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Consistent dynamic equity market code-books
from a practical point of view

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Abstract

Gebräuchliche Aktienpreis- bzw. Martingalmodelle beschreiben die Dynamik des Preises einer Aktie unter einem Martingalmaß; grundlegendes Beispiel ist das *Black-Scholes* Modell. Im Gegensatz dazu zielt ein *Marktmodell* darauf ab die Dynamik eines ganzen Marktes (d.h. Aktienpreis plus abgeleitete Optionen) zu beschreiben. In der vorliegenden Arbeit beschäftigen wir uns mit arbitragefreien Modellen die die Dynamik eines Aktienpreises sowie flüssig gehandelter Derivate beschreiben. (“equity market models” respektive “market models for stock options”).

Die Motivation derartige Modelle zu betrachten, liegt darin, dass europäische Optionen flüssig gehandelt werden und daher Rückschlüsse auf die zugrundeliegende Stochastik erlauben.

Marktmodelle werden auch für Anleihen angewandt; wir verweisen auf [Heath, Jarrow, and Morton \[1992\]](#). In jüngerer Zeit beeinflusst dieser Zugang auch Aktienpreismodelle (siehe beispielsweise [Derman and Kani \[1998\]](#), [Dupire \[1996\]](#)). Derman und Kani schlagen vor, die dynamische Entwicklung von Märkten mittels Differentialgleichungen für Aktienpreis und volatility surface zu modellieren.

Ein anderer Zugang wird von [Schönbucher \[1999\]](#) gewählt; hier ist der Ausgangspunkt die gemeinsame Dynamik von Aktienpreis und impliziten Black-Scholes Volatilitäten. [Carmona and Nadtochiy \[2009\]](#) schlagen ein Marktmodell vor, in dem der Aktienpreis als exponentieller Levy-Prozeß gegeben ist. In diesem Fall wird eine zeitinhomogene Levy-Dichte verwendet um die zusätzlichen, durch Optionspreise gegebenen Informationen miteinzubeziehen.

Der erste Teil dieser Dissertation beschreibt zwei unterschiedliche Zugänge um durch den Markt gegebene Informationen zu beschreiben. Einerseits kann dies durch ein *local volatility* Codebuch geschehen, andererseits kann ein *Levy-Dichte* Codebuch verwendet werden. Wir beschreiben verschiedene Kalibrierungsverfahren; insbesondere vergleichen wir parametrische mit nicht-parametrischen Methoden.

Im zweiten Abschnitt beschreiben wir verschiedene dynamische Modelle, die entstehen, wenn man die beiden obigen Codebücher “in Bewegung setzt”. D.h. wir beschreiben dynamische local-volatility sowie dynamische Levy-Dichte Modelle. Besonderes Augenmerk liegt dabei auf der Konsistenz (d.h. Arbitragefreiheit) dieser Modelle. Aufgrund der zusätzlichen stochastischen Dynamik, erscheint dieser Aspekt durchaus als Herausforderung.

Im letzten Teil dieser Arbeit betrachten wir einen speziellen Typ von Levy-Modellen. Die spezielle Struktur der Modellparameter ist durch eine Kalibrierung an Marktdaten gemäß [Ortega et al. \[2009\]](#) motiviert. Wir berechnen Konsistenzbedingungen für den vorliegenden speziellen Typ und implementieren sie mithilfe eines geeigneten Euler-Schemas.

Abstract

Traditional pricing models, or *martingale models*, are based on the dynamics of the underlying stock price under some martingale measure $\mathbb{Q}$, e.g. as in the *Black-Scholes model*. A *market model* on the other hand, aims at describing the behavior of the whole market (i.e. the stock with derivative prices), not a single stock. In this thesis markets are considered which consist of one underlying together with its liquidly traded derivative instruments. Hence we consider models that describe simultaneously the dynamics of the liquid derivative instruments and the underlying, without introducing arbitrage. We refer to this as *equity market models* or *market models for stock options*.

The motivation for such a model lies first of all in the fact that European vanilla options are liquidly traded on the market, and they contain therefore relevant information on the underlying stochastics. We have that not all observable option price movements can be related to changes of the stock.

Market models have successfully been used in fixed income theory through the approach of [Heath, Jarrow, and Morton \[1992\]](#). More recently this approach has been influencing pricing models for the equity markets, e.g. as advocated by [Derman and Kani \[1998\]](#), [Dupire \[1996\]](#). Derman and Kani propose modeling market dynamics through stochastic differential equations for the stock price process $S = (S_t)_{t \geq 0}$ and the local volatility surface.

Another approach is taken by [Schönbucher \[1999\]](#). He prescribes a stochastic development of the underlying S together with implied Black-Scholes volatilities of a set of given standard options. [Carmona and Nadtochiy \[2009\]](#) propose a market model by prescribing the stock price S as an exponential Lévy process. A time inhomogeneous Lévy density is in this case used to capture the information given by the market option prices.

The first part of this thesis is devoted to describing two approaches of capturing market information; By using the *local volatility code-book* and by using the *Lévy density code-book*. In this part also different calibration methods (of local volatility resp. Lévy models) are discussed in more detail. In particular parametric calibration methods are compared with non parametric ones.

The second part of this thesis discusses dynamic models arising from setting the two code-books above in motion. That is, *dynamic local volatility models* and *dynamic Lévy density models*. In particular consistency (absence of arbitrage) in this type of market models is addressed; As stochastic dynamics for the local volatility (or Lévy density) are introduced, it has to be ensured that this does not give rise to arbitrage opportunities. We notice that this is quite a challenging task.

In the third and last part of this thesis we look at a specific type of a dynamic Lévy market model. Again, consistency is discussed. The specific form of model parameters is motivated by a calibration to market data à la [Ortega et al. \[2009\]](#). We calculate the consistency conditions in our simplified model and implement it using an Euler scheme.

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1 Introduction

Traditional pricing models are based on the dynamics of the underlying stock under a martingale measure $\mathbb{Q}$. For instance, in the classical Black-Scholes framework (Black and Scholes [1973]) the evolution of the underlying asset price $S = (S_t)_{t \geq 0}$ is given by the following stochastic differential equation,

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \quad (1.1)$$

The drift μ and volatility σ are constants, W_t is a standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A *market model* aims at explaining or predicting the behavior of a whole market. In this thesis we consider a market consisting of one underlying asset and its liquidly traded derivative instruments (the European call and put options). We wish to model the dynamics of this market. That is, to describe simultaneously the dynamics of the European options and the underlying asset without introducing arbitrage. We refer to this as a *Equity market model* or a *market model for stock/index options*.

The motivation for equity market models lies first of all in the fact that European vanilla options are liquidly traded on the market and the prices of such heavily traded derivatives have its own independent volatility. We have that not all observable option price movements can be related to changes of the stock. Furthermore, there is an increased interest for derivative products whose payoffs are dependent on the term structure of implied volatility (for implied volatility see Definition 1.1), or that are having other optional features in their payoffs.

Furthermore, we have that the parameters of the traditional pricing models, e.g. the volatility σ of the Black-Scholes model, are in contrary to the model assumptions, *not constant*. Model parameters calibrated to the market today will differ from the parameters calibrated tomorrow. Hence there will be a problem when pricing and hedging the more exotic options mentioned above, when using these “static models”.

At last we also have that certain exotic options are hedged with portfolios of European options. This further motivates the inclusion of the European option prices as stochastic state variables in a pricing model.

Market models have successfully been introduced in fixed income theory through the approach of Heath et al. [1992]. More recently this approach is influencing pricing models in the equity market, e.g. as advocated by Derman and Kani [1998]. Derman and Kani propose modeling market dynamics through stochastic differential equations for the stock price process S and the local volatility surface. (For definition of local volatility see Equation (2.33)). Lyons [1997] propose a market model where prices of derivatives with a fixed payoff function are included as additional stochastic state variables. Lyons prescribes a joint volatility process for the derivatives together with the stock. In the influencing work of Schönbucher [1999] is prescribed the stochastic development of the underlying S together with implied Black-Scholes volatilities of a set of given standard options.

In these models, postulating the evolution of the implied or local volatility will imply specific dynamics for the set of option prices. Certain restrictions have to be fulfilled by the model parameters in order to guarantee that these dynamics do not produce arbitrage opportunities. Deriving and fulfilling these *no arbitrage restrictions* turns out to be a challenging task, especially in the implied volatility case.

Conditions under which absence of arbitrage is equivalent to the existence of an equivalent martingale measure can be found in the *fundamental theorem of asset pricing*, see [Delbaen and Schachermayer \[1994\]](#). Roughly speaking, in a continuous time setting the asset price process is a martingale under the equivalent measure. The martingale is defined through conditional expectation on some informational filtration. Typically this information set will include the entire past path of the asset. Hence, assuring arbitrage free asset prices in an equity market model includes the specification of all possible paths of the stock process and of the European options. One way to simplify the problem has been to allow only for a restricted set of trading strategies, as in [Carr et al. \[2003\]](#). Carr et al. introduces the simplified approach of requiring no *static arbitrage*. The term static in this case refers to that the position taken in the underlying at any time $t > 0$ only depends on time t and the current asset price S_t , not on past prices or path properties. That is, static arbitrage implies arbitrage by trading in one pricing surface. A *dynamic arbitrage* on the other hand, is an arbitrage that requires trading instruments in the future.

The conditions of no *dynamic*- and *static arbitrage* turn out to be analogous to the *drift restriction* and *short rate (or spot) specification* of the classical Heath-Jarrow-Morton theory, ([Heath et al. \[1992\]](#)).

That is, absence of *dynamic arbitrage* will correspond to a drift restriction for the joint dynamics of the derivative prices and the stock price process. Absence of *static arbitrage* implies certain conditions on the derivative prices, or on the "price surface", at each time t . We denote by $\tilde{C}_t(\tau, K)$ the price at time t of a European call option¹ having time to maturity $\tau := T - t$ and strike price K . For absence of *static arbitrage* the price surface $\{\tilde{C}_t(\tau, K)\}_{\tau, K}$ has to fulfill the following at each time t :

$$\begin{aligned}
& \blacktriangleright \quad \{\tilde{C}_t(\tau, K)\}_{\tau, K}, \text{ is increasing in } \tau \\
& \blacktriangleright \quad \{\tilde{C}_t(\tau, K)\}_{\tau, K}, \text{ is nonincreasing and convex in } K \\
& \blacktriangleright \quad \lim_{K \nearrow \infty} \tilde{C}_t(\tau, K) = 0 \\
& \blacktriangleright \quad \lim_{K \searrow 0} \tilde{C}_t(\tau, K) = S_t
\end{aligned} \tag{1.2}$$

see for example [Carr and Madan \[2005\]](#) and the references therein.

The problem of finding no arbitrage restrictions in an implied volatility market model is solved by [Schönbucher \[1999\]](#). This paper handles the case of call options having one fixed strike price K and multiple maturities $\tau > 0$. [Schweizer and Wissel \[2007\]](#) follows the line of Schönbucher but introduces as a counterpart to the *forward rates* in the fixed income market, the *forward implied volatilities*. This somewhat simplifies the conditions needed to assure that no arbitrage is

¹ A European call option with maturity date T and strike price K on an underlying asset S , gives the right to buy a unit of S at time T at the price K .

introduced. Schweizer and Wissel [2008] handles the case of one fix time to maturity and multiple strikes. Derman and Kani [1998] derive no arbitrage conditions in a dynamic local volatility model. Carmona and Nadtochiy [2008] further develop this work and provide explicit formulae, proofs and numerical examples. As a last example, Carmona and Nadtochiy [2009] propose a market model by modeling the stock price as a pure jump martingale. In this approach a time inhomogeneous Lévy density captures the information given by the market prices. No arbitrage restrictions are derived that make the two processes depend on each other.

Market code-books

The information given by the market prices of European options is commonly translated into terms of implied volatilities. For the definition of implied (Black-Scholes) volatility we first need the theoretical price of a European call option in the Black-Scholes model (1.1).

We denote by $C_t(T, K)$ the price at time t of a European call option on the underlying S , having maturity time $T > t$ and strike price K . The theoretical value of a derivative is its discounted expected future value under the martingale measure $\mathbb{Q}$. For the call option, having payoff function $u(S_t) := (S_t - K)^+ = \max\{S_t - K, 0\}$ we have,

$$C_t(T, K) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ | \mathcal{F}_t]. \quad (1.3)$$

$r \in \mathbb{R}^+$ is the interest rate,² $\mathbb{Q}$ is an $\mathbb{P}$ equivalent martingale measure and $\mathcal{F}_t$ is the filtration generated by S up to time t . In the Black-Scholes model, using Itô's lemma and no-arbitrage arguments,³ this gives us the following price of a European call option having maturity time T and strike price K ,

$$C_t^{BS}(T, K, \sigma) = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2). \quad (1.4)$$

Here Φ is the standard normal cumulative distribution function, $\tau := T - t$, $d_1 := \frac{\log(S_t/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$, and $d_2 := d_1 - \sigma\sqrt{\tau}$.

Definition 1.1. Given a call option quote $C_t(T, K)$, its market implied volatility is the unique number Σ such that $C_t(T, K) = C_t^{BS}(T, K, \Sigma)$. $C_t^{BS}(T, K, \Sigma)$ is the price according to the Black-Scholes model, Equation (1.4).

Since $C_t^{BS}(T, K, \Sigma)$ is an increasing function of Σ , we have a one-to-one correspondence between market prices and implied volatilities,

$$\{C_t(T, K); T > t, K > 0\} \Leftrightarrow \{\Sigma_t(T, K); T > t, K > 0\}. \quad (1.5)$$

The implied volatility Σ is used as a measure of the option's relative value. Inspired by Carmona [2005] we refer to this one-to-one mapping as the *implied volatility code-book*. By using this code-book, the no arbitrage restrictions on the call price surface given in (1.2) can be translated into properties of the implied volatility surface $\{\Sigma_t(T, K); T > t, K > 0\}$. This is the approach of

²For simplicity the interest rate is assumed to be constant.

³See the pricing PDE in Section 2.2.1.

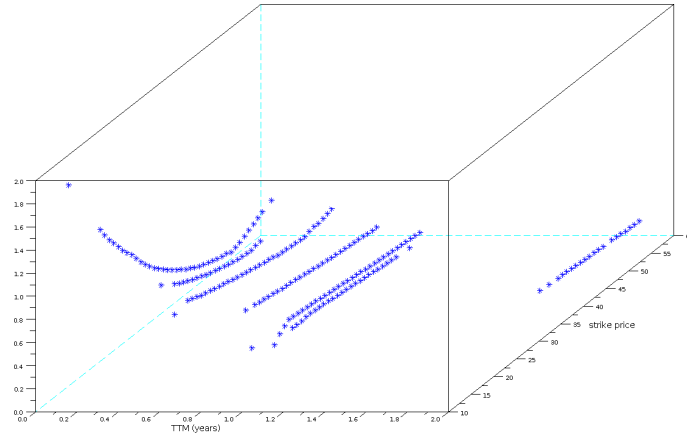


Figure 1.1: Market implied volatilities.

Schönbucher [1999] and Schweizer and Wissel [2007,2008] as mentioned above. However, inverting the Black-Scholes pricing formula leads to quite technical conditions on the surface of implied volatilities. Other code-books may be better suited to represent option prices in equity market models.

Outline

The first part of this thesis is dedicated to describing two market code-books, as alternatives to the implied volatility code-book; The *local volatility code-book* in Chapter 2 and the *Lévy density code-book* in Chapter 3.

Local volatility is a successful way to capture what is commonly referred to as the smile of the implied volatility. In contrary to the Black-Scholes assumption of constant volatility, the volatility implied by the market typically has the structure as shown in Figure 1.1.^{4,5} A local volatility model is able to capture the smile effect of the market by allowing the volatility of the stock price to depend on time and current level of stock. Under certain assumptions, to be made clear later on, we have a one-to-one correspondence between local volatility and European call/put -option prices. This one-to-one correspondence is referred to as the *local volatility code-book*.

Lévy measures (see Definition 3.4) give us another way to capture the information contained in the market prices of options. The use of Lévy processes in financial modeling is increasing in popularity. One of the reasons being that Lévy processes provides a more realistic description of the stock price dynamics since it allows for jumps or spikes of the underlying. Models which are based on a Brownian motion alone, can in most cases not reproduce the empirical distributions of asset returns. Also, it is argued that the stock price dynamics must

⁴ Implied volatilities of options on QQQQ. The QQQQ is a exchange traded fund designed to correspond generally to the performance, before fees and expenses, of the Nasdaq-100 index. The fund holds all the stocks in the Nasdaq-100 index. Source: <http://finance.yahoo.com/q?s=QQQQ>.

⁵ An exchange-traded fund, or ETF, is an investment product representing a basket of securities that track an index such as the Nasdaq-100 or Standard & Poor's 500 Index. ETFs, which are available to individual investors only through brokers and advisers, trade like stocks on an exchange. Source: http://preview.bloomberg.com/markets/etfs/etf_about.html

allow for jumps to be able to describe short time to maturity implied volatility dynamics, see [Gatheral \[2005\]](#). In the case of a pure jump Lévy process, the Lévy measure uniquely characterizes the process. The Lévy density of a pure jump Lévy process calibrated to the market will then provide us with another market code-book, the *Lévy density code-book*.

In the second part of this thesis, equity market models built on the two code-books mentioned above are discussed. We wish to find out, which code-book is better suited to use in an equity market model, from a practical point of view. Certain emphasis is put on consistency (no arbitrage restrictions) in the two different approaches of equity market modeling.

In [Chapter 4](#) we discuss the approach of using the local volatility code-book in a dynamic setting. We follow [Carmona and Nadtochiy \[2008\]](#). In this case the market is modeled by stochastic differential equations for the stock together with the local volatility surface.

In [Chapter 5](#) the approach of setting the Lévy density code-book in motion is discussed. In this case, the stock is modeled by a pure jump martingale and the evolution of call option prices is given by a dynamic time-inhomogeneous Lévy density. The time-inhomogeneous Lévy density evolves according to a stochastic differential. This type of models, the *Tangent Lévy market models*, are due to [Carmona and Nadtochiy \[2009\]](#).

The third and last part of this thesis is dedicated to the concrete implementation of an equity market model. More precisely, we implement a simplified example of a Tangent Lévy market model, ([Chapter 6](#)). Our specific choice of model parameters is motivated by a calibration to market data, à la "layman type", see [Ortega, Pullirsch, Teichmann, and Wergieluk \[2009\]](#). We calculate the Tangent Lévy consistency conditions in our simplified case, and approximate the market model in an Euler scheme. We finally check that resulting call option surfaces are indeed free of arbitrage. [Chapter 7](#) discusses the results and concludes.

Part I

Market code-books

2 The local volatility code-book

In this Section we discuss the theory of local volatility models. This leads us to the concept of the *local volatility code-book*. This code-book can be used as an alternative to the implied volatility code-book, see Definition 1.1.

Furthermore, the inverse problem of calibrating local volatility to market data is discussed and a review on different methods of calibration is given. We start however with reviewing some methods of solving one dimensional stochastic differential equations, since this type of equations plays a big part in financial modeling, and in local volatility modeling in particular. As a reference on this topic we recommend Øksendal [2006].

2.1 1-dim stochastic differential equations

2.1.1 The Itô formula

In this section we present the 1-dimensional Itô formula, as it is our main tool when we want to solve 1-dimensional stochastic differential equations. The Itô formula gives us a rule how to “differentiate” an expression of the form $g(t, X_t)$, when g is a differentiable function and X_t a Itô process, see Definition 2.1 below.

We start by introducing a family of stochastic integrals, named *Itô processes*.

Definition 2.1. Let B_t be a 1-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A 1-dimensional *Itô process* is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ with the following form,

$$X_t = X_0 + \int_0^t u(s, \omega) ds + \int_0^t v(s, \omega) dB_s. \quad (2.1)$$

Here v is in the special class of processes $\mathcal{W}_{\mathcal{H}}$, see Definition 2.2 below. Also, it is assumed that u is $\mathcal{H}_t$ -adapted, where $\mathcal{H}_t$ is as in Definition 2.2 below. Furthermore,

$$P\left[\int_0^t u(s, \omega)^2 ds < \infty\right] = 1, \quad \forall t \geq 0. \quad (2.2)$$

In short we write (2.1) in the following way,

$$dX_t = udt + vdB_t. \quad (2.3)$$

Definition 2.2. $\mathcal{W}_{\mathcal{H}}(S, T)$, $0 \leq S < T$, is the class of processes $f(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathbb{R}$, such that

- (i) $(t, \omega) \rightarrow f(t, \omega)$ is $\mathcal{B} \otimes \mathcal{F}$ -measurable. $\mathcal{B}$ denotes the Borel σ -algebra.
- (ii) There exists an increasing family of σ -algebras $\mathcal{H}_t$, $t \geq 0$, such that B_t is a martingale with respect to $\mathcal{H}_t$ and $f(t, \omega)$ is $\mathcal{H}_t$ adapted.

$$(iii) \quad P\left[\int_S^T f(s, \omega)^2 ds < \infty\right] = 1.$$

Furthermore, $\mathcal{W}_{\mathcal{H}} := \bigcap_{T>0} \mathcal{W}_{\mathcal{H}}(0, T)$.

Remark 2.3. Note that having $f \in \mathcal{W}_{\mathcal{H}}$, one can show that for each t , there exists a sequence of step functions $\{\phi_n\} \in \mathcal{W}_{\mathcal{H}}$,⁶ such that $\int_0^T |\phi_n - f|^2 ds \rightarrow 0$, in probability as $n \rightarrow \infty$. For the sequence $\{\phi_n\}$ one has that $\int_0^T \phi_n(s, \omega) dB_s(\omega)$ converges in probability to a random variable.⁷ The limit does only depend on f and not on the particular choice of $\{\phi_n\}$. Therefore, we may define

$$\int_0^t f(s, \omega) dB_s(\omega) = \lim_{n \rightarrow \infty} \int_0^t \phi_n(s, \omega) dB_s(\omega), \quad (2.5)$$

for $f \in \mathcal{W}_{\mathcal{H}}$. The limit is in probability. Also, there exists a t -continuous version of this integral. See [Öksendal \[2006\]](#) Chapter 3 for details.

Having the definitions above, we can state the 1-dimensional Itô formula as follows.

Theorem 2.4. Let $g(t, x) \in C^{1,2}([0, \infty) \times \mathbb{R})$ and X_t be an Itô process given by

$$dX_t = udt + vdB_t. \quad (2.6)$$

Then $Y_t := g(t, X_t)$ is again an Itô process and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2. \quad (2.7)$$

Here, $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed as follows,

$$\begin{aligned} dt \cdot dt &= dt \cdot dB_t = dB_t \cdot dt = 0, \\ dB_t \cdot dB_t &= dt. \end{aligned} \quad (2.8)$$

Proof. We sketch the proof of the Itô formula, following [Öksendal \[2006\]](#) Theorem 4.1.2. See also [Shreve \[2004\]](#), proof of Theorem 4.4.6. Start by substituting $dX_t = udt + vdB_t$ into (2.7), taking use of the relations in (2.8);

$$\begin{aligned} g(t, X_t) &= g(0, X_0) + \int_0^t \left(\frac{\partial g}{\partial s}(s, X_s) + u(s, \omega) \frac{\partial g}{\partial x}(s, X_s) + \frac{1}{2} v(s, \omega)^2 \frac{\partial^2 g}{\partial x^2}(s, X_s) \right) ds \\ &\quad + \int_0^t v(s, \omega) \frac{\partial g}{\partial x}(s, X_s) dB_s. \end{aligned} \quad (2.9)$$

⁶ The step function is defined as $\phi_n = \phi_n(t, \omega) = \sum_j f(t_j^{(n)}, \omega) \mathbf{1}_{[t_j^{(n)}, t_{j+1}^{(n)})}(t)$. Here

$$t_j^{(n)} := \begin{cases} j \cdot 2^{-n}, & \text{if } 0 \leq j \cdot 2^{-n} \leq T \\ 0, & \text{if } j \cdot 2^{-n} < 0 \\ T, & \text{if } j \cdot 2^{-n} > T. \end{cases}$$

⁷ The integral of the elementary function is defined as

$$\int_0^T \phi(t, \omega) dB_t := \sum_{0 \leq t_j \leq t_{j+1} \leq T} f(t_j, \omega) [B_{t_{j+1}} - B_{t_j}](\omega). \quad (2.4)$$

We note that (2.9) is a Itô process as in Definition 2.1.

Now, assume that $g, \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial x^2}$ and $\frac{\partial g}{\partial t}$ are bounded. Furthermore, we may assume that $u(t, \omega)$ and $v(t, \omega)$ are elementary functions. That is, we can write u and v on the form

$$u(t, \omega) = \sum_j e_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t), \quad (2.10)$$

where the function e_j is $\mathcal{F}_{t_j}$ -measurable. That we can make this assumption follows from Remark 2.3 above.

Now, fix $t > 0$ and let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, t]$, that is $0 = t_0 < t_1 < \dots < t_n = t$. The difference between $g(t, X_t)$ and $g(0, X_0)$ can be written as the sum of the changes in $g(t, X_t)$ over each subinterval $[t_j, t_{j+1}]$. Using this and thereafter writing out the Taylor series expansion of $g(t, X_t)$ with respect to both arguments, we get,

$$\begin{aligned} g(t, X_t) &= g(0, X_0) + \sum_j g(t_{j+1}, X_{j+1}) - g(t_j, X_j) \\ &= g(0, X_0) + \sum_j \frac{\partial g}{\partial t}(t_j, X_{t_j})(t_{j+1} - t_j) + \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})(X_{t_{j+1}} - X_{t_j}) \\ &\quad + \frac{1}{2} \sum_j \frac{\partial^2 g}{\partial t^2}(t_j, X_{t_j})(t_{j+1} - t_j)^2 + \sum_j \frac{\partial^2 g}{\partial t \partial x}(t_j, X_{t_j})(t_{j+1} - t_j)(X_{t_{j+1}} - X_{t_j}) \\ &\quad + \frac{1}{2} \sum_j \frac{\partial^2 g}{\partial x^2}(t_j, X_{t_j})(X_{t_{j+1}} - X_{t_j})^2 + \sum_j R_j, \end{aligned} \quad (2.11)$$

where $R_j = O(|t_{j+1} - t_j|^2 + |X_{t_{j+1}} - X_{t_j}|^2)$, $\forall j$.

Next we let $\|\Pi\| \rightarrow 0$, i.e $\max_j |t_{j+1} - t_j| \rightarrow 0$. We have

$$\begin{aligned} \lim_{\|\Pi\| \rightarrow 0} \sum_j \frac{\partial g}{\partial t}(t_j, X_{t_j})(t_{j+1} - t_j) &= \int_0^t \frac{\partial g}{\partial s}(s, X_s) ds, \\ \lim_{\|\Pi\| \rightarrow 0} \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})(X_{t_{j+1}} - X_{t_j}) &= \int_0^t \frac{\partial g}{\partial x}(s, X_s) dX_s. \end{aligned} \quad (2.12)$$

We have that the following two terms of RHS (2.11) tends to 0 as $\|\Pi\| \rightarrow 0$. For instance,

$$\begin{aligned} \lim_{\|\Pi\| \rightarrow 0} \left| \sum_j \frac{\partial^2 g}{\partial t^2}(t_j, X_{t_j})(t_{j+1} - t_j)^2 \right| &\leq \lim_{\|\Pi\| \rightarrow 0} \sum_j \left| \frac{\partial^2 g}{\partial t^2}(t_j, X_{t_j}) \right| \cdot (t_{j+1} - t_j)^2 \\ &\leq \lim_{\|\Pi\| \rightarrow 0} \left[\max_{0 \leq j \leq n-1} (t_{j+1} - t_j) \right] \cdot \lim_{\|\Pi\| \rightarrow 0} \sum_j \left| \frac{\partial^2 g}{\partial t^2}(t_j, X_{t_j}) \right| (t_{j+1} - t_j) \\ &= 0 \cdot \int_0^t \frac{\partial^2 g}{\partial s^2}(s, X_s) ds = 0. \end{aligned} \quad (2.13)$$

In a similar way,

$$\begin{aligned}
& \lim_{\|\Pi\| \rightarrow 0} \left| \sum_j \frac{\partial^2 g}{\partial t \partial x}(t_j, X_{t_j})(t_{j+1} - t_j)(X_{t_{j+1}} - X_{t_j}) \right| \\
& \leq \lim_{\|\Pi\| \rightarrow 0} \sum_j \left| \frac{\partial^2 g}{\partial t \partial x}(t_j, X_{t_j}) \right| \cdot (t_{j+1} - t_j) \cdot |X_{t_{j+1}} - X_{t_j}| \\
& \leq \lim_{\|\Pi\| \rightarrow 0} \left[\max_{0 \leq j \leq n-1} |X_{t_{j+1}} - X_{t_j}| \right] \cdot \lim_{\|\Pi\| \rightarrow 0} \sum_j \left| \frac{\partial^2 g}{\partial t \partial x}(t_j, X_{t_j}) \right| (t_{j+1} - t_j) \\
& = 0 \cdot \int_0^t \frac{\partial^2 g}{\partial s \partial x}(s, X_s) ds = 0.
\end{aligned} \tag{2.14}$$

Now, using that u and v are elementary, for the second last term of (2.11) we get,

$$\begin{aligned}
& \sum_j \frac{\partial^2 g}{\partial x^2}(X_{t_{j+1}} - X_{t_j})^2 = \sum_j \frac{\partial^2 g}{\partial x^2} u(t_j, \omega)^2 (t_{j+1} - t_j)^2 \\
& + 2 \sum_j \frac{\partial^2 g}{\partial x^2} u(t_j, \omega) v(t_j, \omega) (t_{j+1} - t_j) (B_{t_{j+1}} - B_{t_j}) + \sum_j \frac{\partial^2 g}{\partial x^2} v(t_j, \omega)^2 (B_{t_{j+1}} - B_{t_j})^2.
\end{aligned} \tag{2.15}$$

Where the partial derivatives are, as above, evaluated at the points (t_j, X_{t_j}) . The first two terms of (2.15) goes to 0 as $\|\Pi\| \rightarrow 0$. This can be proven in a similar way as in Equations (2.13) and (2.14) above. For the last term, put $a(t) = \frac{\partial^2 g}{\partial x^2}(t, X_t) v^2(t, \omega)$. Consider

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_j a(t_j) (B_{t_{j+1}} - B_{t_j})^2 - \sum_j a(t_j) (t_{j+1} - t_j) \right)^2 \right] \\
& = \sum_{i,j} \mathbb{E} [a(t_i) a(t_j) ((B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)) ((B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j))]
\end{aligned} \tag{2.16}$$

In the case $i < j$ we have, $a(t_i) a(t_j) ((B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i))$ and $((B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j))$ are independent. This gives us that the terms in (2.16) vanishes. The same holds when $j < i$.

In the case $i = j$ we have,

$$\begin{aligned}
(2.16) & = \sum_j \mathbb{E} [a(t_j)^2 ((B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j))^2] \\
& = \sum_j \mathbb{E} [a(t_j)^2] \mathbb{E} [(B_{t_{j+1}} - B_{t_j})^4 - 2(B_{t_{j+1}} - B_{t_j})^2 (t_{j+1} - t_j) + (t_{j+1} - t_j)^2] \\
& = \sum_j \mathbb{E} [a(t_j)^2] (3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2) \\
& = 2 \sum_j \mathbb{E} [a(t_j)^2] (t_{j+1} - t_j)^2 \rightarrow 0, \text{ as } |t_{j+1} - t_j| \rightarrow 0.
\end{aligned} \tag{2.17}$$

That is, we have that

$$\sum_j \frac{\partial^2 g}{\partial x^2}(t_j, X_{t_j}) v^2(t_j, \omega) (B_{t_{j+1}} - B_{t_j})^2 \rightarrow \int_0^t \frac{\partial^2 g}{\partial x^2}(s, X_s) v^2(s, \omega) ds, \quad (2.18)$$

in $L^2(P)$ as $\|\Pi\| \rightarrow 0$. Equation (2.18) is often written in the following way,

$$(dB_t)^2 = dt. \quad (2.19)$$

Using the same arguments will give us that also $\sum_j R_j \rightarrow 0$, as $\|\Pi\| \rightarrow 0$. That is, Equation (2.11) $\rightarrow$ (2.7) in $L^2(P)$ as $\|\Pi\| \rightarrow 0$.

At last we can prove the general case, $g, \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial x^2}$ and $\frac{\partial g}{\partial t}$ unbounded, by approximating g with functions $g_n \in C^{1,2}([0, \infty) \times \mathbb{R})$ such that $g_n, \frac{\partial g_n}{\partial x}, \frac{\partial^2 g_n}{\partial x^2}$ and $\frac{\partial g_n}{\partial t}$ are bounded for each n , and converge uniformly to $g, \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial x^2}$ and $\frac{\partial g}{\partial t}$ on compact subsets of $[0, \infty) \times \mathbb{R}$. See Øksendal [2006] Exercise 4.9. $\square$

2.1.2 Solution methods

In this section we present some solution methods for one dimensional stochastic differential equations. That is, we want to solve equations of the following type,

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad (2.20)$$

where B_t is a standard Brownian motion and $b, \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions. We illustrate a first solution method with an example.

Example 2.1.1. Assume, as in the classical Black-Scholes model, that we have stock price dynamics given by the following equation,

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 \geq 0, \quad (2.21)$$

where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. We have that S_t is an Itô process. We apply Itô's formula (2.7) to the function $g(x) = \log(x)$,

$$\begin{aligned} d(\log(S_t)) &= \frac{1}{S_t} dS_t + \frac{1}{2} \left(-\frac{1}{S_t^2} \right) (dS_t)^2 = \mu dt + \sigma dB_t - \frac{1}{2S_t^2} \sigma^2 S_t^2 dt \\ &= \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dB_t. \end{aligned}$$

That is, $\log(S_t) - \log(S_0) = (\mu - \frac{1}{2} \sigma^2)t + \sigma B_t$, or,

$$S_t = S_0 \exp\left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t \right\}.$$

Another approach to solve equations of the form (2.20), is to try the ansatz that the solution X_t is of the following form;

$$X_t = f(t) \left\{ X_0 + \int_0^t g(s) dB_s \right\}. \quad (2.22)$$

Here f, g are continuous functions. I.e. the solution is a non-centered Gaussian process. Applying the stochastic product rule (or the Itô product rule, see for instance [Öksendal \[2006\]](#), Exercise 4.3), to (2.22) gives us,

$$\begin{aligned} dX_t &= f'(t)\{X_0 + \int_0^t g(s)dB_s\}dt + f(t)g(t)dB_t \\ &= \frac{f'(t)}{f(t)}X_t dt + f(t)g(t)dB_t. \end{aligned} \quad (2.23)$$

Again we illustrate with an example,

Example 2.1.2. Assume that we have the following Ornstein Uhlenbeck type of process,

$$dX_t = -\alpha X_t dt + \sigma dB_t. \quad (2.24)$$

Using the ansatz in (2.22) and Equation (2.23) we have,

$$\begin{aligned} \frac{f'(t)}{f(t)} &= -\alpha \Rightarrow f(t) = e^{-\alpha t}, \text{ and} \\ f(t)g(t) &= e^{-\alpha t}g(t) = \sigma. \end{aligned}$$

This gives us the solution,

$$X_t = e^{-\alpha t}X_0 + \sigma \int_0^t e^{\alpha(s-t)}dB_s \quad (2.25)$$

Another way to reach the solution of equations of the form (2.24), is to directly multiply with the "integrating factor" $e^{\alpha t}$ and then to compare with $d(e^{\alpha t}X_t)$,

$$\exp(\alpha t)dX_t = -\exp(\alpha t)\alpha X_t dt + \exp(\alpha t)\sigma dB_t. \quad (2.26)$$

Applying Itô formula to $g(t, x) = \exp(\alpha t)x$,

$$d(\exp(\alpha t)X_t) = \exp(\alpha t)\alpha X_t dt + \exp(\alpha t)dX_t.$$

But we also have from equation (2.26),

$$\exp(\alpha t)dX_t + \exp(\alpha t)\alpha X_t dt = \exp(\alpha t)\sigma dB_t.$$

This gives us,

$$d(\exp(\alpha t)X_t) = \exp(\alpha t)\sigma dB_t,$$

which also gives us the solution (2.25) above.

As a last and more general example, we can use a similar technique to solve equations of the form

$$dX_t = \alpha(t, \omega)X_t dt + \beta(t, \omega)X_t dB_t. \quad (2.27)$$

Here $\alpha(t, \omega), \beta(t, \omega) \in \mathcal{W}$, as in Definition 2.2. To solve Equation (2.27), we start by defining the following integrating factor,

$$F_t(\omega) := \exp\left(-\int_0^t \beta(s, \omega)dB_s + \frac{1}{2}\int_0^t \beta^2(s, \omega)ds\right). \quad (2.28)$$

Applying Itô's formula to $g(t, x) := F_t x$, leads to,

$$d(F_t X_t) = F_t \alpha(t, \omega) X_t dt. \quad (2.29)$$

Next define, $Y_t(\omega) := F_t(\omega) X_t(\omega)$. That is,

$$X_t = F_t^{-1} Y_t. \quad (2.30)$$

We have that Equation (2.29) can be written as,

$$\frac{dY_t}{dt} = \alpha(t, \omega) Y_t. \quad (2.31)$$

This is a deterministic differential equation for $t \rightarrow Y_t(\omega)$, for each $\omega \in \Omega$. Solving (2.31) gives us $Y_t = \exp(\int_0^t \alpha(s, \omega) ds)$. Now (2.30) gives us,

$$X_t = X_0 \exp\left(\int_0^t \beta(s, \omega) dB_s + \int_0^t \left(\alpha(s, \omega) - \frac{1}{2}\beta^2(s, \omega)\right) ds\right). \quad (2.32)$$

We refer to [Øksendal \[2006\]](#), Chapter 5, for more examples and solution methods.

2.2 Local volatility models

In a local volatility model the price of a financial asset is modeled as a diffusion process $S = (S_t)_{t \geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with dynamics of the form (2.27). More precisely, S_t is assumed to follow the following stochastic differential equation

$$dS_t = \mu(t, S_t) S_t dt + a(t, S_t) S_t dW_t. \quad (2.33)$$

Here W_t is a standard Brownian motion and $\mu : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$, $a : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are deterministic functions. $\mu(t, S_t)$ represents instantaneous drift and $a(t, S_t)$ the instantaneous volatility. The latter is commonly referred to as a *local volatility function*. The concept of local volatility was introduced by [Dupire \[1993\]](#) and by [Derman and Kani \[1994\]](#).

As in the introductory Section 1, we denote by $C_t(T, K)$ the price at time t of a European call option on the underlying S , having maturity time T and strike price K . It should however be clear that this price depends also on the level of the underlying asset price S_t . We assume certain regularity conditions of a and μ , as well as liquidity and absence of arbitrage and transaction costs. Furthermore, since S_t is a Markov process, the price process $C_t(T, K)$ can be represented in the following way,

$$C_t(T, K) := G(t, S_t, T, K). \quad (2.34)$$

2.2.1 Modified Black-Scholes pricing PDE

We would like to derive a valuation equation when the underlying is given by (2.33). By these means, consider T and K fix, apply Itô's formula (2.7) to the function $G(t, S_t, T, K)$, and substitute with Equation (2.33). Setting $S_t = x$, this gives us,

$$dG = \left(\frac{\partial G}{\partial t} + \frac{1}{2} a^2(t, x) x^2 \frac{\partial^2 G}{\partial x^2} + \mu(t, x) x \frac{\partial G}{\partial x} \right) dt + \frac{\partial G}{\partial x} \cdot a(t, x) x dW_t. \quad (2.35)$$

Next we consider a portfolio consisting of a single call option and $-\frac{\partial G}{\partial x}$ number of shares, i.e. the "delta-hedge" portfolio;

$$\Pi = G - x \frac{\partial G}{\partial x}. \quad (2.36)$$

Denote by R the total profit or loss from this portfolio. Assuming that the underlying stock S pays no dividends, we have,

$$dR = dG - \frac{\partial G}{\partial x} dS_t. \quad (2.37)$$

Substituting (2.33) and (2.35) in Equation (2.37) gives,

$$dR = \left(\frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} a^2(t, x) x^2 \right) dt. \quad (2.38)$$

We note that this equation contains no dW_t term, that is, it contains no risk. We now follow the reasoning of [Black and Scholes \[1973\]](#), and conclude that the mean rate of return of the portfolio Π , must be equal to the rate of return on any other risk-free instrument. If not, there would be opportunities of arbitrage. Assuming the risk free interest rate to be constant and equal to r , we have,

$$r\Pi dt = \left(\frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} a^2(t, x) x^2 \right) dt. \quad (2.39)$$

At last we substitute Equation (2.36) in (2.39), and retrieve,

$$r(G - x \frac{\partial G}{\partial x}) dt = \left(\frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} a^2(t, x) x^2 \right) dt. \quad (2.40)$$

We have that $G(t, S_t, T, K)$ satisfies the following backward partial differential equation:

$$\begin{cases} \frac{\partial G}{\partial t} + \frac{1}{2} a^2(t, x) x^2 \frac{\partial^2 G}{\partial x^2} + r(x \frac{\partial G}{\partial x} - G) = 0, & t < T \\ G(T, S_T, T, K) = (S_T - K)^+. \end{cases} \quad (2.41)$$

2.2.2 Dupire/Backing the LV out of the option prices

Alternatively, under the same assumptions as in the previous section (now keeping (t, x) fix), the call option price $G(t, S_t, T, K)$ satisfies the following forward equation in the variables (T, K) , [Dupire \[1993\]](#):

$$\begin{cases} \frac{\partial G}{\partial T} - \frac{1}{2} a^2(T, K) K^2 \frac{\partial^2 G}{\partial K^2} + rK \frac{\partial G}{\partial K} = 0, & T > t \\ G(t, S_t, T, K) = (x - K)^+. \end{cases} \quad (2.42)$$

To derive this equation we start by noting that the risk neutral call option value $G(t, S_t, T, K)$, may be calculated as the following integral,

$$G(t, S_t, T, K) = e^{-r(T-t)} \int_K^\infty (y - K) \varphi(y, T; S_t) dy. \quad (2.43)$$

Here φ is the risk neutral probability density function of the final spot S_T , given the current value S_t . Differentiating (2.43) with respect to K gives us,

$$\frac{\partial G}{\partial K} = -e^{-r(t-T)} \int_K^\infty \varphi(y, T; S_t) dy, \quad \frac{\partial^2 G}{\partial K^2} = e^{-r(t-T)} \varphi(K, T; S_t). \quad (2.44)$$

Furthermore, given the stochastic differential equation (2.33), the risk neutral probability density function $\varphi(y, T; S_t)$ satisfies the Fokker-Planck equation,

$$\frac{\partial \varphi}{\partial T} = \frac{1}{2} \frac{\partial^2}{\partial y^2} (a^2(y, t) y^2 \varphi(y, T; S_t)) - \frac{\partial}{\partial y} (ry \varphi(y, T; S_t)).$$

Risk neutrality gives us the drift $\mu(t, S_t) = r$. Now, using the second equation of (2.44) we get,

$$\begin{aligned} \frac{\partial}{\partial T} e^{r(T-t)} \frac{\partial^2 G(t, S_t, T, y)}{\partial y^2} &= \frac{1}{2} \frac{\partial^2}{\partial y^2} \left(a^2(y, t) y^2 e^{r(T-t)} \frac{\partial^2 G(t, S_t, T, y)}{\partial y^2} \right) \\ &\quad - \frac{\partial}{\partial y} (ry e^{r(T-t)} \frac{\partial^2 G(t, S_t, T, y)}{\partial y^2}) \end{aligned}$$

Now integrating over y twice from K to ∞ , under the following assumptions,

$$G(t, S_t, T, K), K \frac{\partial G}{\partial K}, K^2 \frac{\partial^2 G}{\partial K^2} \rightarrow 0, \text{ as } K \rightarrow \infty, \quad (2.45)$$

leads to,

$$\frac{\partial G}{\partial T} = \frac{1}{2} a^2(K, t) K^2 \frac{\partial^2 G}{\partial K^2} - rK \frac{\partial G}{\partial K}.$$

2.2.3 Solving the pricing PDE

The pricing Equation (2.41) is a parabolic partial differential equation. To simplify this PDE we start with a change of variable, $\tau := T - t$. We get,

$$\frac{\partial \hat{G}}{\partial \tau} = \frac{1}{2} \hat{a}^2(\tau, x) x^2 \frac{\partial^2 \hat{G}}{\partial x^2} + r(x \frac{\partial \hat{G}}{\partial x} - \hat{G}). \quad (2.46)$$

We use the notation $\hat{G}(\tau, x, T, K) := G(t, x, T, K)$, $\hat{a}(\tau, x) := a(t, x)$. Again we have set $S_t = x$ and consider T, K fix. Also, we make the substitution $\hat{u}(\tau, x) := e^{r\tau} \hat{G}(\tau, x, T, K)$. Substitution in (2.46) gives us,

$$\frac{\partial \hat{u}}{\partial \tau} = \frac{1}{2} \hat{a}^2(\tau, x) x^2 \frac{\partial^2 \hat{u}}{\partial x^2} + rx \frac{\partial \hat{u}}{\partial x}. \quad (2.47)$$

In this new notation, we have the following initial value,

$$\hat{u}(0, x) = (x - K)^+. \quad (2.48)$$

Some particular forms of the local volatility function $\hat{a}(\tau, x)$, will make it possible to derive an analytical solution to the pricing Equation (2.47). For instance,

assuming a constant volatility will reduce the model to the standard Black-Scholes model. The PDE (2.47) can then be transformed into a form of the heat equation with standard solution.

As another example, we can drop the dependence of local volatility on the stock level and assume a stock price process driven by the following stochastic differential equation,

$$dS_t = rS_t dt + \sigma(t)S_t dW_t. \quad (2.49)$$

Here $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, is a bounded and continuous function. We now use *Kolmogorov's Backward equation*, see for instance [Öksendal \[2006\]](#). That is we use the following theorem.

Theorem 2.5. Define $w(t, x) = \mathbb{E}^x[f(X_t)]$, where $\mathbb{E}^x$ is the expectation w.r.t. the natural probability law for X_t starting at x . Then

$$\begin{aligned} \frac{\partial w}{\partial t} &= Aw, \quad t > 0, x \in \mathbb{R} \\ w(0, x) &= f(x), \quad x \in \mathbb{R}. \end{aligned} \quad (2.50)$$

A is the generator of X applied to the function $x \rightarrow w(t, x)$, see [Remark 2.6](#).

Proof. See for instance [Öksendal \[2006\]](#) proof of Theorem 8.1.1. $\square$

Remark 2.6. Recall that the generator A , of a Itô diffusion $X_t \in \mathbb{R}^n$ is defined by,

$$Af(x) := \lim_{t \downarrow 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}, \quad x \in \mathbb{R}^n. \quad (2.51)$$

For an Itô diffusion $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ and $f \in C_0^2(\mathbb{R})$ we have,

$$Af(x) = \sum_j b_j(x) \frac{\partial f}{\partial x_j} + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}. \quad (2.52)$$

See [Öksendal \[2006\]](#) Theorem 7.3.3.

For the 1-dimensional process S , defined by (2.49), we have

$$A\hat{u}(\tau, x) = rx \frac{\partial \hat{u}}{\partial x} + \frac{1}{2} \hat{\sigma}^2(\tau) x^2 \frac{\partial^2 \hat{u}}{\partial x^2}. \quad (2.53)$$

Here we use the notation, $\hat{\sigma}(\tau) := \sigma(t)$. Substitution in Equation (2.47), and using Kolmogorov's Theorem, we get

$$\hat{u}(\tau, x) = \mathbb{E}^x[f(S_\tau)], \quad (2.54)$$

where $f(x) = (x - K)^+$, given by (2.48). Now, from Equation (2.32) we know the solution to (2.49),

$$S_t = S_0 \exp\left(\int_0^t \sigma(s) dB_s + \int_0^t \left(r - \frac{1}{2} \sigma^2(s)\right) ds\right). \quad (2.55)$$

Substitution in (2.54) gives,

$$\hat{u}(\tau, x) = \mathbb{E}\left[f\left(x \cdot \exp\left(\int_0^\tau \sigma(s) dB_s + \int_0^\tau \left(r - \frac{1}{2} \sigma^2(s)\right) ds\right)\right)\right]. \quad (2.56)$$

We have that the random variable $\int_0^\tau \sigma(s)dB_s$ is normal distributed with mean 0 and variance $\delta := \int_0^\tau \sigma^2(s)ds$. That is we can calculate the above expectation as,

$$\hat{u}(\tau, x) = \frac{1}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} \left(x \cdot \exp\left(y + \int_0^\tau \left(r - \frac{1}{2}\sigma^2(s)\right)ds\right) - K \right)^+ \exp\left(-\frac{y^2}{2\delta^2}\right) dy \quad (2.57)$$

Or, in the original notation,

$$G(t, S_t, T, K) = \frac{e^{-r(T-t)}}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} \left(S_t \cdot \exp\left(y + \int_0^{T-t} \left(r - \frac{1}{2}\sigma^2(s)\right)ds\right) - K \right)^+ \exp\left(-\frac{y^2}{2\delta^2}\right) dy. \quad (2.58)$$

2.3 Recovering the local volatility surface

For practitioners the pricing PDE (2.42) might seem more interesting then (2.41), since the derivatives with respect to maturity time T and strike price K may be estimated directly from the market. Therefore it is possible to calculate “spot estimates” of the local volatility $a(T, K)$ from (2.42) and to estimate a deterministic local volatility function which is consistent with the observed values. More precisely, reorganizing the PDE (2.42) gives us what is known as Dupire’s formula;

$$a(T, K) = \sqrt{2 \frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{K^2 \frac{\partial^2 C}{\partial K^2}}}. \quad (2.59)$$

The equations (2.42) and (2.59) gives us a *one-to-one correspondence* between the market call option prices and the local volatility. Under the assumption that the call options corresponding to all possible strikes and maturities are priced consistently on the market, it solves what is commonly referred to as the *inverse- or calibration problem* for a local volatility model;

Problem 2.7. *Given (by the market) a set of call option prices, find a function $a : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ s.t.*

$$dS_t = rS_t dt + a(t, S_t)S_t dW_t, \quad S_0 > 0$$

and

$$C_t(T, K) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ | \mathcal{F}_t]$$

matches the given option prices. $\mathcal{F}_t$ represents the information contained in the prices of S up to time t .

The *one-to-one correspondence* between the market prices and the local volatility, with suitable regularity assumptions, implies that at each time t the surface $\{a_t(T, K); T > t, K > 0\}$ can be chosen as a code-book for the information contained in the market,

$$\{C_t(T, K); T > t, K > 0\} \rightleftharpoons \{a_t(T, K); T > t, K > 0\}.$$

Clearly, since the market prices are changing with time, the local volatility surface $\{a_t(T, K); T > t, K > 0\}$ calibrated at time t will differ from the one

calibrated at a later time s , this motivates the subscript t of $a_t(T, K)$. An important property of the local volatility code-book is that the conditions in (1.2), sufficient to assure absence of static arbitrage, is reduced to keeping the local volatility surface smooth and non negative. This is to be compared with the more complicated implied volatility cases, see for example [Schönbucher \[1999\]](#) and [Schweizer and Wissel \[2007,2008\]](#). See also Section 6.9 below.

2.3.1 The calibration problem

Due to insufficient option price data from the market, it is well known that the local volatility calibration Problem 2.7 is *ill-posed*. Recall the definition by [Hadamard \[1902\]](#), that a problem is *well posed* if there exists a solution, this solution is unique, and it depends continuously on the input data. If one or more of these properties is not fulfilled, the problem is referred to as *ill-posed*.

The calibration Problem 2.7 is the real-world problem of computing model parameters out of a (finite) set of option prices. It is the inverse of the pricing problem of calculating option prices from a model with given parameters.

In practice it typically is not possible to find one set of model parameters that are exactly consistent with all given data. Additionally data come with a bid-ask spread, which makes the problem more difficult. The common case is that there exists a set of parameter values that reproduces prices within the bid-ask spread. We have that similar option values may be calculated from completely different local volatilities and if option prices are perturbed, it is hard to say what is the correct local volatility. See for instance [Bouchouev and Isakov \[1998\]](#), [Coleman et al. \[1999\]](#), [Bodurtha and Jermakyan \[1999\]](#), [Berestycki et al. \[2002\]](#), [Crépey \[2003\]](#) or [Egger and Engl \[2005\]](#). In the early attempts to calibrate the local volatility function by Dupire's formula (2.59), difficulties arise since it is not clear how to find the first- and second order derivatives of the call prices needed.

Many methods have been proposed to deduce the local volatility surface (LVS) from market data, ranging from parametric to non-parametric ones. Below follows a brief review on different approaches. For a review with focus on implied trees, see the book of [Fengler \[2005a\]](#).

2.3.2 Interpolating and extrapolating the data

In the early attempts to solve the calibration problem, it is viewed as a problem of differentiation from discrete data, where the implied volatility surface (IVS) is interpolated and extrapolated from the finite set of data. Thereafter, from a smooth IVS all call option prices can be recovered and hence also the derivatives needed in formula (2.59). The approach was introduced by [Dupire \[1993\]](#) and developed by among others [Derman and Kani \[1994\]](#) and [Rubinstein \[1994\]](#).

Due to the ill-posedness of the problem the resulting calibrated LVS tends to be unstable and dependent on choice of interpolation procedure, [Crépey \[2003\]](#). Furthermore, taking numerical derivatives from the interpolated call option surface might further increase the inaccuracies. It is also criticized that potentially erroneous information can be introduced to the data, see for example [Coleman et al. \[1999\]](#). Also, for far in- or out of the money options, the numerator and denominator of the Dupire formula might become very small, causing numerical instability.

2.3.3 Local volatility in terms of implied volatility

Since options are frequently quoted in terms of implied volatilities, it would be convenient to have the Dupire formula (2.59) in terms of implied volatilities $\Sigma(T, K)$. We may write the European call option price as a function of implied volatility, $C_t(T, K) = C_t^{BS}(T, K, \Sigma)$. By using the chain rule, Elder [2002] derive the following formula for local volatility,

$$a(T, K) = \sqrt{\frac{\Sigma^2 + 2\tau\Sigma(\partial_T \Sigma + rK\partial_K \Sigma)}{(1 + Kd_1\sqrt{\tau}\partial_K \Sigma)^2 + \Sigma^2 K^2 \tau (\partial_{KK} \Sigma - d_1(\partial_K \Sigma)^2 \sqrt{\tau})}}. \quad (2.60)$$

As before, $\tau = T - t$, and $d_1 := \frac{\log(S_t/K) + (r + \Sigma^2/2)\tau}{\Sigma\sqrt{\tau}}$. Schmitz [2004]

show that calculating the local volatility using the above formula gives a more accurate and stable result compared with the formula (2.59). The derivation of local volatility in terms of implied volatility can also be found in Gatheral [2005], Chapter 1.

2.3.4 Non parametric methods

Non parametric methods aim at recovering the LVS directly from observed option prices. It is an inverse problem, where one wishes to compute local volatilities which are consistent with observed prices, up to the bid-ask spread.

One motivation for using non parametric methods in finance is that it is often difficult to theoretically justify a specific parameterization for the economical relationship that is being modeled, see for instance Fengler [2005a]. Also, by choosing a parametric form of the local volatility for calibration purposes the inverse problem may become under-determined. The number of parameters will in most cases be significantly smaller than the number of observations. The model might not fit the observed prices with high enough accuracy.

2.3.5 Implied tree/discrete time approaches

Valuation of derivatives by methods based on trees were introduced by Cox, Ross, and Rubinstein [1979]. Under certain regularity conditions, the tree is the discrete time approximation to the diffusion (2.33).

In an implied tree, the properties of each node are recovered from observed option data. For example Rubinstein [1994] uses backward induction to recover all nodes in a binomial tree from a risk neutral distribution at the terminal nodes (time T). In this approach the terminal distribution is chosen as close as possible in a least square sense to the distribution corresponding to a standard CRR tree (as in Cox, Ross, and Rubinstein [1979]) and such that it prices a set of options that expire at T in accordance with the market. Derman and Kani [1994] on the other hand use forward induction to deduce implied trees.

To recover the asset price and transition densities at each node in an implied tree, it is necessary that the prices of European call (or put) options for any strike and time to maturity are available. In practice the option prices has often been found by smoothening, interpolating and/or extrapolating the implied volatility curves available, Fengler [2005a]. As a result the local volatility can be deduced at each node of the implied tree.

Several implied lattice approaches have been introduced after the pioneering works mentioned above, see for example Jackwerth [1997] that generalizes the backward induction approach of Rubinstein [1994], Charalambous et al. [2007] that uses non-recombining implied trees, or Crépey [2003] that calibrates the local volatility in a trinomial tree using Tikhonov regularization in the spirit of Lagnado and Osher [1997], see below.

A disadvantage with calibration using the above approaches is that they all involve solving a large scale optimization problem. The number of variables will roughly equal the number of nodes in the tree, Avellaneda et al. [1997]. Also, Jackson et al. [1998] argue that implied lattice algorithms are difficult to apply to barrier options or multidimensional pricing problems. Furthermore, the instability resulting from interpolating or extrapolating the option data, see Section 2.3.2, is transferred to the lattice, therefore the algorithms are not robust.

2.3.6 A function approximation problem

Due to the ill-posedness of the inverse Problem 2.7 and also because of possible model misspecification, it is often not meaningful in practice to exactly match the market prices. Therefore the calibration problem has often been reformulated as an optimization problem, aiming to minimize the difference between model and market prices. Most often the minimization is done using a least square criterion. In the spirit of Coleman et al. [1999], we formulate the problem as follows:

Problem 2.8. Assume we are given N market option bid/ask-pairs $\{(\text{bid}_j, \text{ask}_j)\}_{j=1}^N$ on the same underlying asset, corresponding to strike prices and expiration times $\{(K_j, T_j)\}_{j=1}^N$. Let

$$C_j(a) := C_t(T_j, K_j; a), \quad j = 1, \dots, N$$

be the call option prices solving the valuation equation (2.41) for maturities T_j and strikes K_j using a volatility function $a(\cdot, \cdot)$.

We want to approximate the function $a : [0, T] \times [0, \infty)$ from the requirement that;

$$\text{bid}_j \leq C_j(a) \leq \text{ask}_j, \quad j = 1, \dots, N. \quad (2.61)$$

Let $\mathcal{H}$ denote the space of measurable functions in $[0, T] \times [0, \infty)$. The inverse Problem (2.61) can then be written as the least square optimization problem:

$$\min_{a \in \mathcal{H}} G(a), \quad G(a) := \sum_{j=1}^N \left(C_j(a) - \overline{C}_j^* \right)^2, \quad (2.62)$$

where $\overline{C}_j^* := \frac{\text{bid}_j + \text{ask}_j}{2}$, $j = 1, \dots, N$ represents the observed prices.

Typically this problem has an infinite number of solutions due to the finite observation data. In order to get a well-posed problem, some regularization need to be introduced. One of the most popular methods of regularization is due to mathematician Andrey Tikhonov.

Remark 2.9. Note that in most cases we will be given a set of option prices (or bid and ask offers) corresponding to strike prices $\{K_j\}$, $j = 1, \dots, N$, and maturity times $\{T_k\}$, $k = 1, \dots, M$, where $M \ll N$. The market data will be given as "strings" of option prices as in Figure 1.1.

That is we have prices $C(T_1, K_j)$, $j = 1, \dots, n_1$, $C(T_2, K_j)$, $j = 1, \dots, n_2$, etc., where $\sum_{i=1}^M n_i = N$. For simplicity we will write $C(T_j, K_j)$, where the maturity times will be given as a vector of length N of the following form, $[T_1 \cdots T_1, T_2 \cdots T_2, \dots, T_M \cdots T_M]$.

2.3.7 Optimizing using Tikhonov Regularization

Lagnado and Osher [1997] rewrites the local volatility calibration Problem 2.7 as a non-linear optimization problem that they regularize using the approach of Tikhonov. Their technique aims at minimizing the L^2 norm of the gradient of the local volatility, subject to a constraint that ensures that solutions of the pricing differential equation (2.41) matches the observed market prices. Equation (2.62) of Problem 2.8 is hence modified as follows,

$$\min_{a \in \mathcal{H}} J(a), \quad J(a) := G(a) + \lambda \|\nabla a\|_2^2.$$

Here $G(a)$ is the function as defined in Problem 2.8, equation (2.62). λ is a regularization parameter and $\|\cdot\|_2$ denotes the L^2 norm. In this way the optimization problem gains numerical stability and uniqueness by making the objective function more convex. The minimization is carried out numerically using a gradient descent procedure implemented in a finite-difference framework.⁸ This approach is for example extended by Jackson et al. [1998].

In the same spirit, Berestycki et al. [2002] proposes another formulation of the calibration problem, which they show to be well posed. Their formulation is motivated by closed-form asymptotic formulae for the implied volatility near expiry and for deep in- and out of- the money options⁹. Their method includes the following minimization problem,

$$\min_{a \in \mathcal{H}} J(a), \quad J(a) := \sum_{j=1}^N \left(\frac{1}{\Sigma_j(a)} - \frac{1}{\bar{\Sigma}_j} \right)^2 + \lambda \left\| \nabla \left(\frac{1}{a} \right) \right\|_2^2.$$

Here $a(\cdot, \cdot)$ is the local volatility function and λ is again a regularization parameter. $\Sigma_j(a)$ is the Black-Scholes implied volatility as a function of the model call option price. $\bar{\Sigma}_j$, $j = 1, \dots, N$, are implied Black-Scholes volatilities from the market. For a survey on the use of Tikhonov regularization in quantitative finance see Crépey [2008].

Even though the regularized optimization algorithms are more robust than the non regularized Problem 2.8, a disadvantage is that the regularization parameter (λ in the above approaches) has to be chosen upon implementation. This choice determines the trade-off between accuracy and regularity in the method, and may not be easy. See for example Coleman et al. [1999].

⁸ In a gradient descent algorithm, one takes steps proportional to the negative gradient of the objective function ($-\nabla J(a)$ in the current setting) at the current step of the algorithm.

⁹ For a call option in (out of) the money refers to the case when $S_t > K$ ($S_t < K$).

Furthermore, the calculation of the variational derivatives in the gradient descent minimization requires solving a non-homogeneous Black-Scholes PDE for each strike and maturity and for each point on the finite difference grid. This will make the implementation of the gradient methods computationally expensive, see [Jackson et al. \[1998\]](#), [Coleman et al. \[1999\]](#) and the original paper of [Lagnado and Osher \[1997\]](#).

Another disadvantage is pointed out by [Cont and Tankov \[2004\]](#), who argues that in the typical case the solution of Problem 2.8 is not unique. Therefore, with or without regularization the gradient based methods will at best locate one local, or global, minima of the original fitting criterion. They point out that the solution of a regularized method will say nothing about the uncertainty due to multiplicity of solutions of the original non regularized Problem 2.8.

2.3.8 Other methods

In most of the existing literature on optimization of the local volatility surface, the optimization is done in a deterministic framework.

[Cont and Ben Hamida \[2005\]](#) on the other hand propose a probabilistic approach. They outline an evolutionary stochastic algorithm for generating a family of calibrated local volatility surfaces.

In general, evolutionary algorithms involves a search from a population of solutions and not from a single point as is the case in more traditional optimization procedures. Each iteration of an evolutionary algorithm involves a competitive selection that in the end results in new solutions. Selection is done by choosing solutions which results in good fits of the optimization problem. The selected solutions are recombined with other “good” solutions by swapping parts of a solution with another. Also small changes are done to single elements of the good solutions. In this way the new solutions will be biased towards other “good” solutions, [Kalle et al. \[2001\]](#).

This optimization approach is applied by [Cont and Ben Hamida \[2005\]](#) to the local volatility calibration Problem 2.8. The authors points out that in the evolutionary approach, model uncertainty is reflected in heterogeneity of the generated family of solutions. See for references to earlier probabilistic approaches to the local volatility calibration problem in [Cont and Ben Hamida \[2005\]](#). Worth noting is that in this approach, since the evolutionary algorithm does not require differentiability, the common choice of minimizing the quadratic pricing error (2.62) is altered for instead minimizing the absolute pricing error.

[Avellaneda et al. \[1997\]](#) employs yet another approach to calibrate a local volatility model to the market. [Avellaneda et al.](#) regularize the inverse Problem 2.8 by adding a constraint which minimizes the relative entropy distance to a prior diffusion. The relative entropy (Kullback-Leibler) distance $\mathcal{E}(\mathbb{Q}|\mathbb{P})$ is a measure of distance between the two probability measures $\mathbb{Q}$ and $\mathbb{P}$. $\mathcal{E}(\mathbb{Q}|\mathbb{P})$ describes the amount of inefficiency introduced by assuming that the distribution of a random variable is $\mathbb{Q}$ when the true distribution is $\mathbb{P}$.

By adding the constraint of minimizing the relative entropy to a prior, the calibration solution will be biased toward this prior. [Avellaneda et al.](#) approximates the stock price process S_t by a trinomial tree, for which the relative entropy can be calculated as the discretization step goes to zero. A functional

form for the entropy is found. An advantage of this method is that one may include information from empirical distributions in the choice of the prior. Since this approach involves the calculation of an implied tree, the remarks in the Section 2.3.5 above applies.

2.3.9 Parametric and semiparametric methods

Another solution to the ill-posedness of the local volatility calibration problem has been to choose a functional form for the local volatility function. In this case the non linear function minimization in Problem 2.8 may be simplified.

Problem 2.10. *Assume we are given N market option bid/ask-pairs $\{(\text{bid}_j, \text{ask}_j)\}_{j=1}^N$ on the same underlying asset, corresponding to strike prices and expiration times $\{(K_j, T_j)\}_{j=1}^N$. Let*

$$C_j(\Theta) := C_t(T_j, K_j; \Theta), \quad j = 1, \dots, N$$

be the call option prices solving equation (2.41) for maturities T_j and strikes K_j using a local volatility function $a_\Theta(\cdot, \cdot)$. $\{\Theta\}$ is a vector of model parameters. We wish to approximate the parameters $\{\Theta\}$ from the requirement that,

$$\text{bid}_j \leq C_j(\Theta) \leq \text{ask}_j, \quad j = 1, \dots, N. \quad (2.63)$$

The inverse Problem (2.63) can then be written as the least square optimization problem:

$$\min_{\Theta} G(\Theta), \quad G(\Theta) := \sum_{j=1}^N \left(C_t(T_j, K_j; \Theta) - \overline{C}_j^* \right)^2, \quad (2.64)$$

where $\overline{C}_j^ := \frac{\text{bid}_j + \text{ask}_j}{2}$, $j = 1, \dots, N$ represents the observed prices.*

In particular cases the price $C_t(T_j, K_j; \Theta)$ will be known analytically. (See Section 2.2.3.) The minimization is carried out over the parameters $\{\Theta\}$ and not over all measurable functions $a : [0, T] \times [0, \infty)$ as in Problem 2.8. As a result, surfaces from parametric approaches appear to be more stable than the ones resulting from fully non-parametric calibrations, Fengler [2005a].

An early example of a parameterization of the instantaneous volatility is the Constant Elasticity of Variance (CEV) model due to Cox and Ross [1976]. Many other parameterizations have followed, for example modifications of the CEV model e.g. as in Andersen and Andreasen [2000] or the one-factor minimal market model (MMM), due to Platen [2001]. Carr et al. [1999] describes a general class of processes that yield closed-form option pricing formulas and also give some specific examples. Another approach is to model the instantaneous volatility as quadratic in the current value of the underlying e.g. as in Ingersoll [1989] and Rady [1995]. Brown and Randall [1999] on the other hand proposes a sum of trigonometric function as parameterization of the local volatility function.

The local volatility calibration problem has also been approached with semi-parameterization of the local volatility function. For example Coleman et al. [1999] uses cubic splines to fit the local volatility function to market data. Jackson et al. [1998] uses natural cubic splines across strikes and fits piecewise linear functions in the time space. McIntyre [2001] employs Hermite polynomials and

[Fengler \[2005b\]](#) thin plate splines.

The above examples of parameterizations of the local volatility function are by no means intended to be a comprehensive list of the literature in the subject, but to give a view on some of the different directions.

2.3.10 Mixture density models

A last example of a local volatility parameterization, that is also investigated more thoroughly, is the *mixture density approach*. This approach is based on the assumption of a particular parametric risk-neutral distribution of the underlying, depending on several possibly time-dependent parameters. An early example of this approach is the work by [Shimko \[1993\]](#). The remaining part of this section presents the *mixture density models* as prescribed by [Brigo and Mercurio \[2002\]](#), see also [Brigo, Mercurio, and Rapisarda \[2004\]](#).

In a mixture density model the marginal density of the stock price S , under a risk-adjusted measure $\mathbb{Q}$, will be equal to the weighted average of known densities of N given diffusions. Let the dynamics of the stock process S be given by

$$dS_t = \mu S_t dt + a(t, S_t) S_t dW_t, \quad (2.65)$$

with initial value S_0 . Let the dynamics of the N diffusions S^i be given by

$$dS_t^i = \mu S_t^i dt + v_i(t, S_t^i) dW_t, \quad i = 1, \dots, N, \quad (2.66)$$

with initial values S_0^i . $v_i(t, y)$ and $a(t, y)$ are real valued functions satisfying some regularity conditions. In particular they satisfy the following linear growth conditions,

$$\begin{aligned} a^2(t, y) y^2 &\leq L(1 + y^2) \\ v_i^2(t, y) &\leq L_i(1 + y^2), \quad i = 1, \dots, N, \end{aligned} \quad (2.67)$$

uniformly in t and for positive constants L, L_i . The drift μ will be determined by the risk neutral measure $\mathbb{Q}$.

Denote by $\varphi_t^i(\cdot)$ the density function of S_t^i . The aim is to derive $a(t, S_t)$ such that the $\mathbb{Q}$ -density of S satisfies the following,

$$\varphi_t(y) := \frac{d}{dy} \mathbb{Q}\{S_t \leq y\} = \sum_{i=1}^N p_i \frac{d}{dy} \mathbb{Q}\{S_t^i \leq y\} = \sum_{i=1}^N p_i \varphi_t^i(y). \quad (2.68)$$

Each S_0^i is set to S_0 and the p_i 's are positive constants such that $\sum_{i=1}^N p_i = 1$.

We may think of the densities φ^i as defining different volatility scenarios, each having a different probability p_i .

To derive the form of $a(t, S_t)$, we follow [Brigo and Mercurio \[2002\]](#), and start with applying the Fokker-Planck equation¹⁰ to $\varphi_t(\cdot)$,

$$\frac{\partial}{\partial t} \varphi_t(y) = -\frac{\partial}{\partial y} (\mu y \varphi_t(y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (a^2(t, y) y^2 \varphi_t(y)). \quad (2.69)$$

¹⁰I.e the Kolmogorov forward equation, see for instance [Öksendal \[2006\]](#), Example 8.3.

Using equation (2.68) and linearity of the derivative operator we have,

$$\underbrace{\frac{\partial}{\partial t} \sum_{i=1}^N p_i \varphi_t^i(y) + \frac{\partial}{\partial y} (\mu y \sum_{i=1}^N p_i \varphi_t^i(y))}_{= \sum_{i=1}^N p_i \left[\frac{\partial}{\partial t} \varphi_t^i(y) + \frac{\partial}{\partial y} (\mu y \varphi_t^i(y)) \right]} = \frac{1}{2} \frac{\partial^2}{\partial y^2} (a^2(t, y) y^2 \sum_{i=1}^N p_i \varphi_t^i(y)) \quad (2.70)$$

$$= \sum_{i=1}^N p_i \left[\frac{\partial}{\partial t} \varphi_t^i(y) + \frac{\partial}{\partial y} (\mu y \varphi_t^i(y)) \right] \quad (2.71)$$

Now, applying the Fokker-Planck equation to each density $\varphi_t^i(y)$ in the LHS of (2.71) (note that S_t and each S_t^i have the same drift μ and starting value S_0) and using the linearity of the second order derivative operator leads to:

$$\frac{\partial^2}{\partial y^2} \left[\sum_{i=1}^N p_i \left(\frac{1}{2} v_i^2(t, y) \varphi_t^i(y) \right) \right] = \frac{\partial^2}{\partial y^2} \left[\frac{1}{2} a^2(t, y) y^2 \sum_{i=1}^N p_i \varphi_t^i(y) \right]. \quad (2.72)$$

Now, we can find the general solution to the above second order differential equation for $a(t, \cdot)$:

$$a^2(t, y) y^2 \sum_{i=1}^N p_i \varphi_t^i(y) = \sum_{i=1}^N p_i v_i^2(t, y) \varphi_t^i(y) + A_t y + B_t, \quad (2.73)$$

with A_t and B_t real functions of time. The regularity conditions in (2.67) imply that the LHS of (2.73) goes to zero as $y \rightarrow \infty$. This implies $A_t = B_t = 0, \forall t$. Finally, solving for $a(t, S_t)$ leads to the following equation for the local volatility,

$$a(t, S_t) = \sqrt{\frac{\sum_{i=1}^N p_i v_i^2(t, S_t) \varphi_t^i(S_t)}{\sum_{i=1}^N p_i S_t^2 \varphi_t^i(S_t)}}. \quad (2.74)$$

Hence we have found the function $a(t, S_t)$ s.t. Equation (2.68) is fulfilled.

Next, to find the model call option price, we note that (at time $t = 0$), the following holds,

$$C(T, K) := e^{-rT} \mathbb{E}^{\mathbb{Q}} \{ (S_T - K)^+ \} = e^{-rT} \int_0^\infty (y - K)^+ \sum_{i=1}^N p_i \varphi_T^i(y) dy. \quad (2.75)$$

We have assumed risk neutrality, that the underlying pays no dividends, and that the interest rate r is fix, i.e. we have $\mu = r$. Equation (2.75) can now be expressed in terms of the option prices $C_i(T, K)$, associated to equation (2.66). We have,

$$C(T, K) = \sum_{i=1}^N p_i e^{-rT} \int_0^\infty (y - K)^+ \varphi_T^i(y) dy = \sum_{i=1}^N p_i C_i(T, K).$$

From here on, we choose the volatility of the N diffusions of (2.66) to be of the form,

$$v_i(t, y) = \sigma_i(t) y. \quad (2.76)$$

And also, each density $\varphi_t^i(\cdot)$ to be standard log-normal. This will result in the model price (again at time $t = 0$) being the following combination of modified Black-Scholes prices,

$$C(T, K) = \sum_{i=1}^N p_i \left[S_0 \Phi \left(\frac{\log(\frac{S_0}{K}) + (r + \frac{1}{2}\eta_i^2)T}{\eta_i \sqrt{T}} \right) - K e^{-rT} \Phi \left(\frac{\log(\frac{S_0}{K}) + (r - \frac{1}{2}\eta_i^2)T}{\eta_i \sqrt{T}} \right) \right]. \quad (2.77)$$

Φ is the standard normal cumulative distribution function and

$$\eta_i(T) := \sqrt{\frac{1}{T} \int_0^T \sigma_i^2(s) ds}. \quad (2.78)$$

This follows from that each C_i solves a generalized Black-Scholes pricing PDE, as in Section 2.2.3.

One approach to get a more flexible parameterization is to introduce a shift of the underlying process. [Brigo and Mercurio \[2002\]](#) proposes the following transformation of S to model the dynamics of the underlying,

$$A_t = A_0 \alpha e^{\mu t} + S_t. \quad (2.79)$$

A_0 and α are real constants. This preserves the correct drift (risk neutrality) of the underlying. The transformation (2.79) results in an almost parallel shift of the implied volatility surface and moves the strikes where the implied volatility reaches its minimum. [Brigo and Mercurio \[2002\]](#) states that this is the most general affine transformation of S for which the drift rate is μ . Choosing $\alpha > 0$ (< 0) will result in a minimum attained at strikes lower (higher) than at the money forward price.

Furthermore, option prices on the underlying (2.79) are easily calculated using equation (2.77) since the following hold,

$$\mathbb{E}^{\mathbb{Q}}\{(A_T - K)^+\} = \mathbb{E}^{\mathbb{Q}}\{(A_0 \alpha e^{\mu T} + S_T - K)^+\} = \mathbb{E}^{\mathbb{Q}}\{(S_T - (K - A_0 \alpha e^{\mu T}))^+\}. \quad (2.80)$$

The authors also suggest a parameterization of the integrated volatilities of each basic diffusion. For instance, the following Nelson -Siegel type ([Nelson and Siegel \[1987\]](#)) of parameterization,

$$\eta_i(T) = a_i + b_i \left[1 - \exp\left(-\frac{T}{\tau_i}\right) \right] \frac{\tau_i}{T} + c_i \exp\left(-\frac{T}{\tau_i}\right). \quad (2.81)$$

[Carmona and Nadtochiy \[2008\]](#) proposes the use of the following time dependent volatilities,

$$\begin{cases} \sigma_i^2(t) = \sigma^2, & i = 0, \\ \sigma_i^2(t) = \theta_i + (\sigma^2 - \theta_i)e^{-\zeta_i t}, & i = 1, 2. \end{cases} \quad (2.82)$$

to be used in a log-normal mixture model, $\left(\eta_i(T) = \sqrt{\frac{1}{T} \int_0^T \sigma_i^2(s) ds}\right)$.

2.4 Implementing a local volatility model

2.4.1 The log-normal mixture model

To understand better the “log-normal mixture model” as described above in Section 2.3.10, we have calibrated it to market data.

More precisely, we model the underlying as the process S_t defined in (2.65). We assume (2.76) holds, and that local volatilities (of (2.66)) are parameterized by (2.82). Hence model prices can be calculated using formula (2.77).

We start with calibrating this model to market option data for a fix time to maturity ($TTM = 0.1$ years). We choose $N = 3$. This gives us a model parameterized by the vector $\Theta = [\sigma, \theta_1, \theta_2, \zeta_1, \zeta_2, p_1, p_2]$. The market data consists of options on an exchange traded fund following the Nasdaq 100 index, (QQQQ).¹¹ At the given quotation day (2010-02-11), there were 55 bid/ask-pairs on the underlying having time to maturity $T = 0.1$.

The calibration is implemented in Scilab, using a standard non linear optimization routine (leastsq()) based on the Quasi-Newton method. The parameters are approximated by minimizing the objective function $G(\Theta)$ as follows,

$$\min_{\Theta} G(\Theta), \quad G(\Theta) := \sum_{j=1}^{55} \left(\frac{\partial C}{\partial \Sigma}(\bar{\Sigma}_j) \right)^{-2} \left(C(T, K_j; \Theta) - \bar{C}_j^* \right)^2. \quad (2.83)$$

$\bar{C}_j^* := \frac{\text{bid}_j + \text{ask}_j}{2}$, $j = 1, \dots, 55$ represents the observed call option prices. $C(T, K_j; \Theta)$ are option prices according the local volatility model, calculated from (2.77), using the parameters Θ .

We here choose to do a “vega-weighting” of the available data, see Cont and Tankov [2003], Chapter 13. That is, we minimize the square differences of option prices, weighted by the Black-Scholes vegas ($\frac{\partial C}{\partial \Sigma}$), evaluated at the implied volatilities of the market option prices. We have,

$$\frac{\partial C}{\partial \Sigma}(\bar{\Sigma}_j) = S\varphi\left(\frac{\log \frac{S}{K_j} + (r + \frac{\bar{\Sigma}_j^2}{2})T}{\bar{\Sigma}_j \sqrt{T}}\right). \quad (2.84)$$

S denotes the current value of the underlying. $\varphi(\cdot)$ denotes the standard normal probability density function. $\bar{\Sigma}_j$ denotes the mid market implied volatility corresponding to strike price K_j and time to maturity $T = 0.1$. This kind of weighting is motivated by the following approximation,

$$\begin{aligned} \sum_{j=1}^N (\Sigma(C(T, K_j; \Theta)) - \bar{\Sigma}_j)^2 &\approx \sum_{j=1}^N \left(\frac{\partial \Sigma}{\partial C}(\bar{\Sigma}_j) (C(T, K_j; \Theta) - \bar{C}_j^*) \right)^2 \\ &= \sum_{j=1}^N \left(\frac{\partial C}{\partial \Sigma}(\bar{\Sigma}_j) \right)^{-2} \left(C(T, K_j; \Theta) - \bar{C}_j^* \right)^2. \end{aligned} \quad (2.85)$$

Here we denote by $\Sigma(C(T, K))$, the Black-Scholes implied volatility as a function of option price. We have that weighting the objective function as above rescales errors in price into errors of implied volatilities. Hence it results in a better fit of model implied volatilities to market data. In Figure 2.2, the resulting implied volatilities are plotted. Mid market implied volatilities are marked with blue asterisks.

If we were to minimize directly over differences in implied model and market volatilities, we would have to first calculate model option prices, and thereafter

¹¹Quotation date 2010-02-11, source: <http://finance.yahoo.com/q?s=QQQQ>.

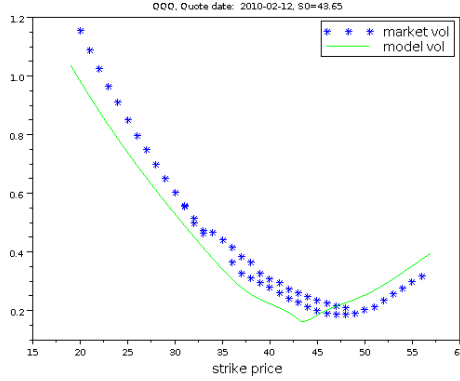


Figure 2.2: Implied volatilities of mixture-model, fit to (vega weighted) option prices (QQQQ) having fix time to maturity (0.1 years).

Table 2.1: Calibrated Mixture-model parameters, TTM = 0.1, $N = 3$

σ	p_1	p_2	η_1	η_2	θ_1	θ_2
0.21	0.0025	0.55	17.11	0.02	19.11	0

solve the inverse problem of finding the implied volatilities. This for each strike-maturity couple, and at each step in the search algorithm for finding the optimal parameter.

Next we fit the model parameters to the market prices of options across strikes and time to maturities. Again the market data consists of options on the exchange traded fund following the Nasdaq 100 index, (QQQQ).

At the given quotation day (2010-02-11), there were 151 bid/ask-pairs on the underlying, corresponding to strike prices $\{K_j\}_{j=1}^{151}$ and expiration times $\{T_k\}_{k=1}^7$. (We have in total 7 different maturity times; 0.1, 0.18, 0.35, 0.63, 0.88, 0.94 and 1.88 years. As in Remark 2.9 we construct a vector of length 155 of the maturity times, $TTM = [T_1 \cdots T_1, \dots, T_7 \cdots T_7]$. We do not include options with time to maturity less then one week.)

Now we approximate the parameters Θ from the following,

$$\min_{\Theta} G(\Theta), \quad G(\Theta) := \sum_{j=1}^{151} \left(\frac{\partial C}{\partial \Sigma}(\Sigma_j) \right)^{-2} \left(C(T_j, K_j; \Theta) - \overline{C}_j^* \right)^2. \quad (2.86)$$

$$\overline{C}_j^* := \frac{\text{bid}_j + \text{ask}_j}{2}, \quad j = 1, \dots, 151.$$

In Figure 2.3 and Figure 2.4 the resulting call option prices, respectively implied volatilities are plotted. In both figures the mid market data is marked with diamonds, and the plotted lines/surface represents the model. The calibrated parameters are as in Table 2.2.

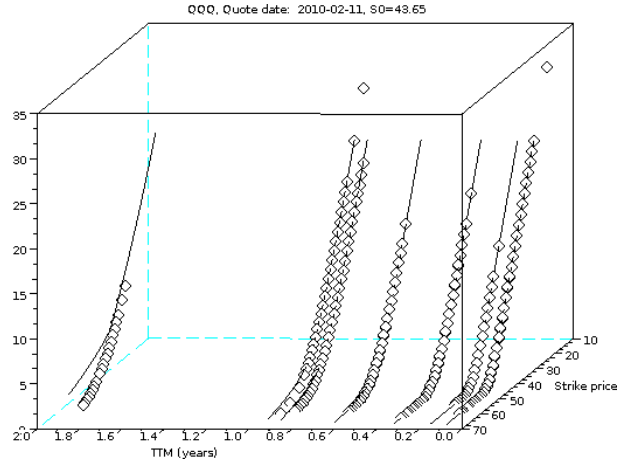


Figure 2.3: Call option prices of mixture-model, fit to (vega weighted) option prices (QQQQ) across strikes and maturities.

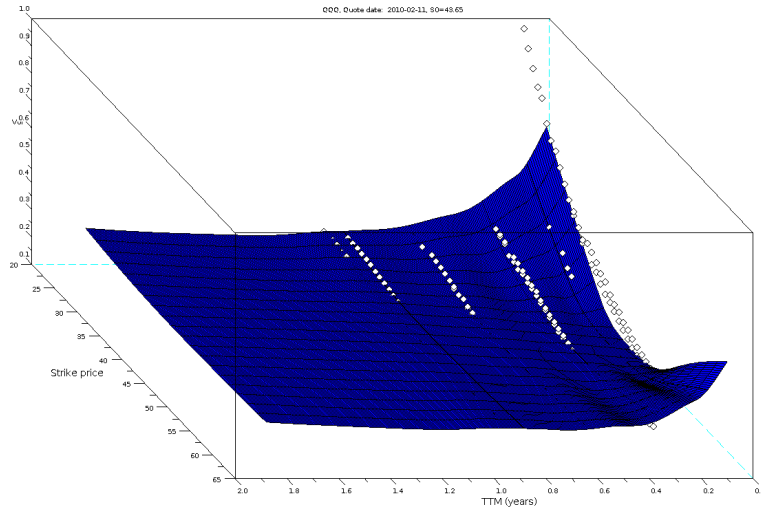


Figure 2.4: Implied volatilities of mixture-model, fit to (vega weighted) option prices (QQQQ).

Table 2.2: Calibrated Mixture-model parameters, $N = 3$

σ	p_1	p_2	η_1	η_2	θ_1	θ_2
0.18753	0.6	0.00162	0.38683	28.62273	0.39502	20.11167

3 The Lévy density code-book

3.1 Introduction to Lévy processes

This section provides some knowledge on Lévy processes that is important when setting up the *Lévy density code-book* and later when setting the code-book in motion, see Section 5. Good references on the theory of stochastic processes in general are Protter [2003] and Jacod and Shiryaev [2003]. See also Sato [1999] for the theory of infinitely divisible distribution processes. For applications of Lévy processes in finance see for instance Cont and Tankov [2003].

Definition 3.1. Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, $\mathcal{F} = \mathcal{F}_T$ and $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be a filtered probability space satisfying the usual conditions¹². A *Lévy process* is a adapted, $\mathbb{R}^d$ -valued stochastic process $X = (X_t)_{t \in [0, T]}$ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, $X_0 = 0$ a.s., with the following properties;

- (i) *Independence of increments*: for every increasing sequence of times $t_0, \dots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent. I.e., $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s$.
- (ii) *Stationarity of increments (or time homogeneity)*: $\forall s, t \geq 0$ the law of $X_{t+s} - X_t$ does not depend on t . That is, $X_{t+s} - X_t \stackrel{d}{=} X_s$.
- (iii) *Stochastic continuity*: $\forall t \geq 0$ and $\forall \varepsilon > 0$, $\lim_{s \rightarrow t} \mathbb{P}(|X_t - X_s| > \varepsilon) = 0$.

The properties (i)-(iii) above imply that there exists a càdlàg modification of X . In the notation of Jacod and Shiryaev [2003] the Lévy process equals a PIIS, i.e. a (càdlàg, adapted, $\mathbb{R}^d$ -valued) stochastic process with independent and stationary increments on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$.¹³

The property (ii) simply means that increments of the process X with equally long time intervals are identically distributed. For applications in finance this may cause problems, something that is discussed later in Sections 3.3 and 3.4. The property (iii) implies that for a fixed time t the probability of a jump is zero, i.e. there are no deterministic times of discontinuities. This property is well suited for financial modeling. Another and very important property that follows from (i)-(ii) is that the distribution of the Lévy process is *infinitely divisible*. This means that for each $t > 0$ and $n \in \mathbb{N}$, the random variable X_t is identical in law with a sum of n i.i.d random variables. The property of infinitely divisibility leads to a specific form of the characteristic function of any Lévy process X . The form is shown in Section 3.1.3 below, and plays an important role in the pricing formulas of financial models based on Lévy processes. Finally we also have that every Lévy process is a semimartingale, see for instance Protter [2003], Chapter II Thm. 9 and Chapter I, Thm. 40. From here on we will consider one dimensional Lévy processes X , taking values in $\mathbb{R}$.

¹²For definition see e.g. Protter [2003], Chapter I.

¹³ Remark: It follows from the càdlàg and stationarity properties that a PIIS has no fixed times of discontinuities, which also implies the property (iii) above, see Jacod and Shiryaev [2003] II 4.3. Hence the property (iii) may be omitted.

3.1.1 The random measure of jumps

An important tool for the analysis of Lévy processes is the *random measure of jumps*. To define this measure, we first need the *jump process* associated to a stochastic process X .

Definition 3.2. The *jump process* $\Delta X = (\Delta X_t)_{0 \leq t \leq T}$, associated to the Lévy process X is defined as,

$$\Delta X_t := X_t - X_{t-}, \quad \text{where } X_{t-} = \lim_{s \uparrow t} X_s.$$

Next, consider a set $\Lambda \subset \mathbb{R}$ with $0 \notin \Lambda$ and let $0 \leq t < T$.

Definition 3.3. The *random measure of jumps of X* is defined as,

$$\mu^X(\omega; t, \Lambda) := \#\{0 \leq s \leq t; \Delta X_s(\omega) \in \Lambda\} = \sum_{s \leq t} \mathbf{1}_\Lambda(\Delta X_s(\omega)).$$

This means, $\mu^X(\omega; t, \Lambda)$ counts the number of jumps of X with size in Λ up to time $t < T$. By stationarity and independence of increments of the process X , μ^X is a Poisson counting measure. See for instance Protter [2003] Chapter I, Theorem 35.

We continue with defining the Lévy measure of the process X .

Definition 3.4. Let $\mu^X : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be the random measure of jumps of the Lévy process X . The *Lévy measure* of X is defined as,

$$\nu(\Lambda) := \mathbb{E}[\mu^X(\omega; 1, \Lambda)] = \mathbb{E}\left[\sum_{s \leq 1} \mathbf{1}_\Lambda(\Delta X_s(\omega))\right]. \quad (3.1)$$

It follows that ν is a measure. Also, $\nu(dx)dt$ is called the *compensator* of μ^X . Intuitively $\nu(\Lambda)$ is the expected number of jumps of X with size in Λ in a time interval of length $t = 1$. In general, ν need not be a finite measure, but must satisfy

$$\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty.$$

This means that the Lévy process will always have a finite number of “big” jumps with size $|x| > \varepsilon$, for any $\varepsilon > 0$.¹⁴ We note that the Lévy measure has no mass at zero, i.e $\nu(\{0\}) = 0$.

A Lévy process is said to have *finite activity* if $\lambda := \int_{\mathbb{R}} \nu(dx) < \infty$. In this case λ is the expected number of jumps. The distribution of the jump size x will then be given by $\frac{\nu(dx)}{\lambda}$.

If λ is infinite, the Lévy process is said to have *infinite activity*. Finally, if the Lévy measure is of the form $\nu(dx) := f(x)dx$, then $f(x)$ is called the *Lévy density* of X .

¹⁴More precisely, for each fix ω and t , the sample path $X_t(\omega)$ of a Lévy process $(X_t)_{t \geq 0}$, has only finitely many jumps on $[0, t]$, with absolute value bigger than any $\varepsilon > 0$. This is a consequence of the càdlàg property of Lévy processes.

3.1.2 The Lévy-Itô decomposition

Theorem 3.5. *Every Lévy process $X = (X_t)_{t \geq 0}$ can be decomposed into independent Lévy processes as follows,*

$$\begin{aligned} X_t = & \gamma t + \sigma B_t + \int_0^t \int_{\{|x| \geq 1\}} x \mu^X(ds, dx) \\ & + \int_0^t \int_{\{|x| < 1\}} x(\mu^X - \nu)(ds, dx). \end{aligned} \quad (3.2)$$

Here, μ^X denotes the random measure of jumps of the process X and ν is its Lévy measure. $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $(B_t)_{t \geq 0}$ is a standard Brownian motion.

Proof. A proof of this decomposition can be found in Sato [1999], Chapter 4. We sketch the proof, following the outline of Cont and Tankov [2003], Proposition 3.7. This approach is close to the one used by Lévy [1934].

We start by noting that the first part of the decomposition (3.2),

$$X_t^{(1)} := \gamma t + \sigma B_t, \quad (3.3)$$

is continuous and Gaussian. Furthermore, every continuous Gaussian Lévy process can be written in this form.

For the remainder, the Poisson counting measure μ^X on $[0, T] \times \mathbb{R}$, constructed from the jumps of X (as described above in Section 3.1.1), is needed. The intensity of μ^X is equal to $\nu(dx)dt$, see Cont, Tankov [2004] proposition 3.5. We have that the measure ν satisfies the integrability condition

$$\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty. \quad (3.4)$$

We now construct the second part of the decomposition (3.2) out of the “big” jumps of X ,

$$X_t^{(2)} := \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| \geq 1} = \int_0^t \int_{\{|x| \geq 1\}} x \mu^X(ds, dx), \quad t \geq 0. \quad (3.5)$$

As stated above (Section 3.1.1) X has only a finite number of jumps greater than 1. Therefore the sum in (3.5) contains almost surely a finite number of terms. It defines a compound Poisson process with arrival rate $\nu(\mathbb{R} \setminus (-1, 1))$ and jump magnitude $\frac{\nu(dx)}{\nu(\mathbb{R} \setminus (-1, 1))} \mathbf{1}_{\{|x| \geq 1\}}$, see for instance Cont and Tankov [2003] Definition 3.3.

The third and last part of (3.2) is constructed out of the small jumps of X . However, since X may have infinitely many small jumps, we need the following compensated process,

$$\begin{aligned} X_t^{(3, \varepsilon)} &= \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{1 > |\Delta X_s| > \varepsilon\}} - t \left(\int_{\{1 > |x| > \varepsilon\}} x \nu(dx) \right) \\ &= \int_0^t \int_{\{1 > |x| > \varepsilon\}} x \mu^X(dx, ds) - t \left(\int_{\{1 > |x| > \varepsilon\}} x \nu(dx) \right). \end{aligned} \quad (3.6)$$

The first sum of (3.6) represents the jumps having absolute size between ε and 1, which is a finite number. As mentioned above, X may have infinitely many small jumps and the sum will in general not converge as $\varepsilon \rightarrow 0$. In order to obtain convergence, the sum in (3.6) is compensated (or centered) by $t(\int_{1 > |x| > \varepsilon} x\nu(dx))$. This integral is the average increase of the process X , on an unit interval, when jumps with absolute sizes smaller than ε or larger than 1 are not included. We have that, in the general case, also this integral does not have a finite limit as $\varepsilon \rightarrow 0$.

We have that $X^{(3,\varepsilon)}$ is a *compensated compound Poisson process*. Also, as $\varepsilon \rightarrow 0$, this process converges; Consider a sequence $\{\varepsilon_n\} \downarrow 0$ and let $Y_n := X_t^{(3,\varepsilon_{n+1})} - X_t^{(3,\varepsilon_n)}$. All Y_n have zero mean, and using the integrability condition (3.4), it follows that $\sum \text{Var}(Y_n) < \infty$. From Kolmogorov's three series Theorem, see for instance [Loève \[1963\]](#) Section 16.3, it follows that $\sum Y_n$ converges almost surely. This gives us that also $X_t^{(3,\varepsilon)}$ converges almost surely as $\varepsilon \rightarrow 0$. We call the limit process $X_t^{(3)} := \lim_{\varepsilon \downarrow 0} X_t^{(3,\varepsilon)}$. It follows from Kolmogorov's maximum inequality, see for instance [Billingsley \[1995\]](#) Theorem 22.4, that the convergence is uniform in t .

The two processes $X_t^{(2)}$ and $X_t^{(3)}$ incorporate all jumps (the discontinuous part) of the process X . However, without loss of generality, we now assume that all jumps of X are smaller than 1 in absolute value.

Continue with defining the process $X_t^c := X_t - X_t^{(3,\varepsilon)}$. By Lemma 3.6 below, X_t^c is a Lévy process which is independent from $X_t^{(3,\varepsilon)}$. We have that $\lim_{\varepsilon \rightarrow 0} X_t^c$ is a continuous process, since $X_t^{(3,\varepsilon)}$ converges uniformly in t , we can interchange limits. Finally, the Gaussian convergence theorem due to Feller-Lévy, see [Kallenberg \[1997\]](#) Theorem 4.15, implies that $\lim_{\varepsilon \rightarrow 0} X_t^c$ is Gaussian. Hence it can be written on the form (3.3). We have

$$X_t = X_t^{(1)} + X_t^{(3)}, \quad (3.7)$$

and $X_t^{(1)}$ and $X_t^{(3)}$ are independent. $\square$

Lemma 3.6. *Let (X_t, Y_t) be Lévy processes. If (Y_t) is compound Poisson and (X_t) and (Y_t) never jump together, then they are independent.*

Proof. For a proof, see [Cont and Tankov \[2003\]](#), Lemma 3.2. $\square$

3.1.3 Lévy-Khintchine formula

Having the Lévy-Itô decomposition (3.2), we can retrieve the celebrated *Lévy-Khintchine formula*, see [Lévy \[1934\]](#).

Theorem 3.7. *The characteristic function of an $\mathbb{R}$ -valued Lévy process $X = (X_t)_{t \geq 0}$ has the following form,*

$$\begin{aligned} \mathbb{E}[e^{iuX_t}] &= \exp \left[t \left(iu\gamma - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}})\nu(dx) \right) \right] \\ &:= \exp(t\psi(u)). \end{aligned} \quad (3.8)$$

Here $\gamma \in \mathbb{R}$ $\sigma \in \mathbb{R}^+$, and $\nu(dx)$ (defined on $\mathbb{R}$) is the Lévy measure of X .

Proof. From the Lévy Itô decomposition (3.2) we know that, for each t , we have the following almost sure convergence; $X_t^{(1)} + X_t^{(2)} + X_t^{(3,\varepsilon)} \rightarrow X_t$ as $\varepsilon \rightarrow 0$. Here $X_t^{(1)}$, $X_t^{(2)}$, $X_t^{(3,\varepsilon)}$ are defined as in (3.3), (3.5) and (3.6) respectively.

The almost sure convergence implies convergence in distribution, and the characteristic function of $X_t^{(1)} + X_t^{(2)} + X_t^{(3,\varepsilon)}$ converges to the characteristic function of X_t . Due to the independence of $X_t^{(1)}$, $X_t^{(2)}$ and $X_t^{(3,\varepsilon)}$,

$$\mathbb{E}[e^{iu(X_t^{(1)} + X_t^{(2)} + X_t^{(3,\varepsilon)})}] = \mathbb{E}[e^{iuX_t^{(1)}}] \mathbb{E}[e^{iuX_t^{(2)}}] \mathbb{E}[e^{iuX_t^{(3,\varepsilon)}}]. \quad (3.9)$$

We have that

$$\mathbb{E}[e^{iuX_t^{(1)}}] = \exp\left[t\left(-\frac{1}{2}u^2\sigma^2 + iu\gamma\right)\right].$$

The characteristic function of the compound Poisson process $X_t^{(2)}$,

$$\mathbb{E}[e^{iuX_t^{(2)}}] = \exp\left\{t \int_{\{|x| \geq 1\}} (e^{iux} - 1) \nu(dx)\right\},$$

see Cont and Tankov [2003] Proposition 3.4. At last, results from Poisson point processes gives that the characteristic function of $X^{(3,\varepsilon)}$ is

$$\mathbb{E}[e^{iuX_t^{(3,\varepsilon)}}] = \int_{\{\varepsilon < |x| < 1\}} (e^{iux} - 1 - iux) \nu(dx).$$

As $\varepsilon \rightarrow 0$ (3.9) converges to (3.8) for each u . $\square$

We have that a Lévy process is uniquely characterized through its triplet (γ, σ, ν) . This will lead to the *the Lévy density code-book* which is discussed in Section 3.5.

3.1.4 The Itô formula for Lévy processes

In this section we state the Itô formula for Lévy processes, as it is a generalization of the Itô formula presented in Section 2.1.1.

Note that since the diffusion part of the Lévy process is independent of the jump part (recall the Lévy-Itô decomposition in Section 3.1.2), the classical Itô formula is complemented by terms which come from the jump part of X , but no mixed terms have to be considered.

Theorem 3.8. *Let $g \in C^{1,2}([0, \infty) \times \mathbb{R})$ and X_t be a Lévy process characterized by a triplet (γ, σ, ν) , $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and ν is a Lévy measure. Then we have,*

$$\begin{aligned} g(T, X_T) = & g(0, X_0) + \int_0^T \frac{\partial g}{\partial t}(t, X_t) dt + \int_0^T \frac{\partial g}{\partial x}(t, X_{t-}) dX_t + \frac{\sigma^2}{2} \int_0^T \frac{\partial^2 g}{\partial x^2}(t, X_t) dt \\ & + \sum_{0 \leq t \leq T} \left[g(t, X_t) - g(t, X_{t-}) - \frac{\partial g}{\partial x}(t, X_{t-}) \Delta X_t \right]. \end{aligned} \quad (3.10)$$

For a proof see for instance Cont and Tankov [2003], Proposition 8.15.

3.1.5 Example of Lévy processes

If the distribution generating the Lévy process $(X_t)_{t \geq 0}$ has finite expectation then $\int_{\{|x| \geq 1\}} x \nu(dx) < \infty$,¹⁵ and the Lévy-Itô decomposition (3.2) can be simplified to

$$X_t = \gamma' t + \sigma B_t + \int_0^t \int_{\mathbb{R}} x \mu^X(ds, dx) - t \int_{\mathbb{R}} x \nu(dx). \quad (3.11)$$

Or in another way,

$$X_t = \gamma' t + \sigma B_t + Z_t \quad (3.12)$$

where $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, B_t is a standard Brownian motion and Z_t is a pure jump process (a purely discontinuous martingale independent of B_t). Note that γ' differs from the drift term γ in (3.2).¹⁶

Example 3.1.1 (Brownian motion). $(X_t)_{t \geq 0}$ can be continuous and consist simply of a Brownian motion with a linear drift,

$$X_t = \gamma' t + \sigma B_t. \quad (3.13)$$

Example 3.1.2 (Compound Poisson). $(X_t)_{t \geq 0}$ can be a purely discontinuous compound Poisson process, (we have $\gamma' = \sigma = 0$),

$$X_t = \sum_{i=1}^{N_t} Y_i = \int_0^t \int_{\mathbb{R}} x (\mu^X - \lambda F(x))(dx, ds). \quad (3.14)$$

Here N_t is a Poisson process with parameter $\lambda > 0$ and $\{Y_i\}$ is a sequence of independent random variables with equal distribution $F(x)$. μ^X is the random measure of jumps of X , see Definition 3.3.

Example 3.1.3 (Jump diffusion). As another example, $(X_t)_{t \geq 0}$ can be a combination of a compound Poisson process and a Brownian motion with drift,

$$X_t = \gamma' t + \sigma B_t + \sum_{i=1}^{N_t} Y_i = \gamma' t + \sigma B_t + \int_0^t \int_{\mathbb{R}} x (\mu^X - \lambda F(x))(dx, ds), \quad (3.15)$$

see Figure 3.1.¹⁷ This last process is an example of a *Lévy jump diffusion*, see Section 3.2 below. Merton [1977] propose modeling asset prices as the exponential of the process (3.15), where the random variables Y_i are standard normal i.i.d., see Section 3.3.1.

Each of the three example processes given above have no, or at most a finite number of jumps, i.e. they are examples of *finite activity* Lévy processes with $\int_{\mathbb{R}} \nu(dx) < \infty$.

As stated earlier, if $\int_{\mathbb{R}} \nu(dx) = \infty$ we have what is known as an *infinite activity* process. Each path of such a process has almost surely infinitely many jumps. Most of the jumps are very small and there is only a finite number of big jumps.

¹⁵See Sato [1999], Thm. 25.3.

¹⁶ $\gamma' = \gamma + \int_{\{|x| \geq 1\}} x \nu(dx)$.

¹⁷Note that plots of this type have been published in for instance in Tankov [2007] and Papapantoleon [2005].

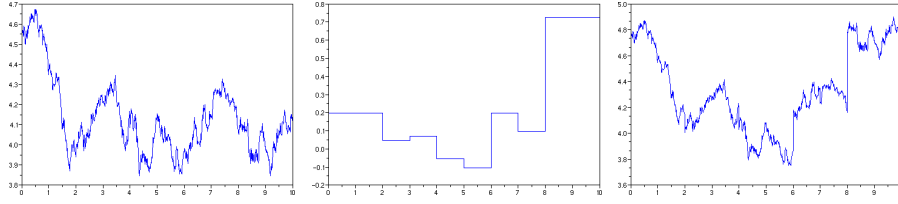


Figure 3.1: a) $X_t = \gamma t + \sigma B_t$ b) $X_t = \sum_{i=1}^{N_t} Y_i$ c) $X_t = \gamma t + \sigma B_t + \sum_{i=1}^{N_t} Y_i$

Example 3.1.4 (Normal Inverse Gaussian). One example of an infinite activity Lévy process is the Normal Inverse Gaussian (henceforth denoted by NIG) process. The NIG distribution is a subclass of the generalized hyperbolic distributions, introduced by [Barnorff-Nielsen \[1977\]](#). The generalized hyperbolic distribution has density dependent on 5 parameters; $\lambda, \alpha, \beta, \delta, \mu$. It is given by

$$f_{GH}(x) = a(\lambda, \alpha, \beta, \delta, \mu) (\delta^2 + (x - \mu)^2)^{(\lambda - \frac{1}{2})/2} e^{\beta(x - \mu)} \times K_{\lambda - \frac{1}{2}}(\alpha \sqrt{\delta^2 + (x - \mu)^2}), \quad (3.16)$$

where

$$a(\lambda, \alpha, \beta, \delta, \mu) = \frac{(\alpha^2 - \beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi} \alpha^{\lambda - \frac{1}{2}} \delta^\lambda K_\lambda(\delta \sqrt{\alpha^2 - \beta^2})}. \quad (3.17)$$

K_λ denotes the modified Bessel function of the third kind with index λ . The parameter $\alpha > 0$ determines the shape, $0 \leq |\beta| < \alpha$ the skewness, $\mu \in \mathbb{R}$ location and $\delta > 0$ scaling of the distribution. $\lambda \in \mathbb{R}$ influences the size of mass contained in the tails. The class of generalized hyperbolic processes is discussed further for instance in [Eberlein and Keller \[1995\]](#), [Eberlein, Keller, and Prause \[1998\]](#) and [Eberlein and Hammerstein \[2004\]](#).

Choosing $\lambda = -\frac{1}{2}$ will result in the NIG distribution. A NIG-Lévy process $X = (X_t)_{t \geq 0}$ is defined as having NIG distributed increments. For each increment of length s , the NIG-process X satisfies $(X_{t+s} - X_t) \sim NIG(\alpha, \beta, \delta s, \mu s)$. $NIG(\alpha, \beta, \delta s, \mu s)$ being the NIG distribution with parameters $\alpha, \beta, \delta s, \mu s$. The Lévy-Itô decomposition of the NIG-process is given by

$$X_t = \gamma^{\text{NIG}} t + \int_0^t \int_{\mathbb{R}} x(\mu^X - \nu^{\text{NIG}})(ds, dx), \quad (3.18)$$

where $\gamma^{\text{NIG}} = \mu + 2 \frac{\delta \alpha}{\pi} \int_0^1 \sinh(\beta x) K_1(\alpha x) dx$, and

$$\nu^{\text{NIG}}(dx) = e^{\beta x} \frac{\delta \alpha}{\pi |x|} K_1(\alpha |x|) dx. \quad (3.19)$$

K_1 denotes the modified Bessel function of the third kind with index $= 1$. For a derivation of the NIG-Lévy measure, see for instance [Raible \[2000\]](#). The characteristic triplet is hence given by $(\gamma^{\text{NIG}}, 0, \nu^{\text{NIG}})$. See [Eberlein and Hammerstein \[2004\]](#).

NIG is the only subclass of the generalized hyperbolic distributions that is closed under convolution, i.e. sums of independent NIG distributed random variables are NIG distributed. The following hold,

$$\begin{aligned} X &\sim NIG(\alpha, \beta, \delta_X, \mu_X), \quad Y \sim NIG(\alpha, \beta, \delta_Y, \mu_Y) \\ \Rightarrow X + Y &\sim NIG(\alpha, \beta, \delta_X + \delta_Y, \mu_X + \mu_Y). \end{aligned} \quad (3.20)$$

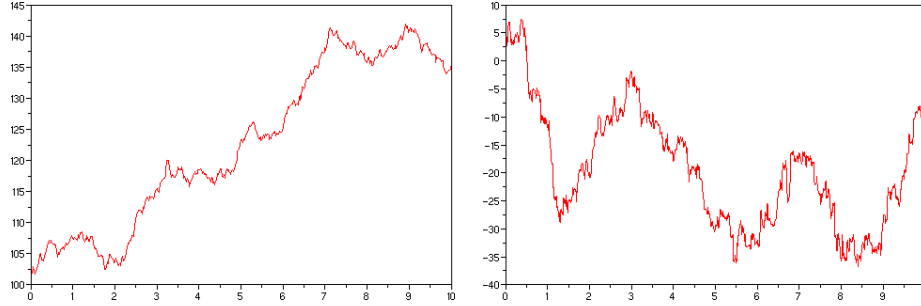


Figure 3.2: a) NIG sample

b) VG sample

Figure 3.2 a) shows a sample path of a process with NIG distributed increments.¹⁸

Example 3.1.5 (Variance gamma). Another example of an infinite activity Lévy process is the variance gamma (henceforth VG) process. A VG-process is obtained by evaluating a Brownian motion at a random time given by a Gamma process,

$$X_t = B(\gamma_t). \quad (3.21)$$

Here $B(t)$ is a standard Brownian motion and γ_t is a Gamma process, i.e. a process having independent, Gamma distributed, increments. The VG-process was introduced for the purpose of financial modeling by Madan and Seneta [1990]. Madan and Milne [1991] and Madan, Carr, and Chang [1998] extend the model by time changing a Brownian motion with constant drift and volatility,

$$X_t = B(\gamma_t), \quad \text{where} \quad B(t) := \theta t + \sigma W_t. \quad (3.22)$$

That is, $B(t)$ is now a Brownian motion with drift θ and volatility σ . We have the Lévy measure of this VG-process as follows,

$$\nu^{VG}(dx) := \kappa^{VG}(x)dx = \frac{\exp(\theta x/\sigma^2)}{\nu|x|} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma}|x|\right)dx, \quad (3.23)$$

see Madan et al. [1998]. This second case is referred to as the *asymmetric* VG-process and the case without drift as the *symmetric* VG-process. An example of a sample path of the symmetric case is given in Figure 3.2 b).

In the same way as the VG-process can be viewed as a gamma time changed Brownian motion, the NIG-process can be viewed as a Brownian motion time changed by an inverse Gaussian process.

¹⁸Again, note that similar plots have been published e.g. in in Tankov [2007] and Papapanoleon [2005].

3.2 Lévy processes in financial models

Commonly used in finance are models based on the exponential of a Lévy process. In such a model the stock price $(S_t)_{t \in [0, T]}$ is assumed to follow the process,

$$S_t = S_0 e^{rt + X_t}. \quad (3.24)$$

Here, r is the interest rate and $X = (X_t)_{t \in [0, T]}$ is a Lévy process determined by a triplet (γ, σ, ν) . The process is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$. $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}$ is the filtration generated by the history of the prices. $\mathbb{Q}$ is a probability measure. We assume absence of arbitrage. Hence we may assume without loss of generality that $\mathbb{Q}$ is an equivalent local martingale measure, see below.

The model in (3.24) reflects the assumption that log returns of the asset evolve independently with identical distributions for equal time steps. It also assures positivity. See for instance Cont and Tankov [2003], Cont and Voltchkova [2005] or Belomestny and Reiß [2006a]. As an alternative to (3.24) the stock price can be modeled by using the stochastic (Doléans-Dade) exponential of a Lévy process X ,

$$dS_t = rS_{t-}dt + S_{t-}dX_t. \quad (3.25)$$

$(X_t)_{t \geq 0}$ is again a Lévy process defined by a triplet (γ, σ, ν) . That is we have replaced the Brownian motion in the Black-Scholes model with a more general Lévy process. Since in this case the asset prices may become negative, we will stay with the exponential Lévy model defined by (3.24).

3.2.1 Martingale measures in Lévy models

Loosely speaking, under the hypothesis of no arbitrage there exists a measure $\tilde{\mathbb{Q}}$ equivalent to the measure $\mathbb{Q}$ such that the discounted prices of all traded assets are local martingales under $\tilde{\mathbb{Q}}$.¹⁹

Note that unlike the classical Black-Scholes model, or the local volatility models presented in Section 2.2, the model defined by (3.24) is incomplete. Recall that a market is said to be *complete* if there exists only one measure $\tilde{\mathbb{Q}}$ equivalent to the real measure $\mathbb{Q}$ under which the discounted prices of all traded assets are martingales. There exists many ways to derive equivalent martingale measures in incomplete markets. For instance we have the *minimal martingale measure*, Föllmer and Schweizer [1991], the *Esscher martingale measure*, the *mean correcting martingale measure* e.g. Gerber and Shiu [1994], or the *minimal entropy martingale measure* e.g. Hubalek and Sgarra [2006], etc.

See Selivanov [2005] for a study on the existence and uniqueness of martingale measures in exponential Lévy models.

Pure jump vs. Jump diffusion in financial models

Theoretically the asset price may be modeled as an (exponential) Lévy process with a diffusion term and an infinite activity jump part. Nevertheless, in most models presented in the literature the underlying asset is assumed to follow either a *jump-diffusion process* or a *infinite activity pure jump process*. A *jump-diffusion* is a process with non zero diffusion part ($\sigma \neq 0$) combined with a finite

¹⁹Prices are discounted by the value of a savings account at time t . Having fix interest rate r this value equals e^{rt} .

activity jump process ($\int_{\mathbb{R}} \nu(dx) < \infty$). In a *infinite activity pure jump model* on the other hand, the price process is assumed to follow a purely discontinuous process ($\sigma = 0$) having frequent small jumps ($\int_{\mathbb{R}} \nu(dx) = \infty$).

Empirically asset prices display frequent and small changes that may be modeled as a diffusion (as in the Black-Scholes model) or as an infinite activity jump process. Roughly speaking we may say that if we have a process that is able to produce infinitely many jumps on any finite interval, the need for a diffusion term is no longer apparent. In fact, it is hard to distinguish an infinite activity pure jump process from a diffusion process if the jumps are not large. On the other hand, the market is also exposed to large and rare events. These events are naturally captured by a (finite activity) compound Poisson process, (part three of the Lévy Itô decomposition (3.2)). This gives one explanation to the distinction between the two model types. We may try to model an asset price as a diffusion together with a finite activity jump process (the later capturing large and rare events). Alternatively, we try to capture asset dynamics by an infinite activity pure jump process. In this case small jumps occur with (very) high probability, but there may also be large jumps occurring with low probability.

It has indeed been recognized that both rare large moves and frequent small moves that are observed in asset prices can be captured by an *infinite activity pure jump process*, Mandelbrot [1997]. Also Eberlein and Keller [1995] find by statistical tests on 30 German index shares over a three year period that Hyperbolic distributions can be fit to empirical distributions with a high accuracy. This indicates the sufficiency of financial modeling by pure jump processes. Also, Carr, Geman, Madan, and Yor [2002] employs empirical investigation to market prices over a five year period to find that index returns tend to be pure jump processes. They also note that the diffusion components estimated in individual equity returns appear to be statistically insignificant. This motivates the choice of modeling the underlying as an exponential *infinite activity pure jump process* when setting up a *Lévy density code-book* in Section 3.5.

3.3 The Inverse problem for Lévy models

In the same way that local volatilities are calibrated in the *inverse problem for local volatilities*, Problem 2.7, the Lévy triplet in a financial model based on a Lévy process is calibrated to reproduce market prices.

Problem 3.9. Assume that a set of call option prices $C_t(T_j, K_j)$ corresponding to maturities T_j and strikes K_j , $j = 0, \dots, N$ are available at time t . Find a Lévy triplet (γ, σ, ν) , characterizing a process $(X_t)_{t \geq 0}$ such that the discounted price, $(e^{-rt}S_t)$, is a martingale and we have

- $S_t = S_0 \exp(rt + X_t)$
- the observed prices match the discounted risk neutral expectations;

$$\forall j = 1, \dots, N, \quad C_t(T_j, K_j) = e^{-r(T_j-t)} \mathbb{E}^{\mathbb{Q}}[(S(T_j) - K_j)^+ | S_t = x].$$

Here x is the level of the underlying at time t . Note that the Remark 2.9, concerning the structure of the market data holds. That is, the market data

is given in “strings” of option prices having the same time to maturity. The number of different maturity times will be much smaller than the number of strike prices. The maturity times will be given as a vector of length N having the following form, $TTM = [T_1 \cdots T_1, T_2 \cdots T_2, \dots, T_M \cdots T_M]$, where M is the number of different maturity times as in Remark 2.9.

Using the Lévy-Khintchine formula (3.8), due to independence of increments, the martingale property of $e^{-rt}S_t$ is fulfilled if and only if

$$\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x\mathbf{1}_{\{|x| \leq 1\}}) \nu(dx) = 0. \quad (3.26)$$

Assume for a moment that a continuum of options on the underlying S_t are quoted on the market. The Lévy triplet can then be found by first using the results of Breeden and Litzenberger [1978] to recover the risk-neutral transition density from market option prices. Breeden and Litzenberger showed that given the initial state $S_t = x$ at time t , the following holds for the transition density f of the stock price S ,

$$f(t, x; T, K) = e^{r(T-t)} \frac{\partial^2}{\partial K^2} C_t(K, T). \quad (3.27)$$

By taking the Fourier transform of (3.27) one will retrieve the characteristic function of the stock price S_T . Since γ is given by the martingale condition (3.26) σ and ν remain to be deduced from the characteristic function, e.g. by using the Lévy Khintchine formula (3.8) and certain assumptions on the Lévy measure. This would give us a solution to Problem 3.9.

Clearly, a continuum of quotes will not be available on the market and we do not have a well defined option pricing function $C_t(K, T)$. Just as in the local volatility case, see Section 2.3.1, Problem 3.9 is an ill-posed inverse problem. There may exist no solution, or infinitely many solutions. Therefore, several regularization methods have been introduced. For instance non parametric regularization as in Cont and Tankov [2004] and Belomestny and Reiß [2006a,b] or regularization by parameterization of the distribution as in Eberlein and Keller [1995] and Carr et al. [2002]. Other parameterizations are discussed in more detail in Section 3.3.1 below. The two non parametric regularization methods are discussed in Section 3.3.3.

Another important issue, ill-posedness aside, is due to the time homogeneity property of the Lévy processes. For single maturity times, a Lévy process can be adequately fit to a market implied volatility smile (or price curve). That is, calibration of a Lévy process to market options $C_t(T, K_j)$ having fix maturity T , and strike prices K_j , $j = 0, \dots, N$ works well.

On the other hand, when it comes to calibration to options having several maturities and strikes, as in Problem 3.9, calibration by Lévy processes is no longer as precise. This is noted for instance by Carr et al. [2002] and Cont and Tankov [2003, 2007]. The problem also becomes apparent later in an implementation of a NIG-parameterization in Section 3.3.1, Figure 3.3. The problem is foremost due to the time homogeneity property of Lévy processes, which implies that for any $s, t \geq 0$ the distribution of $X_{t+s} - X_t$ does not depend on t . This means that the returns for a fixed time horizon will always have the same law. See Section 3.4 for one approach to resolve this problem.

3.3.1 Parametric calibration

The most common way to solve the ill-posedness of Problem 3.9 is to choose a parametric form for the Lévy measure.

Having a parameterization, calibration of Lévy processes to market prices is relatively straightforward. This is due to that pricing formulas of European vanilla options can be derived by Fourier inversion as soon as the characteristic function of the return is known analytically. For instance Carr et al. [1999] consider the Fourier transform of the modified option price,

$$\psi_T(v) := \int_{-\infty}^{\infty} e^{ivx} c_t(T, x) dx, \quad (3.28)$$

where $c_t(T, x) := \exp(\alpha x) \tilde{C}_t(T, x)$. Here $\alpha > 0$ and $\tilde{C}_t(T, x)$ is the value of a call option having maturity time T and strike price e^x .

The reason for the modification $c_t(T, x)$ is that the (unmodified) call price $\tilde{C}_t(T, x)$ will tend to a positive value as $x \rightarrow -\infty$ and hence will not be square integrable in the log-strike x . Choosing the right α will make the integrand of equation (3.28) square integrable over the negative axis of x . The effect on the integrand on the positive log-strike axis will however be reversed, and the following will have to be fulfilled to assure integrability also over this side,

$$\mathbb{E}[S_T^{\alpha+1}] < \infty. \quad (3.29)$$

Carr et al. [1999] derive an analytical expression for the Fourier transform $\psi_T(v)$ in terms of the characteristic function of $\log(S_T)$,

$$\psi_T(v) = \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}. \quad (3.30)$$

Having the characteristic function of the underlying specified by the exponential Lévy model, call prices can be obtained using the inverse transform,

$$\tilde{C}_t(T, x) = \frac{\exp(-\alpha x)}{2\pi} \int_{-\infty}^{\infty} e^{-ivx} \psi_T(v) dv. \quad (3.31)$$

The method can be efficiently implemented using fast Fourier transformation (FFT) algorithms, see Remark 3.11 below. Other modifications of the option prices may obtain a faster convergence of the numerical evaluation of the integrand (3.31). See for instance Cont and Tankov [2003] Section 11.1.3.

Example 3.3.1 (NIG). Barndorff-Nielsen [1998] proposes to model asset returns by using a NIG-process. For instance the dynamics of the stock price in a NIG based pricing model can be given by,

$$S_t = S_0 \exp(\omega t + X_t). \quad (3.32)$$

Here $\omega \in \mathbb{R}$ is a drift parameter that will be given by the martingale restriction on the price process, see below. X_t is a NIG-process, defined by the parameters $\alpha > 0$, $0 \leq |\beta| < \alpha$, $\mu \in \mathbb{R}$ and $\delta > 0$, (see Section 3.1.5 Example 3.1.4).

Since any financial model based on stochastic processes with jumps of unknown sizes is incomplete (see Section 3.2), there exists more than one equivalent martingale measure such that the discounted asset price (3.32) is a martingale.

One way to choose such a measure is related to the *mean correcting martingale measure*. In this case we choose the drift ω such that the discounted process is a martingale under the measure $\mathbb{Q}$. We have,

$$E^{\mathbb{Q}}[e^{-rt}S_t] = S_0. \quad (3.33)$$

Having the stock price defined by (3.32), this gives us

$$\phi_1^X(-i) = e^{r-\omega},$$

$\phi_t^X(u) := \mathbb{E}^{\mathbb{Q}}[e^{iuX_t}]$ being the characteristic function of X_t , or equivalently

$$\omega = r + \delta[\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2}] - \mu.$$

We have that the characteristic function of $\log(S_T)$ in the NIG pricing model (3.32) is given by

$$\phi_T(u; \alpha, \beta, \delta, \mu) = \exp\left(iu \log S_0 + T[\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}) + iu(\mu + \omega)]\right). \quad (3.34)$$

Example 3.3.2 (Merton jump diffusion). Merton [1977] proposes modeling asset prices as exponential Lévy jump diffusions. More precisely, that the stock price dynamics is given by

$$S_t = S_0 \exp(\mu^M t + \sigma B_t + \sum_{i=1}^{N_t} Y_i). \quad (3.35)$$

As in Section 3.1.5, see Example 3.1.3, B_t is a standard Brownian motion and N_t is a Poisson process with intensity λ . The jump sizes Y_i are normal i.i.d random variables with mean μ and variance δ^2 .

Again, mean correcting is done such that $\mathbb{E}[e^{-rt}S_t] = S_0$. This leads to $\mu^M = r - \frac{\sigma^2}{2} - \lambda(\exp(\mu + \delta^2/2) - 1)$. We have that the characteristic function of $\log(S_T)$ in the Merton pricing model (3.35) is given by

$$\phi_T(u; \sigma, \mu^M, \lambda, \delta, \mu) = \exp\left\{iu \log S_0 + T\left[-\frac{\sigma^2 u^2}{2} + i\mu^M u + \lambda(e^{-\delta^2 u^2/2 + i\mu u} - 1)\right]\right\}. \quad (3.36)$$

Example 3.3.3 (VG). As a last example the VG-processes, also discussed in Section 3.1.5, may be used to model financial assets. A financial model based on a VG-process may be defined in the following way,

$$S_t = S_0 \exp(\omega t + V_t). \quad (3.37)$$

As in Example 3.3.1, $\omega \in \mathbb{R}$ is a drift parameter that will be given by the martingale restriction on the price proces. V_t is a asymmetric VG-process, which time changing gamma process is assumed to have a mean rate per unit time equal to 1, and a variance rate v . (Recall Example 3.1.5, Section 3.1.5). Choosing

$$\omega = r + \frac{\log(1 - \theta v - \sigma^2 v/2)}{v}, \quad (3.38)$$

will give the discounted price process martingale properties. Finally, the characteristic function of $\log(S_T)$ in the model (3.37) is given by

$$\phi_T(u; \theta, \sigma, v) = \frac{e^{iu(\log S_0 + \omega T)}}{(1 - i\theta v u + \sigma^2 u^2 v/2)^{T/v}}. \quad (3.39)$$

For details see for instance Madan et al. [1998].

3.3.2 Implementing an exponential NIG-Lévy model

As an example of a parametric calibration, we fit an exponential NIG-process to market data. By substituting the characteristic function defined by (3.34) in the ‘‘Carr-Madan Equation’’ (3.30), we obtain the call option price by calculating Equation (3.31). Denote by $C_{\text{NIG}}(T_j, K_j; \alpha, \beta, \delta) := \tilde{C}_t(T_j, x_j)$ this price when the time to maturity equals T_j and the strike price $K_j = e^{x_j}$.

The inverse Problem 3.9 can now be written as the following non linear least square minimization problem.

Problem 3.10. Assume that we are given N market bid and ask prices $\{(\text{bid}_j, \text{ask}_j)\}$, $j = 1, \dots, N$, corresponding to call options having maturities T_j and strike prices K_j . Denote by $\mathcal{H}$ the set of all parameters satisfying the requirements $\alpha > 0$, $0 \leq |\beta| < \alpha$, and $\delta > 0$. Find the parameters $(\alpha, \beta, \delta) \in \mathcal{H}$ minimizing the objective function J :

$$\min_{(\alpha, \beta, \delta) \in \mathcal{H}} J(\alpha, \beta, \delta), \quad (3.40)$$

$$J(\alpha, \beta, \delta) := \sum_{j=1}^N \left(\frac{\partial C}{\partial \Sigma}(\overline{\Sigma}_j) \right)^{-2} \left(C_{\text{NIG}}(T_j, K_j; \alpha, \beta, \delta) - \overline{C}_j^* \right)^2.$$

$\overline{\Sigma}_j$ denotes the mid market implied volatility corresponding to strike price K_j and time to maturity T_j . $\overline{C}_j^* := \frac{\text{bid}_j + \text{ask}_j}{2}$ represents the mid market price of the observations.

As in Section 2.4.1, we minimize vega-weighted price differences.

Calculating the objective function

As in Section 2.4.1, we implemented the calibration procedure in Scilab, using the optimization routine ‘‘leastsq()’’. The objective function is now given by (3.40). We calculate option prices by using a FFT, see for instance Lee [2004], Carr and Madan [1999]. The discussion below follows mainly Borak, Detlefsen, and Härdle [2005].

From Equation (3.31), we have that option prices can be calculated as,

$$\tilde{C}_t(T, x) = \frac{e^{-\alpha x}}{2\pi} \int_{-\infty}^{\infty} e^{-ivx} \psi_T(v) dv \approx \frac{e^{-\alpha x}}{\pi} \int_0^h e^{-ivx} \psi_T(v) dv. \quad (3.41)$$

h is a constant, giving an upper bound of the integral tail. Now we discretize. Set $v_k := \eta k$, $k = 0, \dots, M-1$. $\eta > 0$ is the distance between points on the integration grid, implying $M\eta = h$. M should be a power of 2. We have,

$$(3.41) \approx \frac{e^{-\alpha x}}{\pi} \sum_{k=0}^{M-1} e^{-iv_k x} \psi_T(v_k) \eta. \quad (3.42)$$

We would like to calculate this sum using a FFT.

Remark 3.11. A FFT is an efficient algorithm to calculate the discrete Fourier transform (DFT). That is, a FFT calculates a sum of the following form,

$$F_u = \sum_{k=0}^{M-1} f_k e^{-2\pi i u \frac{k}{M}}, \quad u = 0, \dots, M-1. \quad (3.43)$$

Where f_k is some given vector. Recall the continuous Fourier transform of a function f ,

$$\mathcal{F}[f(t)](\nu) := \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t} dt. \quad (3.44)$$

The DFT is a generalization of (3.44) to the case of a discrete function; Let $f_k := f(t_k)$ where $t_k := \Delta k$ and $k = 0, \dots, M-1$. This leads to a DFT $F_u := \mathcal{F}_k[\{f_k\}_{k=0}^{M-1}](u)$ as in Equation (3.43).

We will now transform the option pricing formula of Equation (3.42) into a sum of the form of (3.43). We are interested in option prices having strike prices close to the spot price S_t . Setting $x_u := -\frac{1}{2}M\zeta + \zeta u + \log(S_t)$ in (3.42) will give us a set of call option prices having log-strikes at an equidistant grid around the log spot price and maturity time T . We get

$$\tilde{C}_t(T, x_u) \approx \frac{e^{-\alpha x_u}}{\pi} \sum_{k=0}^{M-1} e^{-iku\eta\zeta} e^{iv_k(\frac{1}{2}M\zeta - \log(S_t))} \psi_T(v_k) \eta. \quad (3.45)$$

Now, set $\eta\zeta = \frac{2\pi}{M}$ and $f_k = e^{iv_k(\frac{1}{2}M\zeta - \log(S_t))} \psi_T(v_k)$ and we retrieve a sum of the form (3.43). Hence we can use the known, efficient, FFT-algorithm.

Since the FFT-algorithm will result in M option prices, with log-strikes centered around $\log(S_t)$, we retrieve $C_{\text{NIG}}(T_i, K_j; \alpha, \beta, \delta)$ of the objective function (3.40) by interpolating the prices $\tilde{C}_t(T_i, \cdot)$ at $x = \log(K_j)$.

We also have that, choosing a small ζ will result in many prices near the spot. Since we have $\eta = \frac{2\pi}{\zeta M}$, this will give a big η and hence the integration steps gets large. Therefore there is a trade-off between accuracy of the integral, and number of strikes near the spot. Using the trapezoidal rule to calculate the integral will improve the calculations.

Note that option prices calculated by (3.45) have a fix maturity time. When calibrating a NIG-process across strikes and maturities, e.g. to option prices having H different maturities $\{T_1, \dots, T_H\}$, we calculate the objective function (3.40) as follows,

$$J(\alpha, \beta, \delta) := \sum_{i=1}^H \sum_{j=1}^{n_i} \left(\frac{\partial C}{\partial \Sigma}(\bar{\Sigma}_{i,j}) \right)^{-2} \left(C_{\text{NIG}}(T_i, K_j; \alpha, \beta, \delta) - \bar{C}_{i,j} \right)^2. \quad (3.46)$$

n_i is the number of options with time to maturity T_i . $\bar{\Sigma}_{i,j}$ and $\bar{C}_{i,j}$ denotes the mid market implied volatility, resp. mid market price, corresponding to strike price K_j and time to maturity T_i .

Table 3.1: Calibrated NIG parameters, TTM=0.1

δ	α	β
7.4898923	160.00162	-0.2423393

Figures

In Figure 3.3, implied volatilities resulting from calibrating an exponential-NIG process to market prices of a fixed time to maturity is shown. The calibrated parameters are as in Table 3.3.2.

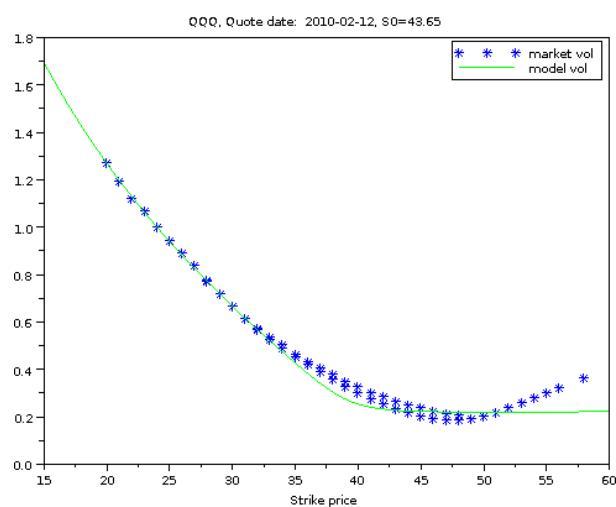


Figure 3.3: NIG- implied volatilities, TTM = 0.1

Figures 3.4 and 3.5 show call prices, resp. implied volatilities, resulting from calibrating an exponential NIG-process across strikes and maturities. That is, the process has been calibrated to all available options on the underlying on a certain quotation day.²⁰ Model parameters are as in Table 3.2.

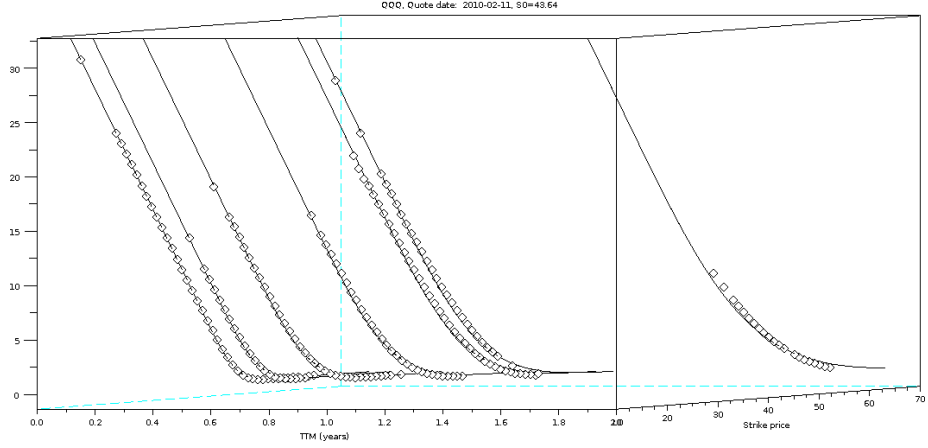


Figure 3.4: NIG - call option prices, fit to market data

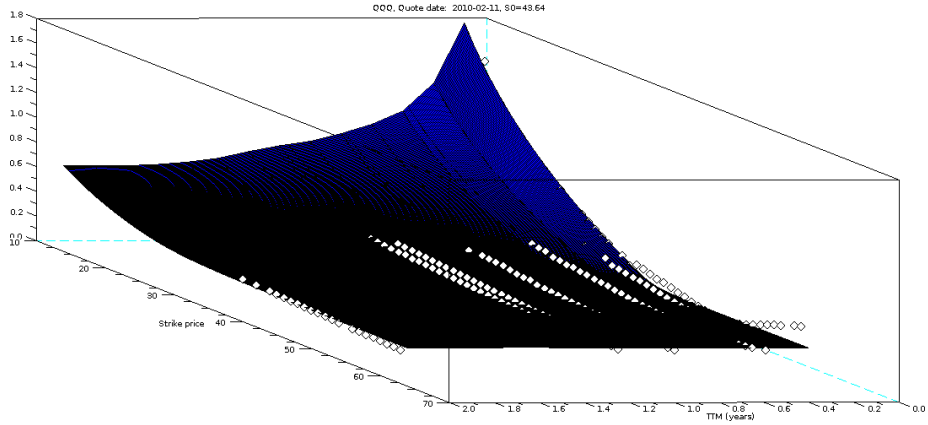


Figure 3.5: NIG - implied volatilities, fit to market data

Table 3.2: Calibrated NIG parameters

δ	α	β
7.9504049	164.99187	-0.4678606

²⁰Again the market data consists of options on the exchange traded fund following the Nasdaq 100 index, (QQQQ). Quotation date 2010-02-11, source: <http://finance.yahoo.com/q?s=QQQQ>

3.3.3 Semi-parametric calibration

Minimizing relative entropy

A non-parametric approach to solve the inverse Problem 3.9 is taken by Cont and Tankov [2004]. They use a least square method and regularize the problem by introducing a penalization by the relative entropy (Kullback-Leibler distance) with respect to an a priori chosen measure. For example the prior probability measure may correspond to a parametric exponential Lévy model, estimated from historical data. In this way they take into account prior information about the distribution. However, the diffusion coefficient of the calibrated process will be fixed. This is due to the fact that the probability measures defined by two Lévy processes are absolutely continuous only if their diffusion components are equivalent, and the relative entropy becomes infinite if a measure is not absolutely continuous with respect to the prior. See Cont and Tankov [2004] Theorem A.1.

Spectral calibration

Another semiparametric approach of calibrating a Lévy process to market prices is the *spectral calibration* method, Belomestny and Reiß [2006a, 2006b]. This method assumes an exponential Lévy stock price process $(S_t)_{t \in [0, T]}$,

$$S_t = S_0 \exp(rt + X_t). \quad (3.47)$$

Here, r is as usual the interest rate which is assumed to be fix. $(X_t)_{t \geq 0}$ is a Lévy process, with characteristic triplet (γ, σ, ν) to be estimated from market data. It is further assumed that $(X_t)_{t \geq 0}$ is a finite activity process $(\int_{\mathbb{R}} \nu(dx) < \infty)$.²¹ The characteristic function of X_T is as before given by the Lévy-Khintchine formula (3.8),

$$\phi_T(u) := \mathbb{E}[e^{iuX_T}] = \exp \left[T \left(i\gamma u - \frac{u^2 \sigma^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux \mathbf{1}_{\{|x| < 1\}}) \nu(dx) \right) \right]. \quad (3.48)$$

It is assumed that at time $t = 0$ prices for European call (and put) options on S_T with different strike prices and possibly different maturities are available.

The method can be applied to option data having one fixed time to maturity, or option data having different maturities can be aggregated to retrieve one Lévy triplet capturing the information in a surface of option prices. As pointed out in Section 3.3, calibrating one Lévy triplet to options of different maturities normally results in poor fitting results due to stationarity of the increments of the Lévy process. Therefore we consider only the case with fix maturity T .

The method relies on the following formula

$$\begin{aligned} \Psi(v) &:= \frac{1}{T} \log \left(1 + iv(1 + iv)\mathcal{F}(O(v)) \right) = \frac{1}{T} \log(\phi_T(v - i)) \\ &= -\frac{\sigma^2 v^2}{2} + i(\sigma^2 + \gamma)v + (\sigma^2/2 + \gamma - \lambda) + \mathcal{F}(e^v \nu(v)). \end{aligned} \quad (3.49)$$

²¹Possible extension to the infinite activity case is discussed in Belomestny and Reiß [2005].

Here $\mathcal{F}((O(v)))$ represents the Fourier inversion of modified option prices, see below, which will be given by the market. $\mathcal{F}(e^v \nu(v))$ is the Fourier inversion of the exponentially weighted Lévy measure. λ is the intensity of the Lévy measure, $\lambda := \int_{\mathbb{R}} \nu(dx)$. Note that up to a shift in the argument, $\Psi(v)$ equals the cumulant generating function of the Lévy process X .

At time t , given observed option prices $C_t(T, K_j)$ having one fixed maturity time T and strikes K_j , $j = 0, \dots, N$, the modified option price $O_t(T, x_j)$, $x_j := \log(K_j/S_t) - rT$, is defined as

$$O_t(T, x_j) := C_t(T, S_t e^{x_j + rT})/S_t - (1 - e^{x_j})^+. \quad (3.50)$$

Since we consider options on the underlying S having one fixed maturity time $T > t$ at time $t = 0$, we write $O(x_j) := O_t(T, x_j)$ for ease of notation.

Now, as a first step of the calibration procedure it is suggested that a natural cubic spline is fit to the observations $O(x_j)$, giving a function $\hat{O}(x)$ representing the observed prices. In the example implemented below, Section 3.3.4, we simply linearly interpolate between available option prices to retrieve $\hat{O}(x)$. Thereafter the Fourier inversion of the function $\hat{O}(x)$ is calculated, giving an estimate,

$$\mathcal{F}(\hat{O}(v)) := \int_{-\infty}^{\infty} \hat{O}(x) e^{ivx} dx. \quad (3.51)$$

Now an estimate of the function Ψ can be retrieved,

$$\hat{\Psi}(v) = \frac{1}{T} \log \left(1 + iv(1 + iv)\mathcal{F}(\hat{O}(v)) \right). \quad (3.52)$$

If we were to calibrate to option prices having different maturities T_k , $k = 1, \dots, M$, the above procedure is to be repeated for each T_k separately. Thereafter the different estimates of Ψ for all T_k should be aggregated.

Having $\hat{\Psi}(v)$, an estimate of the parametric part $(\sigma, \gamma, \lambda)$ is obtained by taking into account the polynomial structure of Ψ . As in Equation (3.49),

$$\Psi(v) = -\frac{\sigma^2 v^2}{2} + i(\sigma^2 + \gamma)v + (\sigma^2/2 + \gamma - \lambda) + \mathcal{F}(e^v \nu(v)).$$

Estimates for σ, γ and λ are obtained as the coefficients of the quadratic polynomial, under the presence of the nonparametric part $\mathcal{F}(e^v \nu(v))$ and possible noise from the market data. The authors compare this with *partial linear models*, but a difference being that the function $\mathcal{F}(e^v \nu(v))$ is not assumed to be smooth, but to be decaying for high frequencies. Higher frequencies are therefore cut off at a level U depending on noise level and smoothness of the Lévy measure. The cut-off by the value U is the way in which the inverse Problem 3.9 is regularized. The choice of U is hence crucial in this method. The estimates are found as

$$\hat{\sigma}^2 := \int_{-U}^U \text{Re}(\hat{\Psi}(u)) w_{\sigma}^U(u) du, \quad (3.53)$$

$$\hat{\gamma} := \int_{-U}^U \text{Im}(\hat{\Psi}(u)) w_{\gamma}^U(u) du, \quad (3.54)$$

$$\hat{\lambda} := \int_{-U}^U \text{Re}(\hat{\Psi}(u)) w_{\lambda}^U(u) du. \quad (3.55)$$

Here $w_\sigma^U, w_\gamma^U, w_\lambda^U$ are weight functions, examples are given by [Belomestny and Reiß \[2006b\]](#).

This step can be seen as an orthogonal projection estimate with respect to a weighted L^2 -scalar product. The weight functions include the choice of a smoothening parameter, reflecting the smoothness of the calibrated Lévy measure. In practice, we start with finding the estimate $\hat{\sigma}$. This is found as the value where (3.53) stabilizes when increasing U . For an example, see Figure 3.6. This will also give us the cut off value $\bar{U}$, chosen as the value where $\hat{\sigma}$ as a function of U stabilizes. E.g. from Figure 3.6 a), we would choose $\bar{U} = 25$. Having $\bar{U}$, we next calculate $\hat{\gamma}$ and $\hat{\sigma}$ from (3.54) and (3.55), setting $U = \bar{U}$.

As a last step, having the parametric estimate at hand, the Fourier inversion of the exponentially weighted Lévy measure is obtained as the remainder. The estimate of the Lévy measure is found as follows,

$$\hat{\nu}(x) := e^{-x} \mathcal{F}^{-1} \left[\left(\hat{\Psi}(v) + \frac{\hat{\sigma}^2}{2}(v-i)^2 - i\hat{\gamma}(v-i) + \hat{\lambda} \right) K(v) \right](x). \quad (3.56)$$

Here $K(v)$ is a compactly supported kernel dependent on a second cut-off parameter U_ν . [Belomestny and Reiß \[2006b\]](#) propose,

$$K(v) := (1 - (\frac{v}{U_\nu})^2)^+.$$

Also U_ν can be chosen in a data driven way, as the value where the measure $\hat{\nu}$ as a function of U_ν stabilizes.

It is important to note that the ill-posedness of the problem depends on the size of the diffusion term σ and time to maturity T and this becomes apparent in the spectral calibration results. For bigger σ and time to maturities, the parametric part in the calibration procedure does not stabilize and hence the method fails. An heuristic explanation to this is that the variance of the diffusion part of X_T increases with larger maturities and larger σ , $\text{Var}[\sigma B_T] = T\sigma^2$. This makes it more difficult to determine σ for longer time to maturities. Also, as time increases it becomes more difficult to separate the diffusion term from the effect of jumps. Having a large increment on a very short time horizon indicates a jump, whereas on longer time horizons it is hard to say if the change is due to a continuous diffusion or many small jumps/few bigger jumps.

3.3.4 Implementing the spectral calibration procedure

We test the spectral calibration approach by applying it to option prices from the Merton jump diffusion model, as defined in Section 3.3.1, equation (3.35). First, we produce 100 option prices using the standard FFT method (as described in Section 3.3.2 in the case of NIG distribution). We choose the options to have a fix maturity time T , and random strike prices centered around moneyness. Thereafter, we modify these prices according to Equation (3.50) and interpolate to retrieve a pricing formula $\hat{O}(x)$. Thereafter, the Fourier transform $\mathcal{F}\hat{O}(x)$ is calculated using a FFT transform and the function $\hat{\Psi}$ is calculated according to Equation (3.52). Having $\hat{\Psi}$, we next calculate $\hat{\sigma}^2$ from Equation (3.53). We choose the function w_σ^U as in [Belomestny and Reiß \[2006b\]](#).

In Figure 3.6 the estimate $\hat{\sigma}$ is plotted as a function of the cut-off parameter U . The “spectral estimate” (green) is compared with an estimate resulting from the spectral calibration procedure as described above, but *where the true characteristic function of the Merton jump model is used in place of the set of option prices*, to retrieve the function Ψ in equation (3.49). That is we use the relation

$$\Psi(v) = \frac{1}{T} \log(\phi_T(v - i)). \quad (3.57)$$

We have that the characteristic function $\phi_T(u)$ is known in the Merton jump diffusion model, see Equation (3.36). This “*true estimate*” is plotted in blue. In the left plot the diffusion term in the Merton jump model equals 0.1 and the option maturity 0.1 years. In the right plot the diffusion term equals 0.1 and the option maturity 0.5 years. In the second case the ill-posedness of the problem becomes apparent and an estimate of the diffusion term using the spectral calibration method becomes very difficult.

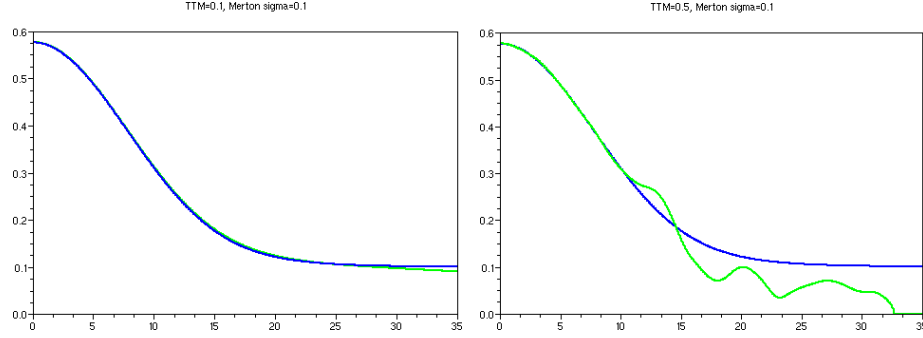


Figure 3.6: “Spectral estimate” of MJD diffusion parameter σ .

a) $\sigma = 0.1$, $TTM = 0.1$

b) $\sigma = 0.1$, $TTM = 0.5$

In the first case however, from Figure 3.6 a), we choose $U = 25$ and calculate $\hat{\sigma}$, $\hat{\gamma}$ and $\hat{\lambda}$ from the formulas (3.53), (3.54), (3.55). Thereafter we can use formula (3.56) to retrieve the Lévy density. Again we use a FFT approximation for the inverse Fourier transform in Equation (3.56). In Figure 3.7 the estimated Lévy measure $\hat{\nu}$ using the same parameters as in figure 3.6 a) is plotted (green). Again for comparison the true Merton Lévy density is plotted (blue).

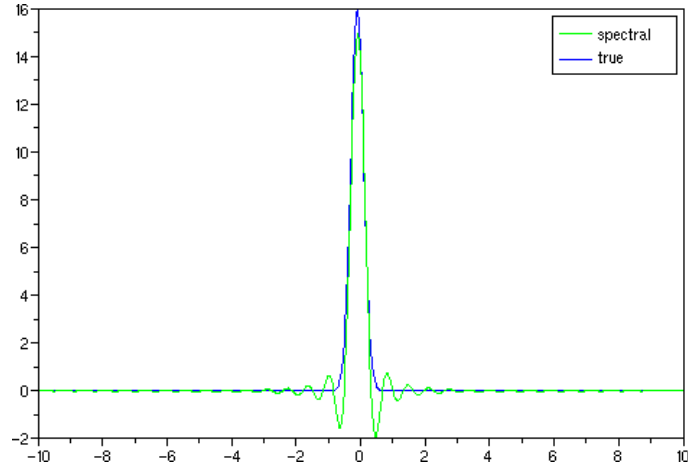


Figure 3.7: Estimated Lévy density $\hat{\nu}$ (green). True Merton Lévy density (blue)

As a second test of the spectral calibration method, we use it to calibrate an exponential Lévy model to market option data.²² Again, the method is applied only to options having one fixed time to maturity. We use the same procedure as in the “Merton-test” above, the only difference being the replacement of the Merton option prices with real market data. In Figure 3.8 a-c estimates of the diffusion term σ as a function of the cut-off parameter U are shown.

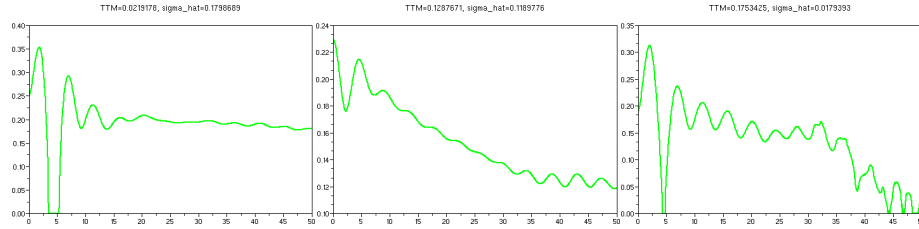


Figure 3.8: a) $TTM = 8 \text{ days}$ b) $TTM = 47 \text{ days}$ c) $TTM = 64 \text{ days}$

The market data consists of call options on the SPDR S&P 500 exchange traded fund (SPY). On the day of observation we were given 156 quotes of options having time to maturity (TTM) = 8 days, 304 quotes having TTM = 47 days and 131 quotes having TTM = 64 days. Again we see that except for very short time to maturities, it is very difficult to estimate the parametric part of the Lévy triplet using the spectral calibration method.

²²Calibration is done to prices of options on an exchange traded fund following the S&P 500 index, (SPY). The quotation day is 2010-02-11. Source: <http://finance.yahoo.com/q?s=SPY>.

3.4 Additive processes

Several approaches extend the “pure Lévy models” (see Section 3.2), in order to solve the problem of not being able to fit one Lévy process to option prices of multiple maturity times simultaneously. This problem is mentioned also in Section 3.3 and is due to the homogeneity property of the Lévy processes. One approach to solve the problem is to model financial assets as a Lévy process with stochastic volatility, for instance as in the model of Bates [1996]. Another approach is to model asset prices with what is known as *additive processes*. An additive process is a time inhomogeneous Lévy process.

Definition 3.12. Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, $\mathcal{F} = \mathcal{F}_T$ and $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be a filtered probability space satisfying the usual conditions.²³ An adapted, $\mathbb{R}^d$ -valued stochastic process $A = (A_t)_{t \in [0, T]}$ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, with $A_0 = 0$ a.s. is called a *additive process* if it has the following properties,

- (i) *Independence of increments*: for every increasing sequence of times $t_0, \dots, t_n$, the random variables $A_{t_0}, A_{t_1} - A_{t_0}, \dots, A_{t_n} - A_{t_{n-1}}$ are independent. That is, $A_t - A_s \perp\!\!\!\perp \mathcal{F}_s$.
- (ii) *Stochastic continuity*: $\forall t \geq 0$ and $\forall \varepsilon > 0$, $\lim_{s \rightarrow t} \mathbb{P}(|A_t - A_s| > \varepsilon) = 0$.

As in Definition 3.1, the properties (i)-(ii) imply that there exists a càdlàg modification of A . Jacod and Shiryaev [2003] (see chapter II def. 4.1) abbreviates this type of process by PIIAC (Process with Independent Increments and Absolutely Continuous characteristics). We will consider only $\mathbb{R}$ -valued additive processes.

We have that an additive process $(A_t)_{t \geq 0}$ allows for deterministic time inhomogeneities. This preserves almost all the tractability we have of the Lévy processes. For example every additive process has infinitely divisible distribution, Sato [1999] Theorem 9.1. It follows that a one dimensional additive process $(A_t)_{t \in [0, T]}$ has characteristic function given by

$$\mathbb{E}[e^{iuA_t}] = \exp \left[\psi(u, t) \right], \quad u \in \mathbb{R} \quad (3.58)$$

$$\psi(u, t) := iu\Gamma(t) - \frac{u^2\Sigma(t)}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x| < 1\}}) \eta(t, dx). \quad (3.59)$$

The triplet $\{\Gamma(t), \Sigma(t), \eta(t, \cdot)\}_{t \geq 0}$ must satisfy the following conditions,

- $\forall t$, $\Sigma(t) \geq 0$ and $\eta(t, \cdot)$ is a positive measure on $\mathbb{R}$ satisfying $\eta(t, \{0\}) = 0$ and $\int_{\mathbb{R}} (|x|^2 \wedge 1) \eta(t, dx) < \infty$.
- $\Gamma(0) = 0$, $\Sigma(0) = 0$, $\eta(0, \cdot) = 0$ and for all $s \leq t$, $\Sigma(t) - \Sigma(s) \geq 0$ and $\eta(t, B) \geq \eta(s, B)$, $\forall$ measurable sets $B \in \mathcal{B}(\mathbb{R})$.
- If $s \rightarrow t$, then $\Gamma(s) \rightarrow \Gamma(t)$, $\Sigma(s) \rightarrow \Sigma(t)$ and $\eta(s, B) \rightarrow \eta(t, B)$, $\forall B \in \mathcal{B}(\mathbb{R})$ s.t. $B \subset \{x : |x| \geq \epsilon\}$, for some $\epsilon > 0$.

Compare with the Lévy Khintchine formula in equation (3.8). See Cont and Tankov [2003] Theorem 14.1 for details.

²³For definition see e.g. Protter [2003], Chapter I.

Remark 3.13. (See [Cont and Tankov \[2003\]](#) Theorem 14.1). One way to construct a triplet $\{\Gamma(t), \Sigma(t), \eta(t, \cdot)\}_{t \geq 0}$ fulfilling the above is as follows;

$$\Sigma(t) := \int_0^t \sigma^2(s) ds \quad (3.60)$$

Here $\sigma : [0, T] \rightarrow \mathbb{R}$ is a continuous function satisfying $\int_0^T \sigma^2(t) dt < \infty$.

$$\eta(t, dx) := \int_0^t \nu(s, dx) ds \quad (3.61)$$

Here $\{\nu(t, dx)\}_{t \in [0, T]}$ is a family of Lévy measures, satisfying $\int_0^T \int_{\mathbb{R}} (1 \wedge |x|^2) \nu(t, dx) dt < \infty$. And at last,

$$\Gamma(t) := \int_0^t \gamma(s) ds, \quad (3.62)$$

where $\gamma : [0, T] \rightarrow \mathbb{R}$ is a deterministic function with finite variation.

In financial models, replacing the Lévy process by an additive (time inhomogeneous) process makes the model more flexible. It is hence possible to reproduce option prices across strikes and maturities with an higher accuracy than in the case of a “classical” Lévy model. Calibration of additive processes may be done using similar methods as described above for Lévy processes. For instance by calibrating one Lévy triplet to each maturity separately, under the requirement that the properties above are fulfilled. For an alternative calibration procedure, see Section 6.1 below.

3.5 A Lévy density code-book definition à la Carmona, Nadtochiy [2009]

Already in Section 3.1.3 it became clear that a triplet (γ, σ, ν) uniquely characterizes a Lévy process. In a similar way, as follows from Equation (3.58), the triplet $\{\Gamma(t), \Sigma(t), \eta(t, \cdot)\}_{t \geq 0}$ characterizes an additive process. But this alone does not explain why a Lévy density may be used as a *code-book* for the information contained in the market prices of options. In this section the *Lévy density code-book* is described. We follow the setup of [Carmona and Nadtochiy \[2009\]](#). For simplicity we assume zero interest rates and a underlying paying no dividends. As [Carmona and Nadtochiy \[2009\]](#), we assume that the stock price follows an exponential additive process,

$$S_t = \exp(X_t). \quad (3.63)$$

More precisely, $X = (X_t)_{t \geq 0}$ is a pure jump time inhomogeneous Lévy process with deterministic compensator $\eta(du, dx) = \kappa(u, x) dx du$,

$$X_T := \log(S_0) - \int_0^T \int_{\mathbb{R}} (e^x - x - 1) \kappa(u, x) dx du + \int_0^T \int_{\mathbb{R}} x (\mu^X(dx, du) - \kappa(u, x) dx du). \quad (3.64)$$

$\mu^X(dx, dt)$ is the random jump measure (Poisson) associated with the jumps of X and $\kappa(T, x)$ is the time inhomogeneous Lévy density of X . Being an *infinite*

activity process (see Section 3.1.1), $\kappa(u, x)dxdu$ will not be finite but has to fulfill the following for each $t > 0$,

$$\int_0^t \int_{\mathbb{R}} (|x| \wedge 1) |x| (1 + e^x) \kappa(u, x) dx du < \infty. \quad (3.65)$$

Applying Itô's formula for jump processes, recall formula (3.10), to $g(x) = \exp(x)$, we find the dynamics of the stock to be governed by a stochastic integral equation;

$$\begin{aligned} S_T &= \exp(X_0) + \int_0^T S_{t-} dX_t + \sum_{0 \leq t \leq T, \Delta X_s \neq 0} [e^{X_{t-} + \Delta X_t} - e^{X_{t-}} - \Delta X_s e^{X_{t-}}] \\ &= S_0 - \int_0^T \int_{\mathbb{R}} S_{t-} (e^x - x - 1) \kappa(u, x) dx du \\ &\quad + \int_0^T \int_{\mathbb{R}} S_{t-} x (\mu^X(dx, du) - \kappa(u, x) dx du) + \int_0^T \int_{\mathbb{R}} S_{t-} (e^x - 1 - x) \mu^X(dx, du) \\ &= S_0 + \int_0^T \int_{\mathbb{R}} S_{u-} (e^x - 1) (\mu^X(dx, du) - \kappa(u, x) dx du). \end{aligned} \quad (3.66)$$

It follows that S_t is a non-negative martingale.

Assume now that pricing is done via expectations under a risk-neutral measure. Denote the level of the underlying at time t by $\bar{S}$. Then the theoretical price of a European call option on S , having maturity time T and strike price e^x is given by

$$\tilde{C}_t(T, x) = \mathbb{E}[(S_T - e^x)^+ | S_t = \bar{S}]. \quad (3.67)$$

The price $\tilde{C}_t(T, x)$ will be determined by the distribution of $(S_u)_{u \in [t, T]}$ conditional on $S_t = \bar{S}$. We know from equation (3.58) Section 3.4 that this distribution is uniquely given by the price level $\bar{S}$ and the measure $\kappa(u, x)dxdu$, as long as this is absolutely continuous. This shows us that we will have a *one-to-one correspondence* between the density $\kappa(T, x)$ and the option prices. In what follows we make this statement more precise.

By following the derivation steps of Carr et al. [2004] or of Cont and Voltchkova [2005]²⁴ we find European call option prices, under the assumption that the asset price follows equation (3.66), to be given as solutions to the following partial integro differential equation,

$$\begin{cases} \partial_T \tilde{C}_t(T, x) = \int_{\mathbb{R}} \psi(\kappa(T, \cdot); x - y) D_y \tilde{C}_t(T, y) dy \\ \tilde{C}_t(t, x) = (\bar{S} - e^x)^+. \end{cases} \quad (3.68)$$

Here $D_y = \partial_y^2 - \partial_y$ and $\psi(f; x)$ is the *double exponential tail* of the function f . $\psi(f; x)$ is introduced in Carr et al. [2004] and defined as follows,

$$\psi(f; x) := \begin{cases} \int_{-\infty}^x e^z \int_{-\infty}^z f(u) du dz, & x < 0 \\ \int_x^{\infty} e^z \int_z^{\infty} f(u) du dz, & x > 0 \end{cases} \quad (3.69)$$

²⁴Carr et al. [2004] and Cont and Voltchkova [2005] derive a pricing PIDE for European call options under the assumption that the price process follows an exponential Lévy process.

or equivalently,

$$\psi(f; x) = \begin{cases} \int_{-\infty}^x (e^x - e^z) f(z) dz, & x < 0 \\ \int_x^{\infty} (e^z - e^x) f(z) dz, & x > 0. \end{cases} \quad (3.70)$$

Note that the double exponential tail of $\kappa(T, \cdot)$ integrates over the jump sizes x of the density $\kappa(T, x)$, keeping the time parameter fix.

Now, differentiate (3.68) with respect to x , in order to make the initial condition of (3.68) integrable on $\mathbb{R}$ and introduce $\Delta(T, x) = -\partial_x \tilde{C}(T, x)$. (3.68) then becomes

$$\begin{cases} \partial_T \Delta_t(T, x) = \int_{\mathbb{R}} \psi(\kappa(T, \cdot); x - y) D_y \Delta_t(T, y) dy \\ \Delta_t(t, x) = e^x \mathbf{1}_{(-\infty, \log(\bar{S})]}(x). \end{cases} \quad (3.71)$$

At last, by using Fourier transformation in x it is possible to solve (3.71). Let “hat” denote values in Fourier space, in particular,

$$\hat{\psi}(f; \xi) := \int_{\mathbb{R}} e^{-2\pi i x \xi} \psi(f; x) dx. \quad (3.72)$$

(3.71) becomes,

$$\begin{cases} \partial_T \hat{\Delta}_t(T, \xi) = \hat{\psi}(\kappa(T, \cdot); \xi) \hat{\Delta}_t(T, \xi) (-4\pi^2 \xi^2 - 2\pi i \xi) \\ \hat{\Delta}_t(t, \xi) = \frac{e^{\log(\bar{S})(1-2\pi i \xi)}}{1-2\pi i \xi}. \end{cases} \quad (3.73)$$

(3.73) gives a mapping from call prices (as given by $\hat{\Delta}$) to the density κ (as given by $\hat{\psi}$). Furthermore, solving (3.73) leads to an analytic expression for call prices in terms of κ . See [Carmona and Nadtochiy \[2009\]](#) for details.

We have that at each time t , the density $\kappa(T, x)$ can be chosen as an alternative code-book for the information contained in the surface of market prices,

$$\{\tilde{C}_t(T, x); T > t, x \in \mathbb{R}\} \rightleftharpoons \{\kappa_t(T, x); T > t, x \in \mathbb{R}\}. \quad (3.74)$$

We think of $\kappa_t(T, x)$ as the nonnegative function such that the prices given by (3.67) (or equivalently by (3.73)) matches the prices given by the market. Since the market prices change, so will the function $\kappa_t(T, x)$. This motivates the subscript t . This is our third code-book example and the one that will be implemented in a dynamic setting in Section 6. In Section 6.7 we show an alternative way of calculating option prices from $\kappa_t(T, x)$ by using the Lévy Khintchine formula and an FFT-algorithm.

Part II

Dynamic

code-books/Market-models

A market model aims at explaining or predicting the behavior of a complete market, as opposed to explaining or predicting the behavior of a single stock. We consider a market consisting of one underlying asset and its liquidly traded derivative instruments (the European call and put options)²⁵. On this market we wish to model simultaneously the dynamics of the European options and the underlying, without introducing arbitrage. As mentioned already in Section 1, we call this a *Equity market model* or a *market model for stock options*. The model will use the set of observable option prices $\{C_t(T, K)\}_{T \geq t, K \geq 0}$ as fundamental market data in addition to the underlying stock (or index).

This kind of models have greatly been inspired by the Heath-Jarrow-Morton approach in the fix income theory. Heath, Jarrow, and Morton [1992] model the evolution of the instantaneous *forward rate curve*, as opposed to directly model bond prices. And just as is the case in the interest rate theory, a direct modeling of the additional variable in a market model for stock options will be difficult. In other words, we do not want to construct a market model by postulating dynamics for the stock price $(S_t)_{t \geq 0}$ and call prices $(C_t(T, K))_{t \geq 0}$ directly. E.g.

$$\begin{cases} dS_t = S_t \sigma_t dB_t^1 \\ dC_t(T, K) = C_t(T, K)[\alpha_t(T, K)dt + \beta_t(T, K) \cdot dB_t], \end{cases} \quad (3.75)$$

where $(\alpha_t(T, K))_{t \geq 0}$, $(\beta_t(T, K))_{t \geq 0}$ and $(\sigma_t)_{t \geq 0}$ are adapted processes, assumed to fulfill proper integrability assumptions. B_t is a Brownian motion.

Difficulties will arise in this type of models due to the requirement of no arbitrage. In particular no static arbitrage implies non trivial constraints on the option prices which will be difficult to fulfill in a dynamic setting.

By using a code-book to represent the option prices, it is possible to simplify the requirements needed for the model to fulfill conditions of no arbitrage. Schönbucher [1999] and Schweizer and Wissel [2007, 2008] proposes using *implied volatilities* as the code-book to be used in a market model for stock options. In the following two sections (Section 4 and 5) we discuss the approach of using the *local volatility code-book*, and the *Lévy density code-book* in a dynamic setting. We follow the work of Carmona and Nadtochiy [2008], [2009].

²⁵Since we can use the put-call parity to represent European put option prices in terms of European call option prices we will consider only the European call option prices.

4 Dynamic local volatility/setting Dupire in motion

4.1 Local volatility dynamic models

In this section we discuss the approach of setting local volatility in motion. I.e., to use the local volatility code-book in place of the call option prices $C_t(T, K)$ in equation (3.75). Thereby, we rely on the type of models presented in Section 2.2 and the one-to-one correspondence between European option prices and local volatilities,

$$\{C_t(T, K); T > t, K > 0\} \rightleftharpoons \{a_t(T, K); T > t, K > 0\}. \quad (4.1)$$

$C_t(T, K)$ is as before the price of a European call option on an underlying S having maturity time T and strike price K . $a_t(T, K)$ is the local volatility (recall Equation (2.33)) of S at time t . This relation stems from the Dupire formula and the pricing formula (see Equation (4.4)), which tells us how to calculate local volatilities from market prices and vice versa. We have the Dupire formula as follows,

$$a_t(T, K) = \sqrt{2 \frac{\frac{\partial}{\partial T} C_t(T, K) + rK \frac{\partial}{\partial K} C_t(T, K)}{K^2 \frac{\partial^2}{\partial K^2} C_t(T, K)}}. \quad (4.2)$$

When basing a market model on the *local volatility code-book*, we assume that the underlying follows a “regular enough” Itô process, in the sense of Gyöngy [1986]. That is, the underlying is a real-valued one-dimensional Itô process, with dynamics of the following form,

$$dX_t = \alpha(t, \omega)dt + \beta(t, \omega)dW_t. \quad (4.3)$$

Here $X_0 = 0$ and W_t is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. $\alpha(t, \omega)$ and $\beta(t, \omega)$ satisfy the regularity conditions of an Itô process, see Definition 2.1. $\omega \in \Omega$ represents the dependence on some arbitrary variables.

The idea behind *local volatility dynamic models*, is to use the code-book (4.1) and postulate market dynamics through stochastic differential equations for the stock price and the local volatility surface. This approach is taken for instance by Derman and Kani [1998] and Carmona and Nadtochiy [2008]. In this section we discuss the model due to Carmona and Nadtochiy [2008]. Hence the market consists of one underlying asset, with price given by $S = (S_t)_{t \geq 0}$, and its liquidly traded derivatives (the European call options). All derivatives included in the model are written on the underlying S . For simplicity it is assumed both zero interest rate and that the underlying S pays no dividends. The notations $\hat{C}_t(\tau, x)$ and $\hat{a}_t(\tau, x)$ are used for the call price $C_t(T, K)$ respectively the local volatility $a_t(T, K)$ expressed in the variables $\tau = T - t$ and $x := \log(K)$. This notation corresponds to the Musiela parameterization in the interest rate theory, see Musiela [1993]. The pricing Equation (2.42), Section 2.2, defining together with (4.2) the *local volatility code-book* is in this notation,

$$\begin{cases} \partial_\tau \hat{C}_t(\tau, x) = \frac{1}{2} \hat{a}_t^2(\tau, x) (\partial_{xx}^2 \hat{C}_t(\tau, x) - \partial_x \hat{C}_t(\tau, x)), \\ \hat{C}_t(0, x) = (S_t - e^x)^+. \end{cases} \quad (4.4)$$

Now, under a risk neutral measure $\mathbb{Q}$, market dynamics in *local volatility dynamic models* are modeled by the following stochastic differential equations,

$$\begin{cases} dS_t = S_t \sigma_t dB_t^1, & S_0, \\ d\hat{a}_t^2(\tau, x) = \hat{a}_t^2(\tau, x) [\hat{\alpha}_t(\tau, x) dt + \hat{\beta}_t(\tau, x) \cdot dB_t], & \hat{a}_0^2(\tau, x). \end{cases} \quad (4.5)$$

Here, for all fix $\tau > 0$, $x \in \mathbb{R}$ the processes $\hat{\alpha}(\tau, x) = (\hat{\alpha}_t(\tau, x))_{t \geq 0}$, $\hat{\beta}(\tau, x) = (\hat{\beta}_t^1(\tau, x), \dots, \hat{\beta}_t^m(\tau, x))_{t \geq 0}$ and $\sigma = (\sigma_t)_{t \geq 0}$ are adapted to the filtration generated by the m -dimensional Brownian motion $\mathbb{B} = (B^1, \dots, B^m)$. In addition, in order to assure that all forthcoming objects are well defined, it is assumed that the processes fulfill certain regularity assumptions. For instance the spot volatility process σ is assumed to have almost surely continuous paths, and for any $t \geq 0$,

$$\mathbb{E} \left[\int_0^t S_u^2 \sigma_u^2 du \right] < \infty.$$

This leads to that S is a continuous $\mathbb{Q}$ -martingale with strictly positive paths. Further regularity assumptions are set on the processes $\hat{\alpha}(\tau, x)$ and $\hat{\beta}(\tau, x)$ and their derivatives, see [Carmona and Nadtochiy \[2008\]](#), Section 2 for details.

To be able to implement the market model specified by (4.5) the stochastic spot volatility process $(\sigma_t)_{t \geq 0}$ and the stochastic processes $(\hat{\alpha}_t(\tau, x))_{t \geq 0}$ and $(\hat{\beta}_t^1(\tau, x), \dots, \hat{\beta}_t^m(\tau, x))_{t \geq 0}$ have to be specified. Furthermore, initial values S_0 and $\hat{a}_0^2(\tau, x)$ have to be inferred from the market (i.e. the initial local volatility surface has to be calibrated). Thereafter sample paths of the stock price process and the local volatility surface can be simulated. The results of [Carmona and Nadtochiy \[2008\]](#) show that only $\hat{\beta}(\tau, x)$ will be a free parameter if this equity market model shall remain consistent.

4.2 Consistent dynamic local volatility surfaces

We have that the stock price process S is a martingale under the measure $\mathbb{Q}$. Therefore, assuring an arbitrage free model (or a consistent one) is reduced to additionally assuring that for each fixed log-strike $x \in \mathbb{R}$ and time to maturity $\tau > 0$, the process of option prices $(\hat{C}_t(\tau, x))_{t \in [0, T]}$, represented by $(\hat{a}_t(\tau, x))_{t \in [0, T]}$, is a martingale under $\mathbb{Q}$. [Carmona and Nadtochiy \[2008\]](#) shows that this is equivalent to that the following conditions are satisfied almost surely and for all $t > 0$, $\tau > 0$, $x \in \mathbb{R}$,

$$\hat{\alpha}_t(\tau, x) = \partial_\tau \log(\hat{a}_t^2(\tau, x)) - \frac{\hat{\beta}_t(\tau, x) \cdot D_x \hat{\zeta}_t(\tau, x)}{D_x \hat{C}_t}, \quad (4.6a)$$

$$\hat{a}_t^2(0, \log(S_t)) = \sigma_t^2. \quad (4.6b)$$

Here $D_x := \frac{1}{2}(\partial_{xx} - \partial_x)$ and $(\hat{\zeta}_t^i(\tau, x))_t$, $i = 1, \dots, m$, are predictable processes such that for any (t, τ, x) , $\int_0^t \hat{\zeta}_u(\tau, x) \cdot dB_u$ is the local martingale component in the semimartingale decomposition of the call price process $\hat{C}_t(\tau, x)$. Provided that $\hat{a}(\tau, x)$ and $\hat{\beta}(\tau, x)$ are known, $(\hat{\zeta}_t^i(\tau, x))_t$, $i = 1, \dots, m$ can be found

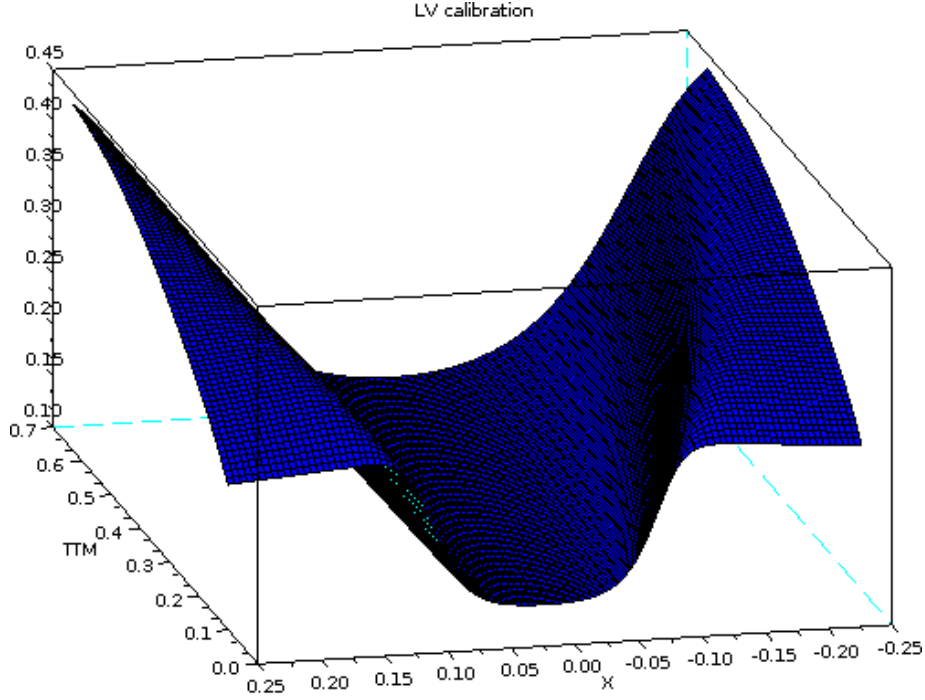


Figure 4.1: Local volatility surface fit to QQQQ-option prices of 2010-02-11. Calibrated using the log normal mixture density parameterization due to [Brigo and Mercurio \[2002\]](#), see Section 2.3.9.

numerically as solutions to the following initial-value problem,²⁶

$$\begin{cases} \partial_\tau \hat{\zeta}_t^i(\tau, x) = \hat{a}_t^2(\tau, x) D_x \hat{\zeta}_t^i(\tau, x) + \hat{\beta}_t^i(\tau, x) \hat{a}_t^2(\tau, x) D_x \hat{C}_t(\tau, x) \\ \hat{\zeta}_t^1(0, x) = S_t \sigma_t \mathbf{1}_{(-\infty, \log(S_t)]}(x) \\ \hat{\zeta}_t^i(0, x) = 0, \quad i = 2, \dots, m. \end{cases} \quad (4.7)$$

The restriction (4.6b) is a *spot volatility restriction*. It gives us a direct link between the two equations of the market model. It corresponds to that for each $t > 0$ the value of the spot volatility σ_t equals the value of the local volatility surface “at the money”²⁷ and for time to maturity $\tau = 0$.²⁸

As an example, see figure 4.1 where a local volatility surface is plotted. This surface has been fit to prices of options on the ETF QQQQ at closing time (4 p.m. standard eastern time) of February 11 2010. We used the “mixture-model” calibration method of Section 2.4.1. We set $\hat{a}_0(\tau, x)$ to be equal to this surface. Restriction (4.6b) tells us that at this time, the value of the spot volatility σ_t in a market model for options on QQQQ, equals the value of the surface at moneyness $\log(S/K) = 0$ and at $\tau = 0$, which is approximately 0.15.

²⁶See [Carmona and Nadtochiy \[2008\]](#), Lemma 3.2 and proof in Appendix A, Part 2.

²⁷ At the money = the stock price level S_t equals the strike price K .

²⁸ Compare with [Dupire \[1996\]](#) or [Derman and Kani \[1998\]](#) where local variance is derived to be equal to the risk neutral expectation of the instantaneous variance conditional on the final stock price equals the strike price.

The restriction (4.6a) will determine the drift process $\hat{\alpha}_t(\tau, x)$ and can be seen as an analog to the *drift restriction* in the Heath-Jarrow-Morton theory for interest rates, (Heath et al. [1992]). At each $t > 0$, it gives us the drift of the local volatility surface $\hat{\alpha}_t(\tau, x)$, given the other parameters of the model. After change of variables (4.6a) corresponds to the drift condition in the stochastic implied trees proposed by Derman and Kani [1998].

The restrictions (4.6a)-(4.6b) imply that the only free parameter in an arbitrage free dynamic local volatility model is $\hat{\beta}(\tau, x)$.

4.3 Remarks on the local volatility dynamic models

4.3.1 Implementing the local volatility dynamic models

The main difficulty to take care of if to implement the *Local volatility dynamic models* as described in Section 4.1-4.2, clearly is to assure that the model stays consistent. That is, at each time $t > 0$ the conditions (4.6a) and (4.6b) have to be fulfilled. This involves solving the Problem (4.7) and (4.4) numerically (both $\hat{\zeta}_t^i(\tau, x)$, $i = 1, \dots, m$ and $\hat{C}_t(\tau, x)$ are needed to calculate the drift (4.6a)), for example by using explicit Euler schemes. It also involves calculating the partial derivatives of $\hat{\zeta}_t^i(\tau, x)$, $i = 1, \dots, m$ and $\hat{C}_t(\tau, x)$ (in the operator D_x) at each time step Δt in the proposed schemes. There will also be a difficulty due to the incomplete set of option market data (there will not be quotes available for each $\tau > 0$ and $x \in \mathbb{R}$), which implies that it is not straightforward how to find the partial derivatives needed and the initial surface $\hat{\alpha}_0^2(\tau, x)$. See Section 2.3. For example a parameterization of the local volatility can be chosen to regularize the inverse problem of finding the initial local volatility surface.

Furthermore, the volatility diffusion processes $(\hat{\beta}_t^1(\tau, x), \dots, \hat{\beta}_t^m(\tau, x))_{t \geq 0}$ has to be specified. This can for example be done by using historical observations. See Section 7 of Carmona and Nadtochiy [2008] where an example of a parametric form of the drift surface is given.

At last it remains an open question whether it is possible to specify values of $\hat{\beta}(\tau, x)$, or if there exists a parametric family for the surface $\hat{\beta}(\tau, x)$, for which it is possible to prove existence of arbitrage free dynamic local volatility models as specified by (4.5) - (4.6b).

4.3.2 Assumptions on the underlying

As stated earlier, a local volatility dynamic model (since it take use of the *local volatility code-book*) assumes that the underlying follows a “regular enough” Itô process, in the sense of Gyöngy [1986]. This restricts the model since jumps or spikes of the underlying, frequently observed on the market, are not included. This is of course a shortcoming of the local volatility dynamic models, which speaks in favor of choosing another code-book for use in equity market models.

5 Dynamic Lévy density/setting Lévy in motion

5.1 Tangent Lévy market models

In the same way that local volatility can be set in motion, the *Lévy density code-book*, discussed in Section 3.5, can be used in a dynamic setting. We have that a market model for option prices based on this code-book relies on the following correspondence,

$$\{\tilde{C}_t(T, x); T > t, x \in \mathbb{R}\} \rightleftharpoons \{\kappa_t(T, x); T > t, x \in \mathbb{R}\}, \quad (5.1)$$

and on the assumption that the log-underlying $(\log S_t)$ is infinitely divisible. Here $\tilde{C}_t(T, x)$ is the price of a European call option having maturity time T and strike price e^x . $\kappa_t(T, x)$ is a time inhomogeneous Lévy density producing the same option prices by solving the pricing equation for jump models, as derived for instance by Carr et al. [2004]. See also Section 3.5 above and the example in Section 6.7. We follow the setup of the Tangent Lévy market models (henceforth also denoted by TLM) due to Carmona and Nadtochiy [2009]. From here on, when mentioning the Tangent Lévy models, TLM, or simply Tangent Lévy, we refer to this work of Carmona and Nadtochiy [2009].

By using the relation (5.1), in accordance with the setup in Section 3.5, we assume that there exists a pure jump, additive process, whose marginal distributions coincide with the marginal distributions of the logarithm of the underlying stock price. That is, with each call price surface given by the market (i.e for each t) it is possible to associate an exponential additive process, characterized by its time-inhomogeneous Lévy density $\kappa_t(T, x)$. By using (5.1) we can set up a market model for stock options by postulating the time evolution of the density $\kappa_t(T, x)$ (our code-book) together with an equation for the stock price process.

As before, the market under consideration consists of one underlying asset, with price given by $S = (S_t)_{t \in [0, \bar{T}]}$, and its liquidly traded derivatives (the European call options). From now on, $\bar{T} > 0$ is fixed and it is considered only $t \in [0, \bar{T}]$. It is assumed that the European call options with strikes e^x , $x \in \mathbb{R}$ and maturities $T \in (t, \bar{T}]$ are available at the prices $\tilde{C}_t(T, x)$. These prices are represented by $\kappa_t(T, x)$. For simplicity it is also assumed that S pays no dividends and that interest rates are equal to zero.

The terminology “Tangent Lévy model” comes from the assumption that at each fix time (tangent point) the static Lévy model corresponding to the market model²⁹ produces the same call option prices as the true underlying. That is at each point of time the true log-underlying can be represented as a time inhomogeneous Lévy process.

In this sense we may call the local volatility dynamic models of Section 4.1 *Tangent diffusion models* or *Tangent Itô models*, as it is assumed in this case that the underlying follows a real-valued one-dimensional Itô process.

²⁹ I.e the time-inhomogeneous Lévy process with compensator being equal to the current value of the code-book.

As in [Carmona and Nadtochiy \[2009\]](#), under a pricing measure $\mathbb{Q}$, for each $t \in [0, \bar{T}]$, $T \in (t, \bar{T}]$, the dynamics of the stock and the time-inhomogeneous Lévy density are given by,

$$S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_{u-} (e^x - 1) (M(dx, du) - K_u(x) dx du) \quad (5.2)$$

$$\kappa_t(T, x) = \kappa_0(T, x) + \int_0^t \alpha_u(T, x) du + \sum_{n=1}^m \int_0^t \beta_u^n(T, x) dB_u^n. \quad (5.3)$$

It is assumed that almost surely for each $t \in [0, \bar{T}]$ and $T \in (t, \bar{T}]$,

$$\text{ess inf}_{x \in \mathbb{R}} \kappa_t(T, x) \geq 0. \quad (5.4)$$

$(\kappa_t(T, x))_{t \in [0, \bar{T}]}$ is called a *dynamic Lévy density*.

M is an integer valued random measure on $(\mathbb{R} \setminus \{0\}) \times [0, \bar{T}]$ with compensator $K_{t,\omega}(x) dx dt$. See definitions for instance in [Jacod and Shiryaev \[2003\]](#), II, 1.13 and 1.8. It is assumed that $K(x) = (K_t(x))_{t \in [0, \bar{T}]}$ is a predictable integrable stochastic process. Furthermore, the compensator $K(x)$ takes values only in a Banach space $\mathcal{B}^0$ of Borel measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying,

$$\int_{\mathbb{R}} (|x| \wedge 1) |x| (1 + e^x) f(x) dx < \infty. \quad (5.5)$$

From the integrability of $K(x)$, it follows that the random measure M satisfies the following,

$$\int_0^{\bar{T}} \int_{\mathbb{R}} (|x|^2 \wedge 1) M(dx, du) < \infty, \quad \text{and} \quad M((\mathbb{R} \setminus (-\varepsilon, \varepsilon)) \times [0, \bar{T}]) < \infty,$$

for all $\varepsilon > 0$. The solution of (5.2) is found to be a stochastic exponential (see for instance [Jacod and Shiryaev \[2003\]](#), I, Theorem 4.61). We can also find the solution by an application of the Itô formula. See Equation (3.66), Section 3.5. It follows that $(S_t)_{t \in [0, \bar{T}]}$ is a non negative càdlàg local martingale.

For the process κ_t in (5.3), it is assumed that for all fix $T > t$ and $x \in \mathbb{R}$, $\alpha(T, x) = (\alpha_t(T, x))_{t \geq 0}$ is a measurable integrable stochastic process and $\beta(T, x) = (\beta_t^1(T, x), \dots, \beta_t^m(T, x))_{t \geq 0}$ is a vector of measurable square integrable stochastic processes. $B = (B_1, \dots, B_m)$ is a m -dimensional Brownian motion. $\kappa(T, \cdot)$ and $\alpha(T, \cdot)$ are assumed to be continuous processes satisfying (5.5).³⁰ More precisely, it is assumed that $\kappa(T, \cdot)$ and $\alpha(T, \cdot)$ take values only in a Banach space $\mathcal{B}$ of absolutely continuous functions $f : [0, \bar{T}] \rightarrow \mathcal{B}^0$ satisfying

$$\|f\|_{\mathcal{B}} := \|f(0)\|_{\mathcal{B}^0} + \int_0^{\bar{T}} \left\| \frac{d}{du} f(u) \right\|_{\mathcal{B}^0} du < \infty.$$

At last it is assumed that $\beta(T, \cdot)$ takes values only in a Hilbert space $\mathcal{H}$, subspace of the Banach space $\mathcal{B}$, such that Itô calculus is applicable. $\beta(T, x)$ and its derivatives are also assumed to fulfill certain regularity assumptions. See [Carmona and Nadtochiy \[2009\]](#) Section 3.1-3.2 for details. See also Remark 5.2 below.

³⁰We say that a function f of two variables $(t, x) \mapsto f(t, x)$ belongs to $\mathcal{B}^0$ (resp. $\mathcal{B}$) if the function $\tilde{f}(t) := f(t, \cdot)$, $\forall t$ belongs to $\mathcal{B}^0$ (resp. $\mathcal{B}$).

Definition 5.1. A *Tangent Lévy model* is defined as a pair $(S_t, \kappa_t(T, x))_{t \in [0, \bar{T}]}$ satisfying,

$$\tilde{C}_t(T, x) = \mathbb{E}[(S_T - e^x)^+ | \mathcal{F}_t].$$

$(S_t)_{t \in [0, \bar{T}]}$ is the càdlàg martingale given by (5.2). $(\kappa_t(T, x))_{t \in [0, \bar{T}]}$ is a dynamic Lévy density fulfilling (5.3) and positivity (5.4). $\tilde{C}_t(T, x)$ is the option price produced by this density, recall the code-book (5.1), assuming that at time t the value of the underlying stock price equals S_t . The initial value $\tilde{C}_0(T, x)$ is given by the market.

As in Section 3.5, we have that $\tilde{C}_t(T, x)$ can be found as the solution of the following PDE,

$$\begin{cases} \partial_T \hat{\Delta}_t(T, \xi) = \hat{\psi}(\kappa_{t, \omega}(T, \cdot); \xi) \hat{\Delta}_t(T, \xi) (-4\pi^2 \xi^2 - 2\pi i \xi) \\ \hat{\Delta}_t(t, \xi) = \frac{e^{\log(S_t)(1-2\pi i \xi)}}{1-2\pi i \xi}. \end{cases} \quad (5.6)$$

Here, using the same notation as in Section 3.5, “hat” denotes values in the Fourier space. $\Delta(T, x)$ is (minus) the x -derivative of the call option price; $-\partial_x \tilde{C}_t(T, x)$. $\psi(f; x)$ is as before the double exponential tail of the function f ;

$$\psi(f; x) := \begin{cases} \int_{-\infty}^x (e^x - e^z) f(z) dz, & x < 0 \\ \int_x^{\infty} (e^z - e^x) f(z) dz, & x > 0. \end{cases} \quad (5.7)$$

That is, $\kappa(T, x)$ in equation (3.73) (Section 3.5) is replaced with the process $(\kappa_{t, \omega}(T, x))_{t \in [0, \bar{T}]}$ defined in (5.3).

We may also calculate the call option price $\tilde{C}_t(T, x)$ by using the Lévy Khintchine formula and FFT pricing. See Section 6.7 below.

Kallsen and Krühner [2010] proposes a more general time evolution in a similar market model for option prices. In their model the time evolution of the “characteristic exponent” of an additive process is modeled, and the evolution is allowed to contain jumps. That is the underlying is not assumed to be a pure jump type process, it may also follow a diffusion or jump diffusion. The driving Brownian motion B above is replaced with a more general Lévy process.

In what follows is discussed conditions that assures that a Tangent Lévy model is consistent. That is, it prices according to the market and the prices are free of arbitrage. As is the case in the Heath-Jarrow-Morton approach of interest rate modeling, and in the *local volatility dynamic models*, see Section 4, this will imply certain restrictions on the drift and diffusion processes $\alpha_t(T, x)$ and $\beta_t(T, x)$, as well as on the compensator process $K_t(x)$.

5.2 Consistent dynamic Lévy densities

Carmona and Nadtochiy [2009] show that a Tangent Lévy market model is consistent if and only if the process of call option prices $(\tilde{C}_t(T, x))_{t \in [0, T]}$, produced by $(\kappa_t(T, x))_{t \in [0, T]}$ is a martingale. It is derived, see Carmona and Nadtochiy

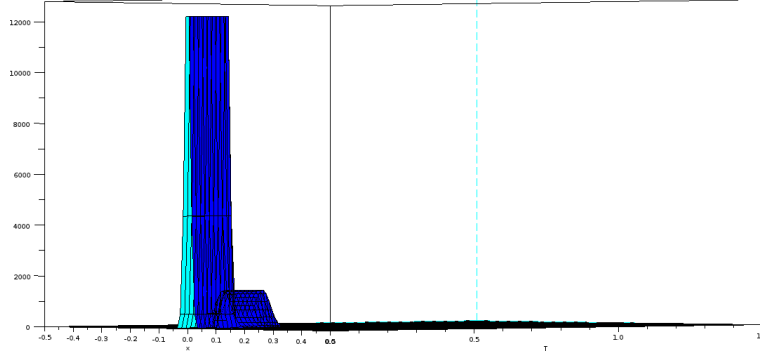


Figure 5.1: A time inhomogeneous Lévy density fit to QQQQ-option prices of 2010-02-11. Calibrated using a compound Poisson process with drift. See Section 6.1, Equation (6.12) below.

[2009] Theorem 12, that this is the case if and only if the following two restrictions are fulfilled,

$$\begin{aligned} \alpha_t(T, x) = & -e^{-x} \sum_{n=1}^m \int_{\mathbb{R}} \partial_y^3 \psi(\bar{\beta}_t^n(T, \cdot); y) \\ & \times \left[\partial_x \psi(\beta_t^n(T, \cdot); x - y) - (1 - y \partial_x) \partial_x \psi(\beta_t^n(T, \cdot); x) \right] \\ & - 2 \partial_y^2 \psi(\bar{\beta}_t^n(T, \cdot); y) \left[\partial_x \psi(\beta_t^n(T, \cdot); x - y) - \partial_x \psi(\beta_t^n(T, \cdot); x) \right] \\ & + \partial_y \psi(\bar{\beta}_t^n(T, \cdot); y) \partial_x \psi(\beta_t^n(T, \cdot); x - y) dy, \\ \bar{\beta}_t^n(T, \cdot) := & \int_{t \wedge T}^T \beta_t^n(u, \cdot) du \end{aligned} \quad (5.8a)$$

and

$$K_t(x) = \kappa_t(t, x), \quad \forall x \in \mathbb{R}, t \geq 0. \quad (5.8b)$$

Here $\psi(f; x)$ is the double exponential tail as in equation (5.7). The restrictions (5.8a)-(5.8b) are derived by first finding a semimartingale representation (of the x -derivative) of the option prices in the Fourier domain. Thereafter, carefully taking the inverse of the Fourier transform and setting the drift to zero will result in the above restrictions.

The restriction given in (5.8b) corresponds to that at each $t > 0$, the compensator process $K_t(x)$ equals the “slice” of the time inhomogeneous Lévy density surface, at the maturity time $T = t$.

As an example, in Figure 5.1 a time inhomogeneous Lévy density is plotted. This surface has been fit to prices of options on the ETF QQQQ at closing time (4 p.m standard eastern time) of February 11 2010. We set $\kappa_0(T, x)$ to be equal to this surface. Restriction (5.8b) tells us that at this time, the value of the compensator $K_t(x)$ in a market model for options on QQQQ, equals the “slice” of this surface at $T = t$.

The restriction (5.8a) tells us that the drift process $(\alpha_t(T, x))_{t \geq 0}$ will be determined by the diffusion processes $(\beta_t^1(T, x), \dots, \beta_t^m(T, x))_{t \geq 0}$, and their derivatives. That is, $(\beta_t^1(T, x), \dots, \beta_t^m(T, x))_{t \geq 0}$ are the only free parameters of the model.

Remark 5.2. We have that the first derivative of the exponential tail function is an integral operator, and the higher derivatives used in Equation (5.8a) may be obtained by straightforward calculations, see Section 6.4.1.

However, to assure that the drift restriction (5.8a) is well defined, in particular the derivatives of the function ψ , it is assumed that the following is fulfilled for each $n \leq m$, almost surely and for almost every $t \in [0, \bar{T}]$.

- For every $T \in [t, \bar{T}]$, the function $\beta_t^n(T, \cdot)$ is absolutely continuous on $\mathbb{R} \setminus \{0\}$.
- For any $\varepsilon > 0$,

$$\sum_{k=0}^1 \sup_{T \in [t, \bar{T}]} \left[\text{ess sup}_{x \in \mathbb{R} \setminus [-\varepsilon, \varepsilon]} (e^x + 1) |\partial_x^k \beta_t^n(T, x)| + \int_{\mathbb{R}} (e^x + 1) |x^3| (|x| \wedge 1)^{k-1} |\partial_x^k \beta_t^n(T, x)| dx \right] < \infty. \quad (5.9)$$

- $\sup_{T \in [t, \bar{T}]} \int_{-1}^1 |x| |\beta_t^n(T, x)| dx < \infty$
- For any $T \in [t, \bar{T}]$, $\int_{\mathbb{R}} (e^x - 1) \beta_t^n(T, x) dx = 0$.

5.2.1 Existence of dynamic Lévy model

A crucial assumption of the Tangent Lévy model is the positivity of the Lévy density, see Equation (5.4). Also, it must be possible to define the drift α by (5.8a). To prove existence of Tangent Lévy market models, Carmona and Nadtochiy [2009] (Section 5), propose to first fix the asymptotic behavior of κ . E.g. they assume that κ is of the following form,

$$\kappa_t(T, x) = e^{-\gamma|x|} (|x| \wedge 1)^{-1-2\delta} \tilde{\kappa}_t(T, x). \quad (5.10)$$

$\gamma > 1$, $0 < \delta < 1$. $\tilde{\kappa}_t(T, \cdot)$ is a continuous function with limits at $\pm\infty$, equipped with the standard sup norm. κ defined in this way will be in $\mathcal{B}$, as in Section 5.1. Hence we may model $\tilde{\kappa}$ in place of κ . Carmona and Nadtochiy now prove existence of consistent Tangent Lévy models by restricting the state space of $\tilde{\kappa}$. This is done by restricting the state space of its diffusion. By this means, motivated by the asymptotic behavior of the drift restriction (5.8a) and order of singularity of $\beta_t^n(T, \cdot)$ and $\bar{\beta}_t^n(T, \cdot)$ at $x = 0$, Carmona and Nadtochiy define a Hilbert space $\tilde{\mathcal{H}}^0$ by the following,

$$\tilde{\mathcal{H}}^0 = \left\{ e^{-\lambda|\cdot|} (|\cdot|^\delta \wedge 1) f(\cdot) \mid f \in \mathcal{G}, \int_{\mathbb{R}} (e^x - 1) e^{-(\gamma+\lambda)|x|} (|x| \wedge 1)^{-1-\delta} f(x) dx = 0 \right\}. \quad (5.11)$$

Here $\lambda > 0$ is a fix real number and $0 < \delta < 1$ as before. $\mathcal{G}$ is a space of absolutely continuous functions on $\mathbb{R}$ whose first derivatives are bounded in $\mathbb{L}^1(\mathbb{R})$ and $\mathbb{L}^\infty(\mathbb{R})$ and furthermore for which the following hold,

$$\|f'\|_{\mathbb{L}^1(\mathbb{R})} + \|f'\|_{\mathbb{L}^\infty(\mathbb{R})} \leq c \|f\|_{\mathcal{G}}.$$

Thereafter, a function space $\tilde{\mathcal{H}}$ is constructed which fulfills,

$$\|f\|_{\tilde{\mathcal{H}}}^2 := \|f(0)\|_{\tilde{\mathcal{H}}^0}^2 + \int_0^{\bar{T}} \left\| \frac{d}{du} f(u) \right\|_{\tilde{\mathcal{H}}^0}^2 du < \infty.$$

Choosing $\tilde{\beta}(T, x)$ from the space $\tilde{\mathcal{H}}$ will assure that the drift, $\tilde{\alpha}(T, x)$, can be defined by (5.8a). Next define the following stopping time,

$$\tilde{\tau}_0 := \inf \left\{ t \geq 0 : \inf_{T \in [t, \bar{T}], x \in \mathbb{R}} \tilde{\kappa}_t(T, x) \leq 0 \right\}. \quad (5.12)$$

By stopping the diffusion and drift processes of $\tilde{\kappa}_t$ at $\tilde{\tau}_0$, positivity of κ_t is assured.

Remark 5.3 (Stopping κ_t). For the market model to be consistent, the price process produced by inserting κ_t into the pricing Equation (5.6) has to be a martingale. We also have that the Lévy density process has to be non-negative.

The process $\kappa_{t \wedge \tilde{\tau}_0}$ is almost surely non negative. Up to time $\tilde{\tau}_0$, by construction the drift and the diffusion processes fulfill the consistency conditions (5.8a)-(5.8b). It follows that $\tilde{C}_{t \in [0, \tilde{\tau}_0]}$ is a martingale. By stopping the drift and diffusion of $\tilde{\kappa}_t$ at $\tilde{\tau}_0$ we stop also the price process $\tilde{C}_t$, which then remains a martingale. Hence the model stays consistent.

By letting the diffusion term depend on $\tilde{\kappa}$, Carmona and Nadtochiy [2009] Section 6, provides an example where the stopping time $\tilde{\tau}_0 \rightarrow \infty$ a.s.

Now, given the dynamics of $\tilde{\kappa}_t(T, x)$, and hence $\kappa_t(T, x)$, the compensator $K_t(x)$ is defined by (5.8b), i.e. $K_t(x) = \kappa_t(t, x)$. For the definition of S , we next need to construct an integer valued jump measure M having compensator $\kappa_t(t, x)$. How this may be done is shown by Carmona and Nadtochiy [2009], Section 5.2.

In Section 6.3.2 and 6.3.3 we provide parameterizations of $\beta(T, x)$, fulfilling the regularity conditions described above. Hence we may calculate the drift using Equation (5.8a). Together with the consistency restriction, this defines a simplified example of a Tangent Lévy market model. We implement the model using an Euler scheme. To assure positivity of the Lévy density, we follow the example given by Carmona and Nadtochiy [2009] Section 6, and stop the diffusion and drift processes when $\kappa_t \leq 0$.

Part III

Implementation/results

6 Implementation

6.1 A time-inhomogeneous Poisson jump model

In this section we implement a pricing model based on the compound Poisson process, see Example 3.1.2, Section 3.1.5. More precisely, we calibrate the following exponential additive process to a call option surface,

$$S_t := S_0 \exp(Y_t). \quad (6.1)$$

Here we define the process Y_t as having independent compound Poisson distributed increments, with normally distributed jump sizes. We also include a drift. Furthermore, we assume that the parameters of Y_t are piecewise constant. That is, in the interval $[t_0, t_1)$ Y_t is given by a set of parameters $\Theta_1 = \{\mu_1, \delta_1, \lambda_1\}$. In the interval $[t_{k-1}, t_k)$ Y_t is given by the parameters $\Theta_k = \{\mu_k, \delta_k, \lambda_k\}$. We have that the following holds for the characteristic function of Y_t ,³¹

$$\begin{aligned} \mathbb{E}[e^{iuY_t}] &= e^{t\psi_1(u)}, & t \leq t_1 \\ \mathbb{E}[e^{iuY_t}] &= \mathbb{E}[e^{iu(Y_{t_1} + (Y_t - Y_{t_1}))}] = e^{t_1\psi_1(u) + (t-t_1)\psi_2(u)}, & t_1 < t \leq t_2 \\ &\vdots \\ \mathbb{E}[e^{iuY_t}] &= \exp\left\{\sum_{j=1}^{k-1} (t_j - t_{j-1})\psi_j(u) + (t - t_{k-1})\psi_k(u)\right\}, & t_{k-1} < t \leq t_k. \end{aligned} \quad (6.2)$$

Here we set $t_0 = Y_0 = 0$, and $\psi_j(u)$ denotes the characteristic exponent of the j -th increment, $(Y_j - Y_{j-1})$. That is,

$$\psi_j(u) = i\omega_j u + \lambda_j \left(\exp\left(-\frac{\delta_j^2 u^2}{2} + i\mu_j u\right) - 1 \right). \quad (6.3)$$

As discussed in Section 3.3.1 (see Example (3.3.2)), the drift parameter ω_j will be given by the martingale condition,

$$\mathbb{E}[e^{-rt} S_t] = S_0. \quad (6.4)$$

For the first increment $(Y_{t_1} - Y_{t_0})$, this gives the “usual” condition,

$$\omega_1 = r - \lambda_1 (\exp(\mu_1 + \delta_1^2/2) - 1), \quad (6.5)$$

where r is the interest rate. For the k -th increment, the martingale condition in Equation (6.4) gives us,

$$\mathbb{E}[\exp\{\sum_{j=1}^k (Y_{t_j} - Y_{t_{j-1}})\}] = \exp\{\sum_{j=1}^k (t_j - t_{j-1})\psi_j(-i)\} = \exp(rt_k). \quad (6.6)$$

³¹All expectations are taken under the martingale measure $\mathbb{Q}$.

Or,

$$(t_k - t_{k-1})\psi_k(-i) + \sum_{j=1}^{k-1} (t_j - t_{j-1})\psi_j(-i) = rt_k. \quad (6.7)$$

We have $\psi_j(-i) = \omega_j + \lambda_j \left(\exp\left(\frac{\delta_j^2}{2} + \mu_j\right) - 1 \right)$. That is, the drift will be given by,

$$\omega_k = \frac{rt_k - \sum_{j=1}^{k-1} (t_j - t_{j-1})\psi_j(-i)}{(t_k - t_{k-1})} - \lambda_k \left(\exp\left(\frac{\delta_k^2}{2} + \mu_k\right) - 1 \right). \quad (6.8)$$

6.1.1 The time inhomogeneous Lévy measure

For reasons that will become clear later in Section 6.3.2, we look closer at the time-inhomogeneous Lévy measure of the process Y_t . Simply comparing the characteristic function of Y_t ,

$$\mathbb{E}[e^{iuY_{t_k}}] = \exp\left\{ \sum_{j=1}^k (t_j - t_{j-1})\psi_j(u) \right\},$$

with the Lévy Khinchine formula for additive processes, see Equations (3.58), (3.61) we have, for $t_{k-1} < t \leq t_k$,

$$\eta^Y(t, dx) = \sum_{j=1}^{k-1} (t_j - t_{j-1})f_j(x)dx + (t - t_{k-1})f_k(x)dx. \quad (6.9)$$

$\eta^Y(t, dx)$ is the time-inhomogeneous Lévy measure of Y_t . $t_1 < \dots < t_k$ are the "calibration points" of the process, i.e the points where the model parameters change. Also, having the increments of Y_t given by the compound Poisson process, with normally distributed jump sizes, we have $f_j(x)$ as follows,

$$f_j(x) = \frac{\lambda_j}{\sqrt{2\pi\delta_j^2}} \exp\left(-\frac{(x - \mu_j)^2}{2\delta_j^2}\right). \quad (6.10)$$

Note that $\eta^Y(t, dx)$ is the integrated (time dependent) Lévy density of the process Y_t . In the sections that come we will use the following notation for this density,

$$\kappa(T, x) := f_j(x), \quad T_{j-1} < T \leq T_j. \quad (6.11)$$

6.1.2 Calibrating a process with piecewise constant parameters

We calibrate the process S_t , given in Equation (6.1), to market data consisting of a call option surface with M different maturity times; $T_1 < T_2 < \dots < T_M$. As stated in the previous section, we assume that S_t has piecewise constant parameters.

We choose to divide the available call option surface into three parts. More precisely we separate options with maturity times $\text{TTM} \in (0, 0.5]$, $\text{TTM} \in (0.5, 1]$ and $\text{TTM} \in (1, 2]$ years. Thereafter we calibrate three Lévy densities $f_j(x)$ using the options in each interval. The calibration procedure is described

in the following paragraphs. This results in a time-inhomogeneous Lévy density of the following form,

$$\kappa(T, x) = f_j(x), \quad T_{j-1} < T \leq T_j, \quad (6.12)$$

where $j = 1, 2, 3$, $T_0 = 0$, $T_1 = 0.5$, $T_2 = 1$, $T_3 = 2$ years and $f_j(x)$ is as in Equation (6.10).

We start by calibrating the first Lévy density to the options having time to maturity less than 0.5 years.

Problem 6.1 (“First time slice”). *Assume that (at time $t = 0$), we are given N market option bid/ask-pairs $\{(\text{bid}_{i,j}, \text{ask}_{i,j})\}$ corresponding to strike prices $\{K_j\}_{j=1}^N$ and maturity times $\{T_i\}_{i=1}^m$, where $T_1 < T_2 < \dots < T_m < 0.5$ years. The set of parameters $\Theta_1 := \{(\mu_1, \delta_1, \lambda_1)\}$ is then approximated by minimizing the following objective function,*

$$\min_{\Theta_1} G(\Theta_1), \quad G(\Theta_1) := \sum_{i=1}^m \sum_{j=1}^{n_i} \left(\frac{\partial C}{\partial \Sigma}(\bar{\Sigma}_{i,j}) \right)^{-2} \left(\tilde{C}(T_i, x_j; \Theta_1) - \bar{C}_{i,j}^* \right)^2. \quad (6.13)$$

Here n_i represents the number of quotes available with maturity T_i . $\bar{C}_{i,j}^* := \frac{\text{bid}_{i,j} + \text{ask}_{i,j}}{2}$, $j = 1, \dots, n_i$ represents the observed prices with maturities T_i and strike prices K_j . $x_j := \log(K_j)$ is the log-strike price. $\tilde{C}(T, x; \Theta)$ denotes the model price of a call option having time to maturity T and log-strike x .

As in Section 2.4.1 we choose to do a “vega-weighting” of the available data, see Cont and Tankov [2003] Chapter 13. $\left(\frac{\partial C}{\partial \Sigma}(\bar{\Sigma}_{i,j}) \right)$ denotes the Black-Scholes vegas evaluated at the implied volatilities of the market option prices. $\bar{\Sigma}_{i,j}$ denotes the mid market implied volatility corresponding to strike price K_j and time to maturity T_i . The objective function G is minimized using a standard nonlinear minimization routine in Scilab.

Calculating the objective function

Having the characteristic function of the underlying specified by our exponential additive model, (Equation (6.2)), the price $\tilde{C}(T_i, x_j; \Theta_1)$ can easily be calculated using standard FFT methods, as proposed for instance by Carr and Madan [1999]. In this approach European call option prices are derived in terms of the characteristic function of the log-underlying, see Sections 3.3.1-3.3.2 above. As in Equation (3.31) call option prices are obtained from the following,

$$\tilde{C}(T_i, x_j; \Theta_1) = \frac{\exp(-\alpha x_j)}{2\pi} \int_{-\infty}^{\infty} e^{-ivx_j} \psi_{T_i}(v) dv. \quad (6.14)$$

where

$$\psi_T(v) = \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}. \quad (6.15)$$

Here, α is a dampening parameter and is set to 1.25. $\phi_T(u)$ is the characteristic function of the log-underlying. For the first increment of S , $\phi_T(u)$ is given by the first equation of (6.2). We calculate (6.14) using a FFT, as described in Section 3.3.2.

Calibrating the k -th set of parameters

For the k -th "increment" of the process Y_t , the same procedure as described above is used to find the set of parameters Θ_k that reproduce the market data. The only difference is that the characteristic function $\phi_T(u)$ is now given by the last equation of (6.2). The characteristic exponents $\psi_1(u), \dots, \psi_{k-1}(u)$ are known from the earlier "time slices". The same holds for the drift parameters $\omega_1, \dots, \omega_{k-1}$ which will give us the drift ω_k using Equation (6.8).

6.2 Calibration to market data (examples)

As input to the "layman calibration method", see Section 6.3 below, we need a time series of "market Lévy densities". For this purpose, we calibrate the model described in Section 6.1 to call option data from the period 2010.02.11-2011.02.11.³²

As we define our model by a process with piecewise constant parameters, this results in parameters $\Theta_{i,j} := \{\mu_{i,j}, \delta_{i,j}, \lambda_{i,j}\}$, $i = 1, \dots, 365$, $j = 1, 2, 3$. That is we have time-inhomogeneous Lévy densities of the form shown in Equation (6.12).

As an example, the implied volatility surfaces resulting from the fit of the "time-inhomogeneous Poisson" to the call options available at 2010-02-11 and 2010-02-12, are plotted in Figures 6.1-6.2. (This corresponds to the first 2 days of our sample.) As a comparison to the exponential NIG-model of Section 3.3.2, the call option curves of 2010-02-11 are included in Figure 6.3.

The fitted parameters are listed in Table 6.1. The calibration procedure is described in Section 6.1.2. Note that the parameters ω_j are not included in Table 6.1, since these are given by the no arbitrage requirement in Equation (6.8).

Table 6.1: Model parameters, sample day 1 and 2.

	$\Theta_{i,1}$	$\Theta_{i,2}$	$\Theta_{i,3}$
$\mu :$			
day 1	-0.0002272	0.0018152	-0.3109229
day 2	-0.0001080	0.0036179	-0.3267551
$\delta :$			
day 1	0.0224593	0.01	0.0773602
day 2	0.0220219	0.0462497	0.0244331
$\lambda :$			
day 1	77.421579	3.9993331	1.0811477
day 2	79.038372	3.9998962	0.8928825

³²The call option data is on the underlying QQQQ. The QQQQ is a exchange traded fund designed to correspond generally to the performance, before fees and expenses, of the Nasdaq-100 index. The fund holds all the stocks in the Nasdaq-100 index. Source: <http://finance.yahoo.com/q?s=QQQQ>.

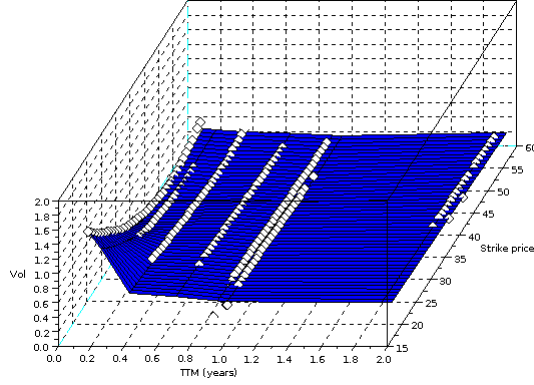


Figure 6.1: Implied volatility fit, 2010-02-11.

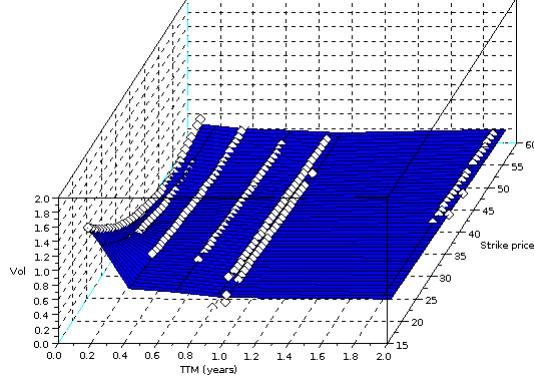


Figure 6.2: Implied volatility fit, 2010-02-12.

6.3 The “layman calibration”

In this section we propose a simplification of the *Tangent Lévy market models* due to [Carmona and Nadtochiy \[2009\]](#). Our suggestion is to choose a specific, simple form of the diffusion component $\beta(T, x)$ of the dynamic Lévy density (see Equation (5.3) and Definition 5.1, Section 5.1). The specific form is motivated by a calibration of “layman type”, see below.

Having the consistency conditions of the Tangent Lévy models at hand (Equation (5.8a)-(5.8b)), specifying the diffusion $\beta(T, x)$ will also determine the drift $\alpha(T, x)$ and the compensator $K_t(x)$. Our choice of $\beta(T, x)$ will hence define a consistent Tangent Lévy market model. The characteristics of the Tangent Lévy model ($\beta(T, x)$) will be directly calculated from observed time series.

6.3.1 Layman’s method

We estimate $\beta(T, x)$ by following the “layman calibration approach” of [Ortega, Pullirsch, Teichmann, and Wergieluk \[2009\]](#). In this paper (*A dynamic approach for scenario generation in risk management*) time evolution of risk factors are given by stochastic differential equations. These SDEs are calibrated to historical data to reproduce stylized facts of time series. For instance risk factors may

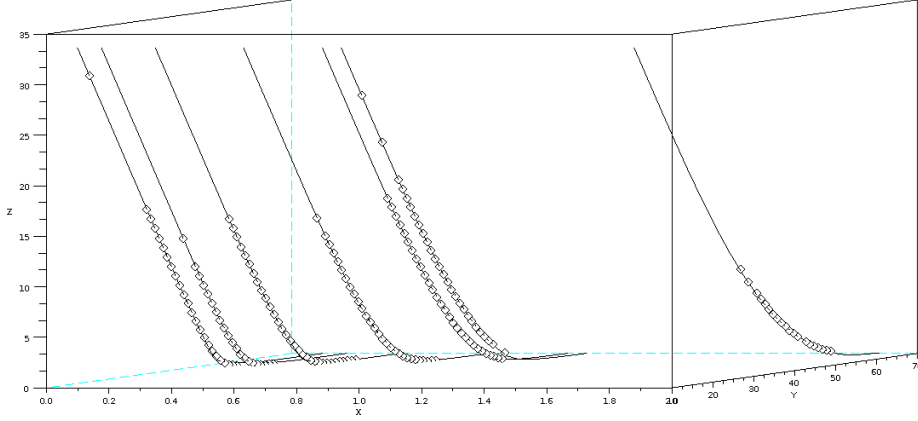


Figure 6.3: Call option fit, 2010-02-11.

be described by the following type of SDE,

$$dY_t = \alpha(Y_t)dt + \sum_{i=1}^m \sigma(Y_t) \cdot \lambda_i dB_t^i. \quad (6.16)$$

Here the drift α will be specified by no arbitrage conditions for each given risk factor, σ is a linear operator acting on λ_i . The λ_i , $i = 1, \dots, m$, are “constant volatility directions” (see Ortega et al. [2009]). B_t^i are standard Brownian motions.

Now, suppose we are given a sample $Y_1, \dots, Y_K$ of (6.16) on an equidistant grid (grid width = Δ) during a period of time T . Then Ortega et al. shows that the process $X_t^{(K)}$ given by following,

$$dX_t^{(K)} = \alpha(X_t^{(K)})dt + \frac{1}{\sqrt{\Delta(K-1)}} \sum_{i=1}^{K-1} \sigma(X_t^{(K)}) \frac{(Y_{i+1} - Y_i)}{\sigma(Y_i)} dW_t^i, \quad (6.17)$$

converges to Y_t in the following sense,

$$\lim_{K \rightarrow \infty} X_t^{(K)} \stackrel{d}{=} Y_t, \quad \forall t \geq 0, \quad (6.18)$$

if $X_0 = Y_0$. W_t^i , $i = 1, \dots, K-1$ are independent standard Brownian motions. The notation $X_t^{(K)}$ shows the dependence of the process X_t on the number of observations K .

6.3.2 Calibrating $\beta(T, x)$, a first example

In our case, we want to calibrate a process $\hat{\kappa}_t$ to a dynamic Lévy density κ_t by using a time series of time-inhomogeneous Lévy densities $\kappa_1, \dots, \kappa_K$. The sample is on an equidistant grid (distance between samples $\Delta = 1$) and we have K days in our sample. As in Equation (5.3),

$$d\kappa_t = \alpha_t dt + \sum_{i=1}^m \beta_t^i dB_t^i. \quad (6.19)$$

To deduce a first example of a "layman calibrated" $\beta(T, x)$, we now make the assumption that the volatility function of Equation (6.16) is equal to 1, $\sigma(\cdot) := 1$. This is also the example that we use in Sections 6.4-6.9 below.

In this case, using the results discussed in Section 6.3.1, and further assuming $\beta_t^i := \beta^i$ leads to,

$$d\hat{\kappa}_t^{(K)} = \hat{\alpha}_t dt + \sum_{i=1}^{(K-1)} \underbrace{\frac{(\kappa_{i+1} - \kappa_i)}{\sqrt{\Delta(K-1)}}}_{:=\hat{\beta}^i} dW_t^i. \quad (6.20)$$

and

$$\lim_{K \rightarrow \infty} \hat{\kappa}_t^{(K)} \stackrel{d.}{=} \kappa_t, \quad \forall t \geq 0. \quad (6.21)$$

For reasons that will become clear when calculating the drift from the Tangent Lévy consistency condition (Equation (5.8a)), see Section 6.4.1, we use the "time-inhomogeneous compound Poisson" parameterization of Section 6.1 to generate a time series $\kappa_1, \dots, \kappa_K$. We calibrate the model as described in Section 6.1.2. This results in densities $\kappa_i(T, x)$ of the following form, for each day i of our sample, and for $T_{j-1} \leq T < T_j$

$$\kappa_i(T, x) = f_j^{(i)}(x), \quad (6.22)$$

where

$$f_j^{(i)}(x) = \frac{\lambda_{i,j}}{\sqrt{2\pi\delta_{i,j}^2}} \exp\left(-\frac{(x - \mu_{i,j})^2}{2\delta_{i,j}^2}\right). \quad (6.23)$$

Recall Equation (6.12). Also, $\Theta_i = \{\lambda_{i,j}, \mu_{i,j}, \delta_{i,j}\}$ are the model parameters corresponding to the j -th increment, measured at day i . As in Section 6.2 we choose $j = 1, 2, 3$, $T_0 = 0$, $T_1 = 0.5$, $T_2 = 1$ and $T_3 = 2$ years.

To assure positivity of $\hat{\kappa}_t^{(K)}$, in accordance with the restriction of Equation (5.4), $\hat{\kappa}_t^{(K)}$ is stopped as it hits zero. That the martingale property of the corresponding call price process is preserved follows from Remark 5.3 above.

6.3.3 Calibrating $\beta(T, x)$, second example

An alternative and possibly more realistic specification of $\beta(T, x)$ is derived from the following assumption on the operator $\sigma(\cdot)$ of Equation (6.16),

$$\sigma(y) = y.$$

In this case, using the results discussed in Section 6.3.1 and as in the previous section assuming $\beta_t^i := \beta^i$ leads to,

$$d\hat{\kappa}_t^{(K)} = \hat{\alpha}_t dt + \sum_{i=1}^{(K-1)} \underbrace{\frac{\hat{\kappa}_t^{(K)}}{\kappa_i} \frac{(\kappa_{i+1} - \kappa_i)}{\sqrt{\Delta(K-1)}}}_{:=\hat{\beta}^i} dW_t^i. \quad (6.24)$$

In this parameterization, $\beta(T, x)$ (and hence $\alpha(T, x)$) will decay as the process $\hat{\kappa}_t^{(K)}(T, x)$ gets close to zero, preventing the process from becoming negative.

As in the example of the previous section we will stop $\hat{\kappa}_t^{(K)}(T, x)$ as it hits zero, which in this case however will be more rare.

A negative property of the parameterization suggested in this section is that calculating the consistency conditions (see Section 6.4.1) will be more complicated. Also, the calculations will be numerically more expensive. For these reasons, in what follows we consider only the simpler case of $\sigma(\cdot) = 1$, described in Section 6.3.2.

6.4 Dynamic Lévy

6.4.1 Calculating the drift

From the layman calibration, Section 6.3, we have estimated the diffusion term of the density process κ_t from a time series $\kappa_1, \dots, \kappa_K$. From Equation (6.20) and Equation (6.22) we have, for $T_{j-1} \leq T < T_j$ and $i = 1, \dots, K-1$,

$$\hat{\beta}^i(T, x) = \frac{\kappa_{i+1}(T, x) - \kappa_i(T, x)}{\sqrt{\Delta(K-1)}} = \frac{f_j^{(i+1)}(x) - f_j^{(i)}(x)}{\sqrt{\Delta(K-1)}}. \quad (6.25)$$

We have that $f_j^{(i)}$ is of the form (6.23).

Now we do the following modification of the "layman calibrated" diffusion function,

$$\begin{aligned} \hat{\beta}^i(T, x) &= (6.25), & |x| \leq 0.4 \\ \hat{\beta}^i(T, x) &= 0, & |x| > 0.4. \end{aligned} \quad (6.26)$$

This modification is justified by the fact that option prices which are far in/out of the money ($|\log(\frac{K}{S_0})| > 0.4$) are not liquidly quoted on the market. This corresponds to jump sizes $x > 0.4$ of the Lévy density $\kappa(T, x)$, see Section 6.7.

Since we use a time series of market data in the layman calibration (Section 6.3.1), the modification in (6.26) filters away errors due to mispricing. At the same time, it prevents the Lévy density process of becoming negative for $|x| > 0.4$. In what follows we define and compute the drift $\alpha(T, \cdot)$ and its components separately on $x \in (-\infty, 0.4)$, $[0.4, 0)$, $(0, 0.4]$, $(0.4, \infty)$.

Now, we can calculate the drift of the *Tangent Lévy model* by using the requirement in (5.8a),

$$\begin{aligned} \hat{\alpha}_t(T, x) &= -e^{-x} \sum_{n=1}^m \int_{\mathbb{R}} \partial_y^3 \psi(\bar{\beta}_t^n(T, \cdot); y) \times \\ &\quad \left[\partial_x \psi(\hat{\beta}^n(T, \cdot); x - y) - \left(1 - y \partial_x\right) \partial_x \psi(\hat{\beta}^n(T, \cdot); x) \right] \\ &\quad - 2 \partial_y^2 \psi(\bar{\beta}_t^n(T, \cdot); y) \left[\partial_x \psi(\hat{\beta}^n(T, \cdot); x - y) - \partial_x \psi(\hat{\beta}^n(T, \cdot); x) \right] \\ &\quad + \partial_y \psi(\bar{\beta}_t^n(T, \cdot); y) \partial_x \psi(\hat{\beta}^n(T, \cdot); x - y) dy. \end{aligned} \quad (6.27)$$

Here, $\bar{\beta}_t^n(T, \cdot) = \int_{t \wedge T}^T \hat{\beta}^n(u, \cdot) du$. We have that this integrand is piecewise constant, with increments at the "calibration points" $\{T_0 < T_1 < T_2 < T_3\}$, (see section 6.1.2).

For $t = T_{k-1} < T = T_j$ and $|x| \leq 0.4$ we now have,

$$\begin{aligned}\bar{\beta}_t^i(T, x) &= \int_{t \wedge T}^T \hat{\beta}^i(s, x) ds = \int_{t \wedge T}^T \frac{(\kappa_{i+1}(s, x) - \kappa_i(s, x))}{\sqrt{\Delta(K-1)}} ds \\ &= \frac{\sum_{l=k}^j (T_l - T_{l-1}) (\kappa_{i+1}(T_l, x) - \kappa_i(T_l, x))}{\sqrt{\Delta(K-1)}}.\end{aligned}\quad (6.28)$$

$\kappa_i(T, x)$ is defined as in Equation (6.22). For $|x| > 0.4$ we have,

$$\bar{\beta}_t^i(T, x) = 0. \quad (6.29)$$

Next, from the Definition (3.69), we have that the first derivative of the exponential tail function is an integral operator,

$$\partial_x \psi(g; x) := \begin{cases} e^x \int_{-\infty}^x g(u) du, & x < 0 \\ -e^x \int_x^{\infty} g(u) du, & x > 0. \end{cases} \quad (6.30)$$

We also retrieve,

$$\partial_x^2 \psi(g; x) := \begin{cases} e^x \left[\int_{-\infty}^x g(u) du + g(x) \right], & x < 0 \\ e^x \left[-\int_x^{\infty} g(u) du + g(x) \right], & x > 0, \end{cases} \quad (6.31)$$

and

$$\partial_x^3 \psi(g; x) := \begin{cases} e^x \left[\int_{-\infty}^x g(u) du + 2g(x) + g'(x) \right], & x < 0 \\ e^x \left[-\int_x^{\infty} g(u) du + 2g(x) + g'(x) \right], & x > 0. \end{cases} \quad (6.32)$$

Now, from Equations (6.25) and (6.30), for $x \in [-0.4, 0)$, we have,

$$\begin{aligned}\partial_x \psi(\hat{\beta}^i(T, \cdot); x) &:= e^x \int_{-\infty}^x \hat{\beta}^i(T, u) du \\ &= e^x \int_{-\infty}^x \left[\frac{\kappa_{i+1}(T, u) - \kappa_i(T, u)}{\sqrt{\Delta(K-1)}} \right] du.\end{aligned}\quad (6.33)$$

Due to the modification in Equation (6.26) the integrand vanishes on the interval $(-\infty, 0.4)$. We have ,

$$(6.33) = e^x \int_{-0.4}^x \left[\frac{\kappa_{i+1}(T, u) - \kappa_i(T, u)}{\sqrt{\Delta(K-1)}} \right] du. \quad (6.34)$$

Recall that since we have chosen the model of Section 6.1 to capture our Lévy densities, for each day i of our sample, we have $\kappa_i(T, x)$ defined by Equation (6.23). We have that the integrand of (6.33) will be a difference of Gaussian probability density functions. Hence we can calculate the integral using the error function.

$$\begin{aligned}(6.34) &= e^x \left[\frac{\left(\frac{\lambda_{i+1,j}}{2} [1 + \operatorname{erf}(\frac{x - \mu_{i+1,j}}{\delta_{i+1,j}\sqrt{2}})] - \frac{\lambda_{i,j}}{2} [1 + \operatorname{erf}(\frac{x - \mu_{i,j}}{\delta_{i,j}\sqrt{2}})] \right)}{\sqrt{\Delta(K-1)}} \right] \\ &\quad - e^x \left[\frac{\left(\frac{\lambda_{i+1,j}}{2} [1 + \operatorname{erf}(\frac{-0.4 - \mu_{i+1,j}}{\delta_{i+1,j}\sqrt{2}})] - \frac{\lambda_{i,j}}{2} [1 + \operatorname{erf}(\frac{-0.4 - \mu_{i,j}}{\delta_{i,j}\sqrt{2}})] \right)}{\sqrt{\Delta(K-1)}} \right].\end{aligned}\quad (6.35)$$

The second derivative of $\psi(\hat{\beta}^i(T, \cdot); x)$, $x \in [-0.4, 0)$, may now be calculated as follows,

$$\partial_x^2 \psi(\hat{\beta}^i(T, \cdot); x) := \partial_x \psi(\hat{\beta}^i(T, \cdot); x) + e^x \hat{\beta}^i(T, x). \quad (6.36)$$

For $|x| > 0.4$ we have,

$$\partial_x \psi(\hat{\beta}^i(T, \cdot); x) = \partial_x^2 \psi(\hat{\beta}^i(T, \cdot); x) = 0. \quad (6.37)$$

To find $\partial_x \psi(\hat{\beta}^i(T, \cdot); x)$ and $\partial_x^2 \psi(\hat{\beta}^i(T, \cdot); x)$ for $x \in (0, 0.4]$, we use the following property of the probability density function;

$$\frac{\lambda}{\sqrt{2\pi\delta^2}} \int_x^\infty \exp\left(-\frac{(u-\mu)^2}{2\delta^2}\right) du = \lambda \left[1 - \frac{1}{\sqrt{2\pi\delta^2}} \int_{-\infty}^x \exp\left(-\frac{(u-\mu)^2}{2\delta^2}\right) du\right]. \quad (6.38)$$

We continue with calculating the derivatives of $\psi(\bar{\beta}_t^i(T, \cdot); x)$. From (6.28) and (6.30) we have, for $x \in [-0.4, 0)$ and $t = T_{k-1} < T = T_j$,

$$\begin{aligned} \partial_x \psi(\bar{\beta}_t^i(T, \cdot); x) &= e^x \int_{-\infty}^x \bar{\beta}_t^i(T, u) du = e^x \int_{-0.4}^x \int_{t \wedge T}^T \hat{\beta}^i(s, u) ds du \\ &= e^x \int_{-0.4}^x \frac{\sum_{l=k}^j (T_l - T_{l-1}) (\kappa_{i+1}(T_l, u) - \kappa_i(T_l, u))}{\sqrt{\Delta(K-1)}} du. \end{aligned} \quad (6.39)$$

Again, having each $f_j^{(i)}(x)$, and hence $\kappa_i(T, x)$ defined by Equation (6.23), we may calculate this integral using the error function. We have,

$$\begin{aligned} (6.39) &= e^x \frac{\sum_{l=k}^j (T_l - T_{l-1}) \left[\frac{\lambda_{i+1,l}}{2} [1 + \operatorname{erf}(\frac{x-\mu_{i+1,l}}{\delta_{i+1,l}\sqrt{2}})] - \frac{\lambda_{i,l}}{2} [1 + \operatorname{erf}(\frac{x-\mu_{i,l}}{\delta_{i,l}\sqrt{2}})] \right]}{\sqrt{\Delta(K-1)}} \\ &\quad - e^x \frac{\sum_{l=k}^j (T_l - T_{l-1}) \left[\frac{\lambda_{i+1,l}}{2} [1 + \operatorname{erf}(\frac{-0.4-\mu_{i+1,l}}{\delta_{i+1,l}\sqrt{2}})] - \frac{\lambda_{i,l}}{2} [1 + \operatorname{erf}(\frac{-0.4-\mu_{i,l}}{\delta_{i,l}\sqrt{2}})] \right]}{\sqrt{\Delta(K-1)}}. \end{aligned} \quad (6.40)$$

Also here we can use the property in (6.38) to calculate the integral for $x \in (0, 0.4]$.

At last we wish to calculate $\partial_x^2 \psi(\bar{\beta}_t^i(T, \cdot); x)$ and $\partial_x^3 \psi(\bar{\beta}_t^i(T, \cdot); x)$. We use the relations in Equations (6.31) and (6.32). For $x \in [-0.4, 0)$ and $x \in (0, 0.4]$ we retrieve,

$$\partial_x^2 \psi(\bar{\beta}_t^i(T, \cdot); x) = \partial_x \psi(\bar{\beta}_t^i(T, \cdot); x) + e^x \bar{\beta}_t^i(T, x), \quad (6.41)$$

$$\partial_x^3 \psi(\bar{\beta}_t^i(T, \cdot); x) = \partial_x \psi(\bar{\beta}_t^i(T, \cdot); x) + 2e^x \bar{\beta}_t^i(T, x) + e^x \partial_x \bar{\beta}_t^i(T, x). \quad (6.42)$$

Here, for $t = T_{k-1} < T = T_j$,

$$\partial_x \bar{\beta}_t^i(T, x) = \frac{\sum_{l=k}^j (T_l - T_{l-1}) \left[\frac{d}{dx} \kappa_{i+1}(T_l, x) - \frac{d}{dx} \kappa_i(T_l, x) \right]}{\sqrt{\Delta(K-1)}}. \quad (6.43)$$

Here $\frac{d}{dx}\kappa_i(T, x)$ follows from our definition of $\kappa_i(T, x)$ (recall Equation (6.23)),

$$\frac{d}{dx}\kappa_i(T, x) = \frac{\lambda_{i,j}(x - \mu_{i,j})}{\delta_{i,j}^2 \sqrt{2\pi\delta_{i,j}^2}} \exp\left(-\frac{(x - \mu_{i,j})^2}{2\delta_{i,j}^2}\right), \quad T_{j-1} \leq T < T_j, \quad (6.44)$$

As before, $\lambda_{i,j}, \mu_{i,j}, \delta_{i,j}$ denotes the model parameters corresponding to the j -th increment, measured at day i . As $\bar{\beta}_t^i(T, x) = 0$ for $|x| > 0.4$, we have

$$\partial_x \psi(\bar{\beta}_t^i(T, \cdot); x) = \partial_x^2 \psi(\bar{\beta}_t^i(T, \cdot); x) = \partial_x^3 \psi(\bar{\beta}_t^i(T, \cdot); x) = 0, \quad |x| > 0.4. \quad (6.45)$$

6.5 On the choice of parameterization

In the previous section, we choose a compound Poisson parameterization to capture a time series of call option surfaces. More precisely, the following Lévy density is calibrated to a time series of call option surfaces,

$$\kappa(T, x) = f_j(x), \quad T_{j-1} < T \leq T_j, \quad (6.46)$$

where $j = 1, 2, 3$, $T_0 = 0$, $T_1 = 0.5$, $T_2 = 1$, $T_3 = 2$ years and $f_j(x)$ is as in Equation (6.10). (See Equation (6.12)). The resulting time series of Lévy densities is then used in a “layman calibration”. The “layman method” is described in Section 6.3.1.

If the purpose would be solely to capture the call option surface and the implied volatility surface at each day of our sample with highest possible accuracy, we would most likely choose an infinite intensity type model. For instance, a time-inhomogeneous NIG-model (i.e. the stock is modeled as a additive process with NIG-distributed increments).

However, this is not the case as we wish also to approximate the diffusion coefficients, $\beta_t^i(T, x)$, as a part of a *market model for stock options*, by using the time series of Lévy densities.

As in the Tangent Lévy models due to Carmona and Nadtochiy [2009], see also Equation (5.3) above, the evolution of the Lévy density is assumed to be given by the following integral equation,

$$\kappa_t(T, x) = \kappa_0(T, x) + \int_0^t \alpha_u(T, x) du + \sum_{n=1}^m \int_0^t \beta_u^n(T, x) dB_u^n. \quad (6.47)$$

We approximate the diffusion coefficients $\beta_t^i(T, x)$ by a layman calibration to a times series of market data. As in Equation (6.25),

$$\beta_t^i(T, x) \approx \hat{\beta}^i(T, x) = \frac{\kappa_{i+1}(T, x) - \kappa_i(T, x)}{\sqrt{\Delta(K-1)}}. \quad (6.48)$$

Here K is the number of days in our sample and $\kappa_i(T, x)$ is the time-inhomogeneous Lévy density corresponding to the call option surface of day i .

Furthermore we wish to calculate the drift term of Equation (6.47) from the relation in Equation (5.8a). It follows that $\hat{\beta}^i(T, x)$ has to fulfill the regularity assumptions in Remark (5.2). This restricts the choice of possible parameterizations of the Lévy densities $\kappa_i(T, x)$.

Example 6.5.1 (NIG parameterization). Say we were to use a time inhomogeneous Normal Inverse Gaussian Lévy density to capture the time series $\kappa_1, \dots, \kappa_K$. From the definition of the NIG-process, we have as in Equation (6.25)

$$\hat{\beta}^i(T, x) = \frac{f_j^{(i+1)}(x) - f_j^{(i)}(x)}{\sqrt{\Delta(K-1)}}, \quad T_{j-1} \leq T < T_j. \quad (6.49)$$

In this case however, the densities $f_j^{(i)}(x)$ are defined as,

$$f_j^{(i)}(x) = e^{\beta_{i,j}x} \frac{\delta_{i,j} \alpha_{i,j}}{\pi|x|} K_1(\alpha_{i,j}|x|). \quad (6.50)$$

See Equation (3.19). Note that $K_1(x)$ denotes the modified Bessel function of the third kind with index = 1. However, the function defined by (6.49)-(6.50) does not fulfill the assumptions of Remark (5.2), as the NIG-process has infinite variation ($\int_{\{|x| \leq 1\}} |x| f_j^{(i+1)}(x) dx = \infty$).

Example 6.5.2 (Variance Gamma). Opposed to the NIG-process, the Variance Gamma process, see Section 3.1.5, has finite variation. Hence this pure jump process seems like a better choice when approximating $\beta_t^i(T, x)$. We have the Lévy density of the Variance Gamma process defined by,

$$f(x) = \frac{\exp(\theta x / \sigma^2)}{\nu|x|} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma} |x|\right). \quad (6.51)$$

See Equation (3.23). Now, a closer look at the VG-density and the regularity assumptions of Remark 5.2 reveals a problem. From Equation (6.25) we know that $\hat{\beta}^i(T, x)$ will include the difference of two VG-Lévy densities. As the VG-process has infinite intensity, $\hat{\beta}^i(T, x)$ defined in this way will not fulfill the regularity assumptions of Remark (5.2). In particular the “symmetry” condition,

$$\int_{\mathbb{R}} (e^x - 1) \beta_t^n(T, x) dx = 0, \quad (6.52)$$

will not be fulfilled. This may be concluded using graphical analysis. As an example, $\hat{\beta}^1(T, x)$, $T = 1$, defined as the difference between two calibrated VG-Lévy densities is plotted in Figure 6.4 a). In Figure 6.4 b) you see $g(x) := (e^x - 1) \beta_t^1(T, x)$.

The “symmetry condition” of Equation (6.52) is needed to assure that the partial derivatives of $\psi(\hat{\beta}^i(T, \cdot), x)$ are well defined globally. In case $\partial_x \psi(\hat{\beta}^i(T, \cdot), x)$ is not well defined, the derivation of the drift restriction (Equation (5.8a)) does not hold. See Carmona and Nadtochiy [2009], proof of Theorem 12.

Example 6.5.3 (Compound Poisson). As a graphical comparison to the parameterizations given in the examples above, in Figure 6.5 a) the function $\beta^1(T, x)$, $T = 1$, when this is defined as described in Section 6.3.2 is plotted. That is, $\beta^1(T, x)$ is defined as the difference between two Lévy densities of the form (6.23). In Figure 6.5 b) $g(x) := (e^x - 1) \beta_t^1(T, x)$ is plotted.

We choose this Lévy density in the “layman calibration” due to it’s nice properties, that allows us to calculate a diffusion and a drift process as described in Sections 6.3.1-6.4.1.

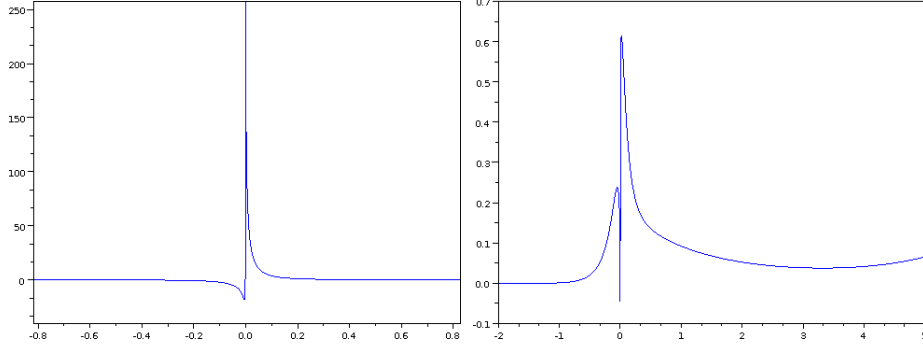


Figure 6.4: a) $\beta^1(1, x)$ defined by VG. b) $g(x) := (e^x - 1)\beta^1(1, x)$.

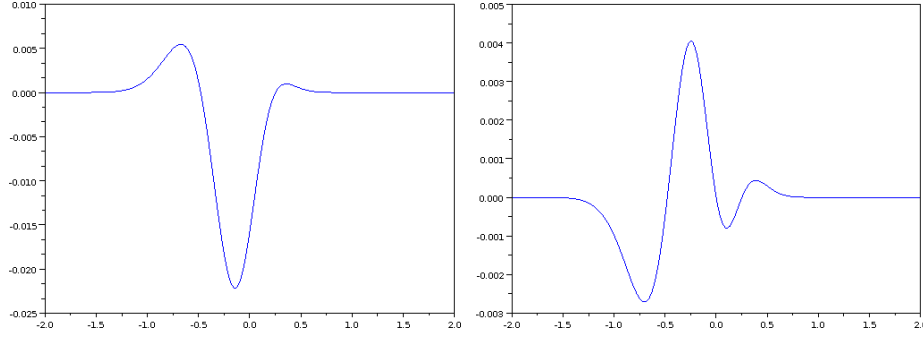


Figure 6.5: a) $\beta^1(1, x)$, as in Section 6.3.2 b) $g(x) := (e^x - 1)\beta^1(1, x)$

6.6 An Euler simulation of the Tangent Lévy model

The results of [Carmona and Nadtochiy \[2009\]](#) show, as is discussed above in Section 5.2, that the diffusion $\beta(T, x)$ is the only free parameter of the Tangent Lévy model.

In Section 6.3.1 is described how to select $\beta(T, x)$ by using the layman-calibration method of [Ortega et al. \[2009\]](#). For numerical reasons, we choose the simple "layman calibrated" $\hat{\beta}(T, x)$ of Section 6.3.2. In Section 6.4.1 is described how to calculate the drift $\hat{\alpha}(T, x)$ when we are given $\hat{\beta}(T, x)$.

Now we are ready to simulate sample paths of the stock price (Equation (5.2)) and the dynamic time-inhomogeneous Lévy density (Equation (5.3)), of a Tangent Lévy model. As in Equation (6.20), for the density we have

$$\hat{\kappa}_t^{(K)}(T, x) = \hat{\kappa}_0(T, x) + \int_0^t \hat{\alpha}_u(T, x) du + \sum_{i=1}^{K-1} \int_0^t \hat{\beta}^i(T, x) dW_u^i. \quad (6.53)$$

K is the number of samples used in our "layman calibration", see Section 6.3. From here on for notational reasons we write $\hat{\kappa}^{(K)}(T, x) = \hat{\kappa}(T, x)$.

6.6.1 Recovering the initial surface

We use the current stock price S_0 , and available call option prices to deduce the initial Lévy density surface $\hat{\kappa}_0(T, x)$. To recover $\hat{\kappa}_0(T, x)$ we use a model

similar to the one presented in Section 6.1. As in Section 6.1 we define the log underlying as having independent compound Poisson distributed increments, but now we assume the jump sizes to be distributed as a mixture of normals. As in Section 6.1 we also include a drift.

More precisely we calibrate the following additive process to the initial call option surface,

$$S_t := S_0 \exp(Y_t).$$

We define the process Y_t as having independent increments given by the compound Poisson distribution with jump sizes distributed as a mixture of 3 normal distributions. We assume also that the parameters of the process Y_t are piecewise constant. That is the following hold for the characteristic function of Y_t ,

$$\begin{aligned} \mathbb{E}[e^{iuY_t}] &= e^{(t-t_0)\psi_1(u)}, & t < t_1 \\ \mathbb{E}[e^{iuY_t}] &= \mathbb{E}[e^{iu(Y_{t_1} + (Y_t - Y_{t_1}))}] = e^{t_1\psi_1(u) + (t-t_1)\psi_2(u)}, & t_1 \leq t < t_2 \\ &\vdots \\ \mathbb{E}[e^{iuY_t}] &= \exp\left\{\sum_{j=1}^k t_j\psi_j(u) + (t-t_{k-1})\psi_k(u)\right\}, & t_{k-1} \leq t < t_k. \end{aligned} \quad (6.54)$$

Here we set $t_0 = Y_0 = 0$, and $\psi_j(u)$ denotes the characteristic exponent of the j -th increment. Having jump sizes given by a mixture of 3 normal distributions, we have,

$$\psi_j(u) = i\omega_j u + \sum_{h=1}^3 \left[\lambda_{j,h} \left(\exp\left(-\frac{\delta_{j,h}^2 u^2}{2}\right) + i\mu_{j,h}u \right) - 1 \right]. \quad (6.55)$$

The drift ω_j will be given by the martingale condition, $\mathbb{E}^\mathbb{Q}[e^{-rt}S_t] = S_0$, as in Equation (6.4). Note that in this sequel only, $\{\lambda_{j,h}, \mu_{j,h}, \delta_{j,h}\}$ denotes the h -th set of (mixture) model parameters of the j -th increment of the initial surface. We use the calibration procedure described in Section 6.1.2. This results in a time-inhomogeneous Lévy density of the following form,

$$\hat{\kappa}_0(T, x) = f_j(x), \quad T_{j-1} \leq T < T_j. \quad (6.56)$$

Now however we have the density f as follows,

$$f_j(x) = \sum_{h=1}^3 \left[\frac{\lambda_{j,h}}{\sqrt{2\pi\delta_{j,h}^2}} \exp\left(-\frac{(x-\mu_{j,h})^2}{2\delta_{j,h}^2}\right) \right]. \quad (6.57)$$

As in Section 6.1.2 we choose $j = 1, 2, 3$, $T_0 = 0$, $T_1 = 0.5$, $T_2 = 1$ and $T_3 = 2$. It follows that we have an initial surface given by 9×3 parameters; $\lambda_{j,1}, \delta_{j,1}, \mu_{j,1}, \lambda_{j,2}, \delta_{j,2}, \mu_{j,2}, \lambda_{j,3}, \delta_{j,3}, \mu_{j,3}$.

Remark 6.2. It is important to note that the initial surface $\hat{\kappa}_0(T, x)$ has an evident effect on the stopping time $\hat{\tau}_0$, which is the first time when the dynamic Lévy density becomes non positive. As in Equation (5.12),

$$\hat{\tau}_0 := \inf\left\{t \geq 0 : \inf_{T \in [t, \bar{T}], x \in \mathbb{R}} \hat{\kappa}_t(T, x) \leq 0\right\}. \quad (6.58)$$

We have that the "layman calibrated" diffusion functions $\hat{\beta}^i(T, x)$, $i = 1, \dots, N$ equals the differences of time dependent compound Poisson densities, calibrated to the call option surfaces of quotation days i and $i - 1$.

As we need the convergence of Equation (6.18) we need a large N . We choose $N = 365$ in our implementation. Most of $\hat{\beta}^i(\cdot, x)$ are centered around zero, as is the initial surface $\hat{\kappa}_0(\cdot, x)$.

However, most likely there will also be some $\hat{\beta}^i(\cdot, x)$ not centered at zero. This leads to that the diffusion (and drift) of $\hat{\kappa}_t(\cdot, x)$ does not go to zero (when $|x|$ gets larger) as fast as the initial surface $\hat{\kappa}_0(\cdot, x)$. It follows that $\hat{\tau}_0$ occurs relatively fast.

One way to prevent $\hat{\kappa}_t(T, x)$ of becoming non positive quickly is to choose an initial surface $\hat{\kappa}_0(T, x)$ which has some density also in the tails. For instance the normal mixture parameterization described above.

6.6.2 Simulating the Lévy density process

We have $\hat{\kappa}_0(T, x)$ as described in Section 6.6.1. Next, given $\hat{\beta}^i(T, x)$, $i = 1, \dots, K - 1$ from Equation (6.25), we compute $\hat{\alpha}_t(T, x)$ using the drift restriction as described in Section 6.4.1.

Having $\hat{\kappa}_0(T, x)$, $\hat{\alpha}_t(T, x)$ and $\hat{\beta}(T, x)$ we use an Euler scheme (see for instance Kloeden and Platen [1995]), to find an approximation $\{\tilde{\kappa}_t(T, x)\}_{t \in [0, \bar{t}]}$.

More precisely, we simulate the process $\tilde{\kappa}_t(T, x)$ on a time interval $t \in [0, \bar{t}]$ where $\bar{t} = 1$ year, we choose a time step $h = 1/365$. That is we have $n = 365$ steps in our simulation. The approximation $\tilde{\kappa}_t(T, x)$ is as follows,

$$\tilde{\kappa}_t(T, x) := \begin{cases} \hat{\kappa}_0(T, x), & 0 \leq t < h, \\ \Delta_1 \tilde{\kappa} + \dots + \Delta_j \tilde{\kappa}, & jh \leq t < (j+1)h, \end{cases} \quad (6.59)$$

where

$$\Delta_j \tilde{\kappa}(T, x) = \hat{\alpha}_{jh}(T, x) \frac{1}{n} + \sum_{i=1}^m \hat{\beta}^i(T, x) N(0, 1) \sqrt{\frac{1}{n}}. \quad (6.60)$$

Here $j = 1, \dots, 364$ and $N(0, 1)$ is a standard normal distributed random variable. We stop the process $\tilde{\kappa}_t(T, x)$ at the following stopping time.

$$\tau_0 = \inf \left\{ t \geq 0 : \inf_{T \in [t, \bar{T}], x \in \mathbb{R}} \tilde{\kappa}_t(T, x) \leq 0 \right\}, \quad (6.61)$$

see Remark 6.2 above.

6.6.3 Simulating the log stock price process

Next we simulate $X_t = \log(S_t)$. We follow the TLM-setup, as in Equation (3.64)

$$X_t = \gamma(t) + \int_0^t \int_{\mathbb{R}} x(\mu^X(dx, du) - K_u(x) dx du). \quad (6.62)$$

where

$$\gamma(t) := \log(S_0) - \int_0^t \int_{\mathbb{R}} (e^x - x - 1) K_u(x) dx du. \quad (6.63)$$

As before, $\mu^X(dx, du)$ is the random jump measure associated with the jumps of X .

We assume that the process X_t has finite intensity.³³ This means that we can write Equation (6.62) in the following way,

$$\begin{aligned} X_t &= b(t) + \int_0^t \int_{\mathbb{R}} x \mu^X(ds, dx), \\ &= b(t) + \sum_{s < t} \Delta X_s \mathbf{1}_{\Delta X \neq 0} \end{aligned} \quad (6.64)$$

where

$$b(t) = \gamma(t) - \int_0^t \int_{\mathbb{R}} x K_u(x) dx du = \log(S_0) - \int_0^t \int_{\mathbb{R}} (e^x - 1) K_u(x) dx du. \quad (6.65)$$

Note that for this to hold we need only finite variation of the process X_t , $(\int_{\mathbb{R}} x K_t(x) dx < \infty, \forall t \in [0, T])$, and not finite intensity.

Since we assume also finite intensity, we have that X_t is a compound Poisson process with the following jump intensity,

$$\lambda_t := \int_{\mathbb{R}} K_t(x) dx, \quad (6.66)$$

and jump size distribution,

$$p_t(x) = \frac{K_t(x)}{\lambda_t}. \quad (6.67)$$

As in Section 6.6.2 above, we simulate the process X_t on a time interval $t \in [0, \bar{t}]$ where $\bar{t} = 1$ year. We put $h = 1/365$, (i.e time step = 1 day). A random walk approximation of $\{X_t\}_{t \in [0, \bar{t}]}$ is now given by,

$$\tilde{X}_t := \begin{cases} X_0 = \log(S_0), & 0 \leq t < h, \\ X_0 + \Delta_1 \tilde{X} + \dots + \Delta_j \tilde{X}, & j \cdot h \leq t < (j+1)h. \end{cases} \quad (6.68)$$

The increments $\Delta_j \tilde{X} = \tilde{X}_{jh} - \tilde{X}_{(j-1)h}$ are random variables with distributions given by the Lévy triplets $\{b(jh), 0, K_{jh}(x)\}$.

Here $K_{jh}(x)$ and the drift $b(jh)$ will change with each step of the random walk, due to the "TLM consistency condition". Recall Equation (5.8b),

$$K_t(x) = \kappa_t(t, x). \quad (6.69)$$

That is, we need the simulated time-inhomogeneous Lévy density of Equation (6.59) to calculate the increments of the process $\tilde{X}_t$.

For each step j of the random walk approximation, we calculate $\Delta \tilde{X}_j$ as follows, (see Cont and Tankov [2003], Algorithm 6.2);

³³ I.e. $\int_{\mathbb{R}} K_t(x) dx < \infty, \forall t \in [0, T]$. This is a reasonable assumption since we use a time dependent mixture of normal distributions as initial value for the density, and we have $K_0(x) = \kappa_0(0, x)$.

1. Simulate a random variable P from the Poisson distribution with parameter $h\lambda_{jh} = h \int_{\mathbb{R}} K_{jh}(x)dx$.
 P corresponds to the number of jumps in the time interval $[jh, (j+1)h)$.
2. Simulate P independent random variables Y_i with law $\frac{K_{jh}(x)}{\lambda_{jh}}$.
For this we use the acceptance/rejection method. See for instance [Devroye \[1986\]](#), Chapter II.3. Y_i , $i = 1, \dots, P$ corresponds to the jump sizes.
3. We have

$$\Delta_j \tilde{X} = \Delta b_j + \sum_{i=1}^P Y_i, \quad (6.70)$$

where

$$\Delta b_j := -\frac{1}{n} \int_{\mathbb{R}} (e^x - 1) K_{jh}(x) dx. \quad (6.71)$$

6.7 Calculating option prices from $\kappa_t(T, x)$

As a last step of the implementation of our version of a *Tangent Lévy model*, we wish to calculate European call option surfaces from the data simulated in Section 6.6. That is, from the sample paths of κ and S (see Equations (6.59) and (6.68)), and for each time step of the realization, we calculate a call option surface/implicit volatility surface.

We calculate the option prices corresponding to the time-inhomogeneous Lévy densities $\tilde{\kappa}_j(T, x)$, $j = 1, \dots, 364$, by using the FFT-approach as described for instance by Carr and Madan [1999]. See also Sections 3.3.1-3.3.2 above. That is, as in Equation (3.45), the price of a European call option having maturity time T and strike price e^{x_u} , is given by,

$$\tilde{C}_t(T, x_u) \approx \frac{e^{-\alpha x_u}}{\pi} \sum_{k=0}^{M-1} e^{-iu\zeta\eta^k} f_k. \quad (6.72)$$

As in Section 3.3.2, α is a scaling factor which we set equal to 1.25 and we have $\eta, \zeta > 0$, $\eta\zeta = \frac{2\pi}{M}$. We choose $\zeta = 0.1$ and $M = 2^{12}$. As in Equation (3.45), we have

$$f_k := e^{iv_k(\frac{1}{2}M\zeta - \log(S_t))} \left[\frac{e^{-rT} \phi_T(v_k - (\alpha + 1)i)}{\alpha^2 + \alpha - v_k^2 + i(2\alpha + 1)v_k} \right] \eta. \quad (6.73)$$

Also,

$$x_u := -\frac{1}{2}M\zeta + \zeta u + \log(S_t), \quad u = 0, \dots, M-1, \quad (6.74)$$

is a grid of log-strike prices, and

$$v_k := \eta k, \quad k = 0, \dots, M-1, \quad (6.75)$$

is an integration grid. Equation (6.72) can be calculated using standard FFT algorithms.

We now have that $\phi_T(u)$, the characteristic function of the log-stock, will be dynamic and given by the simulated process $\tilde{\kappa}_j(T, x)$, $j = 1, \dots, 364$. Recall Equation (6.59). We denote this "dynamic characteristic function" by $\tilde{\phi}_{T,j}(u)$.

We find $\tilde{\phi}_{T,j}(u)$ by using the Lévy-Khinchine formula for additive processes, recall Equation (3.58). We have,

$$\tilde{\phi}_{T,j}(u) = \exp \left[\psi_j(u, T) \right], \quad u \in \mathbb{R}, \quad (6.76)$$

were

$$\psi_j(u, T) := iu\gamma_j(T) + \int_0^T \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x| \leq 1\}}) \tilde{\kappa}_j(s, x) dx ds. \quad (6.77)$$

The martingale restriction on the underlying process (discounted) gives us,

$$\gamma_j(T) := r - \int_0^T \int_{\mathbb{R}} (e^x - x\mathbf{1}_{\{|x| \leq 1\}} - 1) \tilde{\kappa}_j(s, x) dx ds, \quad (6.78)$$

where r is the interest rate. See for instance [Cont and Tankov \[2003\]](#), Proposition 8.20.

Due to the regularity of $\tilde{\kappa}_j(T, x)$ we may calculate (6.77) as follows,

$$\psi_j(u, T) = iur + \int_0^T \int_{\mathbb{R}} (e^{iux} - 1 - iu(e^x - 1)) \tilde{\kappa}_j(s, x) dx ds. \quad (6.79)$$

The integral of Equation (6.79) is calculated numerically using Simpson's 1/3 rule, after taking use of the Euler identity, $e^{iux} = \cos(ux) + i \sin(ux)$.

6.8 Simulated implied volatility surfaces

In this section we show the result of the simulation in Section 6.6, by using the calculations of Section 6.7 above. That is, we use the code-book discussed also in Section 3.5, and transform the simulated Lévy densities into call option prices and implied volatilities. From Section 6.6, Equation (6.59) and (6.68) we have,

$$\tilde{\kappa}_j(T, x), \tilde{S}_j \quad j = 1, \dots, 364.$$

We use the following relations, as in Equations (1.5) and (3.74),

$$\tilde{\kappa}_j(T, x), \tilde{S}_j \Rightarrow \tilde{C}_j(T, x) \Rightarrow \Sigma_j(T, x), \quad j = 1, \dots, 364.$$

Here $\tilde{C}_j(T, x)$ and $\Sigma_j(T, x)$ denotes the European call option price, resp. implied volatility at time step j , corresponding to maturity time T and strike price e^x . Note that we need the values $\tilde{S}_j$ as initial values when calculating the call option prices, (input in Equations (6.73)-(6.74) above).

For computational reasons, we simulate the Lévy densities of Equation (6.59) on a fix grid of jump sizes and time to maturities. We choose 30 jump sizes $x \in [-0.6, 0.6]$ and time to maturities $\tau = [0.15, 0.3, 0.8, 2]$ years. That is we have $\tau := T - t = T - j \cdot h$ fix during all time steps. For each step j of the realizations in Equations (6.59) and (6.68), this results in 4 "slices" of the time-inhomogeneous Lévy density.³⁴

By using the calculations of Section 6.7, this gives us 4 curves of call option prices, one for each time to maturity. These "call option slices" are thereafter interpolated to retrieve a call option surface, resp. an implied volatility surface.

In Figures 6.6-6.7 the initial Lévy density is plotted for the different time to maturities τ . In Figure 6.8 the initial call option curves are plotted. In Figures 6.9-6.10 the time-dependent Lévy densities of the first 6 steps of our simulation are plotted. In Figures 6.11-6.13 we see the corresponding implied volatility surfaces.³⁵ Note that we plot the implied volatility surface as a function of log-moneyness $\hat{x}$. That is, $\hat{x} = \log \frac{K}{S}$. S is the current value of the underlying stock and K is the option strike price.

³⁴For each maturity T and time step j , we interpolate the simulated values $\tilde{\kappa}_j(T, \cdot)$ using cubic splines.

³⁵The initial surface consists of call options on the ETF QQQQ, quoted on the date 2011.02.11. Source: <http://finance.yahoo.com/q?s=QQQQ>.

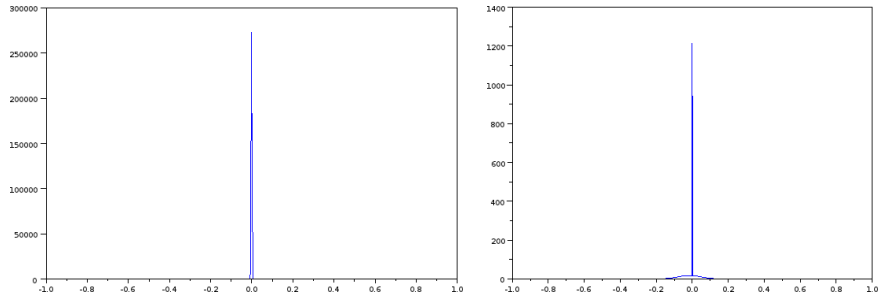


Figure 6.6: a) $\hat{\kappa}_0(0.15, x)$

b) $\hat{\kappa}_0(0.3, x)$

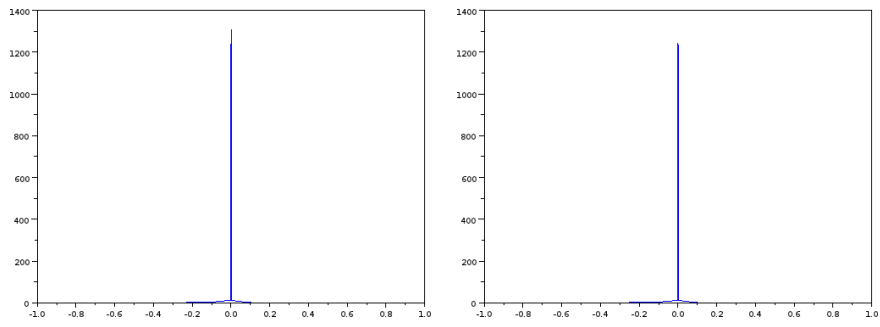


Figure 6.7: c) $\hat{\kappa}_0(0.8, x)$

d) $\hat{\kappa}_0(2, x)$

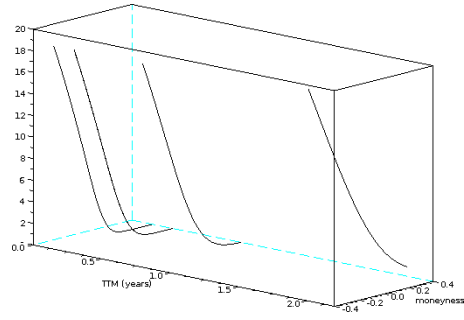


Figure 6.8: European call options $\tilde{C}_j(T, x)$, $j = 1$.

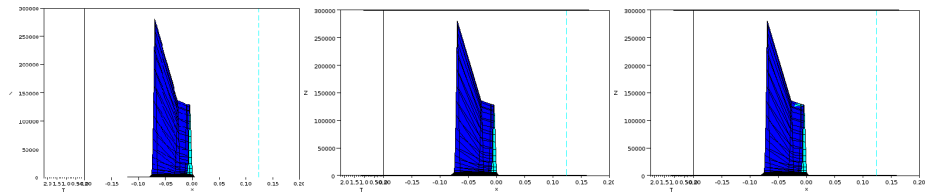


Figure 6.9: Lévy density surfaces, $\tilde{\kappa}_j(T, x)$, $j = 1, 2, 3$.

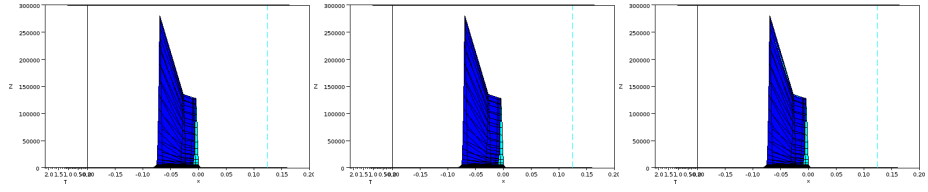


Figure 6.10: Lévy density surfaces, $\tilde{\kappa}_j(T, x)$, $j = 4, 5, 6$.

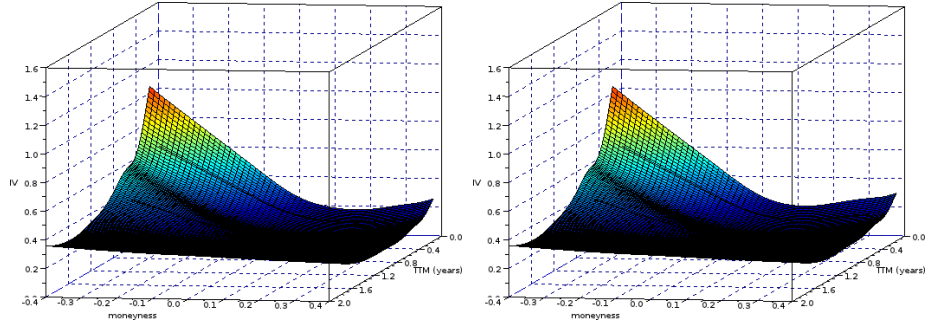


Figure 6.11: Implied volatility surfaces $j = 1, 2$.

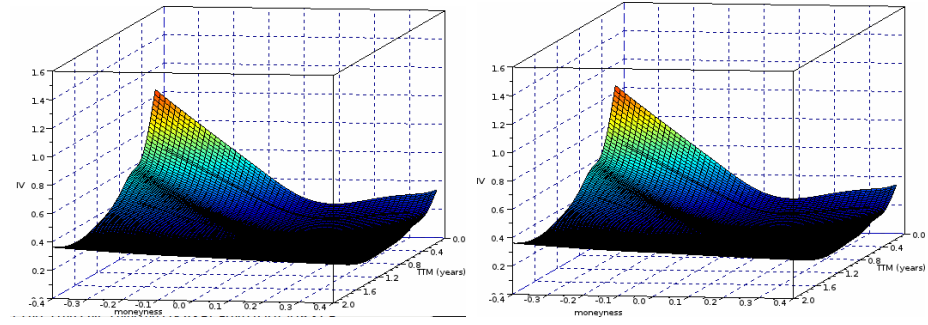


Figure 6.12: Implied volatility surfaces, $j = 3, 4$.

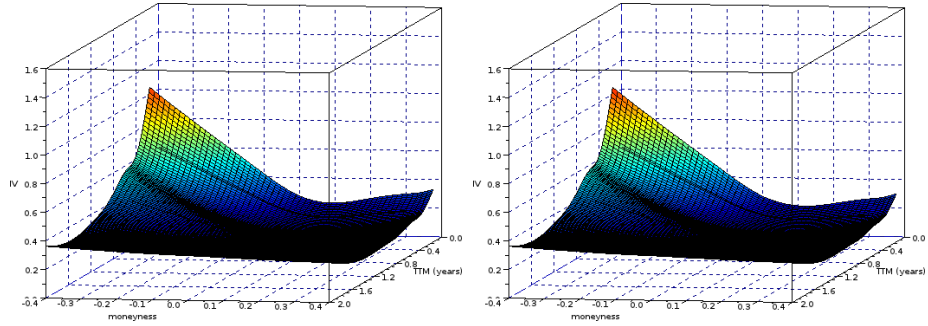


Figure 6.13: Implied volatility surfaces, $j = 5, 6$.

6.9 Check of consistency

After the simulation practice of Section 6.6-6.8, we now like to control that the resulting price/implied volatility surfaces are indeed free of arbitrage. As a first check up we use Dupire's formula.

6.9.1 Calculating the local volatility surface

We have that for a call option surface to be free of arbitrage, the corresponding local volatility surface has to be nonnegative. (Recall the discussion in Section 2.3.) We calculate the local volatility surfaces using formula (2.60),

$$a(T, K) = \sqrt{\frac{\Sigma^2 + 2\tau\Sigma(\partial_T\Sigma + rK\partial_K\Sigma)}{(1 + Kd_1\sqrt{\tau}\partial_K\Sigma)^2 + \Sigma^2K^2\tau(\partial_{KK}\Sigma - d_1(\partial_K\Sigma)^2\sqrt{\tau})}}. \quad (6.80)$$

Here $\Sigma = \Sigma(T, K)$ denotes the implied volatility corresponding to strike price K and maturity time T . As before, $\tau = T - t$, and $d_1 := \frac{\log(S_t/K) + (r + \Sigma^2/2)\tau}{\Sigma\sqrt{\tau}}$.

The partial derivatives needed in Equation (6.80) are approximated by a finite difference method. The FFT-pricing of Section 6.7 gives us call option prices for fix maturities and a grid of log-strikes. We interpolate these model prices in time and strike space using cubic splines. This gives us an approximated pricing function that we use for the numerical derivatives.

Recall however the problem discussed in Section 2.3.2, i.e. that the resulting local volatility surface may be unstable and dependent on the choice of interpolation method. Since we do not intend to deduce “a correct” local volatility model, but merely to check positivity, we believe the above method gives us a good enough approximation of the local volatility surface. We calculate the local volatility for time to maturities $\tau > 0.1$ years.

The resulting local volatility approximations (first 6 time steps of our simulation) are plotted in Figures 6.14-6.16. Observe that we plot the local volatility surface in the variables time to maturity τ and log-moneyness $x = \log \frac{K}{S}$. ($\hat{a}(\tau, x) := a(\tau + t, Se^x)$.)

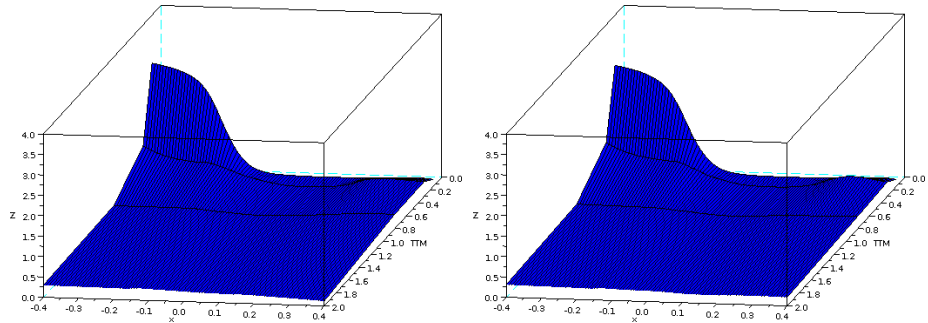


Figure 6.14: Local volatility surface $\hat{a}_j(\tau, x)$, $j = 1, 2$.

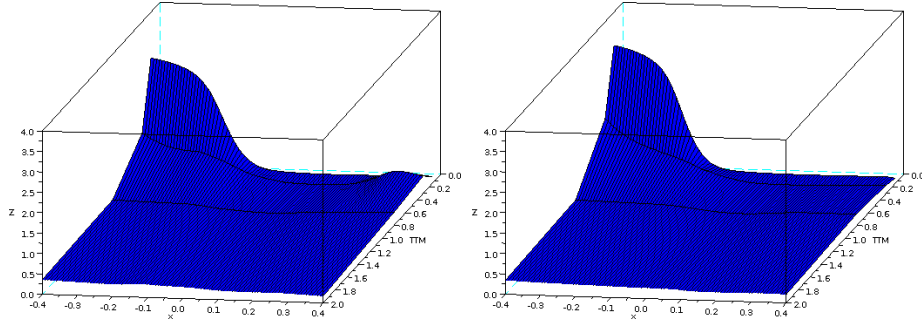


Figure 6.15: Local volatility surface $\hat{a}_j(\tau, x)$, $j = 3, 4$.

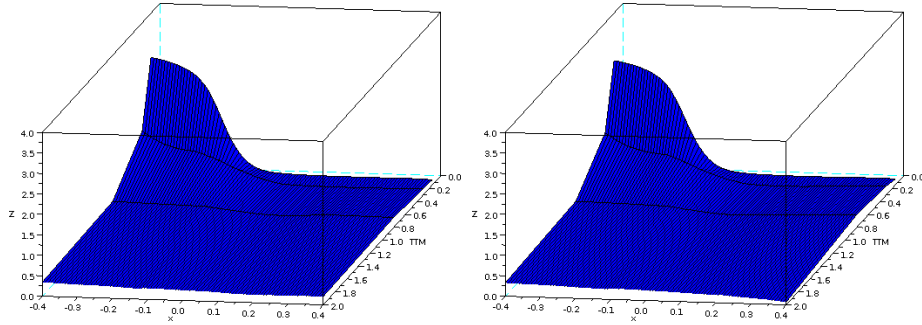


Figure 6.16: Local volatility surface $\hat{a}_j(\tau, x)$, $j = 5, 6$.

6.9.2 Requirements on the implied volatility surface

As a second test of consistency we investigate if the resulting implied volatility surfaces are free of static arbitrage. Recall that static arbitrage refers to that the position taken in the underlying only depends on the current time t and stock price S_t , not on past prices or path properties. That is, static arbitrage implies arbitrage opportunities by trading in one pricing surface.

Roper [2010] finds sufficient and close to necessary conditions on an implied volatility surface to assure that the corresponding pricing surface is free of static arbitrage. For ease of notation Roper introduce the following notation

$$\Xi(\tau, x) := \sqrt{\tau} \hat{\Sigma}(\tau, x). \quad (6.81)$$

Here $\hat{\Sigma}(\tau, x)$ denotes the implied volatility corresponding to time to maturity τ and strike price Se^x .³⁶ Roper [2010] shows that the following has to be fulfilled for the corresponding call option surface to be free of static arbitrage.

1. For every $\tau > 0$, $\Xi(\tau, \cdot)$ is twice differentiable.
2. For every $\tau > 0$, $x \in \mathbb{R}$, $\Xi(\tau, x) > 0$.

³⁶That is, $x = \log \frac{K}{S}$ is the log moneyness, S is the current value of the underlying stock and K is the option strike price.

3. For every $\tau > 0$, $x \in \mathbb{R}$,

$$\mathfrak{D}\Xi(\tau, x) := \left(1 - \frac{x\partial_x \Xi(\tau, x)}{\Xi(\tau, x)}\right) - \frac{1}{4}\Xi(\tau, x)^2 \partial_x \Xi(\tau, x)^2 + \Xi(\tau, x) \partial_{xx} \Xi(\tau, x) \geq 0. \quad (6.82)$$

(Durrleman's condition)

4. For every $x \in \mathbb{R}$, $\Xi(\cdot, x)$ is non decreasing. (Monotonicity in τ)

Note that there are also requirements concerning the behavior of $\Xi(\tau, x)$ at maturity ($\tau = 0$) and for all τ as $x \rightarrow \infty$. See [Roper \[2010\]](#) Theorem 2.9 for details.

We follow [Roper \[2010\]](#) Section 3 and investigate whether our implied volatility surfaces are free of static arbitrage by using graphical analysis. For the first 6 steps of our simulation we plot $\Xi(\tau, x)$ and $\mathfrak{D}\Xi(\tau, x)$.

In each of the plots in [Figure 6.17-6.18](#) we plot $\Xi(\tau, \cdot)$ for $\tau = [0.15, 0.3, 0.8, 2]$. We see that the conditions 1-2 and 4 above are fulfilled for $x \in [-0.6, 0.6]$. As the option prices will go further out of the money, most likely there will be problems with the monotonicity in τ . See for instance plot 3 of [Figure 6.17](#). However, as option quotes with moneyness $|x| > 0.6$ are only very rarely quoted on the market, we do not see this as a major problem.

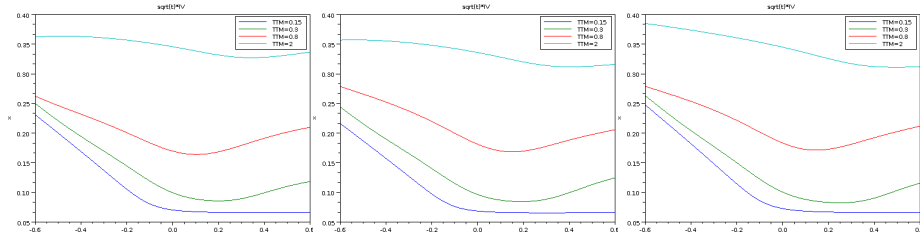


Figure 6.17: $\Xi(\tau, x)$, $\tau = [0.15, 0.3, 0.8, 2]$, $j=1,2,3$.

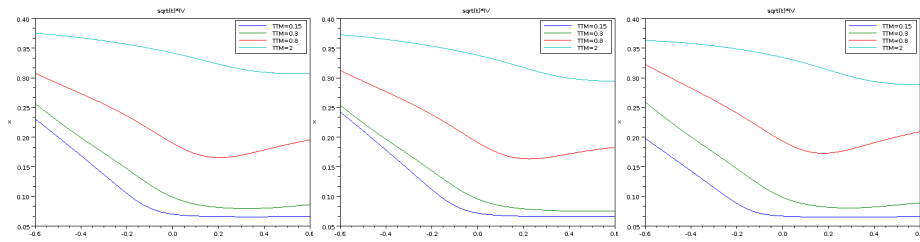


Figure 6.18: $\Xi(\tau, x)$, $\tau = [0.15, 0.3, 0.8, 2]$, $j=4,5,6$.

In Figures 6.19-6.20 we investigate Durrleman's condition and plot $\mathfrak{D}\Xi(\tau, x)$. As in the calculation of the Dupire's formula, we take numerical derivatives when calculating $\mathfrak{D}\Xi(\tau, x)$. We use the same approximation of the pricing function as when calculating the local volatility above. That is we interpolate call option prices in strike (moneyness) space using cubic splines.

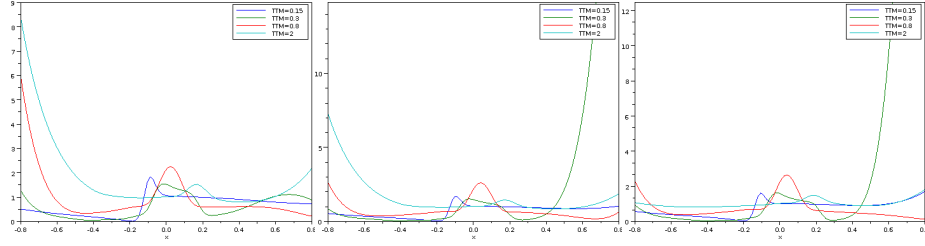


Figure 6.19: $\mathfrak{D}\Xi(\tau, x)$, $j = 1, 2, 3$.

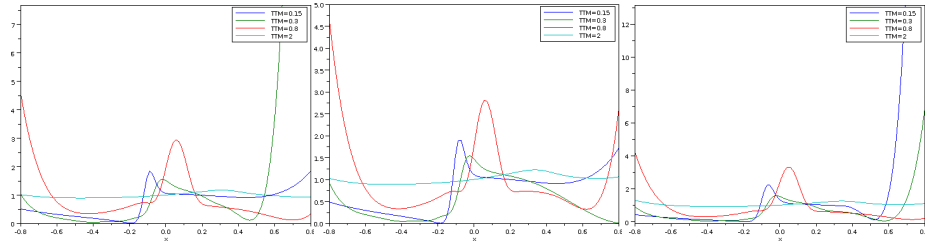


Figure 6.20: $\mathfrak{D}\Xi(\tau, x)$, $j = 4, 5, 6$.

7 Conclusion

In this thesis we have reviewed two different "market code-books", the local volatility code-book and the Lévy density code-book, (Chapter 2-3). The aim of the review was to discuss code-books that are applicable in equity market models, from a practical point of view. See Chapter 4-5 where two examples from Rene Carmona and Sergey Nadtochiy are discussed.

Due to consistency reasons, we found the Lévy density code-book to be better suited for use in an equity market model. The equations to be fulfilled to assure an arbitrage free model, are simply less complicated in the Lévy density case. (Compare Equations (4.6a)-(4.7) and (5.8a)-(5.8b).)

Therefore, we have taken a closer look at the Tangent Lévy models, due to Carmona and Nadtochiy [2009]. In this approach of market modeling, the Lévy density code-book is used in a dynamic setting. More precisely, the evolution of a time-inhomogeneous Lévy density is prescribed by a stochastic differential. Together with a stochastic integral equation for the stock price this describes our market. For the model to be free of arbitrage, the two processes will be coupled through HJM-type consistency conditions; The drift $\alpha(T, x)$ of the Lévy density process $\kappa_t(T, x)$ will be given by the diffusion $\beta(T, x)$. Also, the stock price process will be given by initial values and the process $\kappa_t(T, x)$. We have that the only "free" parameter of the Tangent Lévy model is the diffusion process $\beta(T, x)$.

We choose to look at a simplified case of this type of models. More precisely, we prescribe a specific form of the diffusion $\beta(T, x)$. This simplified model is motivated by a calibration of "layman type", see Ortega et al. [2009]. This means that we use a time series of time-inhomogeneous Lévy densities in our approximation of the diffusion $\beta(T, x)$. See Chapter 6.

We have implemented a specific example of this simplified Tangent Lévy market model. For the "layman calibration" we use market data consisting of options on the ETF QQQQ during the period 2010.02.11-2011.02.11. Given $\beta(T, x)$, we show how to calculate the drift $\alpha(T, x)$ in our model, using the TLM consistency condition. Thereafter we approximate the stock and Lévy density processes using an Euler scheme. From the resulting processes we have calculated European call option prices/implied volatilities.

We note that also when using the Lévy density code-book, the requirement of no arbitrage restricts our choice of model parameters in an evident way. For instance, in our "layman type calibration" we approximate $\beta(T, x)$ by differences of parametrically calibrated Lévy densities. Due to the consistency requirement of the model, $\beta(T, x)$ has to fulfill certain regularity assumptions. Hence we choose a finite intensity Lévy density in the "layman calibration".

Furthermore the question remains, *in which other code-book* it would be more convenient to set up a equity market model. Kallsen and Krühner [2010] proposes the "characteristic exponent" of an additive process to be a suitable code-book. Also, Carmona and Nadtochiy [2010] presents a new class of Tangent Lévy models where the underlying may consist of both continuous and pure jump parts. This of course allows for more flexibility in the market model.

As the model flexibility increases, there is a risk that in practice the model will be more difficult to implement. To investigate this in more detail is however outside the scope of this thesis.

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