DIPLOMARBEIT

Titel der Diplomarbeit

Some remarks on Nagumo's theorem

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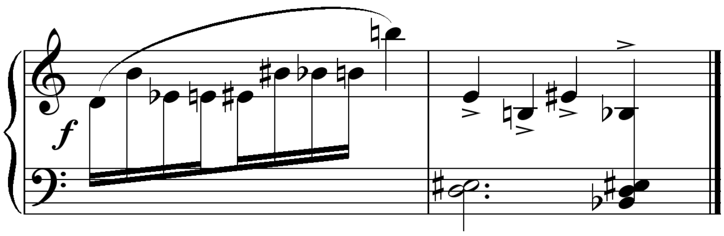
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Abstract

We prove uniqueness and convergence of the successive approximations under more general assumptions than Lipschitz for $x^{(n)} = f(t,x)$. Subsequently we provide a simpler proof for a recent generalization of Nagumo's theorem and we show that not only is the solution unique, but the successive approximations converge to the unique solution. Thus we generalize the Picard-Lindelöf theorem first in the direction initiated by Nagumo and Athanassov, and subsequently in the direction initiated by Constantin.

Chapter 1

Introduction

This thesis concerns the existence and uniqueness of solutions of ordinary differential equations (ODE). We present an easier proof for [Con10], furthermore generalize the work of [Ath90] to higher order ODE.

Differential Equations

If one studies equations, there are two main interests in general. Does the equation have a solution, and if yes, it is unique. Both questions depend on the range where one is looking for solutions.

In this theses we will consider ODE of the form

$$x^{(n)}(t) = f(t, x(t)) \tag{1.1}$$

$$x(0) = x_0, \quad x'(0) = x_1, \cdots, \; x^{(n-1)}(0) = x_{n-1} \;\; x_i \in \mathbb{R}, \tag{1.2}$$

where $x_i \in \mathbb{R}$ and f and x are real valued functions on a appropriate domain of definition. Hence, in general, we will look for continuous differentiable functions as solutions. It will be important, that the ODE depends on x and t. Since the behaviour in the t-variable will be almost singular it is not an option to rewrite (1.1)-(1.2) as an autonomous[1] system.

Finding an explicit solution for a given ODE is difficult in most cases. Because of this in history the attention got turned on conditions, under which an equation must have a (unique) solution. The classic theorems in this connection are the one of Picard and Lindelöf, and the one of Peano. Since that time a lot of generalizations have been proved. The most far reaching is maybe given by Nagumo in [Nag26]. This thesis follows up the papers of Athanassov [Ath90] and Constantin [Con10]. We show, that in both cases the successive approximations converge to the unique solution. The central tool for that will be a Gronwall-type inequality, in the following called "A Integral Inequality". It is proven by adapting the proof in [Ath90]. Uniqueness follows then at once. To show existence we construct a sequence using the successive approximations which satisfies the assumptions needed for the integral inequality. Thus it follows that a certain subsequence of the successive approximations are a solution. Together with uniqueness this yields that every subsequence converges to a solution. We will do this two times. Once to carry on [Win56], the second time

[1] Autonomous means, that the ODE does not depend on the time variable.

following up [Con10]. An advantage of using the successive approximations is, that one gains also a approximate solution, not only existence.

Overview over the Thesis

In Chapter 2 the theorems of Peano and Picard-Lindelöf will be presented. Proofs will not be given, but they can be looked up in the cited references. We will point at a proof using the successive approximations, because we will use them also in the proof of our theorem. Beside, we present a heap of examples. The last mentioned generalizations will be more extensive discussed in chapter 3. Chapter 4 and 5 are the core of this thesis. The long text at the end, is the German translation of this one here.

I would like to thank Prof. Constantin.

Chapter 2

The Standard Existence and Uniqueness result

Theorem of Picard - Lindelöf

In this section we give an overview over the fundamental existence and uniqueness theorems for nonlinear ordinary differential equations

$$x^{(n)}(t) = f(t, x(t)) \tag{2.1}$$

$$x(0) = x_0, \quad x'(0) = x_1, \cdots, \ x^{(n-1)}(0) = x_{n-1} \ \ x_i \in \mathbb{R} \tag{2.2}$$

where $f \in C(D)$, [1] with $D = \{(t, x) : |t| \leq a, |x| < b\}$ and $(0, x_0) \in D$. We may always assume that our initial values are given for the time $t = 0$.

Definition 2.1. $x(t)$ is a *solution of the initial value problem* (2.1)-(2.2) *of order* n *on the interval* $-a \leq t \leq a$ if it is continuous on $[-a, a]$, having n finite derivatives $x^{(n)}(t)$ on $(-a, a)$ and satisfies equations (2.1)-(2.2) on $(-a, a)$.

Example 2.2. Consider the initial value problem

$$x'(t) = x \cdot t \tag{2.3}$$

$$x(0) = 1 \tag{2.4}$$

Then $y(t) = e^{t^2/2}$ is a solution on $\mathbb{R}$. $\boxdot$

We will see, that under the hypothesis that f satisfies a Lipschitz condition, that is "f does not change its values too fast", existence and uniqueness holds for (2.1)-(2.2). This is usually proved using some contracting principle, but we will refer to a proof using Picard's classical method of successive approximations. This is because our proof will essentially use Picard's ideas. The second important theorem we will point at is Peano's existence theorem which deals with the case where f is a continuous function.

Definition 2.3. Let I be an open interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is *locally Lipschitz in* I, if for each point $x_0 \in I$ there is an ε-neighbourhood of x_0, $N = (x_0 - \varepsilon, x_0 + \varepsilon) \subseteq I$ and a constant $L_0 > 0$ such that for all $x, y \in N$

$$|f(x) - f(y)| \leq L_0 \, |x - y| \, .$$

[1] $C^n(D) = \{f : D \to \mathbb{R}, \text{ n-times continuous differentiable}\}$

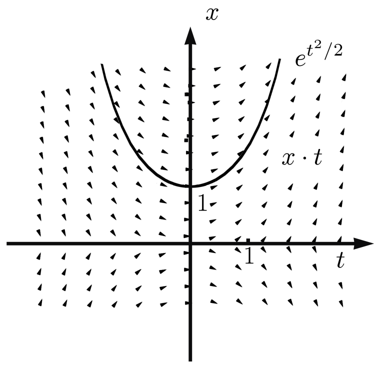

Figure 2.1: Vectorfield of (2.3)-(2.4) with inscribed solution of $x' = x \cdot t$.

The function f is *Lipschitz in* I if there is a constant $L > 0$, such that for all $x, y \in I$

$$|f(x) - f(y)| \leq L\,|x - y|\,.$$

Example 2.4. Examples of Lipschitz functions.

一) $f(x) = x^2$: This function is locally Lipschitz for all $x \in \mathbb{R}$, but not globally.

二) $f(x) = |x|$: This function is Lipschitz on $\mathbb{R}$ with Lipschitz constant 1, but not differentiable in $x = 0$.

三) $f(x) := \begin{cases} 0 & x \leq 0 \\ \sqrt[+]{x} & x > 0 \end{cases}$ f is continuous on $\mathbb{R}$ and continuous differentiable on $\mathbb{R} \setminus \{0\}$. But not locally Lipschitz in $x = 0$.

$\boxdot$

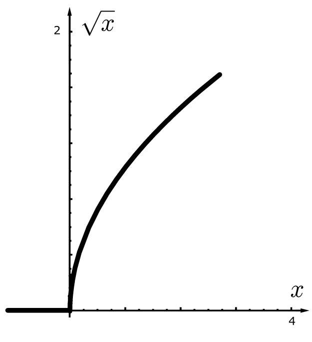

Figure 2.2: Graph of $f(x) = \sqrt{x}$. At $x = 0$ this function has 'infinite' derivative.

Proposition 2.5. *Let $I = (a, b)$ be an open interval in $\mathbb{R}$, and $f : I \to \mathbb{R}$, $f \in C^1$. Then f is locally Lipschitz on I.*

Theorem 2.6. (Picard-Lindelöf)
Assume $f(t,x) : D \to \mathbb{R}$ is a continuous function, Lipschitz in x, $(0, x_0) \in D$. Then there exists an $a > 0$ such that the initial value problem

$$x'(t) = f(t, x(t)) \tag{2.5}$$

$$x(0) = x_0 \tag{2.6}$$

has a unique solution $x(t)$ on the interval $[-a, a]$. A proof of this can be found in [Per93].

Definition and Remark 2.7. The proof uses the method of the successive approximations, also called *Picard iteration*. It is based on the fact that solving the differential equation (2.5)-(2.6) is equivalent to solving this integral equation:

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds. \tag{2.7}$$

Iterated applying of this integral to an initial function, yields a sequence, which converges uniformly to the solution. These iterations are called *successive approximations*. The successive approximations for the the problem (2.1)-(2.2) are defined by the sequence of functions

$$y_0(t) = y(t) \tag{2.8}$$

$$y_{i+1}(t) = p(t) + \frac{1}{(n-1)!} \int_0^t f(s, y_i(s)) \cdot (t-s)^{n-1}\, ds \quad \text{with}$$

$$p(t) := x_0 + x_1 t + x_2 \frac{t^2}{2!} + x_3 \frac{t^3}{3!} + \cdots + \frac{t^{n-1}}{(n-1)!} \tag{2.9}$$

with $i = 1, 2, 3, \cdots$ and $y(t) : I \to \mathbb{R}$ continuous and such that $|y(t)| \leq b$ for all $|t| \leq a$. We will often set $y(t) = p(t)$. $p(t)$ will always, throughout the whole paper, denote this polynomial. We will give some examples after presenting Peano's theorem.

Remark 2.8. Using a simple variable substitution we can always assume that $x_i = 0$ for all $i = 1, \cdots, n-1$. Namely via $y(t) := x(t) - p(t)$. The system (2.1)-(2.2) then transforms to

$$y^{(n)}(t) = f(t, y + p(t))$$
$$y(0) = 0, \;\; y'(0) = 0, \;\; \cdots, \;\; y^{(n-1)}(0) = 0.$$

Theorem of Peano

Peano's theorem only requires continuousness for the right hand side of the differential equation, but deduces only existence of a local solution.

Theorem 2.9. *Assume $f(t,x) : D \to \mathbb{R}$ is a continuous function, $(0, x_0) \in D$. Then the initial value problem*

$$x'(t) = f(t, x(t))$$
$$x(0) = x_0$$

has a solution $y : I \to \mathbb{R}$ defined on a closed interval $I \subseteq \mathbb{R}$. See [CL55] for a proof.

Examples for the Successive Approximations

In this section we will give some examples to become accustomed to the use of the successive approximations, and also to give counterexamples of cases which one could think they are true.

Table of Examples:

2.10 - An example for the successive approximations

2.11 - Example for the use of the successive approximations, for a problem of order 2.

2.12 - Example for the use of the successive approximations with initial value not given for the time $t = 0$.

2.13 - Example for continuous right hand side, but the successive approximations do not converge.

2.14 - Convergence of the successive approximations and continuousness of the right hand side, but no unique solution.

Example 2.10. Consider the following initial value problem:

$$\begin{aligned} x'(t) &= k\,x(t) \\ x(0) &= x_0 = 1 \end{aligned}$$

Let $y_0(t) := 1$, and compute

$$\begin{aligned} y_1(t) &= x_0 + \int_0^t k\,ds = 1 + k\,t \\ y_2(t) &= x_0 + \int_0^t k(1+ks)\,ds = 1 + k\,t + \frac{(k\,t)^2}{2} \\ y_3(t) &= x_0 + \int_0^t k\left(1 + ks + \frac{(ks)^2}{2}\right) ds = 1 + k\,t + \frac{(k\,t)^2}{2} + \frac{(k\,t)^3}{3!} \end{aligned}$$

It follows by induction that

$$y_n(t) = 1 + k\,t + \frac{(k\,t)^2}{2} + \frac{(k\,t)^3}{3!} + \frac{(k\,t)^4}{4!} + \cdots + \frac{(k\,t)^n}{n!} \to e^{k\,t}.$$

One can see that e^{kt} is a solution of our initial value problem. Computing

$$|f(x,t) - f(y,t)| = |k\,x - k\,y| \le |k|\;|x - y|$$

we see that the right hand side is Lipschitz continuous, and so by the theorem of Picard-Lindelöf there are no other solutions to the given initial value problem. $\boxdot$

Example 2.11. An example of an ordinary differential equation of order 2. Consider:

$$\begin{aligned} x''(t) &= -x(t) = f(t,x) \\ x(0) &= 0, \quad x'(0) = 1 \end{aligned}$$

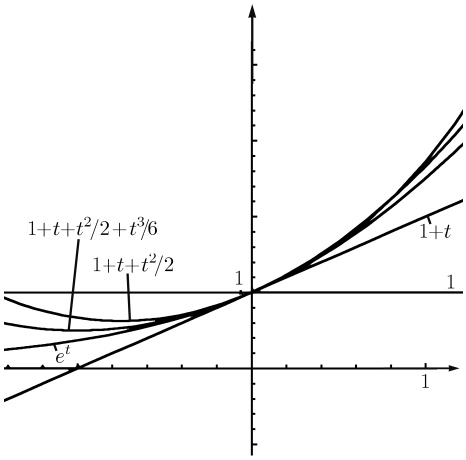

Figure 2.3: Picard-iteration of $x' = k \cdot x$

Because of our initial conditions $p(t) = t$. For our initial function, we make an educated guess and set $y_0(t) := t$. Let us compute the sequence $(y_i(t))_i$ now.

$$
\begin{aligned}
y_1(t) &= p(t) + \frac{1}{(2-1)!}\int_0^t f(s, y_0(s)) \cdot (t-s)^{2-1}\, ds = t + \int_0^t -s \cdot (t-s)\, ds \\
&= t + \int_0^t -st + s^2\, ds = t - \frac{t^3}{2} + \frac{t^3}{3} = t - \frac{t^3}{6} \\
y_2(t) &= t + \int_0^t (-s + \frac{s^3}{6}) \cdot (t-s)\, ds = t + \int_0^t s^2 - \frac{s^4}{6} - st + \frac{s^3 t}{6}\, ds \\
&= t - \frac{t^3}{6} + \frac{t^5}{120} \\
y_3(t) &= t + \int_0^t (s - \frac{s^3}{6} + \frac{s^5}{120}) \cdot (t-s)\, ds = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}
\end{aligned}
$$

It follows by induction that

$$
y_n(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \frac{t^{11}}{11!} + \cdots \pm \frac{t^{2n+1}}{(2n+1)!} \to \sin(t).
$$

Differentiating $\sin(t)$ twice and computing $\sin(0)$ and $\sin'(0)$ we see that $\sin(t)$ is a solution of our initial value problem. Because of the right hand side being Lipschitz, this is the only solution. $\boxdot$

Example 2.12. This is an example which illustrates how to deal with initial conditions not given for time $t = 0$. (And it will not yield to such a easy function like sin or the Exponential).

$$
\begin{aligned}
x'(t) &= x - t \\
x(1) &= x_0 = 2, \quad t_0 = 1
\end{aligned}
$$

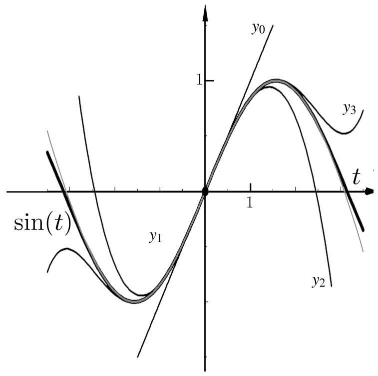

Figure 2.4: Successive Approximations of $x'' = -x$

Define $y(t) := x(t + t_0)$. It follows that $y'(t) = x'(t + t_0) = f(t + t_0, x(t + t_0)) = f(t + t_0, y(t))$ and $y(0) = x(0 + t_0) = x_0$. Hence we can solve this ODE using the successive approximations. We set $y_0 = 0$ and do a little less computation

$$\begin{aligned}
y_0(t) &= 0 \\
y_{i+1}(t) &= x_0 + \int_0^t f(s + t_0, y_i(s))\, ds = 2 + \int_0^t y_i(s) - s - 1\, ds \\
&= 2 - t - \frac{t^2}{2} + \int_0^t y_i(s)\, ds.
\end{aligned}$$

Thus

$$\begin{aligned}
y_1(t) &= 2 - t - \frac{t^2}{2} \\
y_2(t) &= 2 - t - \frac{t^2}{2} + 2t - \frac{t^2}{2} - \frac{t^3}{3} = 2 + t - t^2 - \frac{t^3}{3!} \\
y_3(t) &= 2 + t - \frac{2t^3}{3!} - \frac{t^4}{4!} \\
y_4(t) &= 2 + t - \frac{2t^4}{4!} - \frac{t^5}{5!}.
\end{aligned}$$

It follows by induction that

$$y_n(t) = 2 + t - \frac{2t^n}{n!} - \frac{t^{n+1}}{(n+1)!}.$$

This converges to $y(t) = 2 + t$. Hence $x(t) = y(t - t_0) = 2 + t - 1 = t + 1$, and insertion into the ODE yields that this is indeed a solution. $\boxdot$

Example 2.13. This example shows that the successive approximations need not to converge if the right hand side is not Lipschitz. It is due to Hart-

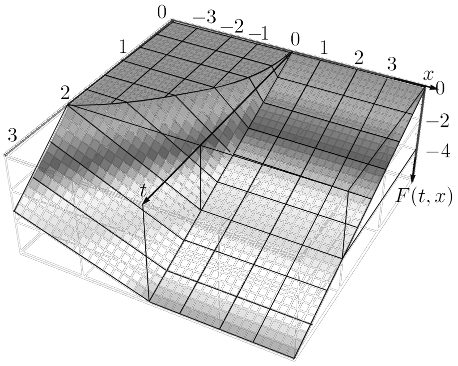

Figure 2.5: Continuous, but not Lipschitz function in $(0,0)$, defined in 2.13

mann [Har73].

$$
\begin{aligned}
x'(t) &= F(t,x) \\
x(0) &= 0 \\
F(t,x) &= \begin{cases} 0 & x \le -t^2 \\ -2x/t - 2t & -t^2 \le x \le 0 \quad \text{(i.e. linear in } x\text{)} \\ -2t & 0 \le x \end{cases}
\end{aligned}
$$

In the point $(0,0)$ the function is not Lipschitz, which can be seen if one sets $x(t) := -t^2$ and $y(t) := 0$:

$$\frac{\left|F(t,-t^2) - F(t,0)\right|}{|-t^2 - 0|} = \frac{|2t|}{t^2} \to +\infty.$$

Defining $x_0(t) := 0$ and computing the successive approximations, one gets

$$
\begin{aligned}
x_1(t) &= \int_0^t F(s,x_0)\,ds = \int_0^t F(s,0)\,ds = \int_0^t -2t\,ds = -t^2 \\
x_2(t) &= \int_0^t F(s,x_1)\,ds = \int_0^t F(s,-s^2)\,ds = \int_0^t 0\,ds = 0.
\end{aligned}
$$

We see that $x_{2n+1}(t) = -t^2$, and $x_{2n}(t) = 0$ for all $n \in \mathbb{N}$. None are solutions because

$$x'_{2n+1}(t) = -2t \neq F(t,-t^2) \quad \text{and} \quad x'_{2n}(t) = 0 \neq F(t,0).$$

Nevertheless this problem has solutions because f is continuous. $\boxdot$

Example 2.14. Also convergence of the successive approximations and continuousness of f do not imply a unique solution. Consider the following differential equation on the positive real axis.

$$
\begin{aligned}
x'(t) &= \sqrt{t} \\
x(0) &= 0
\end{aligned}
$$

It is not hard to see that $x(t) \equiv 0$ and $x(t) = \frac{t^2}{4}$ are solutions, but also the following functions for every positive number $a \in \mathbb{R}^+$.

$$x(t) = \begin{cases} 0 & t \leq a \\ \frac{(t-a)^2}{4} & t \geq a. \end{cases}$$

This shows that there are uncountably many solutions. Nevertheless, the successive approximations converge. Starting with $y_0(t) = 0$ on gets

$$y_1(t) = \int_0^t f(s, y_0(s))\, ds = \int_0^t \sqrt{0} = 0$$

and by induction $y_i(t) = 0$ for all $i = 1, 2, 3, \cdots$. $\boxdot$

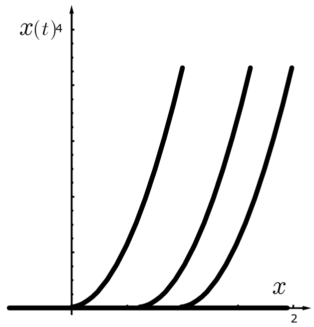

Figure 2.6: Solutions of $x' = \sqrt{t}$

Chapter 3

A Short Overview Over Existing Results

We start again with defining what we mean with *solutions*. We will only consider the positive time axis, and do not require that the ODE is fulfilled in $t = 0$.

Definition 3.1. Consider the differential equation

$$x^{(n)}(t) = f(t, x(t)) \tag{3.1}$$

with initial data

$$x(0) = x_0, \quad x'(0) = x_1, \cdots, \quad x^{(n-1)}(0) = x_{n-1} \quad x_i \in \mathbb{R} \tag{3.2}$$

where $x_i \in \mathbb{R}$, $a < 0$ and $f : (0, a] \times \mathbb{R} \to \mathbb{R}$ is continuous. We may always assume that our initial values are given for the time $t = 0$. $x(t)$ is a *solution of the initial value problem* (3.1)-(3.2) *of order* n *on the interval* $0 \leq t \leq a$ if it is continuous on $[0, a]$, having n finite derivatives $x^{(k)}(t)$ on $(0, a)$ for $1 \leq k \leq n$, and satisfies equations (3.1)-(3.2) on $(0, a)$.

Observe that if $x^{(n)} \in L^\infty[0, a]$, then $x^{(n-1)}$ is continuous on $[0, a]$ so that (3.2) make sense. The reason for this definition is to be consistent with the paper of Athanassov [Ath90]. He sometimes does not require f to be continuous in $(0, 0)$.

Theorem of Nagumo

We now present Nagumo's [Nag26] remarkable theorem.

Theorem 3.2. *The problem* (3.1)-(3.2) *has a unique solution, if* $n = 1$ *and*

$$|f(t, x) - f(t, y)| \leq \frac{1}{t} |x - y| \tag{3.3}$$

for $t \in (0, a]$ *and* $x, y \in \mathbb{R}^n$ *with* $|x|, |y| \leq M$ *for some* $M > 0$.

Remark 3.3. This result improves considerably the classical Lipschitz condition. Assuming that f satisfies the condition for Picard-Lindelöf with Lipschitz constant L, we immediately see that for all $t < \frac{1}{L}$,

$$|f(t, x) - f(t, y)| \leq L |x - y| < 1/t \, |x - y| .$$

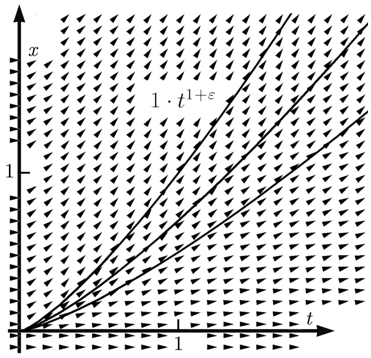

Figure 3.1: The function defined in Example 3.3 together with three inscribed solutions.

This means Nagumo's condition is satisfied. ⊡

Perron proved that the growth of the coefficient $\frac{1}{t}$ as $t \downarrow 0$ is optimal. This means for any $\alpha > 1$ there exist continuous functions f satisfying (3.3) with the right-hand side multiplied by α but for which (3.1)-(3.2) has non trivial solutions. The following example is taken from [Per28].

Example 3.4. Define f as follows

$$\begin{aligned} f &: [0,\infty) \times \mathbb{R} \to \mathbb{R} \\ f(t,x) &:= \begin{cases} (1+\varepsilon) \cdot \frac{x}{t} & 0 < x < t^{1+\varepsilon} \\ (1+\varepsilon) \cdot t^{\varepsilon} & x \geq t^{1+\varepsilon} \\ 0 & x \leq 0 \end{cases} \end{aligned}$$

The function is apparently continuous, and it satisfies (3.3) with an additional multiplication factor $(1+\varepsilon)$ on the right side, as the following computation will show.

$$\left|(1+\varepsilon) \cdot \frac{x}{t} - (1+\varepsilon) \cdot \frac{y}{t}\right| = (1+\varepsilon)\frac{1}{t}\,|x-y| \qquad \checkmark$$

$$\left|(1+\varepsilon) \cdot t^{\varepsilon} - (1+\varepsilon) \cdot \frac{x}{t}\right| = (1+\varepsilon)\left(t^{\varepsilon} - \frac{x}{t}\right) \leq (1+\varepsilon)\frac{1}{t}\,|y-x| \qquad \checkmark$$

Nevertheless, for every $0 \leq C \leq 1$, the following is a solution

$$x(t) = C \cdot x^{1+\varepsilon}.$$

⊡

Wintner [Win56] improved Nagumo's result to differential equations of order n.

Theorem 3.5. *The necessary condition* (3.3) *on f in this case reads*

$$|f(t,x) - f(t,y)| \leq \frac{n!}{t^n}\,|x-y| \tag{3.4}$$

for $t \in (0,a]$, and $x, y \in \mathbb{R}^n$ with $|x|, |y| \leq M$.

Remark 3.6. A reasonable explanation of Nagumo's theorem is the following physically motivated idea[1]: Assume x' is the velocity of some particle moving along the real axis, and x is its position. Then by the standard formula for movement we can interpret the inequality as: "The velocity of the particle cannot be bigger than the distance it covers per time".

Theorem of Athanassov

Among the various generalizations that appeared in the research literature, the most far-reaching were recently obtained by Athanassov in [Ath90] and later on by Constantin in [Con10].

Theorem 3.7. *Uniqueness and convergence of the successive approximations for the problem* (3.1)-(3.2) *of order 1, holds under the following conditions. Assume* $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and satisfies*

$$\frac{f(t, x)}{u'(t)} \to 0 \tag{3.5}$$

as $t \downarrow 0$*, uniformly in* $|x| \leq M$ *for some* $M > 0$*. Furthermore* f *satisfies*

$$|f(t, x) - f(t, y)| \leq \frac{u'(t)}{u(t)} \, |x - y| \,, \tag{3.6}$$

for $t \in (0, a]$ *and* $x, y \in \mathbb{R}^n$ *with* $|x|, |y| \leq M$*, where* u *is an absolutely continuous, nondecreasing function on* $[0, a]$ *with* $u(0) = 0$*. See [Ath90] for a very elaborated discussion about generalizations prior to [Ath90] (for the case* $n = 1$*).*

A Generalization

[Con10] showed that it is possible to generalize condition (3.6) with respect to the modulus of continuity in the spatial variable.

Theorem 3.8. *Condition* (3.6) *can be generalized to*

$$|f(t, x) - f(t, y)| \leq \frac{u'(t)}{u(t)} \omega(|x - y|) \tag{3.7}$$

where ω *is of class* $\mathscr{F}$ *(see Definition 5.2).*

The main goal in this thesis is, to show that in the cases of Wintner and [Con10], the successive approximations also converge to the unique solution. For this we adapt to the present context an approach that was developed in [Ath90] to deal with the classical Nagumo Theorem.

[1] Taken from Wintner [Win56]

Chapter 4

On Wintner's Theorem

The aim of this section is to provide different proofs for our key-role playing *Integral Inequality*. Afterwards we prove that the successive approximations converge to the unique solution. We begin this chapter with a short recapitulation of likely known definitions and theorems.

Definition 4.1. Let $\{f_n(x)\}$ be a sequence of continuous functions from $I \to \mathbb{R}$, where I is an real interval. If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ for all $|x - y| < \delta$ and for all f_n, then $\{f_n(x)\}$ is *equicontinuous on I.*

Example 4.2.

一) A sequence of functions with the same Lipschitz constant is equicontinuous. This is in particular the case, if the set consists of functions with derivatives bounded by the same constant.

二) $f_n(x) := n \cdot x$ with $x \in \mathbb{R}$. This sequence is not equicontinuous.

$\boxdot$

Definition 4.3. Let $\{f_n(x)\}$ be a sequence of continuous functions from $I \to \mathbb{R}$, where I is an real interval. If there exists a constant $K \in \mathbb{R}$ such that $|f_n(x)| < K$ for all $x \in I$ and for all $n = 1, 2, 3, \cdots$, then $\{f_n(x)\}$ is *uniformly bounded on I.*

Example 4.4.

一) Every uniformly convergent sequence of bounded functions is uniformly bounded.

二) The sequence defined by $f_n(x) = \sin nx$, with $x \in \mathbb{R}$ and $n \in \mathbb{N}$ is uniformly bounded by 1. But their derivatives $f_n'(x) = n \cos nx$ are not uniformly bounded. Each f_n' is bounded by $|n|$ but there is no real number M such that $|n| \leq M$ for all $n = 1, 2, 3 \cdots$.

$\boxdot$

Theorem 4.5. (Arzelà-Ascoli)
Let $\{f_n(x)\}$ be a sequence of real valued continuous functions defined on a closed and bounded interval $[a, b] \subseteq \mathbb{R}$. If this sequence is both uniformly bounded and equicontinuous, then there exists a subsequence $\{f_{n_k}(x)\}$ that converges uniformly to a continuous function.

The proof is based on diagonalization and can be found in [Nat61].

A key role in our approach is the following Gronwall-type integral inequality. See [Bel53, Cop65] for the classical Gronwall inequality. We will prove it under various conditions on u, and v. Under these assumptions it is similar to [Ath90].

An Integral Inequality

Lemma 4.6. *Let $u : [0, a] \to \mathbb{R}$ be absolutely continuous, nondecreasing and such that $u(t) > 0$ for $t > 0$. ($u(0)$ not necessary 0). If $v : [0, a] \to \mathbb{R}$ is continuous, nonnegative, such that $v(t) = o(u(t))$ as $t \downarrow 0$, and for $0 < t < a$*

$$v(t) \leq \int_0^t \frac{v(s)}{u(s)} u'(s)\, ds,$$

then v must be identically zero.

Proof. We define $V(t) := \int_0^t \frac{v(s)}{u(s)} u'(s)\, ds$. The hypotheses of the lemma imply the existence of $V(t)$. Since

$$\left(\frac{V}{u}\right)' = \frac{V'u - Vu'}{u^2} = \frac{\frac{v}{u}u'u - Vu'}{u^2} \leq \frac{Vu' - Vu'}{u^2} \leq 0,$$

it follows that $\frac{V}{u}$ is nonincreasing in t. Let $\varepsilon > 0$. Then by hypothesis there exists a $\delta = \delta(\varepsilon)$ such that $\frac{v}{u}(t) \leq \varepsilon$ for $0 < t \leq \delta$. Thus $\lim_{t \downarrow 0} \frac{V(t)}{u(t)} = 0$. Since $V(0^+) = 0$ and $\frac{V}{u}$ is nonnegative and nonincreasing, it follows that $\frac{V(t)}{u(t)} = 0$. This implies $V \equiv 0$, and also $v \equiv 0$. □

Lemma 4.7. (Gronwall's Lemma)
Suppose that $g(t)$ is a continuous, positive function satisfying $g(t) \leq K \int_0^t g(s)\, ds$ for all $t \in [0, a]$, with $K, a > 0$. It then follows for all $t \in [0, a]$ that $g(t) = 0$.

See [Per93] for a proof.

Remark 4.8. If one sets $u(t) = e^{k\, t}$ in Lemma 4.6, one can see that it implies Lemma 4.7.

Lemma 4.9. (Generalization of the *Integral Inequality* 4.6 with respect to the derivative of u.)
Let $u \in C^n[0, a]$, nondecreasing and such that $u^{(i)} \geq 0$ on $0 \leq t \leq a$ for $i = 1, 2, \cdots, n$. If $v : [0, a] \to \mathbb{R}$ is continuous, nonnegative, such that $v(t) = o(u(t))$ as $t \downarrow 0$, and for $t \in (0, a)$

$$v(t) \leq \frac{1}{(n-1)!} \int_0^t \frac{v(s)}{u(s)} u^{(n)}(s) \cdot (t-s)^{n-1}\, ds,$$

then v must be identically zero.

Proof. Assume v is not the zero function. From $\frac{v}{u} \to 0$ as $t \downarrow 0$ it follows that there exists some $\delta \le 1, a$ with $v(t) \le u(t)$ for all $t \le \delta$. Let $\varepsilon := \sup_{t \le \delta} \frac{v}{u}(t)$. Again from $\frac{v}{u} \to 0$ as $t \downarrow 0$ it follows that there exists some $t_1 \le \delta$ with $\varepsilon = \frac{v}{u}(t_1) > \frac{v}{u}(t)$ for all $t < t_1$. We deduce that

$$\begin{aligned} \varepsilon u(t_1) = v(t_1) &\le \frac{1}{(n-1)!} \int_0^{t_1} \frac{v(s)}{u(s)} u^{(n)}(s) \cdot (t-s)^{n-1}\, ds \\ &< \varepsilon \int_0^{t_1} u^{(n)}(s) \cdot (t-s)^{n-1}\, ds = \varepsilon u(t_1) - \varepsilon u(0) \le \varepsilon u(t_1) \end{aligned}$$

which is a contradiction. Thus v is identically zero. □

Remark 4.10. We will present a different proof for the case $n = 2$. Define

$$V(t) := \int_0^t v(s) \frac{u''(s)}{u(s)} (t-s)\, ds.$$

It follows that $V''u = v\frac{u''}{u}u = vu'' \le Vu''$, and consequently $\int V''u \le \int Vu''$. Applying integration by parts two times we see that

$$\int Vu'' \ge \int V''u = \int V'u' - \int V'u' = V'u - Vu' + \int Vu'',$$

and we conclude

$$V'u - Vu' \le 0 \Rightarrow \left(\frac{V}{u}\right)' \le 0.$$

Now argue as in the proof of Lemma 4.6. □

Theorem 4.11. *Assume $f(t,x) : D \to \mathbb{R}$ is continuous and satisfies*

$$|f(t,x) - f(t,y)| \le \frac{u^{(n)}(t)}{u(t)} \cdot |x(t) - y(t)|, \quad \textit{for} \quad (t,x),(t,y) \in D, \quad \textit{and} \tag{4.1}$$

$$f(t,x) = o(u^{(n)}(t)) \quad \textit{as } t \downarrow 0, \quad \textit{uniformly in } |x| < M \le b \tag{4.2}$$

where u is as in Lemma 4.9. Then (3.1)-(3.2) *has at most one solution.*

Proof. The local existence of a solution is guaranteed by Peano's Theorem 2.9. As for uniqueness, let $x(t), y(t)$ be two solutions of (3.1) for $0 < t \le a$. In view of (4.2), given $\varepsilon > 0$, there exists $0 < \delta = \delta(\varepsilon) \le 1, a$, such that $|f(s,x)| \le \varepsilon u^{(n)}(s)$ for $0 < s \le \delta$ and $|x| < b$. For $0 < t \le \delta$ we have

$$\begin{aligned} |x(t) - y(t)| &\le \frac{1}{(n-1)!} \int_0^t \left| f(s,x(s)) - f(s,y(s)) \cdot (t-s)^{n-1} \right| ds \\ &\le 2\varepsilon \frac{1}{(n-1)!} \int_0^t u^{(n)}(s) \cdot (t-s)^{n-1}\, ds \le 2\varepsilon \cdot u(t). \end{aligned}$$

This means that $|x(t) - y(t)| = o(u(t))$ as $t \downarrow 0$. Since

$$\begin{aligned} |x(t) - y(t)| &\le \frac{1}{(n-1)!} \int_0^t |f(s,x(s)) - f(s,y(s))| \cdot (t-s)^{n-1}\, ds \\ &\le \frac{1}{(n-1)!} \int_0^t \frac{u^{(n)}(s)}{u(s)} |x(s) - y(s)| \cdot (t-s)^{n-1} ds, \end{aligned}$$

Lemma 4.9 yields $|x - y| \equiv 0$. □

Convergence of the Successive Approximations

It turns out that the hypotheses (4.1) and (4.2) guarantee not only uniqueness but also the convergence of the successive approximations.

Theorem 4.12. *If the hypotheses of Theorem 4.11 are satisfied, then there exists a sufficiently small interval* $[0, c]$*,* $c > 0$*, on which the successive approximations exist and converge uniformly to the unique solution of* (3.1)-(3.2).

Proof. We can assume all $x_i = 0$ for $i = 1, \cdots, n-1$. Because if we use the variable substitution in Remark 2.8

$$\begin{aligned} y(t) &:= x(t) - p(t) \\ g(t, y) &:= f(t, y + p(t)) \end{aligned}$$

one gets from $|f(t, x_1) - f(t - x_2)| \leq \frac{u^{(n)}}{u} |x_1 - x_2|$ that

$$\begin{aligned} |g(t, y_1) - g(t, y_2)| &= |f(t, y_1 + p(t)) - f(t, y_2 + p(t))| \\ &\leq \frac{u^{(n)}}{u} |y_1 + p(t) - y_2 - p(t)| = \frac{u^{(n)}}{u} |y_1 - y_2| \end{aligned}$$

and from $f = o(u^{(n)})$ with $y_1 + p(t) \leq b$

$$g = o(u^{(n)}).$$

In the latter proof we stick to the use of f and x.

We first prove that the successive approximations $(x_j(t))_j$ are well defined. Let $x_0 : [0, a] \to \mathbb{R}$ be continuous and such that $x_0(0) = 0$, and $|x_0(t)| \leq b$ for $t \in [0, a]$. [1] Define the sequence $(x_j(t))_{j>0}$ recursively by the formula:

$$x_j(t) = p(t) + \frac{1}{(n-1)!} \int_0^t f(s, x_{j-1}(s)) \cdot (t-s)^{n-1} \, ds \quad , j = 1, 2, \cdots . \tag{4.3}$$

From $f = o(u^{(n)})$ it follows that, given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$, $0 < \delta \leq 1, a$ such that $|f(t, x)| \leq \frac{\varepsilon}{2} u^{(n)}(t)$ for $0 < t \leq \delta$, $|x| \leq b$. Denote with $u_0 := \max_{0 \leq t \leq \delta} u(t)$. Then it follows that for $t \in [0, \delta]$

$$\begin{aligned} |x_1(t)| &\leq \frac{1}{(n-1)!} \int_0^t |f(s, x_0(s))| \cdot (t-s)^{n-1} \, ds \\ &\leq \frac{1}{(n-1)!} \frac{\varepsilon}{2} \int_0^t u^{(n)}(s) \cdot (t-s)^{n-1} \, ds \\ &= \frac{\varepsilon}{2} (u(t) - u(0)) \leq \frac{\varepsilon}{2} u_0. \end{aligned}$$

Taking $\varepsilon = \frac{2b}{u_0}$, we obtain

$$|x_1(t)| \leq b \text{ for } 0 \leq t \leq \delta.$$

Suppose now that for $j \geq 1$ the function $x_{j-1}(t)$ is well defined on $0 \leq t \leq \delta$, continuous and satisfies $x_{j-1}^{(i)}(0) = x_i$ for $i = 1, \cdots, n-1$. We then see that

[1] It will be always clear from the context, whether x_i denote the constants in (3.1)-(3.2), or the successive approximations.

$f(t, x_{j-1}(t))$ is well defined, continuous and the integral in (4.3) exists, and its norm does not exceed $\frac{\varepsilon}{2}u_0$. This implies that $x_j(t)$ is also continuous and satisfies

$$x_j^{(i)}(0) = x_i, \;\; |x_j(t)| \leq b \;\text{ for }\; 0 \leq t \leq \delta, \quad i = 1, \cdots, n-1, \quad j = 1, 2, 3 \cdots$$

It follows that the successive approximations are well defined and uniformly bounded on $[0, \delta]$.

Now we prove that the family $\{x_j(t)\}$ is equicontinuous. Let $0 < t_1 < t_2 < \delta$ and $j > 0$ be given. Then[2]:

$$\begin{aligned}
x_j(t_2) - x_j(t_1) &= \frac{1}{(n-1)!}\int_0^{t_2} f(x_{j-1}(s), s) \cdot (t_2 - s)^{(n-1)}\, ds - \int_0^{t_1} \cdots\, ds \\
&= \int_0^{t_2} \underbrace{\int_0^{s_{n-1}} \cdots \int_0^{s_1} f(s_0, x_{j-1}(s_0))\, ds_0 \cdots}_{:=G(s_{n-1})} ds_{n-1} - \int_0^{t_1} \cdots ds_{n-1} \\
&= \int_0^{t_2} G(s_{n-1})\, ds_{n-1} - \int_0^{t_1} G(s_{n-1})\, ds_{n-1} \\
&= \int_{t_1}^{t_2} G(s_{n-1})\, ds_{n-1} \\
|G(s_{n-1})| &\leq \frac{1}{(n-2)!}\int_0^{s_{n-1}} |f(s_0, x_{j-1}(s_0))|\, (t - s_0)^{n-2}\, ds_0 \\
&\leq \frac{1}{(n-2)!}\frac{\varepsilon}{2}\int_0^{s_{n-1}} u^{(n)}(s_0)\, ds_0 = \frac{\varepsilon}{2}\left(u'(s_{n-1}) - u'(0)\right) \\
&\leq \frac{\varepsilon}{2}u'(s_{n-1}) \\
x_j(t_2) - x_j(t_1) &\leq \int_{t_1}^{t_2} u'(s_{n-1})\, ds_{n-1} \\
&\leq \frac{\varepsilon}{2} 2 \max_{0 \leq s \leq \delta} u'(s) \cdot (t_2 - t_1)
\end{aligned}$$

From this and the first calculations it follows that $\{x_j(t)\}$ is both equicontinuous and uniformly bounded on $0 \leq t \leq \delta$. Then by the Arzela-Ascoli theorem, there exists a subsequence $(x_{j_k}(t))_k$ which converges uniformly on $[0, \delta]$ to a continuous function $g(t)$ as $j_k \to \infty$. Since

$$x_{j_k+1}(t) = \frac{1}{(n-1)!}\int_0^t f(s, x_{j_k}(s)) \cdot (t - s)^{n-1}\, ds,$$

by continuity of f, the sequence $(x_{j_k+1}(t))_k$ converges uniformly to a function

$$\tilde{g}(t) = \frac{1}{(n-1)!}\int_0^t f(s, g(s)) \cdot (t - s)^{n-1}\, ds.$$

We shall prove that on $[0, \delta]$ we have

$$\lim_{j \to \infty} x_{j+1}(t) - x_j(t) = 0. \tag{4.4}$$

[2]The second equality can be seen by calculating the n-th derivative on both sides, and comparing the derivatives of lesser order at the point zero.

By (4.3) this yields $g(t) = \tilde{g}(t)$ on $[0, \delta]$. This means that $g(t)$ is a solution of (3.1)-(3.2). Since this solution is unique by Theorem 4.11, every subsequence of $(x_j)_j$ which is convergent will tend to the same solution $g(t)$, and this shows that $(x_j)_j$ converges to $g(t)$ on $[0, \delta]$. Because of the uniform boundedness and the equicontinuity of the sequence this convergence is uniform.

To prove (4.4) we define on $[0, \delta]$ the functions:

$$\begin{aligned} y_j(t) &:= |x_{j+1}(t) - x_j(t)|, \quad j = 1, 2, \cdots \\ m(t) &:= \sup_{0 \le s \le t} \frac{|x_2(s) - x_1(s)|}{u(x)} \\ z_1(t) &:= m(t)u(t) \end{aligned}$$

Then for $t \in [0, \delta]$ we have

$$0 \le m(t) \le \varepsilon$$

so that

$$0 \le z_1(t) \le \varepsilon u(t).$$

Also

$$\begin{aligned} y_j(t) &= |x_{j+1}(t) - x_j(t)| \\ &\le \frac{1}{(n-1)!} \int_0^t |f(s, x_j(s)) - f(s, x_{j-1}(s))| \cdot (t-s)^{n-1}\, ds \\ &\le \frac{2\varepsilon}{2} \int_0^t \left| u^{(n)}(s) \right| \cdot (t-s)^{n-1}\, ds \le \varepsilon u(t), \end{aligned}$$

while

$$\begin{aligned} y_1(t) = |x_2(t) - x_1(t)| &\le \sup_{s \le t} |x_2(s) - x_1(s)| \cdot \frac{u(s)}{u(s)} \\ &\underset{u' > 0}{\le} \sup_{s \le t} \frac{|x_2(s) - x_1(s)|}{u(s)} = m(t)u(t) = z_1(t) \end{aligned}$$

Define now on $[0, \delta]$ the functions z_j with $j \ge 1$ as follows:

$$z_{j+1}(t) := \frac{1}{(n-1)!} \int_0^t \frac{u^{(n)}(s)}{u(s)} z_j(s) \cdot (t-s)^{n-1}\, ds.$$

Since $0 \le z_1(t) \le \varepsilon u(t)$ and $u^{(n)} \in C[0, a]$, the function z_2 is continuous and well defined on $[0, \delta]$ with

$$\begin{aligned} 0 \le z_2(t) &= \frac{1}{(n-1)!} \int_0^t \frac{u^{(n)}(s)}{u(s)} z_1(s) \cdot (t-s)^{n-1}\, ds \\ &\le \int_0^t \varepsilon u^{(n)}(s) \cdot (t-s)^{n-1}\, ds = \varepsilon u(t) \end{aligned}$$

This shows that z_3 is well defined. By induction one has all z_j are well defined and on $t \in [0, \delta]$

$$0 \le z_j(t) \le \varepsilon u(t). \tag{4.5}$$

On the other hand,

$$
\begin{aligned}
y_2(t) &= |x_3(t) - x_2(t)| \\
&\le \frac{1}{(n-1)!}\int_0^t |f(s,x_2(s)) - f(s,x_1(s))| \cdot (t-s)^{n-1}\, ds \\
&\le \frac{1}{(n-1)!}\int_0^t \frac{u^{(n)}(s)}{u(s)} \underbrace{|x_2(s) - x_1(s)|}_{y_1 \le z_1} \cdot (t-s)^{n-1}\, ds \\
&\le \frac{1}{(n-1)!}\int_0^t \frac{u^{(n)}(s)}{u(s)} z_1(s) \cdot (t-s)^{n-1}\, ds = z_2(t).
\end{aligned}
$$

And by induction one gets for $j \ge 1$ and $t \in [0,\delta]$ that $y_j(t) = |x_{j+1}(t) - x_j(t)| \le z_j(t)$.

We now prove by induction that for $j \ge 1$ and $t \in [0,\delta]$ we have

$$
0 \le z_{j+1}(t) \le z_j(t). \tag{4.6}
$$

Indeed,

$$
\begin{aligned}
z_1(t) - z_2(t) &= z_1(t) - \frac{1}{(n-1)!}\int_0^t z_1(s)\frac{u^{(n)}(s)}{u(s)} z_1(s) \cdot (t-s)^{n-1}\, ds \\
&= z_1(t) - \frac{1}{(n-1)!}\int_0^t m(s) u^{(n)}(s) \cdot (t-s)^{n-1}\, ds, \quad \big(m(s) \le m(t)\big) \\
&\ge z_1(t) - \frac{1}{(n-1)!} m(t) \int_0^t u^{(n)}(s) \cdot (t-s)^{n-1}\, ds \\
&\ge z_1(t) - m(t)u(t) + m(t)u(0) \ge z_1(t) - m(t)u(t) \\
&= z_1(t) - z_1(t) = 0.
\end{aligned}
$$

Now assume $z_j(t) \le z_{j-1}(t)$. Then

$$
\begin{aligned}
z_{j+1}(t) &= \frac{1}{(n-1)!}\int_0^t \frac{u^{(n)}(s)}{u(s)} z_j(s) \cdot (t-s)^{n-1}\, ds \\
&\le \frac{1}{(n-1)!}\int_0^t \frac{u^{(n)}(s)}{u(s)} z_{j-1}(s) \cdot (t-s)^{n-1}\, ds = z_j(t)
\end{aligned}
$$

throughout $[0,\delta]$.

Thus the sequence $(z_j(t))_t$ is decreasing and has a limit $z(t) \ge 0$ as $j \to \infty$. By Lebesgue's dominated convergence theorem we get

$$
\begin{aligned}
z(t) &= \lim_{j\to\infty} z_{j+1}(t) = \lim_{j\to\infty} \frac{1}{(n-1)!}\int_0^t \frac{u^{(n)}(s)}{u(s)} z_j(s) \cdot (t-s)^{n-1}\, ds \\
&= \frac{1}{(n-1)!}\int_0^t \lim(\cdots)\, ds = \frac{1}{(n-1)!}\int_0^t \frac{u^{(n)}(s)}{u(s)} \cdot z(s)(t-s)^{n-1}\, ds.
\end{aligned}
$$

Since $z(t) = o(u(t))$ by (4.5), by Lemma 4.9 it follows that $z \equiv 0$. From this and $|y_j| \le z_j$ we deduce that $\lim_{j\to\infty} x_{j+1}(t) - x_j(t) = 0$ and the proof is complete. $\square$

Remark 4.13. We consider again our Example 2.10. We will show that it suffices the conditions (4.1) and (4.2) from the theorem.

$$
\begin{aligned}
x'(t) &= k\,x(t) = f(x,t) \\
x(0) &:= x_0 = 1
\end{aligned}
$$

If we choose $u(t) = e^{\sqrt{t}}$, then $u'(t) = \frac{e^{\sqrt{t}}}{2\sqrt{t}}$ and

$$
\begin{aligned}
|f(s,x) - f(s,y)| = k\,|x-y| \leq \frac{u'(s)}{u(s)}\,|x-y| = \frac{1}{2\sqrt{t}}\,|x-y| \quad \text{for} \quad t \leq 4k^2 \\
\frac{f(x,t)}{u'(t)} = \frac{2k\,x\sqrt{t}}{e^{\sqrt{t}}} \leq \frac{2kb\sqrt{t}}{e^{\sqrt{t}}} \to 0.
\end{aligned}
$$

This means (4.1) and (4.2) are satisfied. It is easy to see that every Lipschitz continuous function satisfies the conditions (4.1), (4.2) with $u(t) = e^{\sqrt{t}}$ for some small interval. ⊡

Remark 4.14. Theorem 4.11 and 4.12 stay also true under the following conditions:

一) f and x are vector valued functions. This is immediate from the proof.

二) The condition on f to be $o(u^{(n)})$ can be replaced by the condition $f = o(u^{(n-1)})$, if $n > 1$. To prove this,it is enough to consider that for $t < 1$ it follows that $(t-s)^{n-1} \leq (t-s)^{n-2}$ for $s \in [0,t]$, and $\frac{1}{(n-1)!} < \frac{1}{(n-2)!}$. In the proof of the existence theorem the functions $m(t), y(t), z(t)$ can be defined on$[0,\delta]$ as follows:

$$
\begin{aligned}
y_j(t) &:= |x_{j+1}(t) - x_j(t)|\,, \quad j = 1,2,\cdots \\
m(t) &:= \sup_{0\leq s\leq t} \frac{|x_2(s) - x_1(s)|}{u(s)} \\
z_1(t) &:= m(t)\cdot u(t) \\
z_{j+1}(t) &:= \frac{1}{(n-1)!}\int_0^t z_j(s)\frac{u^{(n-1)}}{u}(s)\cdot(t-s)^{n-1}\,ds, \quad j = 1,2,\cdots
\end{aligned}
$$

Another, very easy way to see this is, to rewrite the differential equation to a system of lesser order. But this statement is weaker.

三) It is also not hard to see that we can replace condition (4.1) by

$$
f(t,x) = o(u(t)\cdot h(t)) \quad \text{as } t \downarrow 0
$$

where h is a continuous, nonnegative function from $[0,a]$ to $\mathbb{R}$, and $h(t) \leq \frac{u^{(n)}(t)}{u(t)}$ for small t. This approach can be useful for finding functions u which dominate f in the needed way.

Chapter 5

A more general setting for the successive approximations

The aim of this section is to prove the convergence of the successive approximations in the case of the generalization of Nagumos's theorem presented in Section 3. We will consider the system (3.1)-(3.2) to be of first order throughout this chapter.

Class $\mathscr{F}$ Functions

Definition 5.1. Let $f : I \to \mathbb{R}$, I an interval in $\mathbb{R}$, $c \in I$. If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for every $x \in (c, c + \delta) \subset I$ then the function is *right-continuous in c*. If f is right-continuous in every $c \in I$ then it is called *right-continuous.*

Definition 5.2. Let $\omega : [0, a] \to \mathbb{R}^+$, $\omega(0) = 0$, ω strictly monotone increas-

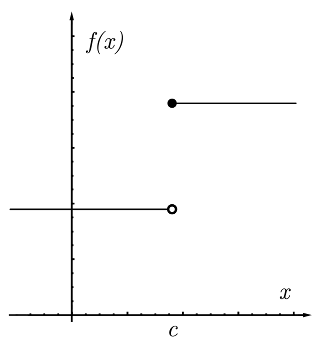

Figure 5.1: A right-continuous function

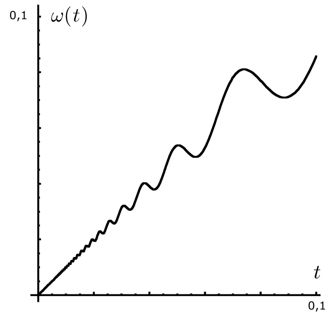

Figure 5.2: Class $\mathscr{F}$ function
Graph of ω defined in 5.4
(4 times super-elevated relative to the $s = s$ axis)

ing, right-continuous and for $t \in (0, a)$

$$\int_0^t \frac{\omega(s)}{s}\, ds \leq t. \tag{5.1}$$

If ω satisfies these assumptions it will be called of *class* $\mathscr{F}$.

Remark 5.3.

一) An easy example of a class $\mathscr{F}$ function is $\omega(s) = s$. Also every (right-)continuous, strictly monotone increasing function which satisfies $w(s) \leq s$ is clearly of class $\mathscr{F}$.

二) If $\omega \in \mathscr{F}$, and using the mean value theorem in equation (5.1) we notice that there must be a sequence $(r_n)_n$, positive, monotone, going to zero, for which $\omega(r_n) \leq r_n$ for all n. Less obvious is, that there can be functions of class $\mathscr{F}$, for which $\omega(r_n) > r_n$ for all n, and appropriate sequences (r_n). The next example given by Constantin in [Con10] will illustrate this.

Example 5.4. Define ω as follows

$$\omega : [0, 1/23\pi) \to \mathbb{R}$$
$$\omega(t) := \begin{cases} t + \frac{1}{2}t^2 \cdot \sin(\frac{1}{t}) - \frac{1}{3}t^2 & \\ & t > 0 \\ 0 & t = 0 \end{cases}$$

We therefore conclude $\omega \in C^1(0, 1/23\pi)$ and

$$\begin{aligned} \omega'(s) &= 1 \cdot \frac{1}{2} 2 \sin\frac{1}{s} + \frac{1}{2}s^2 \cos\frac{1}{s} \cdot \frac{-1}{s^2} - \frac{1}{3}2s \\ &= 1 + s \cdot \sin\frac{1}{s} - \frac{1}{2}\cos\frac{1}{s} - \frac{2}{3}s \geq 1 - s - \frac{1}{2} - \frac{2}{3}s \geq 0 \end{aligned}$$

This means ω is strictly monotone increasing. Defining the following two sequences

$$s_n := \frac{1}{2\pi n}, \quad r_n := \frac{2}{(4n+1)\pi}$$

we conclude that

$$\omega(s_n) = s_n + \frac{1}{2}s_n^2 \cdot \underbrace{\sin\frac{1}{s_n}}_{\sin 2\pi n} - \frac{1}{3}s_n^2 = s_n - \frac{1}{3}s_n^2 \leq s_n$$
$$\omega(r_n) = r_n + \frac{1}{2}r_n^2 \cdot \underbrace{\sin\frac{1}{r_n}}_{\sin 2\pi n + \frac{\pi}{2}} - \frac{1}{3}r_n^2 = r_n + \frac{1}{6}r_n^2 \geq r_n.$$

This means that ω is oscillating around the $s = s$ axis on any interval $(0, \varepsilon)$ with $\varepsilon > 0$. It remains to verify equation (5.1). Let $n \geq 12$. Notice that for $s \in (0, 1/23\pi]$ we have $\sin\frac{1}{s} \geq 0$ only if $\frac{1}{(2n+1)\pi} \leq s \leq \frac{1}{2n\pi}$. Since for any fixed $r \in (0, 1/23\pi]$ there is some integer $N \geq 12$ with

$$\frac{1}{(2N+1)\pi} \leq r < \frac{1}{(2N-1)\pi},$$

we deduce that

$$\begin{aligned}
\int_0^r s \cdot \sin\frac{1}{s}\,ds &\leq \sum_{n\geq N} \int_{\frac{1}{(2n+1)\pi}}^{\frac{1}{2n\pi}} s\sin\frac{1}{s}\,ds \leq \sum_{n\geq N} \int_{\frac{1}{(2n+1)\pi}}^{\frac{1}{2n\pi}} s\,ds \\
&= \sum_{n\geq N} \frac{1}{2}\frac{(2n+1)^2 - 4n^2}{4\pi^2 n^2 (2n+1)^2} < \frac{1}{8\pi^2}\sum_{n\geq N}\frac{1}{n^3} \\
&< \frac{1}{8\pi^2}\int_{N-1}^{\infty}\frac{1}{s^3}\,ds = \frac{1}{16\pi^2(N-1)^2} \\
&< \frac{1}{3\pi^2(2N+1)^2} \leq \frac{1}{3}r^2.
\end{aligned}$$

Consequently

$$\int_0^t \frac{\omega(s)}{s}\,ds = \int_0^t 1 + \frac{1}{2}s\cdot\sin(\frac{1}{s}) - \frac{1}{3}s^2\,ds < \int_0^t 1 + \frac{1}{2}\frac{1}{3}s^2 - \frac{1}{3}s^2\,ds < t$$

which proves that ω is of class $\mathscr{F}$. ⊡

Lemma 5.5. *If $\omega \in \mathscr{F}$ then $\omega(s) < e\,s$ for $s \geq 0$.*

Proof. For $s > 0$ we have

$$s \geq \int_0^s \frac{\omega(r)}{r}\,dr > \int_{s/e}^s \frac{\omega(r)}{r}\,dr > \omega\left(\frac{s}{e}\right)\int_{s/e}^s \frac{1}{r}\,dr = \omega\left(\frac{s}{e}\right)$$

which yields the statement. □

Remark 5.6. The previous result might seem to indicate that we should simply set $\omega(s) = e\cdot s$ in (5.3) and dispense altogether with functions of class $\mathscr{F}$. However, reminding that the coefficient in Nagumo's theorem is optimal, replacing $\omega(s)$ by $s \mapsto e\cdot s$ is not an option.

Example 5.7. This example shows, that there exist functions ω which nearly equal $\omega(s_0) = e \cdot s_0$ for some $s_0 > 0$. Let $0 < \varepsilon < 1$.

$$\omega_{t_0,\varepsilon} : \mathbb{R}^+ \to \mathbb{R}^+$$
$$\omega_{t_0,\varepsilon}(t) := \begin{cases} \varepsilon \cdot t & t < t_0 \\ t_0 \cdot e - t_0 \cdot \varepsilon & t \geq t_0 \end{cases}$$

Let $t > t_0$. And define

$$\begin{aligned}\Omega(t) &= \int_0^t \frac{\omega_{t_0,\varepsilon}(s)}{s}\, ds = \int_0^{t_0} \frac{\varepsilon \cdot s}{s}\, ds + \int_{t_0}^t \frac{t_0 \cdot e - t_0\varepsilon}{s}\, ds \\ &= t_0\varepsilon + (t_0 e - t_0\varepsilon)(\ln(t) - \ln(t_0))\end{aligned}$$

It is easy to see that $\Omega(t_0 e) = t_0 e$. Furthermore $\Omega(t_0 e)' = 1$ and $\Omega''|_{t\in(t_0 e,\infty)} < 0$. This shows that it satisfies (5.2). Thus every strictly increasing, right continuous function which is less or equal $\omega_{t_0,\varepsilon}$ is of class $\mathscr{F}$. $\boxdot$

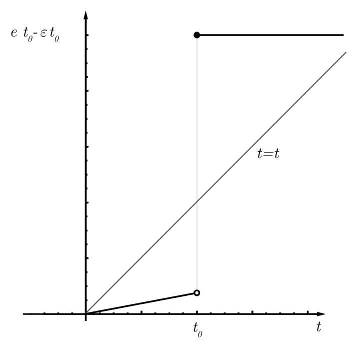

Figure 5.3: The function $\omega_{t_0,\varepsilon}$ defined in Example 5.7

Proposition 5.8. *Properties of class $\mathscr{F}$ functions.*

一) *Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be integrable, right-continuous, strictly monotone increasing and satisfy $\int_0^t \omega(s)\, ds \leq t^2/2$. Then $\omega \in \mathscr{F}$. The converse is wrong in general.*

二) *Let $\omega \in \mathscr{F}$. Then $\int_0^t \omega(s)\, ds \leq t^2$.*

Proof.

一) Defining $\Omega(t) := \int_0^t \omega(s)\, ds$, and doing some preliminary calculations,

$$\Omega(t) \cdot \frac{1}{t} = \int_0^t \omega(s)\, ds \cdot \frac{1}{t} \leq \frac{1}{t}\frac{t^2}{2} = \frac{t}{2} \tag{5.2}$$

we are nearly done. Let $\varepsilon > 0$. Using integration by parts

$$\begin{aligned}\int_\varepsilon^t \omega(s)\frac{1}{s}\, ds &= \Omega(s)\frac{1}{s}\Big|_\varepsilon^t + \int_\varepsilon^t \Omega(s)\frac{1}{s^2}\, ds \\ &\leq \frac{t}{2} - \underbrace{\Omega(\varepsilon)}_{\geq 0}\frac{1}{\varepsilon} + \int_\varepsilon^t \frac{s}{2}\frac{1}{s}\, ds \leq \frac{t}{2} + \frac{1}{2}t - \frac{1}{2}\varepsilon\end{aligned}$$

and letting $\varepsilon \to 0$ going to zero, we get $\displaystyle\int_0^t \frac{\omega(s)}{s}\, ds \le t$. Integration per parts is allowed because we integrated the not continuous function ω.
A counterexample can be constructed using the function

$$\omega_{1,\infty} : \mathbb{R}^+ \to \mathbb{R}^+$$
$$\omega_1(t) := \begin{cases} 0 & t < 1 \\ e & t \ge 1 \end{cases}$$

Computing

$$\int_0^3 \omega_1(s)\, ds = \int_1^3 e\, ds = 2e > 5 > \frac{3^2}{2}.$$

one sees that the inequality is strict, and thus one can construct a counterexample.

二)

$$\int_0^t \omega(s)\, ds \frac{1}{t} \le \int_0^t \frac{\omega(s)}{s}\, ds \le t$$

□

Example 5.9. The previous result allows us to easily find more examples of class $\mathscr{F}$ functions.

一) Define

$$\omega : \mathbb{R}^+ \to \mathbb{R}^+$$
$$\omega(x) = \frac{1}{2^i} \quad \text{for} \quad x \in \left[\frac{1/2^i + 1/2^{i+1}}{2}, \frac{1/2^{i-1} + 1/2^i}{2}\right).$$

Looking at the image, it is easy to see, that

$$\int_0^{t_0} \omega(s)\, ds \le \frac{t_0^2}{2}$$

for all $t_0 > 0$. Because the area marked "A" equals the area marked "B". In view of Proposition 5.8 一 this would yield $\omega \in \mathscr{F}$. But the function is not strictly increasing. In the the next example we will show how to make these functions strictly increasing in a very easy way, and furthermore generalize this idea to arbitrary sentences going to zero.

二) Let $(x_n)_{n\ge1}$ be a positive, strictly decreasing sequence with limit zero. Then the following function suffices (5.1).

$$\omega : (0, x_1) \to \mathbb{R}^+$$
$$\omega(x) = x_n \quad \text{for} \quad x \in \left[\frac{x_n + x_{n+1}}{2}, \frac{x_{n-1} + x_n}{2}\right) \quad \text{and } n \ge 2$$

We show this again using Proposition 5.8 一. Let $t > 0$ and choose $N \in \mathbb{N}$ such that $x_{N+1} \le t < x_N$. Because $t \le x_1$ and (x_n) is strictly decreasing

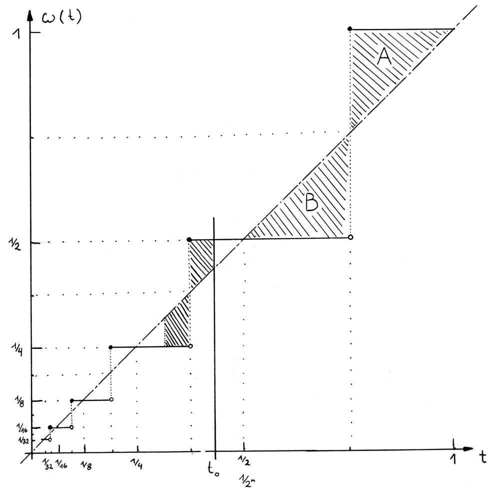

Figure 5.4: The function ω defined in Example 5.9

this number exists and there is only one such number. We compute the following

$$\int_0^t \omega(s)\,ds = \underbrace{\int_0^{x_N} \omega(s)\,ds}_{\text{甲}} + \underbrace{\int_{x_N}^{\frac{x_N+x_{N+1}}{2}} \omega(s)\,ds + \int_{\frac{x_N+x_{N+1}}{2}}^{t} \omega(s)\,ds}_{\text{乙}}$$

We first estimate 甲 , and afterwards 乙 .

$$\begin{aligned}
\text{甲} &= \sum_{i=N}^{\infty} \int_{x_{i+1}}^{x_i} \omega(s)\,ds = \sum_{i=N}^{\infty} \frac{x_i - x_{i+1}}{2} \cdot x_i + \frac{x_i - x_{i+1}}{2} \cdot x_{i+1} \\
&= \sum_{i=N}^{\infty} \frac{x_i^2}{2} - \frac{x_{i+1}^2}{2} = \frac{x_N^2}{2} \\
\text{乙} &= x_N \cdot \frac{x_N + x_{N-1}}{2} + x_{N-1} \cdot \left(t - \frac{x_N + x_{N-1}}{2}\right) \\
&\le x_N \cdot \frac{x_N + x_{N-1}}{2} + \frac{t^2}{2} - \frac{(\frac{x_N+x_{N-1}}{2})^2}{2} + \frac{(\frac{x_N-x_{N-1}}{2})^2}{2} \\
&= \frac{(\frac{x_N+x_{N-1}}{2})^2}{2} - \frac{x_N^2}{2} + \frac{t^2}{2} - \frac{(\frac{x_N+x_{N-1}}{2})^2}{2} = -\frac{x_N^2}{2} + \frac{t^2}{2}
\end{aligned}$$

Summing up we get

$$\int_0^t \omega(s)\,ds = \text{甲} + \text{乙} \le \frac{t^2}{2}$$

Now we will make this function strictly increasing. The idea is to make the jumps Δ smaller for a constant factor K, and then interpolate linear. See the image for an illustration. We will skip the calculations and write directly the formula for the function down. Let $0 < K < 1$. Then the

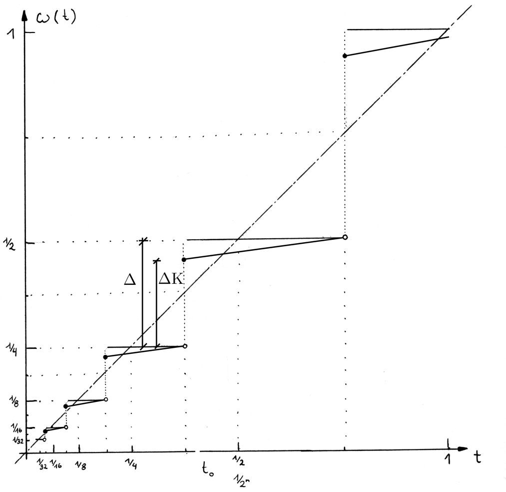

Figure 5.5: Adjustment of the function defined in 5.9 to make it strictly increasing.

following function is of clss $\mathscr{F}$:

$$\tilde{\omega} : (0, x_2) \to \mathbb{R}^+$$

$$\tilde{\omega}(x) = \frac{2(x_n - x_{n+1})(1-K)}{x_{n-1} - x_{n+1}} \cdot x + x_n - \frac{(x_n - x_{n+1})(1-K)(x_n + x_{n-1})}{x_{n-1} - x_{n+1}}$$

$$\text{for} \quad x \in \left[\frac{x_n + x_{n+1}}{2}, \frac{x_{n-1} + x_n}{2}\right)$$

Because $\tilde{\omega} \leq \omega$, this function still satisfies (5.1). Furthermore it is strictly increasing, and for $K = 1$ this functions equals ω. In the latter examples this final step will be always omitted. ⊡

三)

$$\omega : (0, 1) \to \mathbb{R}^+$$

$$\omega(x) = 1/n \quad \text{for} \quad x \in \left[\frac{1/n + 1/(n+1)}{2}, \frac{1/(n-1) + 1/n}{2}\right)$$

is another example with a geometric division of the steps.

Remark 5.10. These examples are not optimal in the sense that their values could be bigger in view of Proposition 5.8 二 . But using this Ansatz it easy to generate better examples. Our aim is to show that for adequate x

$$\int_a^x \frac{a}{s}\, ds + \int_x^b \frac{b}{s}\, ds = b - a$$

equality holds.

Lemma 5.11. *Let $a, b \in \mathbb{R}$ satisfying $0 \le a \le b < \infty$. Define*

$$x := \begin{cases} \frac{a^{\frac{a}{a-b}}}{b^{\frac{b}{a-b}}} \cdot \frac{1}{e} & 0 < a < b < \infty \\ b/e & a = 0 \\ a/e & b = 0 \\ a & a = b \end{cases}$$

Then the following holds:

$$\int_a^x \frac{a}{s}\, ds + \int_x^b \frac{b}{s}\, ds = b - a$$

Proof. If $a = b$ this is trivial. If $a = 0$, we get $x = b/e$ and hence

$$\int_0^{b/e} \frac{0}{s}\, ds + \int_{b/e}^b \frac{b}{s}\, ds = b \overset{!}{=} b - a$$

The same for $b = 0$.

Let us now consider the case $0 < a < b < \infty$. We prove the claim by computing the integrals.

$$\begin{aligned}
\int_a^x \frac{a}{s}\, ds + \int_x^b \frac{b}{s}\, ds &= a \left[\ln \frac{a^{\frac{a}{a-b}}}{b^{\frac{b}{a-b}}} \cdot \frac{1}{e} - \ln a \right] + b \left[\ln b - \ln \frac{a^{\frac{a}{a-b}}}{b^{\frac{b}{a-b}}} \cdot \frac{1}{e} \right] \\
&= a \left[\ln \frac{a^{\frac{a}{a-b}-1}}{b^{\frac{b}{a-b}}} - \ln e \right] + b \left[\ln \frac{b^{\frac{b}{a-b}+1}}{a^{\frac{a}{a-b}}} + \ln e \right] \\
&= (b-a) + a \ \ln \frac{a^{\frac{b}{a-b}}}{b^{\frac{b}{a-b}}} + b \ \ln \frac{b^{\frac{a}{a-b}}}{a^{\frac{a}{a-b}}} \\
&= (b-a) + a \ \ln \left(\frac{a}{b}\right)^{\frac{b}{a-b}} + b \ \ln \left(\frac{b}{a}\right)^{\frac{a}{a-b}} \\
&= (b-a) + \ln \left(\frac{a}{b}\right)^{\frac{ba}{a-b}} + \ln \left(\frac{a}{b}\right)^{\frac{ab}{a-b}} \\
&= (b-a) + \ln 1 = (b-a)
\end{aligned}$$

□

Corollary 5.12. *Let $(x_n)_{n \ge 1}$ be a positive, strictly decreasing sequence with limit zero. Furthermore denote with*

$$W(a,b) := \frac{a^{\frac{a}{a-b}}}{b^{\frac{b}{a-b}}} \cdot \frac{1}{e}$$

for $0 < a < b < \infty$. Then the following function is of class $\mathscr{F}$.

$$\begin{aligned}
\omega &: (0, x_1) \to \mathbb{R}^+ \\
\omega(t) &:= \begin{cases} x_{n+1} & \textit{for } x_{n+1} \le x < W(x_{n+1}, x_n) \\ x_n & \textit{for } W(x_{n+1}, x_n) \le x \le x_n \end{cases}
\end{aligned}$$

Proof. It lacks to proof

$$\int_{x_{n+1}}^{W(x_{n+1},x_n)} \frac{x_{n+1}}{s}\,ds + \int_{W(x_{n+1},x_n)}^{t_0} \frac{x_n}{s}\,ds \leq t_0 - a.$$

We will do it using curve sketching. Let $t \in (W(x_{n+1}, x_n), x_n)$. It follows that

$$f(t) = \int_0^t \frac{\omega(s)}{s}\,ds = \int_0^{x_n} \frac{\omega(s)}{s}\,ds + \int_{x_n}^{W(x_{n+1},x_n)} \frac{x_{n+1}}{s}\,ds + \int_{W(x_{n+1},x_n)}^{t} \frac{x_n}{s}\,ds$$

Consequently

$$f'(t) = \frac{x_n}{t} \quad \text{and}$$
$$f''(t) = \frac{-x_n}{t^2} < 0.$$

In the point $t = x_n$ one gets $f'(x_n) = 1$ and the function has negative curvature. This implies

$$f(t)|_{t \in W(x_{n+1},x_n)} \leq t$$

□

Remark 5.13. *Properties of* W.

一) $W(a,b) = \frac{a^{\frac{a}{a-b}}}{b^{\frac{b}{a-b}}} \cdot \frac{1}{e} = \frac{b^{\frac{b}{b-a}}}{a^{\frac{a}{b-a}}} \cdot \frac{1}{e} = W(b,a)$

二) $a, b \leq W(a,b) \leq \frac{b-a}{2}$.

These equalities follow because the the function defined in 5.12 satisfies (5.2).

Example 5.14. Next we will derive explicit examples. For $x_n := 2^{-n}$ it follows that

$$W(2^{-(n+1)}, 2^{-n}) = \frac{1}{2^{-n+1} \cdot e}.$$

For $x_n := \frac{1}{n}$ one gets

$$W(\frac{1}{n+1}, \frac{1}{n}) = \frac{1}{e} \cdot \frac{(n+1)^n}{n^{n+1}}.$$

See the images for a graph of these two functions. ⊡

Lemma 5.15. (Generalization of the *Integral Inequality* 4.6 with respect to the modulus of v.)
Let $u : [0,a] \to \mathbb{R}$ be absolutely continuous, nondecreasing and such that $u(t) > 0$ for $t > 0$. If $v : [0,a] \to \mathbb{R}$ is continuous, nonnegative, such that $v(t) = o(u(t))$ as $t \downarrow 0$, and for $t \in (0,a)$

$$v(t) \leq \int_0^t \frac{\omega(v(s))}{u(s)} u'(s)\,ds$$

for some $\omega \in \mathscr{F}$, then v must be identically zero.

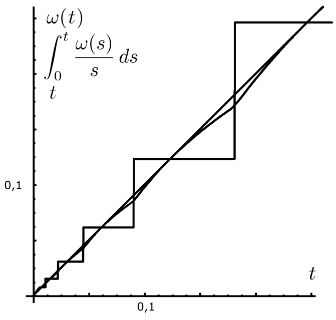

Figure 5.6: Plot of the first function defined in 5.14, together with the $x = x$ axis and its integral.

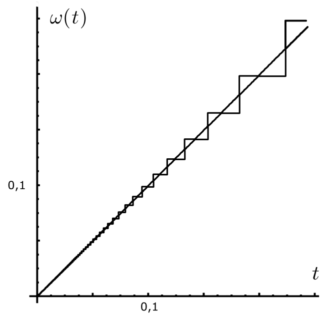

Figure 5.7: Plot of the second function defined in 5.14, together with the $x = x$ axis.

Proof. From Lemma 5.5 it follows that the integral is defined. Assume $v(t)$ is not the zero function. From $\frac{v}{u} \to 0$ as $t \downarrow 0$ it follows that there exists $0 < \delta \leq 1, a$ such that $v(t) \leq u(t)$ for $0 < t \leq \delta$. Let $\varepsilon := \sup_{t \leq \delta} \frac{v}{u}(t)$. Again from $\frac{v}{u} \to 0$ as $t \downarrow 0$ it follows that there exists some $t_1 \leq \delta$ with $\varepsilon = \frac{v}{u}(t_1) > \frac{v}{u}(t)$ for $t < t_1$. We deduce that

$$\begin{aligned}
\varepsilon u(t_1) = v(t_1) &\leq \int_0^{t_1} \omega(v(s)) \frac{u'(s)}{u}(s)\,ds = \int_0^{t_1} \omega\left(\frac{v(s)}{u(s)} u(s)\right) \frac{u'(s)}{u(s)}\,ds \\
&< \int_0^{t_1} \omega(\varepsilon u) \frac{u'(s)}{u(s)}\,ds \underset{r\,:=\,\varepsilon u}{=} \int_{\varepsilon u(0)}^{\varepsilon u(t_1)} \omega(r) \frac{u'(s)}{r/\varepsilon}\,\frac{dr}{\varepsilon u'(s)} \\
&= \int_{\varepsilon u(0)}^{\varepsilon u(t_1)} \frac{\omega(r)}{r}\,dr \leq \int_0^{\varepsilon u(t_1)} \frac{\omega(r)}{r}\,dr \leq \varepsilon u(t_1)
\end{aligned}$$

which is a contradiction. Thus v is identically zero. □

Theorem 5.16. *Assume f is continuous and satisfies*

$$|f(t,x) - f(t,y)| \leq \frac{u'(t)}{u(t)} \cdot \omega(|x-y|), \quad (t,x),(t,y) \in D, \quad \textit{and} \tag{5.3}$$

$$f(t,x) = o(u'(t)) \quad \textit{as } t \downarrow 0 \quad \textit{uniformly in } |x| < M \leq b \tag{5.4}$$

where u is as in Lemma 5.15 and $\omega \in \mathscr{F}$. Then (3.1)-(3.2) *has at most one solution.*

Proof. The local existence of a solution is guaranteed by Peano's Theorem 2.9. Let $x(t), y(t)$ be two solutions of (3.1)-(3.2) on $0 < t \leq a$. Because $f = o(u')$ there exists a $0 < \delta = \delta(\varepsilon) \leq 1, a$, such that $|f(t,x)| \leq \varepsilon u'(t)$ for $0 < t \leq \delta$ and $|x| < b$. Let t be less then δ. Looking at

$$|x(t) - y(t)| \leq \int_0^t |f(s,x(s)) - f(s,y(s))|\,ds \leq 2\varepsilon \int_0^t u'(s)\,ds \leq 2\varepsilon \cdot u(t)$$

we obtain that $|x(t) - y(t)| = o(u(t))$ as $t \downarrow 0$. Since

$$\begin{aligned}|x(t) - y(t)| &\le \int_0^t |f(s, x(s)) - f(s, y(s))| \, ds \\ &\le \int_0^t \frac{u'(s)}{u(s)} \cdot \omega(|x(s) - y(s)|) \, ds\end{aligned}$$

Lemma 5.15 yields that $|x(t) - y(t)| \equiv 0$. □

Convergence of the Successive Approximations

Theorem 5.17. *If the hypotheses of Theorem 5.16 are satisfied, then there exists a sufficiently small interval* $[0, c]$, $c > 0$, *on which the successive approximations exist and converge uniformly to the unique solution of* (3.1)-(3.2).

Proof. Without loss of generality we may assume $x_0 = 0$.

We first prove that the successive approximations are well defined. Let $x_0 : [0, a] \to \mathbb{R}$ be continuous and such that $x_0(0) = 0$, and$|x_0(t)| \le b$ for $t \in [0, a]$. Define the sequence $(x_j(t))_{j>0}$ recursively by the formulas

$$x_j(t) := \int_0^t f(s, x_{j-1}(s)) \, ds \quad, j = 1, 2, 3, \cdots \tag{5.5}$$

From $f = o(u')$ it follows that there exists a $0 < \delta = \delta(\varepsilon) \le 1, a$, $0 < \delta \le a$ such that $|f(t, x)| \le \frac{\varepsilon}{2} u'(t)$ for $0 < t \le \delta$, $|x| \le b$. Define $u_0 := \max_{0 \le t \le \delta} u(t)$. Then it follows that for $t \in [0, \delta]$

$$|x_1(t)| \le \int_0^t |f(s, x_0(s))| \, ds \le \frac{\varepsilon}{2} \int_0^t u'(s) \, ds = \frac{\varepsilon}{2}(u(t) - u(0)) \le \frac{\varepsilon}{2} u(t) \le \frac{\varepsilon}{2} u_0$$

Taking $\varepsilon := \frac{2b}{u_0}$, we obtain

$$|x_1(t)| \le b \quad \text{for} \quad 0 \le t \le \delta.$$

Suppose now that for $j \ge 1$ the function $x_{j-1}(t)$ is well defined on $0 \le t \le \delta$, continuous and satisfies $x(0) = 0$. We then see that $f(t, x_{j-1}(t))$ is well defined, continuous and the integral in (5.5) exists, and its norm does not exceed $\frac{\varepsilon}{2} u_0$. This implies that $x_j(t)$ is also continuous and satisfies on $[0, \delta]$

$$x_j(0) = 0, \;\; |x_j(t)| \le b$$

It follows that the successive approximations are well defined and uniformly bounded on $[0, \delta]$.

Now we prove that the family $\{x_j(t)\}$ is equicontinuous. Let $0 < t_1 < t_2 < \delta$ and $j > 0$ are given. Then

$$\begin{aligned}x_j(t_2) - x_j(t_1) &= \int_0^{t_2} f(s, x_{j-1}(s)) \, ds - \int_0^{t_1} f(s, x_{j-1}(s)) \, ds \\ &= \int_{t_1}^{t_2} f(s, x_{j-1}(s)) \, ds \le \int_{t_1}^{t_2} \varepsilon u'(s) \, ds = \varepsilon(u(t_2) - u(t_1)) \\ &\le \varepsilon(t_2 - t_1) \max_{s \le \delta} u'(s)\end{aligned}$$

From this and the first calculations it follows that $\{x_j(t)\}$ is both equicontinuous and uniformly bounded on $[0,\delta]$. Then by the Arzela-Ascoli theorem, there exists a subsequence $(x_{j_k}(t))_k$ which converges uniformly on $[0,\delta]$ to a continuous function $g(t)$ as $j_k \to \infty$. Since

$$x_{j_k+1}(t) = \int_0^t f(s, x_{j_k}(s))\, ds,$$

by continuity of f, the sequence $(x_{j_k+1})_k$ converges uniformly to a function

$$\tilde{g}(t) = \int_0^t f(s, g(s))\, ds.$$

We shall prove that on $[0,\delta]$ we have

$$\lim_{j\to\infty} x_{j+1}(t) - x_j(t) = 0. \tag{5.6}$$

By (5.5) this yields $g(t) = \tilde{g}(t)$ on $[0,\delta]$. This means that $g(t)$ is a solution of (3.1)-(3.2). Since this solution is unique by Theorem 5.16, every subsequence of $(x_j)_j$ which is convergent will tend to the same solution $g(t)$, and this shows that $(x_j)_j$ converges to $g(t)$ on $[0,\delta]$. Because of the uniform boundedness and the equicontinuity of the sequence this convergence is uniform.

To prove (5.6) we define on $[0,\delta]$ the functions:

$$\begin{aligned} y_j(t) &:= |x_{j+1}(t) - x_j(t)| \quad j = 1, 2, \cdots \\ m(t) &:= \sup_{0\le s\le t} \frac{|x_2(s) - x_1(s)|}{u(x)} \\ z_1(t) &:= m(t)u(t) \end{aligned}$$

Then for $t \in [0,\delta]$ we have

$$0 \le m(t) \le \varepsilon$$

so that

$$0 \le z_1(t) \le \varepsilon u(t).$$

Also

$$\begin{aligned} y_j(t) &= |x_{j+1}(t) - x_j(t)| \\ &\le \int_0^t |f(s, x_j(s)) - f(s, x_{j-1}(s))|\, ds \\ &\le \frac{2\varepsilon}{2} \int_0^t |u'(s)|\, ds \le \varepsilon u(t) \end{aligned}$$

while

$$\begin{aligned} y_1(t) = |x_2(t) - x_1(t)| &\le \sup_{s\le t} |x_2(s) - x_1(s)| \frac{u(s)}{u(s)} \\ &\underset{u'>0}{\le} \sup_{s\le t} \frac{|x_2(s) - x_1(s)|}{u(s)} = m(t)u(t) = z_1(t). \end{aligned}$$

Define now on $[0, \delta]$ the functions z_j with $j \geq 1$ as follows:

$$z_{j+1}(t) := \int_0^t \frac{u'(s)}{u(s)} \omega(z_j(s))\, ds.$$

Since $0 \leq z_1(t) \leq \varepsilon u(t)$ and $u' \in L^1[0, a]$, the function z_2 is continuous and well defined on $[0, \delta]$ with

$$\begin{aligned} 0 \leq z_2(t) &= \int_0^t \frac{u'(s)}{u(s)} \omega(z_1(s))\, ds \leq \int_0^t \frac{u'(s)}{u(s)} \omega(\varepsilon u(s))\, ds \\ &= \int_{\varepsilon u(0)}^{\varepsilon u(t)} \omega(r) \frac{u'(s)}{r/\varepsilon} \frac{dr}{\varepsilon u'(s)} \leq \varepsilon u(t). \end{aligned}$$

This shows that z_3 is well defined. By induction one has all z_j are well defined and on $t \in [0, \delta]$

$$0 \leq z_j \leq \varepsilon u. \tag{5.7}$$

On the other hand,

$$\begin{aligned} y_2(t) = |x_3(t) - x_2(t)| &\leq \int_0^t |f(s, x_2(s)) - f(s, x_1(s))|\, ds \\ &\leq \int_0^t \frac{u'(s)}{u(s)} \omega(|x_2(s) - x_1(s)|)\, ds \leq \int_0^t \frac{u'(s)}{u(s)} \omega(z_1(s))\, ds = z_2(t), \end{aligned}$$

and by induction on gets that for $j \geq 1$ and $t \in [0, \delta]$ that

$$y_j(t) = |x_{j+1}(t) - x_j(t)| \leq z_j(t).$$

We now prove by induction that for $j \geq 1$ and $t \in [0, \delta]$ we have

$$0 \leq z_{j+1}(t) \leq z_j(t). \tag{5.8}$$

Indeed

$$\begin{aligned} z_1(t) - z_2(t) &= z_1(t) - \int_0^t \frac{u'(s)}{u(s)} \omega(z_1(s))\, ds \\ &= z_1(t) - \int_0^t \frac{u'(s)}{u(s)} \omega(m(s)u(s))\, ds\ , \qquad m(s) \leq m(t) \\ &\geq z_1(t) - \int_0^t \frac{u'(s)}{u(s)} \omega(m(t)u(s))\, ds \\ &= z_1(t) - \int_{m(t)u(0)}^{m(t)u(t)} \frac{\omega(r)}{r}\, dr \\ &\geq z_1(t) - \int_0^{m(t)u(t)} \frac{\omega(r)}{r}\, dr \\ &\geq z_1(t) - z_1(t) = 0 \end{aligned}$$

Now assume $z_j(t) \leq z_{j-1}(t)$ and let $t \in [0, \delta]$. Then

$$z_{j+1}(t) = \int_0^t \frac{u'(s)}{u(s)} \omega(z_j(s))\, ds \leq \int_0^t \frac{u'(s)}{u(s)} \omega(z_{j-1}(s))\, ds = z_j(t)$$

throughout $[0,\delta]$. Thus the sequence $(z_j(t))_t$ is decreasing and has a limit $z(t) \geq 0$ as $j \to \infty$. By Lebesgue's dominated convergence theorem , we get

$$\begin{aligned} z(t) &= \lim_{j\to\infty} z_{j+1}(t) = \lim_{j\to\infty} \int_0^t \frac{u'(s)}{u(s)} \omega(z_j(s))\, ds \\ &= \int_0^t \lim_{j\to\infty} \frac{u'(s)}{u(s)} \omega(z_j(s))\, ds \ , \quad \Big(\omega \text{ is right-continuous and (5.8) holds }\Big) \\ &= \int_0^t \frac{u'(s)}{u(s)} \omega(\lim_{j\to\infty} z_j(s))\, ds = \int_0^t \frac{u'(s)}{u(s)} \omega(z(s))\, ds. \end{aligned}$$

Since $z(t) = o(u(t))$ by (5.7), by Lemma 5.15 it follows that $z \equiv 0$. From this and $|y_j| \leq z_j$ we deduce that $\lim_{j\to\infty} x_{j+1}(t) - x_j(t) = 0$ and the proof is complete. $\square$

Remark 5.18. Theorem 5.16 and 5.17 stay also true with the following conditions.

一) f and x are vector valued functions. This is immediate from the proof.

二) It is also not hard to see, that we can replace condition (5.3) by

$$f(t,x) = o(u(t)h(t)) \quad \text{as } t \downarrow 0$$

where h is a continuous, nonnegative function from $[0,a]$ to $\mathbb{R}$, and $h(t) \leq \frac{u'(t)}{u(t)}$ for small t. This approach can be useful for finding functions u which dominate f in the needed way.

QUOD ERAT DEMONSTRANDUM

Chapter 6

Appendix

Abstract (Translation)

Wir beweisen Eindeutigkeit und Konvergenz der Picarditerationen unter allgemeineren Voraussetzungen als Lipschitz für $x^{(n)} = f(t, x)$. Danach präsentieren wir einen einfacheren Beweis für eine kürzlich publizierte Verallgemeinerung von Nagumo's Theorem. Wir zeigen, dass die Lösung eindeutig ist und die Picard-Iterationen[1] gegen die eindeutige Lösung konvergieren. Somit verallgemeinern wir das Theorem von Picard und Lindelöf zuerst in Richtung initiiert von Nagumo und Athanassov, und danach in die Richtung initiiert von Constantin.

Einleitung (Translation)

Diese Diplomarbeit behandelt Existenz und Eindeutigkeit von Lösungen von ODE's.[2] Es wird ein einfacherer Beweis für [Con10] dargebracht, sowie die Arbeit von [Ath90] auf höhere Ordnung verallgemeinert.

Differentialgleichungen

Beim Studium von Gleichungen sind im Allgemeinen zwei Dinge von Interesse. Hat die Gleichung eine Lösung, und wenn ja, ist sie eindeutig. Beide Fragen werden davon beeinflusst in welcher Menge man nach Lösungen sucht.

In dieser Diplomarbeit werden Differentialgleichungen der Form

$$x^{(n)}(t) = f(t, x(t)) \tag{6.1}$$

$$x(0) = x_0, \quad x'(0) = x_1, \cdots, \ x^{(n-1)}(0) = x_{n-1} \ \ x_i \in \mathbb{R}, \tag{6.2}$$

betrachtet. Wobei $x_i \in \mathbb{R}$, und f und x sind reellwertige Funktionen auf einem geeigneten Definitionsbereich sind. Als Lösungen werden wir daher im Allgemeinen stetig differenzierbare Funktionen suchen. Es wird wichtig sein, dass die ODE von t und x abhängt. Da das Verhalten in der Zeit-Variable t fast singulär sein wird, ist das Umschreiben von (6.1)-(6.2) auf ein autonomes[3] System nicht ratsam.

[1] engl.: successive approximations
[2] engl.: ordinary differential equation, dt.: gewöhnliche Differentialgleichung
[3] Autonom bedeutet, dass die ODE nicht von der Zeit abhängt.

Eine explizite Lösung einer ODE anzugeben, gestaltet sich meist als schwierig. Deshalb hat sich das Hauptaugenmerk in der Geschichte darauf verlagert, Bedingungen zu finden, unter denen eine (eindeutige) Lösung existieren muss. Die klassischen Sätze hierzu sind das Theorem von Picard und Lindelöf und jenes von Peano. Inzwischen wurden viele Verallgemeinerungen bewiesen. Die weitreichendste ist wohl jene von Nagumo [Nag26]. Diese Diplomarbeit schließt an die Verallgemeinerung von [Ath90], sowie von [Con10] an. Wir zeigen, dass in beiden Fällen die Picarditertionen[4] zu der eindeutigen Lösung konvergiert. Das Haupthilfsmittel hierzu ist ein Gronwall-artiges Lemma, genannt im folgenden *Integral Inequality* (=Integral Ungleichung), welches durch Adaption des Beweises in [Ath90] gezeigt wird. Eindeutigkeit folgt damit sofort. Für die Existenz zeigen wir, dass wir eine Folge aus den Picarditerationen konstruieren können, die die Voraussetzungen des Lemmas erfüllt. Dadurch folgt dann, dass eine bestimmte Teilfolge der Picarditerationen gegen eine Lösung konvergiert. Zusammen mit der Eindeutigkeit folgt damit, dass jede Teilfolge gegen eine Lösung konvergiert. Diese Vorgangsweise wurde in [Ath90] entwickelt. Dies wird zweimal gemacht. Einmal in Weiterführung an [Win56], dass zweite mal anschließend an [Con10]. Ein Vorteil bei der Verwendung von Picarditerationen ist, dass man dadurch eine Näherungslösung konstruieren kann, und nicht nur eine reine Existenzaussage erhält.

Überblick über die Diplomarbeit

In Kapitel 2 werden die Sätze von Peano und Picard-Lindelöf dargestellt. Beweise werden nicht gebracht und sind in den angegebenen Quellen zu finden. Wir verweisen auf ein Buch welches auch die Picarditerationen verwendet, da diese auch in dem von uns zu beweisendem Theorem von Bedeutung sind. Außerdem wird ein Haufen an Beispielen gezeigt. In Kapitel 3 werden die letztgenannten Verallgemeinerungen ausführlicher behandelt. Die folgenden zwei Kapitel sind dann der Kern der Diplomarbeit. Der lange Text am Anfang ist die englische Übersetzung dieses Textes hier.

Ich möchte mich herzlich bei Prof. Constantin bedanken.

[4] engl.: successive approximations

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[5] University entrance exam