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„Electoral Competition with an arbitrary Number of Participants“

Verfasser

Alexander Rabas, Bakk.rer.soc.oec. Bakk.rer.soc.oec.

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1 Introduction

Since the beginning of democracy at about 600 b.c., it has always been a key question how the voting system affects the policies the candidates propose [4]. How will the voting behavior change the candidates' actions, and how will the position of a candidate influence his number of votes? In later times, more complex questions began to arise, namely what influences the number of political parties or how the outcome of an election is determined by the voter's preferences among policies. As is usual in economics, we try to answer these kinds of questions by forming a model.

A model that is the foundation of many theories of political problems addresses these questions: Hotelling's Electoral Competition Game. In this well-known game, two businesses A and B are situated along a street, both a distance x away from both ends of the street. Buyers are uniformly distributed on this street, and each buyer incurs a cost of transportation which is linear in the distance he has to travel to get the product [7]. Therefore, as the two businesses ask the same price (in the original Hotelling game, the firms not only chose positions but also prices; this is omitted here as we will only look at the problem from a political point of view), a buyer will just go to the nearest business to buy the product. The businesses can then choose to simultaneously alter their positions. This results in the situation that the best response for each business to the position of the other business is to go as near as it can to the other firm's position, as then it maximizes the number of buyers the business attracts. This means that the firms are only satisfied if they are back to back to each other.

This was an economic example for two firms, but Anthony Downs extended the example to fit the context of an election with two parties or candidates [3]. This famous model describes the behavior of these two candidates in a simple majority voting situation, where the voters are uniformly distributed with single-peaked preferences along a [0,1]-interval of possible positions for the candidates. Then also the candidates have an incentive to be as close to the position of the other candidate as possible, up to the point where both of them are in the middle of the interval at exactly the same position. Once they reach this position, they have no incentive to choose another one, as they would get less than half of the votes then and lose the election.

Note that the voters need not be uniformly distributed in this model. If they adopt any kind of distribution, the position both candidates want to be in is naturally the median voter's position; this is why another name for this theorem is the Median Voter Theorem [3].

This behavior by the candidates to adopt the median voter's position has first been analyzed by Duncan Black in his article "On the Rationale of Group Decision-making" [2], and a similar approach was considered even earlier by the Marquis de Condorcet [8]. But the two works that basically shaped all the subsequent research on electoral competition are the works by Downs and Hotelling, and this famous median voter model is therefore also called the Hotelling-Downs-Model.

Many variations of the model have been considered, described for example by Osborne in a survey [10]. He finds that the result of the candidates adopting similar positions is very robust if there are only two candidates, even if the underlying structure is changed massively. The question remains whether this idea survives the natural extension to elections with more than two candidates, as well as variations of the underlying structure. One such example would be a paper by Damien Neven, where he simulates sequential entries of firms in a differentiated industry like Hotelling's [9]. Yet, his work does not allow for a choice whether to participate in the game or not.

In a model about electoral competition, it seems reasonable that it should be possible for the potential candidates to choose not to run for office, as losing the election is usually very costly. Furthermore, in reality, the candidates will not choose their stances on the different policies simultaneously, but rather sequentially.

It should be noted that a conjecture exists about the existence of a unique Nash equilibrium in the game with sequential entry of the candidates, an arbitrary number of participants and an exit choice. A proof in this case has not been found by the time of writing [12]. The conjecture is that this game has a similar Nash-equilibrium to the Hotelling-Downs-Model, namely that only the first and the last player will enter the game at the median voter's position, so that they occupy the same position, and all other players will stay out. This game would then be special in the sense that we know of games where a first mover advantage or a last mover advantage exists, but not both.

Interestingly, this equilibrium would be consistent with the finding of Duverger's Law: A plurality rule, that is, a single-winner voting system, will lead to a two-party system [5]. Although Duverger's law is mainly used to explain parliament compositions like in the US now or in the United Kingdom from 1924 to 1974, where only two parties exist, it can also be used to explain, for example, the US-American presidential election. Such a presidential election is very similar to this context, and the results also align in such a way that rationally, only two candidates should run for office.

Alas, a major shortcoming of the classical median voter model is that its central prediction - convergence of platforms to the median voter's ideal - is seldom observed. One reason is that the classical model neglects the importance of voter expectations as a determinant of votes [1], as analyzed in a paper by M. Bernhardt and D. Ingerman. The party or candidate who first adopts a particular policy will be the one associated with this idea, and any other candidate choosing the same position will be seen as imitating the first candidate. In this sense, one can speak of an incumbent effect in politics. This then clearly leads to the effect that candidates choose distinct policies or positions.

To summarize, in our model we will assume that the candidates choose their policies sequentially, there is an option not to take part in the election, as running incurs a cost, and the candidates do not adopt exactly the same opinions about all policies. Note that this last small but crucial assumption - that no two candidates occupy the same position - changes the game significantly from the one Osborne's Conjecture speaks about; the most striking dissimilarity being that the Nash-equilibrium in the Conjecture is ruled out, as two players occupy the same position. So, the unique Nash-equilibrium in Osborne's game and in the one we will be analyzing cannot be the same.

In the game we will be analyzing, the players are the candidates and their strategy is a number, referred to as a position. We hereby make the abstraction that a policy can be abstracted into a number, thus making policy choice one-dimensional. While this is a very simplifying assumption, it is not without reason, as often political parties or candidates are categorized on a left-right axis [11]. To simplify things even further, we make the standard assumption that the population is uniformly distributed on the policy spectrum; assumptions about the distribution of the population on the policy spectrum seem to have no implication on the outcome of the game, as long as the distribution is nonatomic.

Also, we say that the candidates care only about winning, that is, about not having a player that has more votes than them, and that they have no restrictions to the choice of their position, ideological or otherwise, except that they cannot adopt a position that a previous candidate already chose. Then, as explained before, we find it reasonable to assume that the candidates enter sequentially and have an exit choice.

We then have a conjecture about the unique Nash-Equilibrium in this game, namely that only the first player will enter the game at position $\frac{1}{2}$, the median voter's position, and all other players will stay out. This game is then characterized by a strong first-mover advantage.

In this thesis, we will first give a formal definition of the game in section 2, as well as introduce some notation and definitions in Section 3. Then, we will prove some Lemmata in Section 4 that will be useful throughout the thesis. In section 5, we will set up a conjecture about player 3's strategy that would prove the subgame perfect Nash-equilibrium in this game. We will not be able to prove the conjecture in this thesis, but we are hopeful that in the future a proof can be found. Moreover, if then the assumption about the uniqueness of positions is dropped, Osborne's Conjecture can be proven, too.

We are able to prove, however, parts of our conjecture. To do that, we first derive some Lemmata specifically about the conjecture in section 6, putting them together in section 7 in Theorem 1, one of the main findings in this thesis; in short, it says that two inner intervals, that is intervals with players on both edges, will not both be entered by players under specific assumptions.

The other main finding in this thesis is that player 2 will never enter in a specific interval after player 1 has positioned himself in the middle of the spectrum. To derive this Theorem in section 10, we will need Lemmas provided in sections 8 and 9.

2 The Game

Each of the players $1, \ldots, N$ chooses sequentially a member of the set $[0,1] \cup \{NE\}$, but if a position has already been chosen by a predecessor, it cannot be chosen by any other player. That is, each player either chooses a position on the $[0,1]$-interval that represents the spectrum of possible policies that have not been chosen before, or plays "No Entry", that is, he opts out. The choices are made sequentially starting with player 1, and every player is perfectly informed at all times.

The outcome of the game is then determined as follows: After all players have chosen their actions, each player who has chosen a position receives votes from a continuum of citizens, and the player who receives the most votes wins. If there is a tie between different candidates, a candidate is chosen at random from the candidates who have the most votes to win the game. The distribution of citizens' ideal points is uniform along the interval [0,1]. Each voter's aversion against any position that is not his own is simply given by the distance between his location and the other position. This means that if a voter's favorite position is x^*, he is indifferent between the positions $x^* - s$ and $x^* + s$. This also means that the citizens do not vote strategically, and voting is sincere.

Each player obtains the payoff 0 if he chooses $\{NE\}$, the payoff $1/j$ if he is among the j players who receive the maximal fraction of votes, and -1 if there exists a player who has more votes. That is, each player wants to enter the competition if and only if he has some chance of winning, which in this case means that he has a positive payoff. [12]

So, to summarize: In this game, the players are the candidates $1, \ldots, N$; the set of actions for each player is the set $[0,1] \cup \{NE\}$, and the payoffs are 0 if the player chooses $\{NE\}$, $1/j$ if the player is among the j players who do not receive less votes than any other player, and -1 otherwise.

In this thesis, we will restrict ourselves to the analysis of subgame perfect equilibria. Note that it is clear that in an equilibrium, a player can never get a payoff of -1, as then he can simply choose to play NE and be better off. From this follows that in an equilibrium, all entering players must get the same number of votes.

3 Notation and Definitions

3.1 Notation

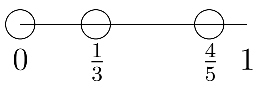
We introduce the following notation:

x_i	...	position of player i
d_1, d_2	...	distances between players
$v(x_i)$	...	number of votes player i receives for entering at position x_i

Note that the number $v(x_i)$ is subject to change when more players enter.

To help with the visualization of the problem, we introduce a graphical representation of the game:

A Candidate's Position



The fractions in the picture below the $[0, 1]$-spectrum denote the specific positions of the players. So, in the example above, we have an interval where one player occupies the left end of the interval, that is $x_a = 0$, one player position $\frac{1}{3}$, that is $x_b = \frac{1}{3}$, and one player the position $\frac{4}{5}$, that is $x_c = \frac{4}{5}$. The votes these players get is calculated as follows: Player a gets no votes from the left of him, and half of the distance to the player to the right of him, so $v(x_a) = 0 + \frac{\frac{1}{3}}{2} = \frac{1}{6}$. Player b gets half of the votes from the interval to the left of him and half of the votes from the interval to the right of him, that is $v(x_b) = \frac{1}{6} + \frac{\frac{4}{5}-\frac{1}{3}}{2} = \frac{2}{5}$. Finally, player c gets half of the votes from the interval to the left of them and all of the votes from the interval to the right, as there is no player occupying any position $x_i > \frac{4}{5}$. So he gets $v(x_c) = \frac{\frac{4}{5}-\frac{1}{3}}{2} + \frac{1}{5} = \frac{13}{30}$. Note that the number of votes must add up to one, so $v(x_a) + v(x_b) + v(x_c) = 1$.

3.2 Definitions

Definition 1. We call a player "winning" if he has a payoff strictly larger than 0 at the end of the game, and we call a player "losing" if he has a payoff of -1.

Definition 2. We will speak of a "voting equilibrium" if all players after the ones already in the game can form an equilibrium in such a way that they are winning.

4 Lemmas and Notes about the Game

In this section, we will derive a number of Lemmas and make observations that will be helpful throughout the thesis.

Lemma 1. If more than two players that want to form a voting equilibrium enter an interval with players on both edges, there can be at most two different alternating distances between the players.

Proof. In a voting equilibrium, all players need to get the same number of votes. For that to happen in an interval where there are players on both edges, there can be at most two different alternating distances between the players, d_1 and d_2. So, the distance between the left edge of the interval and the first player on the left is d_1, between the first and second player d_2, between the second and third player d_1 again, and so on. This ensures that all players get the same number of votes, as the number of votes the players get is always $v = \frac{d_1+d_2}{2}$. Clearly, with more different distances, that would not be possible: If the number of votes player a gets is $2v(x_a) = d_1 + d_2$, and the number of votes player b, who is player a's neighbor, gets is $2v(x_b) = d_2 + d_3$, their votes are the same only if $d_1 = d_3$. □

Note 1. This implies that in a voting equilibrium situation, in an interval with players on both edges, $2v = d_1 + d_2$, where d_1 and d_2 are alternating distances between the players in the interval.

Lemma 2. If more than two players wanting to form a voting equilibrium enter in an interval with players on both edges, and the number of players is odd, the votes they get is $v = \frac{I}{n+1}$, where I is the interval length and n is the number of players entering the interval.

Proof. We know from Note 1 that in this situation, $v = \frac{d_1+d_2}{2}$. Now, the interval length I can then be written as $I = \frac{n+1}{2}d_1 + \frac{n+1}{2}d_2$, which simplifies to $v = \frac{I}{n+1}$. □

Note 2. This also implies that d_1 and d_2 can be written as functions of n if n is odd: $d_1(n) = I - nv$ and $d_2 = (n+2)v - I$.

Lemma 3. If more than two players wanting to form a voting equilibrium enter in an interval with players on both edges, and the number of players is even, the votes they get is $v = \frac{I-d_1}{n}$, where I is the interval length, d_1 is a distance, and n is the number of players entering the interval.

Proof. We know from Note 1 that in this situation, $v = \frac{d_1+d_2}{2}$. Now, the interval length I can then be written as $I = (\frac{n}{2}+1)d_1 + \frac{n}{2}d_2$, which simplifies to $v = \frac{I-d_1}{n}$. □

Lemma 4. It is not possible that an even number of players who want to form a voting equilibrium enter an interval with players on both edges, and both of these edge players are not part of the voting equilibrium.

Proof. First note that if an edge player is not part of the voting equilibrium, he gets strictly less votes than the players forming the equilibrium.
Let us call the edge players players l and r. In this equilibrium we then have two distances d_1 and d_2, separating the players alternatingly, and $2v = d_1+d_2$. As the number of players is even, the first player (player 1) entering the interval must have an interval to the left or to the right of him where an odd number of players enter; we will assume w.l.o.g. it is the right one. If that is the case, the players entering in the right interval always have a possibility to make this player lose without influencing the other parts of the game. For that to happen, the number of votes they get must stay the same, but the number of votes player 1 gets must be reduced. We see that this is always possible by looking at the interval in question: From x_1 to a there is the distance $a - x_1 = s(d_1 + d_2)$, where $s = \frac{n+1}{2}$. The players can now find new $\tilde{d_1}$ and $\tilde{d_2}$ such that $d_1 + d_2 = \tilde{d_1} + \tilde{d_2}$, and $\tilde{d_1} > d_1$ and $\tilde{d_2} > d_2$, thereby making player 1 lose votes and player r gain votes. □

Basically, this means that the first player has no position to enter, as in an interval with an odd number of players the players can shift votes from one end of the interval to the other without changing their own votes. Note the important fact that this deviation does not influence the other players in the game, as the players who deviate do not change their number of votes, and they give votes to a player who already has less votes than them.

Lemma 5. If two players enter in an interval with players on both edges and they want to form a voting equilibrium, they can position themselves in such a way that the infemum of their votes is a quarter of the available votes and the supremum is half of the available votes.

Proof. To see this, consider an interval $[y, z]$ with length I and two players a, b entering who want to form a voting equilibrium, that is they want to get the same number of votes, and more than the edge players. To get the same number of votes they need that $y + d_1 = x_a$ and $z - d_1 = x_b$. If $d_1 \to \frac{I}{2}$, we have that $x_a = x_b$ and $v(x_i) \to \frac{I}{4}$. As d_1 approaches 0, the fraction of votes the players get gets larger and larger, but they can not adopt positions x_y or x_z. We see that $v(x_i) \to \frac{I}{2}$ as $d_1 \to 0$. □

Note 3. We will assume that player 1 always enters at $x_1 = \frac{1}{2}$. The reason is that if no other player enters after him, then player 1 clearly cannot do better.

Note 4. The game is symmetric at $\frac{1}{2}$.

This simply means that if we consider any configuration of positions of players and give any statement concerning that particular configuration, we can reflect the game vertically along the point $\frac{1}{2}$, and we get another configuration for which the same statement holds true.

Later we will often encounter the case that in an interval with one player on the edge (let us call it a side interval), players want to enter and form a voting equilibrium. To see if they can form a voting equilibrium, we consider the following Lemma.

Lemma 6. In an interval with one player on the edge and no player on the other, if players want to form a voting equilibrium, this is only possible if $\exists y \in \mathbb{N} : y = \frac{F}{d_1+d_2}$ for $F = 2+d_2-2v-2p$, where d_1 and d_2 are the distances between the players, p is the position of the edge player, and v is the number of votes the players forming the voting equilibrium get.

Proof. In an interval with one player on the edge at position p, with no player on the right hand side of the interval w.l.o.g., N players enter and form a voting equilibrium. Then, for all players to get the same number of votes, we need that the N players form a sequence $x_1, x_2, \ldots, x_N$ where the position of player x_n is given by

$$x_n = p + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} d_1 + \sum_{\substack{i=2 \\ i \text{ even}}}^{n} d_2, \tag{1}$$

where $p \in [0,1]$ is a fixed position, and d_1 and d_2 are distances.

We need p, and two out of the three following variables d_1, d_2, and the number of votes v the players in the side interval will get, to define this sequence. Note that d_1 and d_2 determine v, or v and one of the distances determines the other distance. Furthermore, this setup fixes the number N of players entering the interval.

While for the players $x_1, \ldots, _{N-1}$ the number of votes equals v by construction, this need not be the case for x_N. This is why we take a close look at x_N and show a way how to easily check whether the number of votes of x_N equals v or not. We assume we know d_1, d_2, v and p. Then we have to analyze two cases: N is odd or N is even.

N is even:
Player N, the one nearest to the edge, has the position

$$x_N = p + \frac{N}{2}d_1 + \frac{N}{2}d_2. \tag{2}$$

Player $N-1$ has the position $x_{N-1} = p + \frac{N}{2}d_1 + (\frac{N}{2} - 1)d_2$, so the distance $x_N - x_{N-1} = d_2$. The number of votes player N gets is $v(x_N) = 1 - x_N + \frac{x_N - x_{N-1}}{2} = 1 - x_N + \frac{d_2}{2}$. We can rewrite this as $x_N = 1 + \frac{d_2}{2} - v$. Now we want to find a conflict in player N's position and the number of votes he gets by inserting x_N into (2). If we then rewrite this expression, we get that $N = \frac{2+d_2-2v-2p}{d_1+d_2}$.

N is odd:
Player N then has the position

$$x_N = p + \frac{N+1}{2}d_1 + \frac{N-1}{2}d_2, \tag{3}$$

and the distance $x_N - x_{N-1} = d_1$. The number of votes player N gets is $v(x_N) = -1 - x_N + \frac{d_1}{2}$. We can rewrite this as $x_N = 1 + \frac{d_1}{2} - v$. Now we insert x_N into (3) and get an expression for N, which is $N = \frac{2+d_2-2v-2p}{d_1+d_2}$; we see that the two expressions coincide no matter if N is even or odd.

We can then rewrite $N = \frac{2+d_2-2v-2p}{d_1+d_2}$ as $N = y + \frac{2+d_2-2v-2p-y(d_1+d_2)}{d_1+d_2}$ for $y \in \mathbb{N}$. This equation is then fulfilled under the condition that $\exists\, y \in \mathbb{N} : y = \frac{F}{d_1+d_2}$ for $F = 2 + d_2 - 2v - 2p$. □

Lemma 7. If the actions $x_1 = \frac{1}{2}, x_2 = NE, x_3 = NE, \ldots, x_N = NE$ form a Nash-equilibrium, it is unique.

Proof. To see that, simply consider the situation when $x_1 \neq \frac{1}{2}$; then, at least one player will enter for sure, because the last player can enter at $x_N = \frac{1}{2}$ and win alone if no other players entered before him. Therefore, there are always at least two players entering the game if $x_1 \neq \frac{1}{2}$, so the maximum payoff player 1 can get is $\frac{1}{2}$. In the above equilibrium, his payoff is 1, so he will always be strictly better off in this equilibrium. □

5 Player 3's Strategy: A Conjecture

In this section, we will give a conjecture about the unique subgame perfect Nash-equilibrium in this game, and then express it in terms of a strategy of player 3. The idea is to formulate a strategy for player 3 which, should player 2 enter the game, will always make player 2 lose and player 3 win, thereby making player 2 play $\{NE\}$. This argument would then in itself prove that the unique subgame-perfect Nash-equilibrium for N players in this game is one where only the first player enters at position $\frac{1}{2}$.

Conjecture 1. The unique subgame perfect Nash-equilibrium in this game is $x_1 = \frac{1}{2}, x_2 = NE, x_3 = NE, \ldots, x_N = NE$.

Note that we see very clearly the difference in Osborne's model and ours here, as the Conjecture about the unique Nash-equilibrium in the game is not the same, as mentioned in the Introduction.

If we can define a strategy by player 3 which, no matter where player 2 enters after $x_1 = \frac{1}{2}$ has been played, will make player 2 lose and will make player 3 win, and thereby player 2 would not want to have entered at all, then this will ensure that Conjecture 1 holds (together with the observation that the last player in the game, if only player 1 has entered so far, will always get less votes than player 1 should he enter, and with Lemma 7).

To find such a strategy, it seems natural to start with a strategy in which player 3 has more votes than players 1 and 2.

So, we have $x_1 = \frac{1}{2}$, and w.l.o.g. $x_2 < \frac{1}{2}$; player 3 then enters at a position $x_3 > \frac{1}{2}$.
Player 1's votes are: $v(x_1) = \frac{x_3 - x_2}{2}$
Player 2's votes are: $v(x_2) = \frac{x_2}{2} + \frac{1}{4}$
Player 3's votes are: $v(x_3) = \frac{3}{4} - \frac{x_3}{2}$
For player 3 to have more votes than player 1, the inequality $v(x_3) > v(x_1)$ has to hold. Inserting the corresponding values and solving for x_3, we get

$$x_3 < \frac{3}{4} + \frac{x_2}{2} \tag{4}$$

For player 3 to have more votes than player 2, the inequality $v(x_3) > v(x_2)$ has to hold. Inserting the corresponding values and solving for x_3, we get

$$x_3 < 1 - x_2 \tag{5}$$

We see that for $x_2 \leq \frac{1}{6}$ equation (5) is the stronger condition and for $x_2 \geq \frac{1}{6}$ equation (4). Therefore, we will try a strategy for a small and irrational number $\varepsilon > 0$

$$x_3 = \begin{cases} 1 - x_2 - \varepsilon & \text{if } x_2 \geq \frac{1}{6}, \\ \frac{3}{4} + \frac{x_2}{2} - \varepsilon & \text{if } x_2 \leq \frac{1}{6}. \end{cases}$$

But, as we are about to see, this strategy is not optimal for $x_2 \leq \frac{1}{14}$:

Lemma 8. If $x_1 = \frac{1}{2}$ and $x_2 \leq \frac{1}{14}$, player 3 will not play $\frac{3}{4} + \frac{x_2}{2} - \varepsilon$ for any $\varepsilon > 0$ and irrational.

Proof. As three players have entered, there are four intervals in the game. We will call these intervals interval I-IV from left to right (so, interval II is the interval $(x_2, \frac{1}{2})$). The number of votes to be gained by player 4 in each interval is:

$$\begin{aligned} v(I) &< x_2 \\ v(II) &= \frac{1}{4} - \frac{x_2}{2} \\ 2v(III) &= x_3 - \frac{1}{2} = \frac{1}{4} + \frac{x_2}{2} - \varepsilon \\ v(IV) &< \frac{1}{4} - \frac{x_2}{2} + \varepsilon \end{aligned}$$

We see that the maximum number of votes in interval IV is always bigger than the number of votes that can be gained in interval II. Note that for the maximum number of votes that can be gained in each interval, $v(IV) > v(II) > v(III) > v(I)$ holds. Note also that by the way x_3 is constructed, one player cannot find a position where he gets more votes than players 1 to 3.

But, two players can always form a voting equilibrium in this game if $x_2 \leq \frac{1}{14}$: The first player (a) entering after player 3 can enter in interval IV in such a way that he gets exactly $v(x_a) = \frac{1}{4} - \frac{x_2}{2}$, namely at $x_a = \frac{3}{4} + \frac{x_2}{2} + \varepsilon$. Then the next player ($b$) can enter in interval II and can only get the same number of votes as player a. An entry in interval II is also the only way to match player a's number of votes, as in all the other intervals, only less votes can be gained maximally.

Now all that remains to show is whether players a and b then have more

votes than the other players in the game:
We know that

$$
\begin{aligned}
v(x_a) &= v(x_b) = \frac{1}{4} - \frac{x_2}{2} && (6)\\
v(x_1) &= \frac{x_3 - x_b}{2} && (7)\\
v(x_2) &= \frac{x_2 + x_b}{2} && (8)\\
v(x_3) &= \frac{x_a - \frac{1}{2}}{2} && (9)
\end{aligned}
$$

If there exists an interval which is a subset of interval II in which player b can enter where he gets more votes than players 1 to 3, players a and b form a voting equilibrium. For this, three inequalities have to hold:

$$
\begin{aligned}
v(x_b) &> v(x_1) && (10)\\
v(x_b) &> v(x_2) && (11)\\
v(x_1) &> v(x_3) && (12)
\end{aligned}
$$

If we insert (7) and (9) into inequality (12) and solve for x_b, we get the condition $x_b < \frac{1}{2}$, which is always true. If we solve (10) for x_b, we get the condition $x_b > \frac{1}{4} + \frac{3x_2}{2} - \varepsilon$. If we solve (11) for x_b, we get the condition $x_b < \frac{1}{2} - 2x_2$.

In Figure 1, we see that for every value of $x_2 \leq \frac{1}{14}$, the interval length is positive, represented by the vertical distance between the two functions. At $x_2 = \frac{1}{14}$, the two lines cross, but as the ε is positive, so is the interval length. Therefore there exists a position for player b where players a and b form a voting equilibrium if $x_2 \leq \frac{1}{14}$, thus making the strategy of player 3 not optimal.

Note that the fact that two players form a voting equilibrium is enough to state that this strategy by player 3 is not optimal. If the strategy was optimal given players a and b enter the game, then player 3 would have to get the same number of votes as player a by entry of other players. Given that $x_a - x_3 = 2\varepsilon$, a large number of players would have to enter in intervals II and III for players a and 3 to get the same number of votes, and to make the entry of these players profitable. But this is not possible as we will see by Theorem 1. □

So, we alter the statement about player 3's optimal strategy to our new conjecture:

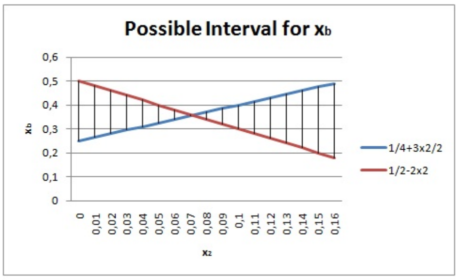


Figure 1: Possible positions where player b can enter.

Conjecture 2. After $x_1 = \frac{1}{2}$ and $x_2 < \frac{1}{2}$, there exists an irrational number $\varepsilon > 0$ such that player 3 can play the strategy

$$x_3 = \begin{cases} 1 - x_2 - \varepsilon & \text{if } x_2 > \frac{1}{6}, \\ \frac{3}{4} + \frac{x_2}{2} - \varepsilon & \text{if } \frac{1}{14} < x_2 \leq \frac{1}{6}, \\ \frac{3}{4} + \frac{x_2}{2} & \text{if } x_2 \leq \frac{1}{14} \end{cases}$$

to keep player 2 out of the game and win.

Lemma 9. Conjecture 1 follows from Conjecture 2.

Proof. If Conjecture 2 holds, this means that there exists a strategy that keeps player 2 out of the game, so player 2 will play $x_2 = NE$. Then, should player $n > 2$ enter the game, player $n+1$ can always play the strategy defined in Conjecture 2 and win alone. Therefore, no player up to player N will enter the game. Then, as $x_1 = \frac{1}{2}$, there exists no position for player N where he can enter such that he does not get less votes than player 1, so he will also play $x_N = NE$, which is consistent with Conjecture 1. □

Unfortunately, we will not be able to prove Conjecture 2 fully, but we will be able to prove part of it, specifically that x_2 will never play any action in interval $(\frac{1}{14}, \frac{1}{4})$. To do that, we will prove a number of new Lemmata concerning entry in the two intervals $\{(x_2, \frac{1}{2})$ and $(\frac{1}{2}, x_3)\}$ in the next chapter.

6 Lemmata about the Conjecture

In this section, we will present Lemmas for the specific setup of Conjecture 2; they will be needed to prove Theorem 1 and 2, the main findings of this thesis.

6.1 General Lemmata

The following is assumed for all Lemmas of section 6.1: $x_1 = \frac{1}{2}$, $x_2 < \frac{1}{2}$ and $x_3 = f(x_2) - \varepsilon$, where f is a linear function and ε is irrational such that if x_2 is rational, x_3 is irrational and vice versa.

Lemma 10. If there is entry in both intervals $(x_2, \frac{1}{2})$ and $(\frac{1}{2}, x_3)$, at least one of the players 1-3 must be part of the voting equilibrium.

Proof. By Lemma 2, if the number of n players entering an interval with players on both edges is odd, the number of votes they get is $v = \frac{I}{n+1}$, where I is the interval length. As one of the intervals has rational length and the other one irrational length, the case where both intervals are entered by an odd number of players is ruled out.
Then, by Lemma 4, if an even number of players enters an interval with players on both edges, at least one of the edge players must be part of the voting equilibrium. □

Lemma 11. If there is entry in both intervals $(x_2, \frac{1}{2})$ and $(\frac{1}{2}, x_3)$, and player 1 is part of the voting equilibrium, an even number of players will enter in both intervals.

Proof. We have seen by Lemma 2 that the number of players entering both intervals cannot be both odd, as then the number of votes the players get would have to be both rational and irrational. Now, if one of the numbers is odd and the other one even, player 1 can be interpreted as the first player of an even number of players entering the interval (x_2, x_3), which is ruled out by Lemma 4 as both players on the edge are not part of the voting equilibrium. □

Lemma 12. It is not possible that there is entry in both intervals $(x_2, \frac{1}{2})$ and $(\frac{1}{2}, x_3)$, and either player 2 or player 3, but not both, are part of the voting equilibrium.

Proof. First, let us look at the case that player 3 is part of the voting equilibrium. Then players 1 and 2 are not, and therefore by Lemma 4 the number of players entering in the left interval must be odd. In this case, we know

by Lemma 2 that the number of votes the players get is rational as the left interval is of rational length. From this we then know that the number of players entering in the right interval must be even, because otherwise they would always get an irrational number of votes, also by Lemma 2.
Now, by Lemma 6, we can check whether the rightmost player in the interval $[x_3, 1]$ gets the correct number of votes. We recall that for this to happen the condition was $\exists\, y \in \mathbb{N} : y = \frac{F}{d_1+d_2}$ for $F = 2 + d_2 - 2v - 2p$. Here we have that $d_2 = 2v - d_1 = 2v - I + nv = 2v - x_3 + \frac{1}{2} + nv$ by Note 2 and $p = \frac{1}{2}$. If we plug this in we get that

$$y = \frac{nv + \frac{3}{2} - x_3}{2v}, \tag{13}$$

and we see that y can never be a natural number if x_3 is irrational.

If player 1 is part of the voting equilibrium, we use the same method and get that the number of votes must be irrational as an odd number of players must enter in the right interval, and an even number of players must enter in the left interval. We then get for the condition that $d_2 = (n+2)v - \frac{1}{2} + x_2$, and $p = \frac{1}{2}$. Then we get that the condition is fulfilled if $y = \frac{\frac{3}{2} - x_2 + nv}{2v}$, which can only be a natural number (if x_2 is rational) if $x_2 = \frac{3}{2}$, which is not possible.
So far we have assumed that x_3 is irrational and x_2 is rational. We can also assume it the other way around and choose ε in such a way that it is irrational and such that x_3 is rational, and if we then take $x_2 = x_3$, the proof still holds. $\square$

Lemma 13. It is not possible that there is entry in both intervals $(x_2, \frac{1}{2})$ and $(\frac{1}{2}, x_3)$, and both player 2 and player 3 are part of the voting equilibrium.

Proof. In this case the condition from Lemma 6 has to hold twice, once for the left side and once for the right side. To recall, the condition is that $\exists\, y \in \mathbb{N} : y = \frac{F}{d_1+d_2}$ for $F = 2 + d_2 - 2v - 2p$.
Right side: Here we have that $p = \frac{1}{2}$, $d_2 = (m+2)v - I$ by Note 2 and $d_1 + d_2 = 2v$. We then get $y = \frac{1+mv-I}{2v}$, where I is the only irrational number, so for this equation to hold the number of votes v must be irrational.
Left side: $p = \frac{1}{2}$, $d_2 = -(n+2)v + I$. We get that $y = \frac{1-(n+4)v+I}{2v}$. We see that this equation can be fulfilled if v is irrational if $I + 1 = 0$, which is not possible. So, from the left side we get that v must be rational, which is a contradiction. $\square$

Lemma 14. It is not possible that there is entry in both intervals $(x_2, \frac{1}{2})$ and $(\frac{1}{2}, x_3)$, and player 2 or player 3 are part of the voting equilibrium.

Proof. This follows directly from Lemmas 12 and 13, as we know from these Lemmas that neither both nor one of them can be part of the voting equilibrium. □

6.2 Lemmata for specific values of x_3

Lemma 15. If $x_1 = \frac{1}{2}$, $x_2 < \frac{1}{2}$ and $x_3 = 1 - x_2 - \varepsilon$, where ε is irrational such that if x_2 is rational, x_3 is irrational and vice versa, it is not possible that there is entry in both intervals $(x_2, \frac{1}{2})$ and $(\frac{1}{2}, x_3)$, and player 1 is part of the voting equilibrium.

Proof. By Lemma 11, we know that the number of players entering both inner intervals is even. If we say that n players enter the interval $(x_2, \frac{1}{2})$ and m players the interval $(\frac{1}{2}, x_3)$, and if w.l.o.g. $n > m$, we get that $\varepsilon = d_1 - d_2 + s(d_1 + d_2)$, where $s = \frac{n-m}{2}$. Note that here we know the specific connection between ε, d_1 and d_2 because we assume a value for x_3, unlike in the Lemmata before, and by this assumption we know the difference in length of the two inner intervals.
Now, by Note 2, we know that in such an interval with an even number of players entering, $d_1 = I - Nv$ and $d_2 = (N + 2)v - I$, where N is the number of players entering that interval. So, from this we can calculate that $\varepsilon = 2I - (2N + 2)v + \frac{n-m}{2}v$. If we express this in terms of v, we get that $v = \frac{2I-\varepsilon}{2N+2+n-m}$.
Then, by Lemma 2, as the interval (x_2, x_3) is entered by an odd number of players, we know that the number of votes they get is $v = \frac{1-2x_2-\varepsilon}{n+m+2}$. Now, if we look at the interval $(\frac{1}{2}, x_3)$, we can check whether the two values for v coincide for any ε. We get that they do if

$$\frac{2(\frac{1}{2} - x_2 - \varepsilon) - \varepsilon}{2m + 2 + \frac{n-m}{2}} = \frac{1 - 2x_2 - \varepsilon}{n + m + 2}. \tag{14}$$

If we express this in terms of ε, we get that

$$\varepsilon = \frac{(1 - 2x_2)(n + m + 2) - (1 - 2x_2)(\frac{3m}{2} + \frac{n}{2} + 2)}{4 + \frac{5n}{2} + \frac{3m}{2}}, \tag{15}$$

which is clearly a contradiction to the fact that ε is an irrational number if x_2 is rational.
If x_2 is irrational, we can do the same thing by reflecting the game along the point $\frac{1}{2}$. Then x_3 is rational and x_2 is irrational with $x_2 = 1 - x_3 + \varepsilon$, and the above equation then holds for $x_2 = x_3$ because x_3 is rational. □

Lemma 16. If $x_1 = \frac{1}{2}$, $x_2 < \frac{1}{2}$ and $x_3 = \frac{3}{4} + \frac{x_2}{2} - \varepsilon$, where ε is irrational such that if x_2 is rational, x_3 is irrational and vice versa, it is not possible that there is entry in both intervals $\{(x_2, \frac{1}{2})$ and $(\frac{1}{2}, x_3)\}$, and player 1 is part of the voting equilibrium.

Proof. This proof is very similar to the one before. By Lemma 11, we know that the number of players entering both intervals is even. If we say that n players enter the interval $(x_2, \frac{1}{2})$ and m players the interval $(\frac{1}{2}, x_3)$, and if w.l.o.g. $n > m$, we get that $\varepsilon = d_1 - d_2 + s(d_1 + d_2) - \frac{1}{4} + \frac{3x_2}{2}$. Now, by Note 2, we know that in such an interval with an even number of players entering, $d_1 = I - Nv$ and $d_2 = (N+2)v - I$, where N is the number of players entering that interval. So, from this we can calculate by plugging in that $\varepsilon = 2I - (2N+2)v + \frac{n-m}{2}v - \frac{1}{4} + \frac{3x_2}{2}$, as $s = \frac{n-m}{2}$. If we express this in terms of v, we get that $v = \frac{2I - \varepsilon - \frac{1}{4} + \frac{3x_2}{2}}{2N+2+\frac{n-m}{2}}$.
Then, by Lemma 2, as the interval (x_2, x_3) is entered by an odd number of players, we know that the number of votes they get is $v = \frac{I}{n+m+2} = \frac{\frac{3}{4} - \frac{x_2}{2} - \varepsilon}{n+m+2}$.
Now, if we look at the interval $(\frac{1}{2}, x_3)$, we can check whether the two values for v coincide for any ε. We get that they do if

$$\frac{2(\frac{1}{4} + \frac{x_2}{2} - \varepsilon) - \varepsilon - \frac{1}{4} + \frac{3x_2}{2}}{2m + 2 + \frac{n-m}{2}} = \frac{\frac{3}{4} - \frac{x_2}{2} - \varepsilon}{n + m + 2}. \tag{16}$$

If we express this in terms of ε, we get that

$$\varepsilon = -\frac{(\frac{3}{4} - \frac{x_2}{2})(\frac{3m}{2} + \frac{n}{2} + 2) - (\frac{1}{4} + \frac{7x_2}{2})(n + m + 2)}{4 + \frac{5n}{2} + \frac{3m}{2}}, \tag{17}$$

which is clearly a contradiction to the fact that ε is an irrational number if x_2 is rational.
If x_2 is irrational, we can do the same thing by reflecting the game along the point $\frac{1}{2}$. Then x_3 is rational and x_2 is irrational with $x_2 = \frac{1}{2} + 2x_3 - 2\varepsilon$, and the above equation then holds for $x_2 = x_3$ because x_3 is rational. $\square$

7 Putting the Lemmata together

In this section we will use the Lemmas presented in Section 6 to prove Theorem 1.

Theorem 1. If $x_1 = \frac{1}{2}$, $x_2 < \frac{1}{2}$ and $x_3 = f(x_2) - \varepsilon$, where f is a linear function and either $f(x_2) = 1 - x_2$ or $f(x_2) = \frac{3}{4} + \frac{x_2}{2}$ and if ε is irrational such that if x_2 is rational, x_3 is irrational and vice versa, there cannot be entry in both intervals $(x_2, \frac{1}{2})$ and $(\frac{1}{2}, x_3)$.

Proof. In this case, we know from Lemma 10 that at least one of the players 1-3 must be part of the voting equilibrium formed by the new players. From Lemma 14 we know that it cannot be the players 2 or 3. Now, for our specific values of x_3, we have that player 1 cannot be part of the voting equilibrium by Lemmas 15 and 16. So, we have found a contradiction in that none of the players can be part of the equilibrium, but at least one of them has to be. □

8 Limit to the Maximum Number of Players

In this section and the next, we will prove Lemmas to help in deriving Theorem 2. Here, we will give an upper bound to the number of players that are able to enter after player 3.

Throughout this section, we will assume that player 3 plays the strategy

$$x_3 = \begin{cases} 1 - x_2 - \varepsilon & \text{if } x_2 > \frac{1}{6}, \\ \frac{3}{4} + \frac{x_2}{2} - \varepsilon & \text{if } \frac{1}{14} < x_2 \leq \frac{1}{6}, \\ \frac{3}{4} + \frac{x_2}{2} & \text{if } x_2 \leq \frac{1}{14} \end{cases}$$

according to our Conjecture 2.

8.1 Limit to the Maximum Number of Players if $x_2 > \frac{1}{6}$

Lemma 17. The maximum number of players that can enter the game after $x_1 = \frac{1}{2}$, $\frac{1}{6} < x_2 < \frac{1}{2}$, and $x_3 = 1 - x_2 - \varepsilon$ for ε small and irrational, is played is $\frac{\frac{1}{2}+x_2+\varepsilon}{\frac{1}{4}-\frac{x_2}{2}-\frac{\varepsilon}{2}}$.

Proof. From Theorem 1, we know that only one of the two intervals $\{(x_2, \frac{1}{2}), (\frac{1}{2}, x_3)\}$ is entered. Let us have a closer look at the maximum number of players that can theoretically enter the game after player 3:

First, let us assume w.l.o.g. that the interval entered is the larger of the two, which is the interval $(x_2, \frac{1}{2})$. As then no player enters the interval $(\frac{1}{2}, x_3)$, player 1 gets a minimum number of votes, namely half of the interval $(\frac{1}{2}, x_3)$. So, we know that $v(x_1) > \frac{x_3 - \frac{1}{2}}{2} = \frac{1}{4} - \frac{x_2}{2} - \frac{\varepsilon}{2}$. We also know that the supremum of the votes the players that enter the game after player 3 can get are all votes except the ones to be gained in interval $(\frac{1}{2}, x_3)$, so if we denote by v_n the number of votes left in the game, $v_n < \frac{1}{2} + x_2 + \varepsilon$. All n new players can now get maximally as the supremum $\frac{v_n}{n}$ votes. These $\frac{v_n}{n}$ must be larger than $v(x_1)$, so $\frac{v_n}{n} > \frac{1}{4} - \frac{x_2}{2} - \frac{\varepsilon}{2}$. If we solve this for n, we get that $n < \frac{\frac{1}{2}+x_2+\varepsilon}{\frac{1}{4}-\frac{x_2}{2}-\frac{\varepsilon}{2}}$. □

To see a graphical representation of this, see Figure 2. On the x-axis there is the value of x_2, and on the y-axis you see the corresponding number of additional players that are able to enter the game given the number of votes that are still available. In the graph we disregard ε as it is small.

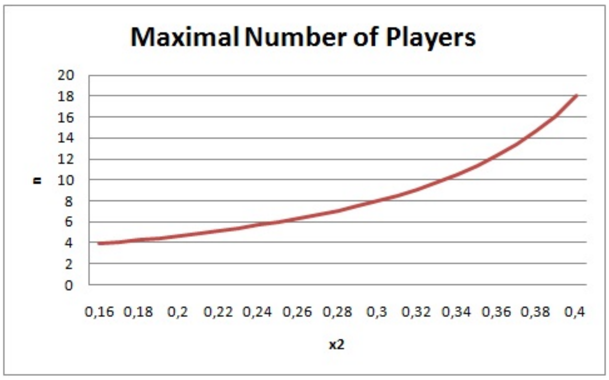


Figure 2: Maximal number of further players for $x_2 > \frac{1}{6}$ fixed and $x_3 = 1 - x_2 - \varepsilon$.

8.2 Limit to the Maximum Number of Players if $\frac{1}{14} < x_2 \leq \frac{1}{6}$

Lemma 18. The maximum number of players that can enter the game after $x_1 = \frac{1}{2}$, $\frac{1}{14} < x_2 \leq \frac{1}{6}$, and $x_3 = 1 - x_2 - \varepsilon$, for ε small and irrational, is played, is $\frac{\frac{3}{4} - \frac{x_2}{2} + \varepsilon}{\frac{1}{8} + \frac{x_2}{4} - \frac{\varepsilon}{2}}$.

Proof. From Theorem 1, we again know that only one of the two intervals $\{(x_2, \frac{1}{2}), (\frac{1}{2}, x_3)\}$ is entered. Let us again assume w.l.o.g. that the interval entered is the larger of the two, which is the interval $(x_2, \frac{1}{2})$. As then no player enters the interval $(\frac{1}{2}, x_3)$, player 1 gets a minimum number of votes, namely half of the interval $(\frac{1}{2}, x_3)$. So, we know that $v(x_1) > \frac{1}{8} + \frac{x_2}{4} - \frac{\varepsilon}{2}$. We also know that in this case, $v_n = \frac{3}{4} - \frac{x_2}{2} + \varepsilon$. All n new players can get a maximum of $\frac{v_n}{n}$ votes, so $\frac{v_n}{n} > \frac{1}{8} + \frac{x_2}{4} - \frac{\varepsilon}{2}$. If we solve this for n, we get that $n < \frac{\frac{3}{4} - \frac{x_2}{2} + \varepsilon}{\frac{1}{8} + \frac{x_2}{4} - \frac{\varepsilon}{2}}$. □

To see a graphical representation of this, see Figure 3. As above, we omit the ε.

8.3 Results

We see that for values of x_2 that are far away from $\frac{1}{2}$, the Lemmas 17 and 18 give a very low bound on the maximum number of additional players; specifically, if we can form a statement saying that at least 6 more players are needed to form a voting equilibrium, entry is not possible in the interval $(\frac{1}{14}, \frac{1}{4})$ by this argument. But, as the x_2 gets closer to $\frac{1}{2}$, the bound grows exponentially, making Lemma 17 less effective for values close to $\frac{1}{2}$. We can

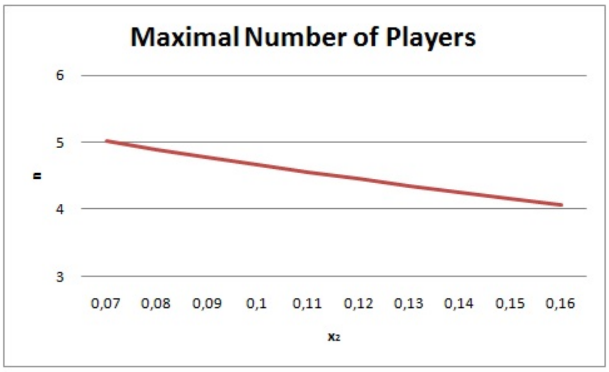


Figure 3: Maximal number of further players for $\frac{1}{14} < x_2 \leq \frac{1}{6}$ fixed and $x_3 = \frac{3}{4} + \frac{x_2}{2} - \varepsilon$.

also see this in Table 1.

x_2	Max.
0,07	5,01
0,1	4,67
0,16	4,06
0,2	4,66
0,25	6,00
0,3	8,00
0,34	10,50
0,4	18,00
0,42	23,00
0,45	38,00
0,49	198,00

Table 1: Maximal Number of Players that can enter the Game after Player 3.

Still, if we can put a lower bound on the maximum number of players that can form an equilibrium, we can exclude certain values that x_2 can take. Note that we also did not take into account the votes player 1 gets from the interval $(x_2, \frac{1}{2})$; this will limit the maximum number of additional players even further.

9 Minimal Number of Players that can form a Voting Equilibrium

We will now have a look at the minimal number of players that need to enter the game such that they can form a voting equilibrium after three players entered the game as described in Conjecture 2. If we can limit the number of people that are able to form a voting equilibrium from below, then, by our considerations before, we can give an interval for x_2 that will not be played by player 2. Because in that case, player 3 then has a strategy he can play where a maximum of n players can enter, as we have seen before, and if we can show that one needs at least $n+1$ players to form a voting equilibrium, then the strategy by player 3 is optimal, and player 2 will not enter the game at all.

In this section, we will give the proof of two Lemmas with the goal to state a minimum number of players that can form a voting equilibrium, and therefore limit the range of positions where x_2 can enter.

For the proofs, we will again have a look at the entries in four intervals and label them I-IV from left to right, namely the intervals $[0, x_2), (x_2, \frac{1}{2}), (\frac{1}{2}, x_3), (x_3, 1]$. The maximum number of attainable votes by all players entering after player 3 are:

$$\begin{aligned} v(I) &< x_2 \\ v(II) &< \frac{1}{2} - x_2 \\ v(III) &< x_3 - \frac{1}{2} \\ v(IV) &< 1 - x_3 \end{aligned}$$

From now on, if a players enter interval I, b players interval II, c players interval III and d players interval IV, we will write (a, b, c, d).

Lemma 19. After $x_1 = \frac{1}{2}$, $\frac{1}{14} < x_2 \leq \frac{1}{6}$ and $x_3 = \frac{3}{4} + \frac{x_2}{2} - \varepsilon$, no voting equilibrium can be found by any number of players.

Proof. Let us first have a look at the maximum number of votes players can attain if they enter the different intervals:

$$
\begin{aligned}
v(I) &< x_2 \\
v(II) &< \frac{1}{2} - x_2 \\
v(III) &< x_3 - \frac{1}{2} = \frac{1}{4} + \frac{x_2}{2} - \varepsilon \\
v(IV) &< 1 - x_3 = \frac{1}{4} - \frac{x_2}{2} + \varepsilon
\end{aligned}
$$

By Theorem 1, we know that only one of the two inner intervals is entered. Note that one of the inner intervals is going to be entered for sure, as they both have the largest number of maximal votes; if intervals II and III were not entered, then player 1 would get the most votes, no matter how the other players enter.

Now we assume w.l.o.g. that interval II is entered, as it is the larger of the two. It has to be entered by at least 2 players: As we have seen in section 5, one player entering in interval II and one player entering in interval IV cannot form a voting equilibrium, because either player 2 or player 1 will have more votes. To further reduce number 1's votes, a player would have to enter in interval III, which is not possible by assumption, and to reduce player 2's votes, a player would have to enter in interval I. But the player entering interval I cannot match the votes of the player entering interval II. So, we must have at least $(0, 2, 0, 0)$, that is to say, we know that at least two players will enter interval II, and at least one other player will enter any of the other intervals.

$(0, 2, 0, 1)$:
Then, we will assume that the next largest interval that can be entered will be entered, namely interval IV. Note that normally the next largest interval would be interval III, but we cannot enter there by Theorem 1. So, we have that $(0, 2, 0, 1)$. We see that in this case $x_a = x_b = \frac{\frac{1}{2}-x_2}{2} + x_2$ and $x_c = x_3 + \frac{\varepsilon}{2}$, where we denote by players a and b the players entering interval II, and by player c the player entering interval IV. We get this solution as in interval IV, roughly half of the votes of interval II can be gained. It is clear that player 1 gets more votes than these players, as $v(x_1) = \frac{x_3 - x_a}{2} = \frac{1}{2} - x_2$, and that is the maximum number of votes attainable in interval II.
So, for a voting equilibrium to form, players 1's votes need to be diminished further; the only way this is possible is to let more players enter in interval II.

$(0, n, 0, 1)$:

The n players entering interval II get $v(x_n) = \frac{(\frac{1}{2}-x_2)}{n+1}$ votes, and player 1 then gets $v(x_1) = \frac{x_3-\frac{1}{2}}{2} + \frac{1}{2(n+1)}(\frac{1}{2} - x_2)$ votes. After some transformations we get that the number of votes of the players entering interval II is larger than the number of votes of player 1 if $x_2 < \frac{1-n}{6+2n}$, which is impossible for any $n > 0$.

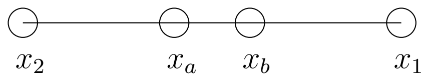
So, this case is ruled out for any number of players as player 1 always has more votes than the players entering after player 3. $\square$

Lemma 20. After $x_1 = \frac{1}{2}$ and $\frac{1}{6} < x_2 < \frac{1}{2}$, at least six more players are needed to form a voting equilibrium.

Proof. Here, the maximum number of attainable votes per interval are

$$\begin{aligned} v(I) &< x_2 \\ v(II) &< \frac{1}{2} - x_2 \\ v(III) &< x_3 - \frac{1}{2} = \frac{1}{2} - x_2 - \varepsilon \\ v(IV) &< 1 - x_3 = x_2 + \varepsilon \end{aligned}$$

Interval II is the largest one, so it will be entered by at least one player. As a similar argument as the one just given, exactly one player cannot enter interval II in a voting equilibrium, as no player in interval I can get his number of votes, but a player would be needed in interval I to reduce player 2's votes. So, we have at least $(0, 2, 0, 0)$.



If player a were to enter first, player 1 would have to be part of the voting equilibrium, but that is not possible as he gets half of the votes of interval III. So, player b has to enter first. Then player 2 must be part of the voting equilibrium, and we get that

$$\begin{aligned} 2v(x_a) &= x_b - x_2 \\ 2v(x_b) &= \frac{1}{2} - x_a \\ 2v(x_2) &= x_a + x_2 \end{aligned}$$

which has the solution $x_a = \frac{1}{4} - \frac{x_2}{2}$. We see that $x_a < x_2$ if $x_2 > \frac{1}{6}$, which always holds true, so we need another player to enter in interval I.

$(1, 2, 0, 0)$:
It is clear that in this case player 3 has too many votes, so one player needs to enter in interval IV. Note that no player can enter in interval III by Theorem 1

$(1, 2, 0, 1)$:
Here, if we label the players with $a - d$ from left to right, we get the system of equations

$$\begin{aligned} 2v(x_a) &= x_a + x_2 \\ 2v(x_b) &= x_c - x_2 \\ 2v(x_2) &= x_b - x_a \\ 2v(x_c) &= \frac{1}{2} - x_b \\ v(x_d) &= 1 - \frac{x_d}{2} - \frac{x_3}{2} \end{aligned}$$

which has the solution $\{x_a = \frac{1}{6} - \frac{2x_2}{3}, x_b = \frac{1}{3} - \frac{x_2}{3}, x_c = \frac{1}{6} + \frac{4x_2}{3}, x_d = \frac{5}{6} + \frac{2x_2}{3} + \frac{\varepsilon}{2}, v = \frac{1}{12} + \frac{x_2}{6}\}$. We get that $v(x_3) > v$ if $x_2 > -\frac{1}{2} - \frac{3\varepsilon}{2}$, so this case is also ruled out because player 3 gets too many votes.

$(1, 2, 0, 2)$:
Here, we get the equations

$$\begin{aligned} 2v(x_a) &= x_a + x_2 \\ 2v(x_b) &= x_c - x_2 \\ 2v(x_2) &= x_b - x_a \\ 2v(x_c) &= \frac{1}{2} - x_b \\ 2v(x_d) &= x_e - 1 + x_2 + \varepsilon \\ 2v(x_e) &= 2 - x_e - x_d \end{aligned}$$

which has the solution $\{x_a = \frac{1}{6} - \frac{2x_2}{3}, x_b = \frac{1}{3} - \frac{x_2}{3}, x_c = \frac{1}{6} + \frac{4x_2}{3}, x_d = \frac{2}{3} + \frac{x_2}{3} - \varepsilon, x_e = \frac{7}{6} - \frac{2x_2}{3} + \varepsilon, v = \frac{1}{12} + \frac{x_2}{6}\}$. Here, we get that $v(x_3) > v$ if $\frac{1}{6} - \varepsilon > \frac{1}{6}$.

So, we see that at least one more player is needed to form a voting equilibrium. $\square$

10 Putting it all together

In this section, we will prove the second main finding of this thesis, Theorem 2

Theorem 2. After $x_1 = \frac{1}{2}$, player 2 will not play an $x_2 \in (\frac{1}{14}, \frac{1}{4})$.

Proof. We will differentiate two cases: $\frac{1}{14} < x_2 \leq \frac{1}{6}$ and $\frac{1}{6} < x_2 < \frac{1}{4}$.

$\frac{1}{14} < x_2 \leq \frac{1}{6}$:
Here player 3 can play the action $x_3 = \frac{3}{4} + \frac{x_2}{2} - \varepsilon$, and by Lemma 19 we know that no voting equilibrium can exist.

$\frac{1}{6} < x_2 < \frac{1}{4}$:
Here player 3 can play the action $1 - x_2 - \varepsilon$ to win alone: By Lemma 17 we know that the maximum number n of players entering after player 3 is $\frac{\frac{1}{2}+x_2+\varepsilon}{\frac{1}{4}-\frac{x_2}{2}-\frac{\varepsilon}{2}} = f(x_2, \varepsilon)$. If we take the first derivative with respect to x_2, we get that $\frac{\partial f}{\partial x_2} = \frac{1}{2(\frac{1}{4}-\frac{x_2}{2}-\frac{\varepsilon}{2})^2}$, which is always positive, therefore the function is increasing in x_2. Hence, the maximum number of players is largest at $x_2 = \frac{1}{4}$. For $x_2 = \frac{1}{4} - \delta$, where $\delta > 0$, we get that $f(\frac{1}{4} - \delta, \varepsilon) = \frac{\frac{3}{4}-\delta+\varepsilon}{\frac{1}{8}+\frac{\delta}{2}-\frac{\varepsilon}{2}}$. We can calculate that this function is lower than 6 if $\varepsilon < \delta$. We see that for any positive δ, we can find an ε such that this inequality is fulfilled, as we can choose our ε arbitrarily small as long as it is irrational. If $\delta = 0$, the inequality does not hold, but it holds for an arbitrarily close number to $x_2 = \frac{1}{4}$, which is why we cannot include the position $x_2 = \frac{1}{4}$ in our Theorem.

Finally, by Lemma 20, we know that we need at least 6 more players to form a voting equilibrium in this case. As the minimum number is 6, and the maximum number is smaller than 6, we see that no voting equilibrium can be found in this case. □

11 Synopsis

In this thesis, we analyze a game that at first glance looks shallow, but as it turns out, analyzing it requires rather deep arguments. While we were not able to give a proof about the unique subgame perfect equilibrium, we were able to give a Conjecture about its existence in section 5, namely Conjecture 1, which says that the unique subgame perfect equilibrium of this game is the action set $\{x_1 = \frac{1}{2}, x_2 = NE, \ldots, x_N = NE\}$.

There, we also gave Conjecture 2, which, as we have shown, describes a strategy by player 3 that would give rise to Conjecture 1. Sadly, we are not able to prove that this strategy is optimal for all actions of player 2, but we are able to show that it is for some.

We then derive a very useful result in section 7, Theorem 1, by use of the Lemmas proved in earlier sections where we see that if players enter the game like in Conjecture 2, players will not enter in both inner intervals.

In section 10 we then calculate, with the help of Lemmas proven in sections 8 and 9, an interval $(\frac{1}{14}, \frac{1}{4})$, where it is clearly suboptimal for player 2 to enter should player 1 enter at $\frac{1}{2}$; this result is stated in Theorem 2.

The work done in this thesis will hopefully be expanded in the future. Places to start would of course be completing the proof for Conjecture 2. To do this, one could try extending the lower bound necessary in Lemma 20, as this would expand Theorem 2 and possibly complete the proof. Then, one could continue by dropping the assumption about uniqueness of the player's positions, as this would be helpful in generalizing the results found here.

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Abstract

In this Thesis, we extend "Hotelling's Electoral Competition Game" to fit a political framework. We assume a sequential choice of postition by the players, who represent candidates in an election, an exit choice, and an incumbent effect – that is, only one player can adopt a specific policy on the policy spectrum. We then find that the subgame-perfect Nash-equilibrium in this game seems to be consistent with many findings in politics nowadays and in the past, for example Duverger's Law or results for the behavior of candidates in the US primary elections. A conjecture exists about the subgame-perfect Nash-equilibrium in this game for *N* players, and two Theorems concerning this conjecture were formulated and proven in this Thesis.

Zusammenfassung

In dieser Arbeit betrachten wir das Modell, das Harold Hotelling in seinem Artikel „Stability in Competition" beschrieben hat, in einem politischen Zusammenhang. Wir nehmen an, dass die Spieler sequentiell ihre Positionen wählen, wobei die Spieler Kandidaten in einer Wahl darstellen. Weiters nehmen wir an, dass es eine Möglichkeit gibt, gar keine Position einzunehmen, und außerdem einen sogenannten „incumbent effect", der besagt, dass nur der erste Spieler, der eine bestimmte Position einnimmt, Stimmen von den Wählern bekommen kann. Dann ergeben sich Parallelen zu politischen Ergebnissen, zum Beispiel das Gesetz von Duverger oder das Verhalten der Kandidaten bei den US primary elections. Es existiert eine Vermutung über das teilspielperfekte Gleichgewicht in diesem Spiel für *N* Spieler, und in dieser Arbeit werden zwei Theoreme bezüglich dieser Vermutung aufgestellt und bewiesen.

Curriculum vitae

Name: Alexander Rabas
Adresse: 1070 Wien, Lerchenfelderstr. 71/1/1
e-mail: a0202663@univie.ac.at
Geburtstag: 30. Mai 1984

Ausbildung

April 2010 – Februar 2011:	Masterstudium der **Volkswirtschaftslehre** an der Universität Wien, Abschluss mit Auszeichung
Okt. 2006 – März 2010:	Bakkalaureatsstudium der **Volkswirtschaftslehre** an der Universität Wien
Okt. 2002 – Oktober 2007:	Bakkalaureatsstudium der **Statistik** an der Universität Wien
Sept. 1994 – Juni 2002:	Gymnasium Amerlingstraße 6, 1060 Wien, Maturaabschluss als Jahrgangsbester mit ausgezeichnetem Erfolg