^{\alpha'}_{\mu} L^{\beta'}_{\nu} T^{\mu\nu} \tag{1.2.5}$$

This is essential because it is needed later when the transformation of the radiation moments is calculated. (1.2.5) can be written in matrix form as

$$\mathbf{T'} = \mathbf{LTL'} = \mathbf{LTL} \tag{1.2.6}$$

because $\mathbf{L}$ is symmetric.
The inverse transformation can be written just as well

$$T^{\alpha\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} T'^{\mu\nu} = L^{\alpha}_{'\mu} L^{\beta}_{'\nu} T'^{\mu\nu} \tag{1.2.7}$$

and in matrix form

$$\mathbf{T} = \mathbf{L^{-1} T' (L^{-1})^{t}} = \mathbf{L^{-1} T L^{-1}} \tag{1.2.8}$$

1.3 Generalization of the Lagrangian Derivative

The last section gave a brief overview of transformation laws that will be considered closer from the physical point of view. As a relativistic formulation of the equations of radiation hydrodynamics is sought, first of all a relativistic generalization of the common classical expressions has to be found. This in turn requires that before starting calculating and generalizing some thought should be given to frameworks.

The considerations in the previous section implied transformations between inertial systems. But in general it can't be assumed that inertial frames are always there.
In general, speaking of velocities of a fluid means the velocity which is measured in the lab frame (i.e. an inertial system) which again is a function of $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$.

However, if switching to the comoving frame of a fluid package (i.e. the frame in which the observer or in this case the particle is at rest and hence $v = 0$), it is obvious that this is no longer an inertial system because the fluid can be accelerated during its (the fluid packages') motion. This change of systems implies that the Lorentz transformation has to be applied to relate quantities between the lab frame and the comoving frame. This is an embarrassing situation because strictly speaking the Lorentz transformation must not be applied on non-inertial systems. In other words, it can only be applied on inertial systems which show constant velocity to each other.
But thanks to special relativity this problem can be solved: it hypothesises namely that a comoving system of a fluid package can be considered as a sequence of inertial frames. These inertial frames have velocities which instantaneously have the velocity of the comoving system. And it is supposed that the Lorentz transformation can be applied between those sequences of inertial systems and the lab frame.
To describe the motion of a fluid and its time, a generalization (i.g. covariant formulation) of the Lagrangian derivative has to be found, which writes, in ordinary three-space as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} \tag{1.3.1}$$

The Newtonian Lagrangian derivative $\frac{D}{Dt}$ is the time derivative evaluated towards the motion of the fluid (lab frame). Hence, $\left(\frac{D}{Dt}\right)_0$ is the time derivative evaluated in the comoving frame.

For relativistic generalization the ordinary three-space time t has to be transformed into the proper time τ. This means that a four-dimensional generalization of our common time is needed for a covariant formalism, since ordinary three-space time is not an invariant in the four-space (as seen above (1.1.4)).
The following definition gives the relation between the proper time τ and the common time t of the Euler frame (lab frame)

$$d\tau = \left\{1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right\}^{1/2} dt = \underbrace{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}_{\gamma^{-1}} dt \tag{1.3.2}$$

The covariant generalizations of velocity and acceleration are naturally the four-velocity and the four-acceleration

$$V^\alpha \equiv \left(\frac{dx^\alpha}{d\tau}\right) = \gamma(c, \mathbf{v}) \tag{1.3.3}$$

$$A^\alpha \equiv \left(\frac{dV^\alpha}{d\tau}\right) = \gamma \frac{d}{dt}\left[\gamma(c, \mathbf{v})\right] \tag{1.3.4}$$

With this knowledge the covariant form of the Lagrangian derivative can be simply written as

$$\frac{D}{D\tau} = \underbrace{\left(\frac{dt}{d\tau}\right)}_{\gamma = V^0 \frac{1}{c}} \frac{\partial}{\partial t} + \underbrace{\left(\frac{dx^i}{d\tau}\right)}_{V^i} \frac{\partial}{\partial x^i} = V^0 \frac{1}{c}\frac{\partial}{\partial t} + V^i \frac{\partial}{\partial x^i} = V^\alpha \left(\frac{\partial}{\partial x^\alpha}\right) \tag{1.3.5}$$

In addition, it should be said that this form is true for Cartesian coordinates.
For curvilinear coordinates the derivation has to be replaced by the covariant derivation

$$\frac{Df}{D\tau} = V^\alpha f_{;\alpha} \tag{1.3.6}$$

And for flat space time this is also true for common curved three-dimensional space (e.g. spherical coordinates).

CHAPTER 2

Covariant Formulation of Radiation Quantities

This chapter touches gradually on radiation hydrodynamics while deriving transformation laws for quantities like frequency, propagation direction, specific intensity, emission and absorption coefficients ([E.A03], [MWM84], [Cas04]).
In addition, covariant generalizations of the radiation moments are found and hence pave the way for the derivation of the equations of relativistic radiation hydrodynamics.

2.1 Lorentz transformation of photon frequency and propagation

Starting with the Lorentz transformation of the photon frequency and propagation direction the photon four momentum writes

$$M^{\alpha} = \left(\frac{h\nu}{c}\right)(1, \mathbf{n}) \tag{2.1.1}$$

where ν is the frequency of the photon, $h\nu$ the photon's energy and $\mathbf{n}$ the propagation direction (see appendix for further information).
The photon-four-propagation-vector is given by

$$K^{\alpha} = \left(\frac{2\pi\nu}{c}\right)(1, \mathbf{n}) \tag{2.1.2}$$

If now a photon in the lab frame is considered which moves in a direction $\mathbf{n}$ with a frequency ν, then this photon would have some other frequency ν_0 and direction $\mathbf{n}_0$ in its rest frame.
Because of the fact that M^{α} is a four-vector and its components are related by a Lorentz

transformation between the two systems, the frequencies and direction propagation vectors transform ([MWM84]) as

$$\nu_0 = \gamma\nu\left(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}\right) \tag{2.1.3}$$

$$\nu = \gamma\nu_0\left(1 + \frac{\mathbf{n}_0\cdot\mathbf{v}}{c}\right) \tag{2.1.4}$$

and

$$\mathbf{n}_0 = \left(\frac{\nu}{\nu_0}\right)\left\{\mathbf{n} - \gamma\left(\frac{\mathbf{v}}{c}\right)\left[1 - \frac{\left(\gamma\frac{\mathbf{n}\cdot\mathbf{v}}{c}\right)}{(\gamma+1)}\right]\right\} \tag{2.1.5}$$

$$\mathbf{n} = \left(\frac{\nu_0}{\nu}\right)\left\{\mathbf{n}_0 + \gamma\left(\frac{\mathbf{v}}{c}\right)\left[1 - \frac{\left(\gamma\frac{\mathbf{n}_0\cdot\mathbf{v}}{c}\right)}{(\gamma+1)}\right]\right\} \tag{2.1.6}$$

Only one dimension is used and the other two are neglected for reasons of simplification and later numerical considerations. Furthermore, spherical geometry is considered and therefore the above transformation can be written from Cartesian into spherical geometry.

For completeness it must be said that the standard spherical metric is

$$ds^2 = -c^2dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{2.1.7}$$

and therefore the result is

$$\left(\nu_0, \nu_0 n_{x0}, \nu_0 n_{y0}, \nu_0 n_{z0}\right) = \left[\gamma\nu(1 - n_z\beta), \nu n_x, \nu n_y, \gamma\nu(n_z - \beta)\right] \tag{2.1.8}$$

And for spherical geometry with $\mu = \cos\theta$ and $\beta = \frac{v}{c}$:

$$\left(\nu_0, \mu_0, (1-\mu_0^2)^{1/2}, \Phi_0\right) = \left[\gamma\nu(1-\beta\mu); \frac{(\mu-\beta)}{(1-\beta\mu)}; \frac{(1-\mu^2)^{1/2}}{\gamma(1-\beta\mu)}; \Phi\right] \tag{2.1.9}$$

The inverse transformation is

$$\left(\nu, \mu, (1-\mu^2)^{1/2}\right) = \left[\gamma\nu_0(1-\beta\mu_0); \frac{(\mu_0+\beta)}{(1+\beta\mu_0)}; \frac{(1-\mu_0^2)^{1/2}}{\gamma(1+\beta\mu_0)}\right] \tag{2.1.10}$$

The latter two relations describe the Dopplershift and abberation between two systems.

Furthermore it follows that

$$d\nu_0 = \frac{\nu_0}{\nu} d\nu \tag{2.1.11}$$

$$d\mu_0 = \left(\frac{\nu}{\nu_0}\right)^2 d\mu \tag{2.1.12}$$

$$d\Phi_0 = d\Phi \tag{2.1.13}$$

and using $d\omega = \sin\theta \, d\theta \, d\phi = -d\mu \, d\phi$ the Lorentz invariant is:

$$\nu \, d\nu \, d\omega = \nu_0 \, d\nu_0 \, d\omega_0 \tag{2.1.14}$$

2.2 Transformation laws for the specific intensity, emission and absorption coefficient

For future considerations, the invariant forms of the specific intensity, the emission and absorption coefficient have to be determined.
Furthermore, invariant expressions of these quantities must be derived by counting photons because the photon number is the same in all systems (including knowledge from fundamental definitions of the specific intensity, photon distribution function, emissivity and opacity which are described in more detail in the appendix).

2.2.1 The Invariant Specific Intensity

Based on the lab frame a photon number N is counted by considering photons passing through an area element dS which is perpendicular to the z axis. Furthermore they are going into a solid angle $d\omega$ along an angle $\theta = \cos^{-1}\mu$ to the z axis in a time interval dt and frequency $d\nu$.
The photon number therefore is

$$N = \left(\frac{I(\mu,\nu)}{h\nu}\right)(d\omega \, d\nu)(dS \cos\theta \, dt) \tag{2.2.1}$$

But to count photons in the rest frame which itself is moving with a velocity v along the z axis while dS is moving in the opposite direction with negative v, an additional term has to be taken into account, namely the volume swept out by dS in a time dt_0.
In this frame the count would be

$$N_0 = \left(\frac{I_0(\mu_0,\nu_0)}{h\nu_0}\right)(d\omega_0 \, d\nu_0)(dS \cos\theta_0 \, dt_0 + \left(\frac{v}{c}\right) dS dt_0) \tag{2.2.2}$$

Taking now the following relations

$$dt_0 = \gamma dt \tag{2.2.3}$$

$$\nu = \gamma \nu_0 (1 + \beta \mu_0) \tag{2.2.4}$$

$$\nu\, d\nu\, d\omega = \nu_0\, d\nu_0\, d\omega_0 \tag{2.2.5}$$

$$\mu = \frac{\mu_0 + \beta}{1 + \beta \mu_0} \tag{2.2.6}$$

the invariant intensity can be calculated by setting $N = N_0$ (because both systems need to have the same number of photons).

$$\left(\frac{I(\mu, \nu)}{h\nu} \right) (d\omega\, d\nu)(dS \cos\theta\, dt) = N_0 = \left(\frac{I_0(\mu_0, \nu_0)}{h\nu_0} \right) (d\omega_0\, d\nu_0) \tag{2.2.7}$$

By reducing and applying the transformation of time and knowledge that $\mu = \cos\theta$

$$\frac{I(\mu, \nu)}{\nu}\, d\omega\, d\nu\, \mu\, dt = \frac{I_0(\mu_0, \nu_0)}{\nu_0}\, d\omega_0\, d\nu_0\, (\mu_0 + \beta)\, \gamma dt \tag{2.2.8}$$

and inserting the transformation law for μ

$$\frac{I(\mu, \nu)}{\nu}\, d\omega\, d\nu\, \mu dt = \frac{I_0(\mu_0, \nu_0)}{\nu_0}\, d\omega_0\, d\nu_0\, \mu\, (1 + \beta \mu_0)\, \gamma dt \tag{2.2.9}$$

it becomes immediately obvious that the result on the right-hand side is an expression for $\left(\frac{\nu}{\nu_0}\right)$ and furthermore the Lorentz invariant can be applied $\nu\, d\nu\, d\omega = \nu_0\, d\nu_0\, d\omega_0$.
The result is

$$I(\mu, \nu) = \left(\frac{\nu}{\nu_0} \right)^3 I_0(\mu_0, \nu_0) \tag{2.2.10}$$

Invariant Intensity:

$$\mathscr{I}(\mu, \nu) = \frac{I(\mu, \nu)}{\nu^3} \tag{2.2.11}$$

2.2.2 The Invariant Emissivity

An expression of the invariant emissivity is found (in analogy to the calculations of the invariant specific intensity) by taking the number of photons in both systems (Euler and Lagrange system) which are emitted from a volume element into a solid angle and frequency $d\omega\, d\nu$.
Therefore it would be counted in the lab frame

$$N = \frac{\eta(\mu, \nu)\, d\omega\, \nu\, dV dt}{h\nu} \tag{2.2.12}$$

and in the comoving frame

$$N_0 = \frac{\eta(\nu)_0 \, d\omega_0 \, \nu_0 \, dV_0 dt_0}{h\nu_0} \tag{2.2.13}$$

Because isotropy of material coefficients in the rest frame is assumed, a dependency of the emission coefficient from the frequency only is given. Again putting N and N_0 on the same level and recalling that the space time volume element dV dt is invariant under Lorentz transformation the following is obtained

$$\frac{\eta(\mu, \nu) \, d\omega \, \nu}{\nu} = \frac{\eta(\nu)_0 \, d\omega_0 \, \nu_0}{\nu_0} \tag{2.2.14}$$

Finally $\nu \, d\nu \, d\omega = \nu_0 \, d\nu_0 \, d\omega_0$ is taken and therefore ends up with the invariant emissivity:

$$\eta(\mu, \nu) = \left(\frac{\nu}{\nu_0}\right)^2 \eta_0(\nu_0) \tag{2.2.15}$$

or

$$\mathscr{E} \equiv \frac{\eta}{\nu^2} \tag{2.2.16}$$

2.2.3 The Invariant Opacity

At last the same calculations as above have to be made for the absorption coefficient in order to get an Lorentz invariant expression.
Counting the number of photons which are absorbed in a volume element in the Euler frame

$$N = \frac{\chi(\mu, \nu) I(\mu, \nu)}{h\nu} \, d\omega \, d\nu \, dV dt \tag{2.2.17}$$

and for the Lagrange system

$$N = \frac{\chi(\nu_0) I(\mu_0, \nu_0)}{h\nu_0} \, d\omega_0 \, d\nu_0 \, dV_0 dt_0 \tag{2.2.18}$$

Again setting $N = N_0$ and taking the same relation as above ends up with the invariant opacity

$$\chi(\mu, \nu) = \left(\frac{\nu_0}{\nu}\right) \chi_0(\nu_0) \tag{2.2.19}$$

or

$$\mathscr{A} \equiv \nu \chi \tag{2.2.20}$$

2.3 The Radiation Energy Tensor

So far the covariant formalism for some basic quantities has been derived but now the aim of this thesis, namely the relativistic formulation of the equations of radiation hydrodynamics has to be targeted. For this the covariant generalization of the radiation quantities (i.e. radiation energy, radiation flux and pressure) has to be found, which will be the stress-energy tensor.

To begin with, the appearance of the pressure tensor shall be reviewed, which can be written as

$$P^{ij}(\mathbf{x},t;\nu) = \frac{1}{c}\oint I(\mathbf{x},t;\mathbf{n},\nu)n^i n^j d\omega \tag{2.3.1}$$

or in dyadic notation

$$\mathbf{P}_\nu = \mathbf{P} = (\mathbf{x},t;\nu) = \frac{1}{c}\oint I(\mathbf{x},t;\mathbf{n},\nu)\mathbf{nn}d\omega \tag{2.3.2}$$

This means that this is the rate of transport per unit area which is aligned perpendicularly to the j-th coordinate axis of the i-th component of the impulse.
In one dimensional spherical geometry it would have the form

$$\mathbf{P}_\nu = \mathbf{P} = (r,t;\nu) = \frac{1}{c}\oint I(r,t;\mu,\nu)\mu^2 d\mu \tag{2.3.3}$$

To depict (2.3.2) for a one-dimensional media (recalling that the Intensity I_ν is independent of Φ)

$$\mathbf{P}_\nu = \begin{pmatrix} P_\nu & 0 & 0 \\ 0 & P_\nu & 0 \\ 0 & 0 & P_\nu \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 3P_\nu - E_\nu & 0 & 0 \\ 0 & 3P_\nu - E_\nu & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3.4}$$

is obtained where the scalar P_ν is defined as the Eddington moment (see appendix)

$$P_\nu \equiv \frac{4\pi}{c}K_\nu \tag{2.3.5}$$

In case of an isotropic radiation field (i.e. $P_\nu = \frac{1}{3}E_\nu$ and therefore $K_\nu = \frac{1}{3}J_\nu$) (2.3.4) reduces to

$$\mathbf{P}_\nu = \begin{pmatrix} P_\nu & 0 & 0 \\ 0 & P_\nu & 0 \\ 0 & 0 & P_\nu \end{pmatrix} \tag{2.3.6}$$

which is in one dimension a scalar so that when speaking of the pressure this refers to this scalar.

After recalling the pressure tensor it can be generalized to the four-vector formalism in that way that the following requirements are applied to it: the space components should be the i-th component of the radiation momentum per unit volume and unit area perpendicular to the j-th coordinate axis. In other words it is demanded that the space components are the radiation pressure and the time component is the energy. That is why the stress-energy tensor can be written as

$$R^{ij} = \int f_R M^i c n^i d^3 M \tag{2.3.7}$$

where f_R is the photon distribution function (see appendix).

For photons

$$cn^j = c^2 \left(\frac{h\nu n^j}{c}\right)\left(\frac{1}{h\nu}\right) = c^2 \frac{d^3 M}{e} \tag{2.3.8}$$

the photon momentum $\frac{h\nu}{c} n^i = p$ and the photon energy is e. So it can be rewritten to

$$R^{\alpha\beta} = c^2 \int f_R M^\alpha M^\beta \frac{d^3 M}{e} \tag{2.3.9}$$

This is a four-vector because it is an integral over the outer product of the four-vector M^α with itself times the invariant f_R and $\frac{d^3M}{e}$. A closer look at its components shows that the 00-component is nothing more than the radiation energy

$$R^{00} = \int f_R h\nu d^3 M \tag{2.3.10}$$

The space components are on the one hand the energy-flux-density times $\frac{1}{c}$ (in i-th direction)

$$R^{0i} = \frac{1}{c} \int f_R h\nu c n^i d^3 M \tag{2.3.11}$$

and on the other hand the momentum-density times c (also in i-th direction):

$$R^{i0} = \frac{1}{c} \int f_R n^i \left(\frac{h\nu}{c}\right) d^3 M \tag{2.3.12}$$

Until now only the photon distribution function was used but the intensity is somehow needed in this tensor, so the relation (6.2.10) is taken to get

$$R^{\alpha\beta} = \frac{1}{c} \int_0^\infty d\nu \oint d\omega I(\mathbf{n}, \nu) n^\alpha n^\beta \tag{2.3.13}$$

The reason for covariance is because it is the outer product of the photon four-momentum with itself times the invariant intensity integrated over all solid angles and frequencies. A more demonstrative version is

$$\mathbf{R} = \begin{pmatrix} E & \frac{1}{c}\mathbf{F} \\ \frac{1}{c}\mathbf{F} & \mathbf{P} \end{pmatrix} \tag{2.3.14}$$

where E, **F**, **P** are the known quantities: radiation energy, radiation flux and the pressure tensor.
Taking P_ν in one dimensional spherical geometry **R** it can be written as

$$R^{\alpha\beta} = \begin{pmatrix} E & \frac{1}{c}F & 0 & 0 \\ \frac{1}{c}F & P & 0 & 0 \\ 0 & 0 & \frac{1}{2}\frac{(E-P)}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\frac{1}{2}(E-P)}{r^2 sin^2\theta} \end{pmatrix} \tag{2.3.15}$$

Recalling that this is true for the lab frame the radiation stress-energy tensor $\mathbf{R}_0$ in the comoving frame is obtained if all quantities are measured in the rest frame.
Of course, both tensors, **R** and $\mathbf{R}_0$ are related by a Lorentz transformation (which was seen in chapter 1). If we now apply the Lorentz transformation on **R** and $\mathbf{R}_0$ we find transformation laws for our radiation moments

$$E_0 = \gamma^2 \left\{ E - 2\beta\frac{1}{c}F + \beta^2 P \right\} \tag{2.3.16}$$

$$F_0 = \gamma^2 \left\{ (1+\beta^2)F - v(E+P) \right\} \tag{2.3.17}$$

$$P_0 = \gamma^2 \left\{ P - 2\beta\frac{1}{c}F - \beta^2 E \right\} \tag{2.3.18}$$

and the inverse transformation is

$$E = \gamma^2 \left\{ E_0 + 2\beta F_0 + \beta^2 P_0 \right\} \tag{2.3.19}$$

$$F = \gamma^2 \left\{ (1+\beta^2)F_0 + v(E_0+P_0) \right\} \tag{2.3.20}$$

$$P = \gamma^2 \left\{ P_0 + 2\beta\frac{1}{c}F_0 + \beta^2 E_0 \right\} \tag{2.3.21}$$

2.4 The Four-Force-Density-Vector

After having derived almost all quantities and relations that were needed only one remains to be defined, namely the radiation four-force-density ([MWM84]).
This is because the dynamical equations for the radiation field can be written

$$R^{\alpha\beta}_{;\beta} = -G^\alpha \tag{2.4.1}$$

in analogue to the equations of fluid-dynamics (see appendix (6.2.27)).
Where G^α is the four-force-density of the radiation field. To be more accurate the time

component of G^α represents the radiation energy input into matter whereas the space component depicts the radiation momentum input.

Therefore the four-force-density can be written in terms of absorption and emission coefficients for a one-dimensional spherical configuration

$$cG^0 = 2\pi \int_0^\infty d\nu \int_{-1}^{1} [\chi(\mu,\nu)I(\mu,\nu) - \eta(\mu,\nu)]\, d\mu \tag{2.4.2}$$

$$cG^1 = 2\pi \int_0^\infty d\nu \int_{-1}^{1} [\chi(\mu,\nu)I(\mu,\nu) - \eta(\mu,\nu)]\, \mu\, d\mu \tag{2.4.3}$$

whereas these G's are measured in the lab frame.
To derive expressions for the comoving frame a boost has to be applied and therefore the transformation laws (forward and backward transformation) are

$$G^0 = \gamma(G_0^0 + \beta G_0^1) \tag{2.4.4}$$

$$G^1 = \gamma(G_0^1 + \beta G_0^0) \tag{2.4.5}$$

$$G_0^0 = \gamma(G^0 - \beta G^1) \tag{2.4.6}$$

$$G_0^1 = \gamma(G^1 - \beta G^0) \tag{2.4.7}$$

where the upper indices denote time and space components and the lower index "0" refers to the comoving frame.
Applying these relations and recalling that in the rest frame the material coefficients are isotropic the integration over $d\mu$ and $\mu\, d\mu$ can be carried out immediately and ends up with the following expressions

$$cG_0^0 = \int_0^\infty d\nu_0\, [c\chi_0(\nu_0)E_0(\nu_0) - 4\pi\eta_0(\nu_0)] \tag{2.4.8}$$

$$cG_0^1 = \int_0^\infty d\nu_0 \chi_0(\nu_0)F_0(\nu_0) \tag{2.4.9}$$

CHAPTER 3

Derivation of the Equations of RRHD

After recapitulating basic principles of special relativity and basic radiation theory and finding covariant expressions of these radiation quantities, tools to derive the equations of relativistic radiation hydrodynamics are given.
In this chapter not only are the equations calculated but also reshaped in such a way that they can be worked with in view of numerics. In other words, first the equations are derived in the lab frame and afterwards transformed into the comoving frame. But because of the fact that the equations should not be absolutely Euler or absolutely Lagrange (for reasons of simplicity and numerics), but a sort of mixture, some more transformations will have to be performed on them to end up with the desired form. (Further readings: [MWM84], [JM99], [Pom73], [Cas04], [LL66])

3.1 The General Form of the Equations of RRHD

As seen in the last chapter the covariant form of the dynamical equations of the radiation field can be written as

$$R^{\alpha\beta}_{;\beta} = -G^{\alpha} \tag{3.1.1}$$

It is obvious that because of the covariant formalism (3.1.1) is frame independent and therefore the covariant derivative can be evaluated directly in an inertial system. However, for non-inertial systems a space-time-metric has to be found first and the Christoffel Symbols calculated which are needed for the covariant derivation.

By breaking the above equation down to one-dimension, taking the spherical case and inserting **R** (2.3.15), the classical (non-relativistic) equations for the radiation field (inertial) are obtained, i.e. the radiation energy and the radiation flux

$$\frac{\partial E}{\partial t} + \frac{1}{r^2}\left[\frac{\partial r^2 F}{\partial r}\right] = -cG^0 \tag{3.1.2}$$

$$\frac{1}{c^2}\frac{\partial F}{\partial t} + \left(\frac{\partial P}{\partial r}\right)\frac{(3P-E)}{r} = -G^1 \tag{3.1.3}$$

There are two possible and equivalent physical pictures to obtain the equations for material interacting with the radiation. One is to consider a radiation field which performs an additional four force on the material

$$M^{\alpha\beta}_{;\beta} = F^\alpha + G^\alpha \tag{3.1.4}$$

with $M^{\alpha\beta}_{;\beta}$ is the material stress-energy tensor and the other is to take an external four-force which acts on a radiating fluid

$$\left(M^{\alpha\beta} + R^{\alpha\beta}\right)_{;\beta} = F^\alpha \tag{3.1.5}$$

The latter can also be written as

$$S^{\alpha\beta} = M^{\alpha\beta} + R^{\alpha\beta} \tag{3.1.6}$$

where $S^{\alpha\beta}$ is the total stress-energy tensor of the radiating fluid.
Although both versions are physically equivalent, the first proves to be better in the streaming limit whereas the second one is more satisfying in the diffusion regime.

3.2 Calculation of the Four-Force-Density Vector

Before starting to calculate the equations, the relativistically correct expression of the four-force-density G^α has to be found because the definition given above (in section 2.4) is not sufficient.
Obviously the material coefficients have to be expanded to second order which brings a positive side effect of simple handling of the lab frame angle-frequency dependence. Based on the invariant emissivity (2.2.15), the invariant opacity (2.2.19) and the transformation laws of the frequency (2.1.4) and (2.1.3) both material coefficients are expanded to $O(\frac{v^2}{c^2})$ and gain the following expressions

$$\chi(\mu,\nu) = \chi_0(\nu) - \left(\frac{\mu v}{c}\right)\left[\chi_0(\nu) + \nu\frac{\partial\chi_0}{\partial\nu}\right] + \frac{1}{2}\left(\frac{v}{c}\right)^2\left[\chi_0(\nu) + \nu\frac{\partial\chi_0}{\partial\nu} + 2\mu^2\nu\frac{\partial\chi_0}{\partial\nu} + \mu^2\nu^2\frac{\partial^2\chi_0}{\partial\nu^2}\right] \tag{3.2.1}$$

$$\eta(\mu,\nu) = \eta_0(\nu) + \left(\frac{\mu v}{c}\right)\left[2\eta_0(\nu) - \nu\frac{\partial\eta_0}{\partial\nu}\right] + \left(\frac{v}{c}\right)^2\left[-\eta_0(\nu) + 3\mu^2\eta_0(\nu) + \frac{1}{2}\nu\frac{\partial\eta_0}{\partial\nu} - 2\mu^2\nu\frac{\partial\eta_0}{\partial\nu} + \frac{1}{2}\mu^2\nu^2\frac{\partial^2\eta_0}{\partial\nu^2}\right] \quad (3.2.2)$$

Apparently the material coefficients are all evaluated in the comoving system while the intensity is still a lab frame quantity. This is advantageous because all the material coefficients are isotropic in the comoving frame. Hence, the following integration over these quantities to get radiation moments is simplified.
The next step is to take these expanded quantities and insert them into the transport equation (one-dimension and spherical symmetric) and unite by order

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mu,\nu)}{\partial t} + \mu\frac{\partial I(\mu,\nu)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I(\mu,\nu)}{\partial\mu} &= \eta_0(\nu) - \chi_0(\nu)I(\mu,\nu)\\ &+ \left(\frac{\mu v}{c}\right)\left\{2\eta_0(\nu) - \nu\frac{\partial\eta_0}{\partial\nu} + \left[\chi_0(\nu) + \nu\frac{\partial\chi_0}{\partial\nu}\right]I(\mu,\nu)\right\}\\ &+ \left(\frac{v}{c}\right)^2\left\{-\eta_0(\nu) + 3\mu^2\eta_0(\nu) + \frac{1}{2}\nu\frac{\partial\eta_0}{\partial\nu} - 2\mu^2\nu\frac{\partial\eta_0}{\partial\nu} + \frac{1}{2}\mu^2\nu^2\frac{\partial^2\eta_0}{\partial\nu^2}\right.\\ &\left.- \frac{1}{2}\left[\chi_0(\nu) + \nu\frac{\partial\chi_0}{\partial\nu} + 2\mu^2\nu\frac{\partial\chi_0}{\partial\nu} + \mu^2\nu^2\frac{\partial^2\chi_0}{\partial\nu^2}\right]I(\mu,\nu)\right\} \quad (3.2.3)\end{aligned}$$

where it is clearly seen that the first line is the zeroth-order and therefore this would be the ordinary non-relativistic transfer equation. The second and third lines are the first and second order of the expansion.

Integration over μ and $\mu\, d\mu$ results in the zeroth and first moment of the transfer equation

$$\frac{\partial E}{\partial t} + \frac{1}{r^2}\frac{\partial r^2 F}{\partial t} = \chi_0\,(4\pi\eta_0 - cE) + \left(\frac{v}{c}\right)\chi_0 F - \frac{c}{2}\left(\frac{v}{c}\right)^2\chi_0 E \quad (3.2.4)$$

$$\frac{1}{c^2}\frac{\partial F}{\partial t} + \frac{\partial P}{\partial r} + \frac{3P - E}{r} = -\frac{1}{c}\chi_0\left[F - 4\pi\eta_0\left(\frac{v}{c}\right)\right] + \left(\frac{v}{c}\right)\chi_0 P - \left(\frac{v}{c}\right)^2\frac{1}{2c}\chi_0 F \quad (3.2.5)$$

and therefore

$$-cG^0 = \int_0^\infty\left\{\chi_0\,(4\pi\eta_0 - cE) + \left(\frac{v}{c}\right)\chi_0 F - \frac{c}{2}\left(\frac{v}{c}\right)^2\chi_0 E\right\}d\nu \quad (3.2.6)$$

$$-G^1 = -\int_0^\infty\left\{\frac{1}{c}\left[\chi_0 F - 4\pi\eta_0\left(\frac{v}{c}\right)\right] + \left(\frac{v}{c}\right)\chi_0 P - \left(\frac{v}{c}\right)^2\frac{1}{2c}\chi_0 F\right\}d\nu \quad (3.2.7)$$

Actually, this mixed-frame version of the four-force-density is the wanted one; the radiation moments are in the lab frame and the material coefficients are in the comoving frame. This is the desired version from the perspective of numerics which will be discussed in the next chapter. Further it must be mentioned that with respect to the adaptive grid and because of the fact that relativistic effects are seen when switching systems, this kind of mixed-frame equations is the needed and desired one. Otherwise if all quantities were calculated in the Euler system they would have to be transformed into Lagrange after computation.

3.3 The Equation of Continuity

The derivation of the system of equations starts with the equation of continuity because it is needed for some transformation for the other equations.
Consideration is also given to the classical continuity equation which is of the form

$$\rho_{,t} + \left(\rho v^i\right)_{,i} = 0 \tag{3.3.1}$$

For a covariant version of this equation the density has to be rewritten in some way ([MWM84]).
That is why N_0 particles per volume with a mass m_0 are regarded in the comoving frame. So it can be written

$$\rho_0 = N_0 m_0 \tag{3.3.2}$$

Switching to the lab frame another relation is given. Choosing a volume element δV_0 a particle number density of $N_0 \delta V_0$ is obtained. Counting the particles in both systems leads to the same number, of course:

$$N\delta V = N_0 \delta V_0 \tag{3.3.3}$$

Because of the fact that there is the Lorentz contraction left to apply, which means $\delta V = \frac{\delta V_0}{\gamma}$ and therefore $N = \gamma N_0$, the density in the lab frame can be written as

$$\rho = N m_0 = \gamma \rho_0 \tag{3.3.4}$$

and the density (with a relative mass) measured in the lab frame

$$\rho' = Nm = \gamma^2 N_0 m_0 = \gamma\rho = \gamma^2 \rho_0 \tag{3.3.5}$$

This implies

$$\rho = \gamma \rho_0 \tag{3.3.6}$$

(with the momentum density: $\rho' \mathbf{v}$)

Inserting (3.3.6) into (3.3.1) obtains the relativistic equation of continuity

$$(\gamma\rho_0)_{,t} + \left(\gamma \rho_0 v^i\right)_{,i} = 0 \tag{3.3.7}$$

which in return provides the classical equation for small velocities, i.e. $\gamma \to 1$. Alternatively

$$(\rho_0 V^\alpha)_{,\alpha} = 0 \tag{3.3.8}$$

whereas for curvilinear coordinates the ordinary derivation has to be exchanged with the covariant derivation

$$(\rho_0 V^\alpha)_{;\alpha} = 0 \tag{3.3.9}$$

3.3.1 The Conservative Form of the Continuity Equation

After deriving the relativistic equation of continuity the conservative form and the fluid derivation of it need to be calculated for later considerations. The first step is to check wether the equation is conservative, even if it looks right. Therefore taking (3.3.8), splitting it up with the product rule and recalling that $x^\alpha = (ct, x^i)$, $V^\alpha = \gamma(c, v^i)$

$$\begin{aligned}(\rho_0 V^\alpha)_{,\alpha} &= V^\alpha \frac{\partial \rho}{\partial x^\alpha} + \rho \frac{\partial V^\alpha}{\partial x^\alpha} \\ &= \gamma \frac{\partial \rho}{\partial t} + \gamma \frac{\partial \rho}{\partial x^i} + \rho \frac{\partial \gamma}{\partial t} + \rho \frac{\partial \gamma v}{\partial x^i} \\ &= \frac{\partial}{\partial t}(\gamma \rho) + \frac{\partial}{\partial x^i}(\gamma \rho v)\end{aligned} \tag{3.3.10}$$

it becomes obvious that this is really the needed conservative form.
Next, the equation of continuity is calculated in the Lagrange system, which means that the Lagrangian derivative has to be applied with the proper time τ. Analogously to the above calculation the Lagrangian derivative is split up in separate terms with the product rule

$$\begin{aligned}\frac{D}{D\tau}(\gamma\rho) &= \gamma \frac{D\rho}{D\tau} + \rho \frac{D\gamma}{D\tau} \\ &= \gamma\gamma \frac{\partial \rho}{\partial t} + \gamma\gamma v \nabla \rho + \gamma \rho \frac{\partial \gamma}{\partial t} + \gamma v \rho \nabla \gamma \\ &= \gamma \frac{\partial \gamma \rho}{\partial t} + \gamma v \nabla (\gamma \rho)\end{aligned} \tag{3.3.11}$$

Splitting up (3.3.7) and grouping together again leads to

$$\frac{\partial}{\partial t}(\gamma \rho_0) + \nabla(\gamma \rho_0 v) = \frac{\partial}{\partial t}(\gamma\rho) + v\nabla(\gamma\rho) + \gamma\rho\nabla v \tag{3.3.12}$$

Applying this on the previous equation, the equation of continuity in fluid derivative formalism is obtained

$$\frac{D}{D\tau}(\gamma\rho) + \gamma^2 \rho \nabla v = 0 \tag{3.3.13}$$

To check the latter equation the Lagrangian derivative has to be split up again and reduced by gamma for ending up with the original expression.

3.4 The Momentum Equation

To obtain the equation of motion the calculations of Mihalas ([MWM84], p. 427) are followed to end up with an equation of the form

$$\rho_* \left(\frac{D\mathbf{v}}{D\tau}\right) = \mathbf{f} - \nabla p - c^2 \mathbf{v}\,(p_{,t} + \mathbf{v}\cdot\mathbf{f}) + \mathbf{G} - \frac{\mathbf{v}}{c} G^0 \tag{3.4.1}$$

This is a mixed-frame equation which is also called quasi-Lagrangian because the time and space derivatives have been grouped together to a Lagrangian (Fluid) derivative, whereas the material coefficients are in the comoving frame and the intensity (including its moments) is measured in the lab frame.
Hence seeking a Lagrangian formalism in one dimension (3.4.1) is reduced to one dimension and set $v = 0$ (for reasons of rest frame where the fluid velocity is zero).

$$\rho_* \left(\frac{Dv}{D\tau}\right) = f - \frac{\partial p}{\partial r} + G^1 \tag{3.4.2}$$

where

$$\rho_* = \gamma \left[\rho_0 \left(1 + \frac{e}{c^2}\right) + \frac{p}{c^2}\right] \tag{3.4.3}$$

which contains the specific enthalpy (second and third term).
It is sufficient to take only the first term, i.e. $\rho_* = \gamma\rho_0$. The other two terms can be neglected for the sake of general relativistic effects. Their contribution would be of such a kind that at high velocities the energy would couple on the gravitational field itself. In other words, at high velocities the energy would increase and in turn the mass would increase and this would lead to an increase of the gravitational potential.

To sum up, our equation of motion is

$$\gamma\rho_0 \left(\frac{Dv}{D\tau}\right) = f - \frac{\partial p}{\partial r} + G^1 \tag{3.4.4}$$

with G^1 defined by (3.2.7).

3.4.1 The Conservative Form of the Equation of Motion

The version of the momentum equation is not that satisfying because a conservative form is needed which means that the derivation on the left hand side should be transformed in a way

$$\gamma\rho_0 \frac{Dv}{D\tau} \rightarrow \frac{D}{D\tau}\,(\gamma\rho_0 v) \tag{3.4.5}$$

In order to achieve this the same procedure is applied as with the equation of continuity. Starting with the right hand side, it is split up with the product rule using the equation of continuity in a way that (3.3.13) is

$$\rho\frac{D\gamma}{D\tau}+\gamma\frac{D\rho}{D\tau}=-\gamma^2\rho\nabla v \tag{3.4.6}$$

$$\frac{D}{D\tau}(\gamma\rho v)=\underbrace{\rho v\frac{D\gamma}{D\tau}+\gamma v\frac{D\rho}{D\tau}}_{v\left\{\rho\frac{D\gamma}{D\tau}+\gamma\frac{D\rho}{D\tau}\right\}=v\left(-\gamma^2\rho\nabla\cdot v\right)}+\gamma\rho\frac{Dv}{D\tau} \tag{3.4.7}$$

Shifting terms from one side to the other yields

$$\frac{D}{D\tau}(\gamma\rho v)+\gamma^2 v\rho\nabla v=\gamma\rho\frac{Dv}{D\tau} \tag{3.4.8}$$

This is almost the wanted version except for the gradient which has to be reshaped one more time by writing out the Lagrangian derivative to get $\gamma\nabla(\gamma\rho vv)$

$$\gamma\frac{d}{dt}(\gamma\rho v)+\gamma v\nabla(\gamma\rho v)+v\gamma\gamma\rho\nabla\cdot v=\gamma\rho\frac{Dv}{D\tau} \tag{3.4.9}$$

Applying the product rule

$$\gamma\nabla(\gamma\rho vv)=\gamma v\nabla(\gamma\rho v)+\gamma\gamma\rho v\nabla v \tag{3.4.10}$$

and again shifting terms and reducing them ends up with the desired conservative form

$$\gamma\rho\frac{Dv}{D\tau}=\gamma\frac{d}{dt}(\gamma\rho v)+\gamma\nabla(\gamma\rho vv) \tag{3.4.11}$$

To finish it up this is inserted into (3.4.4) with (3.2.7) and for $f=-\frac{GM\rho}{r^2}$

$$\begin{aligned}\gamma\frac{d}{dt}(\gamma\rho v)+\gamma\nabla(\gamma\rho vv)=&-\frac{GM\rho}{r^2}-\frac{\partial p}{\partial r}\\&+\frac{1}{c}\left[\chi_0 F-4\pi\eta_0\left(\frac{v}{c}\right)\right]+\left(\frac{v}{c}\right)\chi_0 P-\left(\frac{v}{c}\right)^2\frac{1}{2c}\chi_0 F\end{aligned} \tag{3.4.12}$$

Converting the radiation moments into Eddington moments (see appendix 6.2.4) and using $\chi_0\to\kappa_R\rho$ + viscous force and $4\pi\eta_0\to\chi_0 S$ yields the desired form of the relativistic conservative momentum equation

$$\begin{aligned}&\gamma\frac{\partial}{\partial t}(\gamma\rho v)+\gamma\nabla\cdot(\gamma\rho vv)+\rho\nabla\cdot\phi+\nabla P\\&\quad-\frac{4\pi}{c}\kappa_R\rho\left[H-\left(\frac{v}{c}\right)S\right]+\frac{4\pi}{c}\left(\frac{v}{c}\right)\kappa_R\rho K-\left(\frac{v}{c}\right)^2\frac{1}{2c}\kappa_R\rho 4\pi H+\nabla\cdot Q=0\end{aligned} \tag{3.4.13}$$

Checking the equation by taking the limes of small velocities, i.e. $\gamma \to 1$, and dropping terms of first and second order provides the non-relativistic (classical) equation

$$\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v v) + \nabla P + \rho \nabla \phi - \frac{4\pi}{c} \kappa_R \rho H + \nabla \cdot Q = 0 \tag{3.4.14}$$

3.5 The Equation of Internal Energy

As with the momentum equation the calculations of Mihalas ([MWM84], p. 430) are followed to get the equation of energy in order to start further calculations to get a conservative form

$$\rho \left\{ \frac{De}{D\tau} + p \frac{D}{D\tau} \left(\frac{1}{\rho} \right) \right\} = c \left(F_0^0 + G_0^0 \right) \tag{3.5.1}$$

where F_0^0 is an additional external force, such as contributions like nuclear energy, but vanishes for ordinary body forces

$$cF_0^0 = (v \cdot f)_0 = 0 \tag{3.5.2}$$

It is seen clearly that the left-hand side of the (3.5.1) is put in conservative form. This goes in analogy with the momentum equation in a sense that the left-hand side is changed with the continuity equation. The first term is desired to transform like

$$\rho \frac{De}{D\tau} \to \frac{D}{D\tau}(\gamma \rho e) \tag{3.5.3}$$

This is achieved by rewriting the equation of continuity

$$\frac{D\rho}{D\tau} = -\frac{\rho}{\gamma} \frac{D}{D\tau}(\gamma) - \frac{\gamma^2}{\gamma} \rho \nabla v \tag{3.5.4}$$

splitting

$$\frac{D}{D\tau}(\rho e) = e \frac{D\rho}{D\tau} + \rho \frac{De}{D\tau} \tag{3.5.5}$$

and inserting the newly written equation of continuity. Again by writing the Lagrangian derivative out in full form and grouping terms and space derivatives with product rule a conservative form for the first term of (3.5.1) is obtained

$$\rho \frac{De}{D\tau} = \frac{\partial}{\partial t}(\gamma \rho e) + \nabla(\gamma \rho e v) \tag{3.5.6}$$

Next the second term of (3.5.1) has to be transformed that

$$\rho p \frac{D}{D\tau} \left(\frac{1}{\rho} \right) \to \frac{D}{D\tau}(\gamma p) \tag{3.5.7}$$

First the Lagrangian derivative is expressed into its full form

$$\rho p \frac{D}{D\tau}\left(\frac{1}{\rho}\right) = \rho p \gamma \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + \rho p \gamma v \nabla \left(\frac{1}{\rho}\right) \tag{3.5.8}$$

Then this ansatz is made:

$$\nabla\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\nabla\rho \tag{3.5.9}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial t} \tag{3.5.10}$$

and inserted, and after reducing

$$\rho p \frac{D}{D\tau}\left(\frac{1}{\rho}\right) = -\frac{p}{\rho}\left\{\gamma\frac{\partial\rho}{\partial t} + \gamma v \nabla \rho\right\} = -\frac{p}{\rho}\frac{D\rho}{D\tau} \tag{3.5.11}$$

is obtained, where another form of the continuity equation $\frac{D\rho}{D\tau} = -\frac{\rho}{\gamma}\frac{D\gamma}{D\tau} - \frac{\gamma^2}{\gamma}\rho\nabla v$ is used to get

$$-\frac{p}{\rho}\frac{D\rho}{D\tau} = \frac{p}{\gamma}\frac{D\gamma}{D\tau} + p\gamma\nabla v \tag{3.5.12}$$

Finally rewriting

$$\frac{p}{\gamma}\frac{D\gamma}{D\tau} \rightarrow \frac{D}{D\tau}(\gamma p) = p\frac{D\gamma}{D\tau} + \gamma\frac{Dp}{D\tau} \tag{3.5.13}$$

inserting and grouping terms together leads to

$$\frac{p}{\gamma}\frac{D\gamma}{D\tau} + p\gamma\nabla v = \frac{\partial}{\partial t}(\gamma p) + \nabla(\gamma p v) - \gamma\frac{\partial p}{\partial t} - \gamma\nabla(pv) + \gamma p \nabla(v) \tag{3.5.14}$$

Summing up, the left hand side of (3.5.1) can be written in conservative form as

$$\rho_0\left\{\frac{De}{D\tau} + p\frac{D}{D\tau}\left(\frac{1}{\rho_0}\right)\right\} = \frac{\partial}{\partial t}\Big\{\gamma(\rho_0 e + p)\Big\} + \nabla\cdot\Big\{\gamma v(\rho_0 e + p)\Big\} \tag{3.5.15}$$

Collecting all parts together with (3.2.6) obtains

$$\frac{\partial}{\partial t}\Big\{\gamma(\rho_0 e + p)\Big\} + \nabla\cdot\Big\{\gamma v(\rho_0 e + p)\Big\} - \gamma\frac{\partial p}{\partial t} - \gamma\nabla(pv) + \gamma p\nabla(v)$$
$$+ \chi_0\left(4\pi\eta_0 - cE\right) + \left(\frac{v}{c}\right)\chi_0 F - \frac{c}{2}\left(\frac{v}{c}\right)^2\chi_0 E = 0 \tag{3.5.16}$$

In analogy to the momentum equation using $\chi_0 \rightarrow \kappa_R\rho+$ viscous energy and $4\pi\eta_0 \rightarrow \chi_0 S$ and the Eddington moments the relativistic conservative energy equation in second order is obtained

$$\frac{\partial}{\partial t}\Big\{\gamma(\rho e + p)\Big\} + \nabla \cdot \Big\{\gamma v(\rho e + p)\Big\} - \gamma\frac{\partial P}{\partial t} - \gamma\nabla\cdot(Pv) + \gamma P\nabla\cdot(v) - 4\pi\kappa_P\rho(J - S)$$
$$+ \left(\frac{v}{c}\right)4\pi\kappa_P\rho H - \left(\frac{v}{c}\right)^2 2\pi\kappa_P\rho J + Q : \nabla v = 0 \qquad (3.5.17)$$

It is noteworthy that in the relativistic version new terms are gained. First, it is not only the expected $\gamma\rho e$ constellation that appears in the time and space derivatives, but also the pressure. Another remarkable thing is that the third and fourth terms seem to be a kind of continuity equation of the gas pressure.
At last, the equation in the limes of small velocities is checked to see that the additional pressure term decreases and the classical (non-relativistic) energy equation remains

$$\frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho e v) + P\nabla\cdot v - 4\pi\kappa_P\rho(J - S) + Q : \nabla u = 0 \qquad (3.5.18)$$

3.6 The Radiation Transfer Equation and its Moments

In this section the equation of transfer and its moments, i.e. radiation energy and momentum equation, are derived geometrically in the comoving frame. This is accomplished by starting out from the equation of transfer in the lab frame and transforming it into the rest frame. Afterwards integrating it over μ and $\mu\, d\mu$ will lead to the moment equations.

3.6.1 The Radiation Transfer Equation

As before Mihalas' ([MWM84]) calculations are followed to derive the equation of transfer in the comoving frame.
Because of the fact that mixed-frame equations are wanted the intensity will be transformed back into the Euler frame. Then the radiation terms on the right-hand side are substituted with the previously derived definition of the four-force density.
Based on (3.2.3) the invariant intensity (2.2.10), the invariant emissivity(2.2.15) and the invariant opacity (2.2.19) are applied to obtain

$$\left(\frac{\nu}{\nu}\right)\left[\frac{1}{c}\frac{\partial I_0(\mu_0,\nu_0)}{\partial t} + \mu\frac{\partial I_0(\mu_0,\nu_0)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_0(\mu_0,\nu_0)}{\partial \mu}\right]$$
$$-3\left(\frac{\nu}{\nu_0}\right)\left[\frac{1}{c}\frac{\partial \nu_0}{\partial t} + \mu\frac{\partial \nu_0}{\partial r} + \frac{(1-r^2)}{r}\frac{\partial \nu_0}{\partial \mu}\right] I_0(\mu_0,\nu_0) = \eta_0(\nu_0) - \chi_0(\nu_0) I(\mu_0,\nu_0) \qquad (3.6.1)$$

(2.1.10) is used to transform the derivations of $I_0(\mu_0,\nu_0)$ (because the fluid velocity is not constant, μ_0,ν_0 are not constant either and therefore their variations must be taken into accounted)

$$\frac{\partial}{\partial t}\Big|_{r\mu\nu} = \frac{\partial}{\partial t}\Big|_{r\mu_0\nu_0} + \frac{\partial \mu_0}{\partial t}\Big|_{r\mu\nu}\frac{\partial}{\partial \mu_0} + \frac{\partial \nu_0}{\partial t}\Big|_{r\mu\nu}\frac{\partial}{\partial \nu_0} \tag{3.6.2}$$

$$\frac{\partial}{\partial r}\Big|_{t\mu\nu} = \frac{\partial}{\partial r}\Big|_{t\mu_0\nu_0} + \frac{\partial \mu_0}{\partial r}\Big|_{t\mu\nu}\frac{\partial}{\partial \mu_0} + \frac{\partial \nu_0}{\partial r}\Big|_{t\mu\nu}\frac{\partial}{\partial \nu_0} \tag{3.6.3}$$

$$\frac{\partial}{\partial \mu}\Big|_{rt\nu} = \frac{\partial \mu_0}{\partial \mu}\Big|_{rt\nu}\frac{\partial}{\partial \mu_0} + \frac{\partial \nu_0}{\partial t}\Big|_{rt\nu}\frac{\partial}{\partial \nu_0} \tag{3.6.4}$$

Now (2.1.10) is used to get (further calculations see appendix)

$$\frac{\partial \mu_0}{\partial t} = -\gamma^2(1-\mu_0^2)\left(\frac{\partial \beta}{\partial t}\right) \tag{3.6.5}$$

$$\frac{\partial \nu_0}{\partial t} = -\gamma^2\mu_0\nu_0\left(\frac{\partial \beta}{\partial t}\right) \tag{3.6.6}$$

$$\frac{\partial \mu_0}{\partial r} = -\gamma^2(1-\mu_0^2)\left(\frac{\partial \beta}{\partial r}\right) \tag{3.6.7}$$

$$\frac{\partial \nu_0}{\partial r} = -\gamma^2\mu_0\nu_0\left(\frac{\partial \beta}{\partial r}\right) \tag{3.6.8}$$

$$\frac{\partial \mu_0}{\partial \mu} = \gamma^2(1+\beta\mu_0)^2 \tag{3.6.9}$$

$$\frac{\partial \nu_0}{\partial \mu} = -\beta\gamma^2(1+\beta\mu_0)\nu_0 \tag{3.6.10}$$

Substituting all derivatives into the transfer equation, reducing and rearranging a lot of terms, we arrive at an equation of the form

$$\begin{aligned}
&\frac{\gamma}{c}(1+\beta\mu_0)\frac{\partial I_0(\mu_0,\nu_0)}{\partial t} + \gamma(\mu_0+\beta)\frac{\partial I_0(\mu_0,\nu_0)}{\partial r} + \\
&\quad \frac{\partial}{\partial \mu_0}\left\{\gamma(1-\mu_0^2)\left[\frac{(1+\beta\mu_0)}{r} - \gamma^2(\mu_0+\beta)\frac{\partial \beta}{\partial r} - \frac{\gamma^2}{c}(1+\beta\mu_0)\frac{\partial \beta}{\partial t}\right] I_0(\mu_0,\nu_0)\right\} \\
&\quad - \frac{\partial}{\partial \nu_0}\left\{\gamma\nu_0\left[\frac{\beta(1-\mu_0^2)}{r} + \gamma^2\mu_0(\mu_0+\beta)\frac{\partial \beta}{\partial r} + \frac{\gamma^2}{c}\mu_0(1+\beta\mu_0)\frac{\partial \beta}{\partial t}\right] I_0(\mu_0,\nu_0)\right\} \\
&\quad + \gamma\left\{\frac{2\mu_0+\beta(3-\mu_0^2)}{r} + \gamma^2(1+\mu_0^2+2\beta\mu_0)\frac{\partial \beta}{\partial r} + \frac{\gamma^2}{c}\left[2\mu_0+\beta(1+\mu_0^2)\frac{\partial \beta}{\partial t}\right]\right\} \\
&\qquad = \eta_0(\nu_0) - \chi_0(\nu_0)I_0(\mu_0,\nu_0)
\end{aligned} \tag{3.6.11}$$

3.6.2 The Radiation Energy Equation

The radiation energy equation is obtained by integrating over μ which is already done by Mihalas ([MWM84], p.436). Based on this comoving frame radiation energy equation

$$\gamma\left(\frac{\partial E_0}{\partial t} + \frac{v}{c^2}\frac{\partial F_0}{\partial t}\right) + \gamma\left(\frac{\partial F_0}{\partial r} + v\frac{\partial E_0}{\partial r}\right) + \left[\frac{1}{r}\left(2F_0 + 3vE_0 - vP_0\right)\right.$$
$$\left. + \gamma^2\frac{\partial v}{\partial r}\left(E_0 + P_0 + \frac{2v}{c}F_0\right) + \frac{\gamma^2}{c^2}\frac{v}{t}\left(2F_0 + vE_0 + vP_0\right)\right]$$
$$= \int_0^\infty \left[4\pi\eta_0(\nu_0) - c\chi_0(\nu_0)E_0(\nu_0)\right] d\nu_0 \tag{3.6.12}$$

the wanted conservative form of it will be derived. This equation is in view of mixed-frame structure, i.e. the radiation moments are transformed from Lagrange to Euler with the transformation rules (2.3.16) and the right hand side is replaced with the previously derived four-force-density (3.2.6) to get

$$\gamma\frac{\partial}{\partial t}(\gamma^2 J) - \gamma\nabla(\gamma^2 Jv) + \gamma\frac{\partial}{\partial t}\left\{\gamma^2\left[2\frac{v}{c}H + \left(\frac{v}{c}\right)^2 K\right]\right\} + \left(\frac{v}{c^2}\right)\gamma\frac{\partial}{\partial t}\left\{\gamma^2\left[cH\left(1 + \frac{v^2}{c^2}\right) - v(J+K)\right]\right\}$$
$$+ \gamma\nabla\left\{\gamma^2\left[ch\left(1 + \frac{v^2}{c^2}\right) - vK\right]\right\} + \gamma v\nabla\left\{\gamma^2\left[J - 2\frac{v}{c}H + \frac{v^2}{c^2}K\right]\right\} + \frac{\gamma^3}{r}\left\{2cH\left(1 - \frac{v^2}{c^2}\right)\right.$$
$$\left. - v(J + 3K) + \frac{v^3}{c^2}(J - 3K)\right\} + \gamma^5\nabla(v)\left\{J + K - \frac{v^2}{c^2}(3J + K) + 2vH\left(\frac{v^2}{c^3} - \frac{1}{c}\right)\right\}$$
$$+ \frac{\gamma^5}{c^2}\frac{\partial v}{\partial t}\left\{cH\left(2 - \frac{v^2}{c^2}\right) - v(J + K) + \frac{v^3}{c^2}(K - J)\right\}$$
$$+ c\kappa_R\rho(J - S) - v\kappa_R\rho H + \frac{1}{2}\frac{v^2}{c}\kappa_R\rho J = 0 \tag{3.6.13}$$

Finally the equation is rewritten into conservative form, which implies that the space and time derivatives are transformed with the help of the product rule or by adding and subtracting terms from the equation which are not there but needed. In addition, terms of higher order are dropped, i.e. $O\left(\frac{v^3}{c^3}\right)$, and the rest is collected to obtain the wanted mixed-frame radiation energy equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\gamma^3 J) - \gamma\nabla(\gamma^3 J v) - \left[\frac{\partial}{\partial t}(\gamma) + \nabla\gamma v\right]\left[-\gamma^2\left(J + 2\frac{v}{c}H + \frac{v^2}{c^2}K\right)\right] + \left[\frac{\partial}{\partial t}(\gamma v) + \nabla(\gamma v^2)\right] \\
&\left\{\gamma^2\left[\frac{v}{c^2}(J+K) - \frac{1}{c}H\right]\right\} + \frac{1}{c^2}\left\{\left[\frac{\partial}{\partial t}(\gamma^5 v^2 J) + \nabla(\gamma^5 v^3 J)\right] - v\left[\frac{\partial}{\partial t}(\gamma^5 v J) + \nabla(\gamma^5 v^2 J)\right]\right. \\
&+ 3c\left[\frac{\partial}{\partial t}(\gamma^3 v H) + \nabla(\gamma^3 v^2 H)\right] + 2c\left[\frac{\partial}{\partial t}(\gamma^5 v H) + \nabla(\gamma^5 v^2 H)\right] - 2vc\left[\frac{\partial}{\partial t}(\gamma^5 H) + \nabla(\gamma^5 v H)\right] \\
&\left. - \left[\frac{\partial}{\partial t}(\gamma^5 v^2 K) + \nabla(\gamma^5 v^3 K)\right] - c\left[\frac{\partial}{\partial t}(\gamma^5 v K) + \nabla(\gamma^5 v^2 K)\right]\right\} + \frac{4v}{c}\nabla(\gamma^5 v^2 J) + \gamma^5 J\nabla(v) \\
&- \frac{1}{c}\left[4\nabla(\gamma^3 v^2 H) - 2\gamma^2 v^2 H\nabla(\gamma) - \gamma^3 H\nabla(v^2) + 4\gamma^5 v H\nabla(v)\right] + \gamma^5 K\nabla(v) - \frac{\gamma^3}{r}v(J+3K) \\
&+ \left(1 - \frac{v^2}{c^2}\right)\left[\frac{\gamma^3}{r}2cH - \gamma\left(\nabla(\gamma^2 v K) + \nabla(\gamma^2 v J)\right)\right] + c\kappa_R\rho(J-S) - v\kappa_R\rho H + \frac{v^2}{c}\frac{1}{2}\kappa_R\rho J = 0
\end{aligned}
\tag{3.6.14}
$$

It is seen that in this version high orders of γ arise and also some kinds of continuity equation for the Lorentz factor (zeroth and first order). The reason for not dropping or reducing such high orders is that it cannot be predicted in which way those terms act in different frames. In other words, small terms in one frame can become huge in the other frame. Taking the limes for small velocities this equation reduces itself to the classical form

$$
\frac{\partial}{\partial t}J + \nabla\cdot(vJ) + c\nabla\cdot H + K : \nabla v + c\kappa_P\rho(J-S) = 0 \tag{3.6.15}
$$

3.6.3 The Radiation Momentum Equation

In analogy to the radiation energy equation, the comoving frame radiation momentum equation of Mihalas ([MWM84], p.436) is taken

$$
\begin{aligned}
&\frac{\gamma}{c^2}\left(\frac{\partial F_0}{\partial t} + v\frac{\partial P_0}{\partial t}\right) + \gamma\left(\frac{\partial P_0}{\partial r} + \frac{v}{c^2}\frac{\partial F_0}{\partial r}\right) + \gamma\left[\frac{1}{r}\left(3P_0 + E_0 - \frac{2v}{c^2}F_0\right)\right. \\
&\qquad \left. + \frac{\gamma^2}{c^2}\frac{\partial v}{\partial r}(2F_0 + vE_0 + vP_0) + \frac{\gamma^2}{c^2}\frac{\partial v}{\partial t}\left(E_0 + P_0 + \frac{2v}{c^2}F_0\right)\right] \\
&\qquad = -\frac{1}{c}\int_0^\infty \chi_0(\nu 0)F_0(\nu_0)d\nu_0
\end{aligned}
\tag{3.6.16}
$$

and the same calculations are performed, i.e. transforming the comoving frame radiation moments to get a mixed-frame equation and replacing the right hand side with the four-force-density (3.2.7):

$$
\begin{aligned}
&\frac{\gamma}{c^2}\frac{\partial}{\partial t}\left\{\gamma^2\left[\left(1+\frac{v^2}{c^2}\right)-v(J+K)\right]\right\}+\gamma\frac{v}{c^2}\frac{\partial}{\partial t}\left\{\gamma^2\left[K-2\frac{v}{c}H-\frac{v^2}{c^2}J\right]\right\}+\gamma\frac{\partial}{\partial r}\left\{\gamma^2\left[K-2\frac{v}{c}H-\frac{v^2}{c^2}J\right]\right\}\\
&+\gamma\frac{v}{c^2}\frac{\partial}{\partial r}\left\{\gamma^2\left[\left(1+\frac{v^2}{c^2}\right)cH-v(J+K)\right]\right\}+\frac{\gamma^3}{r}\left\{3K\left(1-\frac{v^2}{c^2}\right)-2\frac{v}{c}H\left(1-\frac{v^2}{c^2}\right)-J\left(1+5\frac{v^2}{c^2}\right)\right\}\\
&+\frac{\gamma^5}{c^2}\nabla(v)\left\{2cH\left(1-\frac{v^2}{c^2}\right)-vJ\left(1+\frac{v^2}{c^2}\right)-vK\left(1-\frac{v^2}{c^2}\right)\right\}\\
&+\frac{\gamma^5}{c^2}\frac{\partial v}{\partial t}\left\{J\left(1-3\frac{v^2}{c^2}\right)-2\frac{v}{c}H\left(1-\frac{v^2}{c^2}\right)+K\left(1-\frac{v^2}{c^2}\right)\right\}\\
&=-\frac{1}{c}\kappa\rho(cH-vS)+\frac{v}{c}\kappa\rho K-\frac{v^2}{c^2}\frac{1}{2}H\kappa\rho
\end{aligned}
\tag{3.6.17}
$$

Finally, the terms are rearranged to obtain the conservative formalism and the terms of higher order are dropped to end up with

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\gamma^3H)+\nabla(\gamma^3vH)+\left[\frac{\partial}{\partial t}(\gamma)+\nabla(\gamma v)\right]\left\{\gamma^2\left[-H\left(1+\frac{v^2}{c^2}\right)+\frac{v}{c}(J+K)\right]\right\}+\left[\frac{\partial}{\partial t}(\gamma v)+\nabla\gamma v^2\right]\\
&\left[\gamma^2\left(2\frac{v}{c}H-\frac{1}{c}K\right)\right]+\frac{1}{c}\Big\{-\left[\frac{\partial}{\partial t}(\gamma^3vH)+\nabla(\gamma^3v^2H)\right]-2\left[\frac{\partial}{\partial t}(\gamma^5v^2H)+\nabla(\gamma^5v^3H)\right]\\
&+2vH\left[\frac{\partial}{\partial t}(\gamma^5vH)+\nabla(\gamma^5v^2H)\right]-c\Big[\frac{\partial}{\partial t}(\gamma^3vJ)+\nabla(\gamma^3v^2J)\Big]+c\left[\frac{\partial}{\partial t}(\gamma^5vJ)+\nabla(\gamma^5v^2J)\right]\\
&-vc\left[\frac{\partial}{\partial t}(\gamma^5J)+\nabla(\gamma^5vJ)\right]+c\left[\frac{\partial}{\partial t}(\gamma^5vK)+\nabla(\gamma^5v^2K)\right]-vc\left[\frac{\partial}{\partial t}(\gamma^5K)+\nabla(\gamma^5vK)\right]\Big\}\\
&+\frac{1}{c}\left\{\gamma^2v^2J\nabla(\gamma)-\nabla(\gamma^3v^2J)-2\nabla(\gamma^5v^2J)+2v\nabla(\gamma^5vJ)\right\}-2\Big[\nabla(\gamma^3vH)-\gamma v^2\nabla(\gamma^2vH)\\
&-\gamma^5H\nabla(v)\Big]-\frac{1}{c}\left\{\gamma v^2\nabla(\gamma^2K)+2\nabla(\gamma^5vK)+2v\nabla(\gamma^5vK)+c^2\gamma\nabla(\gamma^2K)\right\}+3\frac{\gamma^3}{r}cK\left(1-\frac{v^2}{c^2}\right)\\
&-\frac{\gamma^3c}{r}\left[2\frac{v}{c}H+J\left(1-5\gamma^2\frac{v^2}{c^2}\right)\right]+c\kappa_R\rho H-\kappa_R\rho v(S+K)+\frac{1}{2}\frac{v^2}{c}\kappa_R\rho H=0
\end{aligned}
\tag{3.6.18}
$$

And for small velocities the ordinary non-relativistic radiation flux equation is obtained

$$
\frac{\partial}{\partial t}H+\nabla\cdot(vH)+c\nabla\cdot K+H\cdot\nabla v+c\kappa_R\rho H=0 \tag{3.6.19}
$$

3.7 The Poisson Equation

Almost all equations needed for the system of equations are derived except for the Poisson equation which describes the gravitational potential. Since the assumption is to neglect general relativistic effects, a flat space-time configuration is dealt with and therefore the ordinary gravitation potential for self-gravitation is given

$$\Delta\phi = 4\pi G\rho \qquad (3.7.1)$$

3.8 The Complete Set of RRHD Equations

Equation of Continuity

$$\frac{\partial}{\partial t}(\gamma\rho_0) + \nabla\left(\gamma\rho_0 v^i\right) = 0 \tag{3.8.1}$$

Equation of Motion

$$\gamma\frac{\partial}{\partial t}(\gamma\rho v) + \gamma\nabla\cdot(\gamma\rho v v) + \rho\nabla\cdot\phi + \nabla P$$
$$- \frac{4\pi}{c}\kappa_R\rho\left[H - \left(\frac{v}{c}\right)S\right] + \frac{4\pi}{c}\left(\frac{v}{c}\right)\kappa_R\rho K - \left(\frac{v}{c}\right)^2\frac{1}{2c}\kappa_R\rho 4\pi H + \nabla\cdot Q = 0 \tag{3.8.2}$$

Equation of Energy

$$\frac{\partial}{\partial t}\Big\{\gamma(\rho e + p)\Big\} + \nabla\cdot\Big\{\gamma v(\rho e + p)\Big\} - \gamma\frac{\partial P}{\partial t} - \gamma\nabla\cdot(Pv) + \gamma P\nabla\cdot(v) - 4\pi\kappa_P\rho(J - S)$$
$$+ \left(\frac{v}{c}\right)4\pi\kappa_P\rho H - \left(\frac{v}{c}\right)^2 2\pi\kappa_P\rho J + Q : \nabla v = 0 \tag{3.8.3}$$

Poisson Equation

$$\Delta\phi = 4\pi G\rho \tag{3.8.4}$$

Equation of Radiation Energy

$$\frac{\partial}{\partial t}(\gamma^3 J) - \gamma\nabla(\gamma^3 J v) - \left[\frac{\partial}{\partial t}(\gamma) + \nabla\gamma v\right]\left[-\gamma^2\left(J + 2\frac{v}{c}H + \frac{v^2}{c^2}K\right)\right] + \left[\frac{\partial}{\partial t}(\gamma v) + \nabla(\gamma v^2)\right]$$
$$\left\{\gamma^2\left[\frac{v}{c^2}(J + K) - \frac{1}{c}H\right]\right\} + \frac{1}{c^2}\left\{\left[\frac{\partial}{\partial t}(\gamma^5 v^2 J) + \nabla(\gamma^5 v^3 J)\right] - v\left[\frac{\partial}{\partial t}(\gamma^5 v J) + \nabla(\gamma^5 v^2 J)\right]\right.$$
$$+ 3c\left[\frac{\partial}{\partial t}(\gamma^3 v H) + \nabla(\gamma^3 v^2 H)\right] + 2c\left[\frac{\partial}{\partial t}(\gamma^5 v H) + \nabla(\gamma^5 v^2 H)\right] - 2vc\left[\frac{\partial}{\partial t}(\gamma^5 H) + \nabla(\gamma^5 v H)\right]$$
$$\left. - \left[\frac{\partial}{\partial t}(\gamma^5 v^2 K) + \nabla(\gamma^5 v^3 K)\right] - c\left[\frac{\partial}{\partial t}(\gamma^5 v K) + \nabla(\gamma^5 v^2 K)\right]\right\} + \frac{4v}{c}\nabla(\gamma^5 v^2 J) + \gamma^5 J\nabla(v)$$
$$- \frac{1}{c}\left[4\nabla(\gamma^3 v^2 H) - 2\gamma^2 v^2 H\nabla(\gamma) - \gamma^3 H\nabla(v^2) + 4\gamma^5 v H\nabla(v)\right] + \gamma^5 K\nabla(v) - \frac{\gamma^3}{r}v(J + 3K)$$
$$+ \left(1 - \frac{v^2}{c^2}\right)\left[\frac{\gamma^3}{r}2cH - \gamma\left(\nabla(\gamma^2 v K) + \nabla(\gamma^2 v J)\right)\right] + c\kappa_R\rho(J - S) - v\kappa_R\rho H + \frac{v^2}{c}\frac{1}{2}\kappa_R\rho J = 0 \tag{3.8.5}$$

Equation of Radiation Flux

$$\frac{\partial}{\partial t}(\gamma^3 H) + \nabla(\gamma^3 vH) + \left[\frac{\partial}{\partial t}(\gamma) + \nabla(\gamma v)\right]\left\{\gamma^2\left[-H\left(1+\frac{v^2}{c^2}\right) + \frac{v}{c}(J+K)\right]\right\} + \left[\frac{\partial}{\partial t}(\gamma v) + \nabla\gamma v^2\right]$$
$$\left[\gamma^2\left(2\frac{v}{c}H - \frac{1}{c}K\right)\right] + \frac{1}{c}\Big\{-\left[\frac{\partial}{\partial t}(\gamma^3 vH) + \nabla(\gamma^3 v^2 H)\right] - 2\left[\frac{\partial}{\partial t}(\gamma^5 v^2 H) + \nabla(\gamma^5 v^3 H)\right]$$
$$+ 2vH\left[\frac{\partial}{\partial t}(\gamma^5 vH) + \nabla(\gamma^5 v^2 H)\right] - c\left[\frac{\partial}{\partial t}(\gamma^3 vJ) + \nabla(\gamma^3 v^2 J)\right] + c\left[\frac{\partial}{\partial t}(\gamma^5 vJ) + \nabla(\gamma^5 v^2 J)\right]$$
$$- vc\left[\frac{\partial}{\partial t}(\gamma^5 J) + \nabla(\gamma^5 vJ)\right] + c\left[\frac{\partial}{\partial t}(\gamma^5 vK) + \nabla(\gamma^5 v^2 K)\right] - vc\left[\frac{\partial}{\partial t}(\gamma^5 K) + \nabla(\gamma^5 vK)\right]\Big\}$$
$$+ \frac{1}{c}\left\{\gamma^2 v^2 J\nabla(\gamma) - \nabla(\gamma^3 v^2 J) - 2\nabla(\gamma^5 v^2 J) + 2v\nabla(\gamma^5 vJ)\right\} - 2\Big[\nabla(\gamma^3 vH) - \gamma v^2\nabla(\gamma^2 vH)$$
$$- \gamma^5 H\nabla(v)\Big] - \frac{1}{c}\left\{\gamma v^2\nabla(\gamma^2 K) + 2\nabla(\gamma^5 vK) + 2v\nabla(\gamma^5 vK) + c^2\gamma\nabla(\gamma^2 K)\right\} + 3\frac{\gamma^3}{r}cK\left(1-\frac{v^2}{c^2}\right)$$
$$- \frac{\gamma^3 c}{r}\left[2\frac{v}{c}H + J\left(1 - 5\gamma^2\frac{v^2}{c^2}\right)\right] + c\kappa_R\rho H - \kappa_R\rho v(S+K) + \frac{1}{2}\frac{v^2}{c}\kappa_R\rho H = 0 \quad (3.8.6)$$

3.8.1 Additional Notes on the System of Equations of RRHD

For the sake of completeness the source function is mentioned

$$S = \frac{\sigma}{\pi}T^4 \quad (3.8.7)$$

as well as the viscous pressure tensor

$$Q = -\left[q_1 l_{visc}\rho c_s + q_2^2 l_{visc}^2 \rho \max(-\nabla\cdot v, 0)\right]\left[\left(\nabla v_s - \frac{1}{3}\nabla\cdot v\right)\right] \quad (3.8.8)$$

In addition, the equation of state goes into the system of equations over the temperature and gas pressure

$$T = T(\rho, e) \quad (3.8.9)$$
$$P = P(\rho, e) \quad (3.8.10)$$

which are calculated by the ideal gas equation

$$T = e_{int}(\gamma_{adexp} - 1)\frac{x_{mol}}{R_{gas}} \quad (3.8.11)$$
$$P = \rho e_{int}(\gamma_{adexp} - 1) \quad (3.8.12)$$

Not to forget the two opacities, the Planck mean $\kappa_P = \dfrac{\int_0^\infty \kappa_\nu B_\nu d\nu}{\int_0^\infty B_\nu d\nu}$ and the Rosseland mean $\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T_\nu} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T_\nu} d\nu}$, whereas the Rosseland mean is used because of the lack of tables of the Planck mean.

At last, the Eddington approximation is used to close our system of equations because each moment equation contains the next moment so that always $n + 1$ unknown variables are present.
The closure relation is the Eddington-factor

$$\mathbf{f}_{eddi} = \frac{K}{J} \tag{3.8.13}$$

which is the ratio between the second and zeroth moment (i.e. radiation pressure and energy).
Usually it is possible to take this ratio to be $\frac{1}{3}$, whereas the assumption is an isotropic radiation field (and is adequate to first order anisotropy, i.e. $I = I_0 + I_1\mu$). But because of the fact that relativistic effects are dealt with, where the radiation field is highly anisotropic, the dilemma to calculate the Eddington-factor along with the equations is our daily routine.

CHAPTER 4

Numerics of the RRHD Equations

The previous chapters were analytic in order to derive a satisfying formulation of the equations of relativistic radiation hydrodynamics. Satisfying in such a way that expressions were obtained which can be discretized directly without further calculations.
The discretization and the calculation of derivatives of the equations will be the topic of this chapter which will also take a look at the structure of the used RHD code.

4.1 General Remarks on the Numerical Technique

It has already been seen that the quantities to be calculated are implicit in the system of equations. The question looked at in the following is, why is it chosen to be implicit and how can it be solved?
First of all, astrophysical processes are characterized by high propagation speeds and fast flows, e.g. supersonic flows. Further, most processes span over several magnitudes. This suggests that different time and length scales have to be dealt with.
On the one hand, different time scales arise because stars evolve on the nuclear time scale and are disturbed by short-time events such as pulsations. On the other hand, various length scales are given because astrophysical objects often have some spatial extension, whereas their physical processes happen on small scales. In other words, steep gradients and large time steps are the problems which have to be dealt with. This can only be done by implicit schemes because their time steps are not limited by the Courant-Friedrich-Levy (CFL) condition

$$\delta t \leq \min_{\text{all cells}} \frac{\Delta x}{|\mathbf{u}| + c_s} \tag{4.1.1}$$

where $\mathbf{u}$ is the fluid velocity, c_s the sound velocity and Δx a typical length scale. All explicit schemes have to fulfill this relation.

But with an implicit scheme a non-linear system of equations for the unknown variables at the new time step has to be dealt with. As seen in the previous chapter, the system of RRHD equations contains all kinds of differential equations: hyperbolic (hydrodynamic), elliptic (poisson equation), parabolic (viscosity). This has to be solved iteratively with a Newton-Raphson Iteration. As mentioned above, implicit time steps do not underlie the CFL condition but have some limitations which are set by the convergence radius of the Newton iteration and also by accuracy considerations.
Furthermore, it turned out to be advantageous to work with an adaptive grid (first introduced by Winkler (1975)). The adaptive grid uses a constant number of grid points which redistribute themselves to high gradients to optimize accuracy of such features. By using an adaptive grid, an additional equation - the grid equation - which gives the spatial distribution of the grid points, is introduced into the existing system of equations. This grid equation has to be solved simultaneously with the physical equations.

Newton-Raphson Iteration

The implicit non-linear system is a system of N × M unknowns (N equations, M grid points) and is solved, as already stated, by a Newton iteration which expands the system in a taylor series around the estimated variable at the new time level X^{n+1}, first:

$$G(X^{n+1}) = G(X^n) + \frac{\partial G}{\partial X^n}(X^{n+1} - X^n) = 0 \tag{4.1.2}$$

where G denotes the discrete version of the equations and X^{n+1} is assumed to be a solution of the system. The second part in the equation is used to calculate the correction by inverting the linear system

$$\delta X = X^{n+1} - X^n \tag{4.1.3}$$

$$= -\left(\frac{\partial G}{\partial X^n}\right)^{-1} G(X^n) \tag{4.1.4}$$

where $\frac{\partial G}{\partial X^n}$ is the Jacobi matrix. This procedure is repeated until a desired accuracy is achieved. The fact that in the Jacobi matrix derivatives appear due to all variables, leads to the necessity of providing of these derivatives (that is why the last part of this chapter deals with the calculation of these derivatives).

4.2 The RHD Code

The calculations or computations to be performed in this work do not require writing a new program for the simulations. The existing RHD code written by Alexander Stoekl is used and modified to meet the requirements. It is sufficient to take a brief look at the code structure and to refer to the dissertation of A. Stoekl([Sto06]) for further reading. As has been shown a system of coupled equations is given which can only be solved implicitly. This in turn implies that the Jacobi matrix has to be calculated.
The basic structure of the code is as follows:

1. **The Initial Model**
   For reasons of dealing with an initial problem the solution of the equations is needed to start a temporal evolution. The initial model is obtained by iterative numerical integration. For simple models as the shock tube there is no need for integrating an initial model, it is sufficient to write the starting parameters directly into the code.

2. **The Initial Grid**
   The next step is the generation of an initial grid, i.e. adapting the grid on the given physical structure. In other words, the grid equation is solved with the initial parameters. For the solution of the grid equation the same iteration (Newton-Raphson integration) is used as for the real simulation, with the difference that only one equation is solved, namely the grid equation, where the values of the variables are calculated on the new grid points.

   There are two possibilities for this grid adaption: The first one uses an equidistant or logarithmic equidistant (e.g. for spherical geometry) grid as starting grid in which a parameter is defined to control the adaption rate of the grid. This parameter is changed to a given value, and simultaneously the grid equation is solved to obtain a grid which is adapted slowly on the physical structure of the initial model.

   The second method is called pseudo-time evolution where a temporal damping parameter is used in the grid equation. Here the maximum of the damping equals the fact that the new grid is the old grid. This method uses step-by-step reduction of the damping parameter for adaption (recalling that with each step the grid equation is solved). This technique has the advantage of not needing an equidistant starting grid.

3. **Relaxation**
   In this step numerical inaccuracies are smoothed by interpolation between the initial grid and the adaptive grid. This ensures that the initial model with its boundary conditions is a real static solution of the discretized equation. The relaxation procedure equals in principle the time-dependent calculation with one

exception, namely that dynamical processes are oppressed by artificial viscosity (i.e. only small movements are possible). The process of relaxation is finished when a static configuration is achieved through all significant time steps.
As said above, this is only necessary if the initial model is a static solution. For simple test models as the shock tube, relaxation can be neglected.

4. **Simulation**
Finally, the temporal evolution of the whole set of equations is done by Newton iteration (as mentioned above). During this simulation the calculation of the Jacobi matrix and its inversion by Gauss elimination are the crucial steps. Because of the implicit nature of the system the time step only depends on the convergence of the method and no longer on the CFL condition. This allows for larger time steps. If the method finds a root, the time step is minimized until convergence is achieved. This is repeated until the iteration procedure arrives at a terminating condition.
If the method of solution does not lead any solution, there are two reasons: first, the Newton iteration fails because of no convergence or second, the matrix inversion fails because of bad conditioning of the matrix.

4.3 The Discretization Scheme

To compute the equations they need to be transformed from a continuous version to a discrete version. This is achieved by dividing the physical domain into a number of cells in which the physical quantities are assumed to be constant. Flows through the cell borders are described by advection flows. The discrete version of the equations is obtained by integration of the equations over the cell volumes. In other words, the derivatives are going to be differences of sums (over all boundaries). Furthermore, a distinction has to be made between scalar and vectorial variables because they are treated in different ways.
Only the essential part of the discretization scheme is given. Further readings are ([E.A05]) and [Sto06].

4.3.1 The Discretization of Scalar Quantities

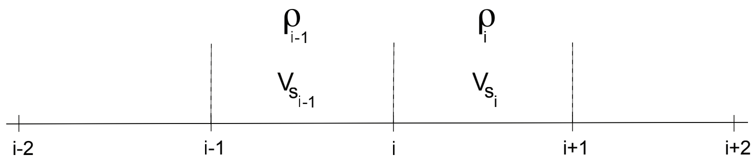

Figure 4.1: Scalar Discretization

Scalar quantities are treated in such a way that they are centered in the cell with whole-number boundaries $[i, i+1]$. The Volume is given by

$$V_{S_i} = \frac{4\pi}{3}\left(r_{i+1}^3 - r_i^3\right) \tag{4.3.1}$$

The advected volume, which is transported over the cell border, has two contributions. The first contribution comes from the flux of the fluid and the second results from the motion of the grid points.
The first contribution can be calculated at the surface of the envelope at r_i with a velocity u_i at grid point i.
Hence, the time integrated volume flux at the old and new time level is needed and has to be calculated in the following way

$$\int_{t^n}^{t^{n+1}} 4\pi r_i^2 u_i dt \simeq 4\pi r_i^2 u_i \delta t \tag{4.3.2}$$

The second contribution can be calculated over the differences of volumes of the sphere with r_i at new and old time level

$$\frac{4\pi}{3}\left[r_i^{n+1}\right]^3 - \frac{4\pi}{3}\left[r_i^n\right]^3 \tag{4.3.3}$$

From this follows that the volume flux over the cell boundary is given by

$$\Delta Vol_s = 4\pi r_i^2 u_i \delta t - \frac{4\pi}{3}\left(r_i^3 - r_i^{old^3}\right) \tag{4.3.4}$$

These flux values are provided by the advection scheme that is used.

4.3.2 The Discretization of Vectorial Quantities

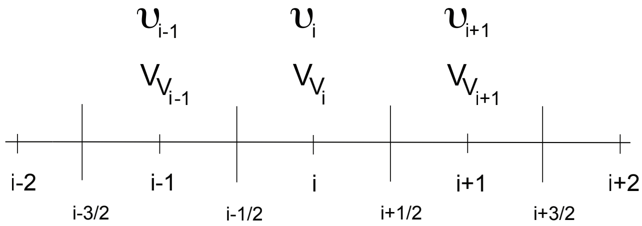

Figure 4.2: Vectorial discretization

In analogy to the scalar quantities the discretization scheme for vectorial quantities is defined in a way that they are shifted by half a cell to the scalar cell, i.e. the cell boundaries are $[r_{i-\frac{1}{2}}, r_{i+\frac{1}{2}}]$. This means the averaged radii $\bar{r}_i$ are of the form

$$r^3_{i+\frac{1}{2}} = \frac{1}{2}\left(r_i^3 + r_{i+1}^3\right) = \bar{r}_i^3 \tag{4.3.5}$$

From this follows

$$V_{V_i} = \frac{4\pi}{3}\left(r^3_{i+\frac{1}{2}} - r^3_{i-\frac{1}{2}}\right) = \frac{4\pi}{3}\frac{1}{2}\left(r^3_{i+1} - r^3_{i-1}\right) \tag{4.3.6}$$

and hence the relation between scalar and vectorial volume is obvious

$$V_{V_i} = \frac{1}{2}\left(V_S\mid_i + V_S\mid_{i-1}\right) \tag{4.3.7}$$

Similar to the scalar advection flux the vectorial advection flux is calculated assuming flux conservation

$$\Delta Vol_V = 4\pi\frac{1}{2}\left(r_i^2 u_i + r_{i+1}^2 u_{i+1}\right)\delta t - 4\pi\frac{1}{2}\left(r_i^3 + u_{i+1}^3 - r_i^{old^3} - r_{i+1}^{old^3}\right) \tag{4.3.8}$$

4.3.3 The Discrete Equations of RRHD

In the following the discrete equations of RRHD are summarized with the subsequent notation: $\widetilde{\ }$ stands for advected quantities and $\bar{\ }$ are averaged quantities.

Equation of Continuity

$$\delta(\gamma_S \rho V_S) + \Sigma\widetilde{(\gamma_S \rho)}\Delta Vol_S = 0 \tag{4.3.9}$$

Equation of Motion

$$\begin{aligned}
&\gamma_V \delta\left(\gamma_V \bar{\rho} v V_V\right) + \gamma_V \Sigma\widetilde{(\gamma_V \bar{\rho} v)}\Delta Vol_V + \frac{Gm}{r^2}\bar{\rho}\delta t V_V + 4\pi r^2 \Delta(P) \\
&- \frac{4\pi}{c}\overline{\kappa_R \rho} H \delta t V_V + \frac{4\pi}{c}\left(\frac{v}{c}\right)\overline{\kappa_R \rho S}\delta t V_V + \frac{4\pi}{c}\left(\frac{v}{c}\right)\overline{\kappa_R \rho K}\delta t V_V \\
&- \left(\frac{v}{c}\right)^2 \frac{2\pi}{c}\overline{\kappa_R \rho} H \delta t V_V - \frac{8\pi}{3r}\Delta\left\{\mu_q \bar{r}^3\left(\frac{\Delta v}{\Delta r} - \frac{\bar{v}}{\bar{r}}\right)\right\}\delta t = 0
\end{aligned} \tag{4.3.10}$$

Equation of Energy

$$\begin{aligned}
&\delta\left\{\gamma_S(\rho e + P)V_S\right\} + \Sigma\left\{\gamma_S \widetilde{(\rho e + P)}\right\}\Delta Vol_S - \gamma_S \delta(P V_S) - \gamma_S \Sigma\widetilde{(P)}\Delta Vol_S \\
&+ \gamma_S P 4\pi \delta t \Delta(r^2 v) - 4\pi\kappa_R \rho(J - S)\delta t V_S + \left(\frac{\bar{v}}{c}\right) 4\pi\delta t \kappa_R \rho \bar{H} V_S \\
&- \left(\frac{\bar{v}}{c}\right)^2 2\pi\kappa_R \rho J \delta t V_S - \mu_q \frac{2}{3}\left(\frac{\Delta v}{\delta r} - \frac{\bar{v}}{\bar{r}}\right)^2 \delta t V_S = 0
\end{aligned} \tag{4.3.11}$$

Poisson Equation

$$\Delta m = \rho V_S \tag{4.3.12}$$

Radiation Energy Equation

$$
\begin{aligned}
&\delta(\gamma_s^3 J V_s) + \Sigma(\widetilde{\gamma_s^3 \bar{J}}) + \left\{\delta(\gamma_s V_s) + \Sigma(\widetilde{\gamma_s})\Delta Vol_s\right\}\left\{-\gamma_s^2\left[J + 2\frac{\bar{v}}{c}\bar{H} + \frac{\bar{v}^2}{c^2}K\right]\right\} + \left\{\delta(\gamma_s \bar{v} V_s) + \Sigma(\widetilde{\gamma_s \bar{v}})\Delta Vol_s\right\} \\
&\left\{\gamma_s^2\left[\frac{\bar{v}}{c}(J+K) - \frac{1}{c}H\right]\right\} + \frac{1}{c^2}\Big\{\delta(\gamma_s^5 \bar{v}^2 J V_s) + \Sigma(\widetilde{\gamma_s^5 \bar{v}^2}\bar{J})\Delta Vol_s - \bar{v}\left\{\delta(\gamma_s^5 \bar{s} J V_s) + \Sigma(\widetilde{\gamma_s^5 \bar{v}}\bar{J})\Delta Vol_s\right\} \\
&+ 3c\left\{\delta(\gamma_s^3 \bar{v}\bar{H} V_s) + \Sigma(\widetilde{\gamma_s^3 \bar{v}\bar{H}})\Delta Vol_s\right\} + 2c\left\{\delta(\gamma_s^5 \bar{v}\bar{H} V_s) + \Sigma(\widetilde{\gamma_s^5 \bar{v}\bar{H}})\Delta Vol_s\right\} \\
&- 2vc\left\{\delta(\gamma_s^5 \bar{H} V_s) + \Sigma(\widetilde{\gamma_s^5 \bar{H}})\Delta Vol_s\right\} - \delta(\gamma_s^5 \bar{v}^2 K V_s) - \Sigma(\widetilde{\gamma_s^5 \bar{v}^2} K)\Delta Vol_s \\
&- \bar{v}\left\{\delta(\gamma_s^5 \bar{v} K V_s) + \Sigma(\widetilde{\gamma_s^5 \bar{v}} K)\Delta Vol_s\right\}\Big\} + 4\frac{\bar{v}}{c^2}4\pi\delta t\Delta(r^2\gamma_V^5 v^2 \bar{J}) + \gamma_s^5 J 4\pi\delta t\Delta(r^2 v) \\
&- \frac{1}{c}\Big\{8\pi\delta t\Delta(r^2\gamma_V^3 v^2 H) - 2\gamma_s^2\bar{v}^2\bar{H}\delta t 4\pi\Delta(r^2\gamma_V) - \gamma_s^3\bar{H}\delta t 4\pi\Delta(r^2 v^2) + 8\gamma_s^5\bar{v}\bar{H}\pi\Delta(r^2 v^2) \\
&+ 8\gamma_s^5\bar{v}\bar{H}\pi\delta t\Delta(r^2 v)\Big\} + \gamma_s^5 K 4\pi\delta t\Delta(r^2 v) - \frac{\gamma^3}{\bar{r}}\bar{v}(J + 3K)\delta t V_s \\
&+ \left(1 - \frac{\bar{v}}{c^2}\right)\left\{\frac{\gamma_s^3}{\bar{r}}2c\bar{H}\delta t V_s - \gamma_s\left[4\pi\delta t\Delta(r^2\gamma_V^2 v\bar{K}) + 4\pi\delta t\Delta(r^2\gamma_V^2 v)\right]\right\} \\
&+ c\kappa_R\rho(J - S)\delta t V_s - \bar{v}\kappa_R\rho\bar{H}\delta t V_s + \frac{\bar{v}^2}{c}\frac{1}{2}\kappa_R\rho J\delta t V_s = 0
\end{aligned}
\tag{4.3.13}
$$

Radiation Momentum Equation

$$
\begin{aligned}
&\delta(\gamma_V^3 H V_V) + \Sigma(\widetilde{\gamma_V^3 H})\Delta Vol_V + \left\{\delta(\gamma_V V_V) + \Sigma(\widetilde{\gamma_V})\Delta Vol_V\right\}\left\{\gamma_V^2\left[-H\left(1 + \frac{v^2}{c^2}\right) + \frac{v}{c}(\bar{J} + \bar{K})\right]\right\} \\
&+ \left\{\delta(\gamma_V v V_V) + \Sigma(\widetilde{\gamma_V v})\Delta Vol_V\right\}\left\{\gamma^2\left[2\frac{v}{c}H - \frac{1}{c}\bar{K}\right]\right\} + \frac{1}{c^2}\Big\{ - \delta(\gamma_V v J V_V) - \Sigma(\widetilde{\gamma_V^3 v^2} H)\Delta Vol_V \\
&- 2\left\{\delta(\gamma_V^5 v^2 H V_V) + \Sigma(\widetilde{\gamma_V^5 v^2} H)\Delta Vol_V\right\} + 2vH\left\{\delta(\gamma_V^5 v H V_V) + \Sigma(\widetilde{\gamma_V^5 v H})\Delta Vol_V\right\} \\
&- c\left\{\delta(\gamma_V^3 v\bar{J} V_V) + \Sigma(\widetilde{\gamma_V^3 v\bar{J}})\Delta Vol_V\right\} + c\left\{\delta(\gamma_V^5 v\bar{J} V_V) + \Sigma(\widetilde{\gamma_V^5 v\bar{J}})\Delta Vol_V\right\} \\
&- vc\left\{\delta(\gamma_V^5 \bar{J} V_V) + \Sigma(\widetilde{\gamma_V^5 \bar{J}})\Delta Vol_V\right\} + c\left\{\delta(\gamma_V^5 v\bar{K} V_V) + \Sigma(\widetilde{\gamma_V^5 v\bar{K}})\Delta Vol_V\right\} \\
&- vc\left\{\delta(\gamma_V^5 \bar{K} V_V) + \Sigma(\widetilde{\gamma_V^5 \bar{K}})\Delta Vol_V\right\}\Big\} + \frac{1}{c}\Big\{\gamma_V^2 v^2 \bar{J} 4\pi r^2\delta t\Delta(\gamma_V) - 4\pi\delta t r^2\Delta(\gamma_V^3 v^2 \bar{J}) \\
&- 8\pi\delta t r^2\Delta(\gamma_V^5 v^2 \bar{J}) + 8\pi v\delta t r^2\Delta(\gamma_V^5 v\bar{J})\Big\} - 2\Big\{4\pi r^2\delta t\Delta(\gamma_V^3 v H) - 4\pi\delta t r^2\gamma_V v^2\Delta(\gamma_V^2 v H) \\
&- \gamma_V^5 H 4\pi\delta t r^2\Delta(v)\Big\} - \frac{1}{c}\Big\{\gamma_V v^2 4\pi r^2\delta t\Delta(\gamma_V^2\bar{K}) + 8\pi r^2\delta t\Delta(\gamma_V^5 v^2\bar{K}) + 8\pi v r^2\delta t\Delta(\gamma_V^5 v\bar{K}) \\
&+ c^2\gamma_V 4\pi\delta t r^2\Delta(\gamma_V^2\bar{K})\Big\} + 3\frac{\gamma_V^3}{r}c\bar{K}\left(1 - \frac{v^2}{c^2}\right)\delta t V_V - \frac{\gamma_V^3}{r}c\left[2\frac{v}{c}H + \bar{J}\left(1 - 5\gamma_V^2\frac{v^2}{c^2}\right)\right]\delta t V_V \\
&+ c\kappa_R^- \rho H\delta t V_V - \kappa_R^- \rho v(\bar{S} + \bar{K})\delta t V_V + \frac{1}{2}\frac{v^2}{c}\kappa_R^- \rho H\delta t V_V
\end{aligned}
\tag{4.3.14}
$$

4.3.4 Supplementary Notes on Stencil and Boundary Conditions

From the discretization scheme follows that for the grid point r_i adjacent points are also included. These grid points form the so-called stencil. Usually it suffices to use a five-point-stencil but for the advection of the Lorentz factor a seven-point stencil is needed. The reason is that the advection scheme requires four points, which means that if one considers a scalar equation, the advection term contains a vectorial quantity like the Lorentz factor that has to be averaged. This implies that one point is missing for being able to calculate the advection of an averaged vectorial quantity.

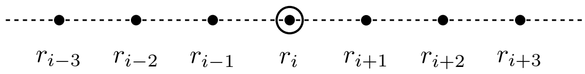

Figure 4.3: 7-point stencil

Because of the fact that the stencil defines the Jacobi matrix, the latter becomes larger as a result of this extension, namely from the previous pentagonal matrix to a heptagonal matrix (see figure (4.3.15) below). As illustrated, most entries of the Jacobi matrix are zero because of the used stencil and only variables on adjacent grid points (i-3, i-2, i-1, i, i+1, i+2, i+3) go into the discrete equations.

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ \cdots & 0 & \frac{\partial G_i}{\partial \vec{x}_{i-3}} & \frac{\partial G_i}{\partial \vec{x}_{i-2}} & \frac{\partial G_i}{\partial \vec{x}_{i-1}} & \frac{\partial G_i}{\partial \vec{x}_{i}} & \frac{\partial G_i}{\partial \vec{x}_{i+1}} & \frac{\partial G_i}{\partial \vec{x}_{i+2}} & \frac{\partial G_i}{\partial \vec{x}_{i+3}} & 0 & \cdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{4.3.15}$$

Finally some notes on the implementation of boundary conditions (figure 4.4): Whereas there are various ways to deal with boundary conditions the so-called pseudo-cell method is used in this code. This means that at the boundaries cells with no physical meaning and no volume are implemented to suffice the stencil at the borders. They only serve to suffice the stencil at the border.

The figure (4.4) illustrates the cells at the inner and outer border of the grid. As can be seen that the first and last three points are used for the pseudo-cells. So, the indicated scalar and vectorial volumes are the first "real" volumes and the two lines above show the range of the actual grid (scalar and vectorial).

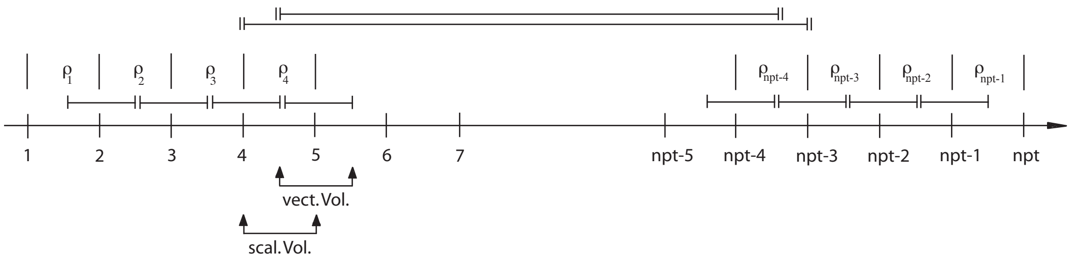

Figure 4.4: Implementation of the boundary conditions via pseudo-cells

4.4 From Paper to Computer via MATHEMATICA

This section deals with the problem of calculating the derivatives of the RRHD equations. As the title of this section implies this is done with a little help from MATHEMATICA ([Res]). The reason for using it is the necessity to get derivatives of the equations to all variables and on relevant stencil-points, which means lots of calculations.
Matthias Kittel ([Kit04]) developed MATHEMATICA packages and notebooks which calculate the derivatives in such a way that the output is in Fortran 90 and therefore can be put directly into the code.

The MATHEMATICA packages contain string substitution rules for defined variables and functions. The purpose of this substitution rules is to reduce the length of the output and make it more readable. Once in a while MATHEMATICA has difficulties in finding strings and therefore does not substitute properly. A reason for this is the canonical order of MATHEMATICA which implies that the string substitution rule has to be in such an order as MATHEMATICA reads it for itself. Usually these notebooks and substitution rules work quite well and don't make any problems. One disadvantage of string replacement rules is that for each new variable new rules have to be written which makes the packages and replacement rules very long and unclear. Therefore writing errors are hard to detect.
Another point is that if there are nested and complex expressions the substitution rules have difficulties working properly. It is therefore useful to work on the lowest level, i.e. the expression level, not with strings, because with expressions it is possible to find patterns which search for special expressions to substitute them.
The following subsection deals with this replacement problem in an alternative way to M. Kittel and shows its pros and cons.

4.4.1 The Structure of the MATHEMATICA Packages

Before the string replacement rules and patterns can be shown it is necessary to understand why the method is used in the way described here. Therefore the structure of the packages will be discussed briefly, referring to the diploma thesis of M. Kittel ([Kit04]) for further reading.
First, it must be mentioned that there are two kinds of variables in the RHD-code itself: primary variables and secondary variables. Primary variables are quantities to be calculated such as density, mass, energy, radius, velocity, radiation energy, radiation flux. Furthermore, there are also averaged quantities like averaged density, radiation energy, radiation momentum and velocity. Originally, there was only one averaged quantity, namely the averaged density. Due to the relativistic advection quantities it turned out to be necessary and helpful to adopt three new averaged quantities. This implies that there are eleven primary variables and hence eleven equations.
Secondary variables are defined to make the equations more readable in a sense that the Fortran form for the equations is a one-to-one form to the physical equation. These quantities are volumes, fluxes and the Lorentz factor.

4.4.2 The Original Packages: String Replacement Rules

Before calculating the derivatives, the equations have to be rewritten in a form that MATHEMATICA is able to read, which means that no indices or underscores are allowed in names.
The equation of continuity is given as an example for an equation in MATHEMATICA style

$$\begin{aligned}G = &\ \mathrm{XSvol\ Di\ gammaSi} - \\ &\ \mathrm{SvolA\ DAi\ gammaSAi} - \mathrm{XSfluxU\ FktgSDadvU} + \mathrm{XSfluxO\ FktgSDadvO}\end{aligned} \quad (4.4.1)$$

where XSvol is the scalar volume, gammaSi is the scalar Lorentz factor (i.e. it contains the averaged velocity). Quantities containing an “A” are variables at the old time level, and functions containing the ending “advO” or “advU” are advection quantities where “O” and “U” indicate the upper or lower advection (i.e. direction of the advection). The MATHEMATICA packages also contain two kinds of variable definitions, the so-called secondary variables, and functions like material functions or advection functions.
The first kind of variables are those whose definitions are known and accessible by MATHEMATICA (i.e. defined in packages). They are marked by an “X” and serve to reduce the length of the output and make it more readable.
When setting up the system of equations all variables and definitions are inserted into one another and then the derivatives are calculated. If the original definition of a variable appears in the derivative, this expression will be replaced by the name of the variable.

Example: scalar volumina

$$\text{XSvol} = \text{z43 pi (Rip1 - Ri)(Rip1}^2 + \text{Rip1 Ri+ Ri}^2\text{);} \tag{4.4.2}$$

XSvol is a secondary variable and therefore marked with an "X" meaning that MATHEMATICA knows the definition. So if this expression appears after the derivative, the program replaces it with the variable name but without the X.
As mentioned above, MATHEMATICA sometimes has problems finding these expressions. It should be noted that only variables at the new time level are derived. Hence, variables at the old time level are not defined.
The second kind of variables are those which describe material and advection functions. They are marked by "Fkt". Derivations due to primary variables are calculated in the code or taken from tables.
These functions are genuine MATHEMATICA functions which means that with these functions the dependencies of the functions are declared.
Example:

$$\text{FktPgasi = PgasZi[Ei,Di];} \tag{4.4.3}$$

From this, MATHEMATICA knows due to which primary variable it has to calculate the derivation. Such a derivation expression would be

$$\text{Derivative(1,0)(Pgasi)(Ei,Di)} \tag{4.4.4}$$

and this would be substituted by dPgasdE(i).

4.4.3 The New Packages: Pattern Matching

The new (alternative) version of the MATHEMATICA packages works with pattern matching (i.e. regular expressions ([BDF05], [Fri98]) and MATHEMATICA patterns). This has the advantage that if a new variable is added, no extra string replacement rule needs to be written; instead, the pattern has to be adjusted. But there is not only a glorious side to pattern matching. The fact that single letters must be looked at and one has to write a replacement module makes it not that general as one would like it to be. In other words, the perfect pattern matching would be to work only with MATHEMATICA patterns but the fact is that this is not sufficient for what is needed in this context. That is why two procedures are mixed: regular expressions and MATHEMATICA patterns. Some definitions do not have any patterns because their appearance is rare and it would be overkill to write patterns for them.
The new packages are made from the original packages which means that there are patterns for time centering, material functions, advection functions, hivek pattern (i.e. volumes, fluxes, Lorentz factor). These patterns exist for so-called "forward substitution" and "back substitution". The forward substitution does the replacement of all variables which are marked with an "X", and the back substitution replaces the output

step by step back into a more readable form. Unfortunately there are not only patterns but string replacement rules as well. The following patterns are called "forward" and "backward" patterns, the "forward" patterns being substitutions before the derivation of the equation and the "backward" ones being the substitutions afterwards.

Forward Time Centering Package

```
GetDefinition[a_] := StringCases[a,
 RegularExpression["X([A-Za-z]+?)Z(i[mp]?[1-3]?)(\\^?[1-9]?)" ] :> "(zz $1$2 + (z1 - zz)

mytimecenteringRules[a_]:= a-> ToExpression[GetDefinition@ToString@a];

MytimecenteringRules = Map[mytimecenteringRules,tclist];


tclist = {XRZi,XRZim1,XRZim2,XRZim3,XRZip1,XRZip2,XRZip3,
 XEZi,XEZim1,XEZim2,XEZim3,XEZip1,XEZip2,XEZip3,
XDZi,XDZim1,XDZim2,XDZim3,XDZip1,XDZip2,XDZip3,
XUZi,XUZim1,XUZim2,XUZim3,XUZip1,XUZip2,XUZip3,
XMZi,XMZim1,XMZim2,XMZim3,XMZip1,XMZip2,XMZip3,
XDavgZi,XDavgZim1,XDavgZim2,XDavgZim3,XDavgZip1,
XDavgZip2,XDavgZip3,XJZi,XJZim1,XJZim2,XJZim3,
XJZip1,XJZip2,XJZip3,XHZi,XHZim1,XHZim2,XHZim3,
XHZip1,XHZip2,XHZip3,XJavgZi,XJavgZim1,XJavgZim2,
XJavgZim3,XJavgZip1,XJavgZip2,XJavgZip3,XHavgZi,
XHavgZim1,XHavgZim2,XHavgZim3,XHavgZip1,XHavgZip2,
XHavgZip3,XUavgZi,XUavgZim1,XUavgZim2,XUavgZim3,
XUavgZip1,XUavgZip2,XUavgZip3};
```

Figure 4.5: Pattern: Forward Time Centering

This package uses (as mentioned before) regular expressions and MATHEMATICA patterns.
The first three lines are an usual MATHEMATICA function (which is also a pattern) whereas the first function includes a regular expression pattern which searches for an expression of the form *X.Zi.* because the time centered variables are defined

$$\text{XRZi} = \text{zz Ri} + (\text{z1} - \text{zz})\ \text{RAi} \tag{4.4.5}$$

with the ending from "i" to "im3" and "ip3". The last bracket term includes the search of exponents. If such a variable is found, it is replaced by the right hand side of the pattern which is the right hand side of the equation above.
The next function "mytimecenteringRules" takes a term "a", transforms it into a string, calls the function "GetDefinition" and checks the pattern. If the pattern matches, the term is replaced by right hand side; otherwise it is ignored and the next term is checked. "MytimecenteringRules" calls the function "mytimecenteringRules" with the "tclist". This list is needed because MATHEMATICA needs to know on which terms it is allowed to apply the pattern. If the list isn't given, Mathematica would replace the whole equation with the right hand side definition and not only the term which fits the pattern.

Forward Hivek Package

Volume-Pattern

```
Volumes[x_]:= Block[{himatch,himatchi,len,l1,l2,l3,l4,prefix,suffix},
himatch = StringCases[x, RegularExpression["X([SV]?)vol([AZ]?)(i?[mp]?)([1-3]?)"]
->{"$1","$2","$3","$4"}];
len = Length@Flatten@himatch;
If[len == 0, Return[],
{l1,l2,l3,l4}= First@First@himatch;

{prefix,suffix}={If[l2=="Z","XRZ"<>l3, "R"<>l2<>l3],l4};
Ripmj[j_]:=ToExpression[prefix<>ToString[ToExpression[suffix]+j]];
Rip1prefix:=ToExpression[prefix<>"ip1"];
Rim1prefix:=ToExpression[prefix<>"im1"];
Riprefix:=ToExpression[prefix<>"i"];
If[l1 == "S",
Switch[prefix,
"Rip" | "RAip"|"XRZip",
z43 pi ( Ripmj[1] - Ripmj[0] ) ( Ripmj[1]^2 + Ripmj[1] Ripmj[0] + Ripmj[0]^2 ),
"R" | "RA"|"XRZ",
z43 pi ( Rip1prefix - Riprefix ) ( Rip1prefix^2 + Rip1prefix Riprefix + Riprefix^2 ),
"Rim" | "RAim"|"XRZim",
z43 pi ( Ripmj[-1]-Ripmj[0] ) ( Ripmj[-1]^2 + Ripmj[-1] Ripmj[0] + Ripmj[0]^2 )]
/.{Rim0->Ri,RAim0->RAi,XRZim0->XRZi},
Switch[prefix,
"Rip" | "RAip"|"XRZip",
z43 pi z12 ( Ripmj[1] - Ripmj[-1] ) ( Ripmj[1]^2 + Ripmj[1] Ripmj[-1] + Ripmj[-1]^2 )
/.{Rip0->Ri,RAip0->RAi,XRZip0->XRZi},
"R" | "RA"|"XRZ",
z43 pi z12 ( Rip1prefix - Rim1prefix )
 ( Rip1prefix^2 + Rip1prefix Rim1prefix + Rim1prefix^2 ),
"Rim" | "RAim"|"XRZim",
z43 pi z12 ( Ripmj[-1] - Ripmj[1] ) ( Ripmj[-1]^2 + Ripmj[-1] Ripmj[1] + Ripmj[1]^2 )
/.{Rim0->Ri,RAim0->RAi,XRZim0->XRZi}]]
]]


GetmyVolumes[x_]:= x -> Volumes[StringSplit[ToString@x,Whitespace]]
MyVolumesRule = Map[GetmyVolumes,hivelist];

hivelist ={XSvolip2,XSvolip1,XSvol,XSvolim1,XSvolim2,XSvolim3,XSvolZip2,XSvolZip1,XSvolZ,
XSvolZim1,XSvolZim2,XSvolZim3,XSvolAip2,XSvolAip1,XSvolA,XSvolAim1,XSvolAim2,XSvolAim3,
XVvolip2,XVvolip1,XVvol,XVvolim1,XVvolim2,XVvolZip2,XVvolZip1,XVvolZ,XVvolZim1,XVvolZim2,
XVvolAip2,XVvolAip1,XVvolA,XVvolAim1,XVvolAim2};
```

Figure 4.6: Forward Volume Pattern

This pattern searches for the volume variable and replaces it with the volume definition if the pattern matches. The basic principle is the same as above. There are two MATHEMATICA functions defined "Volumes" and "GetmyVolumes", the latter calling the first function. In this one it is especially obvious why regular expressions are used. Depending on which letter and ending there is, the definition of the volume will appear, which is checked in the second part (or paragraph) of this pattern where the variables prefix and suffix are defined besides other variables. Then the "Switch" statement searches for the adequate ending and returns the definition of the specific volume. As in the package for time centering a list of upcoming volume variables is needed to let MATHEMATICA know that it should apply the substitution if and only if these variables appear in the list.

Gamma-Pattern

```
gammatrafo[x_] := Block[{len,match,matchi,l1,l2,l3,l4},
match=StringCases[x, RegularExpression["X?gamma([VS]?)([AZ]?)(i[mp]?[1-3]?)(\\^?[1-9]?)"]
len=Length@Flatten@match;
If[len == 0,Return[],
{ l1, l2, l3,l4}=First@First@match;

gammaVA =z1/Sqrt[z1 - ToExpression["UA"<>l3<>"^"<>"2"]/clight^2];
gammaVZ =z1/Sqrt[z1 - ToExpression["XUZ"<>l3<>"^"<>"2"]/clight^2];
gammaV =z1/Sqrt[z1 - ToExpression["U"<>l3<>"^"<>"2"]/clight^2];

gammaSA =z1/Sqrt[z1 - ToExpression["UavgA"<>l3<>"^"<>"2"]/clight^2];
gammaSZ =z1/Sqrt[z1 - ToExpression["XUavgZ"<>l3<>"^"<>"2"]/clight^2];
gammaS =z1/Sqrt[z1 - ToExpression["Uavg"<>l3<>"^"<>"2"]/clight^2];

Switch[l1,
"V", Switch[l2, "A",gammaVA,"Z", gammaVZ, "", gammaV],
"S", Switch[l2,"A",gammaSA,"Z", gammaSZ, "", gammaS]]
]]
```

```
Getmygamma[x_]:= x -> gammatrafo[StringSplit[ToString@x,Whitespace]]
MygammaRules = Map[Getmygamma,gammalist];

gammalist = {gammaVi,gammaVim1,gammaVim2,gammaVip1,gammaVip2,gammaVAi,gammaVAim1,
gammaVAim2,gammaVAip1,gammaVAip2,XgammaVZi,XgammaVZim1,XgammaVZim2,XgammaVZip1,
XgammaVZip2,gammaSi,gammaSim1,gammaSim2,gammaSim3,gammaSip1,gammaSip2,gammaSAi,
gammaSAim1,gammaSAim2,gammaSAim3,gammaSAip1,gammaSAip2,XgammaSZi,XgammaSZim1,
XgammaSZim2,XgammaSZim3,XgammaSZip1,XgammaSZip2};
```

Figure 4.7: Forward Gamma Pattern

At last there is a pattern for the Lorentz factor which is configurationally similar to the last patterns seen. The regular expression searches for an expression like "Xgamma"- or "gamma" + old, new or time centered variable and ending. If found it puts it equal to the definition of the Lorentz factor depending on the letter ("A","Z" or none) and ("V", "S") whether vectorial or scalar version of the gamma.

Forward Material Function Pattern

```
matterfunction[x_]:= Block[{funkmatch,funkmatchi,len,l1, l2, l3, l4, l5, prefix, suffix},
funkmatch = StringCases[x,
RegularExpression["(Fkt)([A-T]+)([gas]*?)([AZ]?)(i[mp]?[1-3]?)"]
->{"$1","$2","$3","$4","$5"}];
len = Length@Flatten@funkmatch;
If[len == 0, Return[],
{l1,l2,l3,l4,l5}= First@First@funkmatch;

{prefix,suffix}={l2 <> l3, l5};
(* neuer zeitschritt *)
fktnameneu := prefix <> suffix;
abhneu1 := "E" <> suffix;
abhneu2:= "D" <> suffix;
rhsfktneu := ToExpression@ToExpression["ToString[fktnameneu(abhneu1, abhneu2)]",
TraditionalForm];
(* alter zeitschritt *)
fktnamealt := prefix <> l4 <> suffix;
abhalt1 := "E" <>  l4 <> suffix;
abhalt2:= "D" <>  l4 <> suffix;
rhsfktalt := ToExpression@ToExpression["ToString[fktnamealt(abhalt1, abhalt2)]",
TraditionalForm];
(* zeitlich zentriert *)
fktnamezz := prefix <>  l4 <> suffix ;
abhzz1 := "XE" <>  l4 <> suffix;
abhzz2:= "XD" <>  l4 <> suffix;
rhsfktzz := ToExpression@ToExpression["ToString[fktnamezz(abhzz1, abhzz2)]",
 TraditionalForm];
If[l4 == "A",rhsfktalt, If[l4 == "Z", rhsfktzz, rhsfktneu]]
]]
```

This function is similar to those above. It searches for a variable which begins with “Fkt”, has upper cases and the word “gas” in it (but not necessarily) and also endings. If the pattern matches, it replaces this variable with its MATHEMATICA version of a function and its dependencies. The function call is the same as with the other patterns and there is also a list of permitted variable names (which is not included as such lists were already shown above and would take up too much room here).
Example: The pattern matches “FktTgasim1” and returns “‘Tgasim1[Eim1,Dim1]”

Forward Advection Pattern

```
advectiondefinition[n_]:=Block[{admatch,admatchi,len,l1, l2, l3, l4, l5, l6},
admatch = StringCases[n, RegularExpression[ "Fkt(g[VS]?[2-5]?)?(FktPgas)?([A-Z][a-z]*
[2-5]*)?([A-Z][a-z]*[2-5]*)?(Fedd[VS]?)?(adv[OU]?)"]
-> {"$1", "$2", "$3", "$4", "$5", "$6"}];
len = Length@Flatten@admatch;
If[len == 0, Return[],
{l1,l2,l3,l4,l5,l6}= First@First@admatch;

stringliste = {l1, l2, l3, l4, l5};

(*holt sich aus der stringliste die nichtleeren strings heraus *)
notEmptyQ[s_]:= StringLength[s]!=0;
listwithoutemptystrings = Cases[stringliste, x_?notEmptyQ];
fktname:=  l1 <> l2<> l3 <> l4 <> l5 <> l6;

(*nimmt aus der nichtleeren stringliste die erste variable; count deswegen, weil
wenn man den schleifenindex nimmt, beschwert sich das mathematica, dass man
mehr positionen nehmen mmöchte, als vorhanden*)
l = Count[listwithoutemptystrings, _];

Do[
Var =Take[listwithoutemptystrings, {k}];
fktend = Last@Take[Flatten@admatch, All];

prefix = "X";
sufim2 = "Zim2";
sufim1 = "Zim1";
sufi = "Zi";
sufip1 = "Zip1";
sufip2 = "Zip2";


      (*nimmt die variablen nacheinander und hängt die jeweiligen endungen an *)
      varim2[a_]:=Block[{Var},
      Var =First@Take[listwithoutemptystrings, {a}];
      If[Var == "gV", Var = "gammaV";
       ptim2 = prefix <>Var <> sufim2,
      If[Var == "gS", Var = "gammaS";
       ptim2 = prefix <>Var <> sufim2,
      If[Var == "gS3", ptim2 = prefix <> "gammaSZim2^3",
      If[Var == "gV3",ptim2 = prefix <> "gammaVZim2^3",
      If[Var == "gV5",ptim2 = prefix <> "gammaVZim2^5",
      If[Var == "gS5", ptim2 = prefix <> "gammaSZim2^5",
      If[Var == "Uavg2", ptim2 = prefix <> "UavgZim2^2",
      If[Var == "U2",ptim2 = prefix <> "UZim2^2",
      If[Var == "Fedd", ptim2 = Var <> "im2",
      If[Var == "FeddSZ", ptim2 = prefix <> Var <> "im2",
      If[Var == "FeddVZ", ptim2 = prefix <> Var <> "im2",
      If[Var == "FktPgas", ptim2 = Var <> sufim2,
      ptim2=  prefix <> Var <> sufim2 ]]]]]]]]]]]]];
```

```
varim1[b_]:=Block[{Var},
Var =First@Take[listwithoutemptystrings, {b}];
If[Var == "gV", Var = "gammaV";
 ptim1 = prefix <>Var <> sufim1,
If[Var == "gS", Var = "gammaS";
 ptim1 = prefix <>Var <> sufim1,
If[Var == "gS3", ptim1 = prefix <> "gammaSZim1^3",
If[Var == "gV3",ptim1 = prefix <> "gammaVZim1^3",
If[Var == "gV5",ptim1 = prefix <> "gammaVZim1^5",
If[Var == "gS5", ptim1 = prefix <> "gammaSZim1^5",
If[Var == "Uavg2", ptim1 = prefix <> "UavgZim1^2",
If[Var == "U2",ptim1 = prefix <> "UZim1^2",
If[Var == "Fedd", ptim1 = Var <> "im1",
If[Var == "FeddSZ", ptim1 = prefix <> Var <> "im1",
If[Var == "FeddVZ", ptim1 = prefix <> Var <> "im1",
If[Var == "FktPgas", ptim1 = Var <> sufim1,
ptim1=  prefix <> Var <> sufim1]]]]]]]]]]]]];
```

```
vari[c_]:=Block[{Var},
Var =First@Take[listwithoutemptystrings, {c}];
If[Var == "gV", Var = "gammaV";
 pti = prefix <>Var <> sufi,
If[Var == "gS", Var = "gammaS";
 pti = prefix <>Var <> sufi,
If[Var == "gS3", pti = prefix <> "gammaSZi^3",
If[Var == "gV3",pti = prefix <> "gammaVZi^3",
If[Var == "gV5",pti = prefix <> "gammaVZi^5",
If[Var == "gS5", pti = prefix <> "gammaSZi^5",
If[Var == "Uavg2", pti = prefix <> "UavgZi^2",
If[Var == "U2",pti = prefix <> "UZi^2",
If[Var == "Fedd", pti = Var <> "i",
If[Var == "FeddSZ", pti = prefix <> Var <> "i",
If[Var == "FeddVZ", pti = prefix <> Var <> "i",
If[Var == "FktPgas", pti = Var <> sufi,
pti=  prefix <> Var <> sufi]]]]]]]]]]]]];
```

```
varip1[d_]:=Block[{Var},
Var =First@Take[listwithoutemptystrings, {d}];
If[Var == "gV", Var = "gammaV";
 ptip1 = prefix <>Var <> sufip1,
If[Var == "gS", Var = "gammaS";
 ptip1 = prefix <>Var <> sufip1,
If[Var == "gS3", ptip1 = prefix <> "gammaSZip1^3",
If[Var == "gV3",ptip1 = prefix <> "gammaVZip1^3",
If[Var == "gV5",ptip1 = prefix <> "gammaVZip1^5",
If[Var == "gS5", ptip1 = prefix <> "gammaSZip1^5",
If[Var == "Uavg2", ptip1 = prefix <> "UavgZip1^2",
If[Var == "U2",ptip1 = prefix <> "UZip1^2",
If[Var == "Fedd", ptip1 = Var <> "ip1",
If[Var == "FeddSZ", ptim2 = prefix <> Var <> "ip1",
If[Var == "FeddVZ", ptim2 = prefix <> Var <> "ip1",
If[Var == "FktPgas", ptip1 = Var <> sufip1,
ptip1=  prefix <> Var <> sufip1]]]]]]]]]]]]]];
```

```
varip2[e_]:=Block[{Var},
Var =First@Take[listwithoutemptystrings, {e}];
If[Var == "gV", Var = "gammaV";
 ptip2 = prefix <>Var <> sufip2,
If[Var == "gS", Var = "gammaS";
 ptip2 = prefix <>Var <> sufip2,
If[Var == "gS3", ptip2 = prefix <> "gammaSZip2^3",
If[Var == "gV3",ptip2 = prefix <> "gammaVZip2^3",
If[Var == "gV5",ptip2 = prefix <> "gammaVZip2^5",
If[Var == "gS5", ptip2 = prefix <> "gammaSZip2^5",
If[Var == "Uavg2", ptip2 = prefix <> "UavgZip2^2",
If[Var == "U2",ptip2 = prefix <> "UZip2^2",
If[Var == "Fedd", ptip2 = Var <> "ip2",
If[Var == "FeddSZ", ptip2 = prefix <> Var <> "ip2",
If[Var == "FeddVZ", ptip2 = prefix <> Var <> "ip2",
If[Var == "FktPgas", ptip2 = Var <> sufip2,
ptip2=  prefix <> Var <> sufip2]]]]]]]]]]]]]];
```

```
(*multipliziert die jeweiligen variablen miteinander, damit
sie in die funktion schreibbar sind*)
depim2 := ToString@Product[varim2[k], {k, 1, l}];
depim1 := ToString@Product[varim1[k], {k, 1, l}];
depi:= ToString@Product[vari[k],{k, 1, l}];
depip1:= ToString@Product[varip1[k],{k, 1, l}];
depip2 := ToString@Product[varip2[k],{k, 1, l}],{k,l}];

advektionOben := ToExpression@ToExpression["ToString[
fktname(depim1, depi, depip1, depip2)]", TraditionalForm];
advektionUnten:= ToExpression@ToExpression["ToString[
fktname(depim2, depim1, depi, depip1)]", TraditionalForm];
```

Figure 4.8: Forward Advection Pattern

This pattern is a bit clumsy, but it works. There is definitely a more elegant way to do this function but for reasons of time it was not developed further. It is similar to the material function pattern because it searches for a variable which begins with "Fkt" and the ending "advO" or "advU". The result for positive matching is for example "JadvU [XJZim2,XJZim1,XJZi,XJZip1] ". Obviously, this pattern is some lines longer than the others because the single letters and syllables have to be put together again with an "X" and "Z".

Forward Variable Eddington Factor Pattern

```
GetSuffix[u_]:=StringCases[u,
RegularExpression["(\\(optdepth) . (\"gamma[VS]Z?\\_i[mp]?\\d?\")
.(\\(U.*[AZ]?(i[mp]?\\d?)\\/clight . z1\\)\\))\\/(\\(\\d.optdepth
 . \"gamma[VS]Z?\\_i[mp]?\\d?\".\\(U.*[AZ]?(i[mp]?\\d?)\\/clight . z1\\))"] :> "$4"];

GetMiddlefix[k_]:= Block[{Middlefix},
Middlefix =StringCases[k,RegularExpression["X?U(.*?)[AZ]?(i[mp]?\\d?)"]:>"$1"];
notEmptyQ[Middlefix] := StringLength[Middlefix] != 0;
 If[First@First@notEmptyQ[Middlefix]== 0, Middlefix = V, Middlefix = S]]

myeddirules[u_]:=  u->ToExpression[ToString@Fedd <>ToString@GetMiddlefix@ToString@u
<> GetSuffix@ToString@Map[InputForm,{u}]];
myeddirulesZ[u_]:=  u->ToExpression[ToString@Fedd <>ToString@GetMiddlefix@ToString@u
<>"Z"<> GetSuffix@ToString@Map[InputForm,{u}]];
myeddirulesA[u_]:=  u->ToExpression[ToString@Fedd <>ToString@GetMiddlefix@ToString@u
<>"A"<> GetSuffix@ToString@Map[InputForm,{u}]];

AlleddisRuleZ = Map[myeddirulesZ,eddilistZ];
AlleddisRuleA = Map[myeddirulesA,eddilistA];
AlleddisRule = Map[myeddirules,eddilist];

SetAttributes[AlleddisRuleZ,Orderless];
SetAttributes[AlleddisRuleA,Orderless];
SetAttributes[AlleddisRule,Orderless];
```

Figure 4.9: Forward pattern for the variable Eddington factor

This pattern for a variable Eddington factor is only used if the Eddington factor of the paper [CA07] is applied on the equations. It works similar to the gamma pattern.

Backward Advection Pattern

```
advection[n_] := Block[{derrelmatch,klammernlinks,klammernrechts,nnew,
nclean,len, l1,l2,l3,l4,l5,l6,l7},
(*Noch bevor nur irgendwie versucht wird etwas zu matchen, werden
die geschwungenen klammern weggeschmissen*)

klammernlinks=StringPosition[n,"{"];
nnew=StringReplacePart[n,"",klammernlinks];
klammernrechts =StringPosition[nnew,"}"];
nclean = StringReplacePart[nnew,"",klammernrechts];


derrelmatch = StringCases[nclean,
 RegularExpression["(Derivative)?(\\[., ., ., .\\])?\\[?(([a-z\\dA-Z\\d?A-Z?a-z]*)
(adv)([OU]?))\\]?(\\[.+\\])"]-> {"$1","$2","$3","$4","$5","$6","$7"}];

len=Length@Flatten@derrelmatch;
If[len == 0,Return[],
{l1, l2, l3,l4,l5,l6,l7}=First@derrelmatch;

If[l1==l2, l4 <>"_" <>l5 <> "_"<>l6,
Switch[l2,
"[1, 0, 0, 0]",Switch[l6,"U",  "d"<>l4<>"_"<>l5<>"_"<>l6<>"_"<>"dim2",
                     "O","d"<>l4<>"_"<>l5<>"_"<>l6<>"_"<>"dim1"],
"[0, 1, 0, 0]",Switch[l6,"U",  "d"<>l4<>"_"<>l5<>"_"<>l6<>"_"<>"dim1",
                     "O","d"<>l4<>"_"<>l5<>"_"<>l6<>"_"<>"di"],
"[0, 0, 1, 0]",Switch[l6,"U",  "d"<>l4<>"_"<>l5<>"_"<>l6<>"_"<>"di",
                     "O","d"<>l4<>"_"<>l5<>"_"<>l6<>"_"<>"dip1"],
"[0, 0, 0, 1]",Switch[l6,"U",  "d"<>l4<>"_"<>l5<>"_"<>l6<>"_"<>"dip1",
                     "O","d"<>l4<>"_"<>l5<>"_"<>l6<>"_"<>"dip2"]]]
]]

getadvection[n_]:= n -> advection@First[ToString/@Map[InputForm,{n}]]

AllmyadvectionbacksubstitutionRules = Map[getadvection, advectionlistRules];
SetAttributes[getadvection, Orderless];
```

Figure 4.10: Advection back substitution pattern

Now, this is a back substitution pattern for advection terms which look like "Derivative(1,0,0,0)(JadvU)(JZim2,JZim1,JZi,JZip1)". The "Switch" assignment checks to which stencil point the derivation is due and assigns the appropriate stencil point for the function name.

Backward Hivek Pattern

Volume Pattern

```
backsubhivek[n_] := Block[{a,b,res,diff,himatch,len,pre,post,a1,a2,a3,a4,a5,a6,
l1,l2,l3,l4,l5,l6,l7,l8,l9},
himatch = StringCases[n, RegularExpression["(.*)?pi.\\((-)?(R[ZA]?)(i[mp]?[1-3]?)
 (.)  (R[ZA]?)(i[mp]?[1-3]?)\\).\\(.+\\).?((z12)?.z43)(.*)?"]
-> {"$1","$2","$3","$4","$5","$6","$7","$8","$9","$10"}];

(*Print["n ="n];*)


len=Length@Flatten@himatch;

{pre,l1, l2, l3,l4,l5,l6,l7,l8,post}= First@himatch;


If[len == 0,Return[pre<>post],


a= l3 ;
b= l6;


diff =  a <>" + "<>b;


res=ToString@If[l7 == "z12*z43",
Switch[diff,
"ip1 + ip3"|"ip3 + ip1",Switch[ l2, "R","V_vol_ip2","RA", "V_volA_ip2","RZ", "V_volZ_ip2"
"i + ip2"|"ip2 + i",Switch[ l2, "R","V_vol_ip1","RA", "V_volA_ip1","RZ", "V_volZ_ip1"],
"im1 + ip1"|"ip1 + im1", Switch[l2, "R","V_vol","RA", "V_volA","RZ","V_volZ"],
"i + im2", Switch[ l2, "R","V_vol_im1","RA","V_volA_im1","RZ","V_volZ_im1"],
"im1 + im3",  Switch[l2, "R","V_vol_im2","RA","V_volA_im2","RZ","V_volZ_im2"]],
If[l7 == "*z43",
Switch[diff,
"ip2 + ip3"|"ip3 + ip2",Switch[ l2, "R","S_vol_ip2","RA","S_volA_ip2","RZ","S_volZ_ip2"],
"ip1 + ip2"|"ip2 + ip1",Switch[ l2, "R","S_vol_ip1","RA","S_volA_ip1","RZ","S_volZ_ip1"],
"i + ip1"|"ip1 + i", Switch[ l2, "R","S_vol","RA", "S_volA","RZ", "S_volZ"],
"i + im1", Switch[ l2, "R","S_vol_im1","RA","S_volA_im1","RZ","S_volZ_im1"],
"im1 + im2", Switch[ l2, "R","S_vol_im2","RA","S_volA_im2","RZ","S_volZ_im2"],
"im2 + im3", Switch[ l2, "R","S_vol_im3","RA","S_volA_im3","RZ","S_volZ_im3"]]]
], Return[pre<>post]];


ToString[pre<>res<>post]
]

VolPattern[n_] := backsubhivek@First[ToString/@Map[InputForm,{n}]];
SetAttributes[VolPattern,Orderless];
```

Figure 4.11: Volume back substitution pattern

The volume pattern is like the inverse function of the forward volume pattern. The problem with this one is that it doesn't always match, in particular if there are not only volume terms but also others; therefore variables have to be singled out first in order for the pattern to find the expression which it searches for.
There are 3 more pattern versions of this volume backward substitution pattern, which

are similar to each other but are effective in different situations. That is the crux of the matter. The point is to single out terms before replacing. MATHEMATICA is able to prescind terms but those terms have to be declared in advance. That causes problems because the derivations can be very long and it is not easy (not to say impossible) to predict all terms which might be multiplied with the volumes to single them out.
Hence, this patterns for back substitution of volume expressions do not work perfectly.

Gamma-Pattern

```
GetSuffix[u_]:=StringCases[u,RegularExpression["X?U.*[AZ]?(i[mp]?\\d?)"]:>"$1"]

GetMiddlefix[k_]:= Block[{Middlefix},
Middlefix =StringCases[k,RegularExpression["X?U(.*?)[AZ]?(i[mp]?\\d?)"]:>"$1"];
notEmptyQ[Middlefix] := StringLength[Middlefix] != 0;
 If[First@First@notEmptyQ[Middlefix]== 0, Middlefix = V, Middlefix = S]]


mygammaRulesA[u_]:=(-(u^2/clight^2)+1)^x_?NumberQ
->ToExpression[ToString@gamma <>ToString@GetMiddlefix@ToString@u <> "A"<> GetSuffix@ToStr

mygammaRulesZ[u_]:=(-(u^2/clight^2)+1)^x_?NumberQ
->ToExpression[ToString@gamma <> ToString@GetMiddlefix@ToString@u <> "Z"<> GetSuffix@ToSt

mygammaRules[u_]:=(-(u^2/clight^2)+1)^x_?NumberQ
->ToExpression[ToString@gamma <>ToString@GetMiddlefix@ToString@u <> GetSuffix@ToString@u]

SetAttributes[mygammaRulesA, Orderless];
SetAttributes[mygammaRulesZ, Orderless];
SetAttributes[mygammaRules, Orderless];
```

Figure 4.12: Gamma back substitution pattern

This is the back substitution pattern for the Lorentz factor which mixes a MATHEMATICA pattern ("mygammaRulesA, mygammaRulesZ, mygammaRules") with regular expressions.
This pattern is one of the core patterns because the advection function would not work without it. The reason is that the relativistic advection quantities contain the Lorentz factor which, after replacements, is in the advection function not in its string form but in its definition form (i.e. root expression). Therefore the original string replacement rules have not worked.
With this pattern matching for gamma it is possible to search for the root-expression of gamma and replace it with its corresponding string expression.

Backward Matter-Fortran Pattern

```
matter[n_] := Block[{klammernlinks,nnew,klammernrechts,nclean,mattermatch,
mattermatchi,len, l1,l2,l3,l4,l5,l6},

klammernlinks=StringPosition[n,"{"];
nnew=StringReplacePart[n,"",klammernlinks];
klammernrechts =StringPosition[nnew,"}"];
nclean = StringReplacePart[nnew,"",klammernrechts];


mattermatch = StringCases[nclean, RegularExpression["(Derivative)?(\\[., .\\])?
(\\[?([A-Za-z]+)(i[mp]?)([1-3]?)\\]?)(\\[., .\\])?"]-> {"$1","$2","$3","$4","$5","$6"}];
len=Length@Flatten@mattermatch;
If[len == 0,Return[],
{l1, l2, l3,l4,l5,l6}=First@mattermatch;

If[l3 == "Derivati", Return[],
If[l1== l2,
Switch[ l5,"i", l4 <> "("<> "i" <>")", "im",l4<>"("<> "i-"<> l6<> ")" ,
"ip",l4<>"("<> "i+" <> l6 <> ")"],
Switch[l2,
"[1, 0]",Switch[ l5,"i", "d"<>l4 <>"dE"<> "("<> "i" <>")", "im",
"d"<>l4<>"dE"<>"("<> "i-"<> l6<> ")" ,"ip","d"<>l4<>"dE"<>"("<> "i+" <> l6 <> ")"],
"[0, 1]", Switch[ l5,"i", "d"<>l4 <>"dD"<> "("<> "i" <>")", "im",
"d"<>l4<>"dD"<>"("<> "i-"<> l6<> ")" ,"ip","d"<>l4<>"dD"<>"("<> "i+" <> l6 <> ")"]]
]
]]]

getmatter[n_]:= n -> matter@First[ToString/@Map[InputForm,{n}]];
AllmymatterbacksubstitutionRules = Map[getmatter, matterbacksublist];
```

Figure 4.13: Matter Function back substitution pattern

The back substitution of the material function is similar to the way it is done for the advection quantities and therefore it is not necessary to go into detail about this.

Backward Time-Centering Pattern

```
timecenteringback[n_] :=Block[{tc,len,l1,l2,l3,l4,l5,l6,l7,l8,tcplus,tcminus },
 tc =StringCases[n, RegularExpression["(-?)\\(?([A-Z](avg)?A)(i[mp]?[1-3]?)
 \\(z1 - zz\\)\\)? (.) ([A-Z](avg)?)(i[mp]?[1-3]?) zz"]
->{"$1","$2","$3","$4","$5","$6","$7","$8"} ];

(*
Print@n;
Print@tc;
*)

len = Length@Flatten@tc;
If[len == 0, Return[],

{l1,l2,l3,l4,l5,l6,l7,l8}= First@tc;

(*
Print["l1 =",l1];
Print["l2 =",l2];
Print["l3 =",l3];
Print["l4 =",l4];
Print["l5 =",l5];
Print["l6 =",l6];
Print["l7 =",l7];
Print["l8 =",l8];
*)


(*If[(l1 && l5)== "-",  l1 <> l6 <>"Z"<>l8, l6 <>"Z"<>l8]*)

tcplus = l6 <>"Z"<>l8;
tcminus = l1 <> l6 <>"Z"<>l8;

(*
Print@tcplus;
Print@tcminus;
*)

Switch[l1,
"-",Switch[l5,"-",ToExpression@n ->ToExpression@tcminus ,"+",Return[]],
"",Switch[l5,"+",ToExpression@n ->ToExpression@tcplus,"-",Return[]]
]
]
]

Getmynewtimecenteringback[n_]:=timecenteringback@ToString@n;
SetAttributes[Getmynewtimecenteringback,Orderless];
AllmynewtcbacksubstitutionRules = Map[Getmynewtimecenteringback,tcbacklistnew];
```

Figure 4.14: Back substitution for time-centered variables

The time-centered variables have to be replaced by their almost original form, and this is done in a similar way as the forward time-centering pattern. The "switch"- statement checks the sign of the expression only to give the right time centered variable back.

Backward Primvar Pattern

```
prime[n_] := Block[
  {primematch, mattermatchi, len, lenl2, l1, l2, l3, l4, l5, plusminus, newavgerageprime,
   oldavgerageprime, timecenteredavgerageprime, newprime, oldprime, timecenteredprime},
primematch = StringCases[n, RegularExpression["([A-Z])(avg)?([AZ]?)(i[mp]?)([1-3]?)"] ->
     {"$1", "$2", "$3", "$4", "$5"}];
len = Length@Flatten@primematch;
If[len == 0, Return[],
{l1, l2, l3, l4, l5} = First@primematch;

lenl2 = Length@l2;

plusminus = Switch[l4,
"i", "i",
"im", "i-",
"ip", "i+"];

newprime = "X(M" <> l1 <> "," <> plusminus <> l5 <> ")";
oldprime = "XA(M" <> l1 <> "," <> plusminus <> l5 <> ")";
timecenteredprime = "XZ(M" <> l1 <> "," <> plusminus <> l5 <> ")";

newavgerageprime = "X(M" <> l1 <> "a" <> "," <> plusminus <> l5 <> ")";
oldavgerageprime = "XA(M" <> l1 <> "a" <> "," <> plusminus <> l5 <> ")";
timecenteredavgerageprime = "XZ(M" <> l1 <> "a" <> "," <> plusminus <> l5 <> ")";

Switch[l3,
"", Switch[l2, "avg", newavgerageprime, "", newprime],
"A", Switch[l2, "avg", oldavgerageprime, "", oldprime] ,
"Z", Switch[l2, "avg", timecenteredavgerageprime, "", timecenteredprime]]
]
]

getprime[n_] := n -> prime[ToString@n];
AllmyprimesRule = Map[getprime, primelist];
```

Figure 4.15: Back substitution pattern for primary variables

Finally, the primary variables have to be transformed in a way that the fortran code is able to work with them, which means that if there is a string or expression "EZi" it will be replaced by "XZ(ME,i)".

Notes

These were the main patterns for the calculation of the derivatives.
The notebook itself has almost its original form, apart from the fact that these pattern-functions are called instead of the original string-replacement packages.
As mentioned in the beginning some more packages exist which contain ordinary string definitions of quantities, but finding a pattern for them would have been an overkill. Furthermore, packages are available which contain string replacement rules for some derivatives of quantities which are common, like the Lorentz factor, fluxes and volumes.

4.4.4 MATHEMATICA Output

After getting a clue on how the new MATHEMATICA notebooks work, some results should be showed. For this purpose, the (classical) continuity equation is regarded and one of its "old" derivative (from the original notebooks) is compared to the "new" one.

$$\begin{aligned} & DG(MD, MR, DG\_i, i) = -(pi * X(MD, i) * (X(MR, i) * *2 + X(MR, i) * X(MR, i+1) \\ & + X(MR, i+1) * *2) * z43) + pi * X(MD, i) * (X(MR, i+1) - X(MR, i)) * (X(MR, i+1) \\ & + X(MR, i) * z2) * z43 - (-(pi * (XA(MR, i) * *2 + XA(MR, i) * X(MR, i) + X(MR, i) * *2) \\ & * z43) - pi * (X(MR, i) - XA(MR, i)) * (XA(MR, i) + X(MR, i) * z2) * z43 \\ & + pi * XZ(MR, i) * tst * XZ(MU, i) * z2 * z4 * zz) * D\_adv\_U \end{aligned} \tag{4.4.6}$$

$$DG(MD, MR, DG\_i, i) = -(dS\_fluxU\_dRi * D_a dv_U) + dS\_vol\_dRi * X(MD, i) \tag{4.4.7}$$

The above two equations show the derivative of the density due to the radius on the point DG_i and the grid index i. The first one is the result from the original MATHEMATICA notebooks, whereas the second output comes from the new pattern-notebooks. The long output in (4.4.6) for expample, is replaced by "dS_fluxU_dRi" and "dS_vol_dRi", which is the derivative of the flux and the volume due to the radius and "D_adv_U" is the advection term. It is seen that (4.4.7) is much more readable and shorter than the old one.
Last, regarding the same output for the relativistic continuity equation:

$$\begin{aligned} DG(MD, MR, DG\_i, i) = & -(dS\_fluxU\_dRi * gSD\_adv\_U) \\ & + dS\_vol\_dRi * gammaS\_i * X(MD, i) \end{aligned} \tag{4.4.8}$$

Here "gSD_adv_U" is the advection term including the Lorentz factor and "gammaS_i" is the scalar version of the Lorentz factor.

CHAPTER 5

Future Prospects

The previous chapters provided not only the analytical and the discrete versions of the equations of RRHD, but also their derivatives for computations. The next step would be to implement these equations into the modified RHD-code. Further, to test them on known problems such as the shock tube problem and then to calculate a point blast explosion (to simulate a gamma-ray burst).

Another important point would be an elegant rounding off of the MATHEMATICA pattern packages to provide better output of the derivatives. This implies some additional work on the volume-patterns, as mentioned in the previous chapter. In other words, a more elegant and simpler way to teach MATHEMATICA to simplify and prescind terms needs to be found.

CHAPTER 6

Appendix

The Appendix is based on the following literature: ([MWM84]), ([E.A03]), ([Cas04]), ([d'I95]).

6.1 Photon-Four-Momentum

Four-Momentum

To derive a definition of the photon-four momentum it is necessary to start out from the four-momentum itself.

In three-space the momentum is known as $\mathbf{p} = m\mathbf{v}$, where $\mathbf{v}$ is the particle's velocity and m the mass of the particle (the particle's rest mass, to be precise).
For the relativistic generalization $\mathbf{v}$ is substituted by the four-velocity V^{α}.
P^{α} is considered as the four-momentum so that in analogy to the Newtonian momentum

$$P^{\alpha} = m_0 V^{\alpha} \tag{6.1.1}$$

is the four-momentum, where m_0 is the rest mass of the particle.

By introducing the relativistic mass

$$m = \gamma m_0 \tag{6.1.2}$$

the four-momentum can be written as

$$P^{\alpha} = \gamma m_0(c, \mathbf{v}) = m(c, \mathbf{v}) = (mc, \mathbf{p}) \tag{6.1.3}$$

Photon-Four-Momentum

When seeking an expression for the photon-four-momentum it becomes clear that (6.1.3) is not reasonable anymore because photons have zero rest mass.
In addition, it is known that photons have energy $e = h\nu$ and momentum $p = \frac{h\nu}{c}$, where ν is the photons' frequency.
Furthermore, the relativistic energy is given as

$$e = mc^2 = \gamma m_0 c^2 \tag{6.1.4}$$

Expanding the Lorentz factor for small velocities ($\frac{v}{c} \ll 1$):

$$\gamma \simeq 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots \tag{6.1.5}$$

and inserting into the energy equation gives

$$e = \underbrace{m_0 c^2}_{\text{rest mass}} + \underbrace{\frac{1}{2} m_0 v^2}_{\text{kinetic energy}} + \underbrace{\frac{3}{8}\frac{m_0 v^4}{c^2} + \frac{5}{16}\frac{m_0 v^6}{c^4} + \cdots}_{\text{relativistic corrections}} \tag{6.1.6}$$

.
Taking (6.1.3) and using the just now arrived expression for the energy (6.1.4) the four-momentum can be rewritten

$$P^\alpha = (mc, \mathbf{p}) = \left(\frac{e}{c}, \mathbf{p}\right) \tag{6.1.7}$$

At last the photon-four-momentum can be written as

$$M^\alpha = \left(\frac{h\nu}{c}\right)(1, \mathbf{n}) = \hbar(k, \mathbf{k}) \equiv \hbar K^\alpha \tag{6.1.8}$$

where $\mathbf{n}$ is the unit vector in the photon's direction of propagation, $k = \frac{2\pi\nu}{c}$ is the wave number and K^α the photon-propagation four-vector.

6.2 Basic Radiation Quantities

6.2.1 Specific Intensity

The specific intensity is defined via the radiation energy (in a frequency interval $\nu + d\nu$) that crosses a space element dS. The normal to this space element encloses an angle α with the direction of propagation of the radiation $\mathbf{n}$. This energy radiates into a solid angle $d\omega$ in a time dt

$$d\mathscr{E} = I(\mathbf{x}, t; \mathbf{n}, \nu) dS \cos\alpha \, d\nu \, dt \tag{6.2.1}$$

with its units

$$[I] = ergs\ cm^{-2}s^{-1}hz^{-1}sr^{-1} \tag{6.2.2}$$

If a one-dimensional spheric symmetric medium is considered, I is independent of the azimuthal angle ϕ and depends only on the polar angle θ: $I = I(r, t; \mu, \nu)$

6.2.2 The Photon Number Density

The above definition of the specific intensity is true for macroscopic considerations but has to be rewritten for photons in such a way that a photon number density is defined

$$\Psi = \Psi(\mathbf{x}, t; \mathbf{n}, v)\ d\nu\ d\omega \tag{6.2.3}$$

This is nothing more than the number of photons per unit volume at a point $(\mathbf{x}, t)$ with a frequency $(\nu + d\nu)$, which moves into an angle $d\omega$ around $\mathbf{n}$.
Therefore the number of photons that crosses an area dS during a time dt through an angle $d\omega$

$$\Psi = \Psi\ \mathbf{n}\ dS(cdt)\ d\nu\ d\omega \tag{6.2.4}$$

Hence, each photon has an energy

$$d\mathscr{E} = \underbrace{ch\nu\Psi(\mathbf{x}, t; \mathbf{n}, v)}_{I(\mathbf{x},t;\mathbf{n},\nu)} dS \cos\alpha\ d\nu\ d\omega\ dt \tag{6.2.5}$$

6.2.3 The Photon Distribution Function

The distribution function f_R is defined in such a way that $f_R(\mathbf{x}, t; \mathbf{n}, p)d^3p$ is the number of photons per unit volume with momentums $(p, p + dp) = \frac{h\nu}{c}(\mathbf{n}, \mathbf{n} + d\mathbf{n})$.
The right hand side looks that way because $E = h\nu$ and

$$E = \sqrt{p^2c^2 + m^2c^4} \rightarrow E = pc \rightarrow p = \frac{E}{c} \tag{6.2.6}$$

where n gives the direction of propagation.
Using $p = \frac{E}{c} = \frac{h\nu}{c} \rightarrow dp = \frac{h}{c}d\nu$ one gets

$$d^3p = p^2dp\ d\omega = \left(\frac{h}{c}\right)^3 \nu^2\ d\nu\ d\omega \tag{6.2.7}$$

and obtains the following relation

$$f_Rd^3p = \left(\frac{h^3\nu^2}{c^3}\right) f_R\ d\nu\ d\omega = \Psi\ d\nu\ d\omega \tag{6.2.8}$$

which implies that

$$\Psi = \left(\frac{h^3 \nu^2}{c^3}\right) f_R \tag{6.2.9}$$

and

$$I = ch\nu\Psi(\mathbf{x}, t; \mathbf{n}, \nu) = \left(\frac{h^4 \nu^3}{c^2}\right) f_R \tag{6.2.10}$$

This photon distribution function is invariant under Lorentz transformation.

6.2.4 The Radiation Moments

Radiation Energy

The radiation energy is the zeroth moment of the transfer equation or in other words it is the zeroth moment of the specific intensity. This means that it is the average of the specific intensity over all solid angles

$$J(\mathbf{x}, t; \nu) = \frac{1}{4\pi} \oint I(\mathbf{x}, t; \mathbf{n}, \nu)\, d\omega \tag{6.2.11}$$

And for one-dimensional spherically-symmetric media

$$J_\nu = \frac{1}{2} \int_{-1}^{1} I_\nu\, d\mu \tag{6.2.12}$$

Another way to define the radiation energy density is via photon number density

$$E(\mathbf{x}, t; \nu) = h\nu \oint \Psi(\mathbf{x}, t; \mathbf{n}, \nu)\, d\omega \tag{6.2.13}$$

and

$$E_\nu = \frac{4\pi}{c} J_\nu \tag{6.2.14}$$

or the total radiation density

$$E = \frac{4\pi}{c} J \tag{6.2.15}$$

where J is called the Eddington moment.

Radiation Flux

The radiation flux **F** is the net rate of the radiant energy through a space element dS per unit frequency interval.

$$\vec{F} = \oint I(\mathbf{x}, t; \mathbf{n}, \nu)\, \vec{n}\, d\omega \tag{6.2.16}$$

with its units [**F**] = $ergs\ cm^{-2}\ s^{-1}\ hz^{-1}$. Similarly

$$H_\nu \equiv \frac{1}{4\pi} F_\nu = \frac{1}{2} \int_{-1}^{1} I(z, t; \mu, \nu)\, \mu\, d\mu \tag{6.2.17}$$

where H is the first Eddington moment.

Radiation Pressure

The scalar radiation pressure K is the second Eddington moment:

$$\begin{aligned} P_\nu &= \frac{2\pi}{c}\int_{-1}^{1} I(z,t;\mu,\nu)\,\mu^2\,d\mu \\ &= \left(\frac{4\pi}{c}\right)\frac{1}{2}\int_{-1}^{1} I_\nu\mu^2 d\mu \\ &= \frac{4\pi}{c}K_\nu \end{aligned} \tag{6.2.18}$$

6.2.5 The Four-Force-Density

To get the covariant equations of relativistic radiation hydrodynamics a covariant generalization of the Newtonian three-force is needed.

The Four-Force

Starting out from the Newtonian definition of the force

$$\mathbf{\Phi} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} \tag{6.2.19}$$

it can be generalized by implementing the four-momentum P^α instead of the three-momentum

$$\Phi^\alpha = \frac{d\mathbf{P}^\alpha}{dt} = m_0\frac{dV^\alpha}{d\tau} = m_0 A^\alpha \tag{6.2.20}$$

where

$$P^\alpha = m_0 V^\alpha = (mc, \mathbf{p}) \tag{6.2.21}$$

$$V^\alpha = \frac{dx^\alpha}{d\tau} = \gamma(c, \mathbf{v}) \tag{6.2.22}$$

$$A^\alpha = \frac{dV^\alpha}{d\tau} = \gamma^2\left[\frac{\gamma^2\mathbf{v}\cdot\mathbf{a}}{c}, \mathbf{a} + \left(\frac{\gamma^2\mathbf{v}\cdot\mathbf{a}}{c^2}\right)\mathbf{v}\right] \tag{6.2.23}$$

has been used with P^α, V^α, A^α the four-momentum, four-velocity and four-acceleration. x^α is a four-vector (as defined in chapter 1.1) with $\alpha = (0, 1, 2, 3)$.

The Four-Force-Density

Now the Four-Force-Density is wanted, i.e. a four-force acting on a finite volume δV_0, and therefore the following is written

$$F^\mu \equiv \frac{\Phi^\mu}{\delta V} = \left(\frac{\gamma}{\delta V_0}\right)\left(\frac{\mathbf{\Phi}\cdot\mathbf{v}}{c}, \mathbf{\Phi}\right) \tag{6.2.24}$$

If the four-force density is written in an arbitrary frame, it gets the form of $f = \Phi/\delta V$, hence

$$F^{\mu} = (\frac{\gamma \delta V}{\delta V_0}) \left(\frac{\mathbf{f} \cdot \mathbf{v}}{c}, \mathbf{f} \right) \tag{6.2.25}$$

Applying the relation $\delta V_0 = \gamma \delta V$ (which is the Lorentz contraction of Volumes) gives

$$F^{\mu} = \left(\frac{\mathbf{f} \cdot \mathbf{v}}{c}, \mathbf{f} \right) \tag{6.2.26}$$

Finally the relativistic fluid-dynamical equations (in covariant form) can be written as

$$M^{\alpha\beta}_{;\beta} = F^{\alpha} \tag{6.2.27}$$

with the $M^{\alpha\beta}_{;\beta}$ the material stress-energy tensor ([MWM84], p. 148).

6.3 Further Calculations

6.3.1 Transformation of Derivations (3.6.5 - 3.6.10)

$$\begin{aligned}
\frac{\partial \mu_0}{\partial t}\Big|_{r\mu\nu} &= \frac{\partial \mu_0}{\partial t}\Big|_{r\mu_0\nu_0} + \frac{\partial \mu_0}{\partial t}\Big|_{r\mu\nu}\frac{\partial \mu_0}{\partial \mu_0} + \frac{\partial \nu_0}{\partial t}\Big|_{r\mu\nu}\frac{\partial \mu_0}{\partial \nu_0} \quad = \quad \frac{\partial \mu_0}{\partial t}\Big|_{r\mu\nu} \\
&= -\left[\frac{(1-\beta\mu) + \mu(\mu-\beta)}{(1-\beta\mu)^2}\right]\frac{\partial \beta}{\partial t} \\
&= -\gamma^2(1-\mu_0^2)\underbrace{\left[\frac{(1-\beta\mu) + \mu(\mu-\beta)}{(1-\beta\mu)}\right]}_{=1}\frac{\partial \beta}{\partial t} \\
&= -\gamma^2(1-\mu_0^2)\frac{\partial \beta}{\partial t}
\end{aligned} \tag{6.3.1}$$

$$\begin{aligned}
\frac{\partial \mu_0}{\partial r}\Big|_{r\mu\nu} &= \frac{\partial \mu_0}{\partial r}\Big|_{t\mu_0\nu_0} + \frac{\partial \mu_0}{\partial r}\Big|_{t\mu\nu}\frac{\partial \mu_0}{\partial \mu_0} + \frac{\partial \nu_0}{\partial r}\Big|_{t\mu\nu}\frac{\partial \mu_0}{\partial \nu_0} \quad = \quad \frac{\partial \mu_0}{\partial r}\Big|_{t\mu\nu} \\
&= -\left[\frac{(1-\beta\mu) + \mu(\mu-\beta)}{(1-\beta\mu)^2}\right]\frac{\partial \beta}{\partial r} \\
&= -\gamma^2(1-\mu_0^2)\underbrace{\left[\frac{(1-\beta\mu) + \mu(\mu-\beta)}{(1-\beta\mu)}\right]}_{=1}\frac{\partial \beta}{\partial r} \\
&= -\gamma^2(1-\mu_0^2)\frac{\partial \beta}{\partial r}
\end{aligned} \tag{6.3.2}$$

$$\frac{\partial \nu_0}{\partial \mu} = \frac{\partial}{\partial \mu}[\gamma\nu(1-\beta\mu)] = -\beta\gamma\nu = -\beta\gamma^2\nu_0(1+\beta\mu_0) \tag{6.3.3}$$

$$\begin{aligned}\frac{\partial \mu_0}{\partial \mu} &= \frac{\partial}{\partial \mu}\left[\frac{\mu-\beta}{1-\beta\mu}\right] = \frac{(1-\beta\mu)+\beta(\mu-\beta)}{(1-\beta\mu)^2} \\ &= \frac{(1+\beta\mu_0)}{(1-\beta\mu)} = \frac{(1+\beta)\gamma(1-\mu_0)^{1/2}}{(1-\mu_0^2)^{1/2}} \\ &= \gamma^2(1+\beta\mu_0)\end{aligned} \tag{6.3.4}$$

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Zusammenfassung

Diese Arbeit befasst sich mit der Erweiterung der klassischen (nicht-relativistischen) Gleichungen der Strahlungshydrodynamik auf den relativistischen Formalismus. Beginnend mit einer kurzen Darstellung des relevanten Wissens der speziellen Relativitätstheorie, im ersten Kapitel, wird im zweiten Kapitel, ein kovarianter Formalismus der Größen der Strahlungshydrodynamik, erörtert. Dieser theoretische Teil beruht vor allem auf den Arbeiten von Mihalas ([MWM84]) und Castor ([Cas04]). Im Anschluss daran werden die Gleichungen der relativistischen Strahlungshydrodynamik in ihrer konservativen Schreibweise hergeleitet. Damit sind sie bereit für die Diskretisierung und ihre Implementierung in den bestehenden SHD-Code. Zusätzlich zur Beschreibung der Diskretisierung und des verwendeten SHD-Codes (Kapitel 4) wird ein weiteres Hauptaugenmerk auf die Berechnung der Ableitungen der Gleichungen gelegt. Dies wird mit Hilfe von MATHEMATICA bewerkstelligt, wobei ein alternativer Weg zur ursprünglichen Version (von Matthias Kittel) vorgestellt wird.

Curiculum Vitae

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(Sabine Ertl)