# DIPLOMARBEIT

Titel der Diplomarbeit

# On symmetry of steady periodic water waves with constant vorticity

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## Magister der Naturwissenschaften (Mag. rer. nat.)


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# Abstract

It was recently proved that a steady periodic wave train monotonic between crests and troughs is symmetric, if the vorticity function fulfills a certain inequality. This proof uses the assumption that there are no stagnation points throughout the fluid. This thesis presents the assumptions usually made to model water waves and derives the corresponding system of partial differential equations. After a discussion of the maximum principles used during the proof of symmetry, the proof mentioned above is extended to allow stagnation inside the fluid.

# 1 Introduction

The existence of water is fundamental for the origin of life, which has developed over nearly 4 billion years to present. Perhaps this fundamentality is the reason for the admiration and interest of humans into water, and for the desire to understand its motion and its properties.

The great Leonardo da Vinci was one of the first to give a scientific explanation of the arisal of water waves. But it was not before the $18^{th}$ century, when Euler found the equations of motion for fluids, which was the beginning of a rigorous theoretical investigation of water. In 1847, Stokes published a text on irrotational periodic water waves, in which he mainly seeked for the application in shipping, but did not prove all of his conjectures (cf.[St1847]). Before Toland came to prove some of Stokes' statements (cf. [To1978]), Garabedian proved symmetry for steady periodic gravity waves for the case of irrotational flow (cf. [Ga1965]) using a variational approach.

To understand the motion of water, e.g., in oceans, the case of irrotational flow is not enough, since since this model does not cover the appearence of a non-uniform current. Therefore, it is necessary to introduce vorticity, which can be constant (the simplest case) or have any distribution throughout the fluid. The problem of existence of waves in this case was first considered by Dubreil-Jacotin in 1934 (cf.[Du1934]), but many results on this topic were not found until the last few years (cf. [BS1998], [CS2002]).

The development of a global bifurcation theory by Crandall and Rabinowitz (e.g. [CR1971]) turned out to provide important tools for the investigation of existence of waves (cf.[KN1978],[CS2002],[CV2011]).

Furthermore, a topic of importance is the existence of stagnation points throughout the fluid and the consequences on the flows inside. The existence of stagnation points was assumed by Thomson (Lord Kelvin), who also gave a rough picture of the streamlines in the neighbourhood of stagnation points (the so-called *cat's eyes*). Some results on this issue are given in [Wa2009].

It was recently proved (cf. [CE2004]) that a steady periodic wave train monotonic between crests and troughs is symmetric, if the vorticity function fulfills a certain inequality. This proof uses the assumption that there are no stagnation points throughout the fluid.

It is the main goal of this thesis to extend this proof allowing stagnation points inside the fluid.

In Section 2, we will present the governing equations for water and derive a system of PDEs that corresponds to a steady periodic gravity water wave. Furthermore, we will transform this system to get a problem in terms of one single function to handle the problem of the free boundary.

In Section 3, we give a short introduction to maximum principles and state the weak maximum principle, the boundary-point lemma and the edge-point lemma, which will be useful for our main result.

Finally, the statement on symmetry will be proved in Section 4.

# 2 Governing Equations

It is a remarkable property of water and water waves not to be confined to a certain shape or a certain type. Wave propagation can have any distribution, there are waves that cannot be seen with the naked eye, waves bigger than a building, there are infinitely many possibilities of flows in the inside etc. For this reason, we have to restrict our attention to a certain type of waves with properties that can (easily) be transformed into mathematical language.

We know that water consist of a huge number of $H_2O$-molecules. Every single atom in the water interacts with all the atoms nearby, which leads to an even larger number of relations to describe the motion of those molecules. Furthermore, it is known from theoretical physics that Newton's N-body-problem has no exact solution for $N > 3$, but we could compute the solution numerically. However, a numerical computation of the parameters of the system (i.e. the water) in terms of the motion of the molecules would take more time than the universe already exists, even for small volumes of water. Furthermore, the assumption that the molecules are discrete points in the volume would be quite unnatural. Therefore, we will use the following assumption: Every (arbitrarily small) volume of water we are looking at contains very many molecules. This assumption is called the *continuum hypothesis* (cf.[Bat1967]). As an immediate consequence, we need not be concerned about a discrete definition of functions defined on the domain, but can assume all these functions to be at least continuous.

We choose a quite simple setting by considering water over a flat area called *bed*, where no interaction between the fluid and the bed takes place. In this case, the bed is called *rigid*. Furthermore, we assume that the boundary on the sides is so far away that there is no influence on the area we are looking at. As an example for this model, one could think of a huge water basin.

With no forces acting on the water (of course apart gravity), there will be no movement in it and the surface will be flat. But as we know from everyday life, there are many reasons for perturbation of this calm state: Wind, water flows, objects entering the water, and many more. The explanation of the appearance of waves is split into two parts. First, the water molecules interact, which leads to a phenomenon called *surface tension*. On the other hand, after a perturbation, water tries to get back to a state with the lowest possible energy, which is the case of a flat surface. It turns out that the best way to do so is the appearance of waves.

Since these two reasons have different origins, it is possible to construct waves due to just one of these phenomena. During this thesis we will neglect surface tension and focus on waves arising from gravity acting on the water. Corresponding to their origin we call them *gravity waves*.

To simplify the analysis of our model we will assume that all parameters of the water are identical in direction orthogonal to the direction of wave propagation. Therefore, we can just look at one plane perpendicular to the crest line, which reduces the domain to two dimensions.

We will now try to express all the stated properties in a mathematical way.

We use Cartesian coordinates $(x, y)$ for this two-dimensional domain. The origin lies on the flat bed and the surface is given by a function $y = \eta(x, t)$. Furthermore, $(u, v)$ denotes the velocity field of the flow inside.

## 2.1 Mass Conservation

We start by looking at a volume $V$ with surface $S$ from the physical point of view. The only changes possible to a quantity inside $V$ are on the one hand to increase or decrease the amount of the quantity inside $V$, which is given by

$$\int_V \rho \, dx, \tag{2.1}$$

where $\rho$ is the density of the given quantity. On the other hand change is possible by a flow in or out (through the surface $S$). To measure the total amount of the quantity flowing out of $V$ per time unit we just integrate over S:

$$\int_S \rho \mathbf{u} \cdot \nu \, dS, \tag{2.2}$$

where $\mathbf{u}$ is the velocity vector of the quantity. The multiplication by $\rho$ is obviously necessary to give us the correct amount having velocity vector $\mathbf{u}$ at any point. Note that the scalar product will give us the component orthogonal to the surface at every point. It is easy to relate those two integrals under the further condition that the quantity is conserved. In this case, it follows that the quantity increases in the volume as much as flows through the surface:

$$\frac{\partial}{\partial t} \int_V \rho \, dx = - \int_S \rho \mathbf{u} \cdot \nu \, dS. \tag{2.3}$$

After a change of the order of integral and derivative on the left hand side and with the help of the Theorem of Gauss on the right hand side we can rewrite this equation to get

$$\int_V \frac{\partial}{\partial t} \rho \, dx = - \int_V \nabla \cdot (\rho \mathbf{u}) \, dx. \tag{2.4}$$

Since those integrals must be equal for all volumes $V$ it follows that the integrands must be equal, which gives

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2.5}$$

This is a fundamental result of physics and is called the *continuity equation.*

The conserved quantity in our model will be mass. A remarkable property of water is that it is nearly incompressible (cf. [Li2003]). From this it follows that $\frac{\partial}{\partial t}\rho = 0$ and therefore equation (2.5) simplifies to

$$\nabla \cdot u = u_x + v_y = 0, \tag{2.6}$$

where $\mathbf{u} = (u, v)$.

## 2.2 The Euler Equations

To set the laws of motion in the fluid, we will use Newtons's Law of Motion, $F = m \cdot a$, where $F$ is a force, $m$ is the mass of a particle and $a$ is the acceleration of the particle. To apply this law, we first have to identify forces and translate the definition of acceleration for fluids.

We can split the total force into two parts. The first one is given by external sources, e.g. gravity, and is assumed to be equal for all the particles. The second one arises from the interaction of the particles. Here, the force only depends on the molecules nearby and can therefore vary throughout the fluid. Moreover, we can further distinguish between an internal force normal and tangential to the surface. The first one can be characterized by a quantity called *pressure*, the latter one is characterized by the so called *viscosity*. Now, another important simplification is made. We will neglect the internal forces tangential to the surface, i.e., we assume water to be *inviscid*.

Now, we can write down the parts of the force and add them up. The total force is given by

$$\int_V \rho F \, dx - \int_S P \cdot n \, dS, \tag{2.7}$$

where the first term represents the external force acting on a volume $V$, and the second term represents the internal forces, where $n$ is the inward normal unit vector. Again, we use the Theorem of Gauss to get

$$\int_V (\rho F - \nabla \cdot P) \, dx, \tag{2.8}$$

We now have to define the acceleration for a fluid. We therefore consider a particle in the fluid moving along the path $x(t)$ with speed $u(x(t), t) = \frac{\partial x}{\partial t}$. Hence, by differentiating again, we get the acceleration of the particle. We obtain

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{D\mathbf{u}}{Dt}, \tag{2.9}$$

which is called *total derivative*. Summarizing, since total mass is given by equation (2.1), we get

$$\int_V \rho \frac{D\mathbf{u}}{Dt} \, dx = \int_V (\rho F - \nabla P) \, dx. \tag{2.10}$$

Again, since the integrals are equal for arbitrary $V$, we conclude that the integrands are equal:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + F. \tag{2.11}$$

Gravity can be written as F=(0,0,-g), and therefore, writing (2.11) componentwise and setting $\rho = 1$ and $\mathbf{u}(x(t),t) = (u,v)$, we get the *Euler Equations*

$$u_t + uu_x + vu_y = -P_x \tag{2.12}$$

$$v_t + uv_x + vv_y = -P_y - g. \tag{2.13}$$

Note the connection to the *Navier-Stokes-Equations*, which describe motion of Newtonian gases and fluids (like water). For incompressible fluids they are given by

$$u_t + uu_x + vu_y = \nu\Delta u - P_x \tag{2.14}$$

$$v_t + uv_x + vv_y = \nu\Delta v - P_y - g \tag{2.15}$$

$$u_x + v_y = 0 \tag{2.16}$$

with $u(x,0) = u_0(x)$. Again, $u \in \mathbb{R}^n$, the velocity vector, and $p \in \mathbb{R}$, the pressure, are unknown and $\nu \in \mathbb{R}$, viscosity, and an external force (in this case: gravity) are given. This is a system of nonlinear partial differential equations of order two. We see that the simplification down to the Euler Equations is just done by formally setting $\nu = 0$ in (2.14) and (2.15), i.e. neglecting viscosity, as we did it in the discussion above.

## 2.3 The Boundary Conditions

As we have derived the equations for the inside of the fluid, we shall now investigate the conditions for the boundary to get a well-posed problem to solve.

We have already mentioned that we are looking at a domain far away from any boundary on the sides, which leaves us with just two boundaries: the bottom and the top. At the bottom, it is easy to find the condition. We said that we do not want any interaction between the water and the flat bed. This means that there is no vertical movement $v$ of the particles at the bottom line, which is given by $\{y = 0\}$, i.e.

$$v = 0 \text{ on } y = 0. \tag{2.17}$$

The treatment of the top boundary, i.e. the surface, is a bit more complicated. First, note that the surface is not given, it is part of the problem and has to be determined to obtain a solution. However, we can write the surface as a function $\eta(x,t)$, on which we will need some smoothness conditions. We can give an equation for the surface by $S(x,y,t) = y - \eta(x,t) = 0$, where $S : \mathbb{R}^3 \to \mathbb{R}$ is $C^k$ (with $k \geq 1$) and $(S_x, S_y) \neq 0$.

We now assume that the surface is always formed by the same particles. This condition prevents any awkward interactions between air and water. An equivalent condition to this assumption is that the total derivative of $S(x,y,t)$ must vanish,

$$\frac{DS}{Dt} = 0. \tag{2.18}$$

Indeed, consider a particle that is located at a point $\mathbf{x}(t_0)$ on the free surface at time $t_0$. If this particle stays on the surface for a time interval $[t_0, t_1)$ (which might be finite, since we do not know anything about the evolution of waves), then we have $S(x(t), y, t) = 0$ for $t \in [t_0, t_1)$. By differentiation with respect to $t$ we get $\frac{DS}{Dt} = 0$, which shows, that this condition is necessary. The proof of sufficiency is a bit more complicated. Here, one has to show that from this equation follows that the flow at the surface is tangential to the surface everywhere. For details see [Jo1997]. We can rewrite equation (2.18) in terms of $u$, $v$ and $\eta$, which yields

$$v = \eta_t + u\eta_x \text{ on } y = \eta(x,t). \tag{2.19}$$

Equations (2.17) and (2.19) are called *kinematic boundary conditions*. Finally, we have to add one more condition on the upper boundary. Since we assumed that there is no surface tension (which means forces with parts being tangential to the surface), there are just orthogonal forces and therefore, the pressure $P$ will be the same everywhere on the surface:

$$P = P_0 \text{ on } y = \eta(x,t). \tag{2.20}$$

This equation is called the *dynamic boundary condition* and completes the system to describe water waves.

Summarizing, we derived the following nonlinear system:

$$\begin{cases} u_t + uu_x + vu_y = -P_x & \text{in } 0 < y < \eta(x,t) \\ v_t + uv_x + vv_y = -P_y - g & \text{in } 0 < y < \eta(x,t) \\ u_x + v_y = 0 & \text{in } 0 < y < \eta(x,t) \\ v = \eta_t + u\eta_x & \text{at } y = \eta(x,t) \\ P = P_0 & \text{at } y = \eta(x,t) \\ v = 0 & \text{at } y = 0 \end{cases} \tag{2.21}$$

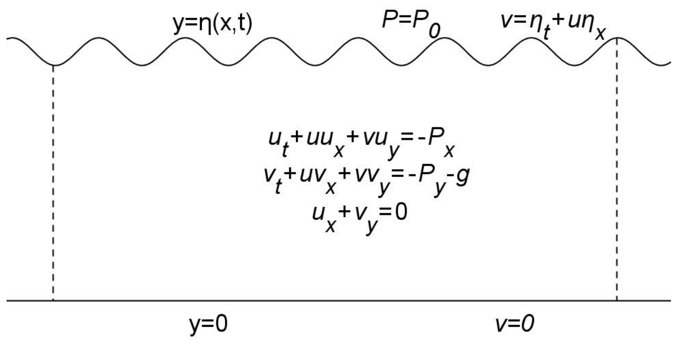


Figure 2.1: The Setting

## 2.4 Vorticity

We will now investigate rotational motion of the particles throughout the fluid. This phenomenon is called *vorticity*, denoted by

$$\omega := \nabla \times \mathbf{u}, \tag{2.22}$$

where $\mathbf{u} = (u, v, 0)$. To understand this definition, consider a disc around some point $p$ normal to some unit vector $\mathbf{n}(p)$ with boundary curve $C$ inside the fluid. We call

$$\int_C \mathbf{u} \cdot dl$$

the *circulation* of $\mathbf{u}$ around $C$, since it is zero if $\mathbf{u}$ is parallel to $\mathbf{n}(p)$ at each point of $C$, and increases with the increase of the angle between $\mathbf{u}$ and $\mathbf{n}(p)$. Now, we can use Stokes' Theorem to get

$$\int_C \mathbf{u} \cdot dl = \int_D \omega(x)\mathbf{n}(x)\, dS = \mathbf{n}(p) \int_D \omega(x)\, dS.$$

Furthermore, taking the radius $R \to 0$, we assume $\omega$ to be constant throughout the disc, which yields

$$\mathbf{n}(p)\omega(p) = \frac{1}{\pi R^2} \int_C \mathbf{u} \cdot dl.$$

The quantity $\mathbf{n}(p)\omega(p)$ is called *circulation density*. By the considerations above, it is maximal if $\mathbf{u}$ is normal to $\mathbf{n}(p)$. This allows us to regard $\omega$ as a measure for the local rotation of the particles. In [CS2004], it is shown that in the absence of stagnation points we can write $\omega$ as a function of the stream function $\psi$ (see below), $\omega = \gamma(\psi)$.

In the case of $\gamma \equiv 0$, we speak of *irrotational flow*, which means that there is no local roation throughout the fluid. For $\gamma = const \neq 0$, we speak of the case of *constant vorticity*. Then, the currend inside the fluid is given by a linear function of the depth, see Figure 2.22. This case corresponds to tidal currents (see [Co2011]).

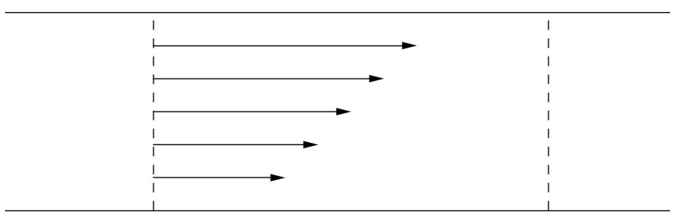

Figure 2.2: Constant Vorticity

The case of constant vorticity will be considered throughout this thesis.

## 2.5 Equivalent Formulation of the Problem

The system in its form (2.21) is quite complicated to handle, since it is nonlinear and involves an unknown free boundary $\eta(x,t)$. We will therefore introduce a new function $\psi : \mathbb{R}^2 \to \mathbb{R}$, in terms of which we will reformulate the problem (cf. [CS2004]).

As a preparing step note that our problem is time-dependent. Indeed, if we consider waves travelling with speed $c$, the space-time-dependency of our unknowns will have the form $(x-ct)$. Thus, we change the frame we are looking at from $(x-ct,y)$ to $(x,y)$. The new frame also moves with speed $c$, and so, time is eliminated.

Next, we can measure the mass flux across a line $\{x = x_0\}$ relative to the flow at speed $c$ by computing

$$\int_0^{\eta(x_0)} [u(x_0,\xi) - c]\, d\xi.$$

It follows from the boundary conditions that this relative mass flux is independent of $x_0$, and so we can define this quantity by

$$m := -\int_0^{\eta(x)} [u(x,\xi) - c]\, d\xi. \tag{2.23}$$

Let us now consider the vector field $U = (-v, u-c, 0)$. Because of mass conservation its curl vanishes:

$$\nabla \times U = \nabla \times \begin{pmatrix} -v \\ u-c \\ 0 \end{pmatrix} = u_x + v_y = 0.$$

Now, note that if the domain of a vector field with vanishing curl is simply connected, it follows that there exists a well-defined potential function $\psi$ with $\nabla\psi = U$.

This is exactly the function we are looking for. It is defined via its derivatives

$$\psi_x = -v \quad , \quad \psi_y = u - c \tag{2.24}$$

and we can give an explicit formula:

$$\psi(x,y) = \psi_0 + \int_0^y [u(x,\xi) - c]\, d\xi,$$

which is uniquely determined for a given constant $\psi_0$.

Using (2.24) and $\cdot_t = -c\cdot_x$, system (2.21) becomes

$$\begin{cases} \psi_y\psi_{yx} - \psi_y\psi_{yy} = -P_x & \text{in } 0 < y < \eta(x) \\ \psi_y\psi_{xx} + \psi_x\psi_{xy} = P_y - g & \text{in } 0 < y < \eta(x) \\ \psi_x = -\psi_y\eta_x \text{ at } y = \eta(x) & \text{at } y = \eta(x) \\ P = P_0 \text{ at } y = \eta(x) & \text{at } y = \eta(x) \\ \psi_x = 0 & \text{at } y = 0 \end{cases} \tag{2.25}$$

By this insight, it is sufficient to find a function $\psi$ with suitable properties, from which we can compute all the unknowns $u, v, P$ via (2.25). Two boundary conditions for $\psi$

are already clear: $\psi = 0$ on $y = \eta(x)$ and $\psi = m$ on $y = 0$. Moreover, we use Bernoulli's Law, which states that

$$E := \frac{|\nabla\psi|^2}{2} + gy + P - \Gamma(\psi)$$

is constant throughout the fluid. The dynamic boundary condition yields that

$$\frac{|\nabla\psi|^2}{2} + gy =: Q = const$$

at the surface. Finally, we compute $\Delta\psi$:

$$\Delta\psi = \psi_{xx} + \psi_{yy} = -v_x + u_y = -\omega.$$

In [CS2004], it is shown that $\omega$ is at least locally representable as a function of $\psi$, $\omega = \gamma(\psi)$. This function is the vorticity function which we discussed in section 2.4. There, we decided to just consider the case of constant vorticity, i.e. a constant function $\gamma(\psi) \equiv \gamma$.

Summarizing, we have to solve

$$\begin{cases} \Delta\psi = \gamma & \text{in } 0 < y < \eta(x) \\ \frac{|\nabla\psi|^2}{2} + gy = Q & \text{at } y = \eta(x) \\ \psi = 0 & \text{at } y = \eta(x) \\ \psi = m & \text{at } y = 0, \end{cases} \tag{2.26}$$

that is, find $\psi(x, y) \in C^2(\mathbb{R}^2)$ and $\eta \in C^3(\mathbb{R}^2)$, which solve problem (2.26).

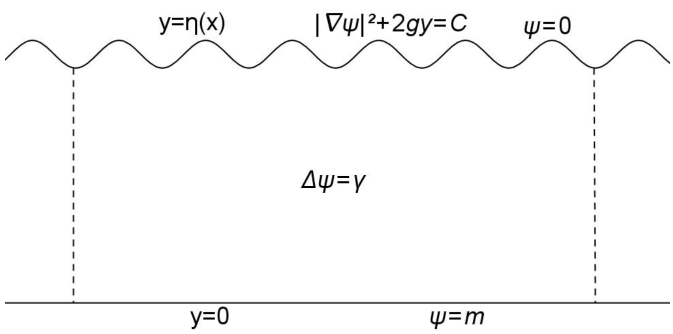


Figure 2.3: The Reformulated Setting

# 3 Maximum Principles

In this chapter, we devote ourselves to a very useful tool in the theory of partial differential equations and its applications by relating derivatives and extrema of a function. Here, we follow the books of *Fraenkel*, [Fr2000] and *Evans*, [Ev1998].

## 3.1 Definitions

Of course, we first have to clarify what we mean by a maximum.

**Definition 3.1**

1. *A function $f$ defined on $D$ has a* local maximum *at a point $x_0$, if*

$$\exists \delta > 0 : f(x_0) > f(x)\ \forall x \in B_\delta(x_0) \cap D.$$

2. *If $f(x_0) > f(x)\ \forall x \in D$, then the maximum is called* global*.*
3. $\underset{x\in D}{min}\{f(x)\} = -\underset{x\in D}{max}\{-f(x)\}$

Relation 3. establishes that we can w.l.o.g. investigate just the case of maxima, since minima can be transformed into maxima by taking $-f$ and $-\max\limits_{x\in D}$.

We know from standard analysis that if a function of one variable has a maximum (resp. minimum) at some point $x_0$, then the first derivative is zero and the second derivative is negative (resp. positive). This fact relates second derivatives and extrema in a very useful way: If the second derivative of a function of one variable is positive everywhere in an open set, it follows that it cannot have a maximum there.

**Example 3.2**

The function $f(x) = x^2$ has second derivative $f''(x) = 2 > 0$ and therefore cannot have a maximum in any open interval, see Figure 3.1.

△

Of course, a function $f$ with $f'' > 0$ has a maximum on a closed interval $[a, b]$. However, this maximum can only be at the boundary points $a$ or $b$, since all points in the inside

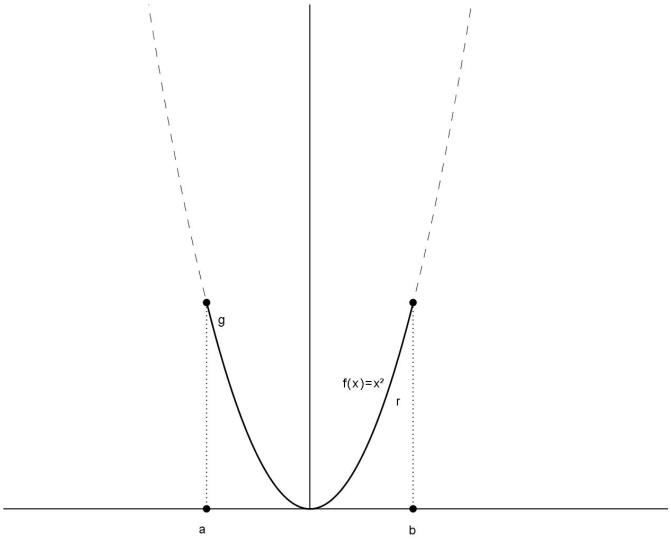


Figure 3.1: The function $x^2$

are out of the question by the discussion above. The existence of maxima at the boundary in Example 3.2 is obvious.

In the case of several variables a function has a maximum (resp. minimum) at some point $x_0$, if all the first derivatives at $x_0$ in any direction are zero and all the second derivatives at $x_0$ in any direction are smaller (resp. bigger) than zero. The first part of this statement is fulfilled if and only if the gradient of the function $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})^T$ is zero at $x_0$, while the second part is equivalent to the Hesse-matrix of the function

$$Hf := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

being negative (resp. positive) definite at $x_0$. In the case of $2 \times 2$-matrices $Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ (note that the Hesse-matrix is always symmetric if the function is twice continuously differentiable), the matrix $Hf$ is positive definite if and only if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ and negative definite if and only if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$. From this, we see that at an extremal point $f_{xx}$ and $f_{yy}$ always have the same sign. We conclude the following

**Lemma 3.3** *Let $f \in C^2(\mathbb{R}^2)$ with $\Delta f > 0$ in an open set $D$. Then $f$ has no maximum in $U$.*

*Proof.* Suppose that $f$ has a maximum at some point $x_0$. Then $f_{xx}(x_0) < 0$ and $f_{yy}(x_0) < 0$ and thus $\Delta f(x_0) = f_{xx}(x_0) + f_{yy}(x_0) < 0$, which is a contradiction.

□

The Laplacian used in Lemma 3.3 is a partial differential operator of order two. It belongs to a certain class of partial differential operators:

**Definition 3.4** *A linear partial differential operator $L$ of order two,*

$$L = \sum_{i,j=1}^{N} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{N} b_i(x)\partial_i + c(x)$$

*with smooth coefficients $a_{ij}(x)$, $b_i(x)$ and $c(x)$ for $x \in D$, is called* elliptic *at $x \in D$, if and only if there is a number $k(x) > 0$ with*

$$\sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \geq k(x)|\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^n. \tag{3.1}$$

The operator $L$ is called elliptic in $D$, if it is elliptic for every $x \in D$ and *uniformly elliptic* if there is a number $k_0 > 0$ with $k(x) \geq k_0\ \forall x \in D$.

In the following section, we state some far reaching results on the existence and location of maxima, whose main step in the proof is the quite simple maximum principle for the Laplacian from Lemma 3.3 above.

## 3.2 Maximum Principles for Elliptic Operators

It is a quite special case to assume that $Lf \equiv 0$ throughout a certain domain. Therefore, results for this case would not be applicable very often, so we have to use other conditions, namely that $Lf$ does not change sign throughout the domain of interest. We give the following

**Definition 3.5** *A function $f$ is called* subsolution *with respect to a linear partial differential operator $L$ and a domain $D$, if and only if $f \in C^2(D)$ and $Lf \geq 0$ in $U$. It is called* supersolution *if and only if $f \in C^2(D)$ and $Lf \leq 0$ in $D$.*

We now state the most important results for subsolutions. Note that they can easily be transferred to the case of supersolutions by taking $-f$ because of the linearity of $L$. Let us start by the *weak maximum principle.*

**Theorem 3.6** *(Weak maximum principle)*

*Suppose that $f$ is a subsolution with respect to an elliptic partial differential operator*

$$L = \sum_{i,j=1}^{N} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{N} b_i(x)\partial_i \tag{3.2}$$

*and a bounded domain $D$ and continuous on $\overline{D}$. Then $f$ has a maximum which attained on the boundary.*

The weak maximum principle for partial differential operators with $c(x) \leq 0$ is not that powerful, since it just gives a relation between the maximum of $f$ on $\overline{D}$ and the maximum on the boundary of the nonnegative part of $f$, $f^+$. Furthermore, the assumption $c(x) \leq 0$ is crucial. For details see [Fr2000].

Anyway, since we will not have to deal with operators with $c(x) \neq 0$, we state the maximum principles just for the case $c(x) \equiv 0$, even if they hold with some slight changes for the case $c(x) \leq 0$.

**Theorem 3.7** *(Strong maximum principle)*

*Suppose that $f$ is a subsolution with respect to an operator as in (3.2) and an open and connected domain $D$. Then, if $f$ has a maximum in $D$, $f$ is constant in $D$.*

The strong maximum principle is one of the most important tools in the theory of partial differential equations, since it ensures the uniqueness of a solution to elliptic problems (cf.[Ev1998]).

The following theorems describe directional derivatives at a maximum point under certain conditions. We start by looking at a ball.

**Theorem 3.8** *Let $f$ be a subsolution with respect to an operator as in (3.2) and some ball $B \subseteq D$. Suppose that $f(Q) > f(x)\, \forall x \in B$ for some $Q \in \partial B$. Then*

$$\lim_{t \searrow 0} \frac{f(Q) - f(Q - tm)}{t} > 0$$

*for any outward unit vector $m$ normal to $\partial B$ at $Q$, if this limit exists.*

As one might expect, this result can be extended to domains with nearly arbitrary shape. The only condition the domain has to fulfill is the so called *interior ball property* at $Q \in \partial D$, which means that we can find a ball $B$ in $D$ so that $Q \in \partial B$, too. In addition, we can state even more:

**Theorem 3.9** *(Hopf maximum principle)*

*Let $f$ be a subsolution with respect to an operator as in (3.2) and some domain $D$. Suppose that $f \in C(D \cup \{Q\})$ and $f(Q) = \max_{x \in D} f(x)$ for some point $Q \in \partial D$ and that $D$ has the interior ball property at $Q$. Then $\frac{\partial f}{\partial m}(Q) > 0$ for some outward unit vector $m$ at $Q$ unless $f$ is constant throughout $D$.*

Obviously, this statement is a combination of Thm 3.8 and the strong maximum principle, which are therefore the central arguments in the proof (cf. [Fr2000], [Ho1952]).

Unfortunately, the Hopf maximum principle does not cover all boundary points occuring throughout this thesis, since the smoothness condition on the boundary is crucial. Hence, the last result of this section deals with non-smooth boundary, in particular with corners which will occur in the domains we come to deal with.

Since we just look at two-dimensional domains, the points of interest are corners and not edges. Therefore, the name *edge-point-lemma* might be a bit misleading in this situation, but comes from the case of higher dimensions, where, e.g. for three dimensions, the points of interest lie on an edge.

**Definition 3.10** *We say that a point $Q \in \partial D$ is an* edge point *if and only if for some neighbourhood $U$ around $Q$ $D \cap U$ can be represented by the intersection of two domains $D_1, D_2$, $D \cap U = D_1 \cap D_2 \cap U$, where $D_1$ and $D_2$ have $C^2$-boundary, $Q \in \partial D_1 \cap \partial D_2$ and $-1 < n_1(Q) \cdot n_2(Q) < 1$, where $n_1, n_2$ are normal outward unit vectors of the respective domain.*

Since we just look at two-dimensional domains, the points of interest are corners and not edges. Therefore, the name *edge-point-lemma* might be a bit misleading in this situation, but comes from the case of higher dimensions, where, e.g. for three dimensions, the points of interest lie on an edge.

To clarify this definition see the following

**Example 3.11**

We consider the first quadrant of $\mathbb{R}^2$, $D = [0, \infty) \times [0, \infty)$ and $Q = (0, 0)$. Then, we can choose $D_1 = \mathbb{R} \times [0, \infty)$, $D_2 = [0, \infty) \times \mathbb{R}$ with $n_1(Q) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $n_2(Q) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. $D_1$ and $D_2$ have $C^2$-boundary and $n_1(Q) \cdot n_2(Q) = 0$ and therefore, the origin is an edge-point.

$\triangle$

To benefit from the definition of edge-points, we have to describe the domains $D_1, D_2$ in terms of suitable functions. We say that a $C^2$-function $\varphi : D \to \mathbb{R}$ is *admissible relative to the domain* $D$, if and only if $|\nabla \varphi| > 0$ for $x \in D$ and

$$\varphi(x) = \begin{cases} < 0 & x \in D \\ = 0 & x \in \partial D \\ > 0 & x \notin \overline{D} \end{cases}$$

for all $x \in U$.

With such functions $\varphi$, one can construct a so called *bluntness function* by

$$B(x) = \sum_{i,j=1}^{n} \partial_i \varphi_1(x) a_{ij}(x) \partial_j \varphi_2(x) \text{ for } x \in \overline{D} \cap U,$$

where $a_{ij}(x)$ are the leading coefficients of an elliptic partial differential operator of order two.

For a better understanding, we continue the example from above.

**Example 3.12**

We can choose the following admissible functions for the first quadrant of $\mathbb{R}^2$:

$$\varphi_1(\mathbf{x}) = -x,\ (\nabla\varphi_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \neq 0)$$

$$\varphi_2(\mathbf{x}) = -y,\ (\nabla\varphi_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \neq 0).$$

Note that the requirements on the signs of $\varphi_1, \varphi_2$ are fulfilled. Considering the Laplacian $\Delta$, where $a_{ij}(x) = \delta_{ij}$,we get

$$B(\mathbf{x}) = 0 \text{ for } x \in [0,\infty) \times [0,\infty).$$

$\triangle$

**Theorem 3.13** *(edge-point-lemma)*
*Let $f$ be a subsolution with respect to an operator as in (3.2) and some domain $D$. Let $Q \in \partial D$ be an edge point and suppose $f \in C^1(D \cap \{Q\})$ and $f(Q) > f(x)\ \forall x \in D$. Furthermore, suppose that the leading coefficients of the operator are $C^2$ in a neighbourhood $U$ of $Q$ and that there exists a bluntness function $B(x)$ for the edge point $Q$ which satisfies $B(Q) = 0$ and the directional derivative of $B$ at $Q$ being zero for every direction tangential to the boundary at $Q$.*
*Then either*

$$\frac{\partial f}{\partial m}(Q) > 0 \qquad or \qquad \frac{\partial^2 f}{\partial m^2}(Q) < 0$$

*for any outward unit vector at $Q$, if these derivatives exist.*

**Corollary 3.14** *If we have $f(Q) = \max_{x \in \overline{D}} f(x)$ instead of $f(Q) > f(x)\ \forall x \in D$, then the above statement holds unless $f$ is constant throughout $\overline{D}$.*

## 3.3 Suitable Reformulations

In the last section we derived a number of versions of maximum principles. All statements on the conditions and the results are given in a very general and quite technical way. Hence, we want to reformulate and adjust them to our setting them so that they can easily be applied during our proof of symmetry.

Before we start, let us concretize a domain $D$ to apply these theorems. It clearly will be sufficient to look at one wave period. Therefore, $D$ is given by

$$D := \{(x,y) \in \mathbb{R}^2 : -L/2 < x < L/2, 0 < y < \eta(x)\},$$

where $\eta : [-L/2, L/2] \to (0, \infty)$ is a function in $C^2(\mathbb{R})$. Hence, $D$ is bounded and has $C^2$-boundary. Moreover, the four boundary points $(\pm L/2, 0), (\pm L/2, \eta(-L/2))$ are edge points. Therefore, we need not be worried on the conditions on the domain any more throughout the following discussion.

**Lemma 3.15** *Let $f \in C^2(D) \cap C(\overline{D})$ with $Lf \leq 0$ in $D$ and $f \geq 0$ on $\partial D$. Then either $f > 0$ in $D$ or $f \equiv 0$ throughout $\overline{D}$.*

*Proof.* This Lemma follows from the weak maximum principle stated in Theorem 3.6. The smoothness properties of $f$ remain unchanged. By taking $-f$ we are looking at a supersolution, so Thm 3.6 tells us that the infimum is attained at the boundary; it is 0. Furthermore, the infimum cannot be attained in the interior of $D$ unless $f$ is constant throughout $\overline{D}$ by the strong maximum principle, so either $f > 0$ in $D$ or $f \equiv 0$ in $\overline{D}$.

□

**Lemma 3.16** *(cf.[GNN1979]) Let $f \in C^2(D) \cap C(\overline{D})$ with $f \geq 0$ in $\overline{D}$, $Lf \leq 0$ in $D$ and $f = 0$ at some boundary point $Q$. Suppose that $D$ satisfies the interior ball property at $Q$. Then the outer normal derivative of $f$ at $Q$ (if it exists) is strictly smaller than zero, $\frac{\partial f}{\partial m}(p) < 0$, unless $f \equiv 0$ throughout $\overline{D}$.*

*Proof.* This is the reformulation of the Hopf maximum principle. The properties of smoothness and the boundary properties again remain unchanged. By $-f$ we get a supersolution and the conditions $f \geq 0$ in $\overline{D}$ and $f = 0$ at $Q$ exactly mean that $f(Q) = \inf_{x \in D} f$. Finally, $\frac{\partial f}{\partial m}(Q) < 0$ for a supersolution is equivalent to $\frac{\partial f}{\partial m}(Q) > 0$ for a subsolution.

□

**Lemma 3.17** *(cf.[Se1971]) Let $T$ be a line normal to the top boundary $\eta(x)$ at some point $Q$. Choose a part of $D$ lying on a particular side of the line $T$ and denote it by $D_0$. If $f \in C^2(\overline{D_0})$ with $Lf \leq 0$ and $f \geq 0$ in $D_0$ and $f = 0$ at $Q$. Then either $\frac{\partial f}{\partial \mu}(Q) > 0$ or $\frac{\partial^2 f}{\partial \mu^2}(Q) > 0$, where $\mu$ is a non-tangential inward vector at $Q$, unless $f \equiv 0$ throughout $\overline{D_0}$.*

*Proof.* This is a consequence from Corollary 3.14. Like before, the smoothness properties do not change and by taking $-f$ we again get a supersolution with infimum attained at $Q$. Since we take inward unit vectors, we get $\frac{\partial f}{\partial \mu}(Q) > 0$ or $\frac{\partial^2 f}{\partial \mu^2}(Q) > 0$ unless $f \equiv 0$ in $\Omega$.

□

# 4 Symmetry of the wave profile

## 4.1 The assumption $u < c$

Throughout this thesis, we assume that the horizontal part of the particle velocity field $u$ is not necessarily smaller than the wave speed $c$. This generalization of the setting will cause a little trouble at first sight, but a few calculations will show that the effects from that can be fitted into our model.

As one might presume, the problem arises from the fact that the difference of these two parameters is equal to the derivative of the stream function with respect to $y$: $\psi_y = u - c$. So the assumption $u < c$ guarantees $\psi_y$ to be smaller than zero throughout the fluid. In particular, this fact establishes injectivity of the stream function in vertical direction, which turns out to be quite important for conclusions on the shape of $\psi$.

According to this, it looks like the assumption of $u$ being bigger than $c$ throughout the fluid will not have any effect on our arguments. As it will turn out, this is the case; for this setting we would only have to change some signs in the proof. The thing that causes problems here is the possibility of $u$ being equal to $c$ somewhere in the fluid, which means that the particles have the same horizontal speed as the wave itself has at some point. Physically, this means that critical layers appear in the flow.

It is well known that a function that has an extremum in the inside of the domain fails to be injective. This is the fate of our candidate $\psi$, which will not be injective if $u = c$ somewhere in the domain.

Hoping for a possibility to save the proof of symmetry, we will gather information about existence, location and consequences of stagnation points.

## 4.2 Laminar Flows

In this section we will discuss the existence of trivial solutions of our system and the existence of points with no particle movement.

Let us consider a wave train with a flat surface subject to the following parameters:

- $h > 0$ is the height over the flat bed
- $k > 0$ is the wavenumber, it is related to the wavelength via $k = 2\pi/\lambda$

- $\gamma \in \mathbb{R}$ is the vorticity; like always in this thesis, it is constant
- $m \in \mathbb{R}$ is the relative mass flux; we have $m = -\int_0^h [u(x,\xi) - c]\, d\xi$

It can be shown that there exist solutions for a system for those parameters with no particle movement in the vertical direction and, as a consequence, all the streamlines being parallel to the flat bed. Flows with these properties are called *laminar flows*.

The stream function for this case will have the following form:

$$\psi(x,y) = -\frac{\gamma}{2}y^2 + \left(\frac{m}{h} + \frac{\gamma h}{2}\right) y - m. \tag{4.1}$$

The velocity field $(u,v) = (\psi_y, -\psi_x)$ is therefore

$$(\psi_y, -\psi_x) = \left(-\gamma y + \frac{m}{h} + \frac{\gamma h}{2}, 0\right). \tag{4.2}$$

Recalling from section that the horizontal speed $u$ is given by $\gamma(h-y)$, we try to write $\psi_y$ in the form $u - c$. We obtain $\psi_y = \lambda = \frac{m}{h} - \frac{\gamma h}{2} + \gamma(h-y)$. Setting $\lambda := -c$, we get

$$\psi_y = \lambda + \gamma(h-y), \tag{4.3}$$

and $\lambda$ is the wave speed at the horizontal surface.

In general, $\psi_y$ will not be zero everywhere, so we have horizontal movement of the particles in the fluid, i.e. a laminar flow. But clearly, there might exist points with $\psi_y = 0$, which means that there is no particle movement at all. Points with $(u,v) = (0,0)$ are called *stagnation points*.

## 4.3 Waves of Small Amplitude

In this section we will discuss the appearance of waves of small amplitude. To establish this we will make use of the quite technical Theorem of Crandall and Rabinowitz.

The Theorem of Crandall and Rabinowitz states that some values of $\lambda$ and $m$ are good for the appearance of nontrivial solutions for the water wave problem. In fact, these nontrivial solutions represent waves with small amplitudes, which can be seen as a small perturbation of the flat surface. For the exact statement and the proof, see [CR1971]. These triggering values of $\lambda$ and $m$ are the following:

$$\lambda_\pm = -\frac{\gamma \tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2 \tanh^2(kh)}{4k^2} + g\frac{\tanh(kh)}{k}} \tag{4.4}$$

$$m_\pm = \frac{\gamma h^2}{2} - \frac{\gamma\, h \tanh(kh)}{2k} \pm h\sqrt{\frac{\gamma^2 \tanh^2(kh)}{4k^2} + g\frac{\tanh(kh)}{k}} \tag{4.5}$$

In Section 4.2, we saw the existence of stagnation points for laminar flows. Thus, we are interested in their existence and location in the case of possible wave appearance.

First of all, note that these points must lie on a line $y = y_0$ parallel to the flat bed, since we have no vertical movement in the fluid. Looking at equation (4.3) with $\lambda = \lambda_\pm$, we will have to distinguish three cases: $\gamma < 0, \gamma = 0$ and $\gamma > 0$.

In the case of irrotational flow, we obtain that the existence of stagnation points is impossible, since $\lambda_+ > 0$ and $\lambda_- < 0$ in any case.

For $\gamma > 0$ it follows that stagnation points can only occur for $\lambda_-$. In this case we can compute the distance of the line of stagnation points to the surface, which yields

$$h - y = -\frac{\lambda_-}{\gamma} = \frac{\tanh(kh)}{2k} + \sqrt{\frac{\tanh^2(kh}{4k^2} + \frac{g\tanh(kh)}{\gamma^2 k}}. \tag{4.6}$$

Since $\tanh(kh) > 0$ for $h > 0$ and tanh is continuous, we conclude that $h - y$ will always be bigger than some $\varepsilon > 0$. Thus, if the wave amplitude is small enough (the existence of such waves is ensured by the local bifurcation approach in [CV2010]) the surface will not cross this line of stagnation and so there is a small neighbourhood around $\eta(x)$ where $\psi$ is injective in vertical direction.

In case of $\gamma < 0$, stagnation points are only possible for $\lambda_+$ and we have

$$h - y = -\frac{\lambda_+}{\gamma} = \frac{\tanh(kh)}{2k} - \sqrt{\frac{\tanh^2(kh)}{4k^2} + \frac{g\tanh(kh)}{\gamma^2 k}}. \tag{4.7}$$

For $-\infty < \gamma$ we get the same result as above, namely, that for small enough wave amplitudes, there will be a neighbourhood around $\eta(x)$, in which $\psi$ is injective with respect to $y$. Note that we have to exlude $\gamma = -\infty$ here, because else, equation (4.7) yields $h - y = 0$, i.e. the stagnation points are on the surface.

Summarizing these considerations, we obtained the following

**Lemma 4.1** *If points of stagnation occur in the fluid, they are all on a line $y = y_0$ with $h - y > \varepsilon$ for some $\varepsilon > 0$.*

## 4.4 The Domain to Deal With

Before we tackle the main question of this text, let us take a look on the domain to deal with. As we already said, the frame we are looking at covers one period of the wave train. Remember that we let the frame move with wave speed $c$ to eliminate time dependence. Furthermore, we assume that the wave amplitude is less than $\epsilon > 0$ determined by Lemma 4.1, which is possible by the discussion above. Due to these assumptions the domain to deal with is

$$D = \{(x, y) \in \mathbb{R}^2 | - L/2 < x < L/2, 0 < y < \eta(x)\}.$$

We now fix $x_* \in (-L/2, 0]$ and consider the map

$$\varphi_{x_*} : D_* \to D_*^R, (x, y) \mapsto (2x_* - x, y),$$

where $D_* = \{(x,y) \in \mathbb{R}^2 | -L/2 < x < x_*, 0 < y < \eta(x)\} \subseteq D$ and $D_*^R = \operatorname{ran}(\varphi_*)$. This is a reflection of $D_*$ at the line $\{x = x_*\}$ which is normal to the bottom.

To make use of the periodicity of the system, we want to map the left and the right boundary of $D$ into the same line. We see that if $x_* = 0$, the left boundary is mapped into the right one and everything is fine. But if $x_* < 0$, the line $\{x = -L/2\}$ is mapped into the line $\{x = 2x_* + L/2 =: x_1\}$ with $x_1 < L/2$. Therefore, we need to map the right boundary into the same line. As it turns out, this is done by a reflection at the line $\{x = x_* + L/2 =: x_2\}$.

We have now to think about the right choice of $x_*$. Assuming that the wave profile is monotonic between crests and troughs, we can choose $x_*$ in a way that $D_*^R$ is a subset of $D$. It is clear that in this case we also have $D_{**}^R \subseteq D$ for all $x_{**} \in (-L/2, x_*]$. Moreover, there exists $x_0 := \max\{x_* | D_*^R \subseteq D\} \leq 0$. Let $D_0$ denote the domain of $\varphi_{x_0}$. Of course, we will choose this maximum to be the reflection line.

There are three different reasons for $x_0$ being maximal. We will investigate them to gain useful information for the proof of symmetry. We have to distinguish the following cases:

(a) $x_0 = 0$, which is the maximum by definition;
(b) the point $(x_0, \eta(x_0))$ is the crest point;
(c) the reflection maps a boundary point $(\xi_0, \eta(\xi_0))$ into another boundary point; this means that $\partial D_*^R$ is tangent to $\partial D$.

Before we continue, note that in cases (b) and (c), where we have $x_0 < 0$, the reflection of the set $\{(x,y) \in \mathbb{R} | x_2 < x < L/2, 0 < y < \eta(x)\}$, i.e. the part between the reflection line $\{x = x_2\}$ and the right boundary is also a subset of $D$.

We are now ready for the main result of this thesis.

**Theorem 4.2** *The stream function of a system described in (2.25) is symmetric.*

*Proof.* The proof of this Theorem is very similar to the proof in [CE2004]. After our preparatory considerations it remains to show symmetry of the stream function for each of the three cases stated above. Case (a):

We define a function $w : D_0 \to \mathbb{R}$,

$$w(x,y) = \begin{cases} \psi(\varphi_{x_0}(x,y)) - \psi(x,y) = \psi(-x,y) - \psi(x,y) & \text{if } \psi_y|_{\eta(x)} < 0 \\ \psi(x,y) - \psi(\varphi_{x_0}(x,y)) = \psi(x,y) - \psi(-x,y) & \text{if } \psi_y|_{\eta(x)} > 0 \end{cases},$$

which clearly is in $C^2(\overline{D_0})$, as $\psi$ is. We prove symmetry for case (a) by showing that $w \equiv 0$ throughout $\overline{D_0}$.

We will do this with the help of the weak maximum principle (Lemma 3.15) and the edge-point lemma (Lemma 3.17). To use the weak maximum principle, we note that $\Delta w = \Delta\psi - \Delta\psi = \gamma - \gamma = 0$ in $D_0$. Next, we check $w \geq 0$ on $\partial D_0$. Clearly, $w = 0$ on the bottom, since $\psi \equiv m$ there. By L-periodicity of the system, we obtain $w = 0$ on

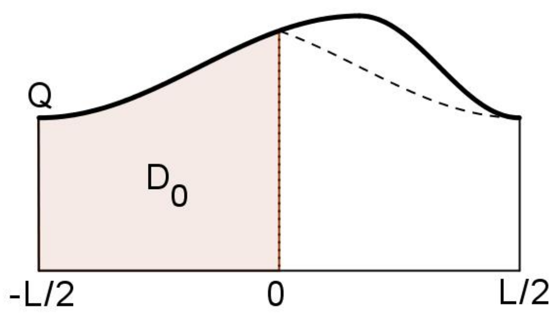


Figure 4.1: Case(a)

$\{x = -L/2\}$ and definition of $x_0$ yields $w = 0$ on $\{x = 0\}$. To compute the values of $w$ on the top, note that $\psi(x, \eta(x)) = 0$ and that $\psi(x, y) > 0$ between the surface and the line of stagnation, if $\psi_y|_{\eta(x)} < 0$ and $\psi(x, y) < 0$ between the surface and the line of stagnation, if $\psi_y|_{\eta(x)} > 0$. Moreover, the sign of $\psi_y$ on the surface cannot change since $\psi$ is twice continuously differentiable and $\psi_y \neq 0$ there. Since we stated that the reflected surface is contained in $\overline{D_0}$, definition of $w$ yields $w \geq 0$ on the surface. Summarizing, we obtained $w \geq 0$ on $\partial D_0$. Application of the weak maximum principle therefore establishes $w \geq 0$ in $D_0$ unless $w \equiv 0$ on $\overline{D_0}$.

By this finding, choosing $Q := (-L/2, \eta(-L/2))$ and $T = \{x = -L/2\}$, we see that all conditions for the edge-point lemma are fulfilled, since furthermore $w(Q) = 0$. It is sufficient to compute the partial derivatives of $w$ up to order two to establish $w \equiv 0$ on $\overline{D_0}$.

- From $w \equiv 0$ on $\{x = -L/2\}$ we conclude that $w_y(Q) = w_{yy}(Q) = 0$.
- To compute $w_x$, take a look at $\psi(x, \eta(x)) = 0$. By differentiation with respect to $x$ we get $\psi_x + \psi_y \eta' = 0$. Now observe that $\eta'(Q) = 0$ since $Q$ is the wave trough and therefore we get $\psi_x(Q) = 0$. From this, it follows $w_x(Q) = (-\psi_x(\mp x, y) - \psi_x(\pm x, y))|_Q = -2\psi_x(Q) = 0$.
- Derivation of $w_x$ with respect to $x$ gives $w_{xx}(Q) = [(-1)(-\psi_{xx}(\mp x, y)) - \psi_{xx}(\pm x, y)]|_Q = \psi_{xx}(Q) - \psi_{xx}(Q) = 0$.
- Finally, differentiate the nonlinear boundary condition $|\nabla\psi|^2 + 2gy = C$ with respect to $x$ to get $2\psi_x(\psi_{xx} + \psi_{xy}\eta') + \psi_y(\psi_{xy} + \psi_{yy}\eta') + 2g\eta' = 0$ on $y = \eta(x)$. Evaluating this at $Q$, we are left with $\psi_y(Q)\psi_{xy}(Q) = 0$ (remember $\eta'(Q) = \psi_x(Q) = 0$). But we have $\psi_y(Q) \neq 0$, which leads to $\psi_{xy}(Q) = 0$ and further $w_{xy}(Q) = -\psi_{xy}(-x, y) - \psi_{xy}(x, y)|_Q = 0$.

This proves that $w(x, y) \equiv 0$ on $\overline{D_0}$, which states $\psi(x, y) = \psi(-x, y)$ throughout $\overline{D_0}$, i.e. symmetry of the stream function in case (a).

Case (b):

Here, we have $x_0 < 0$ and therefore, we have to deal with two reflections at $x_0$ and

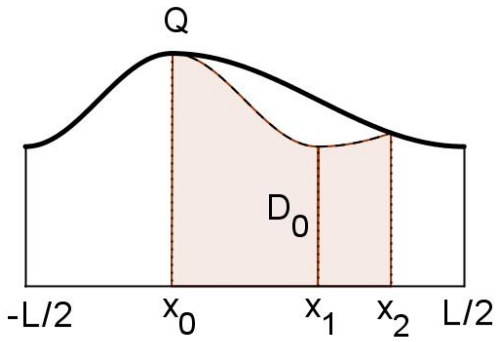


Figure 4.2: Case(b)

$x_2 = x_0 + L/2$. Furthermore, the definition of our auxiliary function $w$ is a bit more complicated in this case:

$$w(x,y) = \begin{cases} \pm\psi(x,y) \mp \psi(2x_0 - x, y) & \text{for } x_0 \leq x \leq x_1, 0 \leq y \leq \widetilde{\eta}(x) \\ \pm\psi(x,y) \mp \psi(2x_2 - x, y) & \text{for } x_1 \leq x \leq x_2, 0 \leq y \leq \widetilde{\eta}(x) \end{cases} \tag{4.8}$$

where $w$ is defined on $D_0 = \{(x,y) \in \mathbb{R}^2 : x_0 < x < x_2, 0 < y < \widetilde{\eta}(x)\}$ and

$$\widetilde{\eta}(x) = \begin{cases} \eta(2x_0 - x, y) & \text{for } x_0 \leq x \leq x_1 \\ \eta(2x_2 - x, y) & \text{for } x_1 \leq x \leq x_2 \end{cases}.$$

Note that the sign of $w$ is chosen so that $w \geq 0$ in a neighbourhood of the surface by the same arguments as in case (a).

Again, we want to show $w \equiv 0$ on $D_0$ repeating the procedure used above. Clearly, $w \in C^2(\overline{D_0})$ and $w = 0$ on $\{y = 0\}$, $\{x = x_0\}$ and $\{x = x_2\}$. Application of the weak maximum principle yields $w \geq 0$ in $D_0$ unless $w \equiv 0$ on $\overline{D_0}$. Thus, we use the edge-point lemma with $Q = (x_0, \eta(x_0)$, the crest point, and $T = \{x = x_0\}$. Again, we have to compute the partial derivatives up to order two to obtain $w \equiv 0$:

- As before, in cause of constance of $w$ on $\{x = x_0\}$, we get $w_y(Q) = w_{yy}(Q) = 0$.
- We differentiate $\psi(x, \eta(x)) = 0$ and get $\psi_x + \psi_y \eta' = 0$. Noting that $\eta'(x_0) = 0$ as $(x_0, \eta(x_0)$ is the wave crest, we get $\psi_x(Q) = 0$ an as before $w_x(Q) = 2\psi_x(Q) = 0$.
- Differentiate $w_x$ with respect to $x$ again to get $w_{xx}(Q) = \psi_{xx}(x,y) + \psi_{xy}(x,y)\eta'(x) - \psi_{xx}(2x_0 - x, y) + \psi_{xy}(2x_0 - x, y)\eta'(x)|_Q = \psi_{xx}(x_0, y) - \psi_{xx}(x_0, y) = 0$, since $\eta'(x_0) = 0$.
- As in case (a), differentiate $|\nabla\psi|^2 + 2gy = C$ with respect to $x$. The same calculation as above shows that it remains $\psi_y \psi_{xy} = 0$ and further $\psi_{xy}(Q) = 0$, as $\psi_y(Q) \neq 0$. We gain $w(Q) = 2\psi_{xy}(Q) = 0$.

Hence, $w \equiv 0$ on $\overline{D_0}$ and so

$$\psi(x,y) = \begin{cases} \psi(2x_0 - x, y) & \text{for } x_0 \leq x \leq x_1 \\ \psi(2x_2 - x, y) & \text{for } x_1 \leq x \leq x_2 \end{cases},$$

which proves case (b).

Case (c):
This is the case where the reflected surface is tangential to $\overline{D_0}$ at some point $(\xi_1, \eta(\xi_1)$.

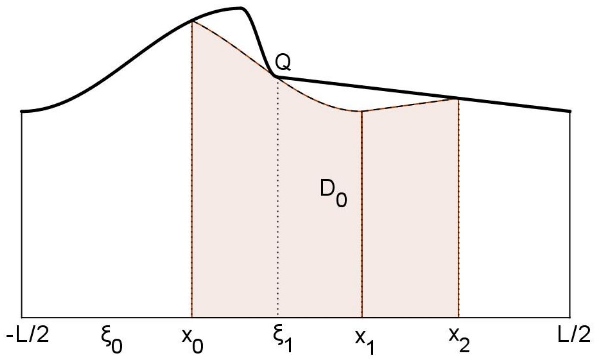


Figure 4.3: Case(c)

Here, we can use the auxiliary function from the previous case (4.8) defined on the same domain $D_0$. A third time, it is our goal to show that $w \equiv 0$ on $\overline{D_0}$.

Again, the weak maximum principle shows us that either $w \geq 0$ in $D_0$ or $w \equiv 0$ on $\overline{D_0}$. Now we check the conditions to apply the boundary-point lemma (Lemma 3.16). Choosing $Q$ to be the tangent point $(\xi_1, \eta(\xi_1))$, we see that $w(Q) = \pm\psi(\xi_1, \eta(\xi_1)) \mp \psi(2x_0 - \xi_1, \eta(2x_0 - \xi_1)) = 0$ since both points are on the surface of the fluid. Moreover, as a consequence of the tangency of the surface, the interior sphere property is fulfilled at $Q$. Now it just remains to compute the derivative in the outward normal direction to establish our claim. Clearly, it is sufficient to compute the partial derivatives. In order to do so, note that we have (setting $\xi_0 := 2x_0 - \xi_1$)

$$\eta(\xi_0) = \eta(\xi_1) \text{ and } \eta'(\xi_0) = -\eta'(\xi_1), \tag{4.9}$$

which follows from the tangent property of the reflected surface. We now differentiate $\psi(x, \eta(x)) = 0$ with respect to x yielding $\psi_x(x, \eta(x)) + \psi_y(x\eta(x))\eta'(x) = 0$. Evaluation at $x = \xi_0$ and $x = \xi_1$ gives

$$\frac{\psi_x(\xi_0, \eta(\xi_0))}{\psi_y(\xi_0, \eta(\xi_0))} = -\eta'(\xi_0) = \eta'(\xi_1) = -\frac{\psi_x(\xi_1, \eta(\xi_1))}{\psi_y(\xi_1, \eta(\xi_1))}.$$

Using the nonlinear boundary condition on $y = \eta(x)$ we get

$$|\nabla\psi|^2(\xi_0, \eta(\xi_0)) = C - 2g\eta(\xi_0) = C - 2g\eta(\xi_1) = |\nabla\psi|^2(\xi_1, \eta(\xi_1)).$$

Combining these two equations, we get $\psi_x(\xi_0, \eta(\xi_0)) = \pm\psi_x(\xi_1, \eta(\xi_1))$ and $\psi_y(\xi_0, \eta(\xi_0)) = \mp\psi_y(\xi_1, \eta(\xi_1))$. But - is not possible in the second equation, since $\psi_y$ must have the same sign in a neighbourhood of the surface and so, we have

$$\psi_x(\xi_0, \eta(\xi_0)) = -\psi_x(\xi_1, \eta(\xi_1))$$

$$\psi_y(\xi_0, \eta(\xi_0)) = \psi_y(\xi_1, \eta(\xi_1)).$$

Using these relations, we compute $w_x$ and $w_y$.

$$\begin{aligned} w_x(Q) &= \psi_x(\xi_1, \eta(2x_0 - \xi_1)) + \psi_y(\xi_1, \eta(2x_0 - \xi_1))(-1)\eta'(2x_0 - \xi_1) \\ &\quad -(-1)\psi_x(2x_0 - \xi_1, \eta(2x_0 - \xi_1)) - \psi_y(2x_0 - \xi_1, \eta(2x_0 - \xi_1))(-1)\eta'(2x_0 - \xi_1) \\ &= \psi_x(\xi_1, \eta(\xi_1)) + \psi_y(\xi_1, \eta(\xi_1))\eta'(\xi_1) + \psi_x(\xi_0, \eta(\xi_0)) + \psi_y(\xi_0, \eta(\xi_0))\eta'(\xi_0) \\ &= 0, \end{aligned}$$

$$\begin{aligned} w_y(Q) &= \psi_y(\xi_1, \eta(2x_0 - \xi_1)) - \psi_y(2x_0 - \xi_1, \eta(2x_0 - \xi_1)) \\ &= \psi_y(\xi_1, \eta(\xi_1)) - \psi_y(\xi_0, \eta(\xi_0)) \\ &= 0. \end{aligned}$$

This assures $w \equiv 0$ on $\overline{D_0}$ in case (c) and completes the proof.

$\square$

**Remark 4.3** *In particular, this means that the surface of the fluid is symmetric.*

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# Deutsche Zusammenfassung

Vor einigen Jahren wurde gezeigt, dass zwischen Wellenbergen und -talen monotone Profile von Wasserwellen in Abwesenheit von Stagnation unter bestimmten Bedingungen an die Vortizität im Inneren des Wassers symmetrisch sind. Diese Arbeit erklärt zu Beginn die Annahmen, die typischerweise für die Modellierung von Wasserwellen getroffen werden und leitet das zugehörige System von partiellen Differentialgleichungen her. Nach einer Vorstellung der im Symmetriebeweis benötigten Maximumprinzipien wird die Symmetrie des Profils von zwischen Wellenbergen und -talen monotonen Wasserwellen für konstante Vortizität gezeigt, wenn im Wasser Stagnation auftritt. Dazu wird der oben genannte Beweis erweitert.

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