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"Optimal Fiscal Policy and Commitment"

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Nomenclature

$\bar{r}$	net of taxes rental rate for capital
$\bar{w}$	net of taxes rental rate for labor
β	discount rate
δ	depreciation rate
τ	labor income taxes
θ	capital income tax
b	government bonds
c	per capita consumption
g	government consumption
H	aggregate labor supply
h	hours worked (per capita)
K	aggregate capital stock
k	capital stock (partially the individual capital stock, indicated in the text)
R	gross return on government bonds
r	factor price for capital
S	transfer payments
v	capital utilization rate
w	factor price for labor

z	factor productivity
y	economic output

1 Introduction and Preliminaries

1.1 The roots of optimal fiscal policy research and the problem of time inconsistency

Research on the properties of optimal taxation in the form we know it today reaches back to Frank P. Ramsey[1]. The idea of Ramsey, that a government can optimize social welfare by setting taxes under the constraint of allocations achieved through perfect market equilibria laid the groundwork for a whole stream of successive literature up to now. Authors as Anthony Atkinson, Agnar Sandmo[2], Joseph Stiglitz[3], Christophe Chamley[4], Kenneth Judd[5], Robert Barro[6], Varadarajan Chari, Lawrence Christiano and Patrick J. Kehoe[7] have developed and extended the approach by Ramsey. The major part of these quoted works examines optimal fiscal policy in a framework in which the government sets taxes in order to finance a stream of expenditures and to maximize the welfare of infinitely lived agents with perfect foresight, who are allocating their resources on perfectly competitive markets.

[1] Ramsey, Frank Plumpton, 1927. "A Contribution to the Theory of Taxation," Economic Journal 37, pp. 47-61.

[2] Atkinson, Anthony B. & Sandmo, Agnar, 1980. "Welfare Implications of the Taxation of Savings," Economic Journal, Royal Economic Society, vol. 90(359), pp. 529-549.

[3] Stiglitz, Joseph E., 1986. "Some Aspects of the Taxation of Capital Gains," NBER Working Papers 1094, National Bureau of Economic Research, Inc.

[4] Chamley, Christophe, 1985. "Efficient Taxation in a Stylized Model of Intertemporal General Equilibrium," International Economic Review, Department of Economics, University of Pennsylvania and Osaka University Institute of Social and Economic Research Association, vol. 26(2), pp. 451-468. or Chamley, Christophe, 1985. "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives," Econometrica, Econometric Society, vol. 54(3), pp. 607-622.

[5] Judd, Kenneth L., 1985. "Redistributive taxation in a simple perfect foresight model," Journal of Public Economics, Elsevier, vol. 28(1), pp. 59-83. or Judd, Kenneth L, 1987. "The Welfare Cost of Factor Taxation in a Perfect-Foresight Model," Journal of Political Economy, University of Chicago Press, vol. 95(4), pp. 675-709.

[6] Barro, Robert J., 1991. "Government Spending in a Simple Model of Endogenous Growth," NBER Working Papers 2588, National Bureau of Economic Research, Inc.

[7] Chari, Varadarajan & Christiano, Lawrence & Kehoe, Patrick J., 1994. "Optimal Fiscal Policy in a Business Cycle Model," Journal of Political Economy, University of Chicago Press, vol. 102(4), pp. 617-652.

One property however all this works mentioned above have in common is the implicit assumption that planners have a commitment technology which allows them to credibly promise to stick to a once taken policy decision. As this assumption however excludes the possibility of deviating and reassessing policy measures, even as incentives to do so would turn up, such an approach is denoted as time-inconsistent. The issue of commitment and time-consistency was first addressed by Finn Kydland and Edward Prescott.[8] In their seminal work from 1977 they showed that policymakers are often confronted with a time-inconsistency problem: Past policy measures or the announcement of future policy strategies strongly shape expectations of individuals and have implications on their behavior already in the present. If the policymaker however is by some reason forced to deviate from its initial policy plan or, differently put, he is unable to commit to this initial plan, outcomes might be worse than if individuals had been left in the dark over future policy measures. Thereafter many authors picked up the ideas by Kydland/Prescott and tried to factor time-(in)consistency and therefore the role of commitment in their models. Subsequent literature tried to substitute for commitment or indirectly control for the time-consistency problem. For instance Robert Lucas and Nancy Stokey[9] simulate the problem of time-inconsistency by allowing each government to bequeath its successor a state contingent basket of debts with specific maturity dates. Varadarajan Chari and Patrick J. Kehoe[10] apply reputational mechanisms in order to model and enforce commitment. Other authors addressing the issue of commitment are Jess Benhabib and Aldo Rustichini[11], who open up the possibility for planners to revise plans but introduce a punishment mechanism for default, and Catarina Reis[12], who chooses a similar approach as Benhabib/Rustichini. Klein, Ríos-Rull and Krusell[13] directly model the problem of commitment/default by allowing for a sequential decision mechanism which evolves in non-cooperative games between different planners in different periods. Later works as that of Davide Debortoli and Ricardo Nunes refine the approach by Klein, Ríos-Rull and Krusell by considering a case where default happens with some probability

[8]Kydland, Finn E. & Prescott, Edward C., 1977. "Rules Rather Than Discretion: The Inconsistency of Optimal Plans," Journal of Political Economy, University of Chicago Press, vol. 85(3), pp. 473-491.

[9]Lucas, Robert Jr. & Stokey, Nancy L., 1983. "Optimal fiscal and monetary policy in an economy without capital," Journal of Monetary Economics, Elsevier, vol. 12(1), pp. 55-93.

[10]Chari, Varadarajan, & Kehoe, Patrick J., 1990, "Sustainable Plans," Journal of Political Economy, vol. 98, pp. 784-802.

[11]Benhabib, Jess & Rustichini, Aldo, 1997. "Optimal Taxes without Commitment," Journal of Economic Theory, Elsevier, vol. 77(2), pp. 231-259.

[12]Reis, Catarina, 2006. "Taxation without Commitment," MPRA Paper 2071, University Library of Munich, Germany.

[13]Klein, Paul & Ríos-Rull, José-Víctor, 2003. "Time-consistent optimal fiscal policy," International Economic Review, Department of Economics, University of Pennsylvania and Osaka University, Institute of Social and Economic Research Association, vol. 44(4), pp. 1217-1245.; Klein, Paul & Ríos-Rull, José-Víctor & Krusell, Per, 2003. "Time-Consistent Public Expenditures," CEPR Discussion Papers, No. 4582.

rather than in every period as in the model by the aforementioned authors.[14] But what exactly is commitment about and how is this issue connected with the problem of time-(in)consistency? The next section is dedicated to an accurate definition of these two terms which are central for this thesis. Afterwards it will be discussed which concepts and model frameworks are suitable to control for the issue of time-(in)consistency.

1.2 Time-inconsistency and commitment

The terms time-inconsistency or dynamic-inconsistency, as it is alternatively denoted, describe a *"...situation in which the optimal plan of a decision-maker made at one point in time is no longer optimal later in time"*[15] It accounts the very likely case that future planners would find it optimal to reassess and readjust initially taken policy decisions. In behavioral economics the observation of time-inconsistent attitudes in reality is often explained by assuming a sort of intertemporal schizophrenia: Economic agents are assumed to be made up by *"...many different 'selves'. Each self represents the ...[economic agent]... at a different point in time.*[16] These selves are characterized by *"different intertemporal utilities"* and *"are playing a non-cooperative intrapersonal game."*[17] against each other. Kyland and Prescott's famous example to illustrate the problem of time-inconsistency is about the question whether a government should build flooding protections or not:

Since the construction of dikes is very costly, a government might not find it desirable that houses are built in an area, which is constantly menaced by flooding. Therefore the government's ex ante policy might be not to take anti-flooding measures in order not to set wrong incentives. In the absence of laws prohibiting the construction of houses in this area, rational agents would however know that if just enough people built their houses there, the government would be forced to deviate from its initial policy decision and start constructing dams.[18] It is important to note however, this is regarded as the major accomplishment by Kydland and Prescott, that the problem of time-inconsistency cannot be traced back to myopia only. It is rather a result of intratemporal incentives and how these are shaped by expectations. An example, put forward by Stanley Fischer, illustrates this in the context of optimal tax policy, as

[14] Debortoli, Davide & Nunes, Ricardo, 2007. "Loose Commitment," International Finance Discussion Papers, Board of Governors of the Federal Reserve System, Number 916.

[15] Black, John & Myles, Gareth D. & Hashimzade, Nigar, 2009. "A Dictionary of Economics," Oxford University Press, pp. 126-127.

[16] ibid.

[17] Brocas, Isabelle & Carillo, Juan D. & Dewatripont, Mathias, 2004. "Commitment Devices under Self-Control Problems: An Overview", in: Brocas, Isabelle & Carillo, Juan D. (ed.), "The psychology of economic decisions. Volume 2. Reasons and Choices," Oxford University Press, p. 50.

[18] Kydland/Prescott 1977, p. 477

discussed in this thesis: A government has to finance its expenditures in two consecutive periods with either labor-income- or capital-income-taxation. Both fiscal instruments are distortionary: capital income taxation negatively effects capital formation, labor income taxation has the drawback of a reduced labor supply. If the government's objective is to enforce economic activity, it would choose the least distortionary policy mix; that means to tax labor income in the first and capital income in the second period. Economic agents would invest and start to produce in the first period, not being hindered by capital income taxation. In the second period labor supply is needed to fire economic activity. It is therefore optimal to readjust fiscal policy and now tax capital instead of labor income. Such a strategy would of course not work out repeatedly, as the private sector would start to anticipate government's caprices.[19]

Both above given examples first of all reflect the *"temporal dimension"* of policy decisions. But they also underline another important point: Policymakers have a high degree of discretion in their decision taking. Legal enforcement of past policy promises or legal recourse in the case of a deviation from the initially announced policy is for the most part not possible.[20] This leads directly to the second part of the question: Are there strategies to overcome the time-inconsistency problem? If governments want to avoid being subject to incentives leading to policy readjustments which are prior regarded as not desirable, they may want to commit themselves to certain policy measures. In the example put forward by Kydland and Prescott, mentioned above, a useful commitment strategy would simply be to ban by law house construction in areas which are likely to be flooded. In other cases however, beside the fact that policymakers in modern democracies have to move in the scope of a legal framework, appropriate commitment strategies are less obvious. Kenneth Shepsle identifies two sources of commitment: Commitment can either be traced back to a certain motivation, it is then *"incentive compatible and hence self-enforcing."*, or commitment roots in an imperative, which means that discretion is fully or partly limited.[21] In fields like taxation policy and monetary policy, the discussion in literature mainly focuses on the role of institutions and institutional mechanisms for commitment and policy credibility and therefore on the second source for commitment along the distinction by Shepsle [22]. From a historic perspective such institutions were constitutional arrangements and laws safeguarding basic rights as the property right protecting citizen from arbitrary confiscation of their property through the state: Barry Weingast and Douglass North famously argued that the achievement of binding arrangements securing political rights and regulating social interaction in England after the the revolution of 1688 allowed economic advancements that paved

[19]see Crain, Mark W., 2004. Dynamic Inconsistency, in: Rowley, Charles K. & Schneider, Friedrich (ed.)., "The Encyclopedia of Public Choice. Volume II", Springer, pp. 157-160.
[20]ibid.
[21]North, Douglass C., 1993. "Institutions and Credible Commitment," Journal of Institutional and Theoretical Economics, vol. 149, pp. 11-23.
[22]Crain 2004, pp. 157-160

the way to english world dominance.[23] David Stasavage extended this hypothesis and explained english advantages in the issuance of state securities (in comparison with its main competitor France) with the commitment creating features of democratic representation: checks and balances, party formation and delegation.[24] In political economy literature commitment in taxation policy is often discussed in the context of political budget cycles, which are induced by strategic fiscal policy adjustments prior to elections. The magnitude of this phenomenon can be plausibly interpreted in the light of different political institutions as for example electoral systems: Systems which offer less incentives for strategic fiscal policy adjustments exhibit a higher degree of commitment.[25] Under the classification by Shepsle this might however be considered as a borderline case as here incentives as well as rules take effect. The most important contribution in the discussion about time-inconsistency and commitment however comes from Kydland and Prescott. The two authors suggest that, in the absence of a commitment technology, monetary policy should be rule based in order to find a compromise between the two conflicting goals of price stability and employment as postulated by the Phillipps-curve: The negative relation between unemployment and inflation described by the Philipps-curve, which was empirically observable until the 1960ies, was explained by the higher degree of bargaining power by labor unions in wage negotiations during times of high employment. In politics this empirical relation was understood as a kind of mechanic law, which allowed to control employment by monetary policy. In assuming rigid nominal wages it was thought, that well-directed inflationary pressure on output prices, given constant input prices, would lower real wages and therefore stimulate labor demand. Authors as Friedman, Phelps and Lucas however pointed out that this view neglects the role of expectations. Expected inflation is factored in wage negotiations and therefore renders the concept of stimulating employment via inflation inoperative. Based on this critique of the Philipps-curve Kydland and Prescott showed that expectations behave even more sensitive: Expectations are not only shaped by the evaluation of evidence, if for example some inflationary pressure on prices is already observable, but also by possibility. If the government fails in credibly committing to a policy of price stability the mere possibility that the government could find it optimal to choose an inflationary policy in the future is accounted for in wage negotiations. This in turn forces the government to satisfy expectations for inflation in order to avoid an increase in real wages which would have negative effects on employment. Strangely the existence of action alternatives there-

[23] North, Douglass C. & Weingast, Barry, 1989. "Constitutions and commitment: The evolution of institutions governing public choice in 17th century England," Journal of Economic History, vol. 49(4), pp. 803-832.

[24] Stasavage, David, 2002. Public Debt and the Birth of the Democratic State. France and Great Britain, 1688-1789," Cambridge University Press.

[25] Klašnja, Marko, 2008. "Electoral rules, forms of government, and political budget cycles in transition countries," Panoeconomicus, vol. 55(2), pp. 185-218.; Karpf, Andreas, 2008. "Wechselwirkungen von Demokratie und Finanzpolitik," Thesis, University of Vienna, Austria.

fore limits the possibilities of the government to shape economic development actively. Time-inconsistency and the possibility of discretion assign a more reactive rather than an active role to the government. If policies are assumed to be time-inconsistent they simply play no role in the evaluation of the choice of actions by private agents and the government loses its ability to influence economic development. To overcome the problem of time-inconsistency in policy making Kydland an Prescott recommend that policies should be rule based in order to better prepare economic agents for possible future policy deviations and reestablish a maneuvering room for the government. Kydland and Prescott's ideas were extremely influential. The stability pact of the European monetary union is one out of many examples in which rule based policymaking found it's way into practical politics where true commitment is not achievable.[26]

The matter of time-inconsistency is omnipresent in policymaking. As long as appropriate commitment technologies are not existent, it is plausible to assume that situations will originate in which a deviation from optimal policy might be reasonable or even optimal. How commitment can be enhanced or which mechanisms could be applied to substitute for commitment is however not central to this thesis. These questions were already even though rudimentary addressed in the paragraph above in which the issue of time-inconsistency and commitment was introduced. This thesis first and foremost aims to show how the matter of time-inconsistency is accounted for in optimal taxation literature and which influence the degree of commitment, which will further on be assumed to be given, has on the specific results. The next section discusses how the the problem of time-inconsistency is addressed in the modeling of optimal taxation problems.

1.3 Time-consistent and time-inconsistent equilibria

In the section before the problem of time-inconsistency was discussed in the context of commitment. To understand the different approaches used in the models which are about to be presented in the next chapters and how the factor commitment is controlled for in those models it seems useful to clarify the terms introduced above, which might be a little bit confusing, and to consider the implications of commitment for the process of decision taking and the finding of equilibria.

In the here examined cases the term commitment is just relevant in the context of the government or social planner. Assuming full-commitment therefore implies that a government possesses the technology, however this may look like, to credibly commit itself to a sequence of policy decisions once and for all. Hence, under the assumption of full commitment the policymaker can decide upon a sequence of policies once and for all

[26] Lucke, Bernd, 2004. "Rationales Verhalten und dynamische Makroökonomie. Zur Verleihung des Nobelpreises an Fynn E. Kydland und Edward C. Prescott," online: www.wiso.uni-hamburg.de.

future periods and possesses the commitment technology not to deviate from this policy plan. Although the assumption of commitment doesn't remove the time-inconsistency problem, the government can bind itself to a bundle of policy decisions and becomes immune to incentives to reassess it. Suchlike model solutions are considered as being time-inconsistent and not incentive compatible, as they simply exclude the possibility that over the course of time situations might originate in which the policymaker might be forced to a reevaluation of policies.

A time-consistent approach however, in consideration of the time-inconsistent nature of policy decisions, controls for this dynamic decision phenomenon and allows a deviation from before chosen policy decisions. In optimal policy literature this normally goes along with the assumption that commitment technologies are not available to the government or simply that a new government comes into office periodically. In such a context the decision process upon an optimal policy is not non-recurring, as in the full-commitment scenario, but repeats itself sequentially. In reference to the example cited above, one could say it evolves in the form of a non-cooperative game between different selves of the government.

The assumptions of full- and no-commitment respectively therefore ask for differentiated modeling approaches: In the full-commitment scenario the decision over future policies takes place once and for all periods, while the decision process in a scenario with no commitment is sequential. Equilibria in both scenarios are respectively found in a Ramsey and a Markov approach. The specific solution methods and equilibrium definitions are going to be outlined in the concerning chapters 2 and 3 respectively.

1.4 Structure

The goal of this thesis is to study the role of commitment in the context of optimal fiscal policy. Therefore three different scenarios are considered. In the first scenario full-commitment is assumed. Along the discussion from above the solutions in such an environment are found in a Ramsey equilibrium and are time-inconsistent. In the second scenario it is assumed that the policymaker cannot commit to a certain sequence of policies. The sequential decision process which is characterizing for such an approach is captured in a Markov perfect equilibrium. Solutions found in such an environment are considered to be time-consistent. The third, the loose commitment, scenario is a compromise between the first and the second type. It is assumed that default on past policy promises occurs with a certain probability. Solutions in such an environment are equally found by a Markov approach. The structure of the thesis follows automatically.

Chapter 2 bases on the seminal works by Kenneth Judd[27] and Christophe Chamley[28] which were already mentioned introductorily. Their classical result that capital taxation should be zero in the long run is derived and interpreted. Following the works of the two authors optimal fiscal policy is studied over the dynamic path and it is examined if capital taxes would be an appropriated instrument for redistribution. Further the model is extended to a stochastic environment as well as to a scenario where the government has to honor a balanced budget constraint each period. In chapter 3 on the other hand optimal fiscal policy is examined under the assumption that in contrast to the second chapter the government doesn't have access to a commitment technology. A Markov equilibrium is defined which controls for the resulting sequential decision process and a basic two period model is derived. Its results are discussed and interpreted in the context of those gained in chapter two in order to get some intuitive understanding for the mechanisms in force in the no-commitment environment which lead to policy outcomes different to that in the full commitment case. The second part of the third chapter concentrates on the work by Paul Klein and José-Victor Ríos-Rull who assume an economy in which the government cannot commit on capital income taxation but rather inherits a set of state contingent capital income tax rates from the former office holder. Further alternative approaches to control for a lack of commitment in optimal fiscal policy models are discussed and it is reconsidered if the results in the standard literature on no-commitment in fiscal policy could hinge on the assumption that governments have to honor a balanced budget constraint. Chapter 4 studies the model of Davide Debortoli and Ricardo Nunes. In their setting, commitment and default are subject to a certain probability. At first the probability is exogenously given. In a second step this probability is endogenous and depends on the state of the economy.

The models, it was attempted to keep the model frameworks as similar as possible, are described and derived in detail. For the definition of the equilibria which control for the different degrees of commitment assumed, mentioned above, the reader is referred to the respective chapters. In chapter 5 all results are recapitulated, compared and interpreted. The advantages and drawbacks of the different models are discussed.

[27] Judd 1985

[28] Chamley 1985

2 Optimal Fiscal Policy with Commitment

The following chapter bases on the seminal works by Kenneth Judd[1] and Christophe Chamley[2]. The Chamley-Judd result, as it is often referred to in literature, states that under an optimal policy, capital should not be taxed in the long run. The derivation of this result and a number of possible extensions is conducted under a single discrete time framework in the following chapter. Section 3.1 is dedicated to a detailed description of the model framework. Building on this model framework a Ramsey equilibrium is defined in section 2.2 which takes account of the central full-commitment assumption effective in this chapter. Sections 2.3 and 2.4 derive the central results of Chamley and Judd: In section 2.3 the zero capital tax result in its basic form is derived. Along Judd section 2.4 then concentrates on the redistributive qualities of capital taxation in an economy with heterogenous agents. Section 2.5 examines fiscal policy aside the steady state assumption. Following Chamley it is shown that optimal capital taxation under the assumption of full commitment is characterized by two regimes: There is a regime of high taxation in initial periods which is later on replaced by a regime of zero capital taxation. Section 2.6 formally argues the results gained in section 2.5 by the fact that such an implied front-loading of distortions is strictly welfare increasing. Sections 2.7 and 2.8 extends these examinations to an economy with stochastic shocks and to an environment without public debt respectively.[3]

2.1 The Model Environment and Notation

2.1.1 Preferences, Constraints and Technology

The representative consumer maximizes his utility by choosing streams of consumption and leisure. His preferences are given in equation (2.1.1). The variables c_t and h_t denote

[1]Judd 1985
[2]Chamley 1985
[3]Chapter 2 bases in wide parts on chapter 15 of the standard textbook "*Recursive Macroeconomic Theory*" by Lars Ljungqvist and Thomas J. Sargent; Ljungqvist, Lars & Sargent, Thomas J., 2004. "Recursive Macroeconomic Theory. Second Edition," The MIT Press.

consumption and hours worked respectively. The variable β denotes the discount rate.

$$\sum_{t=0}^{\infty} \beta^t u(c_t, h_t) \quad where \quad \beta \in (0,1) \tag{2.1.1}$$

The utility function (2.1.1) is increasing in c_t and decreasing in h_t and fulfills the INADA conditions $\lim_{c_t \to 0} u_c(c_t, h_t) = \infty$ and $\lim_{c_t \to \infty} u_c(c_t, h_t) = 0$. The output produced in the economy, the input factors are labor h_t and capital k_t, is either consumed, used by the government or used to increase the capital stock. The economy faces the following resource constraint:

$$c_t + g_t + k_{t+1} = F(k_t, h_t) + (1-\delta)k_t \tag{2.1.2}$$

The variable $\delta \in (0,1)$ denotes the depreciation rate of capital. $\{g_t\}_{t=0}^{\infty}$ represents an exogenous stream of government purchases. Concerning the production function it is assumed that the production technology of the economy exhibits constant returns to scale. It therefore holds that:

$$F(k_t, h_t) = F_k(k_t, h_t)k_t + F_h(k_t, h_t)h_t \tag{2.1.3}$$

Further it is assumed that both production factors are complementary, hence that it for example holds that $F(0, h_t) = 0$. The notation $F_k(.)$, in example, implies the partial derivative of the production function $F(.)$ with respect to k. This style of notation will be used throughout the whole thesis.

2.1.2 The government

The government finances the exogenous stream of government expenditures $\{g_t\}_{t=0}^{\infty}$ by the levy of labor income taxes τ_t, capital income taxes θ_t as well as by the issuance of government bonds b_t maturing at the beginning of period t. Returns on government bonds are assumed to be free of taxes. The government budget constraint is denoted as:

$$g_t = \tau_t w_t h_t + \theta_t r_t k_t + \frac{b_{t+1}}{R_t} - b_t \tag{2.1.4}$$

In the government budget constraint, equation (2.1.4), w_t and r_t denote the factor prices for labor and capital respectively. The variable R_t denotes the gross return of a government bond issued in period t and maturing at the beginning of period $t+1$.

2.1.3 The households

Households face the following per period budget constraint:

$$c_t + k_{t+1} + \frac{b_{t+1}}{R_t} = (1-\tau_t)w_t h_t + (1-\theta_t)r_t k_t + (1-\delta)k_t + b_t \tag{2.1.5}$$

The household's problem is therefore to maximize (2.1.1) subject to the budget constraint (2.1.5) with respect to $\{c_t, h_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$.

$$\begin{aligned}
&\max_{c_t,h_t,k_{t+1},b_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t, h_t) \\
&s.t. \\
&c_t + k_{t+1} + \frac{b_{t+1}}{R_t} = (1-\tau_t)w_t h_t + (1-\theta_t)r_t k_t + (1-\delta)k_t + b_t
\end{aligned}$$

This maximization problem can be written in Lagrange form with $\beta^t \lambda_t$ as the Lagrange multiplier on the household's budget constraint (2.1.5).

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, h_t) + \lambda_t \left[(1-\tau_t)w_t h_t + (1-\theta_t)r_t k_t + (1-\delta)k_t + b_t - c_t - k_{t+1} - \frac{b_{t+1}}{R_t} \right] \right\}$$

The first order condition associated to the household problem are thus:

$$\frac{\partial \mathcal{L}}{\partial c_t} : u_c(c_t, h_t) = \lambda_t \tag{2.1.6}$$

$$\frac{\partial \mathcal{L}}{\partial h_t} : u_h(c_t, h_t) = -\lambda_t(1-\tau_t)w_t \tag{2.1.7}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} : \lambda_t = \beta\lambda_{t+1}[(1-\theta_{t+1})r_{t+1} + (1-\delta)] \tag{2.1.8}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} : \frac{\lambda_t}{R_t} = \beta\lambda_{t+1} \tag{2.1.9}$$

Mergin equations (2.1.6) and (2.1.7), (2.1.6) and (2.1.8), as well as equations (2.1.8) and (2.1.9) allows to summarize household's first order conditions in the optimality conditions (2.1.10), (2.1.11) and (2.1.12) respectively.

$$-\frac{u_h(c_t, h_t)}{u_c(c_t, h_t)} = (1-\tau_t)w_t \tag{2.1.10}$$

$$u_c(c_t, h_t) = \beta u_c(c_{t+1}, h_{t+1})[(1-\theta_{t+1})r_{t+1} + (1-\delta)] \tag{2.1.11}$$

$$R_t = (1-\theta_{t+1})r_{t+1} + (1-\delta) \tag{2.1.12}$$

Equation (2.1.10) denotes the intratremporal substitution behavior between consumption and labor. Equation (2.1.11), also denoted as the household's Euler equation, captures the intertemporal choice of the households concerning consumption and capital accumulation (saving). Constraint (2.1.12) rules out arbitrage possibilities in trades between bonds and capital, which would exist if the rates of return on these two assets would be different. The role of this constraint will be discussed in more detail later on. By forward iteration the budget constraint faced by the households in each period can be summarized to one present value budget constraint. This is done by iteratively eliminating the common term representing government debt of budget constraints in two consecutive periods.

$$\underbrace{\begin{aligned} &c_t + k_{t+1} + \frac{b_{t+1}}{R_t} = (1-\tau_t)w_t h_t + (1-\theta_t)r_t k_t + (1-\delta)k_t + b_t \\ &c_{t+1} + k_{t+2} + \frac{b_{t+2}}{R_{t+1}} = (1-\tau_{t+1})w_{t+1}h_{t+1} + (1-\theta_{t+1})r_{t+1}k_{t+1} + (1-\delta)k_{t+1} + b_{t+1} \end{aligned}}$$

$$\begin{aligned} &c_t + \frac{c_{t+1}}{R_t} + \frac{k_{t+2}}{R_t} + \frac{b_{t+2}}{R_t R_{t+1}} = (1-\tau_t)w_t h_t + \frac{(1-\tau_{t+1})w_{t+1}h_{t+1}}{R_t} \\ &+ \left[\frac{(1-\theta_{t+1})r_{t+1} + (1-\delta)}{R_t} - 1\right] k_{t+1} + [(1-\theta_t)r_t + (1-\delta)]k_t + b_t \end{aligned} \tag{2.1.13}$$

The consolidated budget constraint for periods t and $t+1$ clarifies the role of the no-arbitrage condition (2.1.12): The right hand side reflects the household's resources while the left hand side reflects the use of resources, be it for consumption, the accumulation of capital or the acquisition of bonds. In the case the equality (2.1.12) doesn't hold, households would be able to accomplish arbitrage transactions between capital and bonds and could thus overstretch their budget constraint: In the case of $(1-\theta_{t+1})r_{t+1}+(1-\delta) > R_t$ households would have incentives to carry out short sales with bonds and to invest the profits of such transactions into capital. There would be an excess of supply in the bond market. In the other case $(1-\theta_{t+1})r_{t+1} + (1-\delta) < R_t$ bonds would be the dominating investment instrument as they provide higher returns. The demand for bonds would thus be in excess of supply. For the markets of these two assets to be in equilibrium condition (2.1.12) has to hold. With (2.1.12) equation (2.1.13) can be rewritten to the form displayed below. Consequently b_{t+2} is substituted for with the help of the period

$t+2$ budget constraint. Note that condition (2.1.12) holds in each period.

$$\underbrace{\begin{aligned} & c_t + \frac{c_{t+1}}{R_t} + \frac{k_{t+2}}{R_t} + \frac{b_{t+2}}{R_t R_{t+1}} = (1-\tau_t) w_t h_t + \frac{(1-\tau_{t+1}) w_{t+1} h_{t+1}}{R_t} + [(1-\theta_t) r_t + (1-\delta)] k_t + b_t \\ & c_{t+2} + k_{t+3} + \frac{b_{t+3}}{R_{t+2}} = (1-\tau_{t+2}) w_{t+2} h_{t+2} + [(1-\theta_{t+2}) r_{t+2} + (1-\delta)] k_{t+2} + b_{t+2} \end{aligned}}$$

$$\begin{aligned} c_t + \frac{c_{t+1}}{R_t} + \frac{c_{t+2}}{R_t R_{t+1}} + \frac{k_{t+3}}{R_t R_{t+1}} + \frac{b_{t+3}}{R_t R_{t+1} R_{t+2}} = (1-\tau_t) w_t h_t + \frac{(1-\tau_{t+1}) w_{t+1} h_{t+1}}{R_t} \\ + \frac{(1-\tau_{t+2}) w_{t+2} h_{t+2}}{R_t R_{t+1}} + [(1-\theta_t) r_t + (1-\delta)] k_t + b_t \end{aligned} \tag{2.1.14}$$

Expression (2.1.14) can be finally written down in a generalized way. Period $t=0$ is assumed to be the initial period.

$$\begin{aligned} \sum_{t=0}^{T} \left(\prod_{i=0}^{t-1} R_i^{-1} \right) c_t + \left(\prod_{i=0}^{T-1} R_i^{-1} \right) k_{T+1} + \left(\prod_{i=0}^{T-1} R_i^{-1} \right) \frac{b_{T+1}}{R_T} = \\ \sum_{t=0}^{T} \left(\prod_{i=0}^{t-1} R_i^{-1} \right) (1-\tau_t) w_t h_t + [(1-\theta_0) r_0 + (1-\delta)] k_0 + b_0 \end{aligned} \tag{2.1.15}$$

Taking the limit and imposing the following transversality conditions,

$$\lim_{T\to\infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) k_{T+1} = 0 \tag{2.1.16}$$

$$\lim_{T\to\infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) \frac{b_{T+1}}{R_T} = 0 \tag{2.1.17}$$

one can write down the household's present value budget constraint as:

$$\sum_{t=0}^{\infty} \left(\prod_{i=0}^{t-1} R_i^{-1} \right) c_t = \sum_{t=0}^{\infty} \left(\prod_{i=0}^{t-1} R_i^{-1} \right) (1-\tau_t) w_t h_t + [(1-\theta_0) r_0 + (1-\delta)] k_0 + b_0 \tag{2.1.18}$$

The role of transversality conditions Which role do the transversality conditions play? Additional to the household's optimality conditions and the household's budget constraint, optimality demands to impose transversality conditions of the form displayed in (2.1.16) and (2.1.17). In general the Euler equation (2.1.11) defines the optimal allocation path over periods. Without a fixed terminal point however there might be various paths satisfying the Euler equation, but not all of them necessarily have to be optimal. Transversality conditions are therefore imposed as additional constraints to rule out suboptimal paths.

In the here examined example the household's saving/consumption behavior is concerned. At the outset it will be assumed, contrasting to the model under consideration, that the time horizon is finite. Supposable there are two cases: 1. The households could spend too little over their lifecycle. This would lead to an over-accumulation of assets, bonds and capital, which would be connected to a lower than possible lifetime consumption and is therefore not optimal. 2. Spending too much on the other hand would eventually leave back a negative stock of assets at the end of the lifecycle. A scenario like this can however be excluded as it is supposed that nobody would be willing to participate in transactions which would be implied in such a case.

Related to case 1. transversality conditions (2.1.16) and (2.1.17) imply that the present discounted value of assets, bonds and capital, should be zero at infinity. Or, to state it differently, referring to the above assumed finite horizon, an optimal lifetime saving/consumption path wouldn't leave back any assets at the end of the household's life. To translate this intuitive interpretation of transversality conditions into the mathematical formalism in our model economy it is important to note that strictly speaking two additional constraints should have been included in the original derivation of the household's first order conditions: Namely, physical capital cannot become negative, hence $k_t \geqslant 0\ \forall t$, and households are not allowed to decease with a positive stock of debt (a negative stock of bonds), hence $b_T \geqslant 0$. At first a case with finite horizon is examined: The household thus dies at the end of period T. The above mentioned new conditions are to be included into the household's maximization problem via a Kuhn-Tucker approach.

$$
\begin{aligned}
&\max_{c_t,h_t,k_{t+1},b_{t+1}} \sum_{t=0}^{T} \beta^t u(c_t, h_t) \\
&s.t. \\
&c_t + k_{t+1} + \frac{b_{t+1}}{R_t} = (1-\tau_t)w_t h_t + [\overbrace{(1-\theta_t)r_t + (1-\delta)}^{R_{t-1}}]k_t + b_t \\
&0 \leqslant k_t \quad \forall t \quad \text{and} \quad 0 \leqslant b_T
\end{aligned}
$$

Written down in Lagrange form:

$$
\begin{aligned}
\mathcal{K} = \sum_{t=0}^{T} \beta^t \Bigg\{ & u(c_t, h_t) + \lambda_t \left[(1-\tau_t)w_t h_t + [(1-\theta_t)r_t + (1-\delta)]k_t + b_t - c_t - k_{t+1} - \frac{b_{t+1}}{R_t} \right] \\
& + \eta_t [k_t - 0] \Bigg\} + \phi_T [b_T - 0]
\end{aligned}
$$

The Kuhn-Tucker conditions are:

$$\frac{\partial \mathcal{K}}{\partial c_t} = 0; \quad \frac{\partial \mathcal{K}}{\partial h_t} = 0; \quad \frac{\partial \mathcal{K}}{\partial k_{t+1}} = 0; \quad \frac{\partial \mathcal{K}}{\partial b_{t+1}} = 0; \quad \frac{\partial \mathcal{K}}{\partial \lambda_t} = 0$$

as well as

$$k_t \geqslant 0; \quad \beta^t \eta_t \geqslant 0; \quad and \quad \beta^t \eta_t [k_t - 0] = \beta^t \eta_t k_t = 0$$
$$b_T \geqslant 0; \quad \beta^T \phi_T \geqslant 0; \quad and \quad \beta^T \phi_T [b_T - 0] = \beta^T \phi_T b_T = 0$$

The only first order conditions which change are those associated to k and b (in period $t = T$). At first the focus rests upon the derivation of the transversality condition associated to capital. By inclusion of the constraint $k_t \geqslant 0$ the first order condition associated to k_{t+1} now looks as follows:

$$\frac{\partial \mathcal{K}}{\partial k_{t+1}} : -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} R_t + \beta^{t+1} \eta_{t+1} = 0 \tag{2.1.19}$$

Since the Kuhn-Tucker condition $\beta^t \eta_t k_t = 0$ holds, above condition (2.1.19) is only fulfilled if $k_t > 0$ and hence $\eta_t = 0$. As the third term drops out in period T this assertion is true for all periods $t = 1, \ldots, T-1$. What is the intuition behind this? Since production is a necessary precondition for consumption and it was defined that $F(0, h_t) = 0$, households have an incentive to carry at least some capital from one period to another: without capital there is no production and hence no consumption. As defined in section 2.1.1 the utility function fulfills the INADA condition $\lim_{c_t \to 0} u_c(c_t, h_t) = +\infty$. The marginal utility of some small increment of consumption would thus be infinitely high. To consume nothing in some periods can therefore never be optimal. In the last period T in contrast households would have incentives to choose $k_T = -\infty$, because according to the budget constraint in period T,

$$c_T = (1 - \tau_T) w_T h_T + [(1 - \theta_T) r_T + (1 - \delta)] k_T + b_T$$

this would allow them to choose $c_T = +\infty$ (saving in period T would be pointless anyway as there is no period $T + 1$). This is however not possible by the imposed constraint $k_t \geqslant 0$. In the last period the constraint therefore binds and $k_T = 0$. Households dissolve all their asset holdings in the last period in order to maximize consumption. As there is no period $T+1$ it would make no sense to save/invest anymore. The first order condition associated to capital for period T can be written down as follows:

$$\frac{\partial \mathcal{K}}{\partial k_T} : \beta^T \lambda_T R_{T-1} + \beta^T \eta_T = 0 \tag{2.1.20}$$

Expression (2.1.20) is multiplied by k_T and, with the help of Kuhn-Tucker condition

$\beta^T \eta_T k_T = 0$, rewritten as follows:

$$\beta^T \lambda_T R_{T-1} k_T = 0 \tag{2.1.21}$$

Further it can be substituted for λ_T from the first order condition associated to c, equation (2.1.6).

$$\beta^T u_c(c_T, h_T) R_{T-1} k_T = 0 \tag{2.1.22}$$

As argued above the constraint $k_t \geqslant 0$ binds in the last period T, thus $\eta_T > 0$ and $k_T = 0$. Further it was assumed that $u_c(c_T, h_T) > 0$ (see section 2.1.1). If there would be capital left at the end of the household's life, hence $k_T > 0$, this would imply that $\beta^T u_c(c_T, h_T) R_{T-1} k_T > 0$. Obviously this is not optimal. It would be better to consume all capital which is left.

In an infinite horizon case the intuition is slightly different since there is no last period. Nevertheless over-accumulation of assets, capital in the here examined case, cannot be optimal and has to be avoided. Equation (2.1.22) can be rewritten for the infinite horizon case by taking the limit from $T \to \infty$:

$$\lim_{T\to\infty} \beta^T u_c(c_T, h_T) R_{T-1} k_T = 0 \tag{2.1.23}$$

Equation (2.1.23) implies that the present discounted value of capital held in the distant future should be zero. In contrast, if $\lim_{T\to\infty} \beta^T u_c(c_T, h_T) R_{T-1} k_T > 0$, the household would accumulate too much capital. In such a case it would be better to increase consumption at the expense of saving in order to bring down the discounted marginal utility of consumption at a faster rate than capital is growing. How does one now get to the form of the transversality condition which was used in the derivation of the present value budget constraint? Successive Euler equations, see (2.1.8), are used to eliminate the common term λ_{t+j} until arrived in period T. Note, the net-rate of return on capital was substituted for along the no-arbitrage condition (2.1.12).

$$\left.\begin{aligned} \lambda_t &= \beta\lambda_{t+1}R_t \\ \lambda_{t+1} &= \beta\lambda_{t+2}R_{t+1} \end{aligned}\right\} \quad \left.\begin{aligned} \lambda_t &= \beta^2\lambda_{t+2}R_tR_{t+1} \\ \lambda_{t+2} &= \beta\lambda_{t+3}R_{t+2} \end{aligned}\right\} \quad \left.\begin{aligned} \lambda_t &= \beta^3\lambda_{t+3}R_tR_{t+1}R_{t+2} \\ &\dots \end{aligned}\right\}\dots$$

One can finally write the process of forward iteration in a general way.

$$\lambda_t = \beta^T \lambda_{t+T} \prod_{i=t}^{T-1} R_i \tag{2.1.24}$$

Assigning $t = 0$ as the starting period, equation (2.1.24) can be used to substitute for

λ_T in equation (2.1.21).

$$\beta^T \frac{\lambda_0}{\beta^T} \prod_{i=0}^{T-1} R_i^{-1} R_{T-1} k_T = 0 \rightarrow \prod_{i=0}^{T-1} R_i^{-1} R_{T-1} k_T = 0 \tag{2.1.25}$$

Taking the limit $T \rightarrow \infty$, as before, one finally arrives at the form of transversality condition (2.1.16):

$$\lim_{T\to\infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) R_{T-1} k_T = 0 \rightarrow \lim_{T\to\infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) k_{T+1} = 0 \tag{2.1.26}$$

From the perspective of the household, bonds are just an alternative investment instrument, which along the no-arbitrage condition (2.1.12) has to yield the same return as investments in capital. The interpretation and derivation of the transversality condition concerning government bonds is very similar to the derivation of the transversality condition for capital with the difference that there are no restrictions concerning bond holdings of households in periods $t = 1, \ldots, T-1$: Government bonds play the role of an investment and credit instrument but don't have such a crucial role for production as capital does. In the last period T however households would have, equally to the case with capital, incentives to choose a period T stock of bonds equal to $-\infty$, as this would allow, just like in the case of period T capital, to set $c_T = +\infty$. Such implied transactions, it was supposed, are however debarred by market mechanisms as credit markets would not allow households to accumulate debt stocks like that. As in the case of capital it can therefore be supposed that the constraint $b_T \geqslant 0$ binds, which means that $b_T = 0$: Keeping a positive stock of government bonds beyond one's lifetime makes no sense and decreases household's lifetime utility. The period T first order condition associated to b is:

$$\frac{\partial \mathcal{K}}{\partial b_T} : \beta^T \lambda_T + \beta^T \phi_T = 0 \tag{2.1.27}$$

Multiplying expression (2.1.27) with b_T it can be rewritten with the Kuhn-Tucker condition $\beta^T \phi_T b_T = 0$, yielding:

$$\beta^T \lambda_T b_T = 0 \tag{2.1.28}$$

The result is a terminal condition similar to the case of capital, which states that it is not optimal to hold government bonds beyond death. Equation (2.1.28) implies that either $b_T = 0$ or the shadow price of government bonds $\lambda_T = 0$. Since it is however known that $\lambda_T = u_c(c_T, h_T) > 0$ the second possibility can be excluded. This implies that optimally $b_T = 0$: It would be irrational to invest in bonds in period T, as there is no period $T+1$. Taking the limit $T \rightarrow \infty$ gives the infinite horizon version of the terminal/transversality condition:

$$\lim_{T\to\infty} \beta^T \lambda_T b_T = \lim_{T\to\infty} \beta^T u_c(c_T, h_T) b_T = 0 \tag{2.1.29}$$

Equation (2.1.29) implies that the present discounted value of household's stock of government bonds should be zero when $T \to \infty$. Otherwise the households would save too much and $\lim_{T\to\infty} \beta^T u_c(c_T, h_T) b_T > 0$ implying that an increase of consumption at the expense of investment in government bonds would be utility augmenting. As before, forward iteration applied on the Euler equation, this time the one which is associated to bonds (2.1.9)(by the no-arbitrage condition (2.1.12) it is equivalent to the Euler equation associated to capital), can be used to express λ_T in a general way.

$$\lambda_T = \frac{\lambda_0}{\beta^T} \prod_{i=0}^{T-1} R_i^{-1} \tag{2.1.30}$$

Taking the limit $T \to \infty$ of terminal condition (2.1.28) and substituting for λ_T by expression (2.1.30) yields a transversality condition in a general form equivalent to the transversality condition for capital and the form which was used in the derivation of the present value budget constraint (see (2.1.17)).

$$\begin{aligned} &\lim_{T\to\infty} \beta^T \frac{\lambda_0}{\beta^T} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) b_T = 0 \\ &\to \lim_{T\to\infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) b_T = \lim_{T\to\infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) \frac{b_{T+1}}{R_T} = 0 \end{aligned} \tag{2.1.31}$$

From the perspective of the government condition (2.1.31) can be interpreted as a no-ponzi condition ruling out chain-letter behavior by the government, which would mean that ever-increasing amounts of money are raised by bond issuance in order to acquit interest payments: This implies, similar to before, that the present discounted value of debt has to be zero as $T \to \infty$, or that the debt is growing at a smaller rate than there is interest on bonds.[4]

[4]De la Fuente, Angel, 2000. "Mathematical methods and models for economists," Cambridge University Press, p. 572.; Holger, Gerhardt, 2007. "Angewandte Makroökonomie/Konjunktur- und Wachstumspolitik," Lecture Notes, www.wiwi.hu-berlin.de.; Krauth, Brian, 2004. "Macroeconomic Theory", Lecture Notes, www.sfu.ca.

2.1.4 The firms

Taking factor prices for labor w_t and capital r_t as given firms are maximizing their profits.

$$\Pi = F(k_t, h_t) - r_t k_t - w_t h_t \tag{2.1.32}$$

The firms first order conditions are hence,

$$F_k(k_t, h_t) = r_t \tag{2.1.33}$$

$$F_h(k_t, h_t) = w_t \tag{2.1.34}$$

This means that in an equilibrium the price for one additional unit of labor/capital has to equal the marginal product of labor/capital. As it was assumed that the production technology exhibits constant returns to scale, pure profits are zero in an equilibrium: $\Pi = 0$. The size of firms is undetermined and plays no role for the model economy.

2.2 Defining a Ramsey equilibrium

After having sketched the model framework one is about to define the full commitment Ramsey equilibrium.[5]

1. Let $\pi_t = \{g_t, \theta_t, \tau_t, b_t\}$ denote government policies for period t, and π the sequence of government policies for all t.

2. Let $x_t = \{c_t, h_t, k_{t+1}, b_{t+1}\}$ denote household's allocations for period t, and x the sequence of allocations for all t.

3. Let $\{w, r, R\}$denote the economy's price system for all t.

Since it is assumed that the government possesses a commitment technology, it can bind itself to a certain sequence of policy choices once for all at period $t = 0$, anticipating its implications for allocations of the private sector and the price system. Household allocations and the price system are described by rules: On the one hand there are allocation rules, which are sequences of functions $x(\pi) = \{x(\pi_t)\}$ that map government policies π into household's allocations $x(\pi)$. On the other hand there are price rules, which are sequences of functions $w(\pi) = \{w(\pi_t)\}$, $r(\pi) = \{r(\pi_t)\}$, $R(\pi) = \{R(\pi_t)\}$ which map government policies π into the economy's price system.

[5] Compare: Atkeson, Andrew & Chari, Varadarajan V. & Kehoe, Patrick J., 1999. "Taxing Capital Income: A Bad Idea," Federal Reserve Bank of Minneapolis Quarterly Review, vol. 23(3), pp. 3-17.; Chari/Christiano/Kehoe 1994, pp. 617-652.

A Ramsey equilibrium is thus a policy π, an allocation rule $x(\pi)$ as well as price rules $w(\pi)$, $r(\pi)$ and $R(\pi)$, such that

1. π maximizes (2.1.1), subject to the government's budget constraint (2.1.4) taking allocations and prices as given by $x(\pi)$, $w(\pi)$, $r(\pi)$ and $R(\pi)$.

2. The utility (2.1.1) is maximized subject to the household's budget constraint (2.1.5) by the allocation rule $x(\pi')$, for every π' and it's corresponding prices $w(\pi')$, $r(\pi')$ and $R(\pi')$.

3. Prices $w(\pi')$ and $r(\pi')$ satisfy (2.1.33) and (2.1.34) for every π'.

The conducted definition above establishes a competitive Ramsey equilibrium. Should there be multiple competitive equilibria which correspond to a policy, the one which yields the highest utility is selected. Further it is assumed for the moment that θ_0 is fixed. Otherwise no restrictions on θ are imposed at the moment. k_0 and b_0 are assumed to be given.

2.3 The zero capital tax

In the following section the zero capital tax result is derived which is central to the works of Chamley and Judd. Along the formulation of Chamley, it is assumed that the government sets the taxes indirectly by net of taxes rental rates for capital and labor respectively, denoted by:

$$\bar{r}_t \equiv (1-\theta_t)r_t \rightarrow \theta_t r_t \equiv r_t - \bar{r}_t \tag{2.3.1}$$

$$\bar{w}_t \equiv (1-\tau_t)w_t \rightarrow \tau_t w_t \equiv w_t - \bar{w}_t \tag{2.3.2}$$

Next the firms first order conditions, expressions (2.1.33) and (2.1.34), and the definitions from above for net of taxes factor prices are used to express government's tax revenues.

$$\begin{aligned} \theta_t r_t k_t + \tau_t w_t h_t &= (r_t - \bar{r}_t)k_t + (w_t - \bar{w}_t)h_t = (F_k(k_t,h_t) - \bar{r}_t)k_t + (F_h(k_t,h_t) - \bar{w}_t)h_t \\ &= F_k(k_t,h_t)k_t + F_h(k_t,h_t)h_t - \bar{r}_t k_t - \bar{w}_t h_t \end{aligned} \tag{2.3.3}$$

Since the production technology, as assumed in subsection 2.1.1, exhibits constant returns to scale, hence $F(k_t,h_t) = F_k(k_t,h_t)k_t + F_h(k_t,h_t)h_t$, expression (2.3.3) can be further rewritten to give:

$$\theta_t r_t k_t + \tau_t w_t h_t = F(k_t,h_t) - \bar{r}_t k_t - \bar{w}_t h_t \tag{2.3.4}$$

The government's policy choice in order to maximize aggregate welfare is constrained by its budget constraint (2.1.4), the aggregate resource constraint (2.1.2) and the household's first order conditions derived before, (2.1.10) to (2.1.12). The problem of the Ramsey planner can be written down in the form of a Lagrangian. The net of taxes factor prices are substituted for according to definitions (2.3.1) and (2.3.2).

$$\begin{aligned}\mathcal{L} = \sum_{t=0}^{\infty}\beta^t \Big\{ & u(c_t, h_t) \\ & +\psi_t \left[F(k_t, h_t) - \bar{r}_t k_t - \bar{w}_t h_t + \frac{b_{t+1}}{R_t} - b_t - g_t\right] \\ & +\lambda_t \left[F(k_t, h_t) + (1-\delta)k_t - c_t - g_t - k_{t+1}\right] \\ & +\mu_{1t} \left[u_h(c_t, h_t) + u_c(c_t, h_t)\bar{w}_t\right] \\ & +\mu_{2t} \left[u_c(c_t, h_t) - \beta u_c(c_{t+1}, h_{t+1})(\bar{r}_{t+1} + 1 - \delta)\right]\Big\} \end{aligned} \tag{2.3.5}$$

The variables ψ_t, λ_t and μ_{1t}, μ_{2t} are the Lagrange multipliers associated to the government's budget constraint (the tax revenues were replaced by expression (2.3.4)), the economy's resource constraint and the household's first order conditions (here the net of taxes factor prices are replaced with expressions (2.3.1) and (2.3.1)) respectively. Note that the no-arbitrage condition (2.1.12), $R_t = (1-\theta_{t+1})r_{t+1} + (1-\delta) = \bar{r}_{t+1} + (1-\delta)$, holds. As the resource and the government's budget constraint are satisfied through the inclusion in the above maximization problem, it is not necessary to include the household's budget constraint: It is automatically satisfied by Walras' law. The main interest lies upon the the optimal choice for capital taxation which has direct consequences for the accumulation of capital. It is therefore sufficient to derive the first order condition associated to k_{t+1}. Note that dividends on government bonds are not taxed in this model.

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} : \beta^{t+1}\psi_{t+1}[F_k(k_{t+1}, h_{t+1}) - \bar{r}_{t+1}] + \beta^{t+1}\lambda_{t+1}[F_k(k_{t+1}, h_{t+1}) + (1-\delta)] - \beta^t\lambda_t = 0$$

$$\rightarrow$$

$$\lambda_t = \beta\{\psi_{t+1}[F_k(k_{t+1}, h_{t+1}) - \bar{r}_{t+1}] + \lambda_{t+1}[F_k(k_{t+1}, h_{t+1}) + (1-\delta)]\} \tag{2.3.6}$$

Equation (2.3.6) summarizes the culminate effect of a marginal investment in the factor capital. The term $[F_k(k_{t+1}, h_{t+1}) + (1-\delta)]$ represents the quantitative increase of available goods in the economy in period $t+1$ by a marginal capital investment in period t. The social marginal value of this increase is represented by the Lagrange multiplier λ_{t+1}. The term $[F_k(k_{t+1}, h_{t+1}) - \bar{r}_{t+1}]$ gives the increase in capital tax revenues in period $t+1$ caused by the additional capital investment in period t, allowing to lower other taxes, in our model the tax on labor income τ. The social marginal value of an increase in tax revenues or of a possible reduction of other taxation instruments is denoted by the Lagrange multiplier ψ_{t+1}. Both period $t+1$ effects of the increase in capital investment in period t are discounted by the factor β. They equal the social marginal value of capital investment in the initial period t, represented by the Lagrange multiplier λ_t. It is now supposed that the model economy converges to a steady state after a hypothetical period T. Examining this model after this time period, all endogenous variables can be set constant and the time subscripts can be omitted. The first order condition associated to capital k_{t+1} for periods $t \geqslant T$ can thus be rewritten to:

$$\lambda = \beta\{\psi[F_k(k,h) - \bar{r}] + \lambda[F_k(k,h) + (1-\delta)]\} \tag{2.3.7}$$

Substituting by $F_k(k,h) = r$ from (2.1.33) in equation (2.3.7) allows to further rewrite it.

$$\lambda = \beta\{\psi[r - \bar{r}] + \lambda[r + (1-\delta)]\} \tag{2.3.8}$$

The assumption of a steady state from period T on and the identity $\bar{r}_t \equiv (1-\theta_t)r_t$ (see identity 2.3.1) allows to also rewrite the household's Euler equation (2.1.11).

$$\begin{aligned} &u_c(c,h) = \beta u_c(c,h)[(1-\theta)r + (1-\delta)] \rightarrow \\ &1 = \beta[\bar{r} + (1-\delta)] \end{aligned} \tag{2.3.9}$$

Combining (2.3.9) with (2.3.8) finally yields:

$$\begin{aligned} &\lambda[\bar{r} + (1-\delta)] = \psi[r - \bar{r}] + \lambda[r + (1-\delta)] \rightarrow \lambda[\bar{r} - r] = \psi[r - \bar{r}] \rightarrow \\ &(\lambda + \psi)(\bar{r} - r) = 0 \end{aligned} \tag{2.3.10}$$

The marginal social value of goods available to the economy, represented by the Lagrange multiplier λ, is strictly positive. The marginal social value of a reduction of taxes ψ is nonnegative. Therefore to satisfy equation (2.3.10), in steady state it has to hold that $\bar{r} = r$. Following from the identity $\bar{r} \equiv (1-\theta)r$ (2.3.1), the tax rate on capital income θ consequently has to be zero when the economy has converged to a steady state. An interpretation of this result will be discussed later on in this chapter. It is important to note that a balanced budget assumption would not have changed the result. This can be easily seen by setting the terms b_t and b_{t+1} to zero in equation (2.3.5).

2.4 Capital taxation with heterogenous agents and redistribution

2.4.1 An Economy with a finite number of heterogenous agents

In contrast to before it is now assumed that the economy consists of a finite number of heterogenous rather than identical households. It will be shown that the result of a zero tax on capital income in steady state also holds in such an environment. Let there be a finite number N of classes i. Each class is assumed to have the same size. The control variables concerning class i are denoted with a superscripted i: c_t^i, h_t^i, k_t^i, b_t^i. The utility function for class i is hence denoted $u^i(c_t^i, h_t^i)$. The government has the possibility of lump sum transfers $S_t^i \geqslant 0$ to households of type i. This lump-sum transfer consequently enters the government's as well as the household's budget constraint. It is further assumed that the government attributes weights $a^i \geqslant 0$ to the welfare of classes $i = 1, \ldots, N$. All classes share the same discount factor β. Aggregate values are denoted as $x_t \equiv \sum_{i=1}^N x_t^i$ for $x = c, h, k, b, S$. Approach and notation are otherwise the same as in section 2.3. The government problem, analogously to before, can now be written down in the form of a Lagrangian. Since it is assumed that the economy consists of heterogenous households an inclusion of the government's budget constraint and the resource constraint alone is not sufficient. The budget constraints of all N households have to be included as well as their optimality conditions.

$$\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \sum_{i=1}^{N} a^i u^i(c_t^i, h_t^i) \\
& + \psi_t \left[F(k_t, h_t) - \bar{r}_t k_t - \bar{w}_t h_t + \frac{b_{t+1}}{R_t} - b_t - g_t - S_t \right] \\
& + \lambda_t \left[F(k_t, h_t) + (1-\delta) k_t - c_t - g_t - k_{t+1} \right] \\
& + \sum_{i=1}^{N} \epsilon_t^i \left[\bar{w}_t h_t^i + \bar{r}_t k_t^i + (1-\delta) k_t^i + b_t^i + S_t^i - c_t^i - k_{t+1}^i - \frac{b_{t+1}^i}{R_t} \right] \\
& + \sum_{i=1}^{N} \mu_{1t}^i \left[u_h^i(c_t^i, h_t^i) + u_c^i(c_t^i, h_t^i) \bar{w}_t \right] \\
& + \sum_{i=1}^{N} \mu_{2t}^i \left[u_c^i(c_t^i, h_t^i) - \beta u_c^i(c_{t+1}^i, h_{t+1}^i)(\bar{r}_{t+1} + 1 - \delta) \right] \Bigg\}
\end{aligned} \tag{2.4.1}$$

The variables ψ_t, λ_t, ϵ_t^i and μ_{1t}^i, μ_{2t}^i are the Lagrange multipliers associated to the government's budget constraint, the economy's resource constraint, class $i's$ budget constraint and the first order conditions of class i respectively. The tax revenues in the government's budget constraint were replaced along a version of equation (2.3.4) which fits into the here examined model environment: $\theta_t r_t k_t^i + \tau_t w_t h_t^i = F(k_t^i, h_t^i) - \bar{r}_t k_t^i - \bar{w}_t h_t^i$. The net of taxes factor prices were replaced along expressions (2.3.1) and (2.3.2). Note that the no-arbitrage condition (2.1.12), $R_t = (1-\theta_{t+1})r_{t+1} + (1-\delta) = \bar{r}_{t+1} + (1-\delta)$, holds. To better illustrate the conditions which refer to aggregate values or to a class i respectively the Lagrangian (2.4.1) can also be written in the form:

$$\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \sum_{i=1}^{N} a^i u^i(c_t^i, h_t^i) \\
& + \psi_t \left[F(\sum_{i=1}^{N} k_t^i, \sum_{i=1}^{N} h_t^i) - \bar{r}_t \left[\sum_{i=1}^{N} k_t^i \right] - \bar{w}_t \left[\sum_{i=1}^{N} h_t^i \right] + \frac{\left[\sum_{i=1}^{N} b_{t+1}^i\right]}{R_t} - \left[\sum_{i=1}^{N} b_t^i \right] - g_t - \left[\sum_{i=1}^{N} S_t^i \right] \right] \\
& + \lambda_t \left[F(\sum_{i=1}^{N} k_t^i, \sum_{i=1}^{N} h_t^i) + (1-\delta) \left[\sum_{i=1}^{N} k_t^i \right] - \left[\sum_{i=1}^{N} c_t^i \right] - g_t - \left[\sum_{i=1}^{N} k_{t+1}^i \right] \right] \\
& + \sum_{i=1}^{N} \epsilon_t^i \left[\bar{w}_t h_t^i + \bar{r}_t k_t^i + (1-\delta) k_t^i + b_t^i + S_t^i - c_t^i - k_{t+1}^i - \frac{b_{t+1}^i}{R_t} \right] \\
& + \sum_{i=1}^{N} \mu_{1t}^i \left[u_h^i(c_t^i, h_t^i) + u_c^i(c_t^i, h_t^i) \bar{w}_t \right] \\
& + \sum_{i=1}^{N} \mu_{2t}^i \left[u_c^i(c_t^i, h_t^i) - \beta u_c^i(c_{t+1}^i, h_{t+1}^i)(\bar{r}_{t+1} + 1 - \delta) \right] \Bigg\}
\end{aligned} \tag{2.4.2}$$

Deriving the first order order condition associated to k_{t+1}^i and substituting along (2.1.33) yields (the calculation steps are merely the same as before and are therefore omitted):

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}^i} : \lambda_t + \epsilon_t^i = \beta\{\psi_{t+1}[r_{t+1} - \bar{r}_{t+1}] + \lambda_{t+1}[r_{t+1} + (1-\delta)] + \epsilon_{t+1}^i[\bar{r}_{t+1} + (1-\delta)]\} \tag{2.4.3}$$

It is sassumed that the economy converges to a steady state with a hypothetical period T. Hence time subscripts can be omitted for periods $t \geqslant T$. After some manipulations the steady state version of equation (2.4.3) can be written as:

$$\lambda + \epsilon^i[1 - \beta(\bar{r} + (1-\delta))] = \beta[\psi(r - \bar{r}) + \lambda(r + (1-\delta))] \tag{2.4.4}$$

As in the last section, under the assumption of a steady state and using (2.3.1) Euler equation (2.1.11) can be rewritten to the form (compare to 2.3.9):

$$1 = \beta[\bar{r} + (1 - \delta)] \tag{2.4.5}$$

Equations (2.4.4) and (2.4.5) can now be merged. The resulting condition applies to all households and therefore all classes in the steady state. Consequently the term with ϵ^i, which represents class $i's$ social marginal value for an increase in the quantity of available goods in period $t + 1$ caused by a marginal increase of capital investment in period t, disappears. After some manipulations the resulting expression is the same as before in the last section(compare (2.3.10)):

$$(\lambda + \psi)(\bar{r} - r) = 0 \tag{2.4.6}$$

Since λ is strictly positive and ψ is nonnegative, as already argued in the section before, to satisfy above expression (2.4.6) it has to hold that $\bar{r} = r$ and hence that $\theta = 0$. Hence, in an economy which has converged to a steady state, with a finite number of heterogenous agents and the possibility of redistribution via lump sum transfers, the optimal Pareto efficient tax policy is to set taxes on labor income to zero.

2.4.2 An economy with capitalists and workers

In his paper from 1985, on which the here done examinations partly base, Kenneth Judd[6] assumes an extreme case of heterogeneity; there are two classes , workers and capitalists, where the former group doesn't save and consequently owns no capital or bonds, while the latter group doesn't work but owns all of the assets within the economy. The budget constraints for the two respective groups captures this. The budget constraint for the working class, here labeled as class 1, reads as follows:

$$c_t^1 = (1 - \tau_t)w_t h_t^1 + S_t^1 \tag{2.4.7}$$

The budget constraint for the capitalists class is:

$$c_t^2 + k_{t+1}^2 + \frac{b_{t+1}^2}{R_t} = (1 - \theta_t)r_t k_t^2 + (1 - \delta)k_t^2 + b_t^2 + S_t^2. \tag{2.4.8}$$

[6] Judd 1985

Consequently both groups, workers and capitalists respectively, also have different utility functions. As before they however share the same discount rate β^t.

$$\sum_{t=0}^{\infty} \beta^t u(c_t^1, h_t^1) \tag{2.4.9}$$

$$\sum_{t=0}^{\infty} \beta^t u(c_t^2) \tag{2.4.10}$$

Workers maximize their utility (2.4.9) subject to their budget constraint (2.4.7) with respect to $\{c_t^1, h_t^1\}_{t=0}^{\infty}$. Capitalists maximize their utility (2.4.10) subject to their budget constraint (2.4.8) with respect to $\{c_t^2, k_{t+1}^2, b_{t+1}^2\}_{t=0}^{\infty}$. The maximization problems of workers and capitalists respectively written in Lagrangian form therefore read as follows.

workers

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t^1, h_t^1) + \lambda_t \left[(1-\tau_t) w_t h_t^1 + S_t^1 - c_t^1 \right] \right\} \tag{2.4.11}$$

Taking the partial derivatives of $\mathcal{L}$, (2.4.11), with respect to $\{c_t^1, h_t^1\}_{t=0}^{\infty}$ yields:

$$\frac{\partial \mathcal{L}}{\partial c_t^1} : u_c(c_t^1, h_t^1) = \lambda_t \tag{2.4.12}$$

$$\frac{\partial \mathcal{L}}{\partial h_t^1} : u_h(c_t^1, h_t^1) = -\lambda_t (1-\tau_t) w_t \tag{2.4.13}$$

Combining first order conditions (2.4.12) and (2.4.13) and using the definition $\bar{w}_t \equiv (1-\tau_t) w_t$, compare (2.3.2), yields the working class's sole optimality condition:

$$u_h(c_t^1, h_t^1) = -u_c(c_t^1, h_t^1) \bar{w}_t \tag{2.4.14}$$

capitalists

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t^2) + \lambda_t \left[(1-\theta_t) r_t k_t^2 + (1-\delta) k_t^2 + b_t^2 + S_t^2 - c_t^2 - k_{t+1}^2 - \frac{b_{t+1}^2}{R_t} \right] \right\} \tag{2.4.15}$$

Taking the partial derivatives of $\mathcal{L}$, (2.4.15), with respect to $\{c_t^2, k_{t+1}^2, b_{t+1}^2\}_{t=0}^{\infty}$ yields:

$$\frac{\partial \mathcal{L}}{\partial c_t^2} : u_c(c_t^2) = \lambda_t \tag{2.4.16}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}^2} : \lambda_t = \beta \lambda_{t+1} \{(1 - \theta_{t+1}) r_{t+1} + (1 - \delta)\} \tag{2.4.17}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}^2} : \lambda_t = \beta \lambda_{t+1} R_t \tag{2.4.18}$$

By combination of first order conditions (2.4.16) and (2.4.17), (2.4.16) and (2.4.18) as well as (2.4.17) and (2.4.18) one receives the optimality conditions concerning the capitalist's class maximization problem. Definition $\bar{r}_t \equiv (1 - \theta_t) r_t$, compare (2.3.1), is used to substitute for the net of taxes return on capital.

$$u_c(c_t^2) = \beta u_c(c_{t+1}^2)[\bar{r}_{t+1} + (1 - \delta)] \tag{2.4.19}$$

$$u_c(c_t^2) = \beta u_c(c_{t+1}^2) R_t \tag{2.4.20}$$

$$R_t = \bar{r}_{t+1} + (1 - \delta) \tag{2.4.21}$$

The optimality condition (2.4.21) is equivalent to the well known no-arbitrage condition (2.1.12). As (2.4.21) holds only one of the two other capitalist's optimality conditions has to be included in the government's problem. The problem of the government now is to maximize aggregate utility subject to the government's budget constraint (2.1.4), the resource constraint (2.1.2), the optimality conditions of the workers and capitalists respectively, (2.4.14) and (2.4.19) to (2.4.21), as well as the budget constraints of these two classes, (2.4.7) and (2.4.8). Aggregate values are denoted, as before, as $x_t = x_t^1 + x_t^2$ for $x = c, h, k, b, S$. The government attributes the weights $a^1 \geqslant 0$ and $a^2 \geqslant 0$ to the welfare of workers and capitalists respectively. The government's problem can be denoted in the form of a Lagrangian:

$$
\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t & \left\{ a^1 u(c_t^1, h_t^1) + a^2 u(c_t^2) \right. \\
& + \psi_t \left[F(k_t, h_t) - \bar{r}_t k_t - \bar{w}_t h_t + \frac{b_{t+1}}{R_t} - b_t - g_t - S_t \right] \\
& + \lambda_t \left[F(k_t, h_t) + (1-\delta) k_t - c_t - g_t - k_{t+1} \right] \\
& + \epsilon_t^1 \left[\bar{w}_t h_t^1 + S_t^1 - c_t^1 \right] \\
& + \epsilon_t^2 \left[\bar{r}_t k_t^2 + (1-\delta) k_t^2 + b_t^2 + S_t^2 - c_t^2 - k_{t+1}^2 - \frac{b_{t+1}^2}{R_t} \right] \\
& + \mu_t^1 \left[u_h(c_t^1, h_t^1) + u_c(c_t^1, h_t^1) \bar{w}_t \right] \\
& \left. + \mu_t^2 \left[u_c(c_t^2) - \beta u_c(c_{t+1}^2) [\bar{r}_{t+1} + (1-\delta)] \right] \right\}
\end{aligned}
$$

The variables ψ_t and λ_t are the Lagrange multipliers associated to the government's budget constraint and the resource constraint respectively. ϵ_t^1 and ϵ_t^2 are the Lagrange multipliers for the budget constraint of workers and capitalists respectively. μ_t^1 and μ_t^2 are the Lagrange multipliers associated to the optimality conditions of workers and capitalists respectively. Deriving the government's first order condition associated to k_{t+1}^2 and using $F_k(k_t, h_t) = r_t$, compare (2.1.33), yields:

$$
\frac{\partial \mathcal{L}}{\partial k_{t+1}^2} : \lambda_t + \epsilon_t^2 = \beta \{ \psi_{t+1} [r_{t+1} - \bar{r}_{t+1}] + \lambda_{t+1} [r_{t+1} + (1-\delta)] + \epsilon_{t+1}^2 [\bar{r}_{t+1} + (1-\delta)] \} \tag{2.4.22}
$$

It is now assumed that the economy converges to a steady state from the period T on. Hence the government's first order condition (2.4.22) and the capitalists optimality condition (2.4.19) for periods $t \geqslant T$ after some manipulations can be denoted in the form of equations (2.4.23) and (2.4.24) respectively. The calculation steps are the same as in the two sections before and are therefore omitted.

$$
\lambda + \epsilon^2 [1 - \beta(\bar{r} + 1 - \delta)] = \beta [\psi(r - \bar{r}) + \lambda(r + 1 - \delta)] \tag{2.4.23}
$$

$$
1 = \beta(\bar{r} + (1-\delta)) \tag{2.4.24}
$$

Combination of (2.4.23) and (2.4.24) and some manipulations finally yields the well known expression[7]:

$$
(\lambda + \psi)(\bar{r} - r) = 0 \tag{2.4.25}
$$

[7] compare to (2.4.6) and(2.3.10)

Since λ, the marginal social value of available goods, is strictly positive and ψ, the marginal social value of reducing taxes is nonnegative, it has to hold that $\bar{r} = r$, hence that $\theta = 0$ once the economy converged to a steady state. A benevolent social planner maximizing a paretian social welfare function for a heterogenous society with the possibility of lump sum transfers would therefore always set capital taxes to zero in the limiting steady state. This is also true if the planner would only attribute weight to the welfare of workers, hence if he would set $a^1 = 1$ and $a^2 = 0$. Like in the case before it has no influence on the result of a limiting zero capital tax if one allows for government debt or not.

2.5 Capital taxation aside the steady state

2.5.1 The household's problem in a present value formulation

Hitherto tax policy was just analyzed under the assumption that the economy had converged to a steady state. To examine tax policy aside the steady state one has to choose an approach which is slightly different to the one used in the sections above. Namely, we are going to hark back to the present value formulation of the household's problem from section 2.1.3. The household's present value budget constraint (2.1.18) can be rewritten using Arrow-Debreu prices.

$$q_t^0 = \prod_{i=0}^{t-1} R_i^{-1} \quad \forall t \geqslant 1; \quad \text{with the numeraire} \quad q_0^0 = 1 \tag{2.5.1}$$

The Arrow-Debreu price q_t^0 here, in a deterministic version of the model, denotes the price of consumption and net-of-taxes-labor-income in period t in terms of period zero. With (2.5.1) at hand the present value budget constraint (2.1.18) can be denoted as follows.

$$\sum_{t=0}^{\infty} q_t^0 c_t = \sum_{t=0}^{\infty} q_t^0 (1 - \tau_t) w_t h_t + [(1 - \theta_0) r_0 + (1 - \delta)] k_0 + b_0 \tag{2.5.2}$$

Accordingly, the household's maximization problem in a present value formulation reads:

$$\max_{c_t,h_t} \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$
$$s.t.$$
$$\sum_{t=0}^{\infty} q_t^0 c_t = \sum_{t=0}^{\infty} q_t^0 (1-\tau_t) w_t h_t + [(1-\theta_0) r_0 + (1-\delta)] k_0 + b_0$$

In the present value formulation one has to control for consumption and labor supply only. The initial values k_0 and b_0 are assumed to be given and capital and bond holdings in the limit are prescribed by the transversality conditions (2.1.16) and (2.1.17) respectively derived in subsection 2.1.3. As the budget constraints households are facing in all future periods are given in present value terms the discount factor β only applies to the utility function and the time subscripts on the Lagrange multiplier can be omitted. The Lagrangian formulation of the household's maximization problem hence reads as follows:

$$\mathcal{L} = \sum_{t=0}^{\infty} \left\{ \beta^t u(c_t, h_t) + \lambda \left[q_t^0 (1-\tau_t) w_t h_t - q_t^0 c_t \right] \right\} + [(1-\theta_0) r_0 + 1 - \delta] k_0 + b_0 \tag{2.5.3}$$

Taking the partial derivatives with respect to c_t and h_t yields the household's first order conditions.

$$\frac{\partial \mathcal{L}}{\partial c_t} : \beta^t u_c(c_t, h_t) = \lambda q_t^0 \tag{2.5.4}$$
$$\frac{\partial \mathcal{L}}{\partial h_t} : (1-\tau_t) w_t = -\frac{\beta^t u_h(c_t, h_t)}{\lambda q_t^0} \tag{2.5.5}$$

Combining (2.5.4) and (2.5.5) yields:

$$(1-\tau_t) w_t = -\frac{u_h(c_t, h_t)}{u_c(c_t, h_t)} \tag{2.5.6}$$

Rewriting (2.5.4) for period $t = 0$ and using the the definition of the numeraire $q_0^0 = 1$ yields:

$$\lambda = u_c(c_0, h_0) \tag{2.5.7}$$

Substituting for λ by expression (2.5.7) allows to rewrite the original first order condition (2.5.4) to the form of:

$$\frac{\beta^t u_c(c_t, h_t)}{u_c(c_0, h_0)} = q_t^0 \tag{2.5.8}$$

Expressions (2.5.6) and (2.5.8) represent the relevant household's optimality conditions in this formulation of the problem. Analogously to (2.1.12) the no-arbitrage conditions here reads as:

$$\frac{q_t^0}{q_{t+1}^0} = \frac{\prod_{i=0}^{t-1} R_i^{-1}}{\prod_{i=0}^{t} R_i^{-1}} = R_t = (1-\theta_{t+1})r_{t+1} + (1-\delta) \tag{2.5.9}$$

All conditions and constraints concerning the household's problem, which will be relevant for the government in maximizing overall welfare in the next step, can now be collapsed into one single constraint, the so called implementability constraint. Therefore the optimality conditions (2.5.6) and (2.5.8) are substituted into the present value budget constraint (2.5.2) to give.

$$\sum_{t=0}^{\infty} \beta^t \frac{u_c(c_t, h_t)}{u_c(c_0, h_0)} c_t = -\sum_{t=0}^{\infty} \beta^t \frac{u_c(c_t, h_t)}{u_c(c_0, h_0)} \frac{u_h(c_t, h_t)}{u_c(c_t, h_t)} h_t + [(1-\theta_0)r_0 + (1-\delta)]k_0 + b_0 \tag{2.5.10}$$

After some manipulations and substituting along $F_k(k_0, h_0) = r_0$, (2.5.10) can be rewritten to:

$$\sum_{t=0}^{\infty} \beta^t [u_c(c_t, h_t)c_t + u_h(c_t, h_t)h_t] - A = 0 \tag{2.5.11}$$

with

$$A = A(c_0, h_0, \theta_0) = u_c(c_0, h_0)\,\{[F_k(k_0, h_0)(1-\theta_0) + (1-\delta)]k_0 + b_0\}$$

2.5.2 The problem of the Ramsey planner in primal formulation

In the primal formulation of the Ramsey problem the planner chooses allocations consistent with it's budget constraint, the resource constraint, firms optimization behavior and household's optimization behavior, which was sketched in the preceding subsection 2.5.1. The proceeding is pretty much the same as before. In contrast to sections 2.3 and 2.4 however factor prices and tax rates are not summarized in net-of-taxes-factor prizes. Referring to here examined model economy the Ramsey planner maximizes the social welfare function (2.1.1), subject to the implementability constraint (2.5.11), which was derived in the last subsection, and the economy's resource constraint (2.1.2) with respect to $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$. Since the household's budget constraint and the resource constraint are included in the maximization problem it is not necessary and would be redundant to include the government's budget constraint. It is automatically satisfied through Walras' law. One can now define the function:

$$V(c_t, h_t, \phi) = u(c_t, h_t) + \phi[u_c(c_t, h_t)c_t + u_h(c_t, h_t)h_t] \tag{2.5.12}$$

The variable ϕ is the Lagrange multiplier associated to the implementability constraint (2.5.11). The government's problem in Lagrangian form reads:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{V(c_t, h_t, \phi) + \lambda_t[F(k_t, h_t) + (1-\delta)k_t - c_t - g_t - k_{t+1}]\} - \phi A \tag{2.5.13}$$

b_0 and k_0 are assumed to be given and θ_0 to be fixed. λ_t is the Lagrange multiplier of the economy's resource constraint. Taking the partial derivatives of $\mathcal{L}$ with respect to $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$ yields the first order conditions of the government's optimization problem:

$$\frac{\partial \mathcal{L}}{\partial c_t} : V_c(c_t, h_t, \phi) = \lambda_t \quad t \geqslant 1 \tag{2.5.14}$$

$$\frac{\partial \mathcal{L}}{\partial h_t} : V_h(c_t, h_t, \phi) = -\lambda_t F_h(k_t, h_t) \quad t \geqslant 1 \tag{2.5.15}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} : \lambda_t = \beta \lambda_{t+1}[F_k(k_{t+1}, h_{t+1}) + (1-\delta)] \tag{2.5.16}$$

$$\frac{\partial \mathcal{L}}{\partial c_0} : V_c(c_0, h_0, \phi) = \lambda_0 + \phi A_c \tag{2.5.17}$$

$$\frac{\partial \mathcal{L}}{\partial h_0} : V_h(c_0, h_0, \phi) = -\lambda_0 F_h(k_0, h_0) + \phi A_h \tag{2.5.18}$$

Merging expressions (2.5.14), (2.5.16) and (2.5.17); (2.5.14) and (2.5.16); (2.5.14) and (2.5.15); as well as (2.5.17) and (2.5.18) yields the government's optimality conditions respectively:

$$V_c(c_0, h_0, \phi) - \phi A_c = \beta V_c(c_1, h_1, \phi)[F_k(k_1, h_1) + (1-\delta)] \tag{2.5.19}$$

$$V_c(c_t, h_t, \phi) = \beta V_c(c_{t+1}, h_{t+1}, \phi)[F_k(k_{t+1}, h_{t+1}) + (1-\delta)] \quad t \geqslant 1 \tag{2.5.20}$$

$$V_h(c_t, h_t, \phi) = -V_c(c_t, h_t, \phi) F_h(k_t, h_t) \quad t \geqslant 1 \tag{2.5.21}$$

$$V_h(c_0, h_0, \phi) = [\phi A_c - V_c(c_0, h_0, \phi)] F_h(k_0, h_0) + \phi A_h \tag{2.5.22}$$

Optimality conditions (2.5.19) to (2.5.22) together with the implementability constraint (2.5.11) and the economy's resource constraint (2.1.2) would now allow to solve for allocations $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$ and the Lagrange multiplier ϕ. This in turn would allow to solve for prices r_t, w_t and q_t^0 as well as taxes τ_t and θ_t. As the main interest however lies on tax policies only, there are other, easier ways to find solutions.

2.5.3 Capital taxation over the dynamic path

Assuming that the economy converges to a steady state for $t \geqslant T \geqslant 0$ yields exactly the same result as before: In the steady state the variables g_t, c_t, h_t and k_t are time invariant, hence the term $V_c(.)$ is constant. For $t \geqslant T \geqslant 0$ the optimality condition (2.5.20) can thus be rewritten to

$$1 = \beta[F_k(k,h) + (1-\delta)] \tag{2.5.23}$$

Since c_t is assumed to be invariant in the steady state household's first order condition (2.5.8) for $t \geqslant T$ implies that $\frac{q_t^0}{q_{t+1}^0} = \frac{1}{\beta}$. This together with the no-arbitrage condition (2.5.9) gives:

$$1 = \beta[(1-\theta)r + (1-\delta)] \tag{2.5.24}$$

Merging expressions (2.5.23) and (2.5.24) finally yields expression,

$$F_k(k,h) = (1-\theta)r \tag{2.5.25}$$

which can only be satisfied if $\theta = 0$. This result is equivalent to the these derived in the preceding sections: The tax rate on capital income is zero in the steady state. Examining the implications of the taken steady state assumption however delivers a more pronounced result: Since c_t, h_t and k_{t+1} are constant in a steady state it was postulated that $V_c(.)$ has to be constant too. Recalling equation (2.5.12) it is easy to see that it was thereby implicitly assumed that the expenditure elasticity,

$$\frac{V_c(c_t,h_t,\phi)}{u_c(c_t,h_t)} = 1 + \phi[1 + (u_{cc}(c_t,h_t)c_t + u_{hc}(c_t,h_t)h_t)/u_c(c_t,h_t)] \tag{2.5.26}$$

is also constant within a steady state equilibrium, hence:

$$\frac{V_c(c_t,h_t,\phi)}{u_c(c_t,h_t)} = \frac{V_c(c_{t+1},h_{t+1},\phi)}{u_c(c_{t+1},h_{t+1})} \tag{2.5.27}$$

Equality (2.5.27) implies that equation (2.5.20) can be rewritten to:

$$u_c(c_t,h_t) = \beta u_c(c_{t+1},h_{t+1})[F_k(k_{t+1},h_{t+1}) + (1-\delta)] \tag{2.5.28}$$

Combining household's first order condition (2.5.8) and no-arbitrage condition (2.5.9) yields,

$$\frac{q_t^0}{q_{t+1}^0} = \frac{1}{\beta}\frac{u_c(c_t,h_t)}{u_c(c_{t+1},h_{t+1})} = R_t \rightarrow$$
$$u_c(c_t,h_t) = \beta u_c(c_{t+1},h_{t+1})[(1-\theta_{t+1})r_{t+1} + (1-\delta)] \qquad (2.5.29)$$

Equations (2.5.28) and (2.5.29) in turn imply that capital income tax is optimally zero in period $t+1$: θ_{t+1}=0. This is however not necessarily true for period $t = 1$. Even if $V_c(c_0, h_0, \phi) = V_c(c_1, h_1, \phi)$, above established result doesn't hold as equation (2.5.19), which applies for periods $t = 0$ and $t = 1$, contains an additional term. What is the intuitive explanation for the zero capital tax in period $t+1$ especially in the context of our conclusion that capital taxation should be zero in a steady state? In an economy, as examined above, where capital accumulation comes solely from lifecycle savings, levying a capital income tax would imply a permanent intertemporal distortion as it would make consumption tomorrow pricier than consumption today: Taxing capital in period $t+1$ implies to implicitly tax private good consumption at a higher rate in period $t+1$ than in period t. This is equivalent to a permanently increasing implicit taxation of consumption which is not compatible with the assumption of a steady state equilibrium. The literature speaks in such a case of differential commodity taxation instead of a uniform commodity taxation which would be optimal. Referring to the result of a zero capital tax in the steady state, this means: When the economy converges to a steady state in period T, capital taxes are optimally zero for all periods $t \geqslant T+1$, or differently expressed, the decision to accumulate capital is not distorted for all periods $t \geqslant T$.[8] The interpretation of this results will be discussed in more detail in the next section where an alternative formal derivation of the Chamley-Judd result is conducted.

For utility functions of the class,

$$u(c_t,h_t) = \frac{(c_t h_t^{-\alpha})^{1-\sigma}}{1-\sigma} \qquad (2.5.30) \qquad\qquad u(c_t,h_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + V(h_t) \qquad (2.5.31)$$

it can directly be shown that the expenditure elasticity is constant for periods $t \geqslant 1$:

$$\frac{V_c(c_t,h_t,\phi)}{u(c_t,h_t)} = 1 + \phi(\alpha-1)(\sigma-1) \qquad (2.5.32) \qquad \frac{V_c(c_t,h_t,\phi)}{u(c_t,h_t)} = 1 + \phi(1-\sigma) \qquad (2.5.33)$$

Along the above derived results this means that for utility functions of the class (2.5.30) and (2.5.31) capital taxes optimally should be zero for all periods $t \geqslant 2$. By the same argumentation as before this is however not true for periods $t = 0$ and $t = 1$, since the optimality condition (2.5.19) which is dedicated to these periods contains an additional

[8] Atkeson/Chari/Kehoe 1999

term. This result implies that for utility function of the class (2.5.30) and (2.5.31) the decision to accumulate capital is only distorted for period $t = 1$, with an initially positive rate of taxation. The returns on capital received in period $t = 2$ already fall under the zero capital tax regime. To see this it is crucial to differentiate between the date the decision on capital accumulation is taken and the date this accumulated capital is subject to taxation: If there is a positive capital tax in period $t + 1$ capital accumulation is distorted in period t as its returns in the next period will be subject to capital taxation. It can therefore be stated: For the examined class of utility functions it is not optimal to distort capital accumulation for periods $t \geqslant 1$ and therefore capital taxes should be zero for periods $t \geqslant 2$.

To underline this hitherto just intuitively argued assumption that capital initially should be indeed taxed we turn back to the maximization problem of the government (2.5.13) in the last section. The assumption that θ_0 is fixed is relaxed. It is now assumed that the government could freely choose θ_0. To explore the impact of an increased $t = 0$ capital tax rate on social welfare, the Lagrange function $\mathcal{L}$ (2.5.13) is derived with respect to θ_0.

$$\frac{\partial \mathcal{L}}{\partial \theta_0} = -\phi \frac{\partial A}{\partial \theta_0} = \phi u_c(c_0, h_0) F_k(k_0, h_0) k_0 \tag{2.5.34}$$

The sign of the term (2.5.34) hinges on the sign of ϕ which can be interpreted as the social cost of financing government expenditures by the available financing instruments. If the government doesn't have to resort to distortionary financing means this value would be zero, otherwise it is strictly positive. The first scenario can be obviously neglected since both financing instruments the government has at its disposal, capital and labor taxes, are distortionary. What however is the logic behind this derivation to be positive and hence that taxation in the initial period could have positive welfare effects? Note that ϕ is not only the Lagrange multiplier for A but also for the term $[u_c(c_t, h_t)c_t + u_h(c_t, h_t)h_t]$ from the function (2.5.12) which measures the distortions induced by taxation on labor and capital. In period $t = 0$ this term is however not relevant. The idea is now that the government could use high levies on initial capital in order to build up a capital base which allows to to lower the need of distortionary taxation in the future. Along this rationality the government should lower ϕ, the social cost of financing government expenditures, permanently by a short period of very high capital taxes: Following the argumentation in the last paragraph, taxation of initial capital implies no intertemporal distortions. Interest earnings of the capital base built up in the high taxation regime can thereafter be used to at least partially finance government expenditures and substitute for distortionary taxation.

The original paper by Chamley[9] comes to similar if also qualitatively different results

[9] Chamley 1985

to what was derived above: Along Chamley for utility functions of the class (2.5.30) and (2.5.31), capital taxes are positive and constant until they converge to zero after a finite number of periods. This difference hinges on the fact that Chamley imposes an upper limit for capital taxes as an additional constraint in his model. Following Atkeson/Chari/Kehoe[10] Chamley's result shall be reproduced subsequently in a discrete time environment.

As in the original paper by Chamley an upper bound on capital taxes is introduced: $\theta_t \leqslant 1$. Using household's optimality condition (2.5.8) and the no-arbitrage condition (2.5.9) allows to express θ_{t+1}.

$$1 - \left[\frac{u_c(c_t, h_t)}{\beta u_c(c_{t+1}, h_{t+1})} - (1-\delta)\right] \frac{1}{r_{t+1}} = \theta_{t+1} \tag{2.5.35}$$

Imposing the constraint $\theta_{t+1} \leqslant 1$ on expression (2.5.35) and rearranging it yields the constraint.

$$u_c(c_t, h_t) \geqslant \beta u_c(c_{t+1}, h_{t+1})(1-\delta) \tag{2.5.36}$$

This inequality has to be included into the problem of the Ramsey planner as an additional constraint. What does it do? This additional constraint opens up the option to agents to hold their capital without renting it out to firms. This can be understood as a kind of possibility for tax evasion in the case taxes are too high. Agents can hide their capital in order to escape confiscation, as it was shown before the planner has clear incentives to confiscate capital in the initial period, and are then not subject to capital taxation. The constraint can therefore be understood as a lower bound of after-tax returns on capital at $1-\delta$.

The Ramsey problem can be as usually written down in Lagrangian form, where γ_t represents the Lagrange multiplier on the additional constraint (2.5.36). The rest stays the same as in the case without the upper bound on capital taxation.

$$\begin{aligned}\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{&V(c_t, h_t, \phi) + \lambda_t[F(k_t, h_t) + (1-\delta)k_t - c_t - g_t - k_{t+1}] \\ &+\gamma_t[u_c(c_t, h_t) - u_c(c_{t+1}, h_{t+1})(1-\delta)]\} - \phi A\end{aligned} \tag{2.5.37}$$

The first order conditions with respect to consumption and capital respectively are there-

[10]Atkeson/Chari/Kehoe 1999

fore:

$$\frac{\partial \mathcal{L}}{\partial c_t} : \lambda_t = V_c(c_t, h_t, \phi) + [\gamma_t - \gamma_{t-1}(1-\delta)]u_{cc}(c_t, h_t) \tag{2.5.38}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} : \lambda_t = \beta\lambda_{t+1}[F_k(k_{t+1}, h_{t+1}) + (1-\delta)] = 0 \tag{2.5.39}$$

The argumentation of Chamley[11] and Atkeson/Chari/Kehoe[12] hinges on three assumptions:

1. It is impossible that the constraint (2.5.36) doesn't bind in period t, starts to bind in period $t+1$ and is relaxed again later on in period $t+h$.

2. Constraint (2.5.36) cannot bind in every period. Assumption 1. and 2. together imply that constraint (2.5.36) binds at the beginning and ceases to bind forever after a finite number of periods.

3. Being t the last period in which constraint (2.5.36) binds, it is assumed that, along the before derived results for the class of utility functions (2.5.30) and (2.5.31), the rate of capital income tax is zero for periods $s \geqslant t+2$. In period $t+1$ the tax rate might be at some intermediate level.

Note that, as shown before, for the class of preferences (2.5.30) and (2.5.31) it holds that $V(c_t, h_t, \phi)/u_c(c_t, h_t) = V(c_{t+1}, h_{t+1}, \phi)/u_c(c_{t+1}, h_{t+1})$. Subsequently each of the above assumptions will be assessed.

1. In contrast to assumption 1. it is now supposed that constraint (2.5.36) doesn't bind in period t and $t+h$ but binds in periods $t+1, \ldots, t+h-1$. This would imply that $\gamma_t = \gamma_{t+h} = 0$ while $\gamma_{t+1}, \ldots, \gamma_{t+h-1}$ are without exception greater than zero. Now the time period from $t+1$ on, where the constraint starts to bind, up to period $t+h$, where the constraint ceases to bind (doesn't bind anymore), is examined on the first order condition associated to consumption (2.5.38).

$$\lambda_{t+1} = V_c(c_{t+1}, h_{t+1}, \phi) + \gamma_{t+1}u_{cc}(c_{t+1}, h_{t+1}) \tag{2.5.40}$$

$$\lambda_{t+h} = V_c(c_{t+h}, h_{t+h}, \phi) - \gamma_{t+h-1}u_{cc}(c_{t+h}, h_{t+h})(1-\delta) \tag{2.5.41}$$

In these periods the government exhausts its tax limit. The after tax return on capital is at its lower bound $(1-\delta)$. First order conditions (2.5.40) and (2.5.41), for time periods $t+1$ and $t+h$ respectively, together with constraint (2.5.36)(which is binding in the

[11] Chamley 1985
[12] Atkeson/Chari/Kehoe 1999

examined phase) imply:

$$\begin{aligned} &V_c(c_{t+1}, h_{t+1}, \phi) + \gamma_{t+1} u_{cc}(c_{t+1}, h_{t+1}) \\ &\qquad \geqslant \beta^{h-1}(1-\delta)^{h-1}[V_c(c_{t+h}, h_{t+h}, \phi) - \gamma_{t+h-1} u_{cc}(c_{t+h}, h_{t+h})(1-\delta)] \end{aligned} \tag{2.5.42}$$

For time periods $t+1, \ldots, t+h$ the constraint (2.5.36), it is binding in this phase, together with the fact that $V(c_t, h_t, \phi)/u_c(c_t, h_t) = V(c_{t+1}, h_{t+1}, \phi)/u_c(c_{t+1}, h_{t+1})$ implies:

$$V_c(c_{t+1}, h_{t+1}, \phi) = \beta^{h-1}(1-\delta)^{h-1} V_c(c_{t+h}, h_{t+h}, \phi) \tag{2.5.43}$$

Combining equation (2.5.42) and (2.5.43) yields:

$$\begin{aligned} &V_c(c_{t+1}, h_{t+1}, \phi) + \gamma_{t+1} u_{cc}(c_{t+1}, h_{t+1}) \\ &\geqslant V_c(c_{t+1}, h_{t+1}, \phi) - \beta^{h-1}(1-\delta)^h \gamma_{t+h-1} u_{cc}(c_{t+h}, h_{t+h}) \rightarrow \\ &\gamma_{t+1} u_{cc}(c_{t+1}, h_{t+1}) \geqslant -\beta^{h-1}(1-\delta)^h \gamma_{t+h-1} u_{cc}(c_{t+h}, h_{t+h}) \end{aligned} \tag{2.5.44}$$

Since however by definition $u_{cc}(.) < 0$, see section 2.1.1 (the utility functions fulfills the INADA conditions), (2.5.44) is a contradiction. Constraint (2.5.36) therefore cannot not be binding in one period, be binding in a finite number of periods thereafter, and then cease to be binding again.

2. It can easily be shown that it can neither be true that the constraint binds forever. If so the capital stock in the economy would step by step dissolved at the rate $(1-\delta)$.

$$k_{t+1} = (1-\delta)k_t$$

Since by definition $F_k(0, h_t) = 0$ this would finally violate the resource constraint (2.1.2).

3. If, as supposed, the constraint (2.5.36) ceases to bind in period t the same mechanism as discussed before (for the class of preferences (2.5.30) and (2.5.31), without upper bound on the capital tax rate and seen from period zero) lets the capital income tax converge to zero within two periods. Therefore for periods $s \geqslant t+2$ one can write the first order conditions (2.5.38) and (2.5.39) in combination as

$$u_c(c_s, h_s) = \beta u_c(c_{s+1}, h_{s+1})[F_k(k_{s+1}, h_{s+1}) + (1-\delta)] \tag{2.5.45}$$

With $V(c_s, h_s, \phi)/u_c(c_s, h_s) = V(c_{s+1}, h_{s+1}, \phi)/u_c(c_{s+1}, h_{s+1})$ or $V(c_s, h_s, \phi)/V(c_{s+1}, h_{s+1}, \phi) = u_c(c_s, h_s)/u_c(c_{s+1}, h_{s+1})$ respectively expression (2.5.45) implies that the capital income tax is zero for all periods $s \geqslant t+2$, where t is the period in which the constraint $\theta \leqslant 1$ ceases to bind.

In this section the result from the preceding sections that in the steady state the tax on capital income should be optimally zero was confirmed. Moreover it was shown that in the initial period the planner indeed has an incentive to tax capital: In the initial period, in contrast to all periods thereafter, the taxation of capital causes no intertemporal distortions. The government thus has the incentive to tax capital at very high, confiscatory rates in the initial period in order to lower necessary distortionary taxation in the future. Following Chamley[13] it was shown that for the class of preferences (2.5.30) and (2.5.31) and with an imposed upper bound on capital taxes $\theta \leqslant 1$, taxation on capital income is positive for a finite number of periods and thereafter converges to zero. The explanation for the behavior of the government was by now rather intuitive. The next section will provide a more formal and pronounced argument for the front-loading of taxation/distortions.

2.6 The front-loading argument

In this section the intuition that governments front-load taxation/distortions in order to lower tax-caused distortions in the future will be examined in a more formal way. I will therefore follow the argumentation of Stefania Albanesi and Roc Armenter[14]. Consider the Euler equation of households (2.1.11) under the assumption that the economy has converged to a steady state (the steady state is here denoted by the super/subscript ^{ss}).

$$u_c^{ss} = \beta u_c^{ss}[(1-\theta)r_{ss} + (1-\delta)] \tag{2.6.1}$$

In contrast to the results derived before it is now supposed that, although the economy is assumed to be in steady state, capital taxes are positive. Further the return on capital r_{ss} is replaced along (2.1.33) by the marginal product of capital. This implies for equation (2.6.1) that,

$$\beta^{-1} < F_k^{ss} + (1-\delta) \tag{2.6.2}$$

which contrasts to the identity $1 = \beta\{(1-\theta)r_{ss} + 1 - \delta\}$ which formed the basis of our conclusion that capital taxation should be zero in the steady state (compare to equation (2.3.9)). Equation (2.6.2) represents the intertemporal wedge. As long as there are positive capital taxes the intertemporal substitution behavior and therefore the decision over present and future consumption is distorted: As already mentioned

[13]Chamley 1985

[14]Albanesi, Stefania & Armenter, Roc, 2007. "Understanding Capital Taxation in Ramsey Models," Manuscript, Columbia University.

in section 2.5.3, the existence of positive capital taxes implicitly implies a differential taxation of consumption.

Equally one can write down the intratemporal wedge by using the household's optimality condition (2.1.10). Households equate the marginal rate of substitution between labor and consumption with the after-tax labor income. Again it is supposed that the economy has converged to a steady state.

$$-\frac{u_h^{ss}}{u_c^{ss}} = (1-\tau)w^{ss} \tag{2.6.3}$$

A positive tax rate on labor therefore implies that the intratemporal substitution behavior between labor and consumption is distorted. Substituting for the marginal product of labor this implies:

$$-\frac{u_h^{ss}}{u_c^{ss}} \leqslant F_h^{ss} \tag{2.6.4}$$

Distortions implied by equations (2.6.2) and (2.6.4) corresponding to a tax policy regime $\lim_{t\to\infty}\theta_t = \theta \geqslant 0$ and $\lim_{t\to\infty}\tau_t = \tau \geqslant 0$ can obviously not be optimal. The goal is now to find an alternative tax policy $\{\tilde{\tau}_t, \tilde{\theta}_t\}_{t=0}^{\infty}$ which could increase welfare. In the here examined case the focus rests upon the capital tax rate.

The intertemporal wedge (2.6.2) implies that it would be welfare increasing if the government could implement a policy which could encourage households to reallocate consumption from period t to period $t+1$. Note that is exactly the opposite to what a positive capital tax rate implies incentive-wise for the intertemporal substitution behavior of households: Under the assumption of a steady state with a positive capital tax rate the incentives to procrastinate consumption are absent. As it was argued before a positive tax rate on capital implies a differential and therefore an ever-increasing implicit tax on private good consumption and is thus a contradiction to the steady state assumption. The idea behind Albanesi and Armenter's approach is now to change the timing of distortions or, alternatively expressed, to front-load (capital) taxes in order to allow tax cuts and therefore a lower level of distortions in the future. Note that this is exactly what was argued in the preceding section. The forthcoming argumentation more formally demonstrates the rationality behind Chamley's result that governments use front-loading of taxation in order to lower distortionary taxation in the future: It was shown above that capital taxation in the beginning is extremely high, but zero when the economy had finally converged to a steady state. The government can thus achieve budget surpluses in initial periods and invest them in order to lower the necessity of distortionary taxation in the future by the proceeds. With such an approach governments are able to reallocate resources, in example for consumption, to future periods as it is implied by the intertemporal wedge (2.6.2), displayed above. It is important to note however that the only intertemporal distortions stem from capital taxation, as possible labor supply effects caused through a positive tax rate on labor income in two succeeding periods cancel each other out. Now a tax reform is sketched which front-loads distortions along the discussion above and could therefore lead to an improvement of welfare. The reformed system of taxes is denoted by $\{\tilde{\tau}_t, \tilde{\theta}_t\}_{t=0}^{\infty}$ leading to the new allocations $\{\tilde{c}_t, \tilde{h}_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}$ in the competitive equilibrium. The proposed tax reform is tested exemplary at the hypothetical periods t and $t+1$ only. At all dates $t+1$ and thereafter, with the exception $j \neq t$, all allocations and policies are assumed to be identical, therefore $\tilde{c}_j = c_j$ and $\tilde{h}_j = h_j$ etc. Further it is assumed that the economy under the original policy had already converged to a steady state at period t, therefore $c_j = c^{ss}$ and $h_j = h^{ss}$. The reformed policy has to match the equilibrium conditions and the government's budget constraint valid under the original policy at period t. By Walras' law it is redundant to include the budget constraint for the government and the households: As mentioned before, as soon as the private sector budget constraint and the resource constraint are satisfied the government's budget constraint is satisfied too. It is therefore sufficient to examine the present value budget constraint of the households consolidated with the household's optimality conditions denoted as the implementability constraint. This was already done in section 2.1.5 (see equation 2.5.11). The implementability constraint is displayed again below.

$$\sum_{t=0}^{\infty} \beta^t [u_c(c_t, h_t)c_t + u_h(c_t, h_t)h_t] = u_c(c_0, h_0)\{[\overbrace{F_k(k_0, h_0)}^{r_0}(1-\theta_0) + (1-\delta)]k_0 + b_0\}$$

For the sake of simplicity it is assumed that government purchases $g_t = g$ are constant over time. In order to shift consumption from period t to period $t+1$ consumption in period t has to be reduced. Be the reduction of consumption in period t a small increment $\epsilon > 0$.

$$dc_t = -\epsilon \tag{2.6.5}$$

This reduction of consumption corresponds to an intratemporal adjustment in labor supply. Both effects have to satisfy,

$$d[u_c(c_t, h_t)c_t + u_h(c_t, h_t)h_t] = 0 \tag{2.6.6}$$

in order to fulfill the competitive equilibrium conditions captured by the consolidated present value budget constraint of the households (implementability constraint), displayed above.[15] It is assumed that a value α solves the problem in order that (2.6.6)

[15] compare to equation (2.5.11)

holds. One can write:

$$dh_t = \alpha dc_t \tag{2.6.7}$$

It is therefore assumed that α is the ratio at which h_t has to change corresponding to a reduction in c_t. One can therefore denote the induced change in h_t, dh_t, as:

$$dh_t = -\alpha\epsilon \tag{2.6.8}$$

Thus, for the competitive equilibrium condition to hold a reduction in c_t has to lead to an in-/decrease of labor supply. To see which impact this changes have on capital accumulation the resource constraint (2.1.2) has to be examined, displayed again below.

$$c_t + g_t + k_{t+1} = F(k_t, h_t) + (1-\delta)k_t$$

Given the change in consumption, the change in investment corresponds to

$$dk_{t+1} = F_h^{ss} dh_t - dc_t \tag{2.6.9}$$

implying with (2.6.5) and (2.6.8) that:

$$dk_{t+1} = \epsilon - \alpha\epsilon F_h^{ss} = (1 - \alpha F_h^{ss})\epsilon \tag{2.6.10}$$

Next the implications of the shift of resources for period $t+1$ are examined. It should be noted that in the absence of a response of labor supply a reduction in consumption would lead directly to an increase of investment. It is anyway not necessary here to make an assertion over how investment changes in response to a change (reduction) in consumption. As in period t in period $t+1$ a change in consumption has to be accompanied by a change in labor supply in order that the competitive equilibrium conditions holds.

$$d[u_c(c_{t+1}, h_{t+1})c_{t+1} + u_h(c_{t+1}, h_{t+1})h_{t+1}] = 0 \tag{2.6.11}$$

The relation of a change in consumption and labor supply in period $t+1$ is the same as in period t.

$$dh_{t+1} = \alpha dc_{t+1} \tag{2.6.12}$$

Further the economy's resource constraint is examined to see which implications the perturbations in period t have for period period $t+1$. Note, that the consumption in period t was reduced by an increment in order to reallocate resources to period $t+1$. This has implications for the stock of capital in the economy as well as for the labor supply. The change in consumption in period $t+1$ can thus be expressed via the resource constraint in period $t+1$:

$$dc_{t+1} = F_h^{ss} dh_{t+1} + F_k^{ss} dk_{t+1} + (1-\delta) dk_{t+1} = F_h^{ss} dh_{t+1} + (F_k^{ss} + 1 - \delta) dk_{t+1} \tag{2.6.13}$$

Using the derived change of investment in period t, $dk_{t+1} = (1 - \alpha F_h^{ss})\epsilon$, from (2.6.10) and equation (2.6.12) describing the relationship between changes in labor supply and consumption, $dh_{t+1} = \alpha dc_{t+1}$, substituted into the differentiated resource constraint (2.6.13) allows to rewrite it as follows:

$$\begin{aligned} dc_{t+1} &= F_h^{ss} \alpha dc_{t+1} + (F_k^{ss} + 1 - \delta)(1 - \alpha F_h^{ss})\epsilon \\ &= F_h^{ss} \alpha dc_{t+1} - \alpha F_h^{ss}(F_k^{ss} + 1 - \delta)\epsilon + (F_k^{ss} + 1 - \delta)\epsilon \\ &= \alpha F_h^{ss}[dc_{t+1} - (F_k^{ss} + 1 - \delta)\epsilon] + (F_k^{ss} + 1 - \delta)\epsilon \end{aligned} \tag{2.6.14}$$

Hence the change in consumption dc_{t+1} can be pinned down as

$$dc_{t+1} = (F_k^{ss} + 1 - \delta)\epsilon \tag{2.6.15}$$

Equation (2.6.15) represents the shift of resources from period t to period $t+1$ in the form of consumption: The reduction in consumption in period t goes into the capital stock and allows for a higher consumption by the factor $(F_k^{ss} + 1 - \delta)\epsilon$ in period $t+1$ than under the original policy. The welfare effects in periods t and $t+1$ can be summarized as follows:

$$d[u(c_t, h_t) + \beta u(c_{t+1}, h_{t+1})] = u_c^{ss}[dc_t + \beta dc_{t+1}] + u_h^{ss}[dh_t + \beta dh_{t+1}] \tag{2.6.16}$$

As already postulated, the relation $dh_t = \alpha dc_t$ holds in both periods t and $t+1$. One can hence write:

$$dh_t + \beta dh_{t+1} = \alpha[dc_t + \beta dc_{t+1}] \tag{2.6.17}$$

Expression (2.6.17) allows to rewrite equation (2.6.16), which is capturing the overall welfare effects.

$$\begin{aligned} &d[u(c_t, h_t) + \beta u(c_{t+1}, h_{t+1})] = u_c^{ss}[dc_t + \beta dc_{t+1}] + u_h^{ss}\alpha[dc_t + \beta dc_{t+1}] = \\ &[u_c^{ss} + \alpha u_h^{ss}][dc_t + \beta dc_{t+1}] \end{aligned} \tag{2.6.18}$$

The overall welfare effect induced by the reallocation of resources depends on the sign of the term $[u_c^{ss} + \alpha u_h^{ss}]$. Hence there are two possibilities:

1) $[u_c^{ss} + \alpha u_h^{ss}] \leqslant 0$: The central point of the hitherto used argumentation was that it might be welfare increasing to front-load distortions in the form of high taxes on capital

in initial periods which are later on relaxed. Under the assumption that $[u_c^{ss} + \alpha u_h^{ss}] \leqslant 0$ the result is however contrary to expectations. If $[u_c^{ss} + \alpha u_h^{ss}] \leqslant 0$ the intratemporal wedge (2.6.4), representing the substitution behavior between labor and consumption in period t, $-\frac{u_h^{ss}}{u_c^{ss}} \leqslant F_h^{ss}$, implies that $1 \leqslant \alpha F_h^{ss}$ and hence that $dk_{t+1} = F_h^{ss} dh_{t+1} - dc_t = (1 - \alpha F_h^{ss})\epsilon \leqslant 0$ (compare to equation (2.6.10)). This means that shifting distortions to period t is welfare increasing and that it is not necessary to reduce the distortions again later in the next periods: As the derived conditions $dk_{t+1} \leqslant 0$ shows, no increase in the capital stock in period $t+1$ is needed in order to satisfy the economy's resource constraint and thus the other equilibrium conditions. Albanesi and Armenter admit that one could think of such a situation in an economy which is extremely distorted through high taxation, so that a small tax cut would allow gains in tax revenues and a reallocation of resources via capital is obsolete. This case can be regarded as rather unrealistic and can therefore be discarded.

2) $[u_c^{ss} + \alpha u_h^{ss}] > 0$: The more interesting scenario is the case $[u_c^{ss} + \alpha u_h^{ss}] > 0$. Here the welfare change corresponds to the changes in consumption in periods t and $t+1$ induced by the front-loading of distortions via means of capital taxation. Note: The assumption of positive capital taxes in the steady state corresponds to an ever-increasing (differential) taxation on consumption implying intertemporal distortions. The idea was to revert this effect by changing the timing of distortions (capital taxes) in order to reallocate resources (for consumption) from period t to period $t+1$ to show that this delivers a higher welfare (strictly speaking just the effects and consequences of the front-loading of distortions were examined). The changes in consumption and welfare respectively can be denoted in proportions:

$$d[u(c_t, h_t) + \beta u(c_{t+1}, h_{t+1})] \propto dc_t + \beta dc_{t+1} \tag{2.6.19}$$

Using $dc_t = -\epsilon$ from equation (2.6.5) and $dc_{t+1} = (F_k^{ss} + 1 - \delta)\epsilon$ from (2.6.15) allows to rewrite the overall change of consumption induced by the reallocation of resources to the form:

$$dc_t + \beta dc_{t+1} = -\epsilon + \beta(F_k^{ss} + 1 - \delta)\epsilon = (\beta(F_k^{ss} + 1 - \delta) - 1)\epsilon \tag{2.6.20}$$

Result (2.6.20) together with the intertemporal wedge (2.6.2), $\beta^{-1} < F_k^{ss} + (1 - \delta)$, supports the basic, motivational idea that a shift of distortions and therefore a reallocation of resources from period t to period $t+1$ could improve welfare. This leads to two assertions:

1. The original policy incorporated capital taxation in the steady state which lead to intertemporal distortions as it is implying an ever-increasing implicit taxation of consumption outweighing incentives to an otherwise welfare increasing reallocation of resources from period t to $t+1$. The front-loading of distortions (capital taxes) was

simulated by artificially (against the incentives under the original tax policy) decreasing consumption in the initial period t. It was shown that the thereby induced change in capital accumulation, labor supply and in the overall consumption profile (for period t and $t+1$) leads to a strictly higher welfare. Since the welfare improving new consumption profile is not compatible with the original tax policy (positive capital taxes), it can be concluded that an optimal tax policy in the steady state cannot feature positive capital taxes. Vice versa it is not logic to have long run capital subsidies, implying a negative tax rate on capital. This would just mean to use another source of distortions namely labor taxes, to finance distortionary capital subsidies. The argumentation hitherto adduced therefore supports the core Chamley-Judd result, which was already discussed in detail in the previous sections: capital taxes have to be optimally zero in the steady state.

2. Besides this, the exercised front-loading argument gives insight for the second key result of Chamley which was discussed in section 2.5: The government does have incentives to tax capital in the initial period(s): For preferences of the class (2.5.30) and (2.5.31) and an imposed upper bound on capital taxation, the optimal capital tax policy over time is characterized by two regimes: A phase at the beginning where capital is taxed excessively. Governments use this phase to accumulate capital, therefore to reallocate resources and lower distortions in the future. After a certain time a regime switch takes place and capital income taxes permanently converge to zero. How long the first regime lasts depends on the marginal excess burden, meaning the welfare costs caused by distortionary taxation, as well as on a possible upper limit for capital taxes.[16] One has to note however: The ability of the government to save and therefore to transfer resources from the present to the future is crucial for this result.

2.7 Fiscal policy in a stochastic environment

2.7.1 The Model Environment and Notation

In the following section the hitherto gained results will be discussed in a stochastic environment. This is on the one hand done for the sake of completeness, on the other hand it fits to the models which are going to be discussed in chapters 3 and 4 respectively. The basic model setup is the same as in the deterministic case presented before. Here however it is assumed that in each period a finite number of different events $s_t \in S$ can occur. These events represent shocks on production $F(.,.,s_t)$ and government purchases $g_t(s_t)$. The occurrence of events from the initial period up to the present represents a history denoted as $s^t = (s_0, s_1, s_2, \ldots, s_t)$ where s_0 is given. The probability for the realization of a certain history s^t is expressed by $\pi(s^t)$. $c(s^t)$ and $h(s^t)$ denote

[16] Chamley 1985, pp. 616 and section 2.5

consumption and labor supply at period t for history s^t respectively. $k_{t+1}(s^t)$ represents the $t+1$ capital stock, which, as the decision for the capital accumulation was taken in period t, depends on the history s^t. The household's preferences are given by:

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)u(c_t(s^t),h_t(s^t)) \quad \text{where} \quad \beta\in(0,1) \tag{2.7.1}$$

The resource constraint of the economy is standard but accommodates history contingency and the stochastic nature of the economy.

$$c_t(s^t)+g_t(s^t)+k_{t+1}(s^t)=F(k_t(s^{t-1}),h_t(s^t),s_t)+(1-\delta)k_t(s^{t-1}) \tag{2.7.2}$$

The government finances its expenditures with taxes on labor income $\tau_t(s^t)$ and capital income $\theta_t(s^t)$. Further the government issues state-contingent bonds: $b_{t+1}(s_{t+1}|s^t)$ denotes the stock of public debt in period $t+1$ in the case of shock s_{t+1}. Bonds are traded in period t at the price $p_t(s_{t+1}|s^t)$. The budget constraint of the government reads as follows:

$$g_t(s^t)=\tau_t(s^t)w_t(s^t)h_t(s^t)+\theta_t(s^t)r_t(s^t)k_t(s^{t-1})+\sum_{s_{t+1}}p_t(s_{t+1}|s^t)b_{t+1}(s_{t+1}|s^t)-b_t(s_t|s^{t-1}) \tag{2.7.3}$$

The budget constraint of the households reads:

$$\begin{aligned}&c_t(s^t)+k_{t+1}(s^t)+\sum_{s^{t+1}}p_t(s_{t+1}|s^t)b_{t+1}(s_{t+1}|s^t)=\\&(1-\theta_t(s^t))r_t(s^t)k_t(s^{t-1})+(1-\tau_t(s^t))w_t(s^t)h_t(s^t)+(1-\delta)k_t(s^{t-1})+b_t(s_t|s^{t-1})\end{aligned} \tag{2.7.4}$$

The households maximize the utility function (2.7.1) with respect to $\{c_t(s^t),h_t(s^t),k_{t+1}(s^t),$ $b_{t+1}(s_{t+1}|s^t)\forall s^t,s_{t+1}\}_{t=0}^{\infty}$ and subject to the household's budget constraint (2.7.4). For notational simplicity functions will be further on denoted as $u(s^t)$ instead of $u(c_t(s^t),h_t(s^t))$

The maximization process of the households yields the following optimality conditions:

$$-\frac{u_h(s^t)}{u_c(s^t)} = (1-\tau_t(s^t))w_t(s^t) \tag{2.7.5}$$

$$p_t(s_{t+1}|s^t) = \beta\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}\frac{u_c(s^{t+1})}{u_c(s^t)} \tag{2.7.6}$$

$$\begin{aligned} u_c(s^t) &= \beta\sum_{s^{t+1}|s^t}\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}u_c(s^{t+1})[(1-\theta_{t+1}(s^{t+1}))r_{t+1}(s^{t+1})+(1-\delta)] \\ &= \beta E_t\{u_c(s^{t+1})[(1-\theta_{t+1}(s^{t+1}))r_{t+1}(s^{t+1})+(1-\delta)]\} \end{aligned} \tag{2.7.7}$$

Combining optimality condition (2.7.6) and (2.7.7) gives a no-arbitrage condition similar to that obtained in the deterministic version(compare to expression (2.1.12)).

$$1 = \sum_{s_{t+1}} p_t(s_{t+1}|s^t)[(1-\theta_{t+1}(s^{t+1}))r_{t+1}(s^{t+1})+(1-\delta)] \tag{2.7.8}$$

The interpretation of the no-arbitrage condition is the same as in the deterministic case. By forward iteration of the period-by-period budget constraint one obtains the present value budget constraint (compare to section 2.1 and remember the role of the no-arbitrage condition (2.1.12)).

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)[1-\tau_t(s^t)]w_t(s^t)h_t(s^t) + [(1-\theta_0)r_0+(1-\delta)]k_0 + b_0 \tag{2.7.9}$$

where by definition,

$$q_{t+1}^0(s^{t+1}) = p_t(s_{t+1}|s^t)q_t^0(s^t) = \beta^{t+1}\pi_{t+1}(s^{t+1})\frac{u_c(s^{t+1})}{u_c(s^0)} \tag{2.7.10}$$

with the numeraire $q_0^0 = 1$. Along the argumentation in the deterministic case two transversality conditions were imposed prescribing that the present discounted value of bonds and capital held by the households should be zero at the limit.

$$\lim q_t^0(s^t)k_{t+1}(s^t) = 0 \tag{2.7.11}$$

$$\lim \sum_{s_{t+1}} q_{t+1}^0(\{s_{t+1}, s^t\})b_{t+1}(s_{t+1}|s^t) = 0 \tag{2.7.12}$$

Again the private sector optimality conditions (2.7.5) to (2.7.7), together with the no-arbitrage condition (2.7.8) can be used to express prices and taxes in the present value budget constraint (2.7.9) in order to obtain the implementability constraint. Note that as before we assume that factor markets are in equilibrium and that the marginal products of labor and capital equate to their respective factor prices. Thus, $F_k(s^t) = r_t(s^t)$ and $F_h(s^t) = w_t(s^t)$.

$$\sum_{t=0}^{\infty}\sum_{s^t}\left\{q_t^0(s^t)c_t(s^t) + q_t^0(s^t)\frac{u_h(s^t)}{u_c(s^t)}h_t(s^t)\right\} = [(1-\theta_0)r_0 + (1-\delta)]k_0 + b_0 \qquad (2.7.13)$$

Substitute for $q_t^0(s^t)$ from definition (2.7.10) yields:

$$\sum_{t=0}^{\infty}\sum_{s^t}\left\{\beta^t\pi_t(s^t)\frac{u_c(s^t)}{u_c(s^0)}c_t(s^t) + \beta^t\pi_t(s^t)\frac{u_c(s^t)}{u_c(s^0)}\frac{u_h(s^t)}{u_c(s^t)}h_t(s^t)\right\} = [(1-\theta_0)r_0 + (1-\delta)]k_0 + b_0 \qquad (2.7.14)$$

Further simplified, expression (2.7.14) yields an implementability constraint comparable to that in the deterministic case (2.5.11).

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)[u_c(s^t)c_t(s^t) + u_h(s^t)h_t(s^t)] - A = 0 \qquad (2.7.15)$$

with

$$A = A(c_0, h_0, \theta_0) = u_c(s^0)\{[(1-\theta_0)r_0 + (1-\delta)]k_0 + b_0\}$$

As already used before but for the sake of completeness: the factor prices of labor and capital equate, as in the deterministic case, to their marginal products.

$$r_t(s^t) = F_k(s^t) \qquad (2.7.16)$$

$$w_t(s^t) = F_h(s^t) \qquad (2.7.17)$$

2.7.2 The indeterminacy problem of capital taxes and state contingent debt

Suppose that there is a feasible government policy $\{g_t(s^t), \theta_t(s^t), \tau_t(s^t), b_{t+1}(s_{t+1}|s^t); \forall s^t, s_{t+1}\}_{t\geqslant 0}$ and corresponding optimal allocations $\{c_t(s^t), h_t(s^t), k_{t+1}(s^t); \forall s^t\}_{t\geqslant 0}$. The tax on labor income is pinned down by the household's optimality condition (2.7.5) and by (2.7.17). This is however not true for capital tax rates and state contingent debt. To see this consider the household's optimality condition (2.7.7) rewritten in (2.7.18):

$$u_c(s^t) = \beta E_t\{u_c(s^{t+1})[(1-\theta_{t+1}(s^{t+1}))r_{t+1}(s^{t+1}) + (1-\delta)]\} \qquad (2.7.18)$$

It is obvious that the government has the ability to manipulate the current market value of after tax returns on capital and thereby keep the expectations constant across different states of the world and compatible with the optimal allocations. Such a policy however would have consequences for tax receipts which could be offset by constructing a suitable alternative debt policy. Along Zhu[17] it is assumed that $\{\epsilon_t(s^t)\}$ is a random process which satisfies

$$E_t[u_c(s^{t+1})\epsilon_{t+1}(s^{t+1})r_{t+1}(s^{t+1})] = 0 \tag{2.7.19}$$

Now an alternative policy for debt and capital is constructed for all s^t,s_{t+1} and all $t \geqslant 0$ denoted as $\hat{\theta}_t(s^t)$ and $\hat{b}_{t+1}(s_{t+1}|s^t)$. In period $t = 0$ the original and the alternative policy concerning the capital taxation is equivalent: $\hat{\theta}_0 = \theta_0$.

$$\hat{\theta}_{t+1}(s^{t+1}) = \theta_{t+1}(s^{t+1}) + \epsilon_{t+1}(s^{t+1}) \tag{2.7.20}$$

$$\hat{b}_{t+1}(s_{t+1}|s^t) = b_{t+1}(s_{t+1}|s^t) + \epsilon_{t+1}(s^{t+1})r_{t+1}(s^{t+1})k_{t+1}(s^t) \tag{2.7.21}$$

$$\forall t \geqslant 0$$

To assure this alternative policy would be compatible with the optimal allocations as well three conditions have to be fulfilled: 1) The alternative policy mustn't change household's intertemporal consumption/saving behavior captured in optimality condition (2.7.7). 2) The current market value of period t government debt discounted with the equilibrium price for bonds, see optimality condition (2.7.6), mustn't be changed. 3) The capital tax receipts net-of-debt-redemption should be unchanged for any state of the world s^{t+1}.

1. The intertemporal consumption/saving behavior, captured in the optimality condition (2.7.7), under the alternative policy regime concerning capital taxes $\hat{\theta}_t(s^t)$ reads as follows.

$$\begin{aligned} u_c(s^t) &= \beta E_t\{u_c(s^{t+1})[(1-\hat{\theta}_{t+1}(s^{t+1}))r_{t+1}(s^{t+1}) + (1-\delta)]\} \\ &= \beta E_t\{u_c(s^{t+1})[(1-\theta_{t+1}(s^{t+1}) - \epsilon_{t+1}(s^{t+1}))r_{t+1}(s^{t+1}) + (1-\delta)]\} \end{aligned} \tag{2.7.22}$$

If the perturbation $\epsilon_{t+1}(s^{t+1})$ is a random process as defined in equation (2.7.19) the term, taken expectations, is zero. Obviously condition (2.7.22) stays unchanged compared to under the original policy (2.7.7).

2. For point 2) to be satisfied the current market value of government debt issued at period t has to be the same under the original and the alternative policy. Hence,

$$\sum_{s_{t+1}} p_t(s_{t+1}|s^t)\hat{b}_{t+1}(s_{t+1}|s^t) = \sum_{s_{t+1}} p_t(s_{t+1}|s^t)b_{t+1}(s_{t+1}|s^t) \tag{2.7.23}$$

Substitute along the alternative policy $\hat{b}_{t+1}(s_{t+1}|s^t)$ from (2.7.21) and for $p_t(s_{t+1}|s^t)$

[17]Zhu, Xiaodong, 1992. "Optimal Fiscal Policy in a Stochastic Growth Model," Journal of Economic Theory, vol. 58, pp. 250-289.

from the optimality condition (2.7.6) on the left-hand side of equation (2.7.23). This yields:

$$\begin{aligned}
&\sum_{s_{t+1}} p_t(s_{t+1}|s^t)\hat{b}_{t+1}(s_{t+1}|s^t) \\
&= \sum_{s_{t+1}} \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{u_c(s^{t+1})}{u_c(s^t)} [b_{t+1}(s_{t+1}|s^t) + \epsilon_{t+1}(s^{t+1}) r_{t+1}(s^{t+1}) k_{t+1}(s^t)] \\
&= \sum_{s_{t+1}} \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{u_c(s^{t+1})}{u_c(s^t)} b_{t+1}(s_{t+1}|s^t) = \sum_{s_{t+1}} p_t(s_{t+1}|s^t) b_{t+1}(s_{t+1}|s^t) \qquad (2.7.24)
\end{aligned}$$

Equation (2.7.24) is exactly what point 2) demanded to show.

3. The capital tax receipts net of the redemption of government debt for all states of the world s^{t+1} under the alternative policy can be denoted as:

$$\begin{aligned}
&\hat{\theta}_t(s^t) r_t(s^t) k_t(s^{t-1}) - \hat{b}_t(s_t|s^{t-1}) \\
&= \theta_t(s^t) r_t(s^t) k_t(s^{t-1}) + \epsilon_t(s^t) r_t(s^t) k_t(s^{t-1}) - b_t(s_t|s^{t-1}) - \epsilon_t(s^t) r_t(s^t) k_t(s^{t-1}) \\
&= \theta_t(s^t) k_t(s^{t-1}) r_t(s^t) - b_t(s_t|s^{t-1}) \qquad (2.7.25)
\end{aligned}$$

Hence also condition 3) is fulfilled. This means, as there are infinitely many $\{\epsilon_t(s^t)\}$ which would satisfy definition (2.7.19), that there is not a unique policy plan for capital taxes and state contingent government debt which is feasible and compatible with optimal allocations: Instead there are many policy plans which would be suitable. One can therefore consider a scenario in which one of the two policy instruments is predetermined: 1) The government issues only risk free bonds. The capital tax would then be contingent to which state of the world materializes. 2) The government sets an ex-ante capital tax one period ahead leaving government debt be state contingent again.

Scenario 1: The government issues risk-free bonds in period t paying $\bar{b}_{t+1}(s^t)$ in period $t+1$ regardless which state of the world is realized. To be compatible with the optimal allocations and in order to be feasible conditions 1. and 3., which were demanded before, have to be fulfilled. Hence, the market value of period t government bonds discounted with the equilibrium price $p_t(s_{t+1}|s^t)$ has to be the same under the original debt policy and under the policy with risk free bonds.

$$\sum_{s_{t+1}} p_t(s_{t+1}|s^t)\bar{b}_{t+1}(s^t) = \sum_{s_{t+1}} p_t(s_{t+1}|s^t) b_{t+1}(s_{t+1}|s^t) \qquad (2.7.26)$$

Invoking the equilibrium price $p_t(s_{t+1}|s^t)$ from optimality condition (2.7.6) yields:

$$\sum_{s_{t+1}} \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{u_c(s^{t+1})}{u_c(s^t)} \bar{b}_{t+1}(s^t) = \sum_{s_{t+1}} \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{u_c(s^{t+1})}{u_c(s^t)} b_{t+1}(s_{t+1}|s^t) \rightarrow$$
$$E_t u_c(s^{t+1}) \bar{b}_{t+1}(s^t) = E_t u_c(s^{t+1}) b_{t+1}(s_{t+1}|s^t) \rightarrow$$
$$\bar{b}_{t+1}(s^t) = \frac{E_t u_c(s^{t+1}) b_{t+1}(s_{t+1}|s^t)}{E_t u_c(s^{t+1})} \tag{2.7.27}$$

Such a change to risk free government debt consequently has to be outweighed by a change of capital taxes. Equation (2.7.21) implies a perturbation of capital taxes in the form of:

$$\epsilon_{t+1}(s^{t+1}) = \frac{\bar{b}_{t+1}(s^t) - b_{t+1}(s_{t+1}|s^t)}{r_{t+1}(s^{t+1}) k_{t+1}(s^t)} \tag{2.7.28}$$

As it was shown before a perturbation in the form of (2.7.28) satisfies conditions 1. and 3.(that condition 2. is satisfied was a basic assumption in introducing risk free government debt). Such a policy would therefore be feasible and compatible with the optimal allocations.

Scenario 2: Assume $\bar{\theta}_{t+1}(s^t)$ to be the ex-ante capital tax, set in period t, conditional on the information of period t, and valid in period $t+1$. By condition 1. such a policy change must not change the intertemporal consumption/saving behavior or to put it differently, it must not change expectations over the current market value of after tax returns on capital in order to be compatible with the optimal allocations. Along optimality condition (2.7.7) it therefore has to hold that:

$$E_t\{u_c(s^{t+1})[(1-\bar{\theta}_{t+1}(s^t)) r_{t+1}(s^{t+1}) + (1-\delta)]\}$$
$$= E_t\{u_c(s^{t+1})[(1-\theta_{t+1}(s^{t+1})) r_{t+1}(s^{t+1}) + (1-\delta)]\} \tag{2.7.29}$$

Condition (2.7.29) allows to express the ex-ante capital tax rate $\bar{\theta}_{t+1}(s^t)$:

$$\bar{\theta}_{t+1}(s^t) = \frac{E_t u_c(s^{t+1}) \theta_{t+1}(s^{t+1}) r_{t+1}(s^{t+1})}{E_t u_c(s^{t+1}) r_{t+1}(s^{t+1})} \tag{2.7.30}$$

To pin down capital taxes like that, a perturbation of debt policy to offset the policy shift is necessary:

$$\epsilon_{t+1}(s^{t+1}) = \bar{\theta}_{t+1}(s^t) - \theta_{t+1}(s^{t+1}) \tag{2.7.31}$$

As shown before a perturbation of the form of (2.7.31) fulfills conditions 2. and 3. and would therefore be feasible and compatible with equilibrium allocations.

Intuitively it seems reasonable to assume that a government uses preset and therefore not-contingent capital taxes to solve for itself the indeterminacy problem described above. Hence it is the only way to analyze capital tax policy (note that the tax rate on labor income is uniquely determined by optimality condition (2.7.5) and (2.7.17)). Merging optimality condition (2.7.6) with equation (2.7.30) gives:

$$\bar{\theta}_{t+1}(s^t) = \frac{\sum_{s_{t+1}} p_t(s_{t+1}|s^t)\theta_{t+1}(s^{t+1})r_{t+1}(s^{t+1})}{\sum_{s_{t+1}} p_t(s_{t+1}|s^t)r_{t+1}(s^{t+1})} \tag{2.7.32}$$

Equation (2.7.32) expresses the ex-ante tax rate on capital as the ratio of the current market value of capital tax receipts to the current market value of capital income.

2.7.3 The primal formulation under uncertainty

Along the procedure in the deterministic case, compare to section 2.5.2, an objective function with the utility function (2.7.1) and the implementability constraint (2.7.15) is defined.

$$\begin{aligned} &V(c_t(s^t), h_t(s^t), \phi) \\ &= u(c_t(s^t), h_t(s^t)) + \phi[u_c(c_t(s^t), h_t(s^t))c_t(s^t) + u_h(c_t(s^t), h_t(s^t))h_t(s^t)] \end{aligned} \tag{2.7.33}$$

The variable ϕ is the Lagrange multiplier on the implementability constraint (2.7.15). The problem of the government now is to maximize expression (2.7.33) subject to the economy's resource constraint (2.7.2). Written in Lagrange form the problem of the Ramsey planner is:

$$\begin{aligned} \mathcal{L} = &\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t)\{V(c_t(s^t), h_t(s^t), \phi) \\ &+ \lambda_t(s^t)[F(k_t(s^{t-1}), h_t(s^t), s_t) + (1-\delta)k_t(s^{t-1}) - c_t(s^t) - g_t(s^t) + k_{t+1}(s^t)]\} - \phi A \end{aligned} \tag{2.7.34}$$

The Lagrange function (2.7.34), with k_0, b_0 and θ_0 being given, is derived with respect to $\{c_t(s^t), h_t(s^t), k_{t+1}(s^t); \forall s^t\}_{t \geqslant 0}$. Combination of the respective first order conditions finally yields the following optimality conditions. The derivation of the first order conditions resembles the procedure in the deterministic case and is therefore omitted.

$$
\begin{aligned}
V_c(s^t) &= \beta \sum_{s^{t+1}|s^t} \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} V_c(s^{t+1})[F_k(s^{t+1}) + (1-\delta)] \\
&= \beta E_t V_c(s^{t+1})[F_k(s^{t+1}) + (1-\delta)] && (2.7.35) \\
V_h(s^t) &= -V_c(s^t) F_h(s^t) && (2.7.36) \\
V_h(s^0) &= -[V_c(s^0) - \phi A_c] F_h(s^0) + \phi A_h && (2.7.37) \\
V_h(s^0) &= [\phi A_c - V_c(s^0)] F_h(s^0) + \phi A_h && (2.7.38)
\end{aligned}
$$

2.7.4 Capital taxation in the stochastic economy for a certain class of preferences

As in the deterministic case it is easy to show that for a certain class of utility functions, namely the stochastic versions of the utility functions (2.5.30) and (2.5.31), it is optimal not to distort capital accumulation for periods $t \geqslant 1$ respectively not to tax capital for periods $t \geqslant 2$. To see this compare optimality condition (2.7.35) and the household's optimality condition (2.7.7) rewritten slightly changed in equations (2.7.39) and (2.7.40) respectively.

$$1 = \beta \sum_{s^{t+1}|s^t} \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{V_c(s^{t+1})}{V_c(s^t)} [F_k(s^{t+1}) + (1-\delta)] \tag{2.7.39}$$

$$1 = \beta \sum_{s^{t+1}|s^t} \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{u_c(s^{t+1})}{u_c(s^t)} [(1-\theta_{t+1}(s^{t+1}))r_{t+1}(s^{t+1}) + (1-\delta)] \tag{2.7.40}$$

For the deterministic case it was shown that utility functions of the class (2.5.30) and (2.5.31) exhibit constant expenditure elasticities for all periods $t \geqslant 1$. This is also true for the stochastic scenario. Hence:

$$\frac{V_c(s^t)}{u_c(s^t)} = \frac{V_c(s^{t+1})}{u_c(s^{t+1})} \quad \text{or} \quad \frac{V_c(s^{t+1})}{V_c(s^t)} = \frac{u_c(s^{t+1})}{u_c(s^t)} \tag{2.7.41}$$

With expression (2.7.41) substituted into equation (2.7.39) it can be merged with equation (2.7.40) to yield after some manipulations:

$$\sum_{s^{t+1}|s^t} \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{u_c(s^{t+1})}{u_c(s^t)} [F_k(s^{t+1}) - (1-\theta_{t+1}(s^{t+1}))r_{t+1}(s^{t+1})] = 0 \tag{2.7.42}$$

Substituting for the equilibrium price of government debt $p_t(s_{t+1}|s^t)$ from household's optimality condition (2.7.6) in equation (2.7.42) and using the firm's first oder condition $F_k(s^t) = r_t(s^t)$ to express the marginal product of capital by its factor price yields:

$$\sum_{s_{t+1}} p_t(s_{t+1}|s^t)\theta_{t+1}(s^{t+1})r_{t+1}(s^{t+1}) = 0 \tag{2.7.43}$$

Resulting equation (2.7.43) equals the numerator of the before derived ex-ante capital tax rate from equation (2.7.32): It can therefore be concluded that for the class of utility functions (2.5.30) and (2.5.31) it is not optimal to distort capital accumulation for periods $t \geqslant 1$, hence the ex-ante tax rate on capital should be zero for periods $t \geqslant 2$.[18] If one had imposed an exogenous upper bound on capital taxes, as it was done in the deterministic case, the mechanism would be equivalent: This means that there would be a finite number of periods with high capital taxation and after one period of transition the optimal tax rate on capital would converge to zero. Anyhow this case is not considered here. The introduction of an upper bound on capital tax rates in the stochastic case would lead to a constraint similar to (2.5.36) in the deterministic economy.

2.7.5 Stationarity and Capital Taxation

In an economy with preferences unlike represented through the utility functions (2.5.30) and (2.5.31) one has to turn back to a concept similar to the steady state assumption in the deterministic economy: In a stochastic environment one speaks of stationarity. For a stationary equilibrium to materialize the underlying stochastic process has to fulfill certain criteria. Therefore the following definitions are undertaken:[19]

1. Be the stochastic process $\{s_t\}$ of Markov property where only the most recent of all foregone events is relevant in predicting the future.

$$Prob(s_{t+1}|s_t, s_{t-1}, \ldots, s_{t-k}) = Prob(s_{t+1} = s'|s_t = s) = \pi(s'|s) \tag{2.7.44}$$

2. The Markov chain is time invariant, hence it holds that,

$$Prob(s_{t+1} = j|s_t = i) = Prob(s_{t+2} = j|s_{t+1} = i) \tag{2.7.45}$$

for all t and $(i, j) \in S^2$ where $S = (1, 2, \ldots, n)$ represents the n-dimensional state space.

[18] Chari/Christiano/Kehoe 1994

[19] Ljungqvist/Sargent 2004, pp. 29.; Stein, Luke C. D., 2009. "Economics 210. Section 5," Lecture notes, Stanford University, http://www.stanford.edu/ lstein/ [25/11/2010].

3. The transition probabilities can therefore be summarized in a stochastic $n \times n$ matrix P with elements $P_{ij} = Prob(s_{t+1} = j|s_t = i)\ \forall i, j, t$ where $\sum_{j=1}^{n} P_{ij} = 1\ \forall i$.

4. Be π_0 a $n \times 1$ vector whose ith element defines the $Prob(s_0 = i)$ of being in state i at period zero with $\sum_{j=1}^{n} \pi_{01} = 1$. This can be used to make assertions over the unconditional probability distributions in other periods for example $s_1 \in S$ with $\pi_{1i} = Prob(s_1 = i)$:

$$\pi_{1i} = Prob(s_1 = i) = \sum_{j=1}^{n} \underbrace{Prob(s_1 = i|s_0 = j)}_{P_{ji}} \underbrace{Prob(s_0 = j)}_{\pi_{01}} \tag{2.7.46}$$

One can write this as $\pi_1 = P'\pi_0$ or in general $\pi_{t+1} = P'\pi_t$. Forward iteration leads to $\pi_t = (P')^t\pi_0$. For the probability distribution to be stationary it has to hold that $\pi_{t+1} = \pi_t$. Along the law of motion for the unconditional probability it therefore has to hold that $\pi = P'\pi$ or differently expressed $(I - P')\pi = 0$. Hence the stationary distribution π is an eigenvector associated to a unitary eigenvalue of P' (with $\sum_{i=1}^{n} \pi_i = 1$). As P was defined to be a stochastic matrix with $\sum_{j=1}^{n} P_{ij} = 1 \forall i$ and nonnegative elements. This means there is a positive probability for the transition from any state to any other state, hence P is an ergodic distribution. From the law of motion of unconditional probability one can define the limiting distribution across states as:

$$\lim_{t\to\infty} \pi_t = \lim_{t\to\infty} (P')^t\pi_0 \equiv \pi_\infty(\pi_o) \tag{2.7.47}$$

If the limiting distribution is equal for all initial distributions π_0 then one speaks of an asymptotically stationary Markov process with a unique invariant distribution, where the limiting distribution π_∞ is denoted as an ergodic distribution.

What implications does this have for the here examined outcomes of the stochastic Ramsey model? The optimality conditions (2.7.35) and (2.7.36), displayed again below, are revisited.

$$V_c(s^t) = \beta E_t V_c(s^{t+1})[F_k(s^{t+1}) + (1 - \delta)] \tag{2.7.48}$$
$$V_h(s^t) = -V_c(s^t)F_h(s^t) \tag{2.7.49}$$

If the stochastic process as described in 1. is of a Markov property equations (2.7.48) and (2.7.49) imply that for periods $t \geqslant 1$ allocations for consumption, labor and capital accumulation are time invariant functions of s and k (variables in the Ramsey planner's problem should be understood to be aggregate): $c(s, k)$, $h(s, k)$, $k'(s, k)$. If the stochastic Markov process $\{s_t, k_t\}$ now is stationary and ergodic, as described above, there exists a unique invariant distribution on the compact set $S \times [0, \bar{k}]$ ($\bar{k}$ is the maximal sustainable capital stock, S is the state space), P^{∞}, with:

$$Prob\{(k_t, s_t)\} = P^{\infty} \quad \forall t \tag{2.7.50}$$

$$\lim_{j\to\infty} Prob\{(k_{t+j}, s_{t+j})|k_t, s_t\} = P^{\infty} \tag{2.7.51}$$

In the case of a stationary Ramsey equilibrium there are two possible outcomes regarding the ex-ante tax rate on capital.

$$1. \quad P^{\infty}(\bar{\theta}_t = 0) = 1 \tag{2.7.52}$$

$$2. \quad P^{\infty}(\bar{\theta}_t > 0) > 0 \quad \text{or} \quad P^{\infty}(\bar{\theta}_t < 0) > 0 \tag{2.7.53}$$

Deciding for the sign of the ex-ante capital tax rate is its numerator, see equation (2.7.32): $\bar{\theta}_{t+1}(s^t) \geqslant (\leqslant) 0$ if and only if:

$$\sum_{s_{t+1}} p_t(s_{t+1}|s^t)\theta_{t+1}(s^{t+1})r_{t+1}(s^{t+1}) \geqslant (\leqslant) 0 \tag{2.7.54}$$

Merging equation (2.7.54) with the no-arbitrage condition (2.7.8) implies:

$$1 \leqslant (\geqslant) \sum_{s_{t+1}} p_t(s_{t+1}|s^t)[r_{t+1}(s^{t+1}) + (1-\delta)] \tag{2.7.55}$$

Substituting for the equilibrium price of bonds by the household's optimality condition (2.7.6) and for the factor price of capital by its marginal product yields:

$$1 \leqslant (\geqslant) \sum_{s^{t+1}|s^t} \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{u_c(s^{t+1})}{u_c(s^t)} [F_k(s^{t+1}) + (1-\delta)] \tag{2.7.56}$$

or respectively if $\bar{\theta}_{t+1}(s^t) \geqslant (\leqslant) 0$ then

$$u_c(s^t) \leqslant (\geqslant) \beta E_t u_c(s^{t+1})[F_k(s^{t+1}) + (1-\delta)] \tag{2.7.57}$$

Defining $H(s^t) \equiv \frac{V_c(s^t)}{u_c(s^t)}$ and substituting it into the optimality condition (2.7.48) of the Ramsey planner yields:

$$H(s^t)u_c(s^t) = \beta E_t H(s^{t+1})u_c(s^{t+1})[F_k(s^{t+1}) + (1-\delta)] \tag{2.7.58}$$

Defining $\omega(s^{t+1}) \equiv u_c(s^{t+1})[F_k(s^{t+1}) + (1-\delta)]$ and substituting for in equations (2.7.57) and (2.7.58) gives equations (2.7.59) and (2.7.60) respectively.

$$u_c(s^t) \leqslant (\geqslant)\beta E_t \omega(s^{t+1}) \tag{2.7.59}$$

$$H(s^t)u_c(s^t) = \beta E_t H(s^{t+1})\omega(s^{t+1}) \tag{2.7.60}$$

Merging these two equations finally yields:

$$H(s^t) \geqslant (\leqslant)\frac{E_t\omega(s^{t+1})H(s^{t+1})}{E_t\omega(s^{t+1})} \quad \text{if} \quad \bar{\theta}_{t+1}(s^t) \geqslant (\leqslant)0 \tag{2.7.61}$$

As it was assumed that the stationary Ramsey equilibrium, as the stochastic process is Markov, can be described by time-invariant allocation rules $c(s,k)$, $h(s,k)$ and $k'(s,k)$ this also has to be true for $\bar{\theta}_{t+1}(s^t)$, $H(s^t)$ and $\omega(s^t)$. The transition probability is denoted by $\pi(s'|s)$(see 1. from the definition of the stochastic process). Under the assumption of a stationary equilibrium equation (2.7.61) can be denoted as:

$$H(s,k) \geqslant (\leqslant)\frac{\sum_{s'} \pi(s'|s)\omega(s',k'(s,k))H(s',k'(s,k))}{\sum_{s'} \pi(s'|s)\omega(s',k'(s,k))} \equiv \Gamma H(s,k) \tag{2.7.62}$$
$$\text{if} \quad \bar{\theta}(s,k) \geqslant (\leqslant)0$$

The operator Γ is defined as the weighted average of $H(s',k'(s,k))$. Γ by definition satisfies $\Gamma H* = H*$ for any constant $H*$. The stationary equilibrium is not fixed but oscillates, hence one can assume that there is a minimum $H^-(s^-,k^-) = H^-$ and a maximum $H^+(s^+,k^+) = H^+$. Hence the limiting distribution for the following two cases has to be one. $H(s,k)$ fluctuates between the two extremities:

$$P^\infty[H(s,k) \geqslant H^-] = 1 \tag{2.7.63}$$

$$P^\infty[H(s,k) \leqslant H^+] = 1 \tag{2.7.64}$$

Equation (2.7.62) suggests something similar for the two possible outcomes:

$$P^\infty[H(s,k) \geqslant \Gamma H(s,k)] = 1 \tag{2.7.65}$$

$$P^\infty[H(s,k) \leqslant \Gamma H(s,k)] = 1 \tag{2.7.66}$$

Considering case (2.7.65) and assume a state $(s,k) = (s^-,k^-)$ with the possible state $\{s', k'(s,k); \forall s' \in S\}$ in the next period. Equation (2.7.63) also holds in the next period hence $H(s',k') \geqslant H^-$. By the fact that $H(s,k) = H^-$ and (2.7.65) this implies that $H(s',k') = H^-$. Forward iteration of this argument finally yields $H* = H^-$ as it was initially assumed $\{s_t, k_t\}$ to be a stationary ergodic stochastic process. The same argumentation can be applied for the pair of equations (2.7.64) and (2.7.66). Given the state $(s,k) = (s^+,k^+)$ there are possible states $\{s', k'(s,k); \forall s' \in S\}$ for the next period. By equation (2.7.64) it has to hold that $H(s',k') \leqslant H^+$. $H(s,k) = H^+$ together with

(2.7.66) implies that $H(s', k') = H^{+}$. Further applying this argumentation to successive periods finally yields $H* = H^{+}$. Hence it can be concluded that there is a constant $H*$ for which it holds that:

$$P^{\infty}[H(s,k) = H*] = 1 \tag{2.7.67}$$

Expression (2.7.67) and the inequality (2.7.62) imply that in a stationary equilibrium the optimal ex-ante capital tax rate oscillates around zero, hence $P^{\infty}(\bar{\theta}_t > 0) > 0$ and $P^{\infty}(\bar{\theta}_t < 0) > 0$.

2.8 Fiscal policy under a balanced budget assumption

Another assumption to which the hitherto derived result of an optimally zero capital tax in the steady state or, in a stochastic environment, in a stationary equilibrium, is robust to, is that of a period-by-period balanced budget constraint. It can easily be seen and it was also admitted that the result doesn't change if one sets all variables concerning government debt to zero in sections 2.3 or 2.4. If one chooses the primal approach the procedure is more complicated. In the primal formulation government debt doesn't appear explicitly. Therefore an additional constraint has to be introduced controlling for that. In the case of a balanced budget it obviously has to hold that:

$$g_t = \tau_t w_t h_t + \theta_t r_t k_t \tag{2.8.1}$$

This means the government finances its expenditures solely with the current tax revenues. Substituting for taxes and prices from the household's optimality conditions (2.1.10) and (2.1.11), the firms first order conditions (2.1.33) and (2.1.34) and for government purchases from the economy's resource constraint (2.1.2) in constraint (2.8.1) leads to an implementability constraint similar to that in the standard setting with government debt (compare to equation (2.5.11)).[20]

$$u_c(c_{t-1}, h_{t-1})k_t = \beta[u_c(c_t, h_t)(c_t + k_{t+1}) + u_h(c_t, h_t)h_t] \tag{2.8.2}$$

It would then be the Ramsey problem to maximize the aggregate welfare function subject to (2.8.2) and the economy's resource constraint (2.1.2). The results under the assumption of a balanced budget are qualitatively the same as in the setting with government debt: If the economy is in a steady state or has converged to a stationary allocation the optimal capital tax rate is zero or varies around zero respectively[21]. There are however

[20] Klein/Ríos-Rull 2003, p. 1222; Stockman, David R., 2001. "Balanced-Budget Rules: Welfare Loss and Optimal Policies," Review of Economic Dynamics, Elsevier for the Society for Economic Dynamics, vol. 4(2), pp. 438-459.

[21] compare section 2.3 to 2.5 and Stockman 2001

two differences: First, it was shown above that the capital tax rate is automatically zero for periods $t \geqslant 2$ for the class of utility functions (2.5.30) and (2.5.31). This not true under a balanced budget regime.[22] Second, as the government in the balanced budget setting is unable to save or to borrow and is hence forced to cover its expenditures by current tax revenues, there is no excessive tax levy in the initial period which the government could otherwise use to lower distortionary taxation the future. The front loading argument in the style presented in section 2.6 therefore fails.[23] Albanesi and Armenter however note that in such a case the government *"...manipulate[s] the path of consumption and capital..."* in a way which has the same effects as the front-loading of distortionary taxation.[24]

2.9 Summary and Interpretation

In the chapter 2 optimal fiscal policy was examined under the assumption that the policymaker can fully commit to a sequence of policies for all future periods. The central result was that capital taxes are optimally zero once the economy has converged to a steady state or varies around zero in a stationary equilibrium in a stochastic economy. This result is robust to wether the government is allowed to issue bonds or if it has to stick to a period-by-period balanced budget rule. Further it was shown that this is also true for an economy with heterogenous agents where the revenues from capital taxation could be eventually used to finance monetary transfers from one group of people to another. Even in an economy with a strict separation between the ownership of capital and the supply of labor, hence a capitalist- and a working-class, and where the focus of the policymaker only lies on the welfare of workers the result of a zero capital tax in the steady state holds. It was argued that the existence of a positive capital tax implies a permanent distortionary intertemporal wedge as it amounts to an implicit ever increasing tax on private good consumption which is obviously not compatible to the assumption of a steady state. In the initial period however the government does have the incentives to set capital taxes excessively high: In contrast to all other periods capital in the initial period is already installed, hence inelastically supplied, and it's taxation exhibits no intertemporal distortions for the saving/consumption behavior. Along Chamley it was shown for a certain class of preferences that optimal policy regarding capital taxation is characterized by two regimes: In the first regime capital taxes are very high. This phase, denoted as the tax recovery phase by Chamley, is used by the government to to build up a capital stock which's proceeds it can use to lower distortionary taxation the future.

[22]Stockman 2001, p. 451
[23]Lansing, Kevin J, 1999. "Optimal redistributive capital taxation in a neoclassical growth model," Journal of Public Economics, vol. 73, pp. 423-453.
[24]Albanesi/Armenter 2007, p. 12

The tax recovery phase, depending on wether one assumes an upper bound on capital taxes or not, lasts between two and a finite number of periods. These findings were argued formally by showing that it could be welfare increasing to front-load distortions: As mentioned above, positive capital taxes indirectly imply a differential taxation of private consumption goods, distorting the consumption/saving behavior. This implies that consumption is pricier in the future compared to the present, shifting incentives in favor of consumption than of saving. Along the argumentation of Albanesi/Armenter it was shown that a transfer of resources from the present to the future, this is exactly what the front-loading of distortionary capital taxes would do, leads to strictly positive welfare gains. Since a taxation strategy as described above demands the ability to stick to a decision over a certain sequence of policies it is obviously only applicable for a planner who possesses a commitment technology. This is at the same time the reason why solutions within a Ramsey equilibrium under the assumption of full commitment are time inconsistent: The fact that it is very likely that in the future a situation arises again which resembles that in the initial period, hence capital is already installed and incentives for the planner would be there to tax it, is simply elided. This point will be clarified in the next chapter were it is in contrast assumed that the policymaker doesn't have access to a commitment technology.

3 Optimal Fiscal Policy without Commitment

The model hitherto presented in chapter 2 exhibits an important assumption: Basing on the papers by Judd and Chamley it was presumed that governments possess a technology to credible guarantee or commit to a certain tax policy for all future periods. As Chamley indicates in his conclusions[1] the assumption of full commitment is however not very realistic. A huge range of occasions as unforeseen economic shocks or simply new elections are imaginable, which could tempt future governments to deviated from the initially implemented policy. Judd also addresses this problem, by suggesting that some of his result might strongly depend on his assumption that tax changes are announced and therefore anticipated before they are implemented.[2] After the seminal works by Judd and Chamley, on which chapter 2 bases, a number of papers emerged which try to model a more realistic optimal taxation environment by incorporating the time-inconsistent character of most policy decisions. One of these successive studies is the work by Klein and Ríos-Rull[3], which will be discussed later on in this chapter. Before however, in the next section, a Markov perfect equilibrium is defined which controls for the planners inability to commit itself to future tax policies. In contrast to the Ramsey equilibrium, where the decision over tax policy is taken once for all periods, the decision process in the Markov perfect equilibrium is sequentially. Governments cannot commit to tax policies beyond the current period in which they are in office. After the definition of a Markov equilibrium a simple two period model will be derived. This should give better insight in the decision problems faced by two sequential policy makers and, as a foretaste for the thereafter presented model by Klein/Ríos-Rull, provide some intuitive explanation for the results gained in a no-commitment environment.

3.1 Defining a standard Markov perfect Equilibrium

In an environment in which the government possesses no commitment technology, the equilibrium has to be modeled differently than in the Ramsey approach. In contrast

[1] Chamley 1985, p. 619
[2] Judd 1985
[3] Klein/Ríos-Rull 2003

to the model in chapter 2 policy decisions are not taken at one point in time, but are reevaluated sequentially. In optimizing social welfare the government therefore not only has to consider the best response by the private sector, but also the policy options and therefore possibilities to deviate from the initial policy future governments will have. The possibility of discretion in the future already influences allocation behavior of the private sector in the present. These presumptions are accounted for in a Markov approach. In a Markov setting the government cannot commit to future tax rates. It is however conscious about the fact along which rules deviating policies will be set in successive periods. The government therefore decides upon it's policy taking the current state of the economy and the policy rule for the next period as given and respecting the optimal allocations found on the private sector, which in turn also anticipates the inability of the government to commit to a future tax policy. It turns out that the problem of the planners resembles an infinitely repeated game in which all planners face the same optimization problem.

The basic model and notations The here examined Markov equilibrium bases on the same model layout used in the Ramsey setting in chapter 2 with a period-by-period balanced budget restriction for the government. In contrast to before the planner cannot commit to a future capital income tax policy. He knows however that a future deviation from the current policy takes place along the policy rule $\psi(K_t) = \theta_t$. For the formulation of this problem notations have to be slightly changed. As before perfect factor markets are assumed: $r_t = r(K_t, H_t) = F_K(K, H)$, $w_t = w(K_t, H_t) = F_H(K, H)$. Here k denotes the individual capital stock. The capital letters K and H refer to aggregate values of the capital stock and labor supply respectively.

Step 1 - the "current" government's maximization problem The solution process of a Markov equilibrium resembles backward induction, known from game theory. This is illustrated on two exemplary consecutive periods, denoted as the "former" and the "current" period. According to the backward induction method we are going to start with the "current" period. The household's optimization problem in the "current" period can be denoted as follows.

$$v(k, K; \psi) = \max_{c,k',h} \{u(c, h) + \beta v(k', K'; \psi)\} \tag{3.1.1}$$

$$s.t. \quad c + k' = (1 - \theta)r(K, H)k + (1 - \tau)w(K, H)h + k \tag{3.1.2}$$

$$K' = D_K(K; \psi) \tag{3.1.3}$$

$$H = D_H(K; \psi) \tag{3.1.4}$$

$$\theta = \psi(K) \tag{3.1.5}$$

$$\tau = \phi(K; \psi) \tag{3.1.6}$$

The notation of the social welfare function as a Bellman equation in (3.1.1) underlines the multi-period character of the optimization problem faced by the the households as well as by the government. Governments/households not only have to consider the present, they also have to account for the optimal decision a successive government will take based on the state resulting from the former (current) government's optimal policy decision. The household's optimization process, which demands to find the value function $v(k, K; \psi)$, is subject to the household's budget constraint (3.1.2), the law of motion for capital (3.1.3) and the labor supply function (3.1.4). Equations (3.1.5) and (3.1.6) represent the policy rules along which the government decides upon the capital income tax rate and the labor income tax rate respectively.

The lack of commitment here is explicitly modeled only for the capital income tax rate. Since however a balanced budget constraint applies for the government, this assumption is indirectly also valid for the labor income tax. As the solution strategy here resembles a backward induction approach, ψ is assumed to be exogenously given in this stage of the problem. If $d_h(k, K; \psi) = h$ and $d_k(k, K; \psi) = k'$ represent the solution for the individual problem, an overall equilibrium can then be defined, if

- the set of functions $\{v(k, K; \psi), d_k(k, K; \psi), d_h(k, K; \psi)\}$ solves the individual optimization problem given $\{D_K(K; \psi), D_H(K; \psi), \phi\}$,
- the individual level is representative for the aggregate level: $D_K(K; \psi) = d_k(K, K; \psi)$ and $D_H(K; \psi) = d_h(K, K; \psi)$,
- and the government's budget constraint is satisfied:
  $g = \theta r(K, D_H(K; \psi))K + \phi(K; \psi)w(K, D_H(K; \psi))D_H(K; \psi)$.

Along the solution for the stationary equilibrium from above an aggregate value function $V(K; \psi) = v(K, K; \psi)$ can be defined, which is evaluated by the "current" government, in anticipation of the private sector behavior, in order to maximize social welfare. Viewed from a "current" point of view, separated from other periods, the procedure is similar to the Ramsey approach.

Step 2 - the optimization problem of the "former" government or an optimal response found by backward induction In the next step the problem is examined from the perspective of one period before, resembling the backward induction method used for extensive form games. This is just theoretical however and only serves the purpose of demonstrating how an equilibrium is being found: All planners face exactly the same optimization problem. The planner correctly assumes that its successor honors the policy rule $\psi(K)$ and takes the optimal behavior of his successor, defined in the step above, as given (backward induction). The reference to "current" or "former" governments or periods therefore should only underline the methodological approach.

The problem for the households in the "former" period is similar as in the case before (in the "current" period).

$$\hat{v}(k,K,\theta;\psi) = \max_{c,k',h}\{u(c,h) + \beta v(k',K',\theta')\} \tag{3.1.7}$$

$$s.t. \quad c + k' = (1-\theta)r(K,H)k + (1-\tau)w(K,H)h + k \tag{3.1.8}$$

$$K' = \hat{D}_K(K,\theta;\psi) \tag{3.1.9}$$

$$H = \hat{D}_H(K,\theta;\psi) \tag{3.1.10}$$

$$\tau = \phi(K,\theta;\psi) \tag{3.1.11}$$

$$\theta' = \psi(K') \tag{3.1.12}$$

The laws for labor supply and capital accumulation, $D_H(K;\psi) = \hat{D}_H(K,\theta;\psi)$ and $D_K(K;\psi) = \hat{D}_K(K,\theta;\psi)$ respectively, as well as the function for the labor income tax are the same as in the step presented above. Subcondition (3.1.12) implies that the planner in the "former" period is aware of the fact that the state in which he bequeaths the economy to his successor, the "current" government, influences the policy decision in the next, the "current", period and hence the expectations in the present, the "former", period. Several functions therefore contain θ, the capital tax rate applying in the present period, and $\psi(.)$ the policy rule according to which the "current" planner will decide upon the capital tax rate in the "current" (the next from the perspective of the "former" planner) period.

As before an equilibrium can be defined. The next period's equilibrium $\{v, d_k(k,K;\psi), d_h(k,K;\psi), D_K(K;\psi), D_H(K;\psi), \phi\}$, defined above, and the policy rule ψ are taken as given (are anticipated). It is supposed that $\hat{d}_h(k,K,\theta;\psi) = h$ and $\hat{d}_k(k,K,\theta;\psi) = k'$ represent the solution for the individual problem. Several functions are marked with hats: Although they fulfill the same role as in the step presented above, they now depend on the expectations over the next ("current") government's policy choice for the capital income tax rate. An overall equilibrium can then be defined, if

- the set of functions $\{\hat{v}, \hat{d}_k(k,K,\theta;\psi), \hat{d}_h(k,K,\theta;\psi)\}$ solves the individual optimization problem, given $\{\hat{D}_K(K,\theta;\psi), \hat{D}_H(K,\theta;\psi), \hat{\phi}, \psi\}$,
- the individual level is representative for the aggregate level: $\hat{D}_K(K,\theta;\psi) = \hat{d}_k(K,K,\theta;\psi)$ and $\hat{D}_H(K,\theta;\psi) = \hat{d}_h(K,K,\theta;\psi)$,
- the government's budget constraint is satisfied: $g = \theta r(K, \hat{D}_H(K,\theta;\psi))K + \hat{\phi}(K,\theta;\psi)w(K,\hat{D}_H(K,\theta;\psi))\hat{D}_H(K,\theta;\psi)$.

The aggregate value function, which is the basis for the "former" government's welfare assessment, is defined as $\hat{V}(K,\theta;\psi) = \hat{v}(K,K,\theta;\psi)$. A government which expects the successive government to honor ψ solves for

$$\Psi(K,\psi) = \arg\max_{\theta} \hat{V}(K,\theta,\psi) \tag{3.1.13}$$

If a current ("former") government therefore knows that the future government (the "current" government) follows a certain policy, it prefers the same policy. The equilibrium under the assumption of no-commitment therefore can be regarded as a fixed point problem.

A Markov equilibrium in the here examined environment resembles an infinitely repeated game: All planners mutually give the best responses to the anticipated optimal behavior of the other. As this results into a Nash-equilibrium in every period (every sub-game), it is denoted as sub-game perfect. It can be compared to an infinitely repeated prisoners dilemma, where the optimal Markov-strategy leading to an unique Markov-perfect equilibrium is to defect in every period regardless of past moves.[4]

3.2 An analytical example

Most of the models under a no-commitment assumption are not solvable analytically, especially if the government underlies a balanced budget constraint as it is considered by Klein/Ríos-Rull[5], who's model framework and results will be discussed later on in this chapter[6]. To better understand the solution procedure of a Markov-perfect equilibrium and the intuition behind its result a two-period model by Fernando M. Martin[7] is going to be considered, which gets along without elaborated numerical methods as used by Klein/Ríos-Rull or Debortoli/Nunes[8].

In contrast to the hitherto used model framework in the Ramsey environment and to the sketch of the Markov perfect equilibrium above the model by Martin features endogenous government purchases and full capital depreciation in each period. Further, to arrive at interpretable results, specific preferences and a specific production function have to be

[4]Fudenberg, Drew & Tirole, Jean, 1991. "Game Theory," MIT Press, p. 502.

[5]Klein/Ríos-Rull 2003

[6]a paper which solves for a Markov perfect equilibrium analytically and allows for government borrowing is: Azzimonti-Renzo, Marina & Sarte, Pierre-Daniel & Soares, Jorge, 2006. "Optimal Policy and (the Lack of) Time Inconsistency: Insights from Simple Models," University of Delaware, Department of Economics, Working Papers.

[7]Martin, Fernando M., 2010. "Markov-perfect capital and labor taxes," Journal of Economic Dynamics and Control, vol.34(3), pp. 503-521.

[8]Debortoli/Nunes 2007

defined. The notation is the same as before. Preferences are given by:

$$u(c_t, 1-h_t, g_t) = \alpha_c \ln(c_t) + \alpha_h \ln(1-h_t) + \alpha_g \ln(g_t) \tag{3.2.1}$$

with $\alpha_c = \alpha(1-\alpha_g)$, $\alpha_h = (1-\alpha)(1-\alpha_g)$ and $\alpha \in (0,1)$, $\alpha_g \in (0,1)$, $\alpha_c + \alpha_h + \alpha_g = 1$. The production technology is given by:

$$y_t = F(k_t, h_t) = k_t^{\gamma} h_t^{1-\gamma} \tag{3.2.2}$$

Again perfectly competitive factor markets are assumed: The factor prices of capital and labor thus equal their marginal products: $F_k(k_t, h_t) = r_t$ and $F_h(k_t, h_t) = w_t$. The economy's resource constraint, the government's budget constraint and the household's budget constraint are respectively given by:

$$c_t + g_t + k_{t+1} = y_t \tag{3.2.3}$$
$$g_t = \theta_t r_t k_t + \tau_t w_t h_t \tag{3.2.4}$$
$$c_t + k_{t+1} = (1-\theta_t) r_t k_t + (1-\tau_t) w_t h_t \tag{3.2.5}$$

The optimization process of the households yields standard optimality conditions.

$$u_c(c_t, h_t) = \beta u_c(c_{t+1}, h_{t+1})(1-\theta_{t+1}) r_{t+1} = 0 \tag{3.2.6}$$
$$u_h(c_t, h_t) = u_c(c_t, h_t)(1-\tau_t) w_t \tag{3.2.7}$$

Note that optimality condition (3.2.6) takes into account the full capital depreciation each period. The different signs in optimality condition (3.2.7) in contrast to the model presented in chapter 2 are due to the specific definition of the utility function. Now we are about to characterize a Markov perfect equilibrium in two periods. The equilibrium is solved for by backward induction starting in period two like in the sketch of the equilibrium definition in the preceding section.

Period 2 In a Markov perfect equilibrium the government and the households base their decision over their behavior on fundamentals. As the government lacks of commitment it cannot influence these, current, fundamentals. This is especially true for the second and therefore the last period: The government in period $t = 2$ observes the capital stock k_2 inherited from the last period and sets taxes and government purchases taking into account private sector allocations and factor prices $F_k(k_t, h_t) = r_t$ and $F_h(k_t, h_t) = w_t$. Note that since $t = 2$ is the last period no more capital is accumulated and all resources are used for consumption. The resource constraint for period $t = 2$ is therefore $c_2 = y_2 - g_2$. Period two tax rates on capital and labor income can be expressed using optimality condition (3.2.7), the government budget constraint (3.2.4) as well as the equilibrium factor prices for capital and labor. For notational simplicity functions like

$u_h(c_2, h_2)$ are simply denoted as $u_h(2)$.

$$\tau_2 = 1 - \frac{u_h(2)}{u_c(2)F_h(2)} \tag{3.2.8}$$

$$\theta_2 = \frac{g_2}{F_k(2)k_2} - \frac{F_h(2)h_2\tau_2}{F_k(2)k_2} = \frac{1}{\gamma}\frac{g_2}{y_2} - \frac{1-\gamma}{\gamma}\left[1 - \frac{u_h(2)}{u_c(2)F_h(2)}\right] \tag{3.2.9}$$

The government's problem in the second period, given $k_2 > 0$, can be denoted as:

$$\max_{h_2, g_2} u(y_2 - g_2, 1 - h_2, g_2) \tag{3.2.10}$$

Note that c_2 was substituted for in the utility function with the help of the second period resource constraint. The government's optimality conditions are:

$$u_c(2)F_h(2) - u_h(2) = 0 \tag{3.2.11}$$

$$u_c(2) - u_g(2) = 0 \tag{3.2.12}$$

Optimality condition (3.2.11) together with (3.2.8) implies that $\tau_2 = 0$. Government purchases in period $t = 2$ therefore have to be entirely financed by taxes levied on capital income. The intuition behind this result is similar to the one time capital levy in the first period in the Ramsey framework (compare to section 2.5): The government aims to reduce distortions as much as possible. Since the capital stock k_2 is already installed, taxing it is not distortionary. Labor income taxes are however distortionary regarding the labor (leisure)/consumption decision. The government thus sets labor taxes in period two to zero and finances its expenditures by capital taxes only. Taking this result as given it can be solved for the rest of the period two variables. Using optimality condition (3.2.12) together with the period two resource constraint $c_2 = y_2 - g_2$ allows to express c_2 and g_2. This results together with equation (3.2.9) and the fact that $\tau_2 = 0$ allows to solve for θ_2.

$$c_2 = \frac{\alpha_c}{\alpha_c + \alpha_g} y_2 = (1 - p_2)y_2 \quad \text{with} \quad p_2 \equiv \frac{\alpha_g}{\alpha_c + \alpha_g} \tag{3.2.13}$$

$$g_2 = y_2 p_2 \tag{3.2.14}$$

$$\theta_2 = \frac{p_2}{\gamma} \tag{3.2.15}$$

Equation (3.2.13) together with optimality condition (3.2.11) finally allows to solve for h_2.

$$h_2 = \frac{1-\gamma}{1 - \gamma + \frac{\alpha_h}{\alpha_c}(1 - p_2)} = \frac{1-\gamma}{1 - \gamma + \frac{\alpha_h}{\alpha_g} p_2} \tag{3.2.16}$$

Period 1 The capital stock k_2 is the economy's state variable which the period two government inherits from the period one government. It is determined by the consumption/saving behavior of the households in period one and their anticipation of the capital tax policy in period two. Thus, to derive k_2 the period one Euler equation (3.2.6) is used together with the period one resource constraint (3.2.3), the decision rules for period two consumption (3.2.13) and period two capital taxes (3.2.15) (note that this decision rule is similar to the policy rules $\psi(.)$ in the sketch of the Markov equilibrium in the section before).

$$
\begin{aligned}
&u_c(1) = \beta u_c(2)(1-\theta_2)F_k(2) \rightarrow \\
&\frac{\alpha_c}{c_1} = \beta\frac{\alpha_c}{c_2}(1-\theta_2)F_k(2) \quad \text{with} \quad c_1 = y_1 - g_1 - k_2 \rightarrow \\
&[y_1 - g_1] = k_2\left[1 + \frac{(1-p_2)}{\beta(1-\theta_2)\gamma}\right] \quad \text{with} \quad \theta_2 = p_2/\gamma \rightarrow \\
&k_2 = [y_1 - g_1]\left[1 - \frac{\alpha_c}{\alpha_c[1+\beta\gamma] - \beta\alpha_g[1-\gamma]}\right] = [y_1 - g_1]v_1 \qquad (3.2.17)\\
&\text{with} \quad v_1 \equiv \left[1 - \frac{\alpha_c}{\alpha_c[1+\beta\gamma] - \beta\alpha_g[1-\gamma]}\right]
\end{aligned}
$$

The interpretation of equation (3.2.17) is twofold: First, capital "saved" for period two in period one is represented by a fixed fractional part of period one production minus period one government purchases. Second, one has to note that by definition $F(0,h) = 0$. Thus, if there is no capital transferred from period one to period two the economy would collapse. To get an interior solution a lower bound on the capital transferred to the second period has to be therefore implemented: $k_2 > 0$. As can be easily verified by examining equation (3.2.17) this is only the case if $\theta_2 < 1$ thus $v_1 > 0$. Using this restriction on equation (3.2.15) implies for an interior solution that $\alpha_g \in \left(0, \frac{\gamma\alpha}{1-\gamma(1-\alpha)}\right)$.[9] Without this restriction households would have no incentives to transfer capital from period one to period two. Given $k_2 = v_1(y_1 - g_1)$ from (3.2.17) period one consumption can be expressed using the period one resource constraint (3.2.3): $c_1 = (1-v_1)(y_1 - g_1)$. Using results (3.2.13) and (3.2.14) from period two as well as c_1 and k_2 from above allows

[9] Setting $\theta = p_2/\gamma < 1$ and using $\alpha_c = \alpha(1-\alpha_g)$ after some manipulations gives $\alpha_g < \frac{\gamma\alpha}{1-\gamma(1-\alpha)}$.

to express the period one government's maximization problem:

$$
\begin{aligned}
&\max_{h_1,g_1} u(c_1, 1-h_1, g_1) + \beta u(c_2, 1-h_2, g_2) \qquad (3.2.18)\\
&\text{with}\\
&c_1 = (1-v_1)(y_1 - g_1)\\
&k_2 = v_1(y_1 - g_1)\\
&c_2 = (1-p_2)y_2\\
&g_2 = p_2 y_2
\end{aligned}
$$

Deriving (3.2.18) with respect to h_1 and y_1 under consideration of the interdependencies of the different variables yields the government's first order conditions:

$$u_c(1)(1-v_1)F_h(1) - u_h(1) + \beta v_1 F_k(2) F_h(1)[u_c(2)(1-p_2) + u_g(2)p_2] = 0 \qquad (3.2.19)$$

$$-u_c(1)(1-v_1) + u_g(1) - \beta v_1 F_k(2)[u_c(2)(1-p_2) + u_g(2)p_2] = 0 \qquad (3.2.20)$$

Using first oder condition (3.2.12) from the government's maximization problem of the second period and the household's Euler equation (3.2.6) in the optimality condition (3.2.20) from the government's maximization problem of the first period,

$$v_1 \left[u_c(1) - \frac{u_c(1)}{(1-\theta_2)}\right] - u_c(1) + u_g(1) = 0$$

and substituting along the definition for v_1 yields:

$$
\begin{aligned}
&u_c(1)\Omega_1 = u_g(1) \qquad (3.2.21)\\
&\text{with}\\
&\Omega_1 \equiv \frac{\beta(\alpha_c + \alpha_g) + \alpha_c}{\beta\gamma(\alpha_c + \alpha_g) - \beta\alpha_g + \alpha_c}
\end{aligned}
$$

By the assumption $\alpha_g \in \left(0, \frac{\gamma\alpha}{1-\gamma(1-\alpha)}\right)$ from before it holds that $\Omega_1 > 1$. Expression (3.2.21) is obviously similar to (3.2.12), an optimality condition of the government in the second period. In contrast to the second period however the margin between private and public consumption is distorted by the wedge $\Omega_1 > 1$. The second condition crucial for the problem can be derived by fusion of optimality condition (3.2.19) and (3.2.20):

$$u_g(1)F_h(1) = u_h(1) \qquad (3.2.22)$$

As can be seen by combination of optimality conditions (3.2.11) and (3.2.12) this is the same optimality condition as in the second period. Additionally the government in the first period in contrast to that in the second period is confronted with intertemporal distortions caused by the anticipated positive tax rate on capital income in the second period, which has, as it was shown before, implications for the consumption/saving behavior in period one. Now the remaining period one variables can be computed: Merging equations (3.2.21), (3.2.22) and the household's optimality condition (3.2.7) for period one allows to solve for the period one labor income tax rate:

$$\tau_1 = 1 - \Omega_1 \tag{3.2.23}$$

Using equation (3.2.21) together with the before derived preliminary result allows to solve for g_1:

$$g_1 = \frac{\alpha_g}{\alpha_c}\left[\frac{(\alpha_c+\alpha_g)\beta\gamma+\alpha_c-\alpha_g\beta}{(\alpha_c+\alpha_g)\beta\gamma+\alpha_c}\right]\left[\frac{\alpha_c}{\alpha_c(1+\beta\gamma)-\alpha_g\beta(1-\gamma)}\right][y_1-g_1]$$

$\rightarrow$

$$g_1 = p_1 y_1 \quad \text{with} \quad p_1 \equiv \left[\frac{\alpha_g}{(\alpha_c+\alpha_g)(1+\beta\gamma)}\right] \tag{3.2.24}$$

The solution of g_1 from (3.2.24) allows to complete the preliminary result for c_1 from before:

$$c_1 = (1-v_1)(1-p_1)y_1 \tag{3.2.25}$$

The same is true for k_2:

$$k_2 = v_1(1-p_1)y_1 \tag{3.2.26}$$

With these results in hand, especially the period 2 capital stock k_2, one can solve for the first period's capital income tax rate. Merging (3.2.24) with the period one government budget constraint yields:

$$\theta_1 = \frac{p_1-(1-\Omega_1)(1-\gamma)}{\gamma} \tag{3.2.27}$$

Finally merging (3.2.24) with (3.2.22) leads to the solution of h_1:

$$h_1 = \frac{1-\gamma}{1-\gamma+p_1\frac{\alpha_h}{\alpha_g}} \tag{3.2.28}$$

Concerning the tax policy in the hitherto presented exemplary Markov model one can state summarizing that capital taxes are positive in both periods. In both periods government purchases are financed solely by capital taxes. In the second period labor taxes are zero, in the first period, see (3.2.23), labor income is even subsidized since $\Omega_1 > 1$.

Intuition What is the intuition behind the results of this section? In the full commitment case, compare chapter 2, positive capital taxes were only observed in a finite number of initial periods, the length depending on wether one assumes an upper limit on capital taxes or not. In section 2.6 this was argued by the front loading argument: Every tax exhibits some kind of distortion. Distortions by the labor tax are intratemporal, while those caused through capital taxes are generally intertemporal. The initial period however is an exception. Capital is already installed and current capital taxes can thus have obviously no effect on the capital accumulation behavior (in the preceding period). The government exploits this fact by levying excessively high taxes on capital in the initial period to build up a capital base. The government can then use the proceeds of this capital base to at least partly finance government purchases in the future and thus lower the distortions through taxation. In the full commitment case, it was further argued, that capital taxes were optimally zero, either for a certain class of utility functions[10] for periods $t \geqslant 2$ or under the assumption that the economy had converged to a steady state (or to a stationary equilibrium in the stochastic environment). The second case was reasoned by the fact that positive capital taxes would indirectly imply a differential taxation on private good consumption which is contradictory to the steady state assumption. In the no-commitment scenario in contrast, decisions are not taken once for all periods but sequentially. Each government, it is assumed to be a new one in each period, is thus confronted with a situation similar to the initial period in the full commitment case. Since the capital is perceived to be inelastically supplied in every period, the capital is already installed and because of the lack of commitment the preceding periods are beyond the control of the current planner, current capital taxes are regarded as non-distortionary by the current government. From the perspective of households on the other hand one might state that positive capital taxes are partly unforeseen and therefore not that distortionary. The lack of commitment further has the consequence that planners are unable to front-load distortions as it was observed in the full commitment case. As will be discussed later on, this hinges however on the considered model environment. The interpretation of the labor tax rates in the two periods is as follows: While current capital taxes exhibit no distortions in the no-commitment economy the anticipated future taxes on capital still do. They imply a differential taxation on consumption, just like they would in the full commitment case as described above. In the full commitment environment however the planner had the commitment technology to abolish these intertemporal distortions by eventually setting capital tax rates to zero in the future (compare section 2.5). This is something the Markov planner is unable to do. Instead he tries to reduce the distortions on the consumption/labor(leisure) decision. A positive tax rate on capital in the next period discourages labor supply. The government counters this effect with zero labor taxes or even labor subsidies as in the

[10] see (2.5.30) and (2.5.31)

first period.[11]

3.3 Time-consistent fiscal policy by Klein/Rios-Rull

3.3.1 The basic model and notations

The model by Klein and Ríos-Rull studied in this section is in its basis a standard stochastic growth model like the one examined in section 2.7. A major difference however is the fact that the government doesn't have commitment instruments to guarantee its capital tax policy beyond the next period. In such a setting an equilibrium therefore cannot be found with the standard Ramsey approach which was applied in the models in chapter 2. As governments have to consider the chance of a policy reassessment by successive governments they have to incorporate this in their decision over the optimal policy. Under such circumstances an equilibrium can be found in a Markov setting. A standard form of a Markov equilibrium was sketched in section 3.1 introductorily to this chapter. The here considered concept along the paper by Klein/Ríos-Rull is a little bit different. This is difference is going to be discussed in the next section when the equilibrium for this specific model is defined. Like in section 3.1 and section 3.2, but in contrast to the stochastic Ramsey model presented in section 2.7, Klein/Ríos-Rull assume that the government cannot issue debt. Klein/Ríos-Rull argue this assumption by the fact that it would be inappropriate to introduce a no-ponzi rule for a government. Alternatively a per period upper limit on debt issuance would be necessary. This however would be incompatible to the linear-quadratic approach they use to calculate the numerical results.[12]

In the model economy, the variables g and z stand for government purchases and factor productivity, which are both finite and driven by a Markov process[13]. The economy is subject to the occurrence of possible shocks. This pair of shocks is denoted by $s \in S$, a n-element set. Γ stands for the transition matrix of shocks, capturing the possibility of a future shock with respect to the situation today: Therefore, an element $\Gamma^{ss'}$ of the matrix Γ would represent the probability of a shock s' tomorrow having s today. Just like in section 2.7 Klein/Ríos-Rull use the notation s_t to denote a particular realization of a shock in period t and $s^t = \{s_0, \dots, s_t\}$ for a particular history up to period t. The unconditional probability for history s^t is denoted by $\Gamma(s^t)$.

The representative consumer maximizes his utility by choosing a stream of consumption and leisure. His preferences are therefore given as follows. The variables $c(s^t)$and $h(s^t)$

[11] Martin 2009
[12] Klein/Ríos-Rull 2003, p. 1218
[13] see section 2.7.5

denote consumption and hours worked as functions of the particular history s^t.

$$E\left\{\sum_t \beta^t u(c(s^t), h(s^t))\right\} = \sum_t \beta^t \sum_{s^t} \Gamma(s^t) u(c(s^t), h(s^t)) \tag{3.3.1}$$

The optimization process is subject to a feasibility constraint.

$$F(k(s^{t-1}), h(s^t), z(s^t)) + (1-\delta)k(s^{t-1}) = c(s^t) + k(s^t) + g(s^t) \tag{3.3.2}$$

This constraint states that consumption $c(s^t)$, government purchases $g(s^t)$ and capital stock $k(s^t)$ have to equal the production via a neoclassical production process $F(k(s^{t-1}), h(s^t), z(s^t))$ minus the depreciation of the capital stock available at s^t, $k(s^{t-1})$, by the factor δ. The capital available in period t, $k(s^{t-1})$, was accumulated under consideration of state s^{t-1} in period $t-1$. Households underlie a budget constraint and are subject to labor $\tau(s^t)$ and capital $\theta(s^t)$ taxation.

$$c(s^t) + k(s^t) = (1-\tau(s^t))w(s^t)h(s^t) + [1 + r(s^t)(1-\theta(s^t))]k(s^{t-1}) \tag{3.3.3}$$

The variables $w(s^t)$ and $r(s^t)$ denote the factor prices for labor and capital (net of depreciation) respectively. Since perfect competition on the factor markets is assumed we can express these variables as $F_h(k(s^{t-1}), h(s^t), z(s^t)) = w(s^t)$ and $F_k(k(s^{t-1}), h(s^t), z(s^t)) - \delta = r(s^t)$. The government's budget constraint can be denoted as:

$$g(s^t) = \theta(s^t)r(s^t)k(s^{t-1}) + \tau(s^t)w(s^t)h(s^t) \quad \forall \quad s^t \tag{3.3.4}$$

A stochastic policy vector $\pi = \{\theta(s^t), \tau(s^t)\}$ denotes the policy for capital and labor taxation. A balanced budget equilibrium now would be a set of stochastic processes $c(s^t)$, $h(s^t)$, $k(s^t)$, $r(s^t)$, $\theta(s^t)$ and $\tau(s^t)$ which maximizes the consumer's utility function (3.3.1) subject to the feasibility constraint (3.3.2), the household's budget constraint (3.3.3) and the government's budget constraint (3.3.4), taking perfect factor markets, an initial capital income tax rate θ_0 and the initial capital stock k_0 as given.

The work of Klein and Ríos-Rull focuses on the examination of time-consistent fiscal policy, hence on policy results found within a Markov setting. As a kind of reference model however they consider a full-commitment case of the above presented stochastic economy. The therefore applied procedure is the same as the one presented in section 2.7. The only difference here is that Klein and Ríos-Rull, in contrast to section 2.7, don't allow for government borrowing. The procedure is hence a mix of that in sections 2.7 and 2.8, where an implementability constraint for the balanced budget case was constructed. For the sake of completeness the finding of an equilibrium within the full-commitment case of the above presented model economy is going to be sketched in the following section.

3.3.2 A reference example with full commitment

The process of finding an equilibrium in a Ramsey environment is straightforward and resembles the procedure applied in chapter 2. I therefore refrain from the calculations already conducted extensively in the aforementioned chapter and limit myself to stating the problem of the Ramsey planner in primal form:

$$\max_{c(s^t),h(s^t),k(s^{t-1})} \sum_{t,s^t} \Gamma(s^t)\beta^t u(c(s^t),h(s^t)) \tag{3.3.5}$$

$$s.t.$$

$$u_c(c(s^t),h(s^t))k(s^{t-1}) = \beta E_t \left\{u_c(c(s^{t+1}),h(s^{t+1}))(c(s^{t+1})+k(s^t)) \right.$$
$$\left. +u_h(c(s^{t+1}),h(s^{t+1}))h(s^{t+1})]\right\} \tag{3.3.6}$$

$$F(k(s^{t-1}),h(s^t),z(s^t)) + (1-\delta)k(s^{t-1}) = c(s^t)+k(s^t)+g(s^t) \tag{3.3.7}$$

As defined in section 2.2 the social planner has to find a policy vector $\pi = \{\theta(s^t),\tau(s^t)\}$ which maximize social welfare, rewritten in equation (3.3.5), under consideration of the optimal allocations and prices found on the private sector and the household's budget constraint all captured by the implementability constraint (3.3.6) and the economy's resource constraint (3.3.7). For the construction of the implementability constraint under a period-by-period balanced budget assumption the reader is referred to section 2.8. In assuming that the economy converges to a stationary allocation, in the long run the accumulation of capital is undistorted and capital income is therefore optimally not taxed. This is the same result as gained in chapter 2. A formal derivation of this result in an economy with government borrowing as well a specific definition of the underlying stochastic process was given in section 2.7.

3.3.3 The economy without commitment - Definition of the Equilibrium

In contrast to the setting in chapter 2 with full-commitment analytical solutions in a no-commitment scenario are not feasible in most of the cases and numerical solutions are necessary. A very simplified exception was presented before in section 3.2. Analytical solutions are also inapplicable in the case of the model by Klein and Ríos-Rull. Since the description of the numerical/computational techniques would be beyond the scope of this thesis, I will constrain myself to the definition of the equilibrium. The intuition of the results is however similar to the one in the simple two-period model presented in 3.2. The main idea in the paper of Klein and Ríos-Rull is that each government is just in office for one period. The government currently in office commits itself to a capital income tax θ and observes the realization of possible shocks. At the same time it constitutes a vector of various contingent capital income tax rates corresponding exactly to various economic shocks which could occur in the next period. The government following in office has to stick to the rules for a state contingent capital tax set by the former government.

Hence there a two important differences to the equilibrium sketched in introductorily to this chapter in section 3.1: First, in section 3.1 each of the different successional governments decided upon the tax policy for the respective period in which they are in office while the governments in the model by Klein/Ríos-Rull inherit/bequeath a vector of state contingent capital taxes from/to the former/succeeding government to which the respectively concerned government has to stick. Second, Klein and Ríos-Rull assume in contrast to section 3.1 that the government is subject to stochastic productivity and expenditure shocks. Note, that the assumption of full- or no-commitment is a general one and doesn't rest upon the occurrence of shocks. The incorporation of stochastic shocks has to be rather understood as an additional conceptual instrument to render the process of policy reevaluation (discretion), which is the central feature also in the non-stochastic Markov setting, more realistic.

The private sector as before optimizes its consumption/saving and working behavior. Since the capital tax rate is assumed to be state contingent and the government has to honor a period-by-period balanced budget constraint the control over the labor income tax τ is very limited. The economy is subject to expenditure, g, and productivity, z, shocks summarized in the state vector $s = \{g, z\}$. K and θ denote the aggregate capital stock and the tax rate prescribed by the rules of the former government respectively. Along Klein/Ríos-Rull, these variables are pooled in the state vector $x = \{g, z, K, \theta\}$. The special feature of this economy is that the tax rate on capital income θ evolves in an arbitrary manner depending on the current state x, as well as on a particular future shock s'. The government in office sets, based on the current state of the economy, rules for the capital income tax θ in the next period depending on a set of shocks which could possibly occur. Or, to express it the other way around, the future government inherits a particular capital income tax from its predecessor as a particular shock s' appears. The predecessor, who in turn also inherited a tax rate from his predecessor, had before decided over a set of tax rates fitting to a set of shocks which could occur in the following period when he is not in office anymore. The available information at this point of time was just the particular state of the economy x. The law of motion for the capital income tax rate can therefore be given as $\theta(s') = \psi(x)$. To complete the references for the used notation here: as before k is the capital stock in per capita terms, H and K denote the aggregate values for labor supply and capital stock respectively. Taking all these assumptions as given households solve the following maximization problem:

$$\max_{c,k',h} u(c,h) + \beta \sum_{s'} \Gamma^{ss'} v(x'(s'), k'; \psi) = v(x,k;\psi) \tag{3.3.8}$$

$s.t.$

$$c + k' = (1-\theta) r(z,K,H)k + (1-\tau)w(z,K,H)h + k \tag{3.3.9}$$

$$K' = D_K(x;\psi) \tag{3.3.10}$$

$$H = D_H(x;\psi) \tag{3.3.11}$$

$$\tau = \phi(x;\psi) \tag{3.3.12}$$

$$\theta(s') = \psi(x) \tag{3.3.13}$$

Equation (3.3.9) gives the individual budget constraint. Sub condition (3.3.10) gives the law of motion for capital. Equation (3.3.11) captures the determination of the aggregate labor supply. Equations (3.3.12) and (3.3.13) give the labor income tax rate for the current period and the state depending capital income tax rate for the next period. The variables $r(z,K,H)$ and $w(z,K,H)$ denote the factor prices for capital and labor respectively. ψ is the vector of policy rules set by the former government and has to be obeyed by the current government and is, at least at this step of the solution, exogenously given. The functions $D_K(x;\psi)$ and $D_H(x;\psi)$ and ϕ are to be determined in the equilibrium. Assuming that $h = d_h(x,k;\psi)$ and $k' = d_k(x,k;\psi)$ are the solutions for the individual agents maximization problem (3.3.8), the recursive Markov equilibrium can be defined.

The problem of the "current" government This sections corresponds to the solution step 1 of the Markov equilibrium presented in section 3.1. As in the full-commitment case from chapter 2, the current government takes the optimality conditions of the private sector, d_h and d_k, as well as ϕ as given and maximizes the aggregate welfare with respect to its feasibility and budget constraint. The set of functions $\{v(x,k';\psi), D_K(x;\psi), D_H(x;\psi), \phi(x;\psi), d_k(x,K;\psi), d_h(x,K;\psi)\}$ therefore only describes a stationary equilibrium, if

- $\{v(x,k';\psi), d_k(x,K;\psi), d_h(x,K;\psi)\}$ solve the individual optimization problem (3.3.8) taking $\{D_K(x;\psi), D_H(x;\psi), \phi\}$ as given
- the assumption holds that the individual solution is representative for the aggregate level, therefore that $D_K(x;\psi) = d_k(x,K;\psi)$ and $D_H(x;\psi) = d_h(x,K;\psi)$,
- and the overall solutions satisfy the budget constraint of the government (3.3.14):

$$g = \theta r(z,K,D_H(x;\psi))K + \phi(x;\psi) w(z,K,D_H(x;\psi)) D_H(x;\psi) \tag{3.3.14}$$

The interpretation of the government's budget constraint (3.3.14) is pretty clear. Government expenditures g have to be financed by capital income taxes (the first term on the righthand side) and by labor taxes (the second term on the righthand side). The individual optimal decision over working hours $d_h(.)$ and capital accumulation $d_k(.)$ determines the aggregate optimal labor supply $D_H(.)$ and aggregate capital accumulation $D_K(.)$. It is assumed that the individual level is representative for the aggregate level, which in turn has implications for wages $w(.)$ and the return on capital $r(.)$. In optimizing the social welfare the benevolent "current" planner takes the following aggregate value function.

$$V(x;\psi) = v(x, K; \psi) \tag{3.3.15}$$

The problem of the "former" government and the decision over the bequeathed policy function This section corresponds to step 2 of the solutions of the Markov equilibrium in section 3.1: In the style of backward induction we now view the problem from one period ahead and hence from the position of the "former" government, taking the optimum and therefore the optimal response of the households and the "current" government in the next period as given (derived in the section above).

The terms "current" and "former" government might be misleading. They were chosen to better point out the solution strategy of the equilibrium. The general problem constellation is the same for every government (except the one in period 0 where no policy promises were yet made). It should just help to clarify the mechanisms of mutual best responses by planners in different periods which are not able to commit to a chosen policy. The basic solution concepts were already discussed in the introductorily section 3.1. It seems therefore reasonable to examine the models under consideration with reference to this conceptual outline.

In the here examined case, contrasting to the conceptual presentation in section 3.1, the "former" government not only has to formulate its own optimal response to the discretionary behavior of its "successor" in the sense of a possible policy reassessment under consideration of the current economic situation, but also has to decide over a policy function ψ containing a set of capital income taxes for particular stochastic shocks s' the successive government has to follow. Note that in the exemplary equilibrium from section 3.1 $\psi(.)$ was an anticipated decision rule used by the "current" government rather than a bequeathed basket of state contingent capital taxes. The problem again resembles a non-cooperative game between successive governments as it was already described in the introduction of this chapter. The question therefore arises, how the "former" government will decide over such a policy function knowing that agents will adjust their behavior according to this incertitude in the future as they don't yet know which shock is about to occur. In optimizing the social welfare the government faces two major problems: It has to decide over the next periods state contingent capital income tax and it has to set the current labor income tax rate in order to balance the budget. This is exactly the point where the interaction between future and presence arises which is so crucial in this model. Also the "former" government has inherited a state contingent capital income tax rate from its predecessor to which it is forced to commit and which depends on the actual shock occurring in the present period. This in turn means that it is not really in the position to decide over the labor tax rate since these kind of revenues are restrained to simply balance the budget. The only thing the "former" government can do is to decide over a future state contingent capital income tax, to which the next government, the "current" one in our denomination, has to adhere, and thereby indirectly influence allocation behavior of the private sector in the present.

Individual agents are forward looking and so future policies have implications on their present choice regarding saving, consumption and labor supply. The state contingent capital income tax for tomorrow is given by $\vec{\theta'} = \{\theta(s')\}_{s \in S = \{g,z\}}$ and therefore has to be part of the individual agents decision process. The household's maximization problem in the "former" period can be written down as follows.

$$\max_{c,k',h} u(c,h) + \beta \sum_{s'} \Gamma^{ss'} v(x'(s'), k'; \psi) = \hat{v}(x, \vec{\theta'}, k; \psi) \tag{3.3.16}$$

$s.t.$

$$c + k' = (1-\theta) r(z,K,H) k + (1-\tau) w(z,K,H) h + k \tag{3.3.17}$$

$$K' = \hat{D}_K(x, \vec{\theta'}; \psi) \tag{3.3.18}$$

$$H = \hat{D}_H(x, \vec{\theta'}; \psi) \tag{3.3.19}$$

$$\tau = \hat{\phi}(x, \vec{\theta'}; \psi) \tag{3.3.20}$$

The role of the different conditions is the same as before with the crucial difference, signalized by the hat over them, that this optimization problem is forward looking. The expectation over the future capital income tax rate $\vec{\theta'}$ here is more important than the policy currently in action, set by the former government. Klein and Ríos-Rull call the solution of this problem the intermediate equilibrium. As before the agents take the function $\hat{D}_K(x, \vec{\theta'}; \psi)$ and $\hat{D}_H(x, \vec{\theta'}; \psi)$ as given. The difference here is the intertemporal character of the equilibrium. Nevertheless private sector optimization follows the same principles as in the "current" period. The aggregate private sector optimality conditions therefore have to satisfy:

$$D_K(x; \psi) = \hat{D}_K(x, \psi(x); \psi) \tag{3.3.21}$$

$$D_H(x; \psi) = \hat{D}_H(x, \psi(x); \psi) \tag{3.3.22}$$

In this intertemporal optimization problem the individual agents have to further respect,

or better expressed, anticipate the stationary equilibrium of the next period, the "current" period along the here used denomination, defined in the section above and the policy rule ψ. A set of functions $\{\hat{v}(x,\vec{\theta},k;\psi), \hat{D}_K(x,\vec{\theta}';\psi),$
$\hat{D}_H(x,\vec{\theta}';\psi), \hat{\phi}(x,\vec{\theta}';\psi), \hat{d}_k(x,\vec{\theta}',K;\psi), \hat{d}_h(x,\vec{\theta}',K;\psi)\}$ therefore only describes an equilibrium, if

- $\{\hat{v}(x,\vec{\theta}',k;\psi), \hat{d}_k(x,\vec{\theta}',K;\psi), \hat{d}_h(x,\vec{\theta}',K;\psi)\}$ solve the individual optimization problem taking $\hat{D}_K(x,\vec{\theta}';\psi)$, $\hat{D}_H(x,\vec{\theta}';\psi)$, $\hat{\phi}(x,\vec{\theta}';\psi)$ and ψ as given;
- The assumption holds that the individual solution is representative for the aggregate level, therefore that $\hat{D}_H(x,\vec{\theta}';\psi) = \hat{d}_h(x,\vec{\theta}',K;\psi)$ and $\hat{D}_K(x,\vec{\theta}';\psi) = \hat{d}_k(x,\vec{\theta}',K;\psi)$;
- The overall solution has to satisfy the period-by-period budget constraint of the government:

$$g = \theta r(z,K,\hat{D}_H(x,\vec{\theta}';\psi))K + \hat{\phi}(x,\vec{\theta}';\psi)w(z,K,\hat{D}_H(x,\vec{\theta}';\psi))\hat{D}_H(x,\vec{\theta}';\psi) \qquad (3.3.23)$$

As before the individual value functions can be summed up to the aggregate level, which is used by the benevolent planner to evaluate welfare maximizing policies under anticipation of the private sector's optimal response.

$$\hat{V}(x,\vec{\theta}';\psi) = \hat{v}(x,\vec{\theta}',K;\psi) \qquad (3.3.24)$$

As already mentioned before, in the model by Klein/Ríos-Rull current governments are very limited in there policy making: The current capital income tax rate depends on the state contingent capital income tax rate set by its predecessor in office and the labor tax rate cannot be set freely since governments are obliged to a period-by-period balanced budget. Governments are therefore bounded to the decision over a set of state contingent capital income tax rates for the next period where they will not be in office anymore. Since the agents in the economy are forward looking, however, the expectations about future tax policy already influences allocation behavior in the present. Since "former" governments expect future (the "current") governments to honor their state contingent policy rules, they have to find a set $\vec{\theta}'$ to maximize the aggregate value function constituting the present government's policy, here denoted by a bold Ψ.

$$\Psi(x;\psi) = \arg\max_{\vec{\theta}'} \hat{V}(x,\vec{\theta}';\psi) = \psi(x) \qquad (3.3.25)$$

As in the conceptual outline in section 3.1 the mechanism of mutual best responses reduces the problem to a fixed point problem. The faced problem is the same for every government. If one government knows that its successor follows a certain policy it will find it optimal to choose the same policy.

3.3.4 Calibration

Klein/Ríos-Rull use a linear quadratic approach to calculate numerical results for their model. Describing these techniques however would be beyond the scope of this thesis. I therefore limit myself to describe the calibration of the model and then go on to present the results. Klein/Ríos-Rull choose a standard *constant relative risk aversion utility function*, short CRRA, and a Cobb-Douglas production function.

$$u(c,h) = \frac{(c^{\alpha}(1-h)^{1-\alpha})^{1-\sigma}}{1-\sigma} \tag{3.3.26}$$

$$F(z,K,H) = zAK^{v}H^{1-v} \tag{3.3.27}$$

The factor A in the production function normalizes per period output to 1. The average share of the public sector g of overall production is calibrated to be around 20%. The preferences concerning leisure were calibrated to result into a labor allocation α of 20% with respect to the maximum available time, taking into account a realistic proportion of working age population to total population. The depreciation rate δ is calibrated to be 0.08% and the share of capital K in the production process v is at 0.36%. In the baseline model, the rate of substitution between periods is assumed to be 1. The economy is subject to two different kinds of shocks which eventually occur: A productivity shock on the variable $z \in \{0.976, 1.024\}$ for which the per period probability is at $1-0.946 = 1-\Gamma_z^{11}$ and an expenditure shock on the variable $g \in \{0.184, 0.216\}$ for which the per period probability lies at $1-0.835 = 1-\Gamma_g^{11}$. The length of a period corresponds to one year. Klein/Ríos-Rull choose this period length since, as they say, a readjustment of tax policy is normally carried out on a yearly basis and that the time between legislation and implementation can realistically be assumed to be around one year.[14]

3.3.5 Results

The results given here represent a mere overview. Since the goal of this thesis is to compare the results of the three different approaches with regard to the degree of commitment, it would not make any sense to give detailed numerical values as provided by the study of Klein/Ríos-Rull. This is, in my opinion, particularly reasonable as an analytical approach was used in chapter 2 while Klein/Ríos-Rull and later Nunes/Debortoli achieve their results over numerical techniques. Looking at the results in a more qualitative way does therefore make more sense than the enumeration of quantitative results in absence of any possibility of comparison. At first just the results of the baseline model with standard calibration, quoted above, will be presented. Different cases will be treated afterwards.

[14] Klein/Ríos-Rull 2003, p. 1227

Full Commitment In the full-commitment scenario the results gained by Klein/Ríos-Rull in essence strongly support those gained in chapter 2, particularly in section 2.7. The optimal capital income tax rate is in average at zero, but it is however very volatile. The standard deviation is at 0.18. A standard deviation of 0.13 for the expected capital income tax is in line with this. Labor taxes who constitute the main source of revenues for governments with an average of 31% are in contrast very smooth exhibiting a standard deviation of just 0.009. Klein/Ríos-Rull attribute the difference in volatility between the two taxes to the role of the balanced budget constraint in force in this model economy. Labor taxes therefore finance the major part of government's income while capital income taxes more serve as a shock absorber. This interpretation is also supported by a positive correlation between government purchases and capital income tax rates.[15]

No Commitment In the no-commitment scenario, capital income taxes and labor taxes change roles. Here capital income taxes which are at average at 65% carry the major burden of financing public expenditures. Labor taxes are in contrast just around 12% in mean. Again capital income tax rates are much more volatile than labor income tax rates, showing a standard deviation of 0.11 while the latter exhibit a standard deviation of 0.031. The lower labor income tax rate can be explained by the fact that in an economy without commitment technology, much higher levels of capital income tax rates are sustainable, since here they are partly unforeseen and don't result into such a strong distortion of capital accumulation. The reader is referred to the intuitive interpretation concerning positive capital income taxes in taxation models without commitment in section 3.2. As both types of taxes play different roles they are negatively correlated to each other. This is also reflected in the fact that capital income taxes have a positive correlation with output while the correlation between labor taxes and output is negative. In the full-commitment scenario both taxes appeared to be countercyclical. Klein/Ríos-Rull also suggest that since the autocorrelation of the tax rates is lower than in the full-commitment case and also lower than the autocorrelation of the output, there has to be a connection between capital stock and the particular tax policy. Further in the no-commitment case, mean output is lower in level and variation than in the scenario with full-commitment.

It is also interesting to look at the dynamic evolution of a system where governments are initially able to commit but then loose this ability: One imagines a situation in which the loss of commitment is anticipated to happen in period $t = 0$. Ahead to this loss of commitment, working hours fall and the taxes on labor income eventually go up as the budget still has to be balanced. In period $t = 0$ capital deccumulation starts which is however connected to an inversely hike in capital income taxation. Working hours start to rebound again which leads labor taxes back to pre-shock level and even

[15]Klein/Ríos-Rull 2003, p. 1229-1230

lower. The output level which initially plunged in anticipation of the loss of commitment technology accelerates again but doesn't find its way back to the pre-shock level. Vice-versa, Klein/Ríos-Rull calculated the case in which a government suddenly gains the ability to commit. This would be connected with high welfare gains which can be mainly tracked to an increase in consumption. The authors state that a commitment technology could outweigh a 34% dissolution of capital stock.

The impact of calibration The described results are qualitatively robust. As far as the quantitative impact is concerned they are however sensitive to the particular calibration one chooses. The most sensitive values are therefore the intertemporal elasticity of substitution $1/\sigma$ and the length of periods. Both values were calibrated at 1 in the baseline model.[16] Klein/Ríos-Rull also conduct calculations for cases with a period length of 2 and 4 years and an intertemporal elasticity of substitution valued ceteris paribus 0.2. In an economy with a higher intertemporal elasticity of substitution, capital income taxes are generally lower since work effort and saving behavior react much more sensitively to changes in capital tax policy than in scenarios with a lower intertemporal elasticity of substitution. The same is true vice versa: If the intertemporal elasticity of substitution is low, the cost to increase taxes on capital income is lower and the government is more likely to do so. A lower intertemporal elasticity of substitution however causes a higher volatility in taxation for both capital income taxes and labor taxes. The same is true for the variability of the output which is much higher than in economy's with a high elasticity of substitution.[17]

Intuitively, an extension of the length of periods can be interpreted as a gain in commitment. This is also supported by the figures: The longer the length of the period the lower the taxes on capital income and the higher the taxes on labor income are. In a model economy without commitment but with an extended period length, figures behave much more like in the full-commitment scenario. Implications from period length on the economic volatility are however negligible.[18]

Elasticity of substitution between consumption and leisure The CRRA utility function, see equation (3.3.26), used in the baseline model assumes the elasticity of substitution between consumption and leisure to be one. To examine the effect of other elasticities of substitution deviating from the baseline model, and therefore a different intratemporal substitution behavior, it is necessary to use a different utility function.

[16] since the elasticity of substitution between consumption and leisure σ was calibrated to be one in the baseline model.

[17] Klein/Ríos-Rull 2003, p. 1234 - 1236

[18] Klein/Ríos-Rull 2003, p. 1234 - 1236

The for this purpose specified utility function by Klein/Ríos-Rull is given by equation (3.3.28).

$$u(c,h) = \frac{c^{1-\sigma_1} - 1}{1-\sigma_1} + \gamma \frac{(1-h)^{1-\sigma_2} - 1}{1-\sigma_2} \tag{3.3.28}$$

The variables σ_1 and σ_2 give the curvature for consumption and leisure respectively. If one would set $\sigma_1 = \sigma_2 = 1$, one would receive results corresponding to the results in the baseline model with a period length of 2. To make a comparison with the baseline model meaningful however, it is necessary to recalibrate and fix values as the share of the government expenditures g on total production, which was assumed to be at 20% before, or as the choice for working hours h in the case $\sigma_1 = \sigma_2 = 1$.

Without recalibration, an increase in σ_1 (the curvature for consumption) implies a decrease of capital income taxes leaving the taxes on labor unchanged. An increase of σ_2, which implies an increase of the preference to leisure goes along with an increase in labor taxes as well as capital income taxes and slightly elevates the share of government consumption to total output, g/y. This coherence is also true in the other direction.

Leaving the share of government consumption to total output g/y unchanged however alters the result drastically. Here an increased curvature of consumption σ_1 reduces capital income taxes and increases labor income taxes. The interpretation is clear: A higher σ_1 elevates the preferences with respect to consumption and decreases the willingness to save. Agent's preferences to work, however, stay the same with the difference that they now have to satisfy higher consumption preferences which suggests labor income taxation to be the better source of revenues for the government. An increased curvature for leisure σ_2 implies a reduced willingness to work forcing the government to increase both taxes to furthermore satisfy the balanced budget constraint.

Leaving working hours unchanged to the model with $\sigma_1 = \sigma_2 = 1$, an increase in σ_1 implies lower capital income taxes and slightly higher labor taxes. An increased curvature for leisure σ_2 increases capital income taxation as well as labor taxes.[19]

Separating the shocks Looking at the effect of the shocks on government expenditures, g, and productivity, z, separately allows to better understand the forces at work which are decisive for the particular optimal tax rates. Only looking on the effect of government expenditure, g, shocks, such a shock could be thought of in the case of wars or environmental catastrophes, and setting the productivity z to its average shows merely the same results as before where both sources of fluctuation randomly occurred at the

[19] In this last case the text exhibits a contradiction between the figures and the interpretation given by Klein/Ríos-Rull. I assumed the figures in the table to be correct. See: Klein/Ríos-Rull 2003, p. 1237 - 1239, Table 6.

same time; in the full-commitment scenario the capital income tax is around zero. Nearly the total tax burden with a rate of 31% lies on labor. In the the no-commitment case things are again different; one observes very high capital income tax rates at around 65% in average. Labor here is only taxed with 12% on average. Again the standard deviation for capital income taxes is significantly lower in the no-commitment than in the full-commitment scenario. For labor income taxes the exact opposite is true. The correlation between g and the labor-and capital-tax respectively is in both cases small and negative in the full-commitment scenario. In the no-commitment scenario, τ correlates positively if even on a small scale with g where as the correlation between θ and g is again small and negative. In a dynamic analysis this is reflected by the fact that in the full-commitment scenario expenditure shocks are predominantly financed by capital income taxation. In the no-commitment case the process is more complex. An expenditure shock first leads to a slight decrease in labor taxes and to a strong hike of capital income taxes. In the course of the adjustment process after the shock the additional expenditures are mainly financed by capital income taxes. But as capital deccumulation becomes too strong, taxes on capital income sink again and labor taxes are increased. Capital income tax rates end below and labor income tax rates above their pre-shock levels after the adjustment process.[20]

Only looking at productivity shocks, the general results regarding the average rates for capital income taxes and labor taxes as well as their volatility for the full- and no-commitment scenario are the same as in the cases where the shocks occurred parallel and as in the case with only expenditure shocks described above. The dynamics after a shock are however different. In times of an economic boom triggered by a positive shock on productivity z, the taxes on capital income are cut. In an economy without commitment, this effect is stronger than in a scenario with full-commitment. In an economy without commitment, a boom phase goes along with an increase in labor taxation. In the scenario with full-commitment, labor income taxes eventually stay constant. In the adjustment process after a productivity shock in a no-commitment scenario, taxes on capital income rise again while labor taxes go down. Both taxes end up on a higher and lower level respectively, compared to the pre-shock situation. In the full-commitment scenario, capital income taxation rises again after the shock and ends up on a level higher than before the shock.[21]

3.4 Alternative approaches to model the lack of commitment

Aside the direct approach to study optimal fiscal without commitment in the context of a Markov perfect equilibrium, which is, as discussed before, time-consistent by construc-

[20]Klein/Ríos-Rull 2003, p. 1238 - 1240
[21]Klein/Ríos-Rull 2003, p. 1240 - 1241

tion, also alternative concepts emerged to control for the problem of time-inconsistency as it is evident in the traditional Ramsey optimal taxation models. The perhaps most renowned of these alternative approaches stems from Jesse Benhabib and Aldo Rustichini[22]. Their conceptual approach is frequently used and was gradually refined and extended by other authors as for example Catarina Reis[23] and Begona Dominguez[24]. For the sake of completeness it appears reasonable to briefly present the core ideas of this approach. Beside, this a basic understanding of this concept might be helpful in context of the next section. There one of the shortcomings of the models hitherto presented in this chapter, namely the period-by-period balanced budget assumption, will be discussed. Two of the three papers addressing this problem follow the method of Benhabib/Rustichini.

Benhabib/Rustichini address the problem of time-inconsistency in the traditional Ramsey taxation literature, in contrast to the Markov approach, only indirectly. As was already mentioned several times before the results of a limiting zero capital tax within the Ramsey equilibrium hinges on the fact that the government suppositionally possesses a commitment technology which makes it immune to incentives to deviate from the once chosen sequence of policies. It was however also argued that such a solution is not incentive compatible and therefore obviously also not time-consistent. It is plausible that a situation originates which harbors incentives similar to that in the initial period in the full commitment case, namely to levy high taxes on capital, incidentally in the interest of the private sector, in order to lower distortionary taxation in the future: From the perspective of the respective current planner the existent capital stock is considered as being inelastically supplied as the investments which were necessary for its accumulation are sunk. For a contemporary planner, without the need to choose a whole sequence of policies for all future periods as the fully committed planner has to do it, capital taxation would hence be regarded as being not distortionary. This is were the idea of Benhabib/Rustichini starts off. Instead of allowing for periodical deviations, as the Markov approach does, Benhabib/Rustichini impose an *incentive compatibility constraint* on the government's maximization problem. This *incentive compatibility constraint* overrides the incentives of the government to defect as it serves as a kind of sanctioning mechanism in the case the planner defaults on preannounced policies. Solutions in such an environment are therefore obviously incentive compatible and time-consistent. In the paper by Benhabib/Rustichini the government has to maximize a function of the form,

$$\sum_{t=i}^{\infty} \beta^{t-i} u(c_t, h_t) \geqslant V^D(k_i, b_i)$$

[22]Benhabib/Rustichini 1997
[23]Reis 2006
[24]Dominguez, Begona, 2007. "Public debt and optimal taxes without commitment," Journal of Economic Theory, vol. 135(1), pp. 159-170.

with respect to it's budget constraint, the economy's resource constraint and under consideration of the optimal allocations found in the private sector. The function $V^D(k_i, b_i)$ denotes the deviation value, hence the welfare value of defaulting to the previously announced policy. By definition of the constraint the discounted utility of commitment, which implies a continuation of the previously announced policy, has to be at least as large as the discounted utility of default, which implies a deviation from the preannounced policy. The consequences of default, captured by the deviation value, can be interpreted as a complete loss of reputation which leads the private sector to expect capital taxation to be at the maximum level for all future periods and hence to stop or at least reduce capital accumulation substantially. The government therefore has to ponder costs and benefits of deviation in each period. In example, does it payoff to increase capital taxation in order to lower distortions caused by labor taxation. Benhabib/Rustichini examine this model at the steady state where the incentive constraint is binding, called an "*incentive constrained steady state*", and find that there is a range of compatible levels of the capital stocks below and above the commitment value with respective associated capital tax rates. The commitment value and the associated zero capital tax rate is not sustainable as it is not credible. It turns out that the only sustainable capital tax rate is negative and thus represents a subsidy. The induced higher accumulation rate of capital is widening the inequality above (between the value of commitment and default), lets a one time capital levy in the manner of the full commitment case appear more and more unattractive and thus serves as a commitment device.[25] Hence, as in the case of the Markov perfect equilibrium also this example for time-consistent optimal taxation yields capital tax rates different to zero, even if the results by Klein/Ríos-Rull point in the opposite directions. It has to be stated however that Benhabib/Rustichini's results are calculated for linear preferences only and for an economy where a balanced budget constraint applies. They indeed characterize their model at first under the consideration of public borrowing (see above), the analytical and numerical solutions are however done under a balanced budget assumption. This issue will be addressed in the next section.

3.5 The role of the balanced budget assumption

In sections 2.5 and 2.6 it was shown that a planner who possesses the ability to fully commit to a sequence of policies for all future periods tends to tax capital extremely high in the initial period in order to be able to lower distortionary taxation in later periods. As pointed out prior several times, investments which lead to the capital stock in the initial period are sunk, hence capital is already installed, and its taxation is thus regarded as being not distortionary. The phenomenon of permanently positive capital tax rates in the no-commitment Markov environment was argued conversely:

[25]compare to Benhabib/Rustichini 1997 and Dominguez 2007

First, a contemporary planner regards the present capital stock as installed and hence its taxation as not distortionary, second, a planner without commitment is obviously not able to front load distortions and to transfer resources to later periods as the Ramsey planner does it. The front-loading of distortions and the transfer of resources to later periods, as presented in section 2.6, however hinges essentially on the fact that the government possesses saving respectively borrowing instruments. In the models hitherto examined in the no-commitment environment such instruments are however missing. Klein/Ríos-Rull note that the allowance of public borrowing could indeed change their result.[26] Literature on this issue is unfortunately not very comprehensive. The results of the few available papers however support the intuition that the existence of public saving/borrowing devices is a very relevant factor. To my knowledge there are only three papers controlling for the issue of public borrowing under the assumption that the planner possesses no commitment technology. Two of these papers, one by Catarina Reis[27] and one by Begona Dominguez[28], follow the approach by Benhabib/Rustichini[29] (outlined in section 3.4). The third one by Marina Azzimonti-Renzo, Pierre-Daniel Sarte and Jorge Soares[30] follows a Markov approach and might thus, in the context of this chapter hitherto where we mainly focused on Markov equilibria, be regarded as the most relevant. All three works however come to quite similar results. Dominguez and Reis find that under the assumption that there is no commitment technology available to the planner but the possibility to borrow and to save, capital taxes are optimally zero in the steady and in the long run respectively. Similar to Benhabib/Rustichini they conclude that the accumulation of assets, debt and capital, serves as a commitment device and renders default unattractive. Reis further points out a mechanism which seems similar to the front-loading argument brought forward in section 2.6: In the short run the government accumulate assets, it backloads resources, until, along the same logic as in the paper of Benhabib/Rustichini, the incentive constraint is no longer binding and the economy is back in the full-commitment solution where the tax rate on capital is optimally zero. Azzimonti-Renzo et al. find their solution in a Markov framework as it was mainly discussed hitherto in this chapter. They show that in an economy where the planner is not able to commit but borrowing/saving devices are available capital taxes are zero in a stationary Markov equilibrium. For a finite horizon version of their economy the authors even show that a front-loading of taxation takes place, also in the presence of an upper limit on capital taxes, as it is known from the full-commitment Ramsey environment.[31]

[26] Klein/Ríos-Rull, p. 1243
[27] Reis 2006
[28] Dominguez 2007
[29] Benhabib/Rustichini 1997
[30] Azzimonti-Renzo et al. 2006
[31] Azzimonti-Renzo et al. 2006, pp. 1 - 14

3.6 Summary and Interpretation

In contrast to chapter 2 the central assumption in chapter 3 was that the government doesn't have access to a commitment technology which would allow to stick to an in advance decided upon sequence of policies. As defined introductorily to this chapter and considered in the exemplary two period model, policy decisions under a no-commitment assumption are taken sequentially and lead to completely different policy outcomes than in the case, where the planner has access to a commitment technology. The model framework used by Klein/Ríos-Rull is slightly different. The government bequeaths a vector of state contingent capital taxes to its successor but has no control over tax policy beyond the next period. Since a balanced budget constraint applies and the from the predecessor inherited capital tax rates are state contingent the use of labor taxes is very limited as they simply serve the purpose of balancing the budget. Also here the optimal policies differ considerably to the full-commitment case: Both examined model economy's exhibit strictly positive tax rates on capital income which is in stark contrast to the optimally prescribed zero capital tax result in the full-commitment approach. An intuitive explanation for this was already given in section 3.2. The results can be argued by taking recourse to the argumentation which was used in chapter two to explain the result that capital taxes should be optimally zero in the long run: First it was argued that a positive capital tax amounts to a differential tax rate on private consumption goods which is a contradiction to the steady state assumptions and represents an intertemporal wedge which distorts the saving/consumption behavior. Second, it was reasoned that a planner equipped with a commitment technology is able to front load distortions (capital taxes), transfer resources to future periods and can thus lower distortionary taxation in the future. That capital taxes are positive in the no commitment case can therefore be reasoned by a reversion of these arguments: First, in the absence of a commitment technology each government in every period is confronted with a situation similar to the initial period in the full commitment case. Capital is already installed, hence inelastically supplied, the tax policy in the running period has no influence on the capital stock in the current period, and its taxation is thus regarded as being non distortionary. Second, since the government by assumption possesses no commitment technology it doesn't have the ability to front load distortions as it is practiced by the planner who has the ability to commit to a sequence of tax rates. As however discussed in section 3.5 these results centrally hinge on the balanced budget assumption which is effective in our exemplary two-period model as well as in the model by Klein/Ríos-Rull. If the the planner has access to borrowing/saving devices one can show that the results for the cases of full- and no-commitment are very similar: Most notably in the context of this chapter is the work by Azzimonti-Renzo et al.[32] who, in contrast to Reis and Dominguez, solve for a Markov equilibrium. They find that in an economy where the planner doesn't

[32] Azzimonti-Renzo et al. 2006, pp. 1 - 14

have access to a commitment technology but is able to borrow/save capital taxes are optimally zero in stationary Markov equilibrium. In a version of their model with finite horizon they even observe front-loading of distortions which in its manner is similar to the full commitment case.

4 Optimal Fiscal Policy with loose commitment

The paper by Davide Debortoli and Ricardo Nunes[1], which should be examined subsequently, chooses a middle course compared to the approach in chapters 2 and 3. Their approach reflects the assumption that neither the *full-commitment* nor the *no-commitment* approach renders the situation in reality. Debortoli/Nunes therefore consider a model in which default on past promises, no-commitment, can occur by a certain probability in each period. In the first step this probability is exogenously given. In the second step the authors reformulate the model in a way that the probability for default now depends on endogenous economic variables. The solution process for a Markov equilibrium in the loose commitment case follows, with small differences, a similar concept as in the Klein/Ríos-Rull paper or the definition of the general Markov equilibrium in section 3.1.

4.1 The basic model and notations

4.1.1 Notations

In the model economy described by Debortoli/Nunes, households gain utility by three factors: private consumption c_t, public consumption g_t and leisure $(1-h_t)$. Endowed with one unit of time each period, households take an allocational decisions between leisure $(1-h_t)$ and labor h_t. Households not only supply labor to firms, they also participate on the capital markets by renting out capital k_t to firms. Incomes from labor and capital are subject to the taxes τ_t and θ_t respectively. Further Debortoli/Nunes assume a capital utilization rate v_t, which allows the share of capital rented out to firms to be written as $v_t k_t$. The depreciation of capital $\delta(v_t)$ is a function of this utilization rate. The Output in the model economy is produced according to the production function $F(k_t, v_t, h_t) = (k_t v_t)^\sigma h_t^{1-\sigma}$. As it is assumed that factor markets are perfectly competitive, factor prices for labor and capital can be expressed as $w_t = F_h(k_t, v_t, h_t)$

[1] Debortoli/Nunes 2007

and $F_k(k_t, v_t, h_t) = r_t$ respectively. Taking the taxes τ_t and θ_t as well as the prices w_t and r_t by now as given, we can write down the household's maximization problem.

$$\max_{k_{t+1}, c_t, h_t} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, g_t, h_t) \tag{4.1.1}$$

$s.t.$

$$c_t + k_{t+1} = (1 - \theta_t) r_t v_t k_t + (1 - \tau_t) w_t h_t + (1 - \delta(v_t)) k_t \tag{4.1.2}$$

β^t represents the discount rate applied by agents. E_o is the expectation operator.

4.1.2 Solution procedure and structure

The equilibrium for the here described model properties is to be found in a Markov setting. Although the approach by Debortoli/Nunes follows the same concept as in chapter 3, the specific solution procedure is slightly different. In chapter 3, sections 3.1 to 3.3, it was assumed that the planner can only commit to the tax rates in the present or in the next period respectively. In the model by Debortoli/Nunes the phases of commitment could possibly take longer. One can imagine the concept used in this scenario as a kind of interlaced finite Ramsey sequences. In the standard no-commitment scenario it is assumed that a new government comes into office each period, which means that there is a hundred percent chance that default on past policy promises happens in each period. Each planner therefore also has to take account for the best response of the successive government in the next period regarding the state in which it inherits the economy. In the loose commitment case however the current government has to consider that there is a certain likeliness of default in each period. The actual default however might happen in one, two or even more periods and each government has to consider the reaction of its successor which would then eventually come into office. After a default a new Ramsey sequence begins which last until the next default. In comparison one might state that the Ramsey sequence in the definition of the standard Markov equilibrium of section 3.1 in contrast only lasts one period.

In the following subsection 4.1.3 the private sector optimality conditions are derived. Section 4.3 discusses the problem of the government. As shown in section 4.3.1 the same mechanisms become evident as in the standard Markov formulation: Governments optimize in anticipation of a potential default which might happen with a certain probability in every of the upcoming periods. The problem of the government in such an environment is threefold: First, the government has to consider its own history depending on how long it is in office already. After every default a new government comes into office. In such a situation no promises are to be kept and solely the state of the government is inherited from the predecessor. Second, the planner has to consider the future for the case default happens in the current period and another planner comes into office in the next period. Third, the planner has to consider the future for the case he is able to commit and stays in office. It turns out that, as all other histories have been maximized by other planners already and every planner only considers a single history under the presumption that default has not yet happened. In chapter 4.3.2 the optimizing conditions of the government and the private sector are combined. It is shown that the solutions of the overall optimization problem are "time-invariant and depend on a finite set of states."[2] and are therefore fixed point problems as in the standard Markov formulation. In section 4.4 such an equilibrium is defined and solved for analytically. Section 4.5 examines a case in which the probability to commit is not exogenously given but depends on endogenous state variables. In section 4.6 and 4.7 the calibration is discussed and an overview of the numerical solutions is given respectively.

4.1.3 The private sector optimality conditions

The problem formulated in equations (4.1.1) and (4.1.2) can be written down in the Lagrange form and solved for the first order conditions.

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t, g_t, h_t) + \lambda_t[c_t + k_{t+1} - (1-\theta_t) r_t v_t k_t - (1-\tau_t) w_t h_t - (1-\delta(v_t))k_t]\} \tag{4.1.3}$$

$$\frac{\partial \mathcal{L}}{\partial c_t} : \lambda_t = -u_c(c_t, g_t, h_t) \tag{4.1.4}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} : \lambda_t - \beta E_t \lambda_{t+1}[(1-\theta_{t+1}) r_{t+1} v_{t+1} + (1-\delta(v_{t+1}))] = 0 \tag{4.1.5}$$

$$\frac{\partial \mathcal{L}}{\partial h_t} : u_h(c_t, g_t, h_t) - \lambda_t(1-\tau_t) w_t = 0 \tag{4.1.6}$$

$$\frac{\partial \mathcal{L}}{\partial v_t} : -\lambda_t(1-\theta_t) r_t k_t + \lambda_t \delta_v(v_t) k_t = 0 \rightarrow (1-\theta_t) r_t - \delta_v(v_t) = 0 \tag{4.1.7}$$

The respective combinations of equations (4.1.4) and (4.1.5), as well as (4.1.4) and (4.1.6) and equation (4.1.7) yields the following optimality conditions.

$$u_c(c_t, g_t, h_t) - \beta E_t u_c(c_{t+1}, g_{t+1}, h_{t+1})\{(1-\theta_{t+1}) r_{t+1} v_{t+1} + (1-\delta(v_{t+1}))\} = 0 \tag{4.1.8}$$

$$u_h(c_t, g_t, h_t) + u_c(c_t, g_t, h_t)(1-\tau_t) w_t = 0 \tag{4.1.9}$$

$$(1-\theta_t) r_t - \delta_v(v_t) = 0 \tag{4.1.10}$$

[2]Debortoli/Nunes 2007, p. 15

The interpretation of these optimality conditions is straightforward and similar to the optimality conditions derived in the course of chapter 2: Equation (4.1.8), the household's Euler equation, points out the intertemporal dimension of the household's decision problem. The household's consumption behavior depends on expectations over future variables as for example the expected future return on capital. Equation (4.1.9) states the intratemporal allocation problem between leisure and consumption in terms of opportunity costs. One unit more of leisure has to be indirectly bought for $(1-\tau_t)w_t$ less consumption. Equation (4.1.10) gives the connection between capital income taxation and the capital utilization rate: The higher the taxes on capital income are, the lower the utilization rate of capital is.

The role of the government in this model is standard: In anticipation of the optimal behavior of the households, it has to set tax rates τ and θ and the scale of government expenditures g in order to maximize the social welfare function and to satisfy the government's budget constraint given by equation (4.1.11).

$$g_t = \theta_t r_t v_t k_t + \tau_t w_t h_t \tag{4.1.11}$$

The household's and government's budget constraints, represented by equations (4.1.2) and (4.1.11) respectively, can be combined to give the feasibility constraint (equation (4.1.12)) which is effective in this model economy.

$$\begin{aligned}
c_t + k_{t+1} &= (1-\theta_t)r_t v_t k_t + (1-\tau_t)w_t h_t + (1-\delta(v_t))k_t \\
c_t + k_{t+1} + \underbrace{\theta_t r_t v_t k_t + \tau_t w_t h_t}_{g_t} &= \underbrace{r_t v_t k_t + w_t h_t}_{y_t} + (1-\delta(v_t))k_t \\
y_t &= c_t + k_{t+1} + g_t - (1-\delta(v_t))k_t \qquad (4.1.12)
\end{aligned}$$

Given the above results, Debortoli/Nunes denote the variables v_t[3], c_t[4] and g_t[5] as functions to simplify the notation.

$$\begin{aligned}
v_t &= v(k_t, h_t, \theta_t) \\
c_t &= c(k_{t+1}, k_t, h_t, \theta_t, \tau_t) \\
g_t &= g(k_t, h_t, \theta_t, \tau_t)
\end{aligned}$$

Combining the first order conditions (4.1.8) and (4.1.9) and using the simplifications for v_t, c_t and g_t along Debortoli/Nunes allows to write down the optimality conditions of

[3] using first order condition (4.1.10)
[4] using the household's and government's budget constraint
[5] using the household's and government's budget constraint

the households in a more compact and therefore more manageable form.

$$b_1(x_t(\omega^t), k_t(\omega^t)) + \beta E_t b_2(x_{t+1}(\omega^{t+1}), k_{t+1}(\omega^{t+1})) = 0 \qquad (4.1.13)$$

b_1 and b_2 are function vectors summarizing the dependencies in the first order conditions (4.1.8) and (4.1.9), $x_t \equiv (k_{t+1}, h_t.\theta_t, \tau_t)$ bundles the control variables in the current period, k_t represents the state variable and ω_t stands for the history up to period t.

4.2 The probabilistic model

4.2.1 Supplementary notation

The first step in the *loose commitment* approach is to rewrite the model in order to simulate a situation in which default can appear in every period by an exogenously given probability. The social planner takes his decision in a context in which he cannot be sure whether its promises will be kept or not. This forces him to take into account this indeterminacy and the anticipation of this uncertainty by the households.

The random variable $\{s_t\}_{t=1}^{\infty}$, which is driven by a stochastic Markov process[6], gives the realization of the two possible events *default* D or *commitment* ND in period t. Alternatively expressed, $\bar{s_t} \in \Phi \equiv \{D, ND\}$. $\Omega^t \equiv \{\omega^t = \{D, \{\bar{s_j}\}_{j=1}^t\} : \bar{s_j} \in \Phi, \forall j = 1, \ldots, t\}$ represents the set of possible histories up to time period t, where only histories $\omega = \{D, \bar{s_1}, \bar{s_2}, \ldots, \bar{s_t}\}$ are considered which begin with a default D. In the period after a default a new government comes into office. In this situation there are no promises which have to be kept.

4.2.2 The perspective of the individual agent

The individual agent or household takes the announced policies (promises) of the planner currently in offices for granted. Considering future planners however, agents take into account that there is certain probability at which future policies might deviate from the policies initially promised. Agents therefore anticipate a possibility for default. Since there is a possibility for a default in every successive period, with the exception of periods which directly come after a period where default occurred, it is reasonable to write down future control variables upon which one cannot fully commit as functions of state variables, i.e. $x_{t+1}(\{\omega^t, D\}) = \Psi(k_{t+1}(\{\omega^t, D\})$. Here, the expression $\{\omega^t, D\} = \{\omega^t, \bar{s}_{t+1} = D\}$ represents a situation where the planner defaults on his past promises in the upcoming period. $\Psi(.)$ is a policy vector which was initially expected by rational

[6]For a definition of a stochastic Markov process the reader is referred to section 2.7.5.

agents to be enacted in upcoming periods if there would have been no default. Equation (4.1.13), representing the individual maximization problem, can now be rewritten.

$$\begin{aligned} & b_1(x_t(\omega^t), k_t(\omega^t)) + \beta Prob(\{\omega^t, ND\}|\omega^t) b_2(x_{t+1}(\{\omega^t, ND\}), k_{t+1}(\{\omega^t, ND\}) \\ & + \beta Prob(\{\omega^t, D\}|\omega^t) b_2(\Psi(k_{t+1}(\{\omega^t, D\}), k_{t+1}(\{\omega^t, D\})) = 0 \end{aligned} \tag{4.2.1}$$

The first term in the first line of equation (4.2.1) represents the current decision; the second term reflects the future rationality with respect to probability of non-default. The second line of equation (4.2.1) stands for the rational future decision, taking into account the probability of default in the next period. In this case agents cannot control the variable $x_{t+1}(.)$ anymore, which then solely depends on the state variables as reflected in the notation for $\Psi(.)$ explained above. Since in contrast $k_{t+1}(.)$ is a state variable, the notation for the commitment and no-commitment case stays the same, or alternatively expressed, $k_{t+1}(\{\omega^t, ND\}) = k_{t+1}(\{\omega^t, D\}) \forall \omega^t$. In words, the expression $Prob(\{\omega^t, ND\}|\omega^t)$ gives the probability of non-default in period $t+1$ with respect to the past t periods since the history started after the last default. It can more formally be denoted as $Prob(\{\omega^t, ND\}|\omega^t) = Prob(\{s_j\}_{j=0}^{t+1}) = \{\omega^t, ND\}|\{s_j\}_{j=0}^{t} = \omega^t)$.

4.3 The perspective of the planner

If default occurs, a new planner comes to office who will from then on make the decisions. By distinguishing all histories by the point of time at which default occurred the first time, we can attribute each history to a certain planner. Taking a hypothetical point in time i, we can separated the histories into a subset where only commitment was the case until period i, $\Omega_{ND}^t \equiv \{\omega^t = \{D, \{\bar{s_j}\}_{j=1}^t\} : \bar{s_j} = ND \quad \forall \quad j = 1, \ldots, t\}$ and into a subset where the planners defaulted in period i, $\Omega_{D,i}^t \equiv \{\omega^t = \{D, \{\bar{s_j}\}_{j=1}^t\} : (\bar{s_i} = D) \wedge (\bar{s_j} = ND) \quad \forall \quad j = 1, \ldots, i-1 \quad if \quad i \leqslant t\}$ and $\Omega_{D,i}^t \equiv \emptyset, \quad if \quad i > t$. Expressed in an easier way one can write $\Omega_{ND}^t \cap \{D, \bar{s_1} = ND, \ldots, \bar{s_t} = ND\}$ and $\Omega_{D,i}^t \cap \{D, \bar{s_1} = ND, \ldots, \bar{s_{i-1}} = ND, \bar{s_i} = D\}$. Along this scheme one can express the set of possible histories Ω^t as a partitioned set $\{\Omega_{ND}^t, \Omega_{D,1}^t, \Omega_{D,2}^t, \ldots, \Omega_{D,t}^t\}$.

4.3.1 The problem of the current planner

Taking into account the way of notation denoted above, the general problem of the current planner can be written down, ignoring the constraint he faces as done in equation

(4.3.1).

$$
\begin{aligned}
W(k_0) = \max_{\substack{\{x_t(\omega^t\}_{t=0}^{\infty} \\ \omega^t \in \Omega^t}} & \Bigg[\sum_{t=0}^{\infty} \sum_{\omega^t_{ND}} \beta^t \{Prob(\omega^t) u(x_t(\omega^t), k_t(\omega^t))\} \\
& + \max_{\substack{\{x_t(\omega^t\}_{t=1}^{\infty} \\ \omega^t \in \Omega^t_{D,1}}} \left\{ \sum_{t=1}^{\infty} \sum_{\omega^t \in \Omega^t_{D,1}} \beta^t \{Prob(\omega^t) u(x_t(\omega^t), k_t(\omega^t))\} \right\} \\
& + \max_{\substack{\{x_t(\omega^t\}_{t=2}^{\infty} \\ \omega^t \in \Omega^t_{D,2}}} \left\{ \sum_{t=2}^{\infty} \sum_{\omega^t \in \Omega^t_{D,2}} \beta^t \{Prob(\omega^t) u(x_t(\omega^t), k_t(\omega^t))\} \right\} + \dots \Bigg]
\end{aligned}
\tag{4.3.1}
$$

Clearly, already in the problem of the current planner the behavior of other possible welfare maximizing planners in their administrative periods (after a potential default of the current planner) is included. If $\{\Omega^t_{ND}, \Omega^t_{D,1}, \Omega^t_{D,2}, \dots, \Omega^t_{D,t}\}$ is as said before a partition of the set Ω^t, all possible histories are included in the formulation (4.3.1) of the problem faced by the current planner above. The current planner has to maximize welfare, incorporating the possibility that he stays in office as well as the odds that he might be forced to default and another maximizing planner comes into office, be it in one, two or three (et cetera) periods seen from the respective current period. Since the situation is the same for all planners in a period directly after a default, or expressed more formally $\forall t > i, \quad \Omega^t_{D,i} = \{\omega^i_{D,i}, \{\bar{s_j}\}^t_{j=1}\}$, the probabilities for $\omega^t \in \Omega^t_{D,i}$ can be rewritten.

$$
Prob(\omega^t) = Prob(\omega^i_{D,i} \wedge \omega^t) = Prob(\omega^t | \omega^i_{D,i}) Prob(\omega^i_{D,i}), \quad \forall \quad \omega^t \in \Omega^t_{D,i} \tag{4.3.2}
$$

Expression (4.3.2), which gives the probability for the history ω^t with the probability that history started newly in period i, can be substituted into the equation representing the current planner's problem, equation (4.3.1), as done in equation (4.3.3).

$$
\begin{aligned}
W(k_0) = \max_{\substack{\{x_t(\omega^t\}_{t=0}^{\infty} \\ \omega^t \in \Omega^t}} & \Bigg\{ \sum_{t=0}^{\infty} \sum_{\omega^t_{ND}} \beta^t \{Prob(\omega^t) u(x_t(\omega^t), k_t(\omega^t))\} \\
& + \sum_{i=1}^{\infty} \beta^i Prob(\omega^i_{D,i}) \Bigg[\max_{\substack{\{x_t(\omega^t\}_{t=i}^{\infty} \\ \omega^t \in \Omega^t_{D,i}}} \sum_{t=i}^{\infty} \sum_{\omega^t \in \Omega^t_{D,i}} \beta^{t-i} \{Prob(\omega^t | \omega^i_{D,i}) u(x_t(\omega^t), k_t(\omega^t))\} \Bigg] \Bigg\}
\end{aligned}
\tag{4.3.3}
$$

The current planner therefore maximizes the problem he faces with respect to the maximization of other planners who's tenures might start in upcoming periods after the current planner might default. Under the assumption that any future planner behaves rational as well, one can define a value function:

$$\xi_i(k_i(\omega^i_{D,i})) \equiv \max_{\substack{\{x_t(\omega^t\}^{\infty}_{t=i} \\ \omega^t \in \Omega^t_{D,i}}} \sum_{t=i}^{\infty} \sum_{\omega^t \in \Omega^t_{D,i}} \beta^{t-i} \{Prob(\omega^t|\omega^i_{D,i}) u(x_t(\omega^t), k_t(\omega^t))\} \tag{4.3.4}$$

The value function $\xi_i(k_i(\omega^i_{D,i}))$, defined in equation (4.3.4), gives the decision process after the first default in period i by future planners. The choices of future planners are however independent from each other or more formally expressed $\Omega^t_{D,i} \cap \Omega^j_{D,i} = \emptyset$ for $i = j$. Nevertheless Debortoli/Nunes assume that all planners are facing the same institutional setting. It is therefore debarred that some social planners have, in contrast to the rest, access to a (full-)commitment technology. The value function can therefore be generalized for all agents, $\xi_i(k_i) = \xi(k_i)\ \forall\ i$.

As all histories $\{\Omega^t_{ND}, \Omega^t_{D,1}, \Omega^t_{D,2}, \ldots, \Omega^t_{D,t}\}$ are already maximized by other planners, it is appropriate to assume that the initial planner only takes into account the history $\{\omega^t : \omega^t \in \Omega^t_{ND}\} \equiv \omega^t_{ND}$ instead of all histories $\omega^t \in \Omega^t$. This allows us to further simplify the current planner's problem for the initial period of equation (4.3.3) as done in equation (4.3.5).

$$W(k_0) =$$
$$\max_{\substack{\{x_t(\omega^t_{ND}\}^{\infty}_{t=0} \\ \omega^t \in \Omega^t}} \left\{ \sum_{t=0}^{\infty} \{\beta^t \{Prob(\omega^t_{ND}) u(x_t(\omega^t_{ND}), k_t(\omega^t_{ND}))\}\} + \sum_{i=1}^{\infty} \beta^i Prob(\omega^i_{D,i}) \xi(k_i(\omega^i_{D,i})) \right\} \tag{4.3.5}$$

Under the assumption that the random variable s^t, mentioned introductorily, is independent and identically distributed, Debortoli/Nunes rewrite the formulation from above to the form of Markov processes (compare to section 2.7.5). The probability to commit after history ω^t is denoted by $\pi = Prob(\{\omega^t, ND\}|\omega^t)$, whereas the probability to default after history ω^t is denoted by $(1-\pi) = Prob(\{\omega^t, D\}|\omega^t)$. Along this scheme the expressions $Prob(\omega^t_{ND}) = \pi^t$ and $Prob(\omega^t_{D,t}) = \pi^{t-1}(1-\pi)$ give the probabilities for commitment and default in period t respectively. The part π^{t-1} represents the probability that there was no default up to period $t-1$. With these simplifications we can again rewrite and simplify equation (4.3.5).

$$W(k_0) = \max_{\{x_t(\omega^t_{ND}\}^{\infty}_{t=0}} \sum_{t=0}^{\infty} \beta^t \pi^t \{u(x_t(\omega^t_{ND}), k_t(\omega^t_{ND})) + \beta(1-\pi)\xi(k_{t+1}(\omega^{t+1}_{D,t+1})\} \tag{4.3.6}$$

Along the same procedure we can also substitute for the $Prob(.)$ expressions in the private household's optimality condition, captured by equation (4.2.1).

$$\begin{aligned} & b_1(x_t(\omega_{ND}^t), k_t(\omega_{ND}^t)) + \beta\pi b_2(x_{t+1}(\omega_{ND}^{t+1}), k_{t+1}(\omega_{ND}^{t+1}) \\ & + \beta(1-\pi)b_2(\Psi(k_{t+1}(\{\omega_{ND}^t, D\})), k_{t+1}(\{\omega_{ND}^t, D\})) = 0 \end{aligned} \tag{4.3.7}$$

4.3.2 Recursive Formulation

Using equation (4.3.6) for the planner's problem and equation (4.3.7), which captures the optimality conditions of the private households, now the problem of the planner can be written down in recursive form.

$$\max_{\{x_t(\omega_{ND}^t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \pi^t \{u(x_t(\omega_{ND}^t), k_t(\omega_{ND}^t)) + \beta(1-\pi)\xi(k_{t+1}(\omega_{D,t+1}^{t+1})\} \tag{4.3.8}$$

$$\begin{aligned} & s.t. \\ & b_1(x_t(\omega_{ND}^t), k_t(\omega_{ND}^t)) + \beta\pi b_2(x_{t+1}(\omega_{ND}^{t+1}), k_{t+1}(\omega_{ND}^{t+1}) \\ & + \beta(1-\pi)b_2(\Psi(k_{t+1}(\{\omega_{ND}^t, D\})), k_{t+1}(\{\omega_{ND}^t, D\})) = 0 \end{aligned} \tag{4.3.9}$$

Since the formulation above exhibits future control constraints, namely in the term $\beta\pi b_2(x_{t+1}(\omega_{ND}^{t+1}), k_{t+1}(\omega_{ND}^{t+1}))$ of the sub-condition, a usual condition for Bellman equations is violated. Using the methods put forward by Albert Marcet and Ramon Marimon[7], Debortoli/Nunes rewrite the problem above to a saddle point functional equation in order to show that it could be generalized to a usual Bellman equation.[8] Since the planner always faces the same institutional setting, history dependencies can be dropped. Further, Debortoli/Nunes redefine certain terms as functions in order to display the problem in a more clearly laid out manner.

$$\begin{aligned} r(x_t, k_t) &\equiv u(x_t, k_t) + \beta(1-\pi)\xi(k_{t+1}) \\ g_1(x_t, k_t) &\equiv b_1(x_t, k_t) + \beta(1-\pi)b_2(\Psi(k_{t+1}), k_{t+1}) \\ g_2(x_{t+1}, k_{t+1}) &\equiv b_2(x_{t+1}, k_{t+1}) \end{aligned}$$

[7]Marcet, Albert & Marimon, Ramon, 1994. "Recursive Contracts," Economics Working Papers 337, Department of Economics and Business, Universitat Pompeu Fabra, revised Oct 1998.

[8]Debortoli/Nunes 2007, p. 14

Rewriting the planner's problem along this scheme gives:

$$\max_{\substack{\{x_t(\omega^t_{ND}\}^\infty_{t=0} \\ \omega^t=\omega^t_{ND}}} \sum_{t=0}^{\infty} \beta^t \pi^t \{r(x_t, k_t)\} \tag{4.3.10}$$

$$s.t.$$

$$g_1(x_t, k_t) + \beta\pi g_2(x_{t+1}, k_{t+1}) = 0 \tag{4.3.11}$$

Since $\beta\pi$ can be considered as a discount rate and since the planner only accounts for situations of the kind ω^t_{ND} (as all other histories are already maximized by other planners[9]), the stochastic problem from before has transformed into a non-stochastic one. Along Debortoli/Nunes the problem can therefore be expressed as a saddle point functional equation.

$$W(k,\gamma) = \min_{\lambda \geqslant 0} \max_{x} \{H(x,k,\lambda,\gamma) + \beta(1-\pi)\xi(k') + \beta\pi W(k',\gamma')\} \tag{4.3.12}$$

$$s.t.$$

$$\gamma' = \lambda, \quad \gamma_0 = 0 \tag{4.3.13}$$

where

$$\begin{aligned} H(x,k,\lambda,\gamma) &= u(x,k) + \lambda g_1(x,k) + \gamma g_2(x,k) \\ g_1(x,k) &= b_1(x,k) + \beta(1-\pi) b_2(\Psi(k'),k') \\ g_2(x,k) &= b_2(x,k) \end{aligned}$$

The interpretation of formulation (4.3.12) & (4.3.13) is as follows: Term $H(x,k,\lambda,\gamma)$ captures the problem of the current planner in the present. The second term $\beta(1-\pi)\xi(k')$ displays the behavior of future planners which will come into office after a default. Term $\beta\pi W(k',\gamma')$ represents the current planner's behavior if he is able to commit and therefore able to prolong his tenure. The Lagrange multipliers λ and γ, which work as costate variables here, capture the intertemporal dynamics. Along Debortoli/Nunes, γ' can be interpreted as an aggregation of promises made in the past. Since they stay effective as the current government continues in office, preconditioned the government was able to commit, it is contained in the term $\beta\pi W(k',\gamma')$. In the other term $\beta(1-\pi)\xi(k')$, only the state variable k' is included, as it represents a situation in which default occurred directly before and therefore there are no promises to be kept. Using Marcet and Marimon[10], Debortoli/Nunes show that the solution of the system (4.3.8) & (4.3.9), a policy function $\psi(k,\gamma)$, is time invariant: Again the solution resembles a fixed point

[9] for the argumentation of this see section 4.3.1

[10] Marcet/Marimon 1994

problem as in the standard Markov setting[11].

$$\psi(k,\gamma) \in \arg\min_{\lambda \geqslant 0} \max_{x} \{H(x,k,\lambda,\gamma) + \beta(1-\pi)\xi(k') + \beta\pi W(k',\gamma')\} \tag{4.3.14}$$

$s.t.$

$$\gamma' = \lambda, \quad \gamma_0 = 0 \tag{4.3.15}$$

4.4 Defining an Equilibrium

4.4.1 Preconditions and properties of an Equilibrium

In the recursive formulation of the problem above, it was assumed that all future planners after a default of the current planner face the same institutional setting. This assumption was further extended to the current planner in time period $t = 0$. Taking these assumptions for granted, the solution for the Markov-perfect equilibrium along Debortoli/Nunes[12] has to satisfy the following conditions:

1. Policy function $\Psi(.)$ and value function $\xi(.)$ being given, the system (4.3.8) & (4.3.9) is solved by a sequence $\{x_t\}$.

2. The problems faced by the initial and by future planners have to be equal. The value function $W_{k,\gamma}$ therefore satisfies $\xi(k) = W(k,0) = W(k)$. In the case of a new planner in office, there are no promises which have to be kept. The costate γ which captures the intertemporal dynamics, already mentioned before, is therefore zero. This is always the case after a planner succumbed to the temptation of default.

3. The policy function $\psi(k, \gamma = 0)$, which is expected to be implemented in the case of a default, has to be optimal and consistent with the optimal policy function $\Psi(k)$.

4. The equilibrium is Markov-perfect, as the optimal policy function $\Psi(.)$ solely relies on the state variable k.

5. Individual agents are utility maximizers and they have correct beliefs.

6. In maximizing, the planner takes ψ and $\xi = W$ as given.

[11] compare to sections 3.1 and 3.3.3
[12] Debortoli/Nunes 2007, p. 15 - 16

4.4.2 Analytically solving for the Equilibrium

The basic recursive problem of the planner is given by the formulation of system (4.3.10) & (4.3.11) rewritten in equations (4.4.1) & (4.4.2) respectively.

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (\beta\pi)^t \{u(x_t, k_t) + \beta(1-\pi)\xi(k_{t+1})\} \tag{4.4.1}$$

$$s.t.$$

$$g_1(x_t, k_t) + \beta\pi g_2(x_{t+1}, k_{t+1}) = 0 \tag{4.4.2}$$

Based on the system (4.4.1) and (4.4.2), one can set up the corresponding Lagrangian.

$$\min_{\{\lambda_t\}_{t=0}^{\infty}} \max_{\{x_t\}_{t=0}^{\infty}} \mathcal{L} = \sum_{t=0}^{\infty} (\beta\pi)^t \{u(x_t, k_t) + \beta(1-\pi)W(k_{t+1}) + \lambda_t(g_1(x_t, k_t) + \beta\pi g_2(x_{t+1}, k_{t+1}))\} \tag{4.4.3}$$

At this point it is reasonable to recall the identities $g_1(x, k) = b_1(x, k) + \beta(1-\pi)b_2(\Psi(k'), k')$, $g_2(x, k) = b_2(x, k)$ and $x_t \equiv (k_{t+1}, h_t, \theta_t, \tau_t)$ from before. The argument $\min_{\{\lambda_t\}_{t=0}^{\infty}}$ underlines that it is part of the planner's problem to optimally avoid default; like γ, mentioned in the last section within the preconditions for a Markov-perfect equilibrium, λ here is a costate which captures the intertemporal dynamics. It is only different to zero in the case of default, for which the probability is $(1-\pi)$. Since x_t contains a state variable with k_{t+1}, it cannot be used as a control variable. Based on x_t we define a new variable $z_t \equiv (h_t, \theta_t, \tau_t)$, with respect to which one can now derive the Lagrange function. The first-order conditions for the planner's problem are therefore given by:

$$\frac{\partial \mathcal{L}}{\partial z_t} : u_{z_t,t} + \lambda_t g_{1,z_t,t} + \lambda_{t-1} g_{2,z_t,t} = 0 \tag{4.4.4}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial k_{t+1}} &: u_{k_{t+1},t} + \beta(1-\pi)W_{k_{t+1},t+1} + \lambda_t(g_{1,k_{t+1},t} + \beta\pi g_{2,k_{t+1},t+1}) \\ &+ \beta\pi(u_{k_{t+1},t+1} + \lambda_{t+1} g_{1,k_{t+1},t+1}) - \lambda_{t-1} g_{2,k_{t+1},t} = 0 \end{aligned} \tag{4.4.5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} : g_1 + \beta\pi g_2 = 0 \tag{4.4.6}$$

where

$$g_{1,x_t,t} = b_{1,x_t,t} \qquad g_{1,k_t,t} = b_{1,k_t,t}$$

$$g_{2,x_t,t} = b_{2,x_t,t} \qquad g_{2,k_t,t} = b_{2,k_t,t}$$

$$g_{2,k_{t+1},t} = b_{2,k_{t+1},t}$$

Having derived the results, some words should be said about the notation: Functions contained in the maximization problem are concerned more than once when derived with respect to k_{t+1}. For example, $u(x_t, k_t)$ contains k_{t+1} in period t, since $x_t \equiv (k_{t+1}, h_t, \theta_t, \tau_t)$, and in period $t+1$ via the original k_t. The same is the case for the function $g_1(x_t, k_t)$. Being derived with respect to k_{t+1}, term $g_2(x_{t+1}, k_{t+1})$ is concerned in period $t-1$ via $x_{t+1}(.)$ and in period t via k_{t+1}. Therefore, the subscripts in the first order conditions (4.4.4) to (4.4.6) give the variables with respect to which the functions were derived as well as the respective time period concerned.

One significant property of the first order conditions above is the appearance of the value function $W_{k_{t+1},t+1}$ or $\xi_{k_{t+1},t+1}$ to follow the notation of Debortoli/Nunes in equation (4.4.5). The value function $\xi(k_{t+1})$ gives the welfare of the agents with a new planner coming into office at period $t+1$. A new planner in period $t+1$ however has no influence on the level of the state variable k_{t+1}, whereas the initial planner in period t does influence it strategically. The optimal policies can now be solved for, due to the first order conditions (4.4.4) to (4.4.6). Debortoli/Nunes do so in reducing the problem to a fixed point problem by application of the envelope theorem. This is appropriate, as it was assumed that all planners are facing the same problem or maximizing the same function, $\xi(k_t) = W(k_t)$ or $\xi_{k,t+1} = W_{k,t+1}$, respectively. All variables are therefore solved for with the optimal policy of a planner new in office in period t, presumably after a default, and with a given k_t. The envelope results by Debortoli/Nunes are therefore:

$$\frac{\partial W(k_t)}{\partial k_t} = \frac{\partial u(x_t(k_t), k_t)}{\partial k_t} + \lambda_t g_{1_{k_t,t}} \tag{4.4.7}$$

Using these results, Debortoli/Nunes substitute for $\xi_{k,t+1} = W_{k,t+1}$ within the first order conditions. Remaining is a problem which solely depends on the policy functions $\psi(k_t, \lambda_{t-1})$ and $\Psi(k)$ with $\psi(k_t, 0) = \Psi(k) \quad \forall \quad k$.

4.5 The probability of default is endogenous

In the initial model examined hitherto in this chapter, the probability for default/commitment was given exogenously. In a next step Debortoli/Nunes study an economy where the probability for default/commitment of a respective period depends on the state of economy and therefore on the capital stock k. Instead of π, the probability of commitment in the successive period is now given by the function $P(k_{t+1})$. In the first period, where there are no promises to be kept, $P(.) = 1$. The initial capital stock k_0 in this model is given. Otherwise the probability operator $P(.)$ would depend on a non-existent variable, which would cause time-inconsistency problems as Debortoli/Nunes admit. Private agents, who are aware of the fact that the probability for commitment depends on the capital stock k, take the aggregated capital stock as given and accumulate their capital by optimizing the maximization problem subject to their budget constraints just as before. The optimization problem with endogenous probabilities for commitment can be written down as follows.

$$\max_{\{x_t(\omega^t_{ND}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{\prod_{j=0}^{t}(P(k_j))}{P(k_0)} \{u(x_t(\omega^t_{ND}), k_t(\omega^t_{ND})) + \beta(1 - P(k_{t+1}))\xi(k_{t+1}(\omega^{t+1}_{D,t+1})\} \tag{4.5.1}$$

$s.t.$

$$b_1(x_t(\omega^t_{ND}), k_t(\omega^t_{ND})) + \beta P(k_{t+1}) b_2(x_{t+1}(\omega^{t+1}_{ND}), k_{t+1}(\omega^{t+1}_{ND}) + \beta(1 - P(k_{t+1})) b_2(\Psi(k_{t+1}(\{\omega^t_{ND}, D\})), k_{t+1}(\{\omega^t_{ND}, D\})) = 0 \tag{4.5.2}$$

The term $\frac{\prod_{j=0}^{t}(P(k_j))}{P(k_0)}$ in the planner's problem gives the probability that there was no default up to period t. Clearly in period $t = 0$ this operator would be unity. Again the definitions from above, this time of course with the endogenous probability operator, allow us to rewrite and simplify the notation.

$$\begin{aligned} r(x_t, k_t) &\equiv u(x_t, k_t) + \beta(1 - P(k_{t+1}))\xi(k_{t+1}) \\ g_1(x_t, k_t) &\equiv b_1(x_t, k_t) + \beta(1 - P(k_{t+1})) b_2(\Psi(k_{t+1}), k_{t+1}) \\ g_2(x_{t+1}, k_{t+1}) &\equiv b_2(x_{t+1}, k_{t+1}) \end{aligned}$$

Rewriting the planner's problem along this scheme gives:

$$\max_{\substack{\{x_t(\omega^t_{ND}\}_{t=0}^{\infty} \\ \omega^t = \omega^t_{ND}}} \sum_{t=0}^{\infty} \beta^t \frac{\prod_{j=0}^{t}(P(k_j))}{P(k_0)} \{r(x_t, k_t)\} \tag{4.5.3}$$

$s.t.$

$$g_1(x_t, k_t) + \beta P(k_{t+1}) g_2(x_{t+1}, k_{t+1}) = 0 \tag{4.5.4}$$

Since the problem above again exhibits future control constraints as in the case with exogenous probabilities before, Debortoli/Nunes prove that it also can be rewritten to a saddle point functional equation. The Lagrange formulation of the system (4.5.3) & (4.5.4) shows certain characteristics:

$$\begin{aligned} \min_{\{\lambda_t, \varphi\}_{t=0}^{\infty}} \max_{\{x_t\}_{t=0}^{\infty}} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{ &\kappa_t (u(x_t, k_t) + \beta(1 - P(k_{t+1}))\xi(k_{t+1})) \\ &+ \lambda_t \kappa_t (g_1(x_t, k_t) + \beta P(k_{t+1}) g_2(x_{t+1}, k_{t+1})) + \varphi_t(\kappa_{t+1} - \kappa_t P(k_{t+1}))\} \end{aligned} \tag{4.5.5}$$

The term $\varphi_t(\kappa_{t+1} - \kappa_t P(k_{t+1}))$ in the formulation above represents the evolution of the endogenous probability operator $P(k_{t+1})$, which applies to the other terms of the equation. If a new planner comes into office subsequently to a default and there are no promises which have to be kept, hence $\kappa = 1$. In such a case the Lagrange multiplier λ_t which captures the intertemporal dynamics is zero. Again a new variable $z_t \equiv (h_t, \theta_t, \tau_t)$ is defined which in contrast to the original x_t, on which it is based, doesn't contain the state variable k_{t+1} and can therefore be used as a control variable. The first order conditions to this problem are:

$$\frac{\partial \mathcal{L}}{\partial z_t} : u_{z_t,t} + \lambda_t g_{1,z_t,t} + \lambda_{t-1} g_{2,z_t,t} = 0 \tag{4.5.6}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial k_{t+1}} &: u_{k_{t+1},t} + \beta(1 - P(k_{t+1}))W_{k_{t+1},t+1} + \lambda_t(g_{1,k_{t+1},t} + \beta P(k_{t+1}) g_{2,k_{t+1},t+1}) \\ &+ \beta P(k_{t+1})(u_{k_{t+1},t+1} + \lambda_{t+1} g_{1,k_{t+1},t+1}) - \lambda_{t-1} g_{2,k_{t+1},t} \\ &- \beta P_{k_{t+1}}(k_{t+1}) W(k_{t+1}) - \varphi_t P_{k_{t+1}}(k_{t+1}) + \lambda_t P_{k_{t+1}}(k_{t+1})(b_2(x_{t+1}, k_{t+1}) \\ &- b_2(\Psi(k_{t+1}), k_{t+1})) = 0\end{aligned} \tag{4.5.7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} : g_1 + \beta P(k_{t+1}) g_2 = 0 \tag{4.5.8}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_t} : \kappa_{t+1} - \kappa_t P(k_{t+1}) = 0 \tag{4.5.9}$$

$$\frac{\partial \mathcal{L}}{\partial \kappa_{t+1}} : \beta(u_{t+1} + \beta(1 - P(k_{t+2}))W(k_{t+2})) + \varphi_t - \beta \varphi_{t+1} P(k_{t+2}) = 0 \tag{4.5.10}$$

The first order conditions are valid for $\forall \quad t = 0, \ldots, \infty$ and since there are no promises to be kept in the first period, as already mentioned several times before, the costate $\lambda_{t \leqslant 0} = 0$. The first order conditions (4.5.6) and (4.5.7) have been simplified by substituting for κ_{t+1} from first order condition (4.5.9) and thereafter by dividing by $\beta^t \kappa_t$. Equation (4.5.10) can be unformed to express φ_t. Forward iteration, this means iteratively substituting for φ_{j+1}, finally yields the expression in the general form (4.5.11).

$$\begin{aligned}-\varphi_t &= \beta(u_{t+1} + \beta(1 - P(k_{t+2}))W(k_{t+2})) - \beta \varphi_{t+1} P(k_{t+2}) \\ &= \beta \left[(u_{t+1} + \beta(1 - P(k_{t+2}))W(k_{t+2})) - \varphi_{t+1} P(k_{t+2}) \right] \\ &= \beta P(k_{t+2}) \left[\frac{1}{P(k_{t+2})} (u_{t+1} + \beta(1 - P(k_{t+2}))W(k_{t+2})) - \varphi_{t+1} \right] \\ &\rightarrow\end{aligned}$$

$$-\varphi_t = \beta \sum_{i=t+1}^{\infty} \beta^{i-1} \frac{\prod_{j=0}^{i}(P(k_j))}{P(k_{t+2})} (u(x_i, k_i) + \beta(1 - \beta(1 - P(k_{i+1}))W(k_{i+1})) \tag{4.5.11}$$

New dynamics through endogenous probability Expression (4.5.11) implies that $-\varphi$ equals the value function under commitment, beginning in period $t+1$ times the discount factor β. One can therefore write $-\varphi_t = \beta W(k_{t+1}, \lambda_t)$ for which one can substitute in the first order condition (4.5.7), giving:

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial k_{t+1}} : u_{k_{t+1},t} + \beta(1 - P(k_{t+1}))W_{k_{t+1},t+1} + \lambda_t(g_{1,k_{t+1},t} + \beta P(k_{t+1})g_{2,k_{t+1},t+1}) \\
&+ \beta P(k_{t+1})(u_{k_{t+1},t+1} + \lambda_{t+1}g_{1,k_{t+1},t+1}) - \lambda_{t-1}g_{2,k_{t+1},t} \\
&+ \beta P_{k_{t+1}}(k_{t+1})(W(k_{t+1}, \lambda_t) - W(k_{t+1})) + \lambda_t P_{k_{t+1}}(k_{t+1})(b_2(x_{t+1}, k_{t+1}) \\
&- b_2(\Psi(k_{t+1}), k_{t+1})) = 0
\end{aligned} \tag{4.5.12}$$

The first order condition which was calculated by the derivation of the Lagrange function with respect to capital is insofar interesting as the accumulation of capital is the deciding factor for the probability to commit and therefore also the only parameter which the current government can try to influence in order to increase its commitment probability to stay in office. This is reflected by the fact that this first order condition exhibits additional terms compared to the same first order condition in the case where probabilities where endogenous. The most interesting new term in the first order condition (4.5.12) is the part $\beta P_{k_{t+1}}(k_{t+1})(W(k_{t+1}, \lambda_t) - W(k_{t+1}))$ in the third line. By the accumulation of capital the probability for commitment is increased by the factor $P_{k_{t+1}}$, which would mean a gain in utility in the case of commitment by $W(k_{t+1}, \lambda_t)$, a loss however in the case of default by $W(k_{t+1})$. In the next section I will shortly address the calibration Debortoli/Nunes choose. Thereafter the numerical results will be presented and discussed.

4.6 Calibration

For the numerical solution Debortoli/Nunes use an utility function of the form.

$$u(c_t, g_t, h_t) = (1 - \phi_g)[\phi_c log(c_t) + (1 - \phi_c)log(1 - h_t)] + \phi_g log(g_t) \tag{4.6.1}$$

The depreciation rate of capital is given by the function:

$$\delta(v_t) = \frac{\chi_o}{\chi_1} v_t^{\chi 1} \tag{4.6.2}$$

The production function, as already stated before, is of the form:

$$F(k_t, v_t, h_t) = (k_t v_t)^\sigma h_t^{1-\sigma} \tag{4.6.3}$$

The parameters ϕ_c and ϕ_g represent a measure weighting consumption in relation to leisure and public consumption in relation to private consumption in the preferences of the households respectively. Their values were calibrated to 0.285 and 0.119. The discount factor β and the capital share in the production process σ are determined to have the values 0.96 and 0.36 respectively. The parameters χ_0 and χ_1 are calibrated (to the values 0.171 and $1,521$ respectively) that way that a capital utilization rate v_t of 0.8 corresponds to a depreciation rate δ_t of 0.08. The basis for the calibration of Debortoli/Nunes is annual data of the US economy.

While in the first setting probabilities for commitment where exogenously given, in the second part they are a function of the capital stock k_t and therefore endogenous. The function reflecting these endogenous probabilities is given by:

$$P(k_t) = 1 - \frac{1}{(\frac{k_t}{\tilde{k}})^p + 1} \tag{4.6.4}$$

As already pointed out above, a higher level of capital increases the probability of commitment and therefore the likeliness for a government to stay in office. Parameter $\tilde{k}$ normalizes the function such that $P(\tilde{k}) = 0.5$. p is a parameter reflecting the sensitivity of the function with respect to changes in the level of capital: If $p = 0$ the probability is constant. The higher p is, the stronger it reacts to fluctuations of the capital stock and the higher is the incentive for the planner to encourage capital accumulation with his policies in order to stay in office. The values for $\tilde{k}$ and p are calibrated in order equalize the average capital allocation to the case with an exogenous probability of $\pi = 0.5$ (for commitment), such that the results are better comparable. For the calculation of the specific numerical results Debortoli/Nunes indicate to have used global approximation methods as put forward by Kenneth Judd.[13]

4.7 Results

Table 4.1 in summary displays the numerical results for the model of Debortoli/Nunes: It indicates the long-run average values for the variables in the first column which where calculated for different probabilities in the exogenous probability setting and as well for the endogenous probability case. The probability to commit for the endogenous probability setting corresponds to an average value of 0.799. In the context of the models presented before it makes sense to first look at tax policies, and here especially

[13] Judd, Kenneth L., 1992. "Projection methods for solving aggregate growth models," Journal of Economic Theory, Elsevier, vol. 58(2), pp. 410-452. and Judd, Kenneth L., 2004. "Existence, uniqueness, and computational theory for time consistent equilibria: A hyperbolic discounting example," Stanford University, Manuscript.

π	1.000	0.750	0.500	0.250	0.000	end. prob
k	1.122	0.947	0.899	0.880	0.870	0.932
λ	-0.536	-0.177	-0.080	-0.030	0.000	-0.064
g	0.093	0.076	0.072	0.070	0.069	0.082
c	0.196	0.216	0.220	0.222	0.224	0.209
y	0.378	0.368	0.364	0.363	0.362	0.365
θ	0.000	0.131	0.163	0.178	0.187	0.153
τ	0.384	0.251	0.218	0.203	0.191	0.268
h	0.233	0.245	0.248	0.250	0.250	0.246
u	0.798	0.799	0.801	0.800	0.799	0.790
$\bar{\pi}$	1.000	0.750	0.500	0.250	0.000	0.738

Table 4.1: Numerical Results - Loose Commitment

at the extreme cases of full- and no-commitment. In the case of full-commitment, which here corresponds to a value $\pi = 1.000$, the results calculated by Debortoli/Nunes are completely in line with the findings of chapter 2 and chapter 3: Under the assumption of full-commitment the optimal capital income tax is zero. In this case the tax burden to finance government expenditures fully lies on labor income which is taxed at a rate of 38%. Let aside their specific calibration Klein/Ríos-Rull find a figure which is with 31% not that different to the result of Debortoli/Nunes.[14] In the case of no-commitment the average capital income tax rate lies at around 18.7%. The average labor income tax is with 19.1% just slightly higher. These results are insofar in line with the findings of chapter 3 as the average capital income tax rate is considerably different to zero. In the case of the Klein/Ríos-Rull model however the value for the capital income tax rate is independently from the calibration much higher.[15]

These results go hand in hand with the interpretation of the other variables. Concerning capital income taxation one can state summarizing: the higher the commitment probability the lower capital income taxation is. This has direct consequences for the capital accumulation. One observes a significantly higher capital stock in the cases with a higher commitment probability. As a matter of fact, as capital income taxes are low labor income taxes have to be higher in order to balance the budget as defined introductorily in the basic model assumptions. The figures for τ displayed in Table 4.1 reflect this trivial conclusion. Accordingly labor supply should also decrease with higher income tax rates. The h-values in Table 4.1 support this conclusion.

In the endogenous probability setting the average capital income tax rate lies at around 15.3% in average. Taxation on labor income carries the biggest burden to finance govern-

[14] Klein/Ríos-Rull 2003, p. 1230
[15] Klein/Ríos-Rull 2003, p. 1231

ment expenditures: wages are taxed with 26.8% in average. As already stressed before, in the section about the calibration which Debortoli/Nunes use in their calculations, the endogenous probability model was adjusted to correspond to the $\pi = 0.50$ setting in the exogenous probability model as far as capital allocation is concerned. The $\pi = 0.50$ case is therefore the most interesting reference to compare. The endogenous probability setting in contrast exhibits an average probability to commit of $\bar{\pi} = 0.738$. Eye-catching however is primarily the high average capital stock in the endogenous probability setting. The average value for k here is 0.932. For the $\pi = 0.50$-case on the contrary this value is only at 0.899. This difference can be explained by the forces and incentives which are at work in the endogenous probability setting: Governments in the endogenous probability setting try to implement policies which encourage capital accumulation since a higher capital stock increases the commitment power, namely the probability of commitment $P(k_t)$, and in the same way the likeliness that the current government can stay in office. The instrument of choice for this purpose is a lower capital income tax rate. This line of argumentation is also supported by the figures for θ in Table 4.1: The capital income tax in the $\pi = 0.50$ case is with 16.3% higher than in the endogenous probability setting. As governments can therefore influence their commitment probability, it is not surprising that the overall results in the endogenous probability setting resemble more those of the full-commitment rather than those of the no-commitment scenario.

4.8 Summary and Interpretation

The loose commitment approach by Debortoli/Nunes represents a compromise between the full- and no-commitment assumption effective in chapters 2 and 3 respectively. In contrast to this two extreme cases the authors assume the default on past promises can occur in every period with a certain probability. At first this probability is exogenously given, then it is assumed that planners could enhance it's probability to commit, hence its probability to stay in office, by implementing policies which encourage capital accumulation. Depending on the respective calibrations (of the default probability) the results are in line with the full- and the no-commitment case. Setting the default probability to one, the model resembles a standard Markov perfect equilibrium while setting it to zero it resembles a Ramsey equilibrium. Consequently the intuition regarding the results is the same as in the cases of full- or no-commitment in chapters 2 and 3 respectively.

5 Conclusion

The goal of this thesis was to illustrate the role of commitment for fiscal policy. It was shown that the assumption wether a planner has the ability to commit to a chosen sequence of policies or not, is a central factor in determining the methodic approach as well as the effective policy outcomes. Particularly two approaches were discussed, which control for the factor commitment: In the case of full-commitment the government can decide for a sequence of policies once for all successive periods. Such a situation is captured in a Ramsey equilibrium. Solutions found in such a setting are regarded as being time-inconsistent as they are not incentive compatible over time. If it is however assumed, that governments don't possess a commitment technology, agents incorporate the potential discretion of the successive government. Such an environment resembles an infinitely repeated non-cooperative game in which government policies represent mutual best responses to the past (state of the economy left back by the former government) and the future (expected discretion by its successor). In such a context an equilibrium can be found in a Markov setting, which incorporates the time-inconsistency of once taken policy decisions and therefore delivers time-consistent solutions.

In the full-commitment case the central result is that capital taxation in the limit should be optimally zero. The famous result initially derived by Kenneth Judd and Christophe Chamley hinges on the fact, that capital taxation represents an intertemporal wedge distorting the consumption/saving behavior of households. The planner wants to minimize distortions and sets capital taxes to zero in the long run. In the initial period however the government has incentives to tax capital. Capital in the initial period is already installed and it's taxation is hence considered as being non-distortionary. Capital taxes in the initial period are usually very high and serve the purpose of building up a capital base with which's proceeds the government can lower distortionary taxation in the future. The government therefore front-loads distortions and transfers resources to future periods. This was shown in a converse argument: Positive capital taxes implicitly imply a differential taxation of private good consumption and therefore incentives for households to prepone consumption. Contrary to these incentives it was shown that a transfer of resources to the future, which would be implied by a front-loading of distortionary taxation, is welfare increasing. Such procedures can obviously only be accomplished by a planner who has the ability to fully commit to a once chosen sequence of policies.

In the most extreme form of no-commitment however, as it was presented introductorily to chapter 3, the planner can only decide upon the policy for the actual period. He does this with regard to the state of the economy he inherited from his predecessor and, as he in turn bequeaths the economy to another government in the following period, to the reaction, the *best response*, of his successor. The planner without commitment technology is confronted with a situation each period which is similar to that in the initial period in the full-commitment scenario. Investments which lead to the current capital stock are regarded as being sunk. The taxation of capital is thus viewed as being not distortionary. As discussed in section 3.5 these results however hinge on the assumption that the government has to honor a balanced budget constraint. In the case the government is allowed to issue bonds, and hence has a saving/borrowing instrument, the results gained in a Markov setting resemble those in a Ramsey setting with full commitment: Capital taxes are zero at a stationary Markov equilibrium and planners accomplish transactions similar to the front-loading which was observed in the full-commitment setting. It however has to be noted that the literature on this issue is not very comprehensive yet.

The loose commitment case presented in chapter 4 presents a compromise between the full- and no-commitment setting. In contrast to the other two cases default on past promises can occur with a certain probability in each period. Depending on the respective calibration the results/methods gained/used in the loose commitment environment resemble those in the full- or no-commitment case respectively. The intuitions behind the outcomes within this setting are therefore similar to the full- and no-commitment case.

Although there is a wide variety of literature on the topic of fiscal policy and commitment, it seems that yet all questions haven't been answered. For the full-commitment assumption it is for example known that the zero capital tax result, which might seem universally valid after the foregoing discussions, is for example not robust to assumption of incomplete taxation or non-linear (i.e. progressive) taxation.[1] A variety of still open question lets us expect that optimal fiscal policy in the context of the commitment issue will remain a vivid field of research for the years to come. A pretty common observation however is that commitment is welfare increasing.[2] It would therefore also be essential to look more closely at the institutional causes for commitment and default. Especially interesting in this context is the literature about political budget cycles as for example

[1] Erosa, Andres & Gervais, Martin 2002. "Optimal Taxation in Life-Cycle Economies," Journal of Economic Theory, vol. 105, pp. 338-69.; Mirrlees James A., 1971. "An Exploration in the Theory of Optimum Income Taxation," Review of Economic Studies, Blackwell Publishing, vol. 38(114), pp. 175-208.

[2] i.e. Klein/Ríos-Rull 2003

by Torsten Persson and Guido Tabellini[3] or Kenneth Rogoff [4] for it underlines the significance of institutional settings for the degree of commitment. Bringing these two lines of research together, optimal policy and political economy research, would for sure be fruitful in the future.

[3]Persson, Torsten & Tabellini, Guido 2005. "The Economic Effects of Constitutions," MIT Press Books, The MIT Press, edition 1, volume 1.

[4]Rogoff, Kenneth 1990. "Equilibrium Political Budget Cycles," NBER Working Papers 2428, National Bureau of Economic Research, Inc.; Rogoff, Kenneth & Sibert, Anne, 1988. "Elections and Macroeconomic Policy Cycles," Review of Economic Studies, Blackwell Publishing, vol. 55(1), pp. 1-16.

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Abstract

English The subject of this thesis is to clarify the role of commitment and hence time-consistency in optimal fiscal policy theory. It is shown that the assumption of full-commitment demands a completely different modeling approach compared to the case where it is assumed that the planner doesn't possess a perfect commitment technology. In the case where a planner possesses a commitment technology he decides upon a sequence of policies for all future periods. Possible incentives, which might appear in future periods to reevaluate these policy decisions are simply elided. Such an assumption is captured in a Ramsey equilibrium. Results found in such an environment are however not incentive compatible and time-inconsistent. If the planner in contrast is assumed not to possess a commitment technology, the decision mechanism evolves sequentially rather than at one point in time and resembles a non-cooperative game between planners in consecutive periods. Suchlike assumptions are reflected in a Markov perfect equilibrium, which yields time-consistent results. It turns out, that planners under these two concepts have completely different perceptions of intertemporal distortions, which result into different policy decisions. Around this basic conceptual framework the thesis discusses and presents different varieties of modeling approaches being found in literature.

German Gegenstand dieser Diplomarbeit ist die Rolle von Verbindlichkeit, "*commitment*", und Zeitkonsistenz in der Steuertheorie. Es wird gezeigt, dass unterschiedliche Annahmen über die Verbindlichkeit von Steuerentscheidungen verschiedene Modellierungsmethoden verlangen und zu verschiedenen Ergebnissen führen. Unter der Annahme von vollständiger Verbindlichkeit hat der Planer die Möglichkeit über eine Sequenz von Steuerpolitiken ein für alle Mal zu entscheiden. Mögliche zukünftige Anreize von dieser Entscheidung abzuweichen sind per Annahme ausgeschlossen. Lösungen unter diesen Annahmenkomplex werden in einem Ramseygleichgewicht gefunden. Diese sind jedoch zeitinkonsistent und anreizinkompatibel. Unterstellt man hingegen, dass der Planer nicht die Möglichkeit besitzt sich glaubwürdig an eine Politikentscheidung zu binden erfolgt der Entscheidungsprozess sequentiell in der Form von nicht-kooperativen Spielen zwischen Planern in aufeinander folgenden Perioden, die jeweils die beste Antwort auf die Politiken des jeweils folgenden/vorhergehenden Planers geben. Lösungen unter der Annahme von nicht vollständiger Verbindlichkeit werden in einem Markov perfekten Gleichgewicht gefunden. Diese gelten als zeitkonsistent. Es zeigt sich, dass Planer in beiden Konzepten intertemporale Verzerrungen ausgelöst durch Steuerpolitik in unterschiedlicher Weise wahrnehmen, was sich in der Implementierung unterschiedlicher Steuerpolitiken niederschlägt. Um diese beiden Grundkonzepte werden verschiedene, in der Literatur gängige, Weiterentwicklungen der Modelle diskutiert.

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